Mathematics Year 1, Calculus and Applications I

Portfolio Marks Assessment 1

In this short project you will carry out discretisations to approximate derivatives of functions and then use these to solve numerically some simple differential equations.

You can use any programming language you like, but I would suggest Python or Matlab.

Please produce a typed report (e.g. Latex) that you can submit electronically.

1 Approximating derivatives

We want to approximate the derivative dy/dx or $f'(x)$ for a given function $y = f(x)$. To fix matters, consider the function $y := f(x) = x^3$. The derivative at $x = 1$ is $f'(1) = 3$. We know from class that $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and so $f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = 3$. Consider the following approximations:

(i) $f'(1) \approx \frac{f(1+h)-f(1)}{h} := D_+f$, $h > 0$.

(ii) $f'(1) \approx \frac{f(1+h)-f(1)}{h} := D_-f$, $h < 0$.

(iii) $f'(1) \approx \frac{f(1+h)-f(1-h)}{2h} := Df$, $h > 0$.

The objective is to evaluate how accurate these formulas are compared to the exact value $f'(1) = 3$ (we pick a known function so that the exact value is known; this way we can evaluate the error).

1. In each of the three cases (i)-(iii) take $h = 1/2^n$ for $n = 1, 2, \ldots, 10$ and calculate D_+f, D_-f, Df. Now find the error between computed and exact values, i.e. $\mathcal{E}_1 := |D_+f - f'(1)|$, $\mathcal{E}_2 := |D_-f - f'(1)|$ and $\mathcal{E}_3 := |Df - f'(1)|$. You should have 10 values for each quantity.

2. Now do the following: Produce a plot of $\mathcal{E}_{1,2,3}$ versus h, but instead of a linear plot do a log-log plot, i.e. plot $\log(\mathcal{E}_{1,2,3})$ versus $\log(h)$.

3. Next, make some conclusions from your plots and in particular the slopes of curves represented by your data. What do the results tell you about the accuracy of the schemes (i), (ii) and (iii). Which scheme would you recommend and why?

2 Solving a differential equation numerically

Consider now the problem

$$\frac{dy}{dx} = y, \qquad 0 < x \le 1, \quad y(0) = 1.$$

The solution is well known to you, $y = \exp(x)$. [Confirm this.] Next, define a discretisation of the interval $[0, 1]$ as follows. Take an integer $N > 0$ that measures the number of grid points and define

$$x_k = kh, \qquad h = \frac{1}{N}, \qquad k = 0, 1, 2, \ldots, N, \qquad x_0 = 0.$$

The solution to (1) is approximated by discrete values (y_k, say,to be found) at the grid points $k = 1, \ldots, N$.

1. Use scheme (i) from above, i.e.

$$\frac{dy}{dx}(x_k) \approx \frac{y(x_k) - y(x_{k-1})}{h},$$

to show that the approximation of (1) gives

$$y_k = (1+h)y_{k-1}.$$

2. Write a simple code (this is just a few lines!) to find y_k, $1 \leq k \leq N$, as N varies. Tabulate values of the error $|y_N - y(1)|$ for $N = 2^n, n = 1, \ldots, 10$ and produce log-log plots as above. What can you conclude about the accuracy of the approximation?

3. Show that equation (1) gives $y_k = (1+h)^k y_0$. Hence show that

$$\lim_{N\to\infty} y_N = \mathrm{e}.$$