

Delocalization of Eigenvectors in Random Matrices

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ABSTRACT

In this thesis, we investigate the delocalization properties of eigenvectors in random matrices with a particular focus on the non-mean field model of the random band matrix. We provide an overview of recent advancements that employ the stochastic flow method, placing them in the context of universality classes and Anderson localization. The approach enables rigorous bounds on eigenvector statistics using resolvent identities and diagrammatic perturbation methods. The analysis includes detailed estimates on the drift terms in the dynamical T -equation and applications of martingale theory to the control of the quadratic variation. The final section explores the extension of these results beyond Gaussian distributions using the Lindeberg exchange strategy and provides a Five Moment Theorem that addresses the universality of delocalization behavior.

Acknowledgments

В ПАМЕТ НА БАБА РУМИ

Contents

1	Introduction	1
2	Prerequisites	5
2.1	Resolvent Identities	6
3	Flow Method	8
4	Resolvent Estimates	11
4.1	Off-diagonal G_{xy} estimate	11
4.2	Diagonal $ G_{xx} - m(z) $ estimate	13
5	Bounds on Flow Terms	16
5.1	Estimates for term $\mathcal{E}_t^D(z)_{ab}$	17
5.2	Estimation of $\mathcal{E}_t^M(z)$ term	31
6	Non-Gaussian case	36
6.1	Lindeberg exchange strategy	36
6.2	Four Moment Theorem for Green's function	37
6.3	Application	37
6.4	Old notes	39
6.5	Replacement	41

1 Introduction

The topic of random matrix theory (RMT) was originally motivated by the study of complex quantum systems, described by a Hamiltonian - the operator that models the interactions between constituent particles. Even when all couplings are known, the daunting and often impossible task of diagonalizing the operator to obtain its spectral decomposition prevents calculating its exact energy levels. To tackle this, Eugene Wigner introduced in his seminal 1957 paper [1] the phenomenological model of a random matrix with independent and identically distributed (i.i.d) entries. His goal was to explain the empirical observation that the spectral gaps of large nuclei follow the same statistics, regardless of the material.

The core idea was to ignore all physical details of the system except for the constraint on its symmetry type, defined by the presence of time-reversal and rotational symmetry, or the lack thereof. The key result, now known as the Wigner semicircle law, states that the eigenvalue density of $N \times N$ self-adjoint random matrices with independent entries $H_{ij} \sim [0, 1/N]$, up to the symmetry constraint $H_{ij} = \overline{H_{ji}}$, is given by $\varrho_{SC}(x) = \frac{1}{2\pi} \sqrt{(4 - x^2)_+}$ [2]. What Wigner proved is a type of law of large numbers that holds in the limit $N \rightarrow \infty$ and is independent of the distribution of individual entries. The prediction that the spacing of individual eigenvalues $\lambda_1(H) \leq \dots \leq \lambda_n(H)$ follows universal gap statistics was later refined into the Wigner-Dyson-Mehta (WDM) universality conjecture [3], which asserts the latter hypothesis is true for *Wigner matrices* [4]. This RMT ensemble defines a general class of mean-field integrable models and is part of the active effort of understanding the different universality classes.

Another subject of great importance is the *Poisson universality class*, closely related to the phenomenon of localized eigenstates. While Wigner's original theory had no mention of eigenvectors [12], their properties are essential in studying the behavior of more general random operators. One historically relevant example is the Anderson localization model, introduced by Philip W. Anderson in 1958, which explores how disorder affects quantum transport [6]. The author considered the behavior of a quantum particle moving on a lattice \mathbb{Z}^d with random on-site energies, represented by the random Schrödinger operator $H_{RS} = -\Delta + \lambda V$, where Δ is the discrete Laplacian and λV is a random diagonal perturbation [7]. In his work, Anderson posited that there exists a *mobility edge* that separates a transition between localized and delocalized eigenvector statistics.

Localization for large values of λ was first rigorously demonstrated using multiscale analysis [8] with alternative approaches, such as the fractional moment method [9], later ensuing. The conjectured delocalization for small λ is supported by extensive numerical evidence, yet a rigorous proof has remained out of reach [11]. The primary reason for the scarcity of rigorous results has been in the

general lack of exactly solvable models of sufficient generality. [20]. To address these gaps, it is crucial to explore systems that interpolate between localized and delocalized regimes while preserving analytical tractability. One such ensemble, supported by extensive numerical evidence, provides a promising case study for establishing the coexistence of localized and delocalized regimes. Two examples that we will consider more in depth in this paper are the non-invariant Gaussian random matrix ensemble and the Random Band Matrix model (RBM).

Non-Invariant Gaussian Model

The non-invariant Gaussian random matrix ensemble, called also the critical random matrix ensemble (CRMT) [17], is defined for Hermitian matrices H with independent Gaussian-distributed off-diagonal elements [20]:

$$\langle H_{nm} \rangle = 0, \quad \langle |H_{nm}|^2 \rangle = \begin{cases} \beta^{-1}, & n = m \\ \frac{1}{2} \left[1 + \left(\frac{n-m}{b} \right)^2 \right]^{-1}, & n \neq m \end{cases},$$

where $\beta = 1, 2, 4$ corresponds to the orthogonal (GOE), unitary (GUE), and symplectic (GSE) symmetry classes, respectively, and $b > 0$ is a tunable parameter. The CRMT emerged in the study of the 3D Anderson model as the critical value instantiation of a Power-law RBM (defined below). By mapping the system onto a nonlinear σ -model with nonlocal interaction and using renormalization group (RG) methods, the authors found that for $a = 1$, the model reaches a critical point with multifractal eigenstate behavior, and the spectral statistics exhibit an intermediate regime. The critical nature of CRMT is encoded in the decay $|n - m|^{-2}$, reminiscent of the Anderson model's localization in coordinate space. The parameter b influences the spectrum of fractal dimensions $d_n(b)$, governed by:

$$d_2 = \begin{cases} 1 - c_\beta B^{-1}, & B \gg 1 \\ c_\beta B, & B \ll 1 \end{cases}$$

where $B = b\pi\beta/2$, and c_β is a constant specific to the symmetry class. Notably, this leads to a duality relationship $d_2(B) + d_2(B^{-1}) = 1$ that has been numerically verified with high precision [20]. The level spacing distribution $P(s)$ of CRMT exhibits hybrid behavior, combining a Poisson tail with Wigner-Dyson statistics in the bulk. While this model effectively captures key aspects of multifractality and the coexistence of localized and delocalized phases, it lacks the rigor necessary to fully characterize the critical behaviors and universal properties of these transitions. Particularly helpful will be the aforementioned fact that the CRMT is a critical value instantiation of an random band matrix with power-law entires. Given the rich and active topic of RBMs, it will serve as our main object of study.

Random Band Matrices (RBM)

The Random band matrix model is of great interest by itself because it serves as an interpolation between Wigner matrices and the random Schrödinger operator H_{RS} [2]. A RBM $(H_{xy})_{x,y \in \Gamma}$, with centered entries independent up to symmetry $H_{ij} = \overline{H_{ji}}$, can represent a d -dimensional quantum system on a graph $\Gamma = \llbracket 1, N \rrbracket^d$ with the effective distance being of order $W < \frac{N}{2}$, defined to the band width of the model [12, 28]. What this means is that for $|i - j|_N = \min([x - y]_N, [y - x]_N) > W$ we have $H_{ij} = 0$. It is standard to also normalize the covariance matrix $S_{xy} = \mathbb{E}|H_{xy}|^2$, s.t. $\sum_x S_{xy} = 1$ for any $x \in \Gamma$. The model is conjectured to exhibit both localization for $W \ll W_c$ and delocalization for $W \gg W_c$ w.r.t a critical transition band W_c that depends on the band width W and the dimension d [12]:

$$W_c = \begin{cases} \sqrt{N} & \text{for } d = 1 \\ \sqrt{\log N} & \text{for } d = 2 \\ O(1) & \text{for } d \geq 3 \end{cases}$$

The Anderson model and the RBM are expected to have the same properties when $\lambda \approx \frac{1}{W}$ [12], an observation supported with extensive numerical evidence [22, 23]. For localization, the current best bound is up to $W \ll N^{1/4}$ [24]. On the other side of the transition, there is a long line of work of iterative improvements.

Delocalization and quantum diffusion, explained in detailed below, have been rigorously established for the case of $d \geq 7$ [14, 13, 15, 16] using diagrammatic representations. The first two papers also concluded a strong form of delocalization and GUE statistics for $W \gg N^{3/4}$ [14, 13] in the one-dimensional case. More recently, Dubova and Yang improved on the latter bound for $1d$ Gaussian random band matrices, assuming $W \gg N^{8/11}$ [11], by using the flow method [26, 32]. Sections 2 through 5 are dedicated exclusively to their work. Their contribution of a controlled truncation of observable dynamics is what lead to the conclusive work by Yau and Yin, establishing delocalization for $W \geq N^{1/2+\epsilon}$ in the one-dimensional Gaussian case [18]. The continued success of this approach has allowed for proving delocalization in two dimension when $W \geq N^\epsilon$ [19]. The latter sequence of notable results is the reason for dedicating the bulk of this thesis to the detailed examination of the tools utilized by the stochastic flow method. Furthermore, given the underlying Gaussian assumption in all previous work, we have determined that the question of the non-Gaussian case is a highly motivated one. As such, the remainder of our work (Section 6) will be focused on relaxing the Gaussian assumption by proving a Five Moment Matching theorem. With this said, we can start with our model of interest:

The Model

The matrix model that will be subject to our examination throughout this paper is defined as follows: identify \mathbb{Z}_n with $\Gamma = \{1, \dots, N\}$ and equip it with the aforementioned periodic distance $|i - j|_N$. Now, for a symmetric and compactly supported probability density function (PDF) on \mathbb{R} , define the doubly stochastic matrix as:

$$S_{xy} := Z_{N,W}^{-1} f\left(\frac{|x - y|_N}{W}\right),$$

where $Z_{N,W} \asymp W$ is a normalizing constant bounded deterministically above and below by W . A technical, but necessary assumption is for S to admit a matrix square root $S^{1/2}$, satisfying the same properties for a similarly symmetric and compactly supported PDF f' .

Delocalization and Quantum Diffusion

2 Prerequisites

The main tool used to study both Anderson localization and random matrix statistics is the *Green function* or *resolvent*. For a random matrix H , it is defined as:

$$G(z) := (H - zI_N)^{-1} = (H - z)^{-1},$$

where $z \in \mathbb{C} \setminus \{\lambda_1, \dots, \lambda_N\}$ is the spectral parameter ranging over the complement of the eigenvalues $\lambda_1(H) \leq \dots \leq \lambda_N(H)$ of H . Observe that for any eigenvector \mathbf{u}_α we have:

$$\begin{aligned} H\mathbf{u}_\alpha &= \lambda_\alpha \mathbf{u}_\alpha \Leftrightarrow (H - z)\mathbf{u}_\alpha = (\lambda_\alpha - z)\mathbf{u}_\alpha \\ \Rightarrow G(z)\mathbf{u}_\alpha &= (H - z)^{-1}\mathbf{u}_\alpha = \frac{1}{\lambda_\alpha - z} \frac{(\lambda_\alpha - z)\mathbf{u}_\alpha}{H - z} = \frac{1}{\lambda_\alpha - z} \mathbf{u}_\alpha \end{aligned} \quad (1)$$

This is equivalent to \mathbf{u}_α being eigenvectors for the resolvent with $\frac{1}{\lambda_\alpha - z}$ as the eigenvalues. This also means that we have by spectral decomposition

$$G(z) = \sum_{i=1}^N \frac{\mathbf{u}_i \mathbf{u}_i^*}{\lambda_i - z}.$$

Now, recall that their empirical measure for H is defined as [2]:

$$\varrho_N(dx) := \frac{1}{N} \sum_{j=1}^N \delta(x - \lambda_j) dx,$$

Let $m(z) = m_N(z) = \int_{\mathbb{R}} \frac{\varrho_N(x)}{x - z} dx$ be its Stieltjes transform for $z = E + i\eta$, where $E \in \mathbb{R}$ and $\eta > 0$. By the relationship we found above, we have the following identity, which follows from the definition of the Dirac delta function:

$$m(z) = \int_{\mathbb{R}} \frac{\varrho_N(x)}{x - z} dx = \int_{\mathbb{R}} \frac{1}{N} \frac{\sum_{j=1}^N \delta(x - \lambda_j) dx}{x - z} dx = \frac{1}{N} \sum_{j=1}^N \int_{\mathbb{R}} \frac{\delta(x - \lambda_j)}{x - z} dx = \frac{1}{N} \sum_{j=1}^N \frac{1}{\lambda_j - z} = \frac{1}{N} \text{Tr}(G(z))$$

From the moments method [2] we know that the Stieltjes transform $m_N(z)$ of the empirical distribution converges in probability to the Stieltjes transform $m_{SC} = \frac{1}{2}(-z + \sqrt{z^2 - 4})$ of the semicircle law as $N \rightarrow \infty$. This means that in the limit, with probability one the self-consistent equation holds:

$$m(z) = -\frac{1}{z + m(z)} \Leftrightarrow m(z) + \frac{1}{m(z)} + z = 0 \quad (2)$$

Observe that:

$$m(z)(m(z) + z) = \frac{1}{4}(-z + \sqrt{z^2 - 4})(z + \sqrt{z^2 - 4}) = \frac{1}{4}(-z^2 + z^2 - 4) = -1$$

$$\Rightarrow |m(z)| = \frac{1}{|m(z) + z|^{-1}}$$

Now, take the imaginary part of (2):

$$\text{Im}(m(z)) + \text{Im}\left(\frac{1}{m(z)}\right) + \eta = 0 = \text{Im}(m(z)) + \frac{\text{Im}}{|m(z)|^2} + \eta$$

$$\Rightarrow \text{Im}(m(z)) \left(1 - \frac{1}{|m(z)|^2}\right) = -\eta$$

And since $\text{Im}(m(z)) > 0$ and $\eta > 0$, we must have $1 - \frac{1}{|m(z)|^2} < 0$, i.e $|m(z)| < 1$.

$$\Rightarrow \text{Im } m \asymp 1 - |m|^2 \asymp \eta$$

Hence, our Θ term simplify

$$\Theta \sim \frac{S}{1 - |m(z)|^2 + |m(z)|^2(S - \text{Id})} \sim \frac{S}{\alpha\eta + (S - \text{Id})}$$

2.1 Resolvent Identities

We can define the augmented minor Green function for any index α as $G(z)^{(\alpha)} = (H^{(\alpha)} - w_t)^{-1}$, where:

$$\left(H^{(\alpha)}\right) := H_{ab} \cdot \mathbf{1}(a \neq \alpha) \cdot \mathbf{1}(b \neq \alpha),$$

so the augmented matrix is still $N \times N$ with the α -row and -column set to zero [2]. As a consequence:

$$G_{xy}^{(\alpha)} = \begin{cases} (-z)^{-1} & x = y = i \\ 0 & x = \alpha \text{ XOR } y = \alpha \\ G_{xy}^{[\alpha]} = (H^{[\alpha]} - z)_{xy}^{-1} & x \neq \alpha, y \neq \alpha \end{cases} \quad (3)$$

Before we derive the estimate, let us first establish several identities:

Matrix Identities

Let A and B be arbitrary matrices. Provided that the inverses exist, we can derive the following identities:

$$\begin{aligned} \frac{1}{A} &= A^{-1} = A^{-1}(A+B)(A+B)^{-1} = (A+B)^{-1} + A^{-1}B(A+B)^{-1} = \frac{1}{A+B} - \frac{1}{A}B\frac{1}{A+B} \\ &\Rightarrow \frac{1}{A+B} = \frac{1}{A} - \frac{1}{A}B\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A+B}B\frac{1}{A}, \end{aligned}$$

where the last equality follows from simply swapping the order $A^{-1} = A^{-1}(A+B)^{-1}(A+B)$.

Using this, we can now set $A = H_t^{(\alpha)} - z$ and $B = H_t - H_t^{(\alpha)}$, hence by substituting above, we have:

$$\frac{1}{H_t - z} = G_t(z) = \frac{1}{H_t^{(\alpha)} - z} - \frac{1}{H_t - z} \left(H_t - H_t^{(\alpha)} \right) \frac{1}{H_t^{(\alpha)} - z} = G_t^{(\alpha)} - G_t(H_t - H_t^{(\alpha)})G_t^{(\alpha)}$$

But observe that by definition of the augmented minor, the xy entry of our identity above simplifies as:

$$G_{xy} = G_{xy}^{(\alpha)} - \sum_{j=1}^N \sum_{k=1}^N G_{xj} (H_{jk} - H_{jk}^{(\alpha)}) G_{ky}^{(\alpha)} = G_{xy}^{(\alpha)} - G_{x\alpha} \sum_{k \neq \alpha} H_{\alpha k} G_{ky}^{(\alpha)} = G_{xy}^{(\alpha)} + \frac{G_{x\alpha} G_{\alpha y}}{G_{\alpha\alpha}}$$

Similarly, we can derive our second resolvent identity, by applying again the same properties (3), with the special case of $\alpha = x$, s.t. we get:

$$G_{xy} = G_{xy}^{(x)} - G_{xx} \sum_{k \neq x} H_{xk} G_{ky}^{(x)} = -G_{xx} \sum_{k \neq x} H_{xk} G_{ky}^{(x)} \quad x \neq y$$

Lastly, let us derive the derivatives of the resolvent w.r.t. to the matrix entry H_{ij} using the limit definition. Let us define the perturbed matrix $H' = H(H_{ij} \mapsto H_{ij} + \epsilon)$ at entry H_{ij} , where $G' = (H' - z)^{-1}$ and let $\epsilon_{ij} = \mathbf{0}(0_{ij} \mapsto \epsilon)$ be the perturbed zero matrix, which we can write as $\mathbf{1}_{ij}$ when $\epsilon = 1$. By our matrix identities, we have:

$$\begin{aligned} \partial_{H_{ij}} G(z)_{xy} &= \lim_{\epsilon \rightarrow 0} \frac{G'_{xy} - G_{xy}}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{[G'(H - H')G]_{xy}}{\epsilon} = -\lim_{\epsilon \rightarrow 0} \frac{[G'\epsilon_{ij}G]_{xy}}{\epsilon} = -[G'\mathbf{1}_{ij}G]_{xy} = -G_{xi}G_{jy} \\ \partial_{H_{ij}} \overline{G(z)}_{xy} &= \lim_{\epsilon \rightarrow 0} \frac{[\overline{G'}(H^\top - H'^\top)\overline{G}]_{xy}}{\epsilon} = -\lim_{\epsilon \rightarrow 0} \frac{[\overline{G'}\epsilon_{ji}\overline{G}]_{xy}}{\epsilon} = -[\overline{G'}\mathbf{1}_{ji}\overline{G}]_{xy} = -\overline{G_{xj}} \overline{G_{iy}} \end{aligned}$$

3 Flow Method

Consider the Brownian perturbation of the standard deviation matrix defined as $H(t)_{xy} = \sqrt{S_{xy}}b(t)_{xy}$, where $b(t) \sim \mathcal{N}(0, t)$ represents the Brownian motion. For this system, the matrix resolvent is augmented as $G(t) = (H(t) - w(t))^{-1}$, where $w(t) = -\frac{1}{m(z)} - tm(z)$. In order to study this system and its transition from $t = 0$ with $G(0) = m(z)$, to $t = 1$ with $G(1) = G(z)$, we will employ stochastic calculus. For this purpose, observe that our differentials are $dH(t)_{xy} = \sqrt{S_{xy}}db_{xy}(t)$ and $\partial_t w(t) = -m(z)$. Recall the Itô equation, namely, the statement that a function $f(\vec{b}(t), t)$, dependent on a Brownian random walk $\vec{b}(t) = (b(t)_1, \dots, b(t)_d)$ w.r.t to time t , has derivative:

$$df(\vec{b}(t), t) = \frac{\partial f}{\partial t} dt + \sum_{i=1}^d \frac{\partial f}{\partial b_i} db_i(t) + \frac{1}{2} \sum_{i=1}^d \frac{\partial^2 f}{\partial b_i^2} dt$$

In our case, by taking the function to be $G(t)$ and using restriction of a symmetric matrix and the latter differentials, we get the following expression:

$$dG(t)_{xy} = \frac{\partial G(t)_{xy}}{\partial t} dt + \sqrt{S_{ij}} \sum_{i \leq j} \frac{\partial G(t)_{xy}}{\partial b(t)_{ij}} db(t)_{ij} + \frac{S_{ij}}{2} \sum_{i \leq j} \frac{\partial^2 G(t)_{xy}}{\partial b(t)_{ij}^2} (db(t)_{ij})^2$$

To calculate the individual terms, we will use the definition of derivative:

$$\frac{\partial G(t)_{xy}}{\partial t} = \lim_{\epsilon \rightarrow 0} \frac{(H(t) - w(t + \epsilon))_{xy}^{-1} - (H(t) - w(t))_{xy}^{-1}}{\epsilon},$$

whereas the following matrix identity: $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$ can help us approximate for small perturbations. Namely, for $\Delta w(t) = w(t + \epsilon) - w(t)$, we have:

$$\begin{aligned} & [(H(t) - w(t + \epsilon))^{-1}]_{xy} - [(H(t) - w(t))^{-1}]_{xy} = [G(t)\Delta w(t)G(t)]_{xy} \\ \Rightarrow \frac{\partial G(t)_{xy}}{\partial t} &= \lim_{\epsilon \rightarrow 0} \frac{[G(t)\Delta w(t)G(t)]_{xy}}{\epsilon} = \sum_j G(t)_{xj} \frac{dw(t)}{dt} G(t)_{jy} = -m(z) \sum_j G(t)_{xj} G(t)_{jy} \end{aligned}$$

Similarly, we can carry out the calculation for the second term, using the independence of entries:

$$\begin{aligned} \frac{\partial G(t)_{xy}}{\partial b(t)_{ij}} &= \lim_{\epsilon \rightarrow 0} \frac{(\sqrt{S_{xy}}b(t + \epsilon)_{ij} + w(t))_{xy}^{-1} - (\sqrt{S_{xy}}b(t)_{ij} + w(t))_{xy}^{-1}}{\epsilon} = \\ &= \lim_{\epsilon \rightarrow 0} \frac{[G(t)\sqrt{S}(b(t + \epsilon) - b(t))G(t)]_{xy}}{\epsilon} = \sum_{\alpha, \beta} G(t)_{x\alpha} \sqrt{S_{\alpha\beta}} db(t)_{\alpha\beta} G(t)_{\beta y} \end{aligned}$$

Lastly, carrying out the equivalent summation and using the fact that $(d(b)(t))^2 = dt$, we get:

$$\begin{aligned} dG(t)_{xy} &= -m(z) \sum_j G_{xj} G_{jy} dt + \sum_{\alpha, \beta} G_{x, \alpha} \sqrt{S_{\alpha, \beta}} db_{\alpha, \beta}(t) G_{\beta, t} + \sum_j G_{xj} \left(\sum_y S_{jy} G_{yy} \right) G_{jy} dt = \\ &= \sum_{\alpha, \beta} G_{x\alpha} \sqrt{S_{\alpha, \beta}} db_{\beta}(t) G_{\beta y} + \sum_j G_{xj} (S_{jy} (G_{yy} - m(z))) G_{jy} dt = dI(t) + dII(t)dt \end{aligned}$$

This provides us with the decomposition of a stochastic term $dI(t)$ and a deterministic term $dII(t)dt$.

Dynamical T- equation

Now, define $F_{xy} = |G(t)_{xy}|^2 = G(t)_{xy} \overline{G(t)_{xy}}$. Applying Itô's formula to $F(t)_{xy}$, we have:

$$dF(t)_{xy} = G(t)_{xy} d\overline{G(t)_{xy}} + \overline{G(t)_{xy}} dG(t)_{xy} + dG(t)_{xy} d\overline{G(t)_{xy}}.$$

$$d|G_{xy}|^2 = G_{xy} d\overline{G_{xy}} + \overline{G_{xy}} dG_{xy} + d[G_{xy}, \overline{G_{xy}}]$$

$$d|G_{xy}|^2 = G_{xy} d\overline{G_{xy}} + \overline{G_{xy}} dG_{xy} + d \left[\sum_{\alpha, \beta} G_{x\alpha} dH_{\alpha\beta} \overline{G_{xy}} \right]$$

$$d|G_{xy}|^2 = G_{xy} d\overline{G_{xy}} + \overline{G_{xy}} dG_{xy} + \sum_{\alpha, \beta, \gamma, \delta} G_{x\alpha} G_{\beta\gamma} \overline{G_{xy}} d\overline{G_{\delta\gamma}}$$

To get the T -equation, define it as:

$$T(t) = S^{1/2} F(t) S^{1/2} = \sum_{x,y} S_{xy} F(t)_{xy} = \sum_{x,y} S_{xy} |G(t)_{xy}|^2,$$

On the other hand, by substituting $dG(t)_{xy} = dI(t)_{xy} + dII(t)_{xy}dt$ and its conjugate, and noting that $dI(t)_{xy} d\overline{I(t)_{xy}}$ contributes to the quadratic variation, we obtain:

$$dF(t)_{xy} = dM(t)_{xy} + \Omega(t)_{xy} dt, \quad \text{where}$$

$$dM(t)_{xy} = \overline{G(t)_{xy}} dI(t)_{xy} + G(t)_{xy} d\overline{I(t)_{xy}}.$$

$$\begin{aligned} \Omega(t)_{xy} &= G_{xy} \frac{d\overline{II(t)_{xy}}}{dt} + \overline{G_{xy}} \frac{dII(t)_{xy}}{dt} + \frac{d[G_{xy}, \overline{G_{xy}}]}{dt} = G_{xy}(t) \left(\overline{\sum_j G_{xj}(t) S_{jy} (G_{yy}(t) - m(z)) G_{jy}(t)} \right) + \\ &+ \overline{G_{xy}(t)} \left(\sum_j G_{xj}(t) S_{jy} (G_{yy}(t) - m(z)) G_{jy}(t) \right) + \sum_{\alpha, \beta} |G_{x\alpha}(t)|^2 S_{\alpha, \beta} |G_{x\beta}(t)|^2 \end{aligned}$$

$$\Rightarrow dT = T^2 dt + S^{1/2} dM_t(z) S^{1/2} + S^{1/2} \Omega_t(z) S^{1/2} dt \quad (4)$$

Now, define $\Theta(t) = \Theta_t$ and $\epsilon_t(z)$ as:

$$\Theta(t) = \frac{|m(z)|^2}{1 - t|m(z)|^2 S} \Leftrightarrow d\Theta_t = \Theta_t^2 dt$$

$$\mathcal{E}_t(z) : T_t(z) - \Theta_t$$

$$\begin{aligned} \Rightarrow d\mathcal{E}_t(z) &= T_t(z)^2 dt + \Theta_t^2 dt - S^{1/2} dM_t(z) S^{1/2} + S^{1/2} \Omega_t(z) S^{1/2} dt = \\ &= \{\Theta_t \mathcal{E}_t(z) + \mathcal{E}_t(z) \Theta_t\} dt + \mathcal{E}_t^2(z) dt - S^{1/2} dM_t(z) S^{1/2} + S^{1/2} \Omega_t(z) S^{1/2} dt \end{aligned}$$

by observing that $T_t(z)^2 - \Theta_t^2 = \Theta_t \mathcal{E}_t(z) + \mathcal{E}_t(z) \Theta_t + \mathcal{E}_t(z)^2$. Integrating $\mathcal{E}_t(z)$ over s :

$$\mathcal{E}_t(z) = \int_0^t \{\text{Id} + (t-s)\Theta_t\} \left\{ \mathcal{E}^2(s) ds + S^{1/2} dM(s) S^{1/2} + S^{1/2} \Omega(s) S^{1/2} \right\} \{\text{Id} + (t-s)\Theta_t\} ds,$$

whereas we can split the terms in three separate components, each of which we will individually bound in the following section:

$$\begin{aligned} \mathcal{E}_t^D(z) &= \int_0^t \{\text{Id} + (t-s)\Theta_t\} S^{1/2} \Omega(s) S^{1/2} \{\text{Id} + (t-s)\Theta_t\} ds, \\ \mathcal{E}_t^M(z) &= \int_0^t \{\text{Id} + (t-s)\Theta_t\} S^{1/2} dM(s) S^{1/2} + \{\text{Id} + (t-s)\Theta_t\} ds, \\ \mathcal{E}_t^S(z) &= \int_0^t \{\text{Id} + (t-s)\Theta_t\} \mathcal{E}_s^2(z) \{\text{Id} + (t-s)\Theta_t\} ds, \end{aligned}$$

4 Resolvent Estimates

Throughout this section we will state or derive all of the inequalities and stochastic bounds we will need for control of the resolvent. Starting with a particularly important large deviation estimate (Theorem 7.7 [2]), which we will need as a Lemma for the estimates:

Lemma 1: Large Deviation estimates

Let $(X_i^{(N)})$ and $(Y_i^{(N)})$ are independent families of random variables and $(a_{ij}^{(N)})$ and $(b_{ij}^{(N)})$ be deterministic. Suppose all entrires $X_i^{(N)}$ and $Y_i^{(N)}$ are independent and satisfy:

$$\mathcal{E}X = 0, \quad \mathcal{E}|X|^2 = 1, \quad \|X\|_p := (\mathcal{E}|X|^p)^{1/p} \leq \mu_p$$

for all $p \in \mathbb{N}$ and some constants μ_p . Then, we have the bounds:

$$\sum_i b_i X_i \prec \left(\sum_i |b_i|^2 \right)^{1/2} \quad (1.1)$$

$$\sum_{i,j} a_{ij} X_i Y_j \prec \left(\sum_{i,j} |a_{ij}|^2 \right)^{1/2} \quad (1.2)$$

$$\sum_{i \neq j} a_{ij} X_i Y_j \prec \left(\sum_{i \neq j} |a_{ij}|^2 \right)^{1/2} \quad (1.3)$$

4.1 Off-diagonal G_{xy} estimate

There will be two important observations used throughout the following estimations. First, per the covariance normalization, of the generalized Wigner matrix [2], we have $S_{ab} \lesssim N^{-1}$. Secondly, $\sum_y |B_{xy}|$, $\sum_y |S_{xy}|$ have size $O(1)$ per Lemma 25 [11]. Now, having established that, we can use estimate (1.3) from the Lemma above, along with resolvent identity (3), s.t. we get:

$$\begin{aligned} |G_{xy}| &= \left| G_{xx} \sum_{w \neq x} H_{xw} G_{wy}^{(x)} \right| = |G_{xx}| \left| \sum_{w \neq x} \sqrt{S_{xy}} B(t)_{xy} G_{wy}^{(x)} \right| \prec |G_{xx}| \left(\sum_{w \neq x} S_{xw} |G_{wy}^{(x)}|^2 \right)^{1/2} \quad (x \neq y) \\ &\Rightarrow |G_{xy}|^2 \prec |G_{xx}|^2 \sum_{w \neq x} S_{xw} |G_{wy}^{(x)}|^2 \quad x \neq y \end{aligned} \quad (3.1)$$

On the other hand, we can rewrite identity (3) as

$$G_{wy}^{(x)} = G_{wy} - \frac{G_{wx}G_{xy}}{G_{xx}},$$

s.t. we can bound the latter result using the Cauchy-Schwarz inequality:

$$\begin{aligned} |G_{xy}|^2 &\prec |G_{xx}|^2 \sum_{w \neq x} S_{xw} |G_{wy}^{(x)}|^2 \lesssim |G_{xx}|^2 \sum_{w \neq x} S_{xw} |G_{wy}|^2 + |G_{xx}|^2 \sum_{w \neq x} S_{xw} \frac{|G_{wx}G_{xy}|^2}{|G_{xx}|^2} \\ &\Rightarrow |G_{xy}|^2 \lesssim |G_{xx}|^2 \sum_{w \neq x} S_{xy}^2 |G_{wy}|^2 + |G_{xy}|^2 \sum_{w \neq x} S_{xw} |G_{wx}|^2 \end{aligned} \quad (4)$$

$$|G_{xy}|^2 \lesssim \sum_{w \neq x} S_{xw} |G_{wy}|^2 + N^{-2\delta} |G_{xy}|^2, \quad (4.1)$$

where (4.1) follows from (4) under assumptions $|G_{xx}| \prec 1$ and $\sup_{w \neq x} |G_{wx}|^2 \prec N^{-\delta}$. By the same logic as above, using G^* instead of G for any $w \neq y$, we have:

$$\begin{aligned} |G_{wy}|^2 &= |G_{yw}^*|^2 \prec |G_{yy}^*|^2 \sum_{u \neq y} S_{yu} |G_{uw}^{*(y)}|^2 \lesssim \\ &\lesssim |G_{yy}^*|^2 \sum_{u \neq y} S_{yu} |G_{uw}^*|^2 + |G_{yw}^*|^2 \sum_{u \neq y} S_{yu} |G_{uy}^*|^2 = \\ &= |G_{yy}|^2 \sum_{u \neq y} S_{yu} |G_{wu}|^2 + |G_{wy}|^2 \sum_{u \neq y} S_{yu} |G_{yu}|^2 \end{aligned}$$

Let us now multiply the latter result by S_{xw} and sum over $w \neq x$, isolating the term $w = y$:

$$\sum_{w \neq x} S_{xw} |G_{wy}|^2 \prec |G_{yy}|^2 S_{xy} + \sum_{w \neq x, y} S_{xw} |G_{wy}|^2$$

Now, by using (4), we have:

$$\begin{aligned} \sum_{w \neq x} S_{xw} |G_{wy}|^2 &\lesssim S_{xy} |G_{yy}|^2 + \sum_{w \neq z} S_{xw} |G_{yy}|^2 \sum_{u \neq y} |G_{wu}|^2 S_{uy} + \sum_{w \neq x} |G_{wy}|^2 \sum_{u \neq y} |G_{yu}|^2 S_{uy} \prec \\ &\prec S_{xy} |G_{yy}|^2 + \sum_{w, u} S_{xw} |G_{wu}|^2 S_{uy} + \sum_{w \neq x} |G_{wy}|^2 \sum_{u \neq y} |G_{yu}|^2 S_{uy} \prec \end{aligned}$$

$$\begin{aligned}
& \prec S_{xy} + \sum_{w,u} S_{xw} |G_{wu}|^2 S_{uy} + N^{-2\delta} \sum_{w \neq x} S_{xw} |G_{wy}|^2 = \\
& = S_{xy} + \left(S^{1/2} T S^{1/2} \right)_{xy} + N^{-2\delta} \sum_{w \neq x} S_{xw} |G_{wy}|^2 \\
& \Rightarrow \sum_{w \neq x} S_{xw} |G_{wy}|^2 \prec S_{xy} + (S^{1/2} T S^{1/2})_{xy} \tag{5}
\end{aligned}$$

Hence, by putting together (4) and (5), we get:

$$|G_{xy}|^2 \prec S_{xy} + (S^{1/2} T S^{1/2})_{xy} + N^{-2\delta} |G_{xy}|^2 \Rightarrow \frac{|G_{xy}|^2}{S_{xy} + (S^{1/2} T S^{1/2})_{xy}} \prec 1$$

4.2 Diagonal $|G_{xx} - m(z)|$ estimate

Let us extend the assumption from the previous section for some $\delta? > 0$ as:

$$\max_{a,b} |G_{xy} - m(z) \mathbf{1}_{(x=y)}| = \Psi \leq N^{-\delta}$$

Assume the existence of a family of admissible control parameters Ω_{ij}^2 , s.t $T_{ij} \prec \Omega_{i,j}^2$. Using our earlier estimates, namely (3.1) and (3.2), we have:

$$\begin{aligned}
|G_{xy}|^2 & \prec |G_{xx}|^2 \sum_{w \neq x} S_{xw} |G_{wy}^{(x)}|^2 \prec \sum_{w \neq x} S_{xw} (|G_{wy}|^2 + O_{\prec}(|G_{wx} G_{xy}|^2)) = \\
& = T_{xy} - S_{xx} |G_{xy}|^2 + O_{\prec} \Psi^4 \prec \Omega_{xy}^2 + \Psi^4 \tag{6}
\end{aligned}$$

Using Definition 8.1 [2] for operations $P_i X := \mathcal{E}[X | H^{(i)}]$ and $Q_i = X - P_i X$, we can define the following quantity:

$$Z_x := \sum_{k,l \neq x} Q_x \left(H_{xk} G_{kl}^{(x)} H_{lx} \right) \tag{7}$$

Then for $M := (\max_{i,j} S_{i,j})^{-1} \gtrsim N$, by Lemma 3.8 from [28], we have that:

$$G_{xx} = m + m^2 Z_x + O_{\prec} \left(\Psi^2 + M^{-1/2} \right), \quad Z_i \prec \Psi$$

$$\Rightarrow |G_{xx} - m|^2 \leq |Z_i|^2 + O_{\prec}(\Psi^2 + M^{-1}),$$

using $|m(z)| \leq 1$ from Lemma 6.2 [2]. In order to estimate $|Z_i|^2$, we can rewrite the resolvent identity (3) in a form of an inequality, which holds for $\forall k, \ell$:

$$|G_{k\ell}^{(x)}| \leq |G_{k\ell}| + \frac{|G_{kx}||G_{x\ell}|}{|G_{xx}|}$$

When applied to (6) along with the assumptions $G_{kk}^{(i)} \prec 1$, we get:

$$\Rightarrow \sum_{k, l \neq x, k \neq l} S_{xk} |G_{kl}^{(x)}|^2 S_{lx} \prec \sum_k \Omega_{xk}^2 S_{kx} + \Psi^4 \quad (8)$$

Similarly, applying it to (7) along with (1.1) and (1.2) from the Lemma gives us:

$$|Z_x|^2 \leq \left| \sum_{k \neq x} (|H_{xk}|^2 - S_{xk}) G_{kk}^{(x)} \right|^2 + \left| \sum_{k, \ell \neq x, k \neq \ell} H_{xk} G_{k\ell}^{(x)} H_{\ell x} \right|^2 \quad (9)$$

Hence, by putting (8) and (9) together, we get:

$$\Rightarrow |G_{xx} - m|^2 \prec \sum_k \Omega_{xk}^2 S_{kx} + \Psi^4 + M^{-1} \quad (10)$$

Recall that $|G_{kk}^{(x)}| \prec 1$ and $S_{ab} \lesssim N^{-1}$, hence, we can bound the first term using (1.3) from the Lemma:

$$\left| \sum_{k \neq x} (|H_{xk}|^2 - S_{xk}) G_{kk}^{(x)} \right|^2 \prec \left| \sum_{k \neq x} S_{xk} (|B_{xk}|^2 - 1) \right|^2 \prec \sum_{k \neq x} |S_{xk}|^2 \prec N^{-1}$$

Similarly, we can bound the second term by applying (1.3) from the Lemma:

$$\begin{aligned} \left| \sum_{k, \ell \neq x, k \neq \ell} H_{xk} G_{k\ell}^{(x)} H_{\ell x} \right|^2 &= \left| \sum_{k, \ell \neq x, k \neq \ell} \sqrt{S_{xk}} B_{xk} G_{k\ell}^{(x)} \sqrt{S_{\ell x}} B_{\ell x} \right|^2 \prec \sum_{k, \ell \neq x, k \neq \ell} S_{xk} |G_{k\ell}^{(x)}|^2 S_{lx} \lesssim \\ &\lesssim \sum_{k, \ell \neq x, k \neq \ell} S_{xk} |G_{kl}|^2 S_{lx} + |G_{xx}|^{-2} \sum_{k, \ell \neq x, k \neq \ell} S_{xk} |G_{kx}|^2 |G_{xl}|^2 S_{lx} \lesssim \max_{a,b} |(S^{1/2} T S^{1/2})_{ab}| + \max_{a,b} |S_{a,b}^{1/2}| \end{aligned}$$

$$\Rightarrow \max_x |G_{xx} - m|^2 \prec \max_{a,b} |(S^{1/2} T S^{1/2})_{ab}| + \max_{a,b} |S_{ab}^{1/2}| + \max_{a,b} |G_{ab} - m \mathbf{1}_{a=b}|^4 + N^{-1} \prec$$

$$\prec \max_{a,b} |(S^{1/2}TS^{1/2})_{ab}| + \max_{a,b} |S_{ab}^{1/2}| + \max_{a \neq b} |G_{ab}|^4 + N^{-\delta} \max_x |G_{xx} - m|^2 + N^{-1} \prec$$

$$\max_{a,b} |(S^{1/2}TS^{1/2})_{ab}| + \max_{a,b} |S_{ab}^{1/2}| + N^{-\delta} \max_x |G_{xx} - m|^2 + N^{-1}$$

$$\Rightarrow \max_x |G_{xx} - m|^2 \prec \max_{a,b} |(S^{1/2}TS^{1/2})_{ab}| + \max_{a,b} |S_{ab}^{1/2}| + N^{-1}$$

5 Bounds on Flow Terms

Give backstory [26, 32].

In this section, our goal is to control the three terms of the fluctuations between the T_{xy} and Θ_{xy} matrices. The squared \mathcal{E}_t^S and M -term \mathcal{E}_t^M of the flow SDE can be handled using the resolvent bounds we derived earlier, along with standard martingale calculus. However, the drift term \mathcal{E}_t^D requires extra care. For it, we will follow Dubova and Yang's derivation closely, which employs the graphical perturbation methods developed by Bourgade, Yau and Yin, in their self-energy renormalization paper [30]. More specifically, we will use Definition 17 [11] for the diagrammatic notation of a standard oriented graph:

Diagrammatic notation

- Each vertex will be assigned a label, corresponding to a matrix index $\alpha \in \{1, \dots, N\}$.
- Each blue-colored loop \circlearrowleft , regardless of direction, represents a term $G_s(z)_{\alpha\alpha} - m(z)$;
- A solid blue edge $\alpha \rightarrow \beta$ represents a factor of $G_s(z)_{\alpha\beta}$, whereas a red $\alpha \rightarrow \beta$ one is $\overline{G_s(z)}_{\alpha\beta}$;
- A black wavy edge $\alpha \rightsquigarrow \beta$ represents $S_{\alpha\beta}$, whereas $\alpha \rightsquigarrow \beta$ is $B_{\alpha\beta} = (I - sm(z)^2 S)^{-1}$;
- A double edge $\alpha \equiv \beta$ represents $\{\text{Id} + (t - s)\Theta_t\}S^{1/2}$, which commute as shown before;

Having defined the diagrammatic notation, we can now present the main operation that will allow the bounding of the drift term, namely Lemma 16 [11], which lists the loop expansion (Lemma 3.5 [30]) and the regular vertex expansion (Lemma 3.14 [30]):

Loop Expansion

Consider a differentiable function $f : \mathbb{C}^{N^2} \rightarrow \mathbb{C}$, which represents the remaining subgraph per our diagrammatic notation above. For fixed time $s \in [0, 1]$, we have the following identity of subgraphs, not accounting for the renormalization term:

$$\begin{aligned} (G_s(z))_{vv} - m) f(G_s(z)) &= sm \sum_{\alpha, \beta=1}^N B_{v\alpha} S_{\alpha\beta} ((G_s(z))_{\alpha\alpha} - m) ((G_s(z))_{\beta\beta} - m) f(G_s(z)) \\ &- sm \sum_{\alpha, \beta=1}^N B_{v\alpha} S_{\alpha\beta} (G_s(z))_{\beta\alpha} \partial_{H_{s, \alpha\beta}} f(G_s(z)) = sm \mathcal{G}_{1,s}(z)_{ab} + sm \mathcal{G}_{2,s}(z)_{ab} + sm \mathcal{G}_{3,s}(z)_{ab} + sm \mathcal{G}_{4,s}(z)_{ab}. \end{aligned}$$

Here, $\mathcal{G}_{1,s}(z)_{ab}$ is equivalent to the first term, whereas subgraphs $\mathcal{G}_{2,s}(z)_{ab}$ through $\mathcal{G}_{4,s}(z)_{ab}$ are the result of expanding the partial derivative $\partial_{H_{s, \alpha\beta}} f(G_s(z))$, using the fact that: $\partial_{H_{\beta\alpha}} G_{xy} = -G_{x\beta} G_{\alpha y}$ and $\partial_{H_{\beta\alpha}} \overline{G}_{xy} = -\overline{G}_{x\alpha} \overline{G}_{\beta y}$. Similarly, the regular vertex expansion for a connected pair of edges is:

Regular Vertex Expansion

For a differentiable function $f : \mathbb{C}^{N^2} \rightarrow \mathbb{C}$ and fixed time $s \in [0, 1]$, we have the following identity, not accounting for the renormalization term:

$$\begin{aligned} G_{xu}G_{uy}f(G) &= mB_{uy}G_{xy}f(G) + sm \sum_{\gamma\delta=1}^N B_{u\gamma}S_{\gamma\delta}G_{x\gamma}(G_{\delta\delta} - m)G_{\gamma y}f(G) \\ &+ sm \sum_{\gamma\delta=1}^N B_{u\gamma}S_{\gamma\delta}G_{x\delta}(G_{\gamma\gamma} - m)G_{\delta y}f(G) - sm \sum_{\gamma\beta=1}^N B_{u\gamma}S_{\gamma\delta}G_{x\gamma}G_{\delta y}\partial_{H_{\delta\gamma}}(f(G)) \\ &= m\mathcal{G}_{i0,s} + sm\mathcal{G}_{i1,s} + sm\mathcal{G}_{i2,s} - sm\mathcal{G}_{i3,s}^{(1)} - sm\mathcal{G}_{i3,s}^{(2)} - sm\mathcal{G}_{i3,s}^{(3)}, \end{aligned}$$

where $\mathcal{G}_{i0,s}$, $\mathcal{G}_{i1,s}$ and $\mathcal{G}_{i2,s}$ represent the first three terms, respectively, and $\mathcal{G}_{i3,s}^{(i)}$ are the resulting elements from applying the resolvent derivatives w.r.t. to $H_{\beta\alpha}$. To be clear, the exact number of the latter do not follow from the lemma itself and are rather a feature of our particular diagrammatic structure. The explicit results will be verified when we apply the lemma to each respective term. Now, in order to bound each term from the flow SDE, we will use the following estimates:

Estimates

1. $\sup_x \sum_y |G_{xy}| \lesssim W^{\frac{1}{2}+\epsilon} \eta_s^{-\frac{3}{4}}$ (Result 4.5 [11]).
2. $\max_{x,y} |G_{xy} - \delta_{xy}m(z)| \lesssim W^{-\frac{1}{2}} \eta_s^{-\frac{1}{4}}$ (Result 4.6 [11]).
3. The entries of Θ_t are $O\left(W^{-1}\eta_t^{-\frac{1}{2}}\right)$ per Lemma 23 [11]
4. Per the covariance normalization, $S_{ab} \lesssim W^{-1}$
5. $\sum_y |B_{xy}|, \sum_y |S_{xy}|$ have size $O(1)$ per Lemma 25 [11].
6. $\sum_x [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{ax} \lesssim \eta_t^{-1}\eta_s$ per (3.7) from [11]
7. $(t-s) \lesssim \eta_s$

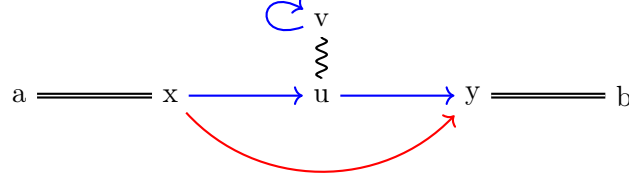
5.1 Estimates for term $\mathcal{E}_t^D(z)_{ab}$

The goal for this subsection is to bound the drift term, which we can expand and rewrite as follows:

$$\begin{aligned} \mathcal{E}_t^D(z)_{ab} &= \int_0^t \{\text{Id} + (t-s)\Theta_t\} S^{\frac{1}{2}} \Omega_s(z) S^{\frac{1}{2}} \{\text{Id} + (t-s)\Theta_t\} \\ &= \int_0^t \{\text{Id} + (t-s)\Theta_t\} \left\{ \overline{G_s(z)}_{xy} G_s(z)_{xu} S_{uv} [G_s(z)_{vv} - m(z)] G_s(z)_{uy} \right\} \{\text{Id} + (t-s)\Theta_t\} + \end{aligned}$$

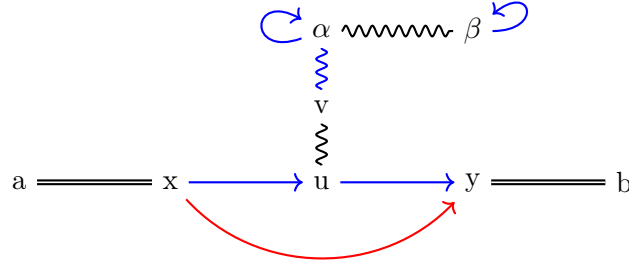
$$\int_0^t \{\text{Id} + (t-s)\Theta_t\} \left\{ G_s(z)_{xy} \overline{G_s(z)}_{xu} S_{uv} \left[\overline{G_s(z)}_{vv} - m(z) \right] \overline{G_s(z)}_{uy} \right\} \{\text{Id} + (t-s)\Theta_t\} =$$

Observe that we can represent the first term with the diagrammatic notation defined earlier, whereas its complex conjugate has the exact same structure, with only the colors of the straight edges being flipped. Since this doesn't change the underlying estimates we have from the previous section, WLOG we can pick the first term and write it out as follows:

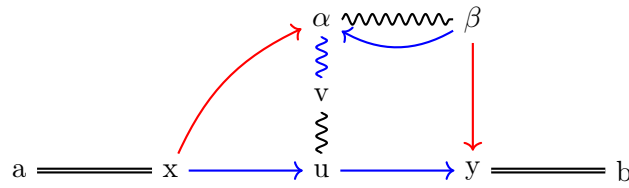


$$\left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_s(z)}_{xy} G_s(z)_{xu} S_{uv} [G_s(z)_{vv} - m(z)] G_s(z)_{uy} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$

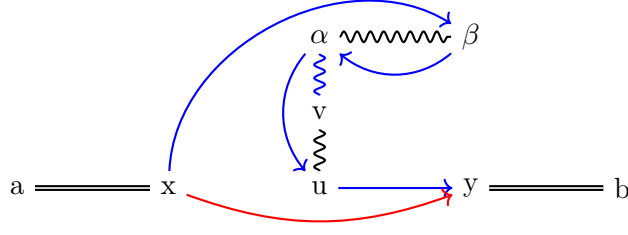
By applying the loop expansion at edge v , we get the following four subgraphs:



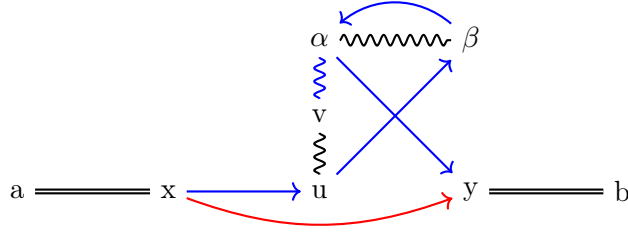
$$\mathcal{G}_{1,s}(z)_{ab} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{xy}} G_{xu} S_{uv} B_{v\alpha} (G_{\alpha\alpha} - m) S_{\alpha\beta} (G_{\beta\beta} - m) G_{uy} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$



$$\mathcal{G}_{2,s}(z)_{ab} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{x\alpha}} \overline{G_{\beta y}} G_{xu} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} G_{uy} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$



$$\mathcal{G}_{3,s}(z)_{ab} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{xy}} G_{x\beta} G_{\alpha u} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} G_{uy} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$



$$\mathcal{G}_{4,s}(z)_{ab} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{xy}} G_{xu} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} G_{u\beta} G_{\alpha y} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$

Observe that in all cases we have a vertex u that has one incoming and one outgoing blue edge, implying that we can again apply the regular vertex expansion w.r.t to u . For both $\mathcal{G}_{1,s}$ and $\mathcal{G}_{2,s}$ the incoming and outgoing edges are the same, starting and ending at x and y , respectively. For $i = 3$ and $i = 4$ we have a different incoming or outgoing edge, so the corresponding expansion will require slight reformulation. Accounting for this fact, we get the following identities:

- For $\mathcal{G}_{1,s}$ and $\mathcal{G}_{2,s}$:

$$\begin{aligned} G_{xu} G_{uy} f(G) &= m B_{uy} G_{xy} f(G) + sm \sum_{\gamma\delta=1}^N B_{u\gamma} S_{\gamma\delta} G_{x\gamma} (G_{\delta\delta} - m) G_{\gamma y} f(G) \\ &+ sm \sum_{\gamma\delta=1}^N B_{u\gamma} S_{\gamma\delta} G_{x\delta} (G_{\gamma\gamma} - m) G_{\delta y} f(G) - sm \sum_{\gamma\beta=1}^N B_{u\gamma} S_{\gamma\delta} G_{x\gamma} G_{\delta y} \partial_{H_{\delta\gamma}} (f(G)) \end{aligned}$$

- For $\mathcal{G}_{3,s}$:

$$\begin{aligned} G_{\alpha u} G_{uy} f(G) &= m B_{uy} G_{\alpha y} f(G) + sm \sum_{\gamma\delta=1}^N B_{u\gamma} S_{\gamma\delta} G_{\alpha\gamma} (G_{\delta\delta} - m) G_{\gamma y} f(G) \\ &+ sm \sum_{\gamma\delta=1}^N B_{u\gamma} S_{\gamma\delta} G_{\alpha\delta} (G_{\gamma\gamma} - m) G_{\delta y} f(G) - sm \sum_{\gamma\delta=1}^N B_{u\gamma} S_{\gamma\delta} G_{\alpha\gamma} G_{\delta y} \partial_{H_{\delta\gamma}} (f(G)) \end{aligned}$$

- For $\mathcal{G}_{4,s}$:

$$\begin{aligned}
G_{xu}G_{u\beta}f(G) &= mB_{u\beta}G_{x\beta}f(G) + sm \sum_{\gamma\delta=1}^N B_{u\gamma}S_{\gamma\delta}G_{x\gamma}(G_{\delta\delta} - m)G_{\gamma\beta}f(G) \\
&+ sm \sum_{\gamma\delta=1}^N B_{u\gamma}S_{\gamma\delta}G_{x\delta}(G_{\gamma\gamma} - m)G_{\delta\beta}f(G) - sm \sum_{\gamma\delta=1}^N B_{u\gamma}S_{\gamma\delta}G_{x\gamma}G_{\delta\beta}\partial_{H_{\delta\gamma}}(f(G))
\end{aligned}$$

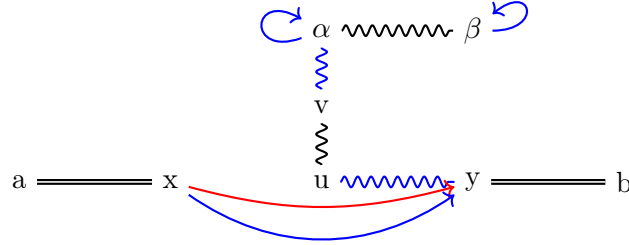
As we noted earlier, each of the former identities can be represented as the sum of independent subgraphs, such that for $i = 1, 2, 3, 4$, we have:

$$\mathcal{G}_{i,s} = m\mathcal{G}_{i0,s} + sm\mathcal{G}_{i1,s} + sm\mathcal{G}_{i2,s} - sm\mathcal{G}_{i3,s}^{(1)} - sm\mathcal{G}_{i3,s}^{(2)} - sm\mathcal{G}_{i3,s}^{(3)},$$

We will begin by focusing our attention on the first three terms for a fixed $i = 1, 2, 3, 4$, namely \mathcal{G}_{ij} for $j = 0, 1, 2$. For brevity, let us define $\eta_s = |\text{Im}w_s| = |\text{Im}[z + (1-t)m(z)]|$. Using the estimates we defined in Section 5, we can simplify each of the resulting subgraph, starting with $j = 0$:

Size estimates for $\mathcal{G}_{10,s}$

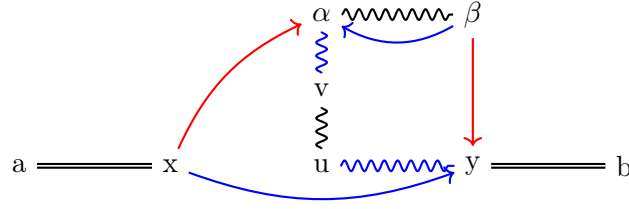
Following diagrammatic notation, each of the $B_{uy_i}G_{x_iy_i}f(G)$ terms becomes:



$$\mathcal{G}_{10,s}(z)_{ab} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{xy}} S_{uv} B_{v\alpha} (G_{\alpha\alpha} - m) S_{\alpha\beta} (G_{\beta\beta} - m) B_{uy} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$

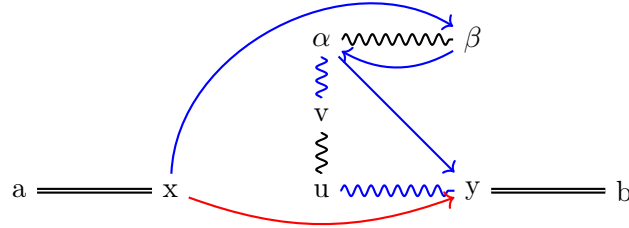
We can apply estimate (2) to the two loops around α and β , as well as estimate (1) to the red edge connecting x and y . From this, we receive three multiplicative terms $W^{-1/2}\eta_s^{-1/4}$, i.e combined it yields a term $W^{-3/2}\eta_s^{-3/4}$. By estimate (5), we can sum all the wavy lines out in order β, α, v , and finally u for a $O(1)$ term, keeping our earlier estimate. We can stop here by leaving the connected component that is a double line from a to x , a blue line from x to y and another double line from y to b , i.e:

$$\mathcal{G}_{10,s} \lesssim W^{-3/2}\eta_s^{-1/4} \times a == x \rightarrow y == b$$

Size estimates for $\mathcal{G}_{20,s}$ 

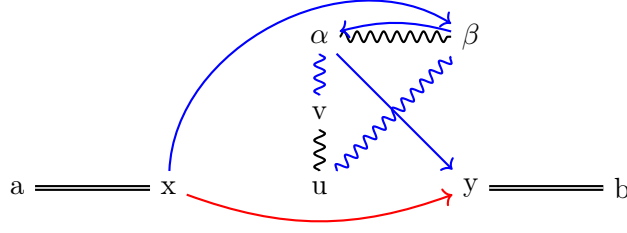
$$\mathcal{G}_{20,s}(z)_{ab} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{x\alpha}} G_{xy} \overline{G_{\beta y}} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} B_{uy} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$

Repeating the same process as before, we bound all three red lines using estimate (2) and sum over the wavy edges in the same order (β, α, v , and u), s.t. we get the same bound as above: $\mathcal{G}_{20,s} \lesssim W^{-3/2} \eta_s^{-1/4} \times$
 $a \equiv x \xrightarrow{\text{blue}} y \equiv b$

Size estimates for $\mathcal{G}_{30,s}$ 

$$\mathcal{G}_{30,s}(z)_{ab} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{xy}} G_{x\beta} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} G_{uy} B_{uy} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$

Here, we eliminate first the edges $\beta \xrightarrow{\text{red}} \alpha$ and $\alpha \xrightarrow{\text{blue}} y$ using estimate (2), giving us a combined $W^{-1} \eta_s^{-1/2}$ term. Due to the edge from $x \xrightarrow{\text{blue}} \beta$, we are not going to be summing $\alpha \rightsquigarrow \beta$ over row/column S_{ab} entries because it instead serves as a multiplicative term to the blue edge $x \xrightarrow{\text{blue}} \beta$. For this reason, we will apply the individual $S_{ab} \lesssim W^{-1}$ bound, s.t now summing over $x \xrightarrow{\text{blue}} \beta$ with estimate (1) gives us a combined bound $W^{3/2} \eta_s^{-3/4}$. Finally, by summing over the remaining wavy edges, starting with α, v and then u , due to the $O(1)$ term (estimate (5)), we get an estimate of the form: $\mathcal{G}_{30,s} \lesssim W^{1/2} \eta_s^{-5/4} \times$
 $a \equiv x \xrightarrow{\text{red}} y \equiv b$

Size estimates for $\mathcal{G}_{40,s}$ 

$$\mathcal{G}_{4,s}(z)_{ab} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{xy}} G_{x\beta} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} B_{u\beta} G_{\alpha y} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$

Here, we bound first the inner edges $\alpha \rightarrow y$ and $\beta \rightarrow \alpha$, we get by estimate (2) a combined term $\lesssim W^{-1}\eta_s^{-1/2}$. Again, by the same logic as above, we have to account for the individual term $\alpha \rightsquigarrow \beta$ instead of summing over it, retaining a $\lesssim W^{-1}$ term. By applying estimate (1) for the edge $x \rightarrow \beta$ and summing over the remaining wavy lines, we get the same bound as above, namely: $\mathcal{G}_{40,s} \lesssim W^{1/2}\eta_s^{-5/4} \times$
 $a = x \rightarrow y = b$

Size estimates for $\max_{i=1,\dots,4} \mathcal{G}_{40,s}$

In order to combine all the latter estimates, we can represent with a pink arrow either blue or red:

$$\max_{i=1,\dots,4} |\mathcal{G}_{40,s}| \lesssim W^{-3/2}\eta_s^{-5/4} \times a = x \rightarrow y = b$$

In order to complete the latter result, we need to consider two subcases w.r.t. to the double lines by splitting the double line $[\{\text{Id} + (t-s)\Theta_t\} S^{1/2}]_{yb}$ into its two components:

- $(t-s)[\Theta_t]_{yb} \lesssim (t-s)W^{-1}\eta_t^{-1/2} \lesssim W^{-1}\eta_t^{-1/2}\eta_s$ (estimate 3 & 7), where by summing over y with estimate 1, we get $O(W^{1/2}\eta_s^{-3/4})$ over x with estimate 6, we have $O(\eta_t^{-1}\eta_s)$, we get a combined term $O(W^{-1/2}\eta_t^{-3/2}\eta_s^{5/4})$
- Id_{yb} is of course $O(1)$, hence we don't have a varying term, which means that we can bound the edge ending at y with estimate 2, s.t. we get $O(W^{-1/2}\eta_s^{-1/4})$ term. Combining it with $O(\eta_t^{-1}\eta_s)$ from estimate 6, we have $O(W^{-1/2}\eta_t^{-1}\eta_s^{3/4})$. With this we achieve a total bound:

$$\max_{i=1,\dots,4} |\mathcal{G}_{i0,s}| \lesssim W^{-2}\eta_t^{-3/2} + W^{-2}\eta_t^{-1}\eta_s^{-1/2}$$

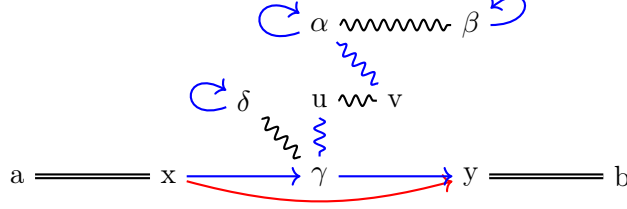
Lastly, we need to integrate over $s \in [0, 5]$ by using a change of variable $\sigma = \eta_s$, s.t. the bounds of the integral are then $\sigma \in [\eta_t, O(1)]$, i.e we have:

$$\max \int_0^t |\mathcal{G}_{i0,s}| ds \lesssim W^{-2}\eta_t^{-3/2} \int_{\eta_t}^{O(1)} d\sigma + W^{-2}\eta_t^{-1} \int_{\eta_t}^{O(1)} \sigma^{-1/2} d\sigma \lesssim$$

$$\lesssim W^{-2}\eta_t^{-3/2} - W^{-2}\eta_t^{-1/2} + 2N^{-2}\eta_t^{-1} - 2W^{-2}\eta_t^{-1/2} \lesssim W^{-2}\eta_t^{-3/2}$$

Size estimates for $\mathcal{G}_{11,s}$

In order to represent $B_{u\gamma}S_{\gamma\delta}G_{x_i\gamma}(G_{\delta\delta} - m)G_{\gamma y_i}f(G)$, we need to add the two new vertices γ and δ :

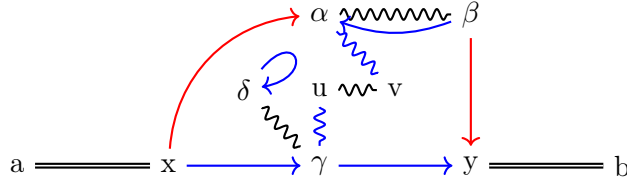


$$\mathcal{G}_{11,s}(z)_{ab} = [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{ax} \overline{G_{xy}} B_{u\gamma} S_{\gamma\delta} G_{x\gamma} (G_{\delta\delta} - m) G_{\gamma y} S_{uv} B_{v\alpha} (G_{\alpha\alpha} - m) S_{\alpha\beta} (G_{\beta\beta} - m) B_{uy} [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{yb}$$

To estimate this graph, we will bound all loops \circlearrowleft and $x \rightarrow y$ using estimate 2 which yields in total $O(W^{-2}\eta_s^{-1})$. Then, we sum the wavy lines over $\beta, \alpha, v, u, \delta$ in that order, which is $O(1)$ by estimate 5.

This means that we get a combined bound: $\mathcal{G}_{11,s} \lesssim W^{-2}\eta_s^{-1} \times a = x \rightarrow \gamma \rightarrow y = b$

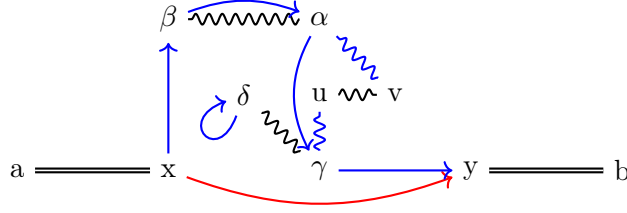
Size estimates for $\mathcal{G}_{21,s}$



$$\mathcal{G}_{21,s}(z)_{ab} = [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{ax} \overline{G_{x\alpha}} G_{x\gamma} G_{\gamma y} \overline{G_{\beta y}} B_{u\gamma} S_{\gamma\delta} (G_{\delta\delta} - m) S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{yb}$$

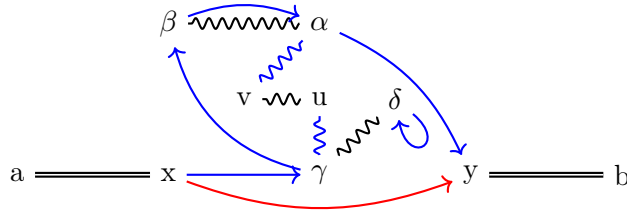
Here, we will repeat an equivalent procedure as above, bounding the loop at δ and the three edges $x \rightarrow \alpha$, $\beta \rightarrow \alpha$ and $\beta \rightarrow y$, getting again a bound $O(W^{-2}\eta_s^{-1})$. Then, we can sum the wavy lines with estimate

(5), getting an equivalent bound as above: $\mathcal{G}_{21,s} \lesssim W^{-2}\eta_s^{-1} \times a = x \rightarrow \gamma \rightarrow y = b$

Size estimates for $\mathcal{G}_{31,s}$ 

$$\mathcal{G}_{31,s}(z)_{ab} = [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{ax} \overline{G_{xy}} G_{x\beta} B_{u\gamma} S_{\gamma\delta} G_{\alpha\gamma} (G_{\delta\delta} - m) G_{\gamma y} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{yb}$$

Let us start by bounding the loop at δ , as well as the $x \rightarrow y$, $\beta \rightarrow \alpha$ and $\alpha \rightarrow \gamma$. In totality, this contributes $O(W^{-2}\eta_s^{-1})$. Here, however, we don't have the same structure as before, since γ no longer has an ingoing edge from x . Regardless, observe that by preserving $x \rightarrow \beta$, when we are performing the total estimation at the end, the wavy lines will be contributing only with $O(1)$ per estimate (5), hence we can expect for the result to be equivalent to the two prior two : $\mathcal{G}_{31,s} \lesssim W^{-2}\eta_s^{-1} \times \text{a} \equiv x \rightarrow \beta \rightarrow y \equiv \text{b}$

Size estimate for $\mathcal{G}_{41,s}$ 

$$\mathcal{G}_{41,s}(z)_{ab} = [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{ax} \overline{G_{xy}} B_{u\gamma} S_{\gamma\delta} G_{x\gamma} (G_{\delta\delta} - m) G_{\gamma\beta} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} G_{\alpha y} [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{yb}$$

Here the logic is the same as above, namely, after eliminating the loop at δ and the edges $x \rightarrow y$, $\gamma \rightarrow \beta$, and $\beta \rightarrow \alpha$ we get a term $O(W^{-2}\eta_s^{-1})$. The remaining graph is of similar form to the first two with additional wavy lines that contribute only $O(1)$ at the end, i.e $\mathcal{G}_{41,s} \lesssim W^{-2}\eta_s^{-1} \times \text{a} \equiv x \rightarrow \gamma \rightarrow y \equiv \text{b}$

Size estimate for $\max_{i=1,\dots,4} \mathcal{G}_{i1,s}$

As such, all our graphs are bounded by:

$$\max_{i=1,\dots,4} \mathcal{G}_{i1,s} \lesssim W^{-2}\eta_s^{-1} \times \text{a} \equiv x \rightarrow \gamma \rightarrow y \equiv \text{b}$$

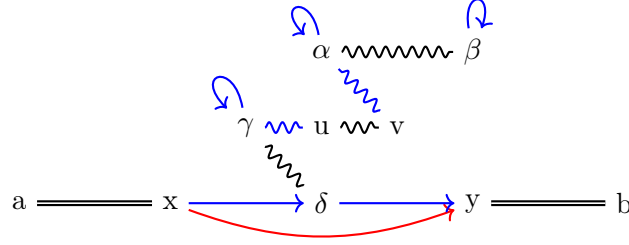
Observe that the latter subgraph is almost identical to the one we calculate for the $\mathcal{G}_{i0,s}$ estimates with the addition of a solid line. This means that we will have an additional term $O(W^{1/2}\eta_s^{-3/4})$ from estimate (1). As such, the subgraph is bounded by the sum of terms $O(\eta_t^{-3/2}\eta_s^{1/2})$ and $O(\eta_t^{-1})$, i.e combined with

the multiplicative term, we get:

$$\begin{aligned} \max_{i=1,\dots,4} |\mathcal{G}_{i1,s}| &\lesssim W^{-2}\eta_t^{-3/2}\eta_s^{-1/2} + W^{-2}\eta_t^{-1}\eta_s^{-1} \\ \Rightarrow \max_{i=1,\dots,4} \int_0^t |\mathcal{G}_{i1,a}| ds &\lesssim W^{-2}\eta_t^{-3/2} \int_{\eta_t}^{O(1)} \sigma^{-1/2} d\sigma + W^{-2}\eta_t^{-1} \int_{\eta_t}^{O(1)} \sigma^{-1} d\sigma = \\ &= 2W^{-2}\eta_t^{-3/2} - 2W^{-2}\eta_t^{-1} - W^{-2}\eta_t^{-1} \ln \eta_t \in O(W^{-2}\eta_t^{-3/2}) \end{aligned}$$

Size estimates for $\mathcal{G}_{12,s}$

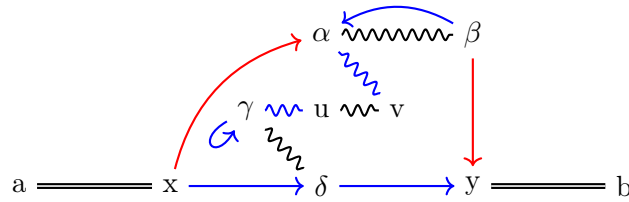
Following the regular vertex expansion, we have to now represent each of the subgraphs that has the form $\mathcal{G}_{i2,s} = B_{u\gamma} S_{\gamma\delta} G_{x_i\delta} (G_{\gamma\gamma} - m) G_{\delta y_i} f(G)$. By applying the graphical representation we get:



$$\mathcal{G}_{12,s} = [\{\text{Id} + (t-s)\Theta_t\} S^{1/2}]_{ax} \overline{G_{xy}} B_{u\gamma} S_{\gamma\delta} G_{x\delta} (G_{\gamma\gamma} - m) G_{\delta y} S_{uv} B_{v\alpha} (G_{\alpha\alpha} - m) S_{\alpha\beta} (G_{\beta\beta} - m) [\{\text{Id} + (t-s)\Theta_t\} S^{1/2}]_{yb}$$

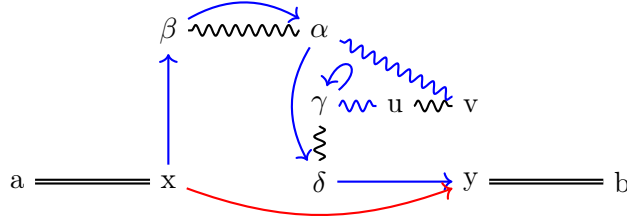
Canceling all three loops \circlearrowleft and the edge $x \rightarrow y$ gives us a combined term $O(W^{-2}\eta_s^{-1})$ per estimate (2). Subsequently, we sum out all the wavy lines, which gives us the same bound as before: $\mathcal{G}_{12,s} \lesssim W^{-2}\eta_s^{-1} \times a \equiv x \xrightarrow{\text{blue}} \delta \xrightarrow{\text{blue}} y \equiv b$

Size estimates for $\mathcal{G}_{22,s}$



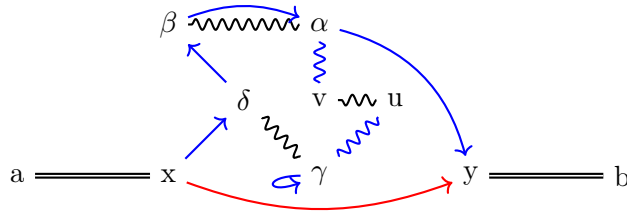
$$\mathcal{G}_{22,s} = [\{\text{Id} + (t-s)\Theta_t\} S^{1/2}]_{ax} \overline{G_{x\alpha}} B_{u\gamma} S_{\gamma\delta} G_{x\delta} (G_{\gamma\gamma} - m) G_{\delta y} \overline{G_{\beta y}} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} [\{\text{Id} + (t-s)\Theta_t\} S^{1/2}]_{yb}$$

Here we cancel the loop at γ and the directed edges $x \rightarrow \alpha$, $\beta \rightarrow y$ and $\beta \rightarrow \alpha$. With this, we are again left with a term $O(W^{-2}\eta_s^{-1})$, s.t. after summing out the wavy lines using estimate (5), we get the same bound: $\mathcal{G}_{22,s} \lesssim W^{-2}\eta_s^{-1} \times a \equiv x \xrightarrow{\text{blue}} \delta \xrightarrow{\text{blue}} y \equiv b$

Size estimates for $\mathcal{G}_{32,s}$ 

$$\mathcal{G}_{32,s} = [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{ax} \overline{G_{xy}} G_{x\beta} B_{u\gamma} S_{\gamma\delta} G_{\alpha\delta} (G_{\gamma\gamma} - m) G_{\delta y} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{yb}$$

For this subgraph, we start by eliminating the loop at γ and the edges $\beta \rightarrow \alpha$, $\alpha \rightarrow \delta$, and $x \rightarrow y$. This means that we have 5 instead of 4 terms per estimate (2), i.e we get a combined term $O(W^{-5/2}\eta_s^{-5/4})$. With this, we have a similar subgraph remaining with a few extra wavy edges that will sum out to $O(1)$ per estimate 5, hence the total estimate will be equivalent to: $\mathcal{G}_{32,s} \lesssim W^{-2}\eta_s^{-1} \times a \equiv x \rightarrow \delta \rightarrow y \equiv b$

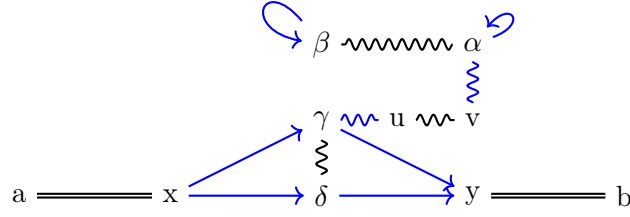
Size estimate for $\mathcal{G}_{42,s}$ 

$$\mathcal{G}_{42,s} = [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{ax} \overline{G_{xy}} B_{u\gamma} S_{\gamma\delta} G_{x\delta} (G_{\gamma\gamma} - m) G_{\delta\beta} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} G_{\alpha y} [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{yb}$$

Here, we eliminate the loop at γ and the edges $\delta \rightarrow \beta$, $\beta \rightarrow \alpha$ and $x \rightarrow y$, all contributing $O(W^{-1/2}\eta_s^{-1/4})$. By summing over $\beta \rightsquigarrow \alpha$ for an $O(1)$ term, we again have a bound that is equivalent to the previous ones, since all the wavy lines sum up to contribute $O(1)$. As such, we have a bound, equivalent to: $\mathcal{G}_{42,s} \lesssim W^{-2}\eta_s^{-1} \times a \equiv x \rightarrow \delta \rightarrow y \equiv b$. But these are exactly the same bounds as for $\mathcal{G}_{i1,s}$, which means that $\max_{i=1,\dots,4} |\mathcal{G}_{i2,s}| \lesssim W^{-2}\eta_t^{-3/2}$

Size estimate for $\mathcal{G}_{13,s}^{(1)}$

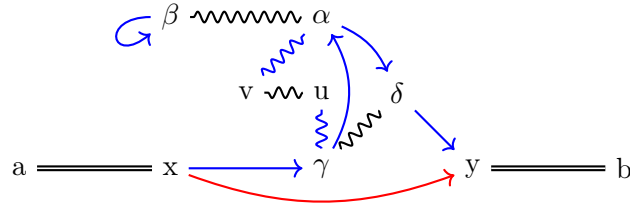
The last non-regularization term from the regular vertex expansion contains a partial derivative: $\mathcal{G}_{i3,s} = B_{u\gamma} S_{\gamma\delta} G_{x_i\gamma} G_{\delta y_i} \partial_{H_{\delta\gamma}}(f(G))$. Recall the resolvent perturbation by which we have that $\partial_{H_{\delta\gamma}} \overline{G_{x_i y_i}} = -\overline{G_{x_i\gamma} G_{\delta y_i}}$ and $\partial_{H_{\delta\gamma}} G_{x_i y_i} = -G_{x_i\delta} G_{\gamma y_i}$. Applying the latter identities results in summation of 3 separate subgraphs for each $G_{i3,s}$. We can again represent each of them individually for $i = 1, 2, 3, 4$:



$$\mathcal{G}_{13,s}^{(1)} = [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{ax} G_{x\delta} G_{\gamma y} S_{uv} B_{v\alpha} (G_{\alpha\alpha} - m) S_{\alpha\beta} (G_{\beta\beta} - m) B_{u\gamma} S_{\gamma\delta} G_{x\gamma} G_{\delta y} [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{yb}$$

Here, we bound the two loops at β, α and then the edges $x \rightarrow \gamma$ and $\gamma \rightarrow y$, getting a combined term $O(W^{-2}\eta_s^{-1})$. Hence, after summing over the wavy lines, we get the same bound as before: $\mathcal{G}_{13,s}^{(1)} \lesssim W^{-2}\eta_s^{-1} \times a \equiv x \rightarrow \delta \rightarrow y \equiv b$

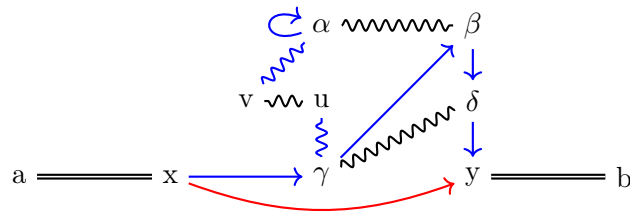
Size estimate for $\mathcal{G}_{13,s}^{(2)}$



$$\mathcal{G}_{13,s}^{(2)} = [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{ax} \overline{G_{xy}} S_{uv} B_{v\alpha} G_{\alpha\delta} G_{\gamma\alpha} S_{\alpha\beta} (G_{\beta\beta} - m) B_{u\gamma} S_{\gamma\delta} G_{x\gamma} G_{\delta y} [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{yb}$$

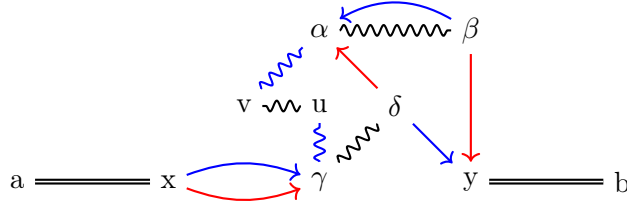
We start by eliminating the loop at β and then the edges $x \rightarrow \gamma$, $\gamma \rightarrow \alpha$, and $\alpha \rightarrow \delta$, getting back $O(W^{-2}\eta_s^{-1})$ by estimate (2). Observe that with this we have almost the same subgraph as before with an additional wavy line $\gamma \rightsquigarrow \delta$, i.e our estimate looks like: $\mathcal{G}_{13,s}^{(2)} \lesssim W^{-2}\eta_s^{-1} \times a \equiv x \rightarrow \gamma \rightsquigarrow \delta \rightarrow y \equiv b$

Size estimate for $\mathcal{G}_{13,s}^{(3)}$



$$\mathcal{G}_{13,s}^{(3)} = [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{ax} \overline{G_{xy}} S_{uv} B_{v\alpha} (G_{\alpha\alpha} - m) S_{\alpha\beta} G_{\beta\delta} G_{\gamma\beta} B_{u\gamma} S_{\gamma\delta} G_{x\gamma} G_{\delta y} [\{\text{Id} + (t-s)\Theta_t\}S^{1/2}]_{yb}$$

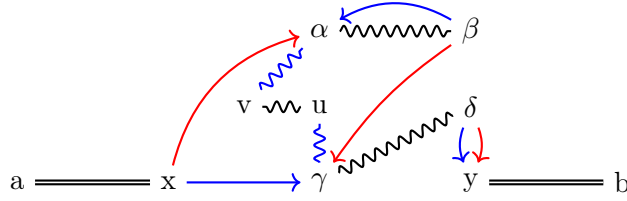
For the last subgraph, we start with the loop α and then apply the second estimate for the edges $x \rightarrow y$, $\gamma \rightarrow \beta$, and $\beta \rightarrow \delta$. We get a combined term $O(W^{-2}\eta_s^{-1})$, s.t. after summing out β, α, v, u , we get a final estimate that looks like the one before: $\mathcal{G}_{13,s}^{(3)} \lesssim W^{-2}\eta_s^{-1} \times a \equiv x \rightarrow \gamma \rightsquigarrow \delta \rightarrow y \equiv b$. And since the partial derivative expansion is equivalent to the summation of the latter three subgraphs, we get a combined bound on $\mathcal{G}_{i3,s}$ that is $O(W^{-2}\eta_t^{-3/2})$ by using our earlier calculations.

Size estimate for $\mathcal{G}_{23,s}^{(1)}$ 

$$\mathcal{G}_{23,s}^{(1)} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{x\gamma} G_{\delta\alpha} G_{\beta\gamma}} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} B_{u\gamma} S_{\gamma\delta} G_{x\gamma} G_{\delta y} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$

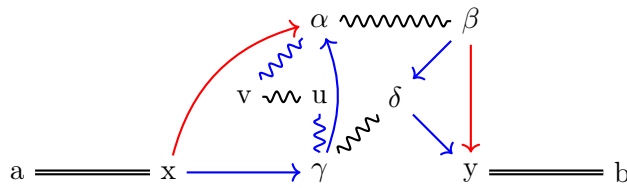
For this subgraph, we can bound the three red edges along with $\beta \rightarrow \alpha$, for which we get combined term $O(W^{-2}\eta_s^{-1})$. After summing over the wavy edges in order β, α, v, u , we get the same bound as before:

$$\mathcal{G}_{23,s}^{(1)} \lesssim W^{-2}\eta_s^{-1} \times a = x \rightarrow \gamma \rightsquigarrow \delta \rightarrow y = b$$

Size estimate for $\mathcal{G}_{23,s}^{(2)}$ 

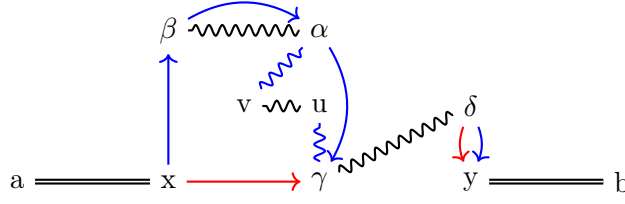
$$\mathcal{G}_{23,s}^{(2)} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{x\alpha} G_{\beta\gamma} G_{\delta y}} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} B_{u\gamma} S_{\gamma\delta} G_{x\gamma} G_{\delta y} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$

Here, we can get bound all the red lines along with $\beta \rightarrow \alpha$, s.t. after the summation over the wavy lines in order β, α, v, u , we get the familiar: $\mathcal{G}_{23,s}^{(2)} \lesssim W^{-2}\eta_s^{-1} \times a = x \rightarrow \gamma \rightsquigarrow \delta \rightarrow y = b$

Size estimate for $\mathcal{G}_{23,s}^{(3)}$ 

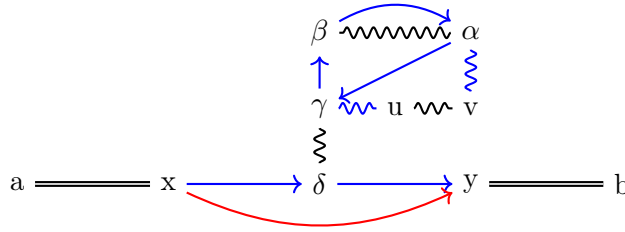
$$\mathcal{G}_{23,s}^{(3)} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{x\alpha} G_{\beta\gamma}} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\delta} G_{\gamma\alpha} B_{u\gamma} S_{\gamma\delta} G_{x\gamma} G_{\delta y} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$

Lastly, by bounding all the red edges along with $\gamma \rightarrow \alpha$ and $\beta \rightarrow \delta$ and summing over the wavy edges in order β, α, v, u , we get the equivalent bound: $\mathcal{G}_{23,s}^{(3)} \lesssim W^{-2}\eta_s^{-1} \times a = x \rightarrow \gamma \rightsquigarrow \delta \rightarrow y = b$, which makes this subgraph subject to the same $O(W^{-2}\eta_t^{-3/2})$ estimate.

Size estimate for $\mathcal{G}_{33,s}^{(1)}$ 

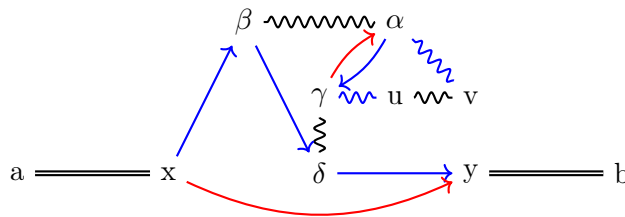
$$\mathcal{G}_{33,s}^{(1)} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{x\gamma} G_{\delta y} G_{x\beta} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} B_{u\gamma} S_{\gamma\delta} G_{\alpha\gamma} G_{\delta y}} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$

By bounding all the blue edges we get a combined $O(W^{-2}\eta_s^{-1})$ term, s.t. after summing over the wavy edges in order β, α, v, u , we get a final bound: $\mathcal{G}_{33,s}^{(1)} \lesssim W^{-2}\eta_s^{-1} \times a \text{ --- } x \xrightarrow{\text{red}} \gamma \text{ --- } \delta \xrightarrow{\text{red}} y \text{ --- } b$

Size estimate for $\mathcal{G}_{33,s}^{(2)}$ 

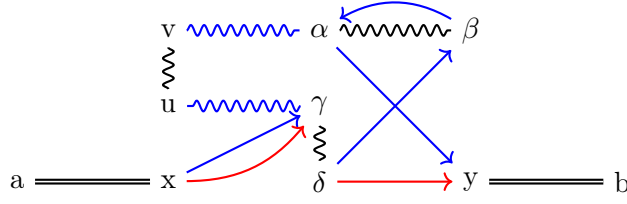
$$\mathcal{G}_{33,s}^{(2)} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{xy} G_{x\delta} G_{\gamma\beta} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} B_{u\gamma} S_{\gamma\delta} G_{\alpha\gamma} G_{\delta y}} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$

For this graph, let us bound the edges $x \xrightarrow{\text{red}} y$, $\gamma \xrightarrow{\text{blue}} \beta$, $\beta \xrightarrow{\text{blue}} \alpha$ and $\alpha \xrightarrow{\text{blue}} \gamma$, s.t. after summing over the wavy lines in order of the indices $\beta, \alpha, v, u, \gamma$, we get a final bound: $\mathcal{G}_{33,s}^{(2)} \lesssim W^{-2}\eta_s^{-1} \times a \text{ --- } x \xrightarrow{\text{blue}} \delta \xrightarrow{\text{blue}} y \text{ --- } b$

Size estimate for $\mathcal{G}_{33,s}^{(3)}$ 

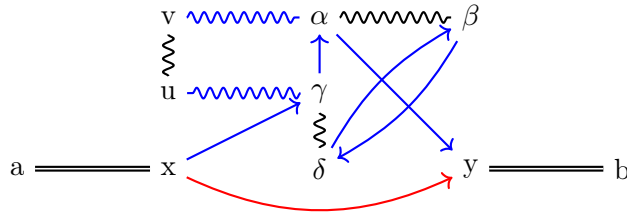
$$\mathcal{G}_{33,s}^{(3)} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{xy} G_{x\beta} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\delta} G_{\gamma\alpha} B_{u\gamma} S_{\gamma\delta} G_{\alpha\gamma} G_{\delta y}} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$

Lastly, by bounding all of the red edges along with $\beta \xrightarrow{\text{blue}} \delta$, $\alpha \xrightarrow{\text{blue}} \gamma$, we get a combined estimate $O(W^{-2}\eta_s^{-1})$. This leaves us with a bound that looks like $\mathcal{G}_{33,s}^{(3)} \lesssim W^{-2}\eta_s^{-1} \times a \text{ --- } x \xrightarrow{\text{blue}} \beta \text{ --- } \delta \xrightarrow{\text{blue}} y \text{ --- } b$, where the red wavy line represents a sequence of black and blue wavy lines (specifically those starting at the indices $\beta, \alpha, v, u, \gamma$). And since we already showed that estimation of these subgraphs doesn't change with extra wavy lines per the $O(1)$ contribution from estimate (5), this set of subgraphs is also $O(W^{-2}\eta_t^{-3/2})$ as the ones before.

Size estimate for $\mathcal{G}_{43,s}^{(1)}$ 

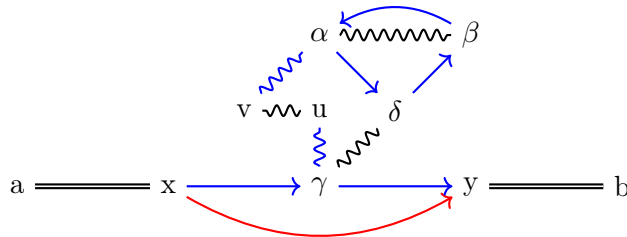
$$\mathcal{G}_{43,s}^{(1)} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{x\gamma} G_{\delta y} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} G_{\alpha\gamma} B_{u\gamma} S_{\gamma\delta} G_{x\gamma} G_{\delta\beta}} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$

By bounding all the blue edges, getting a $O(W^{-1/2}\eta_s^{-1/4})$ term and subsequently summing over the wavy lines in order $\beta, \alpha, v, u, \gamma$, the resulting estimate is $\mathcal{G}_{43,s}^{(1)} \lesssim W^{-2}\eta_s^{-1} \times \text{a} = \text{x} \xrightarrow{\text{red}} \gamma \rightsquigarrow \delta \xrightarrow{\text{red}} \text{y} = \text{b}$

Size estimate for $\mathcal{G}_{43,s}^{(2)}$ 

$$\mathcal{G}_{43,s}^{(2)} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{xy} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\delta} G_{\gamma\alpha} G_{\alpha\gamma} B_{u\gamma} S_{\gamma\delta} G_{x\gamma} G_{\delta\beta}} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$

Bounding $\gamma \rightarrow \alpha$, $\delta \rightarrow \beta$, $\beta \rightarrow \delta$ and $x \rightarrow y$ for a combined $O(W^{-2}\eta_s^{-1})$ term, and summing over β, α, v, u , we get: $\mathcal{G}_{43,s}^{(2)} \lesssim W^{-2}\eta_s^{-1} \times \text{a} = \text{x} \xrightarrow{\text{blue}} \gamma \rightsquigarrow \alpha \xrightarrow{\text{blue}} \text{y} = \text{b}$

Size estimate for $\mathcal{G}_{43,s}^{(3)}$ 

$$\mathcal{G}_{43,s}^{(3)} = \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{ax} \overline{G_{xy} S_{uv} B_{v\alpha} S_{\alpha\beta} G_{\beta\alpha} G_{\alpha\delta} G_{\gamma\gamma} B_{u\gamma} S_{\gamma\delta} G_{x\gamma} G_{\delta\beta}} \left[\{\text{Id} + (t-s)\Theta_t\} S^{1/2} \right]_{yb}$$

Here, we bound the edges $\alpha \rightarrow \delta$, $x \rightarrow y$, $\beta \rightarrow \alpha$, and $\delta \rightarrow \beta$ for a combined term $O(W^{-2}\eta_s^{-1})$. After summing out $\beta, \alpha, v, u, \delta$ in that order, we get $\mathcal{G}_{43,s}^{(3)} \lesssim W^{-2}\eta_s^{-1} \times \text{a} = \text{x} \xrightarrow{\text{blue}} \gamma \xrightarrow{\text{blue}} \text{y} = \text{b}$. With this, we have in fact verified that all of the partial derivative terms satisfy the estimate: $O(W^{-2}\eta_t^{-3/2})$.

Combined bound

5.2 Estimation of $\mathcal{E}_t^M(z)$ term

$$\begin{aligned}
& \int_0^t \{\text{Id} + (t-s)\Theta_t\} S^{1/2} dM(s) S^{1/2} \{\text{Id} + (t-s)\Theta_t\} = \\
& = \int_0^t \{\text{Id} + (t-s)\Theta_t\} \sum_{u,v} [\overline{G_{xy}} G_{xu} dH_{uv}(t) G_{vy}] \{\text{Id} + (t-s)\Theta_t\} + \\
& + \int_0^t \{\text{Id} + (t-s)\Theta_t\} \sum_{u,v} [G_{xy} \overline{G_{xu} dH_{uv}(t) G_{vy}}] \{\text{Id} + (t-s)\Theta_t\}
\end{aligned}$$

After rewriting the fractions as $[\text{Id} + (t-s)\Theta_t]_{ax}$ and with index yb , respectively, the latter two terms can be written in diagrammatically using our familiar notation:

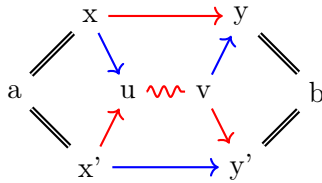
$a = x \xrightarrow{\text{red}} y = b \quad a = x \xrightarrow{\text{blue}} y = b,$

where the purple wavy line represents dH_{uv} and the red wavy line is its complement. In order to estimate the latter term, we will apply the Burkholder-Davis-Gundy (BDG) inequality [27], defined below:

BDG inequality

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} |M_{xy}(s)|^{2p} \right] \leq C_p \mathbb{E} [M_{xy}_t]^p$$

In order to apply the Quadratic Variation (QV) on the RHS, we will need to rewrite our separate graphs into one combined QV graph. To do this, we will use that fact that the Brownian increments $dB_{\alpha\beta}(t)$ are mutually uncorrelated, s.t. we clearly have that the cross-terms in the quadratic variation vanish unless we have $(u, v) = (u', v')$, i.e we can identify the latter pairs, getting a combined graph of the form:



Now, if we are to approach the bounding carelessly, we will quickly stumble into problems. For example, by applying our standard estimates, first eliminating the wavy line, from which we have $\sqrt{S_{uv}} \lesssim N^{-1/2}$ per estimate 4 and another $O(\sqrt{dt})$ per the fact that $\mathbb{E} |dB_{uv}(t)|^2 = dt$. We want to keep a connected component similar to before, so let WLOG this be the path $x' \Rightarrow a \Rightarrow x \rightarrow y \Rightarrow b \Rightarrow y'$. By applying the already familiar estimate 2 to the 4 red lines, we get a combined term $O(N^{-2}\eta_s^{-1})$. Now, to get

to the the aforementioned path, we need to bound the $x' \rightarrow y'$, using estimate 1 (since the edge is not "free") with which we get $O(N^{1/2}\eta_s^{-3/4})$. By observing what remains, we can quickly see this will prove to be unruly due to the fact that we will have to split double lines into the respective Id and $(t-s)\Theta_t$ components, yielding 2^3 separate terms, which will each be of the form $O\left(N^{-\frac{3}{2}}\eta_t^{-\frac{3a+2b}{2}}\eta_s^{\frac{5a+3b}{4}}\right)$, where $a, b \in \{0, 1, 2, 3\}$ and $a+b=3$ (we have an additional estimate from the blue line that is $N^{1/2}\eta_s^{-3/4}$). We can easily observe that by accounting for all of the terms, we have in the worst case scenario a term $\eta_s^{15/4}$, which remains as $\eta_s^{5/4}$ after canceling the other such terms. By the same logic, we can observe that we have at most $\eta_t^{-\frac{9}{2}}$ and at least η_t^{-3} from the splitting terms, hence the bound quickly becomes unruly as we take the limit $\eta_t \rightarrow 0$.

Alternative bound

The latter calculations demonstrate that we need to be more thoughtful with our estimates. Let us come back to our original form of the $S^{1/2}dM(s)S^{1/2}$ term:

$$\begin{aligned} &= \int_0^t [\text{Id} + (t-s)\Theta_t] S^{1/2} \left\{ \sum_{u,v} [\overline{G_{xy}} G_{xu} dH_{uv}(t) G_{vy}] \right\} S^{1/2} [\text{Id} + (t-s)\Theta_t] + \\ &\int_0^t [\text{Id} + (t-s)\Theta_t] S^{1/2} \left\{ \sum_{u,v} [G_{xy} \overline{G_{xu} dH_{uv}(t) G_{vy}}] \right\} S^{1/2} [\text{Id} + (t-s)\Theta_t] = \mathcal{E}_t^{M,1} + \mathcal{E}_t^{M,2} \end{aligned}$$

Observe that we can split each of the components into 4 separate parts by simply expanding the square brackets. We will this explicitly for $\mathcal{E}_t^{M,1}$, whereas the second term is equivalent up to complex conjugation:

$$\begin{aligned} \mathcal{E}_t^{M,1} &= \int_0^t (t-s)^2 S^{1/2} \Theta_t \left\{ \sum_{u,v} [\overline{G_{xy}} G_{xu} dH_{uv}(t) G_{vy}] \right\} S^{1/2} \Theta_t + \\ &+ \int_0^t (t-s) S^{1/2} \Theta_t \left\{ \sum_{u,v} [\overline{G_{xy}} G_{xu} dH_{uv}(t) G_{vy}] \right\} S^{1/2} + \\ &+ \int_0^t (t-s) S^{1/2} \left\{ \sum_{u,v} [\overline{G_{xy}} G_{xu} dH_{uv}(t) G_{vy}] \right\} S^{1/2} \Theta_t + \\ &+ \int_0^t 2 S^{1/2} \left\{ \sum_{u,v} [\overline{G_{xy}} G_{xu} dH_{uv}(t) G_{vy}] \right\} S^{1/2} = \\ &= \mathcal{E}_t^{M,11} + \mathcal{E}_t^{M,12} + \mathcal{E}_t^{M,13} + \mathcal{E}_t^{M,14} \end{aligned}$$

Now, instead of applying the BDG inequality to the entire term, we can bound the individual $\mathcal{E}_t^{M,1i}$ entries:

$$\mathbb{E} \left| \mathcal{E}_{ab}^{M,1i}(s) \right|^{2p} \leq C_p \mathbb{E} \left[\mathcal{E}_{ab}^{M,1i}(s) \right]_t^p$$

Hence, we need to calculate the quadratic variation for each term. To do this, recall from before that the Brownian increments $dB_{\alpha\beta}(t)$ are mutually uncorrelated, i.e our term $dH_s = \sum_{u,v} \sqrt{S_{u,v}} dB_{u,v}(s)$ has covariance structure:

$$\mathbb{E} [dH_s d\overline{H}_s] = \mathbb{E} \left[\sum_{u,u',v,v'} \sqrt{S_{uv}} dB_{uv}(s) \sqrt{S_{u'v'}} dB_{u'v'}(s) \right]_{ab} = S_{ab} ds$$

Hence the QV of each of the latter terms is:

$$\Rightarrow [\mathcal{E}^{M,11}]_{ab} = \int_0^t (t-s)^4 \sum_{x,x',y,y',u,v} (\Theta_t S^{1/2})_{ax} (\Theta_t S^{1/2})_{ax'} \overline{G_{xy}} G_{x'y'} G_{xu} \overline{G_{x'u}} S_{uv} G_{vy} \overline{G_{vy'}} (S^{1/2} \Theta_t)_{yb} (S^{1/2} \Theta_t)_{yb'} ds$$

$$[\mathcal{E}^{M,12}]_{ab} = \int_0^t (t-s)^2 \sum_{x,x',y,y',u,v} (\Theta_t S^{1/2})_{ax} (\Theta_t S^{1/2})_{ax'} \overline{G_{xy}} G_{x'y'} G_{xu} \overline{G_{x'u}} S_{uv} G_{vy} \overline{G_{vy'}} S_{yb}^{1/2} S_{yb'}^{1/2} ds$$

$$[\mathcal{E}^{M,13}]_{ab} = \int_0^t (t-s)^2 \sum_{x,x',y,y',u,v} S_{ax}^{1/2} S_{ax'}^{1/2} \overline{G_{xy}} G_{x'y'} G_{xu} \overline{G_{x'u}} S_{uv} G_{vy} \overline{G_{vy'}} (S^{1/2} \Theta_t)_{yb} (S^{1/2} \Theta_t)_{yb'} ds$$

$$[\mathcal{E}^{M,14}]_{ab} = \int_0^t \sum_{x,x',y,y',u,v} S_{ax}^{1/2} S_{ax'}^{1/2} \overline{G_{xy}} G_{x'y'} G_{xu} \overline{G_{x'u}} S_{uv} G_{vy} \overline{G_{vy'}} S_{yb}^{1/2} S_{yb'}^{1/2} ds$$

Observe that we have the same primary components of the separate terms, differing up to the utmost left and right terms. Hence, we can abstract away four separate matrices, to represent this two-by-two overlap:

$$\Upsilon_{yy}^u := \sum_v S_{uv} G_{vy} \overline{G_{vy'}} (\Theta_t S^{\frac{1}{2}})_{yb} (\Theta_t S^{\frac{1}{2}})_{yb'},$$

$$\Omega_{yy}^u := \sum_{x,x'} \overline{G_{xy}} G_{x'y'} G_{xu} \overline{G_{x'u}} (\Theta_t S^{\frac{1}{2}})_{ax} (\Theta_t S^{\frac{1}{2}})_{ax'}.$$

$$\Gamma_{yy}^u := \sum_v S_{uv} G_{vy} \overline{G_{vy'}} S_{yb}^{\frac{1}{2}} S_{yb'}^{\frac{1}{2}},$$

$$\Xi_{yy}^u := \sum_{x,x'} \overline{G_{xy}} G_{x'y'} G_{xu} \overline{G_{x'u}} S_{ax}^{\frac{1}{2}} S_{ax'}^{\frac{1}{2}}.$$

$$\begin{aligned} \left[\mathcal{E}_t^{M,11}(z)_{ab} \right] &= \int_0^t (t-s)^4 \sum_{u,y} (\Omega^u \Upsilon^{u,*})_{yy'} ds, \quad \left[\mathcal{E}_t^{M,12}(z)_{ab} \right] = \int_0^t (t-s)^2 \sum_{u,y} (\Omega^u \Gamma^{u,*})_{yy'} ds \\ \Rightarrow \left[\mathcal{E}_t^{M,13}(z)_{ab} \right] &= \int_0^t \sum_{u,y} (t-s)^2 (\Xi^u \Upsilon^{u,*})_{yy'} ds, \quad \left[\mathcal{E}_t^{M,14}(z)_{ab} \right] = \int_0^t \sum_{u,y} (\Xi^u \Gamma^{u,*})_{yy'} ds \end{aligned}$$

Now, observe that both Ω^u and Ξ^u are positive-semidefinite, since we can rewrite them as:

$$\begin{aligned} \Omega^u &= \left(\sum_x (\Theta_t S^{1/2})_{ax} G_{xu} \overline{G}_{xy} \right) \left(\sum_{x'} (\Theta_t S^{1/2})_{ax'} G_{x'u} \overline{G}_{x'y'} \right)^* \\ \Xi^u &= \left(\sum_x S_{ax}^{1/2} G_{xu} \overline{G}_{xy} \right) \left(\sum_{x'} S_{ax'}^{1/2} G_{x'u} \overline{G}_{x'y'} \right)^* \end{aligned}$$

This means that we can apply the trace inequality, which states that the trace is bounded above by singular values $|\text{Tr}(AB)| \leq \sum_{i=1}^N \alpha_i \beta_i$, hence by the positive-semidefiniteness, we have that $\left| \sum_y (\Omega^u \Upsilon^{u,*})_{yy} \right| = |\text{Tr}(\Omega^u \Upsilon^{u,*})| \leq \|\Omega^u\|_{op} \sum_y \Omega_{yy}^u ds$. This means that we can bound each of the earlier terms as follows:

$$\left[\mathcal{E}_t^{M,11}(z)_{ab} \right] = \int_0^t (t-s)^4 \sum_{u,y} (\Omega^u \Upsilon^{u,*})_{yy} ds \leq \int_0^t (t-s)^4 \sum_u \|\Upsilon^u\|_{op} \sum_y \Omega_{yy}^u ds$$

$$\left[\mathcal{E}_t^{M,12}(z)_{ab} \right] = \int_0^t (t-s)^2 \sum_{u,y} (\Omega^u \Gamma^{u,*})_{yy} ds \leq \int_0^t (t-s)^2 \sum_u \|\Gamma^u\|_{op} \sum_y \Omega_{yy}^u ds$$

$$\left[\mathcal{E}_t^{M,13}(z)_{ab} \right] = \int_0^t \sum_{u,y} (t-s)^2 (\Xi^u \Upsilon^{u,*})_{yy} ds \leq \int_0^t (t-s)^2 \sum_u \|\Upsilon^u\|_{op} \sum_y \Xi_{yy}^u ds$$

$$\left[\mathcal{E}_t^{M,14}(z)_{ab} \right] = \int_0^t \sum_{u,y} (\Xi^u \Gamma^{u,*})_{yy} ds \leq \int_0^t \sum_u \|\Gamma^u\|_{op} \sum_y \Xi_{yy}^u ds$$

We can estimate the latter bounds by using the fact that $|G|^2 = GG^*$ and $|G|^4 = |G|^2 |G|^2$:

$$\begin{aligned} \sum_{u,y} \Omega_{yy}^u &= \sum_{u,y} \sum_{x,x'} \overline{G}_{xy} G_{x'y'} G_{xu} \overline{G}_{x'u} (\Theta_t S^{\frac{1}{2}})_{ax} (\Theta_t S^{\frac{1}{2}})_{ax'} = \\ &= \sum_{x,x'} |G_{xx'}^2| |G_{x'x}^2| (\Theta_t S^{\frac{1}{2}})_{ax} (\Theta_t S^{\frac{1}{2}})_{ax'} \leq 4 \sum_x |G_{xx}^2| |(\Theta_t S^{\frac{1}{2}})_{ax}|^2, \end{aligned}$$

following from the Schwarz inequality. By the Ward's identity for the resolvent $|G_{xx}|^2 = \frac{\text{Im} G_{xx}}{\eta} = \frac{\text{Im} G_{xx}}{\text{Im} w_s}$ (equation 7.23 in [2]), we have:

$$4 \sum_x |G_{xx}^4| |(\Theta_t S^{\frac{1}{2}})_{ax}|^2 = 4 |\text{Im} w_s|^{-2} \sum_x |\text{Im} G_{xx}|^2 |(\Theta_t S^{\frac{1}{2}})_{ax}|^2 \leq 4 |\text{Im} w_s|^{-2} \max_k |\text{Im} G_{xx}|^2 \sum_x |(\Theta_t S^{\frac{1}{2}})_{ax}|^2 \leq$$

$$? \lesssim |\operatorname{Im} w_s|^{-3} \max_k \operatorname{Im} G_{kk} \sum_x |(\Theta_t S^{\frac{1}{2}})_{ax}|^2 \leq N^{-1} |\operatorname{Im} w_s|^{-3} |\operatorname{Im} w_t|^{-3/2},$$

using estimates (4) and (5). By the same logic, we have that:

$$\begin{aligned} \sum_{u,y} \Xi_{yy}^u &= \sum_{u,y} \sum_{x,x'} \bar{G}_{xy} G_{x'y'} G_{xu} \bar{G}_{x'u} S_{ax}^{\frac{1}{2}} S_{ax'}^{\frac{1}{2}} = \sum_{x,x'} |G_{xx'}|^2 |G_{x'x}|^2 S_{ax}^{\frac{1}{2}} S_{ax'}^{\frac{1}{2}} \leq \\ &\leq 4 \sum_x |G_{xx}|^4 |S_{ax}^{\frac{1}{2}}|^2 \lesssim |\operatorname{Im} w_s|^{-3} \max_k \operatorname{Im} G_{kk} \sum_x |S_{ax}^{1/2}|^2 \lesssim N^{-1} |\operatorname{Im} w_s|^{-3} \end{aligned}$$

Lastly, we need to bound the other two matrices, which we can do by using the fact that the diagonal matrix S^u is of the same order as estimate 6, i.e by bounding with the maximal value and Wald's identity, we have:

$$\|\Upsilon^u\|_{op} \leq \max_{\alpha,\beta} |(\Theta_t S^{1/2})_{\alpha\beta}|^2 N^{-1} |\operatorname{Im} w_s|^{-2} \lesssim N^{-1} |\operatorname{Im} w_s|^{-2} |\operatorname{Im} w_t|^{-1}$$

$$\|\Gamma^u\|_{op} \leq \max_{\alpha,\beta} |S_{\alpha\beta}^{1/2}|^2 N^{-1} |\operatorname{Im} w_s|^{-2} \lesssim N^{-3} |\operatorname{Im} w_s|^{-2}$$

Lastly, recall estimate 9 that $(t-s) \lesssim |\operatorname{Im} w_s|^{-1}$, with which we have all the necessary pieces to complete the estimation of all the QV terms $\mathcal{E}_t^{M,1i}$, by again changing the variable of integration w.r.t $\sigma = |\operatorname{Im} w_s|$:

$$\left[\mathcal{E}_t^{M,11}(z)_{ab} \right] \leq \int_0^t (t-s)^4 \sum_u \|\Upsilon^u\|_{op} \sum_y \Omega_{yy'}^u ds \lesssim N^{-4} |\operatorname{Im} w_t|^{-5/2} \int_{\operatorname{Im} w_t}^0 \sigma^{-1} \lesssim N^{-4} |\operatorname{Im} w_t|^{-5/2} \log |\operatorname{Im} w_t|$$

$$\left[\mathcal{E}_t^{M,12}(z)_{ab} \right] \leq \int_0^t (t-s)^2 \sum_u \|\Gamma^u\|_{op} \sum_y \Omega_{yy'}^u ds \lesssim N^{-4} |\operatorname{Im} w_t|^{-3/2} \int_{\operatorname{Im} w_t}^0 \sigma^{-3} \lesssim N^{-4} |\operatorname{Im} w_t|^{-7/2}$$

$$\left[\mathcal{E}_t^{M,13}(z)_{ab} \right] \leq \int_0^t (t-s)^2 \sum_u \|\Upsilon^u\|_{op} \sum_y \Xi_{yy'}^u ds \lesssim N^{-4} |\operatorname{Im} w_s|^{-1} \int_{\operatorname{Im} w_t}^0 \sigma^{-3} \lesssim N^{-4} |\operatorname{Im} w_t|^{-3}$$

$$\left[\mathcal{E}_t^{M,14}(z)_{ab} \right] \leq \int_0^t \sum_u \|\Gamma^u\|_{op} \sum_y \Xi_{yy'}^u ds \lesssim N^{-4} \int_{\operatorname{Im} w_t}^0 \sigma^{-5} \lesssim N^{-4} |\operatorname{Im} w_t|^{-4}$$

6 Non-Gaussian case

Up to this point, our work on non-mean field models, specifically band matrices, has been focused on the case with Gaussian entries. Given that the delocalization behavior is conjectured to extend beyond the latter case, our last section has t

6.1 Lindeberg exchange strategy

The method developed by Lindeberg in his seminal proof for the generalized CLT [33] and later generalized by Chatterjee [34]. The argument proceeds as follows [35]: Suppose that $X_1, \dots, X_n \sim [\mu, 1]$ are i.i.d, and let $Y_1, \dots, Y_n \sim [\mu, 1]$ be another such set. One would like to show that for any smooth, compactly supported function F :

$$\mathbb{E}F\left(\frac{X_1 + \dots + X_n}{\sqrt{n}}\right) = \mathbb{E}F\left(\frac{Y_1 + \dots + Y_n}{\sqrt{n}}\right) + o(1)$$

by swapping the entries one at a time. Define $S = \frac{X_1 + \dots + X_{n-1}}{\sqrt{n}}$, such that $S + n^{-1/2}X_n = \frac{X_1 + \dots + X_n}{\sqrt{n}}$.

Using the smoothness and compact support of F , we can apply the Taylor expansion:

$$\begin{aligned} F\left(\frac{X_1 + \dots + X_n}{\sqrt{n}}\right) &= F(S) + n^{-1/2}X_n F'(S) + \frac{1}{2}n^{-1}X_n^2 F''(S) + O(n^{-3/2}|X_n|^3) \\ \Rightarrow \mathbb{E}F\left(\frac{X_1 + \dots + X_n}{\sqrt{n}}\right) &= \mathbb{E}F(S) + n^{-1/2}(\mathbb{E}X_n)\mathbb{E}F'(S) + \frac{1}{2}n^{-1}(\mathbb{E}X_n^2)\mathbb{E}F''(S) + O(n^{-3/2}) \\ \Rightarrow \mathbb{E}F\left(\frac{X_1 + \dots + X_{n-1} + Y_n}{\sqrt{n}}\right) &= \mathbb{E}F(S) + n^{-1/2}(\mathbb{E}Y_n)\mathbb{E}F'(S) + \frac{1}{2}n^{-1}(\mathbb{E}Y_n^2)\mathbb{E}F''(S) + O(n^{-3/2}) \\ \Rightarrow \mathbb{E}F\left(\frac{X_1 + \dots + X_n}{\sqrt{n}}\right) &= \mathbb{E}F\left(\frac{X_1 + \dots + X_{n-1} + Y_n}{\sqrt{n}}\right) + o\left(\frac{1}{n}\right) \end{aligned}$$

by the the fact X_i and Y_i have matching moments of second order. If we have higher matching moments, we could continue the Taylor expansion in order to refine the error term. This strategy can naturally be extended to the case of independent Wigner matrices M_n, M'_n with the goal of obtaining bounds like:

$$\mathbb{E}F(M_n) - \mathbb{E}F(\tilde{M}_n) = o(1/n)$$

The equivalent result for the spectral behavior of random matrices is known as the *Four Moment Theorem* [36]:

6.2 Four Moment Theorem for Green's function

Let M_n, M'_n be two Wigner random matrices satisfying **C0** that match to order 4 off the diagonal and to order 2 on the diagonal for some sufficiently large C_0 . Let $z = E + i\eta$ for some $E \in \mathbb{R}$ and some $\eta > 0$. We assume the level repulsion hypothesis that for any $c > 0$, one has with high probability that

$$\inf_{1 \leq i \leq n} |\lambda_i(\sqrt{n}M_n) - nz| \geq n^{-c} \quad (5)$$

and

$$\inf_{1 \leq i \leq n} |\lambda_i(\sqrt{n}M'_n) - nz| \geq n^{-c}. \quad (6)$$

Let $1 \leq p, q \leq n$. Then for any smooth function $G : \mathbb{C} \rightarrow \mathbb{C}$ obeying the bounds $\nabla^j G(x) = O(1)$ for all $x \in \mathbb{C}$ and $0 \leq j \leq 5$, one has

$$\mathbb{E}G\left(\left(\frac{1}{\sqrt{n}}M_n - zI\right)_{pq}^{-1}\right) - \mathbb{E}G\left(\left(\frac{1}{\sqrt{n}}M'_n - zI\right)_{pq}^{-1}\right) = O(n^{-c_0}) \quad (7)$$

for some constant $c_0 > 0$ independent of n .

6.3 Application

In order to apply the Lindeberg expansion strategy to our case, let us consider the following function:

$$F(H) = \left| (H - z)_{ij}^{-1} - m(z)S_{ij} \right|^{2p} = |G_{ij}(z) - m(z)S_{ij}|^{2p} = |W_{ij}|^{2p}$$

We consider replacing the (k, ℓ) entry of the H matrix with the corresponding entry from H' . If we define their difference to be $\Delta = H_{k\ell} - H'_{k\ell}$, the Taylor expansion becomes:

$$F(H') - F(H) = \Delta \partial_{H_{k\ell}} F(H) + \frac{\Delta^2}{2} \partial_{H_{k\ell}}^2 F(H) + \frac{\Delta^3}{3!} \partial_{H_{k\ell}}^3 F(H) + \frac{\Delta^4}{4!} \partial_{H_{k\ell}}^4 F(H) + \frac{\Delta^5}{5!} \partial_{H_{k\ell}}^5 F(H) + R_6$$

When the distributions match up to the 4th moment, we have:

$$\mathbb{E}[F(H') - F(H)] = \frac{\mathbb{E}[\Delta^5]}{2} \mathbb{E}[\partial_{H_{k\ell}}^5 F(H)] + \mathbb{E}[R_6]$$

In order to estimate this expression, we need to calculate the derivatives. Let us start with the first derivative of our function with respect to the matrix entry

First, let me establish the relationship between these matrices:

- H is the base matrix

- G is the resolvent of H , defined as $G = (H - zI)^{-1}$ where z is a complex parameter
- $W_{ij} = G_{ij} - mS_{ij}$ is the difference between the resolvent and some scaled matrix S
- We want to find $\frac{\partial |W_{ij}|^{2p}}{\partial H_{kl}}$

I'll use the chain rule since $|W_{ij}|^{2p}$ depends on H_{kl} through G :

$$\frac{\partial |W_{ij}|^{2p}}{\partial H_{kl}} = \frac{\partial |W_{ij}|^{2p}}{\partial W_{ij}} \cdot \frac{\partial W_{ij}}{\partial G_{ij}} \cdot \frac{\partial G_{ij}}{\partial H_{kl}} + \frac{\partial |W_{ij}|^{2p}}{\partial W_{ij}^*} \cdot \frac{\partial W_{ij}^*}{\partial G_{ij}^*} \cdot \frac{\partial G_{ij}^*}{\partial H_{kl}}$$

The second term appears because we're dealing with complex numbers, and $|W_{ij}|^2 = W_{ij} \cdot W_{ij}^*$. This is why you're seeing real components emerge.

For a complex number W , we have $|W|^{2p} = (W \cdot W^*)^p$

Using complex differentiation:

$$\frac{\partial |W_{ij}|^{2p}}{\partial W_{ij}} = p|W_{ij}|^{2p-2} \cdot W_{ij}^*$$

$$\frac{\partial |W_{ij}|^{2p}}{\partial W_{ij}^*} = p|W_{ij}|^{2p-2} \cdot W_{ij}$$

Since $W_{ij} = G_{ij} - mS_{ij}$:

$$\frac{\partial W_{ij}}{\partial G_{ij}} = 1$$

$$\frac{\partial W_{ij}^*}{\partial G_{ij}^*} = 1$$

This is where the resolvent identity becomes crucial. For a resolvent $G = (H - zI)^{-1}$, we have:

$$\frac{\partial G_{ij}}{\partial H_{kl}} = -G_{ik}G_{lj}$$

This identity follows from differentiating the equation $(H - zI)G = I$ and applying the product rule.

Since we're differentiating with respect to a real parameter H_{kl} (assuming H is Hermitian):

$$\frac{\partial G_{ij}^*}{\partial H_{kl}} = -G_{ik}^*G_{lj}^*$$

Now we can substitute all these derivatives into our chain rule expression:

$$\frac{\partial |W_{ij}|^{2p}}{\partial H_{kl}} = p|W_{ij}|^{2p-2} \cdot W_{ij}^* \cdot 1 \cdot (-G_{ik}G_{lj}) + p|W_{ij}|^{2p-2} \cdot W_{ij} \cdot 1 \cdot (-G_{ik}^*G_{lj}^*)$$

Simplifying:

$$\frac{\partial |W_{ij}|^{2p}}{\partial H_{kl}} = -p|W_{ij}|^{2p-2} \cdot [W_{ij}^* \cdot G_{ik}G_{lj} + W_{ij} \cdot G_{ik}^*G_{lj}^*]$$

This can be rewritten as:

$$\frac{\partial |W_{ij}|^{2p}}{\partial H_{kl}} = -p|W_{ij}|^{2p-2} \cdot [W_{ij}^* \cdot G_{ik}G_{lj} + (W_{ij}^* \cdot G_{ik}G_{lj})^*]$$

$$\frac{\partial |W_{ij}|^{2p}}{\partial H_{kl}} = -2p|W_{ij}|^{2p-2} \cdot \text{Re}(W_{ij}^* \cdot G_{ik}G_{lj})$$

$$\frac{\partial |W_{ij}|^{2p}}{\partial H_{kl}} = -2p|W_{ij}|^{2p-2} \cdot \text{Re}(W_{ij}^* \cdot G_{ik}G_{lj})$$

Where:

- $W_{ij} = G_{ij} - mS_{ij}$
- W_{ij}^* is the complex conjugate of W_{ij}
- $\text{Re}()$ denotes the real part of the complex expression

6.4 Old notes

$H_{k\ell}$:

$$\partial_{H_{k\ell}} F(H) = \partial_{H_{k\ell}} |W_{ij}|^{2p} = 2p|W_{ij}|^{2p-2} \text{Re}(W_{ij} \partial_{H_{k\ell}} \overline{W_{ij}}) = -2p|W_{ij}|^{2p-2} \text{Re}(W_{ij} \overline{G_{ik}G_{\ell j}})$$

Similarly, the second derivative is:

$$\begin{aligned} \partial_{H_{k\ell}}^2 F(H) &= \partial_{H_{k\ell}} [2p|W_{ij}|^{2p-2} \text{Re}(W_{ij} \partial_{H_{k\ell}} \overline{W_{ij}})] = \\ &= -2p \partial_{H_{k\ell}} [|W_{ij}|^{2p-2}] \text{Re}(W_{ij} \overline{G_{ik}G_{\ell j}}) - 2p|W_{ij}|^{2p-2} \partial_{H_{k\ell}} [\text{Re}(W_{ij} \overline{G_{ik}G_{\ell j}})] \end{aligned}$$

The derivative in the first term is:

$$\begin{aligned} \partial_{H_{k\ell}} [|W_{ij}|^{2p-2}] &= (2p-2)|W_{ij}|^{2p-4} \partial_{H_{k\ell}} [|W_{ij}|^2] = (2p-2)|W_{ij}|^{2p-4} (\partial_{H_{k\ell}} W_{ij} \cdot \overline{W_{ij}} + W_{ij} \cdot \partial_{H_{k\ell}} \overline{W_{ij}}) = \\ &= -(2p-2)|W_{ij}|^{2p-4} (G_{ik}G_{\ell j} \cdot \overline{W_{ij}} + W_{ij} \cdot \overline{G_{ik}G_{\ell j}}) = -(4p-4)|W_{ij}|^{2p-4} \text{Re}(W_{ij} \cdot \overline{G_{ik}G_{\ell j}}) \end{aligned}$$

For the second term, we have:

$$\begin{aligned}\partial_{H_{k\ell}} [\text{Re} (W_{ij} \overline{G_{ik} G_{\ell j}})] &= \text{Re} (\partial_{H_{k\ell}} W_{ij} \overline{G_{ik} G_{\ell j}} + W_{ij} \partial_{H_{k\ell}} \overline{G_{ik} G_{\ell j}}) = \\ &= \text{Re} (-G_{ik} G_{\ell j} \overline{G_{ik} G_{\ell j}} + W_{ij} \partial_{H_{k\ell}} [\overline{G_{ik} G_{\ell j}}]) = -\text{Re} (G_{ik} G_{\ell j} \overline{G_{ik} G_{\ell j}} + W_{ij} \overline{G_{ik} G_{kk} G_{\ell j}} + \overline{G_{ik} G_{\ell \ell} G_{\ell j}})\end{aligned}$$

As such, by combining the terms, we get the second derivative expansion:

$$(8p^2 - 8p)|W_{ij}|^{2p-4} \text{Re}^2 (W_{ij} \overline{G_{ik} G_{\ell j}}) + 2p|W_{ij}|^{2p-2} (|G_{ik} G_{\ell j}|^2 + \text{Re} (W_{ij} \overline{G_{ik} G_{kk} G_{\ell j}}) + \text{Re} (\overline{G_{ik} G_{\ell \ell} G_{\ell j}}))$$

In order to derive the third derivative, we need to differentiate each of the terms above. By applying the chain rule and using our previous calculations, we have:

$$\begin{aligned}&-48p(p-1)(p-2)|W_{ij}|^{2p-6} \text{Re}^3 (W_{ij} \overline{G_{ik} G_{\ell j}}) + 24p(p-1)|W_{ij}|^{2p-4} \text{Re} (W_{ij} \overline{G_{ik} G_{\ell j}}) \text{Re} (W_{ij} \overline{G_{ii} G_{kk} G_{\ell j}} + W_{ij} \overline{G_{ik} G_{\ell \ell} G_{\ell j}}) \\ &-4p(2p-2)|W_{ij}|^{2p-4} \text{Re} (W_{ij} \overline{G_{ik} G_{\ell j}}) |G_{ik} G_{\ell j}|^2 - 2p|W_{ij}|^{2p-2} \text{Re} (G_{ik} G_{\ell j} \overline{G_{ii} G_{kk} G_{\ell j}} + G_{ik} G_{\ell j} \overline{G_{ik} G_{\ell \ell} G_{jj}}) - \\ &-2p|W_{ij}|^{2p-2} \text{Re} (W_{ij} \overline{G_{ii} G_{kk} G_{ii} G_{kk} G_{\ell j}}) - 2p|W_{ij}|^{2p-2} \text{Re} (W_{ij} \overline{G_{ii} G_{kk} G_{ik} G_{\ell \ell} G_{jj}}) - \\ &-2p|W_{ij}|^{2p-2} \text{Re} (W_{ij} \overline{G_{ik} G_{\ell k} G_{\ell \ell} G_{jj}}) - 2p|W_{ij}|^{2p-2} \text{Re} (W_{ij} \overline{G_{ik} G_{\ell \ell} G_{jk} G_{\ell j}}) - 2p|W_{ij}|^{2p-2} \text{Re} (W_{ij} \overline{G_{ik} G_{\ell i} G_{kk} G_{\ell j}}) - \\ &-2p|W_{ij}|^{2p-2} \text{Re} (W_{ij} \overline{G_{ii} G_{k\ell} G_{\ell k} G_{\ell j}}) - 2p|W_{ij}|^{2p-2} \text{Re} (W_{ij} \overline{G_{ii} G_{kk} G_{\ell \ell} G_{j\ell}})\end{aligned}$$

Generalization

As we can see, continuing with the explicit calculation of the derivatives is going to be quite unruly due to the factorial growth. For this reason, we would like to instead abstract away the estimates, focusing on the primary terms. Let us first make several observations before we prove them rigorously. We posit that the general form is along the lines of:

$$\partial_{H_{k\ell}}^{(n)} F(H) = \sum_{s=1}^n c_s(p) |W_{ij}|^{2p-2s} \cdot R_s^{(n)}(G, W),$$

where s indexes the power of $|W_{ij}|$, $c_s(p)$ is the polynomial coefficient in p and $T_s(G, W)$ represents the real terms from the product of G and W , i.e. the resolvent and its fluctuation, respectively. Pattern-matching for our earlier calculations we can observe that:

1. First derivative ($s = 1$): $\partial_{H_{k\ell}} F(H) = -2p|W_{ij}|^{2p-2} \text{Re} (W_{ij} \overline{G_{ik} G_{\ell j}})$:

- $c_1(p) = 1$ with $R_1^{(1)} = \text{Re} (W_{ij} \overline{G_{ik} G_{\ell j}})$

2. Second Derivative $\partial_{H_{k\ell}}^{(2)} F(H)$:

- $c_1(p) = 2p$ with $R_1^{(2)} = |G_{ik}G_{\ell j}|^2 + \text{Re}(W_{ij}\overline{G_{ik}G_{kk}G_{\ell j}} + \overline{G_{ik}G_{\ell\ell}G_{\ell j}})$
- $c_2(p) = 8p(p-1)$ with $R_2 = (R_1^{(1)})^2$.

3. Third derivative $\partial_{H_{k\ell}}^{(3)} F(H)$:

- $c_1(p) = -2^1 \cdot 1! \cdot p$ with $R_1^{(3)}$ being the sum of terms of the form $\text{Re}(W_{ij}\overline{G_1}, \overline{G_2}, \overline{G_3}, \overline{G_4})$
- $c_2(p) = 8p(p-1) = 2^2 \cdot 2!(p)_2$ with $R_2^{(3)} = \text{Re}(W_{ij}\overline{G_{ik}G_{\ell j}}) \text{Re}(W_{ij}\overline{G_{ii}G_{kk}G_{\ell j}} + W_{ij}\overline{G_{ik}G_{\ell\ell}G_{jj}})$
- $c_3(p) - 48p(p-1)(p-2) = -2^3 \cdot 3!(p)_3$ with $R_3^{(3)} = (R_1^{(1)})^3$

Let us verify the following coefficient formula rigorously $c_n = (-1)^n 2^n \cdot n!(p)_n$. The base case is clearly satisfied, so assume it holds for n . Observe that it necessarily comes from the first term and as such, we need to worry about only two of its components. Namely, By differentiating the $|W_{ij}|^{2p-2n}$, we get a factor of $-(2p-2n)$ that reduces the power by 2 and differentiating $\text{Re}^n(W_{ij}\overline{G_{ik}G_{\ell j}})$, we get an additional factor. As such, the combined result becomes $c_{n+1} = -2(n+1)(p-n)c_n = (-1)^{n+1} 2^{n+1}(n+1)!(p)_{n+1}$

6.5 Replacement

Let $\rho \in [0, 1]$. Define:

$$H(\rho) = H(H_{k\ell}, H_{\ell k} \mapsto \rho H_{k\ell}, \rho H_{\ell k})$$

$$H'(\rho) = H'(H'_{k\ell}, H'_{\ell k} \mapsto \rho H'_{k\ell}, \rho H'_{\ell k})$$

Let us now consider the Taylor expansion w.r.t. to the latter scaling variable ρ up to exponent n :

$$F(H) = F(H)(0) + \frac{d}{d\rho} F(H(\rho)) \Big|_{\rho=0} + \frac{d^2}{d\rho^2} F(H(\rho)) \Big|_{\rho=0} + \cdots + \frac{d^{n+1}}{d\rho^{n+1}} F(H(\rho)) \Big|_{\rho=0}$$

Focusing on the last term, we have:

$$\frac{d^{n+1}}{d\rho^{n+1}} F(H(\rho)) = \sum_{\alpha=0}^{n-1} H_{k\ell}^\alpha H_{\ell k}^{n+1-\alpha} C_\alpha,$$

where C_α is some function of $F(H(\rho))$. Under the assumption (justify) C_{n+1}

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