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String-inspired cosmologies with Torsion

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Abstract

This thesis investigates two gravitational models with torsion: the Einstein-Cartan-Sciamma-Kibble (ECSK) theory and a low-energy effective string-inspired approach. In the context of the latter, this assignment explores the effects of torsion contribution in the cosmological evolution of the universe.

Starting with the ECSK theory, torsion emerges as a geometrical property of spacetime, where it couples to matter spin. Quantum anomaly corrections to torsion conservation induce an axion-like degree of freedom, and spinspin interactions become significant at high spin densities, replacing the Big Bang singularity with a Big Bounce scenario.

In the string-inspired model, which is the primary focus of this thesis, torsion arises as the field strength of the KalbRamond (KR) antisymmetric tensor. Through the cancellation of gravitational anomalies in the early universe, an axion-like pseudoscalar field is naturally introduced, coupling to gravity via a ChernSimons term. This coupling modifies the Einstein field equations by introducing a Cotton tensor, leading to parity-violating corrections in the propagation of gravitational waves in a perturbed FLRW universe. These perturbations are connected to primordial gravitational waves in this model (rising in the context of a first hill-top inflationary scenario).

Finally, the quantum expectation value of the anomaly-induced condensate contributes a term analogous to H^4 during the stiff-matter era, when the kinetic term of the axion-like field dominates, leading to an inflationary phase similar to the Running Vacuum Model (RVM). During inflation, the anomaly condensate remains approximately constant, resulting in a quaside Sitterlike expansion. This effective energy density can be described by a nearly linear potential arising from shift-symmetry breaking and can be interpreted as a positive cosmological constant driving the accelerated expansion.

Keywords

Torsion, General Relativity, Einstein-Cartan-Schiamma-Kibble theory, QED in contorted spacetime, String theory, Hirzebruch-Pontryagin topological density, Cotton tensor, QFT in curved spacetime, Gravitational waves, cosmology, Gravitational Wave condensate, stiff matter era, inflation.

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Contents

1 Introduction

The current cosmological model, based on the Friedmann-Lemaître-Robertson-Walker (FLRW) solution of General Relativity (GR) with a positive cosmological constant (> 0) and Cold Dark Matter (CDM), accurately describes the universe's large-scale structure and evolution, with satisfying precision to observational data[1]. The FLRW metric assumes a homogeneous and isotropic universe and is given by:

$$ds^2 = -dt^2 + \alpha^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right] \tag{1.0.1}$$

where (t, r, θ, ϕ) are the co-moving coordinates, $\alpha(t)$ is the cosmic scale-factor and has dimensions $[L]$ and $k = -1, 0, 1$ is the spatial curvature. Current cosmological observations (e.g., from the CMB) strongly suggest that the universe today is spatially flat, i.e., k = 0.

The main way to model the universes energy content in a way consistent with the cosmological principle is to assume a cosmological perfect fluid. This simplification allows for various cosmic components, such as dark matter, baryonic or vacuum energy, to be described by the Stress-energy tensor that respects homogeneity and isotropy.

In the standard cosmological evolution of the universe, several distinct eras appear. An early inflationary epoch solves key fine-tuning issues such as the flatness, entropy, and horizon problems. This is followed by the radiation-dominated era, during which high-energy processes such as leptogenesis and nucleosynthesis take place. As the universe expands and cools, it transitions into a matter-dominated phase, enabling the formation of structure. Observations of the late-time accelerated expansion suggest that the universe has entered a dark energy-dominated phase, often modeled by the cosmological constant Λ.

While the ΛCDM model (universe with cosmological constant Λ and Cold Dark Matter) provides an excellent framework for describing the late-time acceleration of the universe, significant questions remain unresolved, particularly concerning the earliest stages of cosmic evolution. The physics of the inflationary epoch, which is invoked to solve the flatness, entropy, and horizon problems, still lacks a definitive theoretical underpinning. In the standard approach, inflation is typically driven by a hypothetical scalar inflaton field with a phenomenologically introduced scalar potential, lacking a deeper theoretical justification.

Furthermore, persistent observational tensions such as the discrepancy in measurements of the Hubble constant H_0 and the amplitude of matter fluctuations σ_8 as well as the unexplained nature of dark matter and dark energy, suggest the need for a more fundamental or generalized cosmological framework.

In this context, the model of this assignment focuses on the very early universe and proposes a string-inspired modification of General Relativity. In this approach, we use the low-energy effective string action, where the role of the Kalb-Ramond (KR) field strength is the role of torsion. Through corrections made at a quantum level, this field induces an $R\tilde{R}$-term and gives rise to an axion-like pseudoscalar degree of freedom $b(x)$.

Notably, the coupling between gravity and the axion-like pseudoscalar field behaves as a topological current, denoted as the Pontryagin density, simplifying the effective theory's total action. This current eventually modifies GR by the introduction of the Cotton tensor in the Einstein field equations, which accounts for the energy exchanged between the graviton and this pseudoscalar axion-like field.

This contribution is visible only in a perturbative FLRW universe, where, due to the Cotton tensor contribution, there are parity-violating corrections in the propagation of gravitational waves (GWs). Specifically, the equations' evolution depends on the helicity mode and introduces the phenomenon of cosmological birefringence. The expectation value of the Chern-Simons term can be computed using the formalism of quantum field theory in curved spacetime, and its explicit form depends on the chosen cosmological era. This condensate emerges at the end of our universe's stiff-matter-dominated era ($w = 1$) and eventually drives an RVM-like (Running Vacuum Model[2]) inflation, during which it remains stable.

In the context of the previously outlined cosmological evolution, the conventional inflationary models included a postulated scalar inflaton field that slowly rolls down a potential, driving inflation. In our model, the role of this inflaton field is performed by the axion-like degree of freedom $b(x)$ that introduces an RVM inflationary phase driven by a gravitational wave (GW) condensate.[3]

This assignment will present the mathematical tools necessary to explore Einstein-Cartan theory in a contorted spacetime, as outlined in [4]. We will demonstrate how axion-like currents arise naturally from the incorporation of torsion into the manifold and discuss the physical implications of these phenomena. Additionally, we will link this theory to the string-inspired theory with torsion, where the latter emerges naturally in the low-energy limit due to the variations of the KR field. We resolve quantum anomalies via the introduction of the Chern-Simons terms and simplify the theory through the Pontryagin 4-current density. After introducing the Cotton tensor in the Einstein equations, we consider metric perturbations in the FLRW universe and study the anisotropic GW equations. Using the method of field quantization in a curved background, a first evaluation of the R_{CS} Chern-Simons condensate is given. Finally, following a brief introduction to the Running Vacuum Model, we analyze the cosmological evolution and track the behavior of the R_{CS} condensate throughout the stiff matter and inflationary phases featured in this model.

2 Torsion in Einstein-Cartan-Sciama-Kibble (ECSK)

Torsion, like metric compatibility, is a fundamental mathematical property of spacetime. While metric compatibility ensures the conservation of inner products under parallel transport, torsion accounts for the rotation of the geometrical objects during parallel transport. What is interesting is that, in the framework of Einstein-Cartan-Schiamma-Kibble, it has been proved that the coupling between torsion and spinors induces a spin-spin self-interaction (4-fermion interaction) which becomes significant in high spin densities, and it can lead to the replacement of the big bang with the so-called big bounce[5].

In this section, we explore the Einstein-Cartan-Sciama-Kibble extension of General Relativity, focusing on how torsion is incorporated into the gravitational action. We also examine the implementation of Quantum Electrodynamics (QED) and the axionic degree of freedom that appears due to quantum anomaly cancellations in this framework.

Before starting the exploration of the properties of a contorted universe, the mathematical framework must be provided. The main parts that compose the framework are the tensor definition, the operations, and the basis, i.e., metric and coordinate system.

2.1 Mathematics

2.1.1 Spacetime properties

It is important to mention and explain the notation used in this assignment. Let $\mathcal{M}$ be a (3+1)-dimensional manifold parametrized by coordinates x^μ, with $\mu = 0, 1, 2, 3$, described by $g_{\mu\nu}$ metric. $T_p\mathcal{M}$ is the corresponding tangent space at point p described by the corresponding η_{ab} Minkowski metric. Using the vielbein notation (analyzed below), one can express the metric as:

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab} \tag{2.1.1}$$

For the vielbein components, $e_\mu{}^a$ are the elements of the matrix that transform objects from the natural coordinate basis ∂_μ to the orthonormal basis e_a. Conversely, we define $e^\mu{}_a$ as the components of the inverse matrix, which maps objects from the orthonormal basis back to the coordinate basis. These matrices satisfy the following relations:

$$e^\mu{}_a e_\nu{}^a = \delta^\mu_\nu \text{ and } e_\mu{}^a e^\nu{}_b = \delta^a_b \tag{2.1.2}$$

In this assignment, we use objects that have Greek and/or Latin indices. Latin indices are locally flat indices that are used on objects that "live" on Lorentzian spacetime or else on a Tangent space described by the Minkowski metric η_{ab}. They are independent of the coordinate system used. Greek indices are used on objects that "live" on a specific chart of our manifold, which is described by the GR metric tensor $g_{\mu\nu}$. The main purpose during the calculations is to eliminate the Greek indices to obtain a more simplified and global notation, easy to handle. The transformation of a vector or a tensor is something like a change of basis:

$$\text{Vector: } V^a = e^a_\mu V^\mu \tag{2.1.3}$$

$$\text{Tensor: } T^a_b = e^a_\mu T^\mu_b = e^a_\mu e^\nu_b T^\mu_\nu \tag{2.1.4}$$

In the two examples above, the components V^a and T^a_μ are the vector and tensor components, as seen from the viewpoint of an observer in a local Minkowski space that is defined by the orthonormal basis $e^a = e^a_\mu dx^\mu$. In other words, it is like a transformation of the objects' components from the curved spacetime coordinate basis (Greek indices) to a locally flat frame described by an orthonormal basis (Latin indices) and the Minkowski metric η_{ab}.

This notation is useful for two reasons. First, we can describe spinor fields, which in general "live" in a Lorentzian space, and calculate their covariant derivatives, adding an extra feature to the Theory. Second, we can think of tensors as tensor-valued differential forms; more specifically, one can think of a p-form defined on a chart of the manifold, whose components are tensors.

2.1.2 Wedge product

Denoted as $\wedge$, the wedge product, or else exterior product, is defined as the anti-symmetric tensor product of the cotangent space basis elements. The mathematical definition is shown below:

$$dx^\mu \wedge dx^\nu = \frac{1}{2}(dx^\mu \times dx^\nu - dx^\nu \times dx^\mu) = -dx^\nu \wedge dx^\mu \tag{2.1.5}$$

One general property of the wedge product for a q-form a_q and a p-form b_p is:

$$a \wedge b = (-1)^{pq}\ b \wedge a \tag{2.1.6}$$

2.1.3 Introduction of Spacetime Torsion-Geometry and explanation

Before introducing the mathematical objects to study the theory of a spacetime with torsion, we must talk about torsion from a more physical point of view.

To begin with, we will explore the concept of Lie brackets of two covariant derivatives. The Lie brackets of two vector fields illustrate how one vector field changes with respect to the other. More specifically, in the expression $[\nabla_\mu, \nabla_\nu]f = \nabla_\mu \nabla_\nu f - \nabla_\nu \nabla_\mu f$, the first term represents how ∇_ν changes in the direction of ∇_μ and vice versa for the second term.

The space where those two covariant derivatives are independent from each other, meaning that they don't change while being parallely transported across each other's flow curves, is torsion-free. On the contrary, in a space that is twisted, there is torsion; each parallel-transported vector across the flow lines changes. This change is the failure to close a standard parallelogram in the case where the space has no torsion, and it is given by this exact Lie bracket operator.

Let us explain what happens in Figure 1. v_P, a vector at point P, is parallel transported to the point Q, and is denoted as $v^{\parallel}_Q$, and the same thing happens with the parallel transport of u_P, which is a vector at point P, to the point R, denoted as $u^{\parallel}_R$. The actual vectors in the vector field look like v_Q and v_R correspondingly. We can measure the difference between $v^{\parallel}_Q$ and v_Q by taking the covariant derivative of v with respect to u, and $u^{\parallel}_R$ and u_R by taking the covariant derivative of u with respect to v. Those two covariant derivatives show us how much the vectors u, v deviate from the parallel transported ones. Now, we can measure the difference between those two covariant derivatives by taking the Lie bracket. Denoted as T, the so-called Torsion tensor represents the difference of the parallel transported vectors $v^{\parallel}_Q$ and $u^{\parallel}_R$. It is found to be:

$$T(\vec{u}, \vec{v}) = \nabla_{\vec{u}}\vec{v} - [\vec{u}, \vec{v}] - \nabla_{\vec{v}}\vec{u} \tag{2.1.7}$$

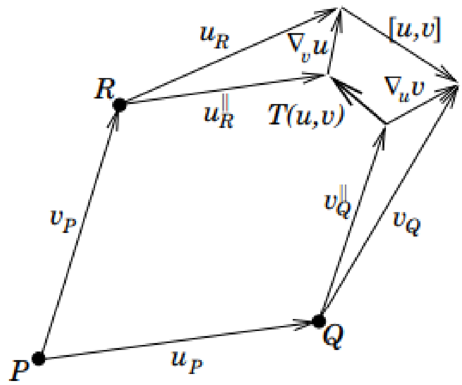

Figure 1: Torsion representation[7].

This is the formula of the Torsion tensor acting on two vector fields u and v. In case the result of this equation is equal to zero, there is no torsion, and we get a closed loop.

In order to analyze further the torsion tensor and the rest of the geometrical tools of our manifold, we need to introduce objects such as the connection, exterior derivatives and the curvature tensors.

2.1.4 Derivatives

To carry out a detailed study of the curvature of a manifold, one must introduce an affine **connection** that relates vectors from tangent spaces at nearby points. In a coordinate (curved) basis, this is achieved through the **Christoffel symbols**, denoted as $\Gamma^{\lambda}_{\mu\nu}$. The Christoffel symbols are used as correction terms in the **covariant derivative** to properly define the parallel transport of a vector in a way that is compatible with the curvature of the manifold.

However, when working in the vielbein formalism, which introduces an orthonormal frame at each point of the manifold, an additional connection must be defined: the **affine spin connection** ${\omega^a}_b$. This spin connection is used for the differentiation of tensor fields that "live" on the orthonormal basis of the tangent space and have flat (Latin) indices.

For the orthonormal coordinates in the tangent space of a point, we define the affine spin connection as a matrix-valued one-form:

$$ {\omega^a}_b = {\omega^a}_{\mu b} dx^{\mu} \tag{2.1.8} $$

This object encapsulates how vielbein vectors rotate as we move through spacetime. Specifically, it shows how the b-th direction on the tangent space rotates into the a-th direction as we move along x^{μ}. Its role is analogous to the Christoffel symbols, but it is adapted to the local Lorentzian frame.

While the covariant derivative of a vector with spacetime coordinates is denoted as:

$$ \nabla_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma^{\nu}_{\mu\lambda} V^{\lambda} \tag{2.1.9} $$

The covariant derivative of a vector expressed in the orthonormal basis (i.e., with Latin indices that "live" in the tangent space) is given by:

$$ \nabla_{\mu} V^{a} = \partial_{\mu} V^{a} + {\omega_{\mu}}^{a}{}_{b} V^{b} \tag{2.1.10} $$

The **exterior derivative** acts on a p-form field and returns the (p+1)-form field that measures the rate of change of the field itself, without accounting for the curvature of the manifold. For a simple real scalar field ϕ (0-form), the exterior derivative $d\phi = \partial_\mu \phi \; dx^\mu$ represents the rate of change of the field along the spacetime directions. The general definition of the exterior derivative is:

$$(dX)_{\mu\nu} = \partial_\mu X_\nu - \partial_\nu X_\mu \tag{2.1.11}$$

Having said that, to include curvature and torsion effects when differentiating, especially tensor-valued one-forms, we will use the **exterior covariant derivative**. Acting on a q-form $Q^{\alpha,\dots}_{b,\dots}$ the exterior covariant derivative combines d with the affine spin connection ω and accounts for local Lorentz symmetry:

$$D(\omega)Q^{a,\dots}{}_{b,\dots} = dQ^{a,\dots}{}_{b,\dots} + \omega^a{}_c \wedge Q^{c,\dots}{}_{b,\dots} + \dots - (-1)^q Q^{a,\dots}{}_{d,\dots} \wedge \omega^d{}_b \tag{2.1.12}$$

One very important property of the tangent space at a point p on the manifold is the metric compatibility. The Minkowski metric describes the tangent space, so the expression for the metric compatibility is:

$$\nabla \eta_{ab} = 0 \tag{2.1.13}$$

Metric-compatible spaces preserve the inner product of parallel-transported vectors or other objects.

Using the **metric compatibility** property in our spacetime, we can prove the antisymmetrization of the affine spin connection one-form, which will be used for calculations later in the assignment:

$$D(\omega)\eta_{ab} = 0 \Rightarrow \cancelto{0}{d\eta_{ab}} - \eta_{cb} \wedge \omega^c{}_a - \eta_{ac} \wedge \omega^c{}_b = 0 \Rightarrow \eta_{cb}(\omega_\mu)^c{}_a = -\eta_{ac}(\omega_\mu)^c_b \Rightarrow \boxed{\boldsymbol{\omega}_{ab} = -\boldsymbol{\omega}_{ba}} \tag{2.1.14}$$

In the next section, we will see how the spin connection naturally enters into the definitions of both the curvature and torsion tensors, providing a geometric foundation for describing gravitational dynamics in spacetimes with torsion.

2.1.5 Mathematical objects used

Based on vielbein notation, we will introduce the main objects we need for our purposes. First, let us introduce the torsion tensor $\boldsymbol{T}^a$ (again with the Greek indices suppressed), the geometric derivation of which will be explained later. For now, we need the algebraic definition to illustrate some influences on the rest of the objects used for the spacetime description. The tensor in vielbein notation is written:

$$\boldsymbol{T}^a = \frac{1}{2} T^a{}_{\mu\nu} dx^\mu \wedge dx^\nu \equiv de^a + \omega^a{}_b \wedge e^b \tag{2.1.15}$$

Considering a spacetime with Torsion (something we will next explain geometrically), we can split this tensor into a **torsion-free one-form**, $\mathring{\omega}$, and **contorsion one-form**, $K^a{}_{b\mu}$. So, the spin connection can be written as:

$$\omega^a{}_{b\mu} = \mathring{\omega}^a{}_{b\mu} + K^a{}_{b\mu} \tag{2.1.16}$$

We can relate the contorsion tensor K_{abc} with the torsion tensor to make the calculations easier in the future. It has been proved that:

$$K_{abc} = -\frac{1}{2}(T_{cab} - T_{abc} - T_{bca}) \Rightarrow T_{[abc]} = -2K_{[abc]} \tag{2.1.17}$$

In this case we chose the notation in which K_{abc}, is antisymmetric in the 1st and 2nd index $K_{abc} = -K_{bac}$ (or else $K_{\mu\nu\lambda} = -K_{\nu\mu\lambda}$). To change the notation from Minkowski to manifold coordinates, we can simply do $K^{a}{}_{bc} = K^{a}{}_{b\mu}e^{\mu}{}_{c}$.

The next object we need to introduce is the two-form of the **generalized Riemann curvature** again with the entire tensor form (with the Greek indices included in bold) $\boldsymbol{R}^a_b$. It is defined as:

$$\boldsymbol{R}^{a}{}_{b} = d\omega^{a}{}_{b} + \omega^{a}{}_{c} \wedge \omega^{c}{}_{b} \tag{2.1.18}$$

Using the exterior covariant derivative definition, we will provide an alternative formula for the generalized Riemann curvature:

$$\begin{aligned}
\boldsymbol{R}^{a}{}_{b} &= d\omega^{a}{}_{b} + \omega^{a}{}_{c} \wedge \omega^{c}{}_{b} \\
&= d(\mathring{\omega}^{a}{}_{b} + K^{a}{}_{b}) + (\mathring{\omega}^{a}{}_{c} + K^{a}{}_{c}) \wedge (\mathring{\omega}^{c}{}_{b} + K^{c}{}_{b}) \\
&= \underbrace{d\mathring{\omega}^{a}{}_{b} + \mathring{\omega}^{a}{}_{c} \wedge \mathring{\omega}^{c}{}_{b}}_{\mathring{R}^{a}{}_{b}} + \underbrace{dK^{a}{}_{b} + \mathring{\omega}^{a}{}_{c} \wedge K^{c}{}_{b} + K^{a}{}_{c} \wedge \mathring{\omega}^{c}{}_{b}}_{D(\mathring{\omega})K^{a}{}_{b} = dK^{a}{}_{b} + \mathring{\omega}^{a}{}_{c} \wedge K^{c}{}_{b} - (-1)^{1} K^{a}{}_{c} \wedge \mathring{\omega}^{c}{}_{b}} + K^{a}{}_{c} \wedge K^{c}{}_{b} \\
&= \boxed{\mathring{R}^{a}{}_{b} + D(\mathring{\omega})K^{a}{}_{b} + K^{a}{}_{c} \wedge K^{c}{}_{b}}
\end{aligned} \tag{2.1.19}$$

The equations 2.1.15 and 2.1.18 are known as **Cartan structure equations**.

Let us derive the affine connection $\Gamma^{\lambda}_{\mu\nu}$ from the torsion tensor and explore at the same time their relation in a contorted spacetime. Using the equations below for the infinitesimal elements:

$$de^{a} = \frac{1}{2}\partial_{\mu}e^{a}{}_{\nu}(dx^{\mu} \wedge dx^{\nu}) + \frac{1}{2}e^{a}{}_{\mu}\underbrace{(dx^{\nu} \wedge dx^{\mu})}_{-(dx^{\mu} \wedge dx^{\nu})} \tag{2.1.20}$$

$$\omega \wedge e^{b} = \frac{1}{2}\omega^{a}{}_{b\mu}e^{b}{}_{\nu}(dx^{\mu} \wedge dx^{\nu}) - \frac{1}{2}\omega^{a}{}_{b\nu}e^{b}{}_{\mu}(dx^{\nu} \wedge dx^{\mu}) \tag{2.1.21}$$

The torsion tensor, using again differential-form notation, appears to be:

$$\begin{aligned}
\boldsymbol{T}^{a} &= [\partial_{\mu}e^{a}{}_{\nu} - \partial_{\nu}e^{a}{}_{\mu}](dx^{\mu} \wedge dx^{\nu}) + [\omega^{a}{}_{b\mu}e^{b}{}_{\mu} - \omega^{a}{}_{b\nu}e^{b}{}_{\nu}](dx^{\mu} \wedge dx^{\nu}) \\
&= [\partial_{\mu}e^{a}{}_{\nu} + \omega^{a}{}_{b\mu}e^{b}{}_{\mu} - (\partial_{\nu}e^{a}{}_{\mu} + \omega^{a}{}_{b\nu}e^{b}{}_{\nu})](dx^{\mu} \wedge dx^{\nu}) \\
&= [\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}](dx^{\mu} \wedge dx^{\nu}) \\
&= \Gamma^{\lambda}{}_{[\mu\nu]}(dx^{\mu} \wedge dx^{\nu})
\end{aligned} \tag{2.1.22}$$

where $[\mu\nu]$ shows the antisymmetrization in the indices of the affine connection. We calculated the abstract one-form object suppressing the Greek indices, so the components of each object defined by the Greek indices will be:

$$T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}{}_{[\mu\nu]} \tag{2.1.23}$$

Having now established the geometric and algebraic framework of spacetimes with torsion using the vielbein formalism, we are equipped with all the necessary tools to analyze gravitational theories beyond General Relativity. In the next section, we will explore how torsion modifies the variational structure of gravity by extending the EinsteinHilbert action. We will derive the field equations resulting from the inclusion of torsion and examine how the dynamics of the spin connection and vielbein lead to a more general theory of spacetime geometry, commonly known as EinsteinCartan theory or RiemannCartan geometry.

2.2 Physics of gravitation with Torsion

Starting with the standard Einstein-Hilbert gravitational action, modifications will be applied based on the new torsion tensor, which was introduced to study the impacts of the torsion component in General Relativity. We shall bring the action to a form convenient for analyzing its impacts on cosmology.

2.2.1 Derivation of the gravitational action

The classical Einstein-Hilbert action can be expressed using the curvature 2-form R^{ab} defined via the affine spin connection $\omega^a{}_b$. This allows us to consider the torsion effects via the contorsion tensor $K^a{}_b$. Working in vielbein formalism, the action reads:

$$
\begin{aligned}
S_{\mathrm{EH}} &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\boldsymbol{R} \\
&\overset{R=R^{ab}{}_{ab}}{=} \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R^{ab}{}_{[\mu\nu]}e^{\mu}{}_a e^{\nu}{}_b \\
&= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\frac{1}{2}R^{ab}{}_{\mu\nu}(e^{\mu}{}_a e^{\nu}{}_b - e^{\nu}{}_a e^{\mu}{}_b) \\
&= \frac{1}{4\kappa^2}\int d^4x\sqrt{-g}R^{ab}{}_{\mu\nu}(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\nu}_{\rho}\delta^{\mu}_{\sigma})e^{\rho}{}_a e^{\sigma}{}_b
\end{aligned}
$$

also, we used reference [8] to calculate the below useful quantities:

- The 4-dimensional oriented volume element in a curved spacetime is defined using the wedge product of the coordinate differentials:

$$\epsilon^{\mu\nu\kappa\lambda}d^4x\sqrt{-g} = dx^{\mu}\wedge dx^{\nu}\wedge dx^{\kappa}\wedge dx^{\lambda} \tag{2.2.1}$$

  Here, $\epsilon^{\mu\nu\kappa\lambda}$ is the Levi-Civita symbol, which encodes the antisymmetrization of the wedge product on the right-hand side, $\sqrt{-g}$ is the determinant of the metric tensor, which accounts for the proper volume scaling in the manifold.

- The contraction of two Levi-Civita symbols yields a combination of Kronecker deltas that projects antisymmetric index combinations:

$$(\delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\nu}_{\rho}\delta^{\mu}_{\sigma}) = \frac{1}{2}\epsilon_{\rho\sigma\kappa\lambda}\epsilon^{\mu\nu\kappa\lambda} \tag{2.2.2}$$

- The Hodge dual operator $\star$ maps a p-form to a $(4-p)$-form in four-dimensional spacetime, using the Levi-Civita symbol to encode orientation. The Hodge dual of a wedge product of two vielbeins is given by:

$$\star e^a \wedge e^b = \frac{1}{2}\epsilon_{\rho\sigma\kappa\lambda}e^{a\rho}e^{b\sigma}dx^{\kappa}\wedge dx^{\lambda} \tag{2.2.3}$$

And the corresponding general form is[8]:

$$\star(\underbrace{e^a \wedge ... \wedge e^b}_{p-form}) = \frac{1}{(4-p)!}\epsilon^{a...b}{}_{c...d}(\underbrace{e^c \wedge ... \wedge e^d}_{(4-p)-form}) \tag{2.2.4}$$

so using 2.2.1 and 2.2.2 at the same time we get:

$$\begin{aligned}
S_{\text{EH}} &= \frac{1}{8\kappa^2}\int (dx^\mu \wedge dx^\nu \wedge dx^\kappa \wedge dx^\lambda) R^{ab}{}_{\mu\nu} e^\rho{}_a e^\sigma{}_b \epsilon_{\rho\sigma\kappa\lambda} \\
&= \frac{1}{2\kappa^2}\int \left(\frac{1}{2}R^{ab}{}_{\mu\nu} dx^\mu \wedge dx^\nu\right) \wedge \left(\frac{1}{2}\epsilon_{\rho\sigma\kappa\lambda} e^{\rho a'} e^{\sigma b'} dx^\kappa \wedge dx^\lambda\right) \eta_{aa'}\eta_{bb'} \\
&\overset{2.2.3}{=} \frac{1}{2\kappa^2}\int \left(\boldsymbol{R}^{ab}\eta_{aa'}\eta_{bb'}\right) \wedge \star\left(e^{a'} \wedge e^{b'}\right) \\
&= \frac{1}{2\kappa^2}\int \left(\boldsymbol{R}_{a'b'}\right) \wedge \star\left(e^{a'} \wedge e^{b'}\right)
\end{aligned}$$

And the final result using the equation 2.1.19 will be, eventually:

$$S_{\text{EH}} = \frac{1}{2\kappa^2}\int (\boldsymbol{R}_{ab}) \wedge \star\left(e^a \wedge e^b\right) = \frac{1}{2\kappa^2}\int (\mathring{\boldsymbol{R}}_{ab} + D(\mathring{\omega})\boldsymbol{K}_{ab} + \boldsymbol{K}_{ac} \wedge \boldsymbol{K}^c{}_b) \wedge \star\left(e^a \wedge e^b\right) \tag{2.2.5}$$

where $\mathring{R}_{ab}$ is the torsion-free Ricci tensor, $\kappa^2 = 8\pi G$, G is Newton's gravitational constant and $D(\mathring{\omega}$ is the exterior covariant derivative of the torsion-free affine spin connection.

From classical general relativity, it is known that we can use the Stokes theorem for the boundary condition in a volume V, to simplify the formula:

$$\int_V D(\mathring{\omega})\boldsymbol{K}_{ab} \wedge \star(e^a \wedge e^b) = 0 \tag{2.2.6}$$

The Einstein-Hilbert action becomes the Einstein-Cartan action after adding the torsion property on the spacetime. This means that it describes the gravitational force in the Einstein-Cartan spacetime. Gravity action can be expressed as:

$$\begin{aligned}
S_{\text{grav}} &= \frac{1}{2\kappa^2}\int (\mathring{\boldsymbol{R}}_{ab} + \boldsymbol{K}_{ac} \wedge \boldsymbol{K}^c{}_b) \wedge \star\left(e^a \wedge e^b\right) \\
&= \frac{1}{2\kappa^2}(C_1 + C_2)
\end{aligned} \tag{2.2.7}$$

In the above equation, torsion seems to be separated. We notice that torsion is purely a topological feature, which allows us to treat the action equations similarly to how we would in classical General Relativity. However, if torsion were to act as a physical source or have dynamical effects, we would need to adopt a different approach.

The C_1, C_2 coefficients are used to simplify the action denoted:

- to calculate C_1, one must follow the reverse procedure that was already shown above. Finally we get: $C_1 = \int d^4x\sqrt{-g}\mathring{\boldsymbol{R}}_{ab}$

- and to calculate C_2:

$$
\begin{aligned}
C_2 &= \int (\boldsymbol{K}_{ac} \wedge \boldsymbol{K}^c{}_b) \wedge \star \left(e^a \wedge e^b\right) \overset{2.2.3}{=} \int (\boldsymbol{K}_{ac} \wedge \boldsymbol{K}^c{}_b) \wedge \left(\frac{1}{2}\epsilon_{\rho\sigma\kappa\lambda} e^{a\rho} e^{b\sigma} dx^\kappa \wedge dx^\lambda\right) \\
&= \int (K_{ac\mu} K^c{}_{b\nu}) \left(\frac{1}{2}\epsilon_{\rho\sigma\kappa\lambda} e^{a\rho} e^{b\sigma} dx^\mu \wedge dx^\nu \wedge dx^\kappa \wedge dx^\lambda\right) \\
&\overset{2.2.1}{=} \int d^4x\sqrt{-g}(K_{ac\mu} K^c{}_{b\nu}) \left(\frac{1}{2}\epsilon_{\rho\sigma\kappa\lambda}\epsilon^{\mu\nu\kappa\lambda}\right) e^{a\rho} e^{b\sigma} \\
&\overset{2.2.2}{=} \int d^4x\sqrt{-g}(K_{ac\mu} K^c{}_{b\nu}) \left(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma\right) e^{a\rho} e^{b\sigma} \\
&= \int d^4x\sqrt{-g}\left(K_{ac\rho} K^c{}_{b\sigma} - K_{ac\sigma} K^c{}_{b\rho}\right) e^{a\rho} e^{b\sigma} \\
&= \int d^4x\sqrt{-g}\left(K_{ac}{}^a K^c{}_b{}^b - K_{ac}{}^b K^c{}_b{}^a\right)
\end{aligned}
$$

Now, we can replace the Latin indices by the Greek ones since they seem to cancel out: $K_{ac}{}^a K^c{}_b{}^b = K_{\lambda\mu}{}^\lambda K^\mu{}_\nu{}^\nu e^\lambda_a e^a_\lambda e^\nu_b e^b_\nu e^\mu_c e^c_\mu = K_{\lambda\mu}{}^\lambda K^\mu{}_\nu{}^\nu$. So we have:

$$
C_2 = \int d^4x\sqrt{-g}\left(K_{\lambda\mu}{}^\lambda K^\mu{}_\nu{}^\nu - K_{\lambda\mu}{}^\nu K^\mu{}_\nu{}^\lambda\right) = \int d^4x\sqrt{-g}\left(K_{\lambda\mu}{}^\lambda K^{\mu\nu}{}_\nu - K^\lambda{}_{\mu\nu} K^{\mu\nu}{}_\lambda)\right)
$$

Using the antisymmetrization $K^{\mu\nu}{}_\lambda = -K^{\nu\mu}{}_\lambda$ and $K_{\lambda\mu}{}^\lambda = -K_{\mu\lambda}{}^\lambda$ we get:

$$
C_2 = \int d^4x\sqrt{-g}\left(K^\lambda{}_{\mu\nu} K^{\nu\mu}{}_\lambda - K_{\mu\lambda}{}^\lambda K^{\mu\nu}{}_\nu\right) \tag{2.2.8}
$$

Next, we set the C_2 term (it is a scalar value) :

$$
\Delta \equiv K^\lambda{}_{\mu\nu} K^{\nu\mu}{}_\lambda - K^{\mu\nu}{}_\nu K_{\mu\lambda}{}^\lambda \overset{2.1.17}{=} T^\nu{}_{\nu\lambda} T^\lambda{}_\lambda{}^\mu - \frac{1}{2} T^\mu{}_{\nu\lambda} T^\nu{}_{\lambda\mu} + \frac{1}{4} T_{\mu\nu\lambda} T^{\mu\nu\lambda} \tag{2.2.9}
$$

The action equation can be simplified into a torsionless and a contorted part:

$$
\boxed{S_{\text{grav}} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(\mathring{\boldsymbol{R}}_{ab} + \Delta)} \tag{2.2.10}
$$

2.2.2 Torsion tensor decomposition

Since we work in a (3+1)-dimensional universe, the torsion tensor may be considered a 4-vector composed of 4×4 antisymmetric matrices ($T_{\mu\nu\rho} = -T_{\mu\rho\nu}$). This means that, from the 64 initial components, the independent ones are only $4 \times 6 = 24$. Specifically, the torsion tensor can be decomposed as follows:

$$
T_{\mu\nu\rho} = \frac{1}{3}(T_\nu g_{\mu\rho} - T_\rho g_{\mu\nu}) - \frac{1}{6}\epsilon_{\mu\nu\rho\sigma} S^\sigma + q_{\mu\nu\rho} \tag{2.2.11}
$$

where we defined the:

- **torsion trace vector**, as the 4-component vector component of the torsion tensor that comes from the contraction with the metric:

$$
T_\mu \equiv T^\nu{}_{\mu\nu} \tag{2.2.12}
$$

- **pseudo-scalar axial vector**, as the 4-component pseudovector component (changes sign under parity transformation) of the torsion tensor that comes from the contraction with the Levi-Civita tensor:

$$S_\mu \equiv \epsilon_{\mu\nu\rho\sigma} T^{\nu\rho\sigma} \tag{2.2.13}$$

- **antisymmetric traceless tensor**, as the 16-component torsion tensor with the rest of the elements:

$$q^{\nu}{}_{\rho\nu} = 0 = \epsilon^{\sigma\mu\nu\rho} q_{\mu\nu\rho} \tag{2.2.14}$$

Now focusing on 2.2.13, one can observe that:

$$\mathbf{S} = \star\mathbf{T} \tag{2.2.15}$$

Since torsion's components can be projected on the orthonormal basis as $T = \frac{1}{3!} T_{abc}(e^a \wedge e^b \wedge e^c)$, the pseudo-axial vector's components, thus, can be expressed accordingly as:

$$\mathbf{S} = \star\mathbf{T} = \frac{1}{6}\, T_{abc} \star (e^a \wedge e^b \wedge e^c) \overset{2.2.4}{\Rightarrow} S_d = \frac{1}{6} \epsilon^{abc}{}_d T_{abc} \tag{2.2.16}$$

Through equations 2.1.17 and 2.2.9, the contorsion tensor K_{abc} and the scalar quantity Δ are directly linked to the torsion tensor. Using the torsion decomposition described above, both quantities can be expressed in terms of the axial (pseudovector) component of torsion S_d as follows:

$$\boldsymbol{K_{abc}} = \frac{1}{2} \epsilon_{abcd} S^d + \hat{\boldsymbol{K}}_{abc} \tag{2.2.17}$$

$$\Delta = \frac{3}{2} S_d S^d + \hat{\Delta} \tag{2.2.18}$$

So by the end of this subsection, we have managed to derive the action for gravity for a spacetime with torsion.

2.3 QED in a contorted spacetime

The S_{grav} term that was derived in the previous section describes the gravity in a contorted spacetime; however, to study further the physical phenomena, a good start is made by adding matter to the universe. To add matter, we start by adding QED. Before proceeding to more complicated equations, we need to consider first the classical QED action for a massless Dirac fermion.

2.3.1 QED action in a contorted spacetime

Let there be a 3+1-dimensional QED with torsion. The action equation is:

$$S_{Torsion-QED} = \frac{i}{2} \int d^4x \sqrt{-g} \left[\bar{\psi}(x) \gamma^\mu D_\mu(\omega, A) \psi(x) - \overline{D_\mu(\omega, A) \psi(x)} \gamma^\mu \psi(x) \right] \tag{2.3.1}$$

To take torsion into account in the above equation, it is necessary to replace the covariant derivative to match our theory; thus, the full covariant derivative (gauge-gravitational with torsion) used in this case is:

$$D_\mu(\omega, A) = D(\omega) - ieA_\mu = \partial_\mu + i\omega^a{}_{b\mu}\sigma_a{}^b - ieA_\mu \tag{2.3.2}$$

and $\sigma^{ab} = \frac{i}{4} = [\gamma^a, \gamma^b]$ the 4x4 Dirac matrices on the tangent space T_pM of the manifold. Eventually, one can rewrite this action using some differential geometry as:

$$S_{Torsion-QED} = S_{Classical-QED} + \underbrace{\frac{1}{8}\int d^4x\sqrt{-g}\bar{\psi}(x)\{\gamma^c, \sigma^{ab}\}K_{abc}\psi(x)}_{\text{Torsion Term}} \tag{2.3.3}$$

The final term is the torsion tensor acting on a fermion. The whole expression between $\bar{\psi}$ and ψ can be considered as an operator:

$$\begin{aligned}
\text{Torsion Term} &= \frac{1}{8}\int d^4x\sqrt{-g}\bar{\psi}(x)\left(2\epsilon^{abc}{}_d\gamma^d\gamma^5\right)K_{abc}\psi(x) \\
&\overset{2.1.17}{=} \frac{1}{8}\int d^4x\sqrt{-g}\bar{\psi}(x)\left(2\epsilon^{abc}{}_d\gamma^d\gamma^5\right)\left(-\frac{1}{2}(T_{cab} - T_{abc} - T_{bca})\right)\psi(x) \\
&= -\frac{1}{8}\int d^4x\sqrt{-g}\bar{\psi}(x)\left(\gamma^d\gamma^5\right)\left(\epsilon^{abc}{}_dT_{cab} - \epsilon^{abc}{}_dT_{abc} - \epsilon^{abc}{}_dT_{bca}\right)\psi(x) \\
&= -\frac{1}{8}\int d^4x\sqrt{-g}\bar{\psi}(x)\left(\gamma^d\gamma^5\right)\left(\epsilon^{abc}{}_dT_{abc}\right)\psi(x) \\
&\overset{2.2.16}{=} -\frac{1}{8}\int d^4x\sqrt{-g}\bar{\psi}(x)\left(\gamma^d\gamma^5\right)(6S_d)\psi(x) \\
&= -\frac{3}{4}\int d^4x\sqrt{-g}\bar{\psi}(x)S_d\gamma^d\gamma^5\psi(x)
\end{aligned} \tag{2.3.4}$$

The form of the QED-action torsionful becomes:

$$S_{Torsion-QED} = S_{Classical-QED} - \frac{3}{4}\int d^4x\sqrt{-g}S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi \tag{2.3.5}$$

In the second term, we notice that we have a vector field S_μ coupling to the fermion's axial current. We may name the latter one:

$$j^{5\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi \tag{2.3.6}$$

2.3.2 Current conservation correction & Torsion-induced axions

Including the Maxwell tensor for QED, we get the final expression for the action:

$$\begin{aligned}
S_{Torsion-QED} &= S_{grav} + S_{Torsion-QED} + S_{Maxwell} \\
&= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(\mathring{\boldsymbol{R}}_{ab} + \Delta) + S_{Classical-QED} - \frac{3}{4}\int d^4x\sqrt{-g}S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi \\
&- \frac{1}{4}\int d^4x\sqrt{-g}F^{\mu\nu}F_{\mu\nu}
\end{aligned} \tag{2.3.7}$$

This action considers gravity and the QED, which represent the content of matter in our universe. Having the full action, we can use it to calculate the stress-energy tensor, and after that, the equations of motion.

The stress-energy tensor can be decomposed as follows:

$$\begin{aligned} T_{\mu\nu} &= T^A_{\mu\nu} + T^\psi_{\mu\nu} + T^S_{\mu\nu} \\ &= F_{\mu\lambda}F^\lambda_\nu - \frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - \frac{i}{2}\left(\bar{\psi}\gamma_{(\mu}D_{\nu)} - \gamma_{(\mu}D_{\nu)}\psi\right) + \frac{3}{4}S_{(\mu}\bar{\psi}\gamma_{\nu)}\gamma^5\psi \\ &- \frac{3}{2\kappa^2}\left(S_\mu S_\nu - \frac{1}{2}g_{\mu\nu}S_\alpha S^\alpha\right) \end{aligned} \tag{2.3.8}$$

The equations of motion after taking the variation of the fully-developed action 2.3.7 are proved to be:

$$T_\mu = 0 \tag{2.3.9}$$

$$q_{\mu\nu\rho} = 0 \tag{2.3.10}$$

$$S = \frac{\kappa^2}{2}j^5 \tag{2.3.11}$$

Now, the final expression for the spin-connection in this contorted universe becomes:

$$\begin{aligned} \omega^{ab}{}_\mu &= \mathring{\omega}^{ab}{}_\mu + K^{ab}{}_\mu \\ &= \mathring{\omega}^{ab}{}_\mu + \frac{\kappa^2}{4}\epsilon^{ab}{}_{cd}e^c{}_\mu j^5 \end{aligned} \tag{2.3.12}$$

For massless fermions, we can denote the equation of motion by taking the $S_{\text{Torsion-QED}}$ and deriving the Dirac equation:

$$\frac{\partial S_{Torsion-QED}}{\partial\bar{\psi}} = 0 \Rightarrow\ i\gamma^\mu D_\mu(\omega, A)\psi - \frac{3}{4}S_\mu\gamma^\mu\gamma^5\psi = 0 \tag{2.3.13}$$

We notice that the pseudo-vector S_μ seems to play the role of an axial source. We also notice conservation laws to be true not only for the fermionic current, but also for the torsion pseudo-vector S:

$$d\star j^5 = 0 \overset{2.3.11}{\Rightarrow} d\star S = 0 \tag{2.3.14}$$

Now, we notice that the action is proportional to the S_μ squared, one could integrate it using a path integral and get the repulsive four-fermion interaction from the torsion term [8]:

$$-\frac{3}{16}\int j^5 \wedge \star j^5 \tag{2.3.15}$$

However, we notice that the conservation of the axial current of a fermion is violated at a quantum level. Specifically, the one-loop path integral of the conservation gives a non-zero value, even in the torsionless limit[8]:

$$d\star j^5 = \frac{e^2}{8\pi}F\wedge F - \frac{1}{96\pi^2}R^a_b \wedge R^a_b \equiv \mathcal{G}(\omega, A) \tag{2.3.16}$$

The violation of torsion conservation due to quantum anomalies can be corrected by imposing the corresponding constraint and performing path integration. To achieve this, one should introduce a δ-functional constraint in the path integral and sum up to all possible torsion fields S_μ on the path. Using this δ-functional, it is ensured that the conservation law $d\star S = 0$ holds at every point of spacetime. The field Φ that is added to the expression is the Lagrangian multiplier and ensures that only the torsion configurations that satisfy this condition will eventually contribute to the path integral.

Since the torsion pseudo-vector S_d changes sign under parity transformation, the Lagrangian multiplier Φ should also be a pseudoscalar field. This guarantees that the term $\Phi d\star S$ in the exponent remains parity-even, preserving the overall symmetry of the action. In this way, Φ guarantees the conservation of both the torsion field S_μ and the axial current j^5_μ even in the presence of quantum anomalies, while maintaining parity invariance. By summing over all possible configurations of the torsion field, we maintain the consistency of the quantum theory.

Let's break down the steps for the path integral calculation.
The δ-functional used is defined:

$$\delta(d\star S) = \int D\Phi \, \exp\left(i\int \Phi d\star S\right) \tag{2.3.17}$$

By integrating 2.3.17 in the path integral, the initial expression is:

$$\begin{aligned} \mathcal{Z} &\propto \int DS\; D\Phi\; \delta(d\star S)\, \exp\left(i\int\left[\frac{3}{4\kappa^2}S\wedge\star S - \frac{3}{4}S\wedge\star j^5\right]\right) \\ &\propto \int DS\; D\Phi\;\; \exp\left(i\int\left[\frac{3}{4\kappa^2}S\wedge\star S - \frac{3}{4}S\wedge\star j^5 + \Phi d\star S\right]\right) \end{aligned} \tag{2.3.18}$$

It is easier to work with Greek notation in this case, so we are going to replace all the wedge products in the same way we did before (using 2.2.1, 2.2.2, and 2.2.3):

- $\int S\wedge\star S = \int d^4x\sqrt{-g}S^\mu S_\mu$
- $\int S\wedge\star j^5 = \int d^4x\sqrt{-g}S^\mu j^5_\mu = \int d^4x\sqrt{-g}S_\mu j^{5\mu}$
- $\int \Phi d\star S = \int d^4x\sqrt{-g}\;\Phi\;\partial_\mu S^\mu$, where Φ and $\partial_\mu S^\mu$ transform as pseudo-scalars

So, the initial expression 2.3.18 transformation is shown below, and it is then simplified using the square completion technique [8]:

$$\begin{aligned} \mathcal{Z} &\propto \int DS^\mu\; D\Phi\;\; \exp\left(i\int d^4x\sqrt{-g}\left[\frac{3}{4\kappa^2}S^\mu S_\mu - \frac{3}{4}S^\mu j^5_\mu + \Phi\;\partial_\mu S^\mu\right]\right) \\ &\propto \int DS^\mu\; D\Phi\;\; \exp\left(i\int d^4x\sqrt{-g}\left[\frac{3}{4\kappa^2}S^\mu S_\mu + S^\mu \underbrace{\left(-\frac{3}{4}j^5_\mu + \Phi\;\partial_\mu\right)}_{J_\mu}\right]\right) \\ &\propto \int DS^\mu\; D\Phi\;\; \exp\left(i\int d^4x\sqrt{-g}\left[\frac{3}{4\kappa^2}S^\mu S_\mu + S^\mu J_\mu\right]\right) \\ &\propto \int DS^\mu\; D\Phi\;\; \exp\left(i\int d^4x\sqrt{-g}\left[\frac{\sqrt{3}}{2^2\kappa^2}S^\mu S_\mu + 2\frac{\sqrt{3}}{2\kappa}\frac{\kappa}{\sqrt{3}}S^\mu J_\mu + \frac{\kappa^2}{3}J_\mu J^\mu - \frac{\kappa^2}{3}J_\mu J^\mu\right]\right) \end{aligned}$$

$$\propto \int DS^\mu \, D\Phi \;\; \exp\left(i \int d^4x\sqrt{-g} \left[\frac{3}{4\kappa^2} \left(S^\mu + \frac{2\kappa^2}{3} J^\mu \right) \left(S_\mu + \frac{2\kappa^2}{3} J_\mu \right) - \frac{\kappa^2}{3} J_\mu J^\mu \right] \right)$$

$$\propto \int DS^\mu \, D\Phi \;\; \underbrace{\exp\left(i \int d^4x\sqrt{-g} \left[\frac{3}{4\kappa^2} \left(S^\mu + \frac{2\kappa^2}{3} J^\mu \right) \left(S_\mu + \frac{2\kappa^2}{3} J_\mu \right) \right] \right)}_{\text{of the form } \int e^{it^2} dt \text{ Gaussian integral over S}} \exp\left(-i \int d^4x\sqrt{-g} \frac{\kappa^2}{3} J_\mu J^\mu \right)$$

The first exponential contracts to a Gaussian integral during the integration over S_μ, so it is a constant which can be absorbed in the norm of S. The value of this constant is calculated from the Gaussian integral over S seems to be $\frac{\sqrt{6\pi}}{3}\kappa(i+1)$. The second exponential has no dependence on S^μ, thus, it remains intact. So we end up with the path integral depending only on the $J^\mu J_\mu$ term:

$$\begin{aligned}
\mathcal{Z} \propto & \int D\Phi \exp\left(-i \int d^4x\sqrt{-g} \frac{\kappa^2}{3} J_\mu J^\mu \right) \\
\propto & \int D\Phi \exp\left(-i \int d^4x\sqrt{-g} \frac{\kappa^2}{3} \left(-\frac{3}{4} j^5_\mu + \Phi \, \partial_\mu \right) \left(-\frac{3}{4} j^{5\mu} + \Phi \, \partial^\mu \right) \right) \\
\propto & \int D\Phi \exp\left(-i \int d^4x\sqrt{-g} \frac{\kappa^2}{3} \left(\frac{9}{16} j^5_\mu j^{5\mu} - \frac{3}{2} j^{5\mu} \partial_\mu \Phi + \Phi^2 \, \partial_\mu^2 \right) \right) \\
\propto & \int D\Phi \exp\left(i \int d^4x\sqrt{-g} \left(-\frac{1}{2} \frac{3\kappa^2}{8} j^5_\mu j^{5\mu} + \frac{\kappa^2}{2} j^{5\mu} \partial_\mu \Phi - \frac{\kappa^2}{3} \left(\partial_\mu \Phi \right)^2 \right) \right)
\end{aligned}$$

Rescaling the Φ pseudo-scalar field as $\Phi = \sqrt{\frac{3}{2\kappa^2}} b$ we get:

$$\mathcal{Z} \propto \int D\Phi \exp\left(i \int d^4x\sqrt{-g} \left(-\frac{1}{2} \frac{3\kappa^2}{8} j^5_\mu j^{5\mu} + \sqrt{\frac{3\kappa^2}{8}} j^{5\mu} \partial_\mu b - \frac{1}{2} \left(\partial_\mu b \right)^2 \right) \right)$$

Finally, we set $\frac{1}{f_\phi} = \sqrt{\frac{3\kappa^2}{8}}$:

$$\mathcal{Z} \propto \int D\Phi \exp\left(i \int d^4x\sqrt{-g} \left(-\frac{1}{2f_\phi} j^5_\mu j^{5\mu} + \frac{1}{f_\phi} j^{5\mu} \partial_\mu b - \frac{1}{2} \left(\partial_\mu b \right)^2 \right) \right) \tag{2.3.19}$$

The final form 2.3.19 of the path integral 2.3.18 reminds us of a kinetic term of a massless axion-like degree of freedom $b(x)$ coupled to fermions, which emerges from torsion through quantum anomaly correction. The constant f_b is the axion decay that determines the interaction strength between the axion and fermions. One can write:

$$\mathcal{Z} \propto \int D\Phi \exp\left(i \int d^4x\sqrt{-g} \left(-\frac{1}{2f_\phi} j^5_\mu j^{5\mu} + \frac{1}{f_\phi} \mathcal{G}(\omega, A) - \frac{1}{2} \left(\partial_\mu b \right)^2 \right) \right) \tag{2.3.20}$$

Where the $\mathcal{G}(\omega, A)$ is the same anomalous term mentioned in 2.3.16. This redefinition shows that QED on a torsionful space is equivalent to QED on a torsionless space coupled to an axion. In this way, torsion becomes dynamical due to quantum anomalies, and the current that is conserved in this case is the $J^5 = j^5 + f_\phi d\phi$ from the axion equations of motion [8].

Since the effective field theory approach we made above guarantees the conservation law for the current

J^5, there should be a conserved "Torsion Charge":

$$Q_S = \int \star S \tag{2.3.21}$$

The model described in this section includes only fermion interactions with Torsion for QED. Torsion is gravitational in nature, which means interactions are allowed with all fermion species. Expanded to other groups, e.g., non-Abelian SU(3) QCD group, one must sum over all possible interactions to find the total axial current. During the QCD cosmological era, instanton configurations of the colour group via chiral anomalies induce a breaking of the action shift-symmetry and induce a potential of the form [9]:

$$V(b) = \int d^4x\sqrt{-g}\Lambda^4_{\mathrm{QCD}}\left[1 - cos\left(\frac{b}{f_b}\right)\right] \tag{2.3.22}$$

From the potential equation, it appears that the action mass term is introduced as $m_b = \frac{\Lambda^2_{\mathrm{QCD}}}{f_b}$, and it can play the role of the dark matter component in the universe. This idea, though, may be further extended in a string-theoretic framework.

3 Torsion in string-inspired framework

In the previous section, we saw that torsion is an additional geometrical property in the context of the Einstein-Cartan-Sciama-Kibble (ECSK) framework (extension of GR), introducing a dynamical pseudoscalar axion-like field. This torsionful geometry leads to modified conservation laws and opens new possibilities for early-universe dynamics. Remarkably, similar structures appear in the low-energy limit of string theory. More specifically, the Kalb-Ramond (KR) field strength, part of the massless bosonic spectrum, may be interpreted as the torsion tensor in a string-inspired background[8]. In the next section, we explore how these two pictures connect and how the string-theoretic framework extends into a cosmological context.

3.1 String-inspired contorted action and derivation of axion-like field

3.1.1 String-inspired gravitational action of the early bosonic universe

Starting from the early universe phase, where a bosonic gravitational theory arises from String theory, three primary fields compose the the framework: the spin-0 dilaton, the spin-2 graviton, and the spin-1 KR tensor field $B_{\mu\nu} = -B_{\nu\mu}$ (it arises naturally in the low-energy limit of String theory). In this context, one can find that torsion is induced naturally from string theory as a tensor field strength, without invoking the geometrical tensor we described above.

The Kalb-Ramond (KR) field with a $U(1)$ gauge symmetry $B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu}\theta_{\nu]}$, accounts for the corresponding electromagnetic field A_μ for strings, only here it is a rank-2 tensor since strings are extended objects. Due to the Abelian-gauge transformation of the KR-field, we can introduce the field strength $H_{\mu\nu\rho}$ in the same way as the electromagnetic tensor $F_{\mu\nu} \equiv dA = \partial_\mu A_\nu - \partial_\nu A_\mu$. This field strength $H_{\mu\nu\rho}$ is defined [8]:

$$\begin{aligned} H_{\mu\nu\rho} \equiv dB &= \frac{1}{2}(\partial_\mu B_{\nu\rho} - \partial_\mu B_{\rho\nu} + \partial_\nu B_{\rho\mu} - \partial_\nu B_{\mu\rho} + \partial_\rho B_{\mu\nu} - \partial_\rho B_{\nu\mu}) \\ &\overset{B_{\mu\nu}=-B_{\nu\mu}}{=} \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} = \partial_{[\mu} B_{\nu\rho]} \end{aligned} \tag{3.1.1}$$

and it reminds us of the typical contorsion tensor definition 2.1.17. At this level, the KR-field strength satisfies the Bianchi identity:

$$dH = 0 \tag{3.1.2}$$

After string compactification, (3+1)-dimensional particle phenomenology is not affected, so one can use the effective gravitational action expansion to the lowest $a' = M_s^2$ order ($\mathcal{O}((\alpha')^0)$), where a' is the Regge slope. Ignoring the dilaton term ($\Phi = 0$), the action can be re-expressed:

$$S_B = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu} + ...\right) \tag{3.1.3}$$

Where we normalized $\mathcal{H}_{\lambda\mu\nu} = \kappa^{-1}H_{\lambda\mu\nu}$, $\kappa = \sqrt{8\pi G}$ and ... represent the higher derivative terms. As mentioned above, comparing the actions from the previous analysis 2.2.10, one observes that $\mathcal{H}_{\lambda\mu\nu}$ is the corresponding contorsion tensor. This means that the KR-strength may describe twists and rotations of the KR-field in the background. The Christoffel symbols have a contributing term $\mathcal{H}_{\lambda\mu\nu}$ in the same way as in the previous analysis:

$$\Gamma^\rho_{\mu\nu} = \mathring{\Gamma}^\rho_{\mu\nu} + \frac{\kappa}{\sqrt{3}}\mathcal{H}^\rho_{\mu\nu} \tag{3.1.4}$$

Where $\overset{\circ}{\Gamma}{}^{\rho}_{\mu\nu}$ is the torsionless Christoffel symbol.

In the analysis before, we confronted quantum anomalies in the ECSK framework; in this case, the gravitational anomaly cancellation has led Green and Schwartz to add appropriate counterterms in the effective action by making a few modifications KR-field strength $\mathcal{H}_{\lambda\mu\nu}$. This was done by adding the so-called Lorentz Ω_{3L} and Yang Mills Ω_{3Y}, Chern-Simons 3-form terms as below:

$$\mathcal{H} = dB + \frac{\alpha'}{8\kappa}\left(\Omega_{3L} - \Omega_{3Y}\right) \tag{3.1.5}$$

$$\Omega_{3L} = \omega^a{}_c \wedge d\omega^c{}_a + \frac{2}{3}\omega^a{}_c \wedge \omega^c{}_d \wedge \omega^d{}_a, \quad \Omega_{3Y} = A \wedge dA + A \wedge A \wedge A \tag{3.1.6}$$

The Bianchi identity is expressed as:

$$d\mathcal{H} = d\left(dB + \frac{\alpha'}{8\kappa}\left(\Omega_{3L} - \Omega_{3Y}\right)\right) \overset{d^2=0}{=} d\left(\frac{\alpha'}{8\kappa}\left(\Omega_{3L} - \Omega_{3Y}\right)\right) \tag{3.1.7}$$

Where the derivative of the :

$$d(\Omega_{3L}) \overset{d^2=0}{=} d\omega^a{}_c \wedge d\omega^c{}_a + \frac{2}{3}(d\omega^a{}_c \wedge \omega^c{}_d \wedge \omega^d{}_a + \omega^a{}_c \wedge d\omega^c{}_d \wedge \omega^d{}_a + \omega^a{}_c \wedge \omega^c{}_d \wedge d\omega^d{}_a)$$

We know that: $d\omega^a{}_c = R^a{}_c - \omega^a{}_c \wedge \omega^c{}_a$. If we replace this equation in the $d(\Omega_{3L})$ and make index rearrangements (detailed calculations are shown in personal notes), we get a term:

$$d(\Omega_{3L}) = R^a{}_c \wedge R^c{}_a = Tr(R \wedge R)$$

We follow the same procedure for $d\Omega_{3Y} = -Tr(F \wedge F)$, where $F = dA + A \wedge A$. The form of the derivative of the field strength can be written:

$$d\mathcal{H} = \frac{\alpha'}{8\kappa} Tr\left(R \wedge R - F \wedge F\right) \tag{3.1.8}$$

The non-zero term on the right-hand side of equation 3.1.8 represents a mixed quantum anomaly, arising from both gauge fields (F) and curvature (R). This type of anomaly indicates a violation of classical symmetries at the quantum level, involving both gauge and gravitational interactions. To address this anomaly, we will employ a similar approach to the one used for resolving quantum anomalies in the case of General Relativity (GR) with torsion, where additional counterterms or constraints are introduced to restore consistency at the quantum level.

In the effective (3+1)-dimensional spacetime of the early low-energy universe, where bosons exist as background fields of the theory, quantum anomalies are still present. Fermions do not emerge until the end of inflation, when they arise from the decay of the false vacuum. Under these conditions, the electromagnetic tensor A (gauge field) can be set to zero (since fermions do not exist yet), allowing the Bianchi identity to be reformulated. This sets the stage for addressing the quantum anomalies through the path integral of the action. Using the Bianchi identity from [8], we simplify its form and finally we get:

$$\varepsilon_{abc}{}^{\mu}\partial_\mu \mathcal{H}^{abc} = \frac{\alpha'}{16\kappa}\sqrt{-g}(R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} - F^{\mu\nu}F_{\mu\nu}) \overset{A=0}{=} -\sqrt{-g}\mathcal{G}(\omega) = \frac{\alpha'}{16\kappa}\partial_\mu\left(\sqrt{-g}\,\mathcal{K}^{\mu}(\omega)\right) \tag{3.1.9}$$

where $\varepsilon = \sqrt{-g}\epsilon$, $\epsilon^{\mu\nu\rho\sigma} = \frac{sgn(g)}{\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}$ and $\tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}R^{\alpha\beta}{}_{\rho\sigma}$ are the dual tensor of the curvature. The $\mathcal{K}^\mu$ tensor expresses the total derivative of the gravitational Chern-Simons anomalous terms.

Having expressed the mixed quantum anomaly given by the Bianchi identity 3.1.9, the next step is to implement it as a δ-functional constraint, exactly as we did in the contorted QED case to resolve the issue of the current conservation. The δ-functional that we are using is defined:

$$\Pi_x\delta\left(\varepsilon_{abc}{}^{\mu}\partial_\mu\mathcal{H}^{abc} - \frac{\alpha'}{16\kappa}\sqrt{-g}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right) \Rightarrow$$

$$\Pi_x\delta\left(\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\mathcal{H}_{\nu\rho\sigma} - \frac{\alpha'}{16\kappa}\sqrt{-g}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right) = \int Db\,\exp\left(i\int d^4x\,\frac{b(x)}{\sqrt{3}}\left(\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\mathcal{H}_{\nu\rho\sigma} - \frac{\alpha'}{16\kappa}\sqrt{-g}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right)\right)$$
$$= \int Db\,\exp\left(-i\int d^4x\,\sqrt{-g}\left(\frac{\partial_\mu b(x)}{\sqrt{3}}\epsilon_{\mu\nu\rho\sigma}\mathcal{H}^{\nu\rho\sigma} + \frac{b(x)}{\sqrt{3}}\frac{\alpha'}{16\kappa}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right)\right)$$

The path integral of the action can be expressed:

$$\mathcal{Z} = \int Dg\;DH\;\delta(\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\mathcal{H}_{\nu\rho\sigma} - \frac{\alpha'}{16\kappa}\sqrt{-g}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma})\;exp\left(i\int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R - \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu}\right]\right)$$
$$= \int Dg\;DH\;Db\;exp\left(i\int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R - \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu} - \frac{\partial_\mu b(x)}{\sqrt{3}}\epsilon_{\mu\nu\rho\sigma}\mathcal{H}^{\nu\rho\sigma} - \frac{b(x)}{\sqrt{3}}\frac{\alpha'}{16\kappa}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right]\right)$$

We use the same mathematical trick as before - complete the square to form a Gaussian integral. But before this, we need to change the indices on the third term:

$$\partial_\mu b\,\epsilon_{\mu\nu\rho\sigma}\mathcal{H}^{\nu\rho\sigma} \xrightarrow[\sigma\rightarrow\nu,\;\mu\rightarrow\rho]{\nu\rightarrow\lambda,\;\rho\rightarrow\mu} \partial_\rho b\,\epsilon_{\rho\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu}$$

Square completion:

$$\mathcal{Z} = \int Dg\;DH\;Db\;\exp\left(i\int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R - \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu} - \frac{\partial_\rho b}{\sqrt{3}}\,\epsilon_{\rho\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu} - \frac{b}{\sqrt{3}}\frac{\alpha'}{16\kappa}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right]\right)$$
$$= \int Dg\;DH\;Db\;\exp\left(i\int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R - \frac{b}{\sqrt{3}}\mathcal{G}(\omega) - \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu} - 2\left(\frac{1}{\sqrt{6}}\right)\left(\frac{\sqrt{6}}{2\sqrt{3}}\right)\partial_\rho b\;\epsilon_{\rho\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu}\right]\right)$$
$$= \int Dg\;DH\;Db\;\exp\left(i\int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R - \frac{b}{\sqrt{3}}\mathcal{G}(\omega) - \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu}\right.\right.$$
$$\left.\left.-2\left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{2}}\right)\partial_\rho b\;\epsilon_{\rho\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu} - \frac{1}{2}\epsilon_{\rho\lambda\mu\nu}\epsilon^{\rho\lambda\mu\nu}\partial_\rho b\,\partial^\rho b + \frac{1}{2}\epsilon_{\rho\lambda\mu\nu}\epsilon^{\rho\lambda\mu\nu}\partial_\rho b\,\partial^\rho b\right]\right)$$

Remember $\epsilon_{\rho\lambda\mu\nu}\epsilon^{\rho\lambda\mu\nu} = 24$:

$$\mathcal{Z} = \int Dg\ DH\ Db\ \exp\left(i\int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R - \frac{b}{\sqrt{3}}\mathcal{G}(\omega) - \left[\frac{1}{\sqrt{6}}\mathcal{H}_{\lambda\mu\nu} + \frac{1}{\sqrt{2}}\partial_\rho b\ \epsilon^{\rho\lambda\mu\nu}\right]\right.\right.$$
$$\left.\left.\left[\frac{1}{\sqrt{6}}\mathcal{H}^{\lambda\mu\nu} + \frac{1}{\sqrt{2}}\partial^\rho b\ \epsilon_{\rho\lambda\mu\nu}\right] + 12\,\partial_\rho b\,\partial^\rho b\right]\right)$$
$$= \int Dg\ DH\ Db\ \exp\left(i\int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R - \frac{b}{\sqrt{3}}\mathcal{G}(\omega) - \frac{1}{6}\left[\mathcal{H}_{\lambda\mu\nu} + \sqrt{3}\partial_\rho b\ \epsilon^{\rho\lambda\mu\nu}\right]^2 + 12\,\partial_\rho b\,\partial^\rho b\right]\right)$$

The second term can be calculated during the path integration over DH, as a Gaussian integral, and gives a constant value which will be multiplied by the rest of the exponential, and this means it can be absorbed in the norm of H and neglected. The rest of the factors do not have any dependence whatsoever on the H, torsion field and by rescaling the b field to be $b' = \sqrt{24}b$, the path integral is simplified to:

$$\mathcal{Z} = \int Dg\ Db\ \exp\left(i\int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R - \frac{b'}{6\sqrt{2}}\mathcal{G}(\omega) + \frac{1}{2}\,\partial_\rho b'\,\partial^\rho b'\right]\right)$$

Now integrating over the rest of the variables, Dg for the graviton and Db for the axion-like pseudoscalar, we get the action:

$$\boxed{S_B^{eff} = \int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R + \frac{\sqrt{2}\alpha'}{192\kappa}b(x)(R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} - F^{\mu\nu}F_{\mu\nu}) + \frac{1}{2}\,\partial_\rho b\,\partial^\rho b + ...\right]} \tag{3.1.10}$$

As noticed from the process above, $b(x)$ is a naturally induced from the string theory axion(one of the axions from string theory) that couples to gravitational Chern-Simons terms making torsion H, a dynamical parameter in the theory $\epsilon_{\mu\nu\rho\sigma}H^{\nu\rho\sigma} = \partial_\mu b(x)$. Since the action is purely bosonic, 4-fermion interactions are not observed, and the anomalies considered are purely gravitational and are not related to fermion anomalies, as we previously saw in QED. This gravitational model is a Chern-Simons modified gravity model.

3.1.2 Hirzebruch-Pontryagin topological density and simplification of the effective action

Before rewriting the previous result 3.1.10 as a composition of three main action terms, it is essential to investigate what the consequences of introducing the Chern-Simons gravitational terms might be.

Chern-Simons terms also appear in a 3-dimensional gauge field (Maxwell's theory). Based on [10], the additional CS-term in Maxwell's theory derives a topological Pontryagin density through its divergence, which contributes to the final form of the Lagrangian. Due to a massive term appearing in the equations of motion, the photon acquires two polarizations with different dispersion relations for the frequency $\omega = \sqrt{|k|^2 \pm \mu|k|}$. This modification leads to physical consequences such as birefringence of the vacuum; however, such phenomena have not been observed in nature based on measurements from distant galaxies.

In this case, the form of the CS-terms is given in 3.1.6. Similarly, a 4-dimensional topological current can be related to those terms, as already shown in 3.1.9, and provides an expression for the corresponding Pontryagin density, which will then be used to simplify the effective action. The related 4-current is given:

$$K^\mu = 2\epsilon^{\mu\alpha\beta\gamma}\left[\frac{1}{2}\Gamma^\sigma{}_{\alpha\tau}\partial_\beta\Gamma^\tau{}_{\gamma\sigma} + \frac{1}{3}\Gamma^\sigma{}_{\alpha\tau}\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma}\right] \tag{3.1.11}$$

Let us now calculate analytically the divergence of the four-dimensional topological current and prove the result noted in Bianchi's identity in 3.1.9. First, we calculate and simplify the partial derivative of 3.1.11:

$$\begin{aligned}
\partial_\mu K^\mu &= 2\partial_\mu \left(\epsilon^{\mu\alpha\beta\gamma} \left[\frac{1}{2} \Gamma^\sigma{}_{\alpha\tau} \partial_\beta \Gamma^\tau{}_{\gamma\sigma} + \frac{1}{3} \Gamma^\sigma{}_{\alpha\tau} \Gamma^\tau{}_{\beta\eta} \Gamma^\eta{}_{\gamma\sigma} \right] \right) \\
&= 2\epsilon^{\mu\alpha\beta\gamma} \left[\frac{1}{2} \partial_\mu \Gamma^\sigma{}_{\alpha\tau} \partial_\beta \Gamma^\tau{}_{\gamma\sigma} + \frac{1}{3} \partial_\mu \Gamma^\sigma{}_{\alpha\tau} \Gamma^\tau{}_{\beta\eta} \Gamma^\eta{}_{\gamma\sigma} + \frac{1}{3} \Gamma^\sigma{}_{\alpha\tau} \partial_\mu \Gamma^\tau{}_{\beta\eta} \Gamma^\eta{}_{\gamma\sigma} + \frac{1}{3} \Gamma^\sigma{}_{\alpha\tau} \Gamma^\tau{}_{\beta\eta} \partial_\mu \Gamma^\eta{}_{\gamma\sigma} \right] \\
&= \epsilon^{\mu\alpha\beta\gamma} \left[\partial_\mu \Gamma^\sigma{}_{\alpha\tau} \partial_\beta \Gamma^\tau{}_{\gamma\sigma} + 2\partial_\mu \Gamma^\sigma{}_{\alpha\tau} \Gamma^\tau{}_{\beta\eta} \Gamma^\eta{}_{\gamma\sigma} \right] \xrightarrow[\gamma\rightarrow\beta]{\nu\rightarrow\alpha,\ \alpha\rightarrow\beta} \\
&= \epsilon^{\mu\nu\alpha\beta} \left[\partial_\mu \Gamma^\sigma{}_{\nu\tau} \partial_\alpha \Gamma^\tau{}_{\beta\sigma} + 2\partial_\mu \Gamma^\sigma{}_{\nu\tau} \Gamma^\tau{}_{\alpha\eta} \Gamma^\eta{}_{\beta\sigma} \right]
\end{aligned} \tag{3.1.12}$$

We will now prove that the above expression is equal to $\frac{1}{2}\tilde{R}^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$. We can rename the indices and make some calculations as follows:

$$\begin{aligned}
\frac{1}{2} \tilde{R}^{\sigma\kappa\mu\nu} R_{\sigma\kappa\mu\nu} &= \frac{1}{2} \tilde{R}^\sigma{}_\tau{}^{\mu\nu} R_{\sigma\kappa\mu\nu} g^{\kappa\tau} = \frac{1}{2} \tilde{R}^\sigma{}_\tau{}^{\mu\nu} R_\sigma{}^\tau{}_{\mu\nu} = \frac{1}{2} \tilde{R}^\tau{}_\sigma{}^{\mu\nu} R^\tau{}_{\sigma\mu\nu} = \frac{1}{4} \epsilon^{\mu\nu\alpha\beta} R^\tau{}_{\sigma\alpha\beta} R^\tau{}_{\sigma\mu\nu} \\
&= \frac{1}{4} \epsilon^{\mu\nu\alpha\beta} \left(\partial_\nu \Gamma^\tau{}_{\mu\sigma} - \partial_\mu \Gamma^\tau{}_{\nu\sigma} + \Gamma^\tau{}_{\nu\eta} \Gamma^\eta{}_{\mu\sigma} - \Gamma^\tau{}_{\mu\eta} \Gamma^\eta{}_{\nu\sigma} \right) \left(\partial_\beta \Gamma^\tau{}_{\alpha\sigma} - \partial_\alpha \Gamma^\tau{}_{\beta\sigma} + \Gamma^\tau{}_{\beta\eta} \Gamma^\eta{}_{\alpha\sigma} - \Gamma^\tau{}_{\alpha\eta} \Gamma^\eta{}_{\beta\sigma} \right) \\
&= \frac{1}{4} \epsilon^{\mu\nu\alpha\beta} \Big(\partial_\nu \Gamma^\tau{}_{\mu\sigma} \partial_\beta \Gamma^\tau{}_{\alpha\sigma} - \partial_\nu \Gamma^\tau{}_{\mu\sigma} \partial_\alpha \Gamma^\tau{}_{\beta\sigma} + \partial_\nu \Gamma^\tau{}_{\mu\sigma} \Gamma^\tau{}_{\beta\eta} \Gamma^\eta{}_{\alpha\sigma} - \partial_\nu \Gamma^\tau{}_{\mu\sigma} \Gamma^\tau{}_{\alpha\eta} \Gamma^\eta{}_{\beta\sigma} \\
&\quad - \partial_\mu \Gamma^\tau{}_{\nu\sigma} \partial_\beta \Gamma^\tau{}_{\alpha\sigma} + \partial_\mu \Gamma^\tau{}_{\nu\sigma} \partial_\alpha \Gamma^\tau{}_{\beta\sigma} - \partial_\mu \Gamma^\tau{}_{\nu\sigma} \Gamma^\tau{}_{\beta\eta} \Gamma^\eta{}_{\alpha\sigma} + \partial_\mu \Gamma^\tau{}_{\nu\sigma} \Gamma^\tau{}_{\alpha\eta} \Gamma^\eta{}_{\beta\sigma} \\
&\quad \Gamma^\tau{}_{\nu\eta} \Gamma^\eta{}_{\mu\sigma} \partial_\beta \Gamma^\tau{}_{\alpha\sigma} - \Gamma^\tau{}_{\nu\eta} \Gamma^\eta{}_{\mu\sigma} \partial_\alpha \Gamma^\tau{}_{\beta\sigma} + \Gamma^\tau{}_{\nu\eta} \Gamma^\eta{}_{\mu\sigma} \Gamma^\tau{}_{\beta\eta} \Gamma^\eta{}_{\alpha\sigma} - \Gamma^\tau{}_{\nu\eta} \Gamma^\eta{}_{\mu\sigma} \Gamma^\tau{}_{\alpha\eta} \Gamma^\eta{}_{\beta\sigma} \\
&\quad - \Gamma^\tau{}_{\mu\eta} \Gamma^\eta{}_{\nu\sigma} \partial_\beta \Gamma^\tau{}_{\alpha\sigma} + \Gamma^\tau{}_{\mu\eta} \Gamma^\eta{}_{\nu\sigma} \partial_\alpha \Gamma^\tau{}_{\beta\sigma} - \Gamma^\tau{}_{\mu\eta} \Gamma^\eta{}_{\nu\sigma} \Gamma^\tau{}_{\beta\eta} \Gamma^\eta{}_{\alpha\sigma} + \Gamma^\tau{}_{\mu\eta} \Gamma^\eta{}_{\nu\sigma} \Gamma^\tau{}_{\alpha\eta} \Gamma^\eta{}_{\beta\sigma} \Big) \\
&= \frac{1}{4} \epsilon^{\mu\nu\alpha\beta} \left(4\partial_\mu \Gamma^\tau{}_{\nu\sigma} \partial_\alpha \Gamma^\tau{}_{\beta\sigma} + 4\partial_\mu \Gamma^\tau{}_{\nu\sigma} \Gamma^\tau{}_{\alpha\eta} \Gamma^\eta{}_{\beta\sigma} + 4\Gamma^\tau{}_{\mu\eta} \Gamma^\eta{}_{\nu\sigma} \partial_\alpha \Gamma^\tau{}_{\beta\sigma} + 4\Gamma^\tau{}_{\mu\eta} \Gamma^\eta{}_{\nu\sigma} \Gamma^\tau{}_{\alpha\eta} \Gamma^\eta{}_{\beta\sigma} \right)
\end{aligned}$$

The last term is antisymmetric when it is multiplied by the Levi-Civita symbol; thus, it vanishes completely from the expression. Finally, we get:

$$\begin{aligned}
\frac{1}{2} \tilde{R}^{\sigma\kappa\mu\nu} R_{\sigma\kappa\mu\nu} &= \epsilon^{\mu\nu\alpha\beta} \left(\partial_\mu \Gamma^\tau{}_{\nu\sigma} \partial_\alpha \Gamma^\tau{}_{\beta\sigma} + \partial_\mu \Gamma^\tau{}_{\nu\sigma} \Gamma^\tau{}_{\alpha\eta} \Gamma^\eta{}_{\beta\sigma} + \partial_\alpha \Gamma^\tau{}_{\mu\eta} \Gamma^\eta{}_{\nu\sigma} \Gamma^\tau{}_{\beta\sigma} \right) \\
&= \epsilon^{\mu\nu\alpha\beta} \left(\partial_\mu \Gamma^\tau{}_{\nu\sigma} \partial_\alpha \Gamma^\tau{}_{\beta\sigma} + 2\partial_\mu \Gamma^\tau{}_{\nu\sigma} \Gamma^\tau{}_{\alpha\eta} \Gamma^\eta{}_{\beta\sigma} \right) \overset{3.1.11}{=} \partial_\mu K^\mu
\end{aligned} \tag{3.1.13}$$

Based on the above result, the effective action of the early bosonic universe with the pseudoscalar degrees of freedom 3.1.10 can be expressed using the topological current K^μ:

$$\begin{aligned}
S_B^{eff} &= \int d^4x \sqrt{-g} \left[-\frac{1}{2\kappa^2} R + \frac{\sqrt{2}\alpha'}{192\kappa} b(x) (R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma}) + \frac{1}{2}\, \partial_\rho b\, \partial^\rho b + ... \right] \\
&= \int d^4x \sqrt{-g} \left[-\frac{1}{2\kappa^2} R + \frac{\sqrt{2}\alpha'}{96\kappa} (u_\mu K^\mu) + \frac{1}{2}\, \partial_\rho b\, \partial^\rho b + ... \right] \\
&= S_{grav} + S_{b-grav} + S_b
\end{aligned} \tag{3.1.14}$$

Where $u_\mu = \nabla_\mu b(x)$.

3.2 Influence of the Axion-like Pseudoscalar on Gravitational Dynamics

3.2.1 Derivation of the modified Einstein's equation

Using the alternative definition of the effective action, one can derive the modified Einstein equation by applying a metric variation.
Some useful expressions for the analysis are:

- the metric determinant variation:
$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} \tag{3.2.1}$$

- the square root of the metric determinant:
$$\delta(\sqrt{-g}) = -\frac{\delta g}{2\sqrt{-g}} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g_{\mu\nu} \tag{3.2.2}$$

- the Christoffel symbol variation with respect to the metric:
$$\delta\Gamma^{\sigma}_{\alpha\tau} = \frac{g^{\sigma\nu}}{2}(\nabla_\alpha \delta g_{\nu\tau} + \nabla_\tau \delta g_{\nu\alpha} - \nabla_\nu \delta g_{\alpha\tau}) \tag{3.2.3}$$

Starting with the gravitational term, it is known from the standard theory that:

$$\begin{aligned} \delta S_{grav} &= \delta\left(\int d^4x \sqrt{-g} \frac{R}{2\kappa^2}\right) = \delta\left(\int d^4x \sqrt{-g} \frac{g^{\mu\nu} R_{\mu\nu}}{2\kappa^2}\right) \\ &= \frac{1}{2\kappa^2} \int d^4x \left(\sqrt{-g}\ \delta g^{\mu\nu} R_{\mu\nu} - \delta(\sqrt{-g}) R\right) \overset{3.2.2}{\Rightarrow} \end{aligned}$$

$$\boxed{\delta S_{grav} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) \delta g^{\mu\nu}} \tag{3.2.4}$$

The variation of the gravitational action term is:

$$\frac{2}{\sqrt{-g}} \left[\frac{\delta S_{grav}}{\delta g_{\mu\nu}}\right] = \frac{1}{\kappa^2} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) \equiv G_{\mu\nu} \tag{3.2.5}$$

Afterwards, metric variations over S_b are calculated below:

$$\begin{aligned} \delta S_b &= \frac{1}{2}\delta\left(\int d^4x \sqrt{-g} \partial_\mu b\ \partial^\mu b\right) = \frac{1}{2}\delta\left(\int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu b\ \partial_\nu b\right) \\ &= \frac{1}{2}\int d^4x \left(\sqrt{-g}\ \delta g^{\mu\nu} \partial_\mu b\ \partial_\nu b - \delta(\sqrt{-g}) \partial_\rho b\ \partial^\rho b\right) \end{aligned}$$

Through 3.2.2:

$$\boxed{\delta S_b = \int d^4x \sqrt{-g} \left(\frac{1}{2} \partial_\mu b\ \partial_\nu b - \frac{1}{4} g_{\mu\nu} \partial_\rho b\ \partial^\rho b\right) \delta g^{\mu\nu}} \tag{3.2.6}$$

The variation of the b-scalar field action term is:

$$\frac{2}{\sqrt{-g}} \left[\frac{\delta S_b}{\delta g_{\mu\nu}}\right] = \partial_\mu b\ \partial_\nu b - \frac{1}{2} g_{\mu\nu} \partial_\rho b\ \partial^\rho b \equiv T^b_{\mu\nu} \tag{3.2.7}$$

Finally, the gravity-pseudoscalar coupling action term will be computed:

$$\delta S_{b-grav} = \frac{\sqrt{2}a'}{96\kappa}\delta\left(\int d^4x\sqrt{-g}\ \partial_\mu b(x)K^\mu\right) = \frac{\sqrt{2}a'}{96}\delta\left(-\int d^4x\sqrt{-g}b(x)\partial_\mu K^\mu\right)$$
$$= \frac{\sqrt{2}a'}{96\kappa}\int d^4x\left(\delta(\sqrt{-g})b(x)\partial_\mu K^\mu - \sqrt{-g}\ b(x)\ \delta(\partial_\mu K^\mu)\right)$$

We will calculate the quantity δK^μ, taking into account the properties of the Levi-Civita tensor and the symmetrization of the Christoffel symbols:

$$\delta K^\mu = \delta\left(2\epsilon^{\mu\alpha\beta\gamma}\left[\frac{1}{2}\Gamma^\sigma{}_{\alpha\tau}\partial_\beta\Gamma^\tau{}_{\gamma\sigma} + \frac{1}{3}\Gamma^\sigma{}_{\alpha\tau}\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma}\right]\right)$$
$$= 2\epsilon^{\mu\alpha\beta\gamma}\left[\frac{1}{2}\delta\Gamma^\sigma{}_{\alpha\tau}\partial_\beta\Gamma^\tau{}_{\gamma\sigma} + \frac{1}{2}\Gamma^\sigma{}_{\alpha\tau}\partial_\beta\delta\Gamma^\tau{}_{\gamma\sigma} + \frac{1}{3}\delta\Gamma^\sigma{}_{\alpha\tau}\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma} + \frac{1}{3}\Gamma^\sigma{}_{\alpha\tau}\delta\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma} + \frac{1}{3}\Gamma^\sigma{}_{\alpha\tau}\Gamma^\tau{}_{\beta\eta}\delta\Gamma^\eta{}_{\gamma\sigma}\right]$$

- The 2^{nd} term becomes:

$$\epsilon^{\mu\alpha\beta\gamma}\Gamma^\sigma{}_{\alpha\tau}\partial_\beta\delta\Gamma^\tau{}_{\gamma\sigma} = \epsilon^{\mu\beta\gamma\alpha}\partial_\beta\Gamma^\sigma{}_{\gamma\tau}\delta\Gamma^\tau{}_{\alpha\sigma} \overset{\sigma\leftrightarrow\tau}{=} \epsilon^{\mu\beta\gamma\alpha}\partial_\beta\Gamma^\tau{}_{\gamma\sigma}\delta\Gamma^\sigma{}_{\alpha\tau} \tag{3.2.8}$$

- The 4^{th} term becomes:

$$\frac{1}{3}\epsilon^{\mu\alpha\beta\gamma}\Gamma^\sigma{}_{\alpha\tau}\delta\Gamma^\tau_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma} \overset{\alpha\leftrightarrow\beta}{=} -\frac{1}{3}\epsilon^{\mu\beta\alpha\gamma}\Gamma^\sigma{}_{\beta\tau}\delta\Gamma^\tau_{\alpha\eta}\Gamma^\eta{}_{\gamma\sigma} \overset{\text{permute}}{=} -\frac{1}{3}\epsilon^{\mu\alpha\beta\gamma}\Gamma^\eta{}_{\beta\sigma}\delta\Gamma^\sigma_{\alpha\tau}\Gamma^\tau{}_{\gamma\eta} \tag{3.2.9}$$

- The 5^{th} term becomes:

$$\frac{1}{3}\epsilon^{\mu\alpha\beta\gamma}\Gamma^\sigma{}_{\alpha\tau}\Gamma^\tau_{\beta\eta}\delta\Gamma^\eta{}_{\gamma\sigma} \overset{\alpha\leftrightarrow\gamma}{=} -\frac{1}{3}\epsilon^{\mu\gamma\beta\alpha}\Gamma^\sigma{}_{\gamma\tau}\delta\Gamma^\tau_{\beta\eta}\Gamma^\eta{}_{\alpha\sigma} \overset{\text{permute}}{=} -\frac{1}{3}\epsilon^{\mu\alpha\beta\gamma}\Gamma^\tau{}_{\gamma\eta}\Gamma^\eta_{\beta\sigma}\delta\Gamma^\sigma{}_{\alpha\tau} \tag{3.2.10}$$

δK^μ becomes:

$$\delta K^\mu = 2\epsilon^{\mu\alpha\beta\gamma}\left[\frac{1}{2}\partial_\beta\Gamma^\tau{}_{\gamma\sigma} - \frac{1}{2}\partial_\gamma\Gamma^\tau{}_{\beta\sigma} + \frac{1}{3}\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma} - \frac{1}{3}\Gamma^\eta{}_{\beta\sigma}\Gamma^\tau{}_{\gamma\eta} - \frac{1}{3}\Gamma^\tau{}_{\gamma\eta}\Gamma^\eta{}_{\beta\sigma}\right]\delta\Gamma^\sigma{}_{\alpha\tau}$$
$$= \epsilon^{\mu\alpha\beta\gamma}\left[\partial_\beta\Gamma^\tau{}_{\gamma\sigma} - \partial_\gamma\Gamma^\tau{}_{\beta\sigma} + \frac{2}{3}\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma} - \frac{4}{3}\Gamma^\tau{}_{\gamma\eta}\Gamma^\eta{}_{\beta\sigma}\right]\delta\Gamma^\sigma{}_{\alpha\tau}$$
$$= \epsilon^{\mu\alpha\beta\gamma}[\underbrace{\partial_\beta\Gamma^\tau{}_{\gamma\sigma} - \partial_\gamma\Gamma^\tau{}_{\beta\sigma} + \Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma} - \Gamma^\tau{}_{\gamma\eta}\Gamma^\eta{}_{\beta\sigma}}_{R^\tau{}_{\sigma\beta\gamma}}$$
$$-\frac{1}{3}\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma} + \Gamma^\tau{}_{\gamma\eta}\Gamma^\eta{}_{\beta\sigma} - \frac{1}{3}\Gamma^\tau{}_{\gamma\eta}\Gamma^\eta{}_{\beta\sigma}]\delta\Gamma^\sigma{}_{\alpha\tau}$$
$$= \epsilon^{\mu\alpha\beta\gamma}\left[R^\tau{}_{\sigma\beta\gamma} - \frac{1}{3}\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma} - \frac{1}{3}\Gamma^\tau{}_{\gamma\eta}\Gamma^\eta{}_{\beta\sigma}\right]\delta\Gamma^\sigma{}_{\alpha\tau}$$

The last two terms cancel out under Levi-Civita permutations:

$$\epsilon^{\mu\alpha\beta\gamma}\left[\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma} + \Gamma^\tau{}_{\gamma\eta}\Gamma^\eta{}_{\beta\sigma}\right] = \epsilon^{\mu\alpha\beta\gamma}\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma} + \epsilon^{\mu\alpha\gamma\beta}\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma} = \epsilon^{\mu\alpha\beta\gamma}\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma} - \epsilon^{\mu\alpha\beta\gamma}\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma} = 0$$

Thus, the final result for the topological current variation will be:

$$\delta K^{\mu} = \epsilon^{\mu\alpha\beta\gamma} R^{\tau}{}_{\sigma\beta\gamma} \delta\Gamma^{\sigma}{}_{\alpha\tau} \tag{3.2.11}$$

Going back to the b-gravity coupling variation:

$$\begin{aligned}
\delta S_{b-grav} &= -\frac{\sqrt{2}a'}{96\kappa} \int d^4x\sqrt{-g}\ \nabla_{\mu} b(x)\ \epsilon^{\mu\alpha\beta\gamma} R^{\tau}{}_{\sigma\beta\gamma} \delta\Gamma^{\sigma}{}_{\alpha\tau} \\
&\overset{3.2.3}{=} -\frac{\sqrt{2}a'}{96\kappa} \int d^4x\sqrt{-g}\ \nabla_{\mu} b(x)\ \epsilon^{\mu\alpha\beta\gamma} R^{\tau}{}_{\sigma\beta\gamma} \frac{g^{\sigma\nu}}{2} (\nabla_{\alpha}\delta g_{\nu\tau} + \nabla_{\tau}\delta g_{\nu\alpha} - \nabla_{\nu}\delta g_{\alpha\tau}) \\
&= -\frac{\sqrt{2}a'}{192\kappa} \int d^4x\sqrt{-g}\ \nabla_{\mu} b(x)\ \epsilon^{\mu\alpha\beta\gamma} R^{\tau\nu}{}_{\beta\gamma} (\nabla_{\alpha}\delta g_{\nu\tau} + \nabla_{\tau}\delta g_{\nu\alpha} - \nabla_{\nu}\delta g_{\alpha\tau}) \\
&= -\frac{\sqrt{2}a'}{192\kappa} \int d^4x\sqrt{-g}\ \nabla_{\mu} b(x)\ \epsilon^{\mu\alpha\beta\gamma} (\underbrace{R^{\tau\nu}{}_{\beta\gamma}\nabla_{\alpha}\delta g_{\nu\tau}}_{\text{does not contribute}} + R^{\tau\nu}{}_{\beta\gamma}\nabla_{\tau}\delta g_{\nu\alpha} - \underbrace{R^{\tau\nu}{}_{\beta\gamma}\nabla_{\nu}\delta g_{\alpha\tau}}_{\tau\leftrightarrow\nu}) \\
&= -\frac{\sqrt{2}a'}{192\kappa} 2 \int d^4x\sqrt{-g}\ \nabla_{\mu} b(x)\ \epsilon^{\mu\alpha\beta\gamma} R^{\tau\nu}{}_{\beta\gamma} \nabla_{\tau}\delta g_{\nu\alpha}
\end{aligned}$$

With partial integration, we get:

$$\begin{aligned}
\delta S_{b-grav} &= -\frac{\sqrt{2}a'}{96\kappa} \int d^4x\sqrt{-g}\ \left(\nabla_{\tau}\nabla_{\mu} b(x)\ \underbrace{\epsilon^{\mu\alpha\beta\gamma} R^{\tau\nu}{}_{\beta\gamma}}_{2\tilde{R}^{\tau\nu\mu\alpha}} \delta g_{\nu\alpha} + \nabla_{\mu} b(x)\ \epsilon^{\mu\alpha\beta\gamma} \nabla_{\tau} R^{\tau\nu}{}_{\beta\gamma} \delta g_{\nu\alpha} \right) \\
&= -\frac{\sqrt{2}a'}{96\kappa} \int d^4x\sqrt{-g}\ \left[2\nabla_{\tau}\nabla_{\mu} b(x)\ \tilde{R}^{\tau\nu\mu\alpha} + \nabla_{\mu} b(x)\ \epsilon^{\mu\alpha\beta\gamma} \nabla_{\tau} R^{\tau\nu}{}_{\beta\gamma} \right] \delta g_{\nu\alpha}
\end{aligned} \tag{3.2.12}$$

Now, using the Bianchi identity, we can write:

$$\begin{aligned}
&\nabla_{\tau} R^{\tau\nu}{}_{\beta\gamma} = \nabla_{\beta} R^{\nu}{}_{\gamma} - \nabla_{\gamma} R^{\nu}{}_{\beta} \Rightarrow \\
&\qquad \epsilon^{\mu\alpha\beta\gamma} \nabla_{\tau} R^{\tau\nu}{}_{\beta\gamma} = \epsilon^{\mu\alpha\beta\gamma} \nabla_{\beta} R^{\nu}{}_{\gamma} - \epsilon^{\mu\alpha\beta\gamma} \nabla_{\gamma} R^{\nu}{}_{\beta} = \epsilon^{\mu\alpha\beta\gamma} \nabla_{\beta} R^{\nu}{}_{\gamma} - \epsilon^{\mu\alpha\gamma\beta} \nabla_{\gamma} R^{\nu}{}_{\beta} \\
&\qquad = 2\epsilon^{\mu\alpha\beta\gamma} \nabla_{\beta} R^{\nu}{}_{\gamma}
\end{aligned} \tag{3.2.13}$$

Through 3.2.13:

$$\begin{aligned}
\delta S_{b-grav} &= -\frac{\sqrt{2}a'}{96\kappa} \int d^4x\sqrt{-g}\ \left[2\nabla_{\tau}\nabla_{\mu} b(x)\ \tilde{R}^{\tau\nu\mu\alpha} + 2\nabla_{\mu} b(x) \epsilon^{\mu\alpha\beta\gamma} \nabla_{\beta} R^{\nu}{}_{\gamma} \right] \delta g_{\nu\alpha} \\
&= -4\frac{\sqrt{2}a'}{192\kappa} \int d^4x\sqrt{-g}\ \left[\nabla_{\tau}\nabla_{\mu} b(x)\ \tilde{R}^{\tau\nu\mu\alpha} + \nabla_{\mu} b(x) \epsilon^{\mu\alpha\beta\gamma} \nabla_{\beta} R^{\nu}{}_{\gamma} \right] \delta g_{\nu\alpha} \\
&= -4\frac{\sqrt{2}a'}{192\kappa} \int d^4x\sqrt{-g}\ \frac{1}{2} \left[\nabla_{\tau}\nabla_{\mu} b(x)\ [\tilde{R}^{\tau\nu\mu\alpha} + \tilde{R}^{\tau\alpha\mu\nu}] + \nabla_{\mu} b(x) [\epsilon^{\mu\alpha\beta\gamma} \nabla_{\beta} R^{\nu}{}_{\gamma} + \epsilon^{\mu\nu\beta\gamma} \nabla_{\beta} R^{\alpha}{}_{\gamma}] \right] \delta g_{\nu\alpha}
\end{aligned}$$

After renaming the indices, a first form of the Cotton tensor can be derived as follows:

$$C_{\mu\nu} = -\frac{1}{2} \left[\nabla_{\tau}\nabla_{\sigma} b(x)\ [\tilde{R}^{\tau\nu\sigma\alpha} + \tilde{R}^{\tau\alpha\sigma\nu}] + \nabla_{\sigma} b(x) [\epsilon^{\sigma\mu\alpha\beta} \nabla_{\alpha} R^{\nu}{}_{\beta} + \epsilon^{\sigma\nu\alpha\beta} \nabla_{\alpha} R^{\mu}{}_{\beta}] \right] \tag{3.2.14}$$

To simplify the expression even more, partial integration is used for the first term of the equation 3.2.12:

$$\int d^4x\sqrt{-g}\ \nabla_\tau \nabla_\mu b(x)\ \tilde{R}^{\tau\nu\mu\alpha} = \int d^4x\sqrt{-g}\ \nabla_\tau\left(\nabla_\mu b(x)\ \tilde{R}^{\tau\nu\mu\alpha}\right) - \int d^4x\sqrt{-g}\ \nabla_\mu b(x)\ \nabla_\tau \tilde{R}^{\tau\nu\mu\alpha} \tag{3.2.15}$$

Thus, the equation 3.2.12 becomes:

$$\delta S_{b-grav} = -4\frac{\sqrt{2}a'}{192\kappa}\int d^4x\sqrt{-g}\ \nabla_\tau\left(\nabla_\mu b(x)\ \tilde{R}^{\tau\nu\mu\alpha}\right)\delta g_{\nu\alpha} \Rightarrow$$

$$\boxed{\delta S_{b-grav} = 2\frac{\sqrt{2}a'}{192\kappa}\int d^4x\sqrt{-g}\ \left[-\frac{1}{2}\nabla_\tau\left(\nabla_\mu b(x)\ [\tilde{R}^{\tau\nu\mu\alpha} + \tilde{R}^{\tau\alpha\mu\nu}]\right)\right]\delta g_{\nu\alpha}} \tag{3.2.16}$$

The definition of the Cotton tensor is now:

$$C^{\mu\nu} = -\frac{1}{2}\ \nabla_\alpha\left(\nabla_\beta b(x)\tilde{R}^{\alpha\mu\beta\nu} + \nabla_\beta b(x)\tilde{R}^{\alpha\nu\beta\mu}\right) \tag{3.2.17}$$

Thus:

$$\frac{2}{\sqrt{-g}}\left[\frac{\delta S_{b-grav}}{\delta g_{\mu\nu}}\right] = \frac{a'\sqrt{2}}{48\kappa}C^{\mu\nu} \tag{3.2.18}$$

The final form of the Einstein equation is:

$$\frac{2}{\sqrt{-g}}\left[\frac{\delta S_{grav}}{\delta g_{\mu\nu}} + \frac{\delta S_b}{\delta g_{\mu\nu}} + \frac{\delta S_{b-grav}}{\delta g_{\mu\nu}}\right] = 0 \Rightarrow$$

$$\boxed{R^{\mu\nu} - \frac{1}{2}g^{\mu\nu} = \kappa^2 T_b^{\mu\nu} + \frac{a'\kappa\sqrt{2}}{48}C^{\mu\nu}} \tag{3.2.19}$$

In standard theories, due to the diffeomorphism invariance, the matter stress-energy tensor should be conserved $\nabla_\mu T_b^{\mu\nu} = 0$. In the above equation, the left-hand side is invariant under general coordinate diffeomorphisms $\nabla_\mu G^{\mu\nu} = 0$ (Bianchi identity), which means that the right-hand side should obey the same rule. However, the derivative of the Cotton tensor is calculated $\nabla_\mu C^{\mu\nu} = -\frac{1}{8}\partial^\nu R^{\alpha\beta\gamma\delta}\tilde{R}_{\alpha\beta\gamma\delta}$ [11]. From equation 3.2.19, it is obvious that the energy conservation is violated:

$$\frac{a'\kappa\sqrt{2}}{48}\nabla_\mu C^{\mu\nu} = -\kappa^2\nabla_\mu T_b^{\mu\nu}$$

The energy violation due to the Cotton tensor non-zero derivative implies the non-trivial interaction between the axion field and gravity. To ensure energy conservation, we introduce a modified stress-energy tensor:

$$\kappa^2\tilde{T}^{\mu\nu}_{b+gCS} = \frac{a'\kappa\sqrt{2}}{48}C^{\mu\nu} + \kappa^2 T_b^{\mu\nu} \Rightarrow \nabla_\mu \tilde{T}^{\mu\nu}_{b+gCS} = 0 \tag{3.2.20}$$

3.2.2 b(x) as a perfect fluid

In a homogeneous and isotropic FLRW universe, the Cotton tensor $C^{\mu\nu}$ is traceless ($C^\mu{}_\mu = 0$) and thus does not contribute directly to the Friedmann equations (Check the appendix B equations 6.2.6, 6.2.7 6.2.8 to confirm). Consequently, in this background, one may study the dynamics of the axion field $b(x)$ as a perfect fluid, but its interaction term with gravity via the gravitational anomaly term $bR_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$ vanishes due to symmetry constraints.

The action that includes the $b(x)$ field yields the stress-energy tensor:

$$T^b_{\mu\nu} = \partial_\mu b\ \partial_\nu b - \frac{1}{2} g_{\mu\nu} \partial_\rho b\ \partial^\rho b$$

Assuming that $b(x)$ is only time dependent and does not have any dependence on spatial coordinates $\partial_i b = 0$ for $i = 1, 2, 3$, the tensor above can be rewritten:

$$T^b_{\mu\nu} = \delta^0_\mu \delta^0_\nu\ \dot{b}^2 - \frac{1}{2} g_{\mu\nu} \dot{b}^2$$

We will use the stress-energy tensor of a perfect fluid in co-moving coordinates $u^\mu = (-1, 0, 0, 0)$:

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu - g_{\mu\nu} P$$

Calculating the components of the stress-energy tensor in the end gives:

- For the 00-component :

$$T^b_{00} = (\partial_0 b)^2 - \frac{1}{2} g_{00}\ \partial_\alpha \partial^\alpha b = \frac{\dot{b}^2}{2} = (\rho_b + P_b) u_0 u_0 - g_{00} P_b = \rho_b \Rightarrow \boxed{\rho_b = \frac{\dot{b}^2}{2}} \tag{3.2.21}$$

- For the ii-component :

$$T^b_{ii} = -\frac{1}{2} g_{ii} \partial_\alpha \partial^\alpha b = -\frac{1}{2} g_{ii} \frac{\dot{b}^2}{2} = -\frac{1}{2} g_{ii} P_b \Rightarrow \boxed{P_b = \frac{\dot{b}^2}{2}} \tag{3.2.22}$$

From 3.2.21 and 3.2.22, the equation of state is found to be:

$$\frac{\dot{b}^2}{2} = P_b = w \rho_b \Rightarrow w = 1 \tag{3.2.23}$$

Using equation 6.2.9 , it follows that the energy density scales $\rho \propto \alpha^{-6}$. Therefore, it becomes clear that, without the additional contributions from the gravitational anomaly term, the $b(x)$ alone cannot directly drive inflation, since the energy density decays rapidly as the universe expands.

This result indicates that, when the $b(x)$ pseudoscalar field dominates the energy content of the universe, it behaves as a stiff matter fluid ($w = 1$). More specifically, this is the **stiff matter era**, when the action reduces to the kinetic term of the $b(x)$ field, without any potential yet, and its dynamics dominate the stress-energy tensor of the universe. As we will see later, the inclusion of the anomalous coupling term $bR_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$, particularly when metric perturbations are considered, leads to non-trivial contributions that can eventually drive inflation.

4 Cosmological evolution of the String-Inspired model with Torsion

Before proceeding, it is important to highlight the role of torsion in cosmological scenarios. Based on the scenario described in [3], during the very early epochs of the universe, right after the Big Bang, a first inflationary phase is assumed due to gravitino torsion in a SUGRA effective theory, which is responsible for the isotropy and homogeneity of the cosmic fluid. The late stage of this pre-inflationary phase is characterized by a stiff matter universe $w = 1$, as derived in the previous section, that results in the dominance of stringy massless axionic degrees. This axion-like degree is induced from the stringy torsion or the KR-field strength. The Chern-Simons anomalous term of the action at this stage plays a non-trivial role in the Einstein equation, especially in the presence of chiral Gravitational Wave (GW) perturbations caused by, e.g. merging of primordial Black holes. A condensate induced from the CS-anomaly drives the universe naturally into a second RVM-like inflation.

4.1 Modified GW equations due to the R_{cs}-anomalous term

As discussed in the previous chapter, the Cotton tensor is traceless; it does not directly contribute to the Friedmann equations, which causes the gravitational Chern-Simons term in the action 3.1.14 to vanish. Considering metric perturbations related to primordial Gravitational Waves (GWs), a condensate of gravitational anomalies is induced, which may act as a linear axion potential and alter the final equations of GWs. In correspondence to the Chern-Simons modification of general relativity[10], the contribution of the CS-anomalous term raises the phenomenon of birefringence, which in our case appears as a difference in the left and right-handed GW frequencies.

4.1.1 Background perturbations

Let us consider metric perturbations in an FRLW universe:

$$ds^2 = dt - \alpha(t)(\delta_{ij} + h_{ij})dx^i dx^j \tag{4.1.1}$$

Now, because of the gauge-like invariance of the perturbation, we can choose to work in the transverse traceless gauge fixing (Appendix C). Thus, the perturbation takes the form of a matrix with two independent non-zero elements:

$$h_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.1.2}$$

Thus, the metric becomes:

$$g_{\mu\nu}^{FLRW\prime} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\alpha(t)(1+h_+) & h_\times & 0 \\ 0 & h_\times & -\alpha(t)(1-h_+) & 0 \\ 0 & 0 & 0 & -\alpha(t) \end{pmatrix} \tag{4.1.3}$$

Where h_+ is the "+" polarization and $h_\times$ is the "×" polarization. Now we can consider the z-axis to be the direction of propagation and we can choose a triad of right-handed orthogonal basis vectors $e_1 = [0\ 1\ 0\ 0]$, $e_2 = [0\ 0\ 1\ 0]$, $e_3 = [0\ 0\ 0\ 1]$. Thus, polarization tensors can be defined as:

$$e_+^{ij} = e_1^i e_1^j - e_2^i e_2^j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.1.4}$$

$$e_\times^{ij} = e_1^i e_2^j - e_2^i e_1^j = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.1.5}$$

The Einstein equation, initially, was calculated in the previous section:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T^b_{\mu\nu} + C_{\mu\nu} \tag{4.1.6}$$

After the metric perturbation, the stress-energy tensor for the b-field has no contribution in the modified Einstein equation, and the Ricci tensor, Cotton tensor, and Ricci scalar are modified. Through Appendix C, in the transverse traceless basis, the right-hand side of the equation above writes:

$$R^{\text{pert}}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{\text{pert}} = -\frac{1}{2} \Box h_{\mu\nu} \tag{4.1.7}$$

Now, given that $\partial_\mu b(x) = (\dot{b}, 0, 0, 0)$, the corresponding Cotton tensor expanded in first-order metric perturbations is:

$$C^{\text{pert}}_{\mu\nu} = -\frac{\dot{b}}{2} \nabla^a [\delta \tilde{R}_{\alpha\mu 0\nu} + \delta \tilde{R}_{\alpha\nu 0\mu}] \tag{4.1.8}$$

4.1.2 Modified GW equations

All tensors can be recalculated, given the perturbation above. The equations of motion for the two polarizations of the waves can be expressed as a wave equation:

$$\Box h_{\times,+} = \mp \frac{4A\kappa^2}{\alpha^2} (2\dot{\alpha}\dot{b} + \alpha\ddot{b}) \partial_t \partial_z h_{\times,+} \mp \frac{4A\kappa^2 \dot{b}}{\alpha} \partial_t^2 \partial_z h_{\times,+} \pm \frac{4A\kappa^2 \dot{b}}{\alpha^3} \partial_z^3 h_{\times,+} \tag{4.1.9}$$

where the d'Alembertian symbol in the perturbed FLRW spacetime is equal to:

$$\Box = -\partial_t^2 - 3\frac{\dot{\alpha}}{\alpha} \partial_t + \frac{1}{\alpha^2} \partial_z^2 \tag{4.1.10}$$

Passing to a helicity basis will clarify the role of the Chern-Simons term in the condensate and gravitational wave equations later on. The helicity basis is composed of the following polarization vectors:

$$e^R_{ij} = \frac{1}{\sqrt{2}} (e^+_{ij} + i e^\times_{ij}) = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left[e^L_{ij}\right]^\dagger \tag{4.1.11}$$

Where i, j denotes only the spatial components. The components of the perturbation in helicity basis, expressed in terms of the elements from the previous basis, are:

$$h_{R,L} = \frac{1}{\sqrt{2}}(h_+ \pm h_\times)$$

Physically, h_L and h_R correspond to waves whose pattern rotates in opposite directions as the wave propagates (analogous to left- and right-circular polarization of light). Under a parity transformation, these two helicity modes swap: a left-handed GW becomes right-handed and vice versa.

The GW equation now writes:

$$\Box h_R = \frac{4iA\kappa^2}{\alpha^2}(2\dot{\alpha}\dot{b} + \alpha\ddot{b})\partial_t\partial_z h_R + \frac{4iA\kappa^2\dot{b}}{\alpha}\partial_t^2\partial_z h_R - \frac{4iA\kappa^2\dot{b}}{\alpha^3}\partial_z^3 h_R \tag{4.1.12}$$

$$\Box h_L = -\frac{4iA\kappa^2}{\alpha^2}(2\dot{\alpha}\dot{b} + \alpha\ddot{b})\partial_t\partial_z h_L - \frac{4iA\kappa^2\dot{b}}{\alpha}\partial_t^2\partial_z h_L + \frac{4iA\kappa^2\dot{b}}{\alpha^3}\partial_z^3 h_L \tag{4.1.13}$$

The gravitational Chern-Simons term, up to second-order metric perturbations, is [3]:

$$R_{CS} = \frac{1}{2}R^{\mu}_{\nu\rho\sigma}\tilde{R}^{\nu}{}_{\mu}{}^{\rho\sigma} = \frac{2i}{\alpha^3}\left[\left(\partial_z^2 h_L\,\partial_z\partial_t h_R + \alpha^2\partial_t^2 h_L\,\partial_z\partial_t h_R + \alpha\dot{\alpha}\,\partial_t h_L\,\partial_z\partial_t h_R\right) - \left(L \leftrightarrow R\right)\right] + \mathcal{O}(h^4) \tag{4.1.14}$$

The Einstein gravitational wave equations for left-handed and right-handed gravitational waves have opposite signs, which indicates the non-triviality of the R_{CS} anomaly. This phenomenon is known as cosmological birefringence. It predicts the rotation of the polarization planes of gravitational waves as they travel over cosmological distances, a concept also discussed in [10]. The polarizations of the h_L and h_R evolve in opposite sign due to the perturbed Cotton tensor contribution in the initial Einstein equation. To investigate more on the evolution of the GWs, one may pass to conformal time $dt = \alpha d\eta$ and consider Fourier modes:

$$h_\lambda(\eta, \vec{x}) = \int \frac{d^3x}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}}\tilde{h}_{\lambda,\vec{k}}(\eta) \quad \Rightarrow \partial_z h_\lambda = ik\ h_\lambda \Rightarrow \partial_z^2 h_\lambda = -k^2 h_\lambda \tag{4.1.15}$$

So:

- $\partial_t h_\lambda = \frac{dh_\lambda}{dt} = \frac{dh_\lambda}{d\eta}\frac{d\eta}{dt} = \alpha^{-1}h'_\lambda$
- $\partial_t^2 h_\lambda = \frac{d}{dt}\left[\frac{dh_\lambda}{dt}\right] = \frac{d}{dt}\left[\alpha^{-1}h'_\lambda\right] = -\alpha'\alpha^{-3}h'_\lambda + \alpha^{-2}h''_\lambda$
- $\dot{\alpha} = \frac{\alpha'}{\alpha^2}$, $\dot{b} = \frac{b'}{\alpha}$ and $\ddot{b} = \frac{b''}{\alpha^2} - \frac{\alpha' b'}{\alpha^3}$
- the box operator becomes:

$$\Box h_{R,L} = \left[-\alpha^{-2}\partial_\eta^2 + \frac{\alpha'}{\alpha^3}\partial_\eta - 3\frac{\alpha'}{\alpha^3}\partial_\eta + \frac{1}{\alpha^2}\partial_z^2\right]h_{R,L} = \left[-\alpha^{-2}\partial_\eta^2 - 2\frac{\alpha'}{\alpha^3}\partial_\eta + \frac{1}{\alpha^2}\partial_z^2\right]h_{R,L} \tag{4.1.16}$$

Thus considering $l_R = 1$ and $l_L = -1$ and $\lambda = R, L$:

$$\Box h_\lambda = \pm \frac{4iA\kappa^2}{\alpha^2}(2\dot{\alpha}\dot{b} + \alpha\ddot{b})\partial_t\partial_z h_\lambda \pm \frac{4iA\kappa^2\dot{b}}{\alpha}\partial_t^2\partial_z h_\lambda \mp \frac{4iA\kappa^2\dot{b}}{\alpha^3}\partial_z^3 h_\lambda \xrightarrow{dt=\alpha d\eta}$$

$$\left[\alpha^{-2}\partial_\eta^2 + 2\frac{\alpha'}{\alpha^3}\partial_\eta - \frac{1}{\alpha^2}\partial_z^2\right]h_\lambda = -l_\lambda\frac{4iA\kappa^2}{\alpha^2}\left[\frac{1}{\alpha^3}(\alpha'b' + \alpha b'')\partial_z h'_\lambda + b'\left(-\frac{\alpha'}{\alpha^3}\partial_z h'_\lambda + \frac{1}{\alpha^2}\partial_z h''_\lambda\right) - \frac{b'}{\alpha^2}\partial_z^3 h_\lambda\right]$$

$$\left[\partial_\eta^2 + 2\frac{\alpha'}{\alpha}\partial_\eta - \partial_z^2\right]h_\lambda = -l_\lambda\frac{4iA\kappa^2}{\alpha^2}\left[\frac{\alpha'}{\alpha}b'\partial_z h'_\lambda + b''\partial_z h'_\lambda - \frac{\alpha'}{\alpha}b'\partial_z h'_\lambda + b'\partial_z h''_\lambda - b'\partial_z^3 h_\lambda\right]$$

$$\left[\partial_\eta^2 + 2\frac{\alpha'}{\alpha}\partial_\eta - \partial_z^2\right]h_\lambda = -l_\lambda\frac{4iA\kappa^2}{\alpha^2}\left[b''\partial_z h'_\lambda + b'\partial_z h''_\lambda - b'\partial_z^3 h_\lambda\right]$$

$$\left[h''_\lambda + 2\frac{\alpha'}{\alpha}h'_\lambda - \partial_z^2 h_\lambda\right] = -l_\lambda\frac{4iA\kappa^2}{\alpha^2}\partial_z\left[b''h'_\lambda + b'h''_\lambda - b'\partial_z^2 h_\lambda\right] \tag{4.1.17}$$

At this stage, we need to decompose the initial wave function into its modes and track their time evolution through 4.1.15:

$$\left[\tilde{h}''_{\lambda,\vec{k}} + 2\frac{\alpha'}{\alpha}\tilde{h}'_{\lambda,\vec{k}} + k^2\tilde{h}_{\lambda,\vec{k}}\right] = l_\lambda l_{\vec{k}}\frac{4kA\kappa^2}{\alpha^2}\left[b''\tilde{h}'_{\lambda,\vec{k}} + b'\tilde{h}''_{\lambda,\vec{k}} + b'k^2\tilde{h}_{\lambda,\vec{k}}\right] \tag{4.1.18}$$

The parameter $l_{\vec{k}} = 1$ ($l_{-\vec{k}} = -1$) has been introduced to keep track of the orientation of the wave vector (momentum) relative to a reference direction. To bring the equation into a more convenient form:

$$\tilde{h}''_{\lambda,\vec{k}}\left[1 - l_\lambda l_{\vec{k}}\frac{4kAb'\kappa^2}{\alpha^2}\right] + \tilde{h}'_{\lambda,\vec{k}}\left[2\frac{\alpha'}{\alpha} - l_\lambda l_{\vec{k}}\frac{4kAb''\kappa^2}{\alpha^2}\right] + \tilde{h}_{\lambda,\vec{k}}\ k^2\left[1 - l_\lambda l_{\vec{k}}\frac{4kA\kappa^2}{\alpha^2}\right] = 0$$

Considering a field redefinition of the form:

$$\tilde{\psi}_{\lambda,\vec{k}}(\eta) = \frac{1}{\kappa}\ \alpha\sqrt{1 - l_\lambda l_{\vec{k}}\frac{4kAb'\kappa^2}{\alpha^2}}\ \tilde{h}_{\lambda,\vec{k}}(\eta) = \frac{z_{\lambda,\vec{k}}}{\kappa}\ \tilde{h}_{\lambda,\vec{k}}(\eta) \Rightarrow \tilde{h}_{\lambda,\vec{k}}(\eta) = \kappa\frac{\tilde{\psi}_{\lambda,\vec{k}}(\eta)}{z_{\lambda,\vec{k}}(\eta)} \tag{4.1.19}$$

The equation 4.1.18 then becomes:

$$\boxed{\tilde{\psi}_{\lambda,\vec{k}}(\eta)'' + \omega^2_{\lambda,\vec{k}}(\eta)\tilde{\psi}_{\lambda,\vec{k}}(\eta) = 0} \tag{4.1.20}$$

Where:

- The frequency: $\omega^2_{\lambda,\vec{k}}(\eta) = k^2 - \dfrac{z''_{\lambda,\vec{k}}}{z_{\lambda,\vec{k}}}$
- The modified scale factor is: $z_{\lambda,\vec{k}} = \alpha\sqrt{1 - l_\lambda l_{\vec{k}} L_{CS}}$,
- The CS-coupling is: $L_{CS} = k\xi$, $\xi = \dfrac{4Ab'\kappa^2}{\alpha^2}$

The standard methodology of finding the equations of motion for complex scalars in curved backgrounds is presented in Appendix D; however, in the case of the chiral-violating nature of the GWs here, the method has proved to be slightly more complicated. Due to the Cotton tensor contribution from the Chern-Simons term, the dispersion relation becomes polarization (l_λ)- and direction ($l_{\vec{k}}$) - dependent. Depending on the wave vector direction and polarization, this birefringence effect leads to different evolutions for left- and right-handed GW modes.

The parameter ξ encodes the contribution of the b-pseudoscalar field in the theory, and has dimensions $[\xi] = [M]^{-1}$, so that L_{CS} is dimensionless. In the case where $b(x)$ does not exist, $\xi = 0 \Rightarrow z_{\lambda,\vec{k}} = \alpha(\eta)$, and the GW equation is the same for both wave helicities. In this limit, the gravitational wave equation reduces to the standard one derived from GR, and parity is restored.

The formula 4.1.20 describes the evolution of tensor perturbations, or gravitational waves, within the context of a dynamic Chern-Simons (CS) coupling, which arises from the rolling scalar field $b(\eta)$. It is the primary equation we will use to investigate the contribution of the CS term during both the stiff-matter era and the inflationary era in our analysis. For now, we must calculate the R_{CS} condensate, using the quantization method of fields in the FLRW background. To proceed with this calculation, we apply the quantization framework laid out in Appendix D, focusing on tensor perturbations in a FLRW spacetime.

4.1.3 Symmetries of the polarizations and metric perturbation quantization

From the change to helicity basis polarization, we know that:

$$h_L^*(\eta, \vec{x}) = h_R(\eta, \vec{x}) \overset{\mathcal{F}}{\Rightarrow} \tilde{h}^*_{L,-\vec{k}}(\eta, \vec{x}) = \tilde{h}_{R,\vec{k}}(\eta, \vec{x})$$

One may notice that the equation 4.1.20 is the same only under simultaneous transformations:

$$L \to R \qquad \text{and} \qquad k \to -k \tag{4.1.21}$$

since $l_R\, l_{\vec{k}} = l_L l_{-\vec{k}} = 1$ and $z_{L,-\vec{k}} = z_{R,\vec{k}} \Rightarrow \omega_{L,-\vec{k}} = \omega_{R,\vec{k}}$

We will now examine the solutions of the GW wave equation. Based on the symmetries mentioned above, there are two possible cases.

1. $\boxed{\tilde{\psi}_{R,\vec{k}} = \tilde{\psi}_{L,-\vec{k}}}$, when $l_R l_{\vec{k}} = l_L l_{-\vec{k}} = 1$

   In this case, the frequency in the GW equations is:

$$\omega_{R,\vec{k}} = \omega_{L,-\vec{k}} = k^2 - \left(\frac{\alpha''(\eta)\alpha(\eta)}{z_{R,\vec{k}}(\eta)} - L_{CS} \frac{(\alpha'(\eta))^2 \sqrt{\alpha(\eta)}}{z_{R,\vec{k}}(\eta)} \right) \tag{4.1.22}$$

   where:

$$z_{R,\vec{k}}(\eta) = \alpha(\eta)\sqrt{1 - L_{CS}} \tag{4.1.23}$$

2. $\boxed{\tilde{\psi}_{L,\vec{k}} = \tilde{\psi}_{R,-\vec{k}}}$, when $l_L l_{\vec{k}} = l_R l_{-\vec{k}} = -1$

   Correspondingly, the frequency denotes:

$$\omega_{L,\vec{k}} = \omega_{R,-\vec{k}} = k^2 - \left(\frac{\alpha''(\eta)\alpha(\eta)}{z_{L,\vec{k}}(\eta)} + L_{CS} \frac{(\alpha'(\eta))^2 \sqrt{\alpha(\eta)}}{z_{L,\vec{k}}(\eta)} \right) \tag{4.1.24}$$

   where:

$$z_{L,\vec{k}}(\eta) = z_{R,-\vec{k}}(\eta) = \alpha(\eta)\sqrt{1 + L_{CS}} \tag{4.1.25}$$

In this case, the phenomenon of cosmological birefringence is clear because of the frequency sign differences for each helicity.

4.2 Quantization of Gravitational Waves and Chern-Simons Condensate

In this section, we perform the quantization of gravitational wave (GW) perturbations in the presence of a dynamical axion-like background that induces a Chern-Simons (CS) coupling. We aim to compute the quantum expectation value of the CS term, $\langle R_{\rm CS}\rangle$. This quantity plays a crucial role in the formation of a vacuum condensate sourced by gravitational wave fluctuations during the early universe.

Given the parity-violating nature of the theory, the GW perturbations are split into left- and right-handed helicity modes, each satisfying distinct equations of motion. In this setup, we need to adopt a semi-classical approach. Specifically, the background spacetime will be treated classically, while the tensor perturbations are promoted to quantum operators. We consider these helicity modes as components of a complex scalar field and express the equations of motion and the action accordingly.

Following the formalism outlined by Mukhanov [12], we expand the complex scalar field in terms of mode functions and associate independent sets of creation and annihilation operators for each helicity state. We then impose the canonical commutation relations and derive Wronskian normalization conditions for the mode functions. These results allow us to compute the key correlators $\langle h_L h_R\rangle$ and $\langle h_R h_L\rangle$, which enter the expression for $\langle R_{\rm CS}\rangle$.

Finally, we evaluate this expectation value under the assumption of an instantaneous vacuum at a fixed conformal time η_0, restricted to subhorizon modes defined by a physical ultraviolet cutoff μ. The resulting condensate reflects the quantum contribution of GW modes in an anisotropic FLRW background with torsion.

4.2.1 Complex scalar field definition

Since full quantization of gravity is a general problem today, we will use a semi-classical approach. We will consider the metric perturbation as a complex scalar field and quantize this quantity.

The full metric perturbation may be written with respect to the helicity and Fourier modes:

$$h_{ij} = \kappa \sum_{\lambda} \int \frac{d^3k}{(2\pi)^{2/3}} e^{i\vec{k}\cdot\vec{x}} \frac{\tilde{\psi}_{\lambda,\vec{k}}(\eta)}{z_{\lambda,\vec{k}}(\eta)} \epsilon^{\lambda}_{ij} \tag{4.2.1}$$

We have shown in the previous section that there are two different solutions for the GW equation, one for the Right-handed GWs and another one for the Left-handed GWs both propagating in the z-direction. Thus, one may define a complex scalar field to describe in a more unifying way the previous solutions:

$$\phi(\eta,\vec{x}) = \psi_L(\eta,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \tilde{\psi}_{L,\vec{k}}(\eta) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \tilde{\phi}_{\vec{k}}(\eta) \tag{4.2.2}$$

$$\phi^*(\eta,\vec{x}) = \psi_R(\eta,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \tilde{\psi}_{R,\vec{k}}(\eta) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \tilde{\phi}^*_{-\vec{k}}(\eta) \tag{4.2.3}$$

Based on the redefinition above, the equations of motion 4.1.20 are expressed:

$$\tilde{\phi}''_{\vec{k}} + \Omega^2_{\vec{k}} \tilde{\phi}_{\vec{k}} = 0 \tag{4.2.4}$$

$$\tilde{\phi}^{*\prime\prime}_{-\vec{k}} + \Omega^2_{-\vec{k}}\tilde{\phi}^*_{-\vec{k}} = 0 \tag{4.2.5}$$

Where $\Omega_{\vec{k}}$ and $\Omega_{-\vec{k}}$ are defined in 4.1.24 and 4.1.22 respectively and $\Omega_{\vec{k}} \neq \Omega_{-\vec{k}}$.

Now starting from equations 4.2.4 and 4.2.5 we may define the Lagrangian density through the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial \tilde{\phi}_{\vec{k}}} = \Omega^2_{-\vec{k}}\tilde{\phi}^*_{-\vec{k}}, \qquad \frac{\partial \mathcal{L}}{\tilde{\phi}^*_{-\vec{k}}} = \Omega^2_{\vec{k}}\tilde{\phi}_{\vec{k}} \quad \& \quad \frac{d}{d\eta}\left(\frac{\partial \mathcal{L}}{\tilde{\phi}^*_{-\vec{k}}}\right) = \tilde{\phi}^{\prime\prime}_{\vec{k}}, \qquad \frac{d}{d\eta}\left(\frac{\partial \mathcal{L}}{\tilde{\phi}_{\vec{k}}}\right) = \tilde{\phi}^{*\prime\prime}_{-\vec{k}}$$

Integrating the above equations and by applying the $-\vec{k} \to \vec{k}$ spatial reflection of the momentum, we get the form of the Lagrangian density with respect to the Fourier modes of the field:

$$S = \int d\eta \int d^3k \left[-\tilde{\phi}^{\prime}_{\vec{k}}\tilde{\phi}^{*\prime}_{\vec{k}} + \Omega^2_{\vec{k}}\ \tilde{\phi}_{\vec{k}}\tilde{\phi}^*_{\vec{k}}\right] \tag{4.2.6}$$

This action encodes information for both left-handed and right-handed GWs using the spatial reflection condition $\vec{k} \to -\vec{k}$.

4.2.2 Second Quantization & Commutation Relations

In the context of an effectively anisotropic background, where the mode-dependent frequencies satisfy $\Omega_{\vec{k}} \neq \Omega_{-\vec{k}}$, we are dealing with a complex scalar field rather than a real one. Therefore, we must introduce two independent sets of creation and annihilation operators. We promote the field modes $\tilde{\phi}_{\vec{k}}$, $\tilde{\phi}^*_{-\vec{k}}$ to quantum operators in the standard way. In the analysis below, in order to prove and work with the commutation relations, it is enough to work with the mode expansions alone (Check [12] for more analytic calculations).

The creation and annihilation operators are associated with the quantum field operators as follows:

- $\hat{a}^{\pm}_{\vec{k}}$ is associated with $\tilde{\phi}_{\vec{k}}$, where the operator $\hat{a}^{+}_{\vec{k}}$ creates a Left-helicity quantum mode or else excitation
- $\hat{b}^{\pm}_{\vec{k}}$ is associated with $\tilde{\phi}^*_{-\vec{k}}$, where the operator $\hat{b}^{+}_{\vec{k}}$ creates a Right-handed mode

From standard Quantum Field Theory for complex scalars, it is true that creation and annihilation operators satisfy the following commutation relations [12]:

$$[\hat{a}^{-}_{\vec{k}}, \hat{a}^{+}_{\vec{k}'}] = [\hat{b}^{-}_{\vec{k}}, \hat{b}^{+}_{\vec{k}'}] = \delta^{(3)}(k - k') \tag{4.2.7}$$

To proceed with quantization, we need to decompose the Fourier modes into mode functions that form a basis in the Fock space where the operators $\hat{a}^{\pm}_{\vec{k}}$, $\hat{b}^{\pm}_{\vec{k}}$ act, creating and annihilating quantum excitations. In particular, one may define two sets of mode functions for each left- and right-field mode:

- $\tilde{\psi}_{L,\vec{k}} = \tilde{\phi}_{\vec{k}}$ can be expressed as a linear combination of the mode function basis: $\{\tilde{u}_{\vec{k}}, \tilde{u}^*_{\vec{k}}\}$. Both mode functions should be solutions to equation 4.2.4 thus we write:

$$\tilde{u}^{\prime\prime}_{\vec{k}} + \Omega^2_{\vec{k}}\ \tilde{u}_{\vec{k}} = 0 \tag{4.2.8}$$

$$\tilde{u}^{*\prime\prime}_{\vec{k}} + \Omega^2_{\vec{k}}\ \tilde{u}^*_{\vec{k}} = 0 \tag{4.2.9}$$

- $\tilde{\psi}_{R,\vec{k}} = \tilde{\phi}^*_{-\vec{k}}$ can be expressed as a linear combination of the mode function basis: $\{u_{\vec{k}}, u^*_{\vec{k}}\}$. Again, the mode functions should solve the equation 4.2.5 thus we write:

$$u''_{\vec{k}} + \Omega^2_{-\vec{k}}\ u_{\vec{k}} = 0 \tag{4.2.10}$$

$$u^{*\prime\prime}_{\vec{k}} + \Omega^2_{-\vec{k}}\ u^*_{\vec{k}} = 0 \tag{4.2.11}$$

However, by performing a spatial transformation $\vec{k} \to -\vec{k}$, for example, in 4.2.8 and comparing it with 4.2.10, the basis mode functions can be related in the way described below:

$$\begin{Bmatrix} \tilde{u}''_{-\vec{k}} + \Omega^2_{-\vec{k}}\tilde{u}_{-\vec{k}} = 0 \\ u''_{\vec{k}} + \Omega^2_{-\vec{k}}u_{\vec{k}} = 0 \end{Bmatrix} \quad \Rightarrow \quad u_{\vec{k}} = \tilde{u}_{-\vec{k}} \tag{4.2.12}$$

We may then express the field operators as a linear combination of the correlated bases:

$$\hat{\tilde{\phi}}_{\vec{k}} = \tilde{\psi}_{L,\vec{k}} = \tilde{u}_{\vec{k}}\hat{a}^-_{\vec{k}} + u^*_{-\vec{k}}\hat{b}^+_{-\vec{k}} \tag{4.2.13}$$

$$\hat{\tilde{\phi}}^*_{-\vec{k}} = \tilde{\psi}_{R,\vec{k}} = u_{\vec{k}}\hat{b}^-_{\vec{k}} + \tilde{u}^*_{-\vec{k}}\hat{a}^+_{-\vec{k}} \tag{4.2.14}$$

The canonical momenta then are:

$$\hat{\pi}_{\vec{k}} = -\left(\hat{\tilde{\phi}}^*_{\vec{k}}\right)' = -u'_{-\vec{k}}\hat{b}^-_{-\vec{k}} - \tilde{u}^{*\prime}_{\vec{k}}\ \hat{a}^+_{\vec{k}} \tag{4.2.15}$$

$$\hat{\pi}^*_{\vec{k}} = -\left(\hat{\tilde{\phi}}_{\vec{k}}\right)' = -\tilde{u}'_{\vec{k}}\hat{a}^-_{\vec{k}} + u^{*\prime}_{-\vec{k}}\hat{b}^+_{-\vec{k}} \tag{4.2.16}$$

To ensure consistency with canonical quantization, we impose:

$$[\hat{\tilde{\phi}}_{\vec{k}}, \hat{\pi}_{\vec{k}'}] = i\delta(k - k') \Rightarrow$$
$$[\hat{\tilde{\phi}}_{\vec{k}}, \hat{\pi}_{\vec{k}'}] = \hat{\tilde{\phi}}_{\vec{k}}\hat{\pi}_{\vec{k}'} - \hat{\pi}_{\vec{k}'}\hat{\tilde{\phi}}_{\vec{k}} \overset{4.2.7}{=} -\tilde{u}_{\vec{k}}\tilde{u}^{*\prime}_{\vec{k}'}\ [\hat{a}^-_{\vec{k}}, \hat{a}^+_{\vec{k}'}] - u^*_{-\vec{k}}u'_{-\vec{k}'}[\hat{b}^+_{-\vec{k}}, \hat{b}^-_{-\vec{k}'}] = [u^*_{-\vec{k}}u'_{-\vec{k}'} - \tilde{u}_{\vec{k}}\tilde{u}^{*\prime}_{\vec{k}'}]\ \delta^{(3)}(\vec{k} - \vec{k}') \Rightarrow$$

$$u^*_{-\vec{k}}u'_{-\vec{k}} - \tilde{u}_{\vec{k}}\tilde{u}^{*\prime}_{\vec{k}} = i \overset{4.2.12}{\Rightarrow} \boxed{\tilde{u}^*_{\vec{k}}\tilde{u}'_{\vec{k}} - \tilde{u}_{\vec{k}}\tilde{u}^{*\prime}_{\vec{k}} = i} \tag{4.2.17}$$

$$[\hat{\tilde{\phi}}^*_{-\vec{k}}, \hat{\pi}^*_{-\vec{k}'}] = i\delta(k - k') \Rightarrow$$
$$[\hat{\tilde{\phi}}^*_{-\vec{k}}, \hat{\pi}^*_{-\vec{k}'}] = \hat{\tilde{\phi}}^*_{-\vec{k}}\hat{\pi}^*_{-\vec{k}'} - \hat{\pi}^*_{-\vec{k}'}\hat{\tilde{\phi}}^*_{-\vec{k}} \overset{4.2.7}{=} -\tilde{u}^*_{-\vec{k}}\tilde{u}'_{-\vec{k}'}\ [\hat{a}^+_{-\vec{k}}, \hat{a}^-_{-\vec{k}'}] - u_{\vec{k}}u^{*\prime}_{\vec{k}'}[\hat{b}^-_{\vec{k}'}, \hat{b}^+_{\vec{k}}] = [\tilde{u}^*_{-\vec{k}}\tilde{u}'_{-\vec{k}'} - u_{\vec{k}}u^{*\prime}_{\vec{k}'}]\ \delta^{(3)}(\vec{k} - \vec{k}')$$

$$\Rightarrow \tilde{u}^*_{-\vec{k}}\tilde{u}'_{-\vec{k}} - u_{\vec{k}}u^{*\prime}_{\vec{k}} = i \overset{4.2.12}{\Rightarrow} \boxed{u^*_{\vec{k}}u'_{\vec{k}} - u_{\vec{k}}u^{*\prime}_{\vec{k}} = i} \tag{4.2.18}$$

The commutation relations between the field modes and their canonical momenta lead to equations 4.2.17 and 4.2.18, which define the Wronskian normalization for the k-mode functions of the basis. Using the field redefinition 4.1.19, we can now express the Fourier modes of the perturbations as:

$$\tilde{h}_{L,\vec{k}} = \kappa\frac{\tilde{\psi}_{L,\vec{k}}(\eta)}{z_{L,\vec{k}}(\eta)} = \kappa\frac{\tilde{u}_{\vec{k}}(\eta)}{z_{L,\vec{k}}(\eta)}\hat{a}^-_{\vec{k}} + \kappa\frac{u^*_{-\vec{k}}(\eta)}{z_{L,\vec{k}}(\eta)}\hat{b}^+_{-\vec{k}} \equiv v_{L,\vec{k}}\hat{a}^-_{\vec{k}} + v^*_{R,-\vec{k}}\hat{b}^+_{-\vec{k}} \tag{4.2.19}$$

$$\tilde{h}_{R,\vec{k}} = \kappa \frac{\tilde{\psi}_{R,\vec{k}}(\eta)}{z_{R,\vec{k}}(\eta)} = \kappa \frac{u_{\vec{k}}(\eta)}{z_{R,\vec{k}}(\eta)} \hat{b}^{-}_{\vec{k}} + \kappa \frac{\tilde{u}^{*}_{-\vec{k}}(\eta)}{z_{R,\vec{k}}(\eta)} \hat{a}^{+}_{-\vec{k}} \equiv v_{R,\vec{k}} \hat{b}^{-}_{\vec{k}} + v^{*}_{L,-\vec{k}} \hat{a}^{+}_{-\vec{k}} \tag{4.2.20}$$

Where the redefined mode functions inherit the properties from the previous basis mode functions, since:

$$v_{L,\vec{k}}(\eta) = \kappa \frac{\tilde{u}_{\vec{k}}(\eta)}{z_{L,\vec{k}}(\eta)} \quad \& \quad v_{R,\vec{k}}(\eta) = \kappa \frac{u_{\vec{k}}(\eta)}{z_{R,\vec{k}}(\eta)} \tag{4.2.21}$$

When $a^{+}_{\vec{k}}$ and $b^{+}_{\vec{k}}$ act on the vacuum state, they create a Left and Right k-mode respectively, while the operators $a^{-}_{\vec{k}}$ and $b^{-}_{\vec{k}}$, while acting on the vacuum, give zero, so:

$$a^{-}_{\vec{k}_1}|0\rangle = 0 \Rightarrow \langle 0|[a^{-}_{\vec{k}_1}, a^{+}_{\vec{k}_2}]|0\rangle = \delta^{(3)}(\vec{k_1} - \vec{k}_2) \Rightarrow \langle 0|a^{-}_{\vec{k}_1} a^{+}_{\vec{k}_2}|0\rangle = \delta^{(3)}(\vec{k_1} - \vec{k}_2) \tag{4.2.22}$$

$$b^{-}_{\vec{k}_1}|0\rangle = 0 \Rightarrow \langle 0|[b^{-}_{\vec{k}_1}, b^{+}_{\vec{k}_2}]|0\rangle = \delta^{(3)}(\vec{k_1} - \vec{k}_2) \Rightarrow \langle 0|b^{-}_{\vec{k}_1} b^{+}_{\vec{k}_2}|0\rangle = \delta^{(3)}(\vec{k_1} - \vec{k}_2) \tag{4.2.23}$$

Now the correlators are easily calculated; the relation above (most of the terms cancel):

$$\langle \tilde{h}_{L,\vec{k}_1} \tilde{h}_{R,\vec{k}_2} \rangle = \langle 0| \left(v_{L,\vec{k}_1} \hat{a}^{-}_{\vec{k}_1} + v^{*}_{R,-\vec{k}_1} \hat{b}^{+}_{-\vec{k}_1} \right) \left(v_{R,\vec{k}_2} \hat{b}^{-}_{\vec{k}_2} + v^{*}_{L,-\vec{k}_2} \hat{a}^{+}_{-\vec{k}_2} \right) |0\rangle = v_{L,\vec{k}_1} v^{*}_{L,-\vec{k}_2} \delta^{(3)}(\vec{k_1} - \vec{k}_2) \tag{4.2.24}$$

$$\langle \tilde{h}_{R,\vec{k}_1} \tilde{h}_{L,\vec{k}_2} \rangle = \langle 0| \left(v_{R,\vec{k}_1} \hat{b}^{-}_{\vec{k}_1} + v^{*}_{L,-\vec{k}_1} \hat{a}^{+}_{-\vec{k}_1} \right) \left(v_{L,\vec{k}_1} \hat{a}^{-}_{\vec{k}_1} + v^{*}_{R,-\vec{k}_2} \hat{b}^{+}_{-\vec{k}_2} \right) |0\rangle = v_{R,\vec{k}_1} v^{*}_{R,-\vec{k}_2} \delta^{(3)}(\vec{k_1} - \vec{k}_2) \tag{4.2.25}$$

These correlators denote the quantum two-point function, which describes how the left and right chiral waves correlate in the vacuum state. In a torsionless General Relativity, those values should be zero since the left and right-handed modes should be the same.

4.2.3 R_{CS} calculation

To express the Gravitational Chern-Simons term with respect to mode functions, we will use the results of the previous section, especially the 4.2.24 and 4.2.25. We remind the R_{CS} term below and we calculate each term separately:

$$\langle R_{\rm CS} \rangle = \frac{2i}{a^4} \left[\langle \partial_z^2 h_L \, \partial_z h'_R \rangle + \langle h''_L \, \partial_z h'_R \rangle - \langle \partial_z^2 h_R \, \partial_z h'_L \rangle - \langle h''_R \, \partial_z h'_L \rangle \right]$$

$$\langle \partial_z^2 h_L(\eta,\vec{x})\,\partial_z h_R'(\eta,\vec{x})\rangle = \langle 0| \left(\partial_z^2 \int \frac{d^3k_1}{(2\pi)^{3/2}} e^{i\vec{k}_1\cdot\vec{x}} \tilde{h}_{L,\vec{k}_1}(\eta)\right)\left(\partial_z \int \frac{d^3k_2}{(2\pi)^{3/2}} e^{i\vec{k}_2\cdot\vec{x}} \tilde{h}'_{R,\vec{k}_2}(\eta)\right)|0\rangle$$

$$= \langle 0| \left(\int \frac{d^3k_1}{(2\pi)^{3/2}} (-\vec{k}_1^2) e^{i\vec{k}_1\cdot\vec{x}} \tilde{h}_{L,\vec{k}_1}(\eta)\right)\left(\int \frac{d^3k_2}{(2\pi)^{3/2}} (i\vec{k}_2) e^{i\vec{k}_2\cdot\vec{x}} \tilde{h}'_{R,\vec{k}_2}(\eta)\right)|0\rangle$$

$$= \int \frac{d^3k_1}{(2\pi)^{3/2}} \int \frac{d^3k_2}{(2\pi)^{3/2}} (-i\vec{k}_1^2\vec{k}_2) e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{x}} \langle 0|\tilde{h}_{L,\vec{k}_1}(\eta)\tilde{h}'_{R,\vec{k}_2}(\eta)|0\rangle$$

$$= \int \frac{d^3k_1}{(2\pi)^{3/2}} \int \frac{d^3k_2}{(2\pi)^{3/2}} (-i\vec{k}_1^2\vec{k}_2) e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{x}} v_{L,\vec{k}_1} v^{*\prime}_{L,-\vec{k}_2} \delta^{(3)}(\vec{k}_1 - \vec{k}_2)$$

$$= \int \frac{d^3k}{(2\pi)^3} (i\vec{k}^3) v_{L,\vec{k}} v^{*\prime}_{L,\vec{k}}$$

$$\langle \partial_z^2 h_R(\eta,\vec{x})\,\partial_z\partial_t h_L(\eta,\vec{x})\rangle = \int \frac{d^3k}{(2\pi)^3} (i\vec{k}^3) v_{R,\vec{k}} v^{*\prime}_{R,\vec{k}}$$

$$\langle h_L''(\eta,\vec{x})\,\partial_z h_R'(\eta,\vec{x})\rangle = \langle 0| \left(\int \frac{d^3k_1}{(2\pi)^{3/2}} e^{i\vec{k}_1\cdot\vec{x}} \tilde{h}''_{L,\vec{k}_1}(\eta)\right)\left(\partial_z \int \frac{d^3k_2}{(2\pi)^{3/2}} e^{i\vec{k}_2\cdot\vec{x}} \tilde{h}'_{R,\vec{k}_2}(\eta)\right)|0\rangle$$

$$= \langle 0| \left(\int \frac{d^3k_1}{(2\pi)^{3/2}} e^{i\vec{k}_1\cdot\vec{x}} \tilde{h}''_{L,\vec{k}_1}(\eta)\right)\left(\int \frac{d^3k_2}{(2\pi)^{3/2}} (i\vec{k}_2) e^{i\vec{k}_2\cdot\vec{x}} \tilde{h}'_{R,\vec{k}_2}(\eta)\right)|0\rangle$$

$$= \int \frac{d^3k_1}{(2\pi)^{3/2}} \int \frac{d^3k_2}{(2\pi)^{3/2}} (i\vec{k}_2) e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{x}} \langle 0|\tilde{h}''_{L,\vec{k}_1}(\eta)\tilde{h}'_{R,\vec{k}_2}(\eta)|0\rangle$$

$$= \int \frac{d^3k_1}{(2\pi)^{3/2}} \int \frac{d^3k_2}{(2\pi)^{3/2}} (\vec{k}_2) e^{i(\vec{k}_1+\vec{k}_2)\cdot\vec{x}} v_{L,\vec{k}_1} v^{*\prime}_{L,-\vec{k}_2} \delta^{(3)}(\vec{k}_1 - \vec{k}_2)$$

$$= \int \frac{d^3k}{(2\pi)^3} (i\vec{k}) v''_{L,\vec{k}} v^{*\prime}_{L,\vec{k}}$$

$$\langle h_R''(\eta,\vec{x})\,\partial_z h_L'(\eta,\vec{x})\rangle = \int \frac{d^3k}{(2\pi)^3} (i\vec{k}) v''_{R,\vec{k}} v^{*\prime}_{R,\vec{k}}$$

$$\langle R_{\text{CS}}\rangle = -\frac{2}{a^4}\int \frac{d^3k}{(2\pi)^3}\left(k^3\left[v_{L,\vec{k}}v^{*\prime}_{L,\vec{k}} - v_{R,\vec{k}}v^{*\prime}_{R,\vec{k}}\right] + k\left[v''_{L,\vec{k}}v^{*\prime}_{L,\vec{k}} - v''_{R,\vec{k}}v^{*\prime}_{R,\vec{k}}\right]\right)$$

At this point, the expression for $\langle R_{\text{CS}}\rangle$ remains general and depends on the quantum states in which it is calculated. Since we are dealing with an FLRW anisotropic universe, the vacuum is evolving and the vacuum states are dynamical as well. The expression above was calculated, let's say, at some specific time η_0 (instantaneous vacuum state [12]).

The instantaneous vacuum is well-defined when the frequencies in the anisotropic universe are positive and have a physical cut-off. From this condition and from equation 4.1.20 we get that $L_{CS} < 1$ which accounts for avoiding the ghost-like modes. Now we know that the physical momenta are defined as:

$$k_{physical} = \frac{k}{\alpha(\eta)} < \mu$$

Where μ is the physical cut-off at the Ultra-Violet cut-off of the momenta of the graviton. This cut-off is obtained at some specific value of the parameter ξ, and this will define the upper limit of μ. In this case, only subhorizon modes are taken into consideration. From the L_{CS} condition mentioned above, we get:

$$L_{CS} < 1 \Rightarrow \mu\alpha\xi < 1 \Rightarrow \mu < \frac{1}{\alpha\xi}$$

Thus, the above integral will be calculated over this physical cut-off as:

$$\langle R_{\rm CS}\rangle = -\frac{2}{a^4}\int^{\alpha\mu}\frac{d^3k}{(2\pi)^3}\left(k^3\left[v_{L,\vec{k}}v^{*\prime}_{L,\vec{k}} - v_{R,\vec{k}}v^{*\prime}_{R,\vec{k}}\right] + k\left[v''_{L,\vec{k}}v^{*\prime}_{L,\vec{k}} - v''_{R,\vec{k}}v^{*\prime}_{R,\vec{k}}\right]\right) \tag{4.2.26}$$

Finding the expectation value of a quantum quantity in a curved spacetime is a difficult task. One should take into account the dynamical evolution of the background geometry and thus, the dynamical behavior of the vacuum state of the field. To calculate explicitly the R_{CS} condensate, we need to choose a vacuum state typically associated with a given cosmological era.

4.3 Running Vacuum Model (RVM) and phenomenology

The existence of an inflationary phase of the universe explains many features of the observable cosmos. Many models have been proposed to describe this phase, with some being ruled out by the latest CMB measurements. Among the viable models is the Starobinsky model, which incorporates higher-order curvature terms to modify Einstein's equation.

In this assignment, we focus on the so-called Running Vacuum Model (RVM), another cosmological model compatible with the observational data. Specifically, we analyze a string-inspired version of the RVM, referred to as the stRVM. This model starts with a Chern-Simons gravity coupled to axionic pseudoscalar degrees of freedom. The inflationary RVM-like phase arises naturally through the condensate of the gravitational Chern-Simons anomaly. Due to the primordial gravitational waves' condensation, the contribution of this CS condensate is non-trivial.

In the context of ΛCDM, the vacuum energy density is $\rho_\Lambda = \frac{\Lambda}{8\pi G}$. The standard RVM suggests that the cosmological constant Λ "runs" smoothly with cosmic time, thus we consider:

$$\rho_\Lambda(t) = \frac{\Lambda(t)}{8\pi G} \tag{4.3.1}$$

Using the Renormalization Group equation proposed for ρ^Λ_{RVM}, where the vacuum energy density can be expressed as a series expansion in even powers of the Hubble parameter H, we derive the final expression for ρ^Λ_{RVM}:

$$\rho^\Lambda_{RVM} = \frac{\Lambda(H)}{\kappa^2} = \frac{3}{\kappa^2}\left(c_0 + \nu H^2 + \tilde{\nu}\frac{H^4}{H_I^2}\right) \tag{4.3.2}$$

In the RVM-type used for this analysis, we consider $c_0 = 0$, $\nu < 0$, and $\tilde{\nu} > 0$, for the vacuum energy density. Below, we will present how the RVM may be implemented phenomenologically in the theory.

4.3.1 Universe evolution in RVM-phenomenology

As discussed previously, the early universe enters a stiff matter phase, dominated by the kinetic term of the b-scalar field, with an equation of state $w = 1$.

To start the analysis of the cosmological eras, one must begin with the derivation of the Friedmann and conservation law equations. First, we will consider the stiff matter era, where the vacuum density is constant, assuming that the GW condensate does not form until the very end of this era. As we will see later, the scale factor of the stiff matter era already has a transition phase from stiff to inflation, which is the period during which the condensate forms. After that, to study the inflationary phase, we will implement the running vacuum model since, according to the theory, the condensate induces RVM-like inflation and we will re-express the Friedmann and conservation law equations.

Considering an FLRW universe, where both stiff matter and vacuum energy are modeled as cosmological fluids, the stress-energy tensor reads:

$$T_{\mu\nu} = T_{\mu\nu}^{\text{stiff}} + T_{\mu\nu}^{vac} \tag{4.3.3}$$

The conservation law for the stiff matter, as analytically shown in the Appendix B, is:

$$\left\{\begin{aligned} &\dot{\rho} + 3\frac{\dot{\alpha}}{\alpha}(\rho + P) = 0 \\ &P = w\rho \overset{w=1}{\Rightarrow} P_s = \rho_s \end{aligned}\right\} \Rightarrow \dot{\rho}_s + 6H\rho_s = 0 \Rightarrow \frac{d\rho_s}{\rho_s} = -6\frac{d\alpha}{\alpha} \Rightarrow$$

$$\rho_s = \rho_{s,0}\left(\frac{\alpha}{\alpha_0}\right)^{-6} \qquad \text{Stiff matter density} \tag{4.3.4}$$

In this theory, we can model the stiff matter density to be the energy density of the axion-like pseudoscalar $b(t)$ field. Using the equations 3.2.21 & 3.2.22:

$$P_b = \rho_b = \frac{\dot{b}^2}{2} = \rho_s$$

Solving for $\dot{b}$ gives the time evolution of this b-field during the stiff matter era:

$$\dot{b} = \frac{\sqrt{\rho_{tot,0}\Omega_{vac,0}}}{\sinh\left(3\sqrt{\Omega_{vac,0}}H_0 t\right)} \Rightarrow b(t) = \frac{\sqrt{2\rho_{tot,0}}}{3H_0} \arctan\left[\cosh\left(3\sqrt{\Omega_{vac,0}}H_0 t\right)\right] \tag{4.3.5}$$

Where $\Omega_{vac,0} = \frac{\rho_{vac,0}}{\rho_{tot,0}}$ is the dimensionless vacuum density and H_0 is the Hubble parameter at the present era.

For the vacuum component, we use the equation of state $w = -1$, i.e., $P_{\text{vac}} = -\rho_{\text{vac}}$. The conservation law gives:

$$\left\{\begin{aligned} &\dot{\rho} + 3\frac{\dot{\alpha}}{\alpha}(\rho + P) = 0 \\ &P = w\rho \overset{w=-1}{\Rightarrow} P_{vac} = -\rho_{vac} \end{aligned}\right\} \Rightarrow \dot{\rho}_{vac} = 0$$

$$\rho_{vac} = \rho_{vac,0} = \text{const.} \qquad \text{Vacuum density} \tag{4.3.6}$$

From the 1st Friedmann equation:

$$\begin{Bmatrix} H^2 = \frac{\kappa^2}{3}(\rho_s + \rho_{vac}) \\ \\ H_0^2 = \frac{\kappa^2}{3}(\rho_{s,0} + \rho_{vac,0}) \end{Bmatrix} \Rightarrow \frac{H}{H_0} = \sqrt{\Omega_{s,0}\left(\frac{\alpha}{\alpha_0}\right)^{-6} + \Omega_{vac,0}}$$

where $\Omega_{s,0} = \frac{\rho_{s,0}}{\rho_{tot,0}}$ is the dimensionless stiff matter density.

With a change of variable $\chi = \frac{\alpha}{\alpha_0} \Rightarrow \frac{d\chi}{dt} = \frac{\dot{\alpha}}{\alpha_0}\frac{\alpha}{\alpha} = H\chi \Rightarrow H = \frac{1}{\chi}\frac{d\chi}{dt}$, the above equation can be integrated to obtain the evolution of the scale factor:

$$\frac{1}{\chi H_0}\frac{d\chi}{dt} = \sqrt{\frac{\Omega_{s,0}}{\chi^6} + \Omega_{vac,0}} \;\Rightarrow\; 3H_0\sqrt{\Omega_{vac,0}}t = \tan\left(\frac{\chi^3\sqrt{\Omega_{vac,0}}}{\sqrt{\Omega_{s,0} + \chi^6\Omega_{vac,0}}}\right) \;\Rightarrow$$

$$\alpha(t) = \tilde{\alpha}_0\left[\sinh\left(3\sqrt{\Omega_{vac,0}}H_0t\right)\right]^{1/3}, \qquad \tilde{\alpha}_0 = \alpha_0\left(\frac{\Omega_{s,0}}{\Omega_{vac,0}}\right)^{1/6} \tag{4.3.7}$$

The scale-factor's behavior is specifically interesting, as its limits for very early or very late times correspond to the stiff and inflationary eras, respectively:

- For $t \simeq 0$, we expand around zero:

$$\alpha(t) = \alpha_0(9H_0^2\Omega_{s,0})^{1/6}\; t^{1/3}$$

  which is the scale factor during a stiff matter era.

- For $t \gg 1$, we take the limit of the above scale factor to be:

$$\alpha(t) = \tilde{\alpha}_0\left[\sinh\left(3\sqrt{\Omega_{vac,0}}H_0t\right)\right]^{1/3} \xrightarrow[x\to+\infty]{e^{-x}=0} \tilde{\alpha}_0 e^{H_0\sqrt{\Omega_{vac,0}}t}$$

  which indicates an exponential de Sitter-like expansion.

Now entering the inflationary era, the vacuum energy density is non-constant; instead, it evolves according to the running vacuum model. This means that the energy-momentum tensor takes the form:

$$T_{\mu\nu} = (\rho+P)u_\mu u_\nu - g_{\mu\nu}P + g_{\mu\nu}\rho_{RVM} = \begin{pmatrix} \rho+\rho_{RVM} & 0 & 0 & 0 \\ 0 & -P+\rho_{RVM} & 0 & 0 \\ 0 & 0 & -P+\rho_{RVM} & 0 \\ 0 & 0 & 0 & -P+\rho_{RVM} \end{pmatrix} \tag{4.3.8}$$

The conservation law and Friedmann equations are easily calculated from the above tensor form:

- Conservation law:

$$\partial_0 T^0_0 + \Gamma^\mu{}_{\mu 0}T^0_0 - \Gamma^\lambda{}_{\mu 0}T^\mu_\lambda = 0 \Rightarrow \dot{\rho} + \dot{\rho}_{RVM} + 3H(\rho+P) = 0 \tag{4.3.9}$$

- 1^{st} Friedmann equation (time component):

$$H^2 = \left(\frac{\dot{\alpha}}{\alpha}\right)^2 = \frac{\kappa^2}{3}(\rho + \rho_{RVM}) \tag{4.3.10}$$

- 2^{nd} Friedmann equation (spatial component):

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\Lambda\right) g^{\mu\nu} = \left(\kappa^2 T_{\mu\nu}\right) g^{\mu\nu} \Rightarrow 6\left(\frac{\ddot{\alpha}}{\alpha} + \frac{\dot{\alpha}^2}{\alpha^2}\right) = \kappa^2(\rho - 3P) + 4\rho_{RVM} \Rightarrow$$

$$2\dot{H} + 4H^2 = \frac{\kappa^2}{3}\rho - \kappa^2 P + \frac{4\kappa^2}{3}\rho_{RVM} \overset{4.3.10}{\Rightarrow} 2\dot{H} + 4H^2 = \kappa^2(\rho_{RVM} - P) \Rightarrow$$

$$2\dot{H} + 4H^2 + \kappa^2(P_{RVM} + P) = 0 \tag{4.3.11}$$

Where $P_{RVM} = -\rho_{RVM}$. In the early universe era, it is true that $\frac{c_0}{H^2} \ll 1$, so this term is negligible [13]. Considering $\dot{H} = \frac{dH}{d\alpha}\frac{d\alpha}{dt} = \frac{dH}{d\alpha}\dot{\alpha} = \frac{dH}{d\alpha}H\alpha$ and that $\nu \ll 1$, the equation 4.3.11 becomes:

$$\int \frac{dH}{H\left(1 - \nu - \tilde{\nu}\frac{H^2}{H_I^2}\right)} = -\frac{3}{2}(1+w)\int \frac{d\alpha}{\alpha} \Rightarrow \frac{1}{1-\nu}\int \frac{dH}{H\left(1 - \frac{\tilde{\nu}}{1-\nu}\frac{H^2}{H_I^2}\right)} = -\frac{3}{2}(1+w)\frac{d\alpha}{\alpha} + C \Rightarrow$$

$$\frac{1}{2}\ln\left(\frac{H^2}{|\tilde{\nu}H^2 - H_I^2(1-\nu)|}\right) = -\frac{3}{2}(1+w)(1-\nu)\ln\alpha + C \;\Rightarrow\; \frac{H^2}{-\tilde{\nu}H^2 + H_I^2(1-\nu)} = D\alpha^{-3(1+w)(1-\nu)}$$

$$-\tilde{\nu} + \frac{H_I^2}{H^2}(1-\nu) = D\alpha^{3(1+w)(1-\nu)} \Rightarrow \frac{H_I^2}{H^2} = \frac{1}{1-\nu}\left(D\alpha^{3(1+w)(1-\nu)} + \tilde{\nu}\right) \Rightarrow H^2 = \frac{H_I^2(1-\nu)}{D\alpha^{3(1+w)(1-\nu)} + \tilde{\nu}}$$

The variable $\tilde{\nu}$ is absorbed in the integration constant $D = \ln C > 0$:

$$H(\alpha) = \sqrt{\frac{1-\nu}{\tilde{\nu}}}\frac{H_I}{\sqrt{C\alpha^{3(1+w)(1-\nu)} + 1}} \tag{4.3.12}$$

It is easy to observe that for very small values of $C\alpha^{-3(1+w)(\nu-1)} \ll 1$, the Hubble parameter becomes approximately constant, implying $\dot{H} = 0$, which is the signature of inflation (de Sitter-like expansion). Therefore, in the context of the RVM, the inflationary phase is triggered naturally, not by an additional inflaton field, but due to the RVM's fourth-order Hubble term. Furthermore, as a consistency check, for $C\alpha^{-3(1+w)(\nu-1)} \gg 1$, the Hubble rate scales as $H(\alpha) = \alpha^4$, corresponding to the expected transition to the radiation-dominated era that follows inflation [13].

The analysis in the previous section involved a toy-model implementation of the Running Vacuum Model (RVM), where the vacuum energy terms were manually inserted into the Friedmann equations. In contrast, the approach that follows does not impose the RVM structure by hand. Instead, the RVM-like contributions will emerge naturally from the quantum expectation value of the Chern-Simons condensate $\langle R_{CS}\rangle$, derived from gravitational wave fluctuations during the exploration of the stiff-matter era.

4.4 Cosmological eras in the string-inspired model

In this model again, the cosmological eras of our interest are the stiff-matter era ($w = 1$) and the RVM-like inflationary era. Based on the form of the scale factor $a(\eta)$, we may use the equation 4.1.20 to examine the behavior of the gravitational waves for each era, the CS-term contribution, and the R_{CS}-condensate that induces the Running Vacuum Model (RVM)-like inflation.

4.4.1 Stiff matter era

In this case, we consider a universe purely dominated by stiff matter, that is, the b-scalar field density. No vacuum energy has yet been introduced, so the Friedmann equations, based on Appendix B and equations 3.2.21 and 3.2.22, take the form:

- Conservation law:
$$\dot{\rho_b} + 3H(\rho_b + P_b) = 0 \Rightarrow \ddot{b} + 3H\dot{b} = 0 \tag{4.4.1}$$

- 1^{st} Friedmann:
$$H^2 = \frac{\kappa^2}{3}\rho_b = \frac{\kappa^2}{6}\dot{b}^2 \tag{4.4.2}$$

- 2^{nd} Friedmann:
$$\frac{\ddot{\alpha}}{\alpha} = -\frac{\kappa^2}{6}(\rho_b + 3P_b) = -\frac{\kappa^2}{3}\dot{b}^2 \tag{4.4.3}$$

Through the 1^{st} Friedmann, we may calculate the evolution of the b-field:

$$\dot{b} = \frac{\sqrt{6}}{\kappa}\frac{\dot{\alpha}}{\alpha} \Rightarrow db = \frac{\sqrt{6}}{\kappa}\frac{d\alpha}{\alpha} \Rightarrow b(\alpha) = \frac{\sqrt{6}}{\kappa}\log(\alpha) + C \tag{4.4.4}$$

From the 2^{nd} Friedmann(or the conservation law), we can confirm the dependence of the scale factor on time:

$$\frac{\ddot{\alpha}}{\alpha} = -\frac{\kappa^2}{3}\dot{b}^2 \Rightarrow 2\frac{\dot{\alpha}^2}{\alpha^2} + \frac{\ddot{\alpha}}{\alpha} = 0 \Rightarrow \frac{d\dot{\alpha}}{\dot{\alpha}} = -2\frac{d\alpha}{\alpha} \Rightarrow \dot{\alpha} = \alpha^{-2} \Rightarrow \alpha(t) = Ct^{1/3}$$

During the stiff matter era, the scaling factor evolves as below:

$$\alpha(\eta) = \sqrt{\frac{\eta}{\eta_0}} \Rightarrow dt = \alpha d\eta = \sqrt{\frac{\eta}{\eta_0}}d\eta \Rightarrow t = 2\frac{\eta^{3/2}}{3\eta_0} \Rightarrow \eta = \left(\frac{3}{2}\eta_0\right)^{2/3} t^{2/3}$$

$$\alpha(t) = \frac{1}{\sqrt{\eta_0}}\left(\frac{3}{2}\eta_0\right)^{1/3} t^{1/3} = C\ t^{1/3} \tag{4.4.5}$$

Again, using equation 4.1.20 with Mathematica, we obtain the full expression for the GW differential equation in the stiff matter era:

- For the Right-handed GWs:

$$\psi''_{R,k} + \left(k^2 + \frac{\eta^4 + 12\sqrt{6}A\kappa\eta_0(k\eta) - 72A^2\kappa^2\eta_0^2k^2}{4\eta^2(\eta^2 - 2\sqrt{6}A\kappa\eta_0 k)^2}\right)\psi_{R,k} = 0 \tag{4.4.6}$$

- For the Left-handed GWs:

$$\psi''_{L,k} + \left(k^2 - \frac{-\eta^4 + 12\sqrt{6}A\kappa\eta_0(k\eta) + 72A^2\kappa^2\eta_0^2k^2}{4\eta^2(\eta^2 + 2\sqrt{6}A\kappa\eta_0 k)^2}\right)\psi_{L,k} = 0 \tag{4.4.7}$$

The above two expressions do not have an analytic solution; however, since the period that interests us the most is the end of the stiff matter era and the beginning of inflation. Considering conformal time values

$\eta \gg 1$, so that terms involving η^4 dominate in both the numerator and the denominator, the expressions above are reduced into one single GW equation:

$$\psi''_{\lambda,k} + k^2\left(1 + \frac{1}{4k^2\eta^2}\right)\psi_{\lambda,k} = 0 \tag{4.4.8}$$

As we may see, this result implies that helicity-violation effects are negligible at the end of the stiff matter era. However, this is not true since the mode functions should be later rescaled, using the modified scale-factor of this specific era for each helicity mode $z_{\lambda,k}(\eta)$. This will separate once again the GW solution.

The solution for the above simplified expression includes Bessel functions of the 1^{st} and 2^{nd} kind and is of the form:

$$\psi(\eta) = \sqrt{\eta}\,(c_1 J_0(k\eta) + c_2 Y_0(k\eta)) \tag{4.4.9}$$

To fix our solution and use it in the R_{CS} calculation, two constraints are used: 1) the Wronskian normalization derived in the previous chapter, and 2) the Bunch-Davies vacuum approximation.

The Bunch-Davis vacuum states that for large values of the conformal time, the GWs become plane waves, and through the Wronskian normalization, their form becomes:

$$\left\{\begin{array}{l} \psi_k = Ae^{ik\eta} \\ \\ \psi_k\psi_k^{*\prime} - \psi_k^*\psi_k' = 1 \end{array}\right\} \Rightarrow\ 2A^2k = 1 \Rightarrow\ \psi_k(\eta) = \sqrt{\frac{1}{2k}}e^{ik\eta} \tag{4.4.10}$$

The asymptotic behavior of the Bessel functions, based on Mathematica, is:

- The first-order Bessel function is:

$$J_0(k\eta) \xrightarrow{\eta\to+\infty} \sqrt{\frac{2}{\pi k\eta}}\cos\left(k\eta - \frac{\pi}{4}\right)$$

- The second-order Bessel function becomes:

$$Y_0(k\eta) \xrightarrow{\eta\to+\infty} \frac{(-1)^{3/4}e^{-ik\eta} - (-1)^{1/4}e^{ik\eta}}{\sqrt{2\pi k\eta}} = \frac{e^{\frac{3i\pi}{4}}e^{-ik\eta} - e^{\frac{i\pi}{4}}e^{ik\eta}}{\sqrt{2\pi k\eta}} = \frac{\cos\left(\frac{3\pi}{4} - k\eta\right) - \cos\left(\frac{3\pi}{4} + k\eta\right)}{\sqrt{2\pi k\eta}}$$
$$= \frac{-2\sin\left(\frac{\pi}{4} - k\eta\right)}{\sqrt{2\pi k\eta}} = \sqrt{\frac{2}{\pi k\eta}}\sin\left(k\eta - \frac{\pi}{4}\right)$$

Using trigonometry, we may write the form of the k-mode function in the form:

$$u_k = \sqrt{\frac{2}{\pi k}}\left[C_1\cos\left(k\eta - \frac{\pi}{4}\right) + C_2\sin\left(k\eta - \frac{\pi}{4}\right)\right] \tag{4.4.11}$$

$$= \sqrt{\frac{2}{\pi k}}\frac{\sqrt{2}}{2}\left[C_1\cos(k\eta) + C_1\sin(k\eta) + C_2\cos(k\eta) - C_2\sin(k\eta)\right] \tag{4.4.12}$$

$$= \sqrt{\frac{1}{\pi k}}\left[(C_1 + C_2)\sin(k\eta) + (C_1 - C_2)\cos(k\eta)\right] \tag{4.4.13}$$

Matching the results with the Bunch-Davies condition for plane waves in 4.4.10, the constants C_1 and C_2

are calculated:

$$\left\{\begin{array}{l}\sqrt{\dfrac{1}{\pi k}}(C_1+C_2)=\sqrt{\dfrac{1}{2k}}\\ \sqrt{\dfrac{1}{\pi k}}(C_1-C_2)=\sqrt{\dfrac{1}{2k}}\end{array}\right\} \Rightarrow \left\{\begin{array}{l}C_1=\dfrac{1+i}{2}\sqrt{\dfrac{\pi}{2}}=\dfrac{\sqrt{\pi}}{2}e^{\frac{i\pi}{4}}\\ C_2=\dfrac{-1+i}{2}\sqrt{\dfrac{\pi}{2}}=\dfrac{\sqrt{\pi}}{2}e^{\frac{3i\pi}{4}}=C_1e^{\frac{i\pi}{2}}=iC_1\end{array}\right. \tag{4.4.14}$$

The mode function redefinition relation 4.2.21 is used to take into consideration the parity-violating effects of the Chern-Simons contribution. The mode function now becomes:

$$v_{L,\vec{k}}=\kappa\frac{e^{i\pi/4}\sqrt{\pi\eta}}{2\sqrt{\dfrac{\eta}{\eta_0}+\dfrac{2\sqrt{6}A\kappa k}{\eta}}}\left[J_0(k\eta)+iY_0(k\eta)\right],$$
$$v_{R,\vec{k}}=\kappa\frac{e^{i\pi/4}\sqrt{\pi\eta}}{2\sqrt{\dfrac{\eta}{\eta_0}-\dfrac{2\sqrt{6}A\kappa k}{\eta}}}\left[J_0(k\eta)+iY_0(k\eta)\right].$$

However, as proved in 4.4.11, the limit of the Bessel functions considering $\eta\to\infty$ can be directly represented in plane waves form. Thus, it is easier to use the mode functions below instead:

$$v_{L,\vec{k}}=\kappa\frac{e^{ik\eta}}{\sqrt{2k}\sqrt{\dfrac{\eta}{\eta_0}+\dfrac{2\sqrt{6}A\kappa k}{\eta}}},$$
$$v_{R,\vec{k}}=\kappa\frac{e^{ik\eta}}{\sqrt{2k}\sqrt{\dfrac{\eta}{\eta_0}-\dfrac{2\sqrt{6}A\kappa k}{\eta}}}$$

The integral will be approximated by using specific constraints:

- $\alpha'=M_s^{-2}$ for the regge slope,
- $\kappa^{-1}=M_{Pl}$ for the gravitational coupling,
- $A=\dfrac{\sqrt{2}}{192}\dfrac{\alpha'}{\kappa}\simeq 10^{-2}\left(\dfrac{M_{Pl}}{M_s^2}\right)$ for the coupling of the axion field versus the Chern-Simons term,
- $\dfrac{k}{\alpha}\simeq M_s=\mu$, for the UV cut-off,
- $L_{CS}=\dfrac{4A\kappa^2 b'k}{\alpha^2}\simeq 10^{-2}\left(\dfrac{M_{Pl}}{M_s^2}\right)M_{Pl}^{-2}\dfrac{\alpha\dot{b}}{\alpha}\dfrac{k}{\alpha}=10^{-2}\left(\dfrac{\dot{b}}{M_sM_{Pl}}\right)\ll 1$, the era-specific L_{CS} estimation.

The condensate is calculated based on the equation 4.2.26, using spherical coordinates to integrate in the whole phase space:

$$\begin{aligned}\langle R_{CS}\rangle &=-\frac{2}{a^4(\eta)}\int^{\alpha\mu}\frac{d^3k}{(2\pi)^3}\left(k^3\left[v_{L,\vec{k}}v^{*\prime}_{L,\vec{k}}-v_{R,\vec{k}}v^{*\prime}_{R,\vec{k}}\right]+k\left[v''_{L,\vec{k}}v^{*\prime}_{L,\vec{k}}-v''_{R,\vec{k}}v^{*\prime}_{R,\vec{k}}\right]\right)\\ &\approx -\frac{2}{a^4(\eta)}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\, d\theta\int^{\alpha(\eta)\mu}\frac{25}{192}\frac{L_{CS}^3(\eta,k)}{A^2\eta_0}dk=-\frac{25\sqrt{6}A\kappa^3\mu^4\eta_0^2}{16\pi^2\eta^6}\end{aligned}$$

According to the assumption that $L_{CS} \ll 1$, a Taylor expansion around 0 was performed in the calculation above, the integral is simplified, and its final estimation is:

$$\langle R_{CS} \rangle = -\frac{25\sqrt{6}A\kappa^3\mu^4}{\pi^2} H^4_{\text{stiff}}(\eta) \tag{4.4.15}$$

where

$$H_{\text{stiff}} = \frac{\sqrt{\eta_0}}{2\eta^{3/2}} \Rightarrow H^4_{\text{stiff}} = \frac{\eta_0^2}{16\eta^6}$$

The Hubble rate during the stiff matter era.

So, at the end of the stiff matter era, the cosmological evolution undergoes a transition phase driven by the emergence of the CS-term. As computed in this section, the expectation value of the CS term acquires a contribution proportional to H^4_{stiff}, where H_{stiff} is the Hubble parameter of the stiff matter era.

Physically, this indicates that the expectation value of the Chern-Simons term is proportional to the universe's expansion rate to fourth order. As we recall from equation 4.3.2, we can say that the condensate can be related to the energy density of the dark sector defined by $\rho^{\Lambda}_{\text{RVM}}$, and as noted in the toy model, this is the term that drives inflation.

4.4.2 Inflation

In the dynamical systems analysis of the paper [3], it is mentioned that a linear effective potential of the form $|V_{eff}| = A\langle R_{CS}\rangle b$ accounts for the dynamics of the axion field. By giving a small perturbation in our universe of the magnitude $y_i = \dfrac{\kappa^2|V_{eff}|}{3H_i^2}$, where H_i is the Hubble rate at the beginning of inflation, inflation is triggered.

Now, in order to determine the magnitude of the L_{CS}, for the inflation period, we will use some critical points:

- $H_I \sim 10^{-5} M_{Pl}$ the Hubble rate at the end of inflation, based on Planck Collaboration Data,
- $H_i \sim 10^{7/2} H_I \approx 10^{-3/2} M_{Pl}$, the ratio corresponds to 50-60 e-foldings from the beginning till the end of inflation,
- $\dot{b}_i = \dfrac{\sqrt{6}}{M_{Pl}^{-1}} H_i \sim 10^{-2} M_{Pl}^2$, is the evolution of the b-field with cosmic time at the end of the stiff era (that at the same time indicates the beginning of inflation),
- $\dot{b}_I \simeq 10^{-1} H_I M_{Pl} \simeq 10^{-6} \dfrac{M_{Pl}}{M_s}$ the evolution of b-field at the end of the inflation (value obtained from the dynamical systems graph),
- $L^i_{CS} = 4A\kappa^2 \dot{b}\frac{k}{\alpha} \sim 4 \cdot 10^{-2} \left(\dfrac{M_{Pl}}{M_s^2}\right) M_{Pl}^{-2} \dot{b}_i M_s \sim 10^{-4} \dfrac{M_{Pl}}{M_s}$, the estimation of L^i_{CS} at the beginning of inflation and
- $L^I_{CS} \sim 10^{-2} \left(\dfrac{M_{Pl}}{M_s^2}\right) M_{Pl}^{-2} \dot{b}_I M_s \sim 10^{-6} \dfrac{M_{Pl}}{M_s}$, the value of L^I_{CS} at the end of the inflationary period.

The string scale cut-off stems from the assumption that $L_{CS,max} \sim 10^{-2}$. The latter expression is consistent with the assumption made earlier, $L_{CS} \ll 1$, to avoid negative frequencies in the GW equations. so in this case:

$$M_s \sim 10^{-1} M_{Pl} \tag{4.4.16}$$

During the inflationary era, the scale factor evolves as described below:

$$\alpha(\eta) = -\frac{1}{H_I \eta} \quad \Rightarrow dt = \alpha d\eta = -\frac{1}{H_I \eta} d\eta \Rightarrow t = -\frac{\ln|\eta|}{H} \overset{\eta \leq 0}{\Rightarrow} \eta = -e^{-Ht} \tag{4.4.17}$$

$$\alpha(t) = \frac{1}{H_I} e^{H_I t} \tag{4.4.18}$$

From 4.1.20 the equations of motion are:

- For the Right-handed GWs:

$$\frac{d^2\psi_R}{d\eta^2} + \left(k^2 - \frac{8 + 12\, CH\ (k\eta) + 3\ C^2 H^2\ (k\eta)^2}{4\eta^2 (1 + CH\ (k\eta))^2}\right)\psi_R = 0 \tag{4.4.19}$$

- For the Left-handed GWs:

$$\frac{d^2\psi_L}{d\eta^2} + \left(k^2 - \frac{8 - 12\, CH\ (k\eta) + 3\ C^2 H^2\ (k\eta)^2}{4\eta^2 (-1 + CH\ (k\eta))^2}\right)\psi_L = 0 \tag{4.4.20}$$

where $C = 4A\kappa^2 \dot{b}$, where $\dot{b}$ is slowly varying.

Since the above solution is rather complex, we approximate the integral by considering plane wave solutions in the limit $\eta \to 0^-$.

Using a technique similar to the one applied in the previous analysis, we adopt the plane wave approximation to evaluate the solutions of the aforementioned differential equations. The mode functions, now, are:

$$v_{L,\vec{k}} = \kappa\, \frac{e^{ik\eta}}{\sqrt{2k}\, z_L(\eta,k)} \qquad \text{with} \qquad z_L(\eta,k) = -\frac{\left(\sqrt{1 - CH_I k\eta}\right)}{H_I \eta}, \tag{4.4.21}$$

$$v_{R,\vec{k}} = \kappa\, \frac{e^{ik\eta}}{\sqrt{2k}\, z_R(\eta,k)} \qquad \text{with} \qquad z_R(\eta,k) = -\frac{\left(\sqrt{1 + CH_I k\eta}\right)}{H_I \eta} \tag{4.4.22}$$

Since, the value of L_{CS} has a range in between $10^{-8} - 10^{-4}\left(\frac{M_{Pl}}{M_s}\right)$ during inflation, we may use the same approximation and perform a Taylor expansion again for $L_{CS}(\eta, k)$, around the value 0. Thus, the integral writes:

$$\begin{aligned}\langle R_{CS}\rangle &= -\frac{2}{a^4(\eta)} \int^{\alpha\mu} \frac{d^3k}{(2\pi)^3} \left(k^3 \left[v_{L,\vec{k}} v^{*\prime}_{L,\vec{k}} - v_{R,\vec{k}} v^{*\prime}_{R,\vec{k}}\right] + k \left[v''_{L,\vec{k}} v^{*\prime}_{L,\vec{k}} - v''_{R,\vec{k}} v^{*\prime}_{R,\vec{k}}\right]\right) \\ &\approx -\frac{2}{a^4(\eta)} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\ d\theta \int^{\alpha(\eta)\mu} \frac{\kappa^2 L^3_{CS}(\eta,k)}{C\eta^3} dk = -\frac{C H_I^3 \kappa^2 \mu^4}{4\pi^2}\end{aligned}$$

By replacing the constant C, we finally get:

$$\langle R_{CS}\rangle = -\frac{A\dot{b}\kappa^4\mu^4}{\pi^2} H_I^3 \tag{4.4.23}$$

The R_{CS} during inflation is constant, and its negative sign is related to a positive sign for the effective cosmological constant $|V_{eff}| = \Lambda_{eff} = A\langle R_{CS}\rangle b \simeq const.$ We know that $b(\eta) < 0$, thus it is true that

$R_{CS} < 0$ too. The GW condensate has already formed, and the condensate's value only depends on $\dot{b}$, which has an almost negligible variation during this era. So, in comparison to the Running Vacuum Model, the term that dominates this era is the constant term of the ρ^{Λ}_{RVM} vacuum energy density ($\propto c_0$).

4.4.3 Sources

In the presence of primordial gravitational waves (GWs), it is important to account for the number of microscopic sources that are responsible for their generation during the stiff matter and inflationary eras. To take this factor into consideration, we introduce two phenomenological parameters, N_S and N_I, which represent the number of independent GW sources active during the stiff and inflationary phases, correspondingly.

These quantities may be interpreted as weights multiplying the expectation value of the gravitational Chern-Simons condensate.

For the stiff matter era:

$$\langle R_{CS,tot}\rangle_{\text{stiff}} = -N_S \frac{25\sqrt{6}A\kappa^3\mu^4}{\pi^2} H^4_{\text{stiff}}(\eta) \tag{4.4.24}$$

And for the inflationary era:

$$\langle R_{CS,tot}\rangle_{\text{infl}} = -N_I \frac{A\dot{b}_I\kappa^4\mu^4}{\pi^2} H^3_I \tag{4.4.25}$$

As shown in the calculation of $\langle R_{CS}\rangle$ during inflation, the condensate remains constant throughout the inflationary era. Therefore, its value at the beginning of inflation must be equal to its value at the end of inflation:

$$\begin{aligned}
&\langle R_{CS,tot}\rangle_{\text{stiff}} = \langle R_{CS,tot}\rangle_{\text{infl}} \Rightarrow \\
&\frac{N_I}{N_S} = \frac{25\sqrt{6}A\kappa^3\mu^4 H^4_{\text{stiff}}(\eta)}{A\dot{b}\kappa^4\mu^4 H^3_I} = \frac{25\sqrt{6}H^4_{\text{stiff}}(\eta)}{\dot{b}_I\kappa H^3_I} \Rightarrow \\
&\frac{N_I}{N_S} = \frac{62}{10^{-1}M_{Pl}M^{-1}_{Pl}} \cdot \left(\frac{H_i}{H_I}\right)^4 \sim 6\cdot 10^{16}
\end{aligned}$$

Physically, this relation tells us that the number of sources increases significantly, by several orders of magnitude, in the transition phase from the stiff era to inflation. This amplification indicates that inflation is triggered by the growing population of chiral GW modes and the effects of the Chern-Simons anomaly.

5 Conclusion

The study of torsion within both geometrical General Relativity (GR) and string-inspired models offers a deep understanding of the potential extensions of classical gravity and cosmology.

In the geometrical framework of GR, based on the Einstein-Cartan-Sciama-Kibble (ECSK) theory, spacetime is equipped with both curvature and torsion. In this setup, torsion, which is already a fundamental mathematical property of the spacetime description, becomes a dynamical variable directly related to the spin density of matter (fermions) and introduces new degrees of freedom. Focusing on QED, it has been shown that torsion interactions with fermionic currents are non-trivial and quite important at high spin densities[5].

On the other hand, the string-inspired model provides a more dynamic and deeper role for torsion. Here, torsion arises from the Kalb-Ramond (KR) field strength, a component of the massless bosonic sector of string theory in the low-energy limit approximation. It is important to note that this component can be linked more deeply with the geometrical decomposed torsion component from the Einstein-Cartan theory.

Throughout the presentation of this model, a key result of this thesis is the realization that the b(x)-field, when coupled to gravity, plays a dual role in physics: (1) as a source of anisotropic gravitational wave propagation causing the phenomenon of cosmological birefringence, and (2) as a dynamical inflaton-like field, giving rise to an RVM-like inflationary phase.

Comparing the RVM to our final result, therefore the equations 4.3.2, 4.4.15 and 4.4.23, the ν

The cosmological evolution presented, includes the stiff matter pre-inflationary phase, whose end denotes the gravitational wave condensation and the triggering of this RVM-like inflationary regime. This model not only provides a mechanism for cosmic inflation but also introduces a potential dark matter candidate, offering a unified framework to explain both the early universe's rapid expansion and the origin of some dark matter components.

One very interesting aspect that can be further explored, is the decay of those axion particles into radiation particles and the reheating process following the exit from the RVM-like inflation. In the RVM framework, it has been shown phenomenologically that the vacuum energy decays slowly and there is a smooth transition from the inflationary era to the radiation era [14]. Thus, in our case of the stringy RVM, it is promising to look for such a transition regime.

In the string-inspired framework, considering world-sheet instantons, a periodic potential is generated and gives light masses to the axion fields of the theory. The b axion-like degree is dominant at early times and induces the Chern-Simons condensate, which drives inflation. However, during the later stages, the world-sheet instantons take over and oscillate slowly around the minimum of the periodic potential, while the RVM vacuum slowly decays. By the decay of the RVM at the end of inflation, photons are produced, and the slow matter-dominated reheating will be succeeded by the also slow radiation-dominated era reheating. The axions will finally couple to the gauge anomaly ($\propto tr[F \wedge F]$), transforming their energy into radiation modes for e.g., photon pairs. So one can notice that there is a prolonged adiabatic period of reheating[15] and that the smooth transition from inflation to radiation-dominated era described in the RVM phenomenology is valid.

Although certain aspects still require further investigation, such as the transition from the stiff matter era to inflation and the detailed dynamics of reheating in the RVM framework described above, the string-inspired model developed in this thesis provides a consistent and promising framework in the context of cosmology.

6 Appendices

6.1 Appendix A - The tetrad formalism

Let us consider a (3+1)-dimensional manifold $\mathcal{M}$ and $p \in \mathcal{M}$. The tangent space at this point is denoted by $T_P\mathcal{M}$, and the natural basis vectors on this space are $\hat{e}_\mu = \partial_\mu$, where $\mu = 0, 1, 2, 3$. Correspondingly, the basis of the cotangent space $T_P^*\mathcal{M}$ is given by $\hat{\theta}^\mu = dx^\mu$.

To simplify later calculations, we can choose a different basis within the tangent space at p: an orthonormal basis denoted by e_a, where $a = 0, 1, 2, 3$.This procedure is effectively a change of basis in $T_P\mathcal{M}$, and the new basis vectors e_a are known as vielbeins (or tetrads in four dimensions).

The metric tensor can be expressed in the coordinate basis as:

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu = g(\partial_\mu, \partial_\nu) dx^\mu \otimes dx^\nu \tag{6.1.1}$$

We now express the components of the metric in the new orthonormal basis in terms of the original coordinate basis (natural):

$$g(e_a, e_b) = g({e_a}^\mu \partial_\mu, {e_b}^\nu \partial_\nu) = {e_a}^\mu {e_b}^\nu g(\partial_\mu, \partial_\nu) = {e_a}^\mu {e_b}^\nu g_{\mu\nu} = g_{ab} = \eta_{ab} \tag{6.1.2}$$

where $\eta_{ab} = diag(-1, 1, 1, 1)$ is the natural basis.

This transformation is depicted in the figure below, where we see the change of basis at the point p. Each vielbein e_a can be written in terms of the coordinate basis as:

$$e^a = {e_a}^\mu \partial_\mu \tag{6.1.3}$$

Here, the Latin index a labels the vielbein (internal Lorentz index), while the Greek index μ refers to the spacetime coordinate index of the manifold.

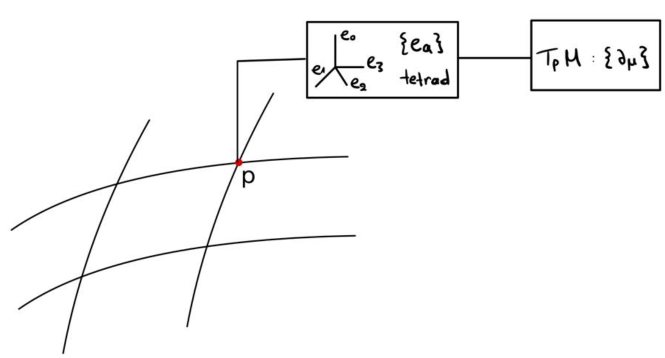

Figure 2: Change of basis inside the Tangent space of the point p.

6.2 Appendix B - FLRW universe with cosmological fluid

The Friedmann-Lemaître-Robertson-Walker (FLRW) metric describes a homogeneous and isotropic universe:

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \quad d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2, \tag{6.2.1}$$

where $a(t)$ is the scale factor, and $k \in \{-1, 0, +1\}$ encodes spatial curvature. For simplicity, we assume $k = 0$ (flat universe):

$$ds^2 = dt^2 - a^2(t)\left(dr^2 + r^2 d\Omega^2\right). \tag{6.2.2}$$

where $d\Omega = r^2(d\theta^2 + sin^2\theta\ d\phi^2)$ The matrix form of the metric is:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\alpha^2(t) & 1 & 0 \\ 0 & 1 & -\alpha^2(t)r^2 & 0 \\ 0 & 0 & 0 & -\alpha^2(t)r^2 sin^2\theta \end{pmatrix} \tag{6.2.3}$$

The inverse metric has elements of the form $g^{ii} = g_{ii}^{-1}$, with $i = 0, 1, 2, 3$.

The non-vanishing components of the Ricci tensor for the FLRW universe are:

$$R_{00} = -3\frac{\ddot{\alpha}}{\alpha} \qquad R_{11} = \ddot{\alpha}\alpha + 2\dot{\alpha}$$
$$R_{22} = r^2(\ddot{\alpha}\alpha + 2\dot{\alpha}) \qquad R_{44} = r^2 sin^2\theta(\ddot{\alpha}\alpha + 2\dot{\alpha})$$

And the Ricci scalar is:

$$R = -6\left(\frac{\ddot{\alpha}}{\alpha} + \frac{\dot{\alpha}^2}{\alpha^2}\right)$$

The universe is filled with a cosmological fluid with energy density ρ, pressure P, and 4-velocity $u^\mu = (1, 0, 0, 0)$ in co-moving coordinates. The stress-energy tensor $T_{\mu\nu}$ describes how energy and momentum are distributed in spacetime. For a perfect fluid, it takes the form:

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu - g_{\mu\nu}P \tag{6.2.4}$$

Einstein's field equations with a cosmological constant Λ build the relationship between spacetime curvature and matter content:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \quad \kappa^2 = 8\pi G. \tag{6.2.5}$$

Where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ the Einstein tensor encodes curvature, Λ the cosmological constant represents the existence of dark energy, and $T_{\mu\nu}$ the stress-energy tensor represents the matter content of the universe.

The Friedman equations are derived as below ($c = 1$):

- 1st Friedman equation:

$$R_{00} - \frac{g_{00}}{2}R - \Lambda g_{00} = \kappa^2 T_{00} \Rightarrow -3\frac{\ddot{\alpha}}{\alpha} + 3\left(\frac{\ddot{\alpha}}{\alpha} + \frac{\dot{\alpha}^2}{\alpha^2}\right) - \Lambda = \kappa^2 T_{00} \Rightarrow \boxed{\frac{\dot{\alpha}^2}{\alpha^2} = \frac{\kappa^2}{3}\rho + \frac{\Lambda}{3}} \tag{6.2.6}$$

- 2nd Friedman equation:

$$\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\Lambda\right) g^{\mu\nu} = \left(\kappa^2 T_{\mu\nu}\right) g^{\mu\nu} \Rightarrow R - 2R - 4\Lambda = \kappa^2 T \Rightarrow$$

$$6\left(\frac{\ddot{\alpha}}{\alpha} + \frac{\dot{\alpha}^2}{\alpha^2}\right) - 4\Lambda = \kappa^2(\rho - 3P) \overset{6.2.6}{\Rightarrow} \boxed{\frac{\ddot{\alpha}}{\alpha} = \frac{\Lambda}{3} - \frac{\kappa^2}{6}(\rho + 3P)} \tag{6.2.7}$$

- Conservation Law:

$$\nabla_\mu T^{\mu 0} = \partial_\mu T^{\mu 0} + \Gamma^{\mu}_{\mu\alpha} T^{\alpha 0} - \Gamma^{\alpha}_{\mu 0} T^{\mu\alpha} = 0 \Rightarrow$$

$$\partial_0 T^{00} + \Gamma^{\mu}_{\mu 0} T^{00} + \Gamma^{i}_{j0} T^{ji} = 0 \Rightarrow$$

$$\boxed{\dot{\rho} + 3H(\rho + P) = 0} \tag{6.2.8}$$

where $H = \frac{\dot{\alpha}}{\alpha}$ the Hubble parameter.

Using the equation of state $P = w\rho$ together with the energy conservation law derived previously, we can determine how the energy density evolves with the expansion of the universe. The relationship between ρ and the scale factor $\alpha(t)$ is obtained as follows:

$$\frac{\dot{\rho}}{\rho} = -\frac{\dot{\alpha}}{\alpha}3(w+1) \Rightarrow \ln(\rho) = \ln(\alpha^{-3(w+1)}) \Rightarrow \rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)} \tag{6.2.9}$$

6.3 Appendix C - Introduction to linearized gravity

6.3.1 weak GW perturbations

In this section, we present an overview of the fundamental aspects of the linearized theory of General Relativity, its invariance under diffeomorphisms, and an introduction to transverse traceless gravitational wave solutions, which are going to be used for the analysis later.

To begin with, considering a weak gravitational field, the metric can be decomposed into the flat Minkowski background and a small perturbation, $|h_{\mu\nu}| \ll 1$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{6.3.1}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{6.3.2}$$

Multiple coordinate systems can be defined on the manifold for which 6.3.1 holds; therefore, the definition is not unique. Specifically, while there are many coordinate systems for which the metric can be decomposed into the Minkowski component and a small perturbation, the form of the perturbation varies and is not uniquely determined. To prove that, we need to choose a reference frame for which 6.3.1 stands for a sufficiently large region. We will examine the behavior of the perturbation under a local transformation:

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x) \tag{6.3.3}$$

for which the metric transforms in a covariant way:

$$g^{\mu\nu} \to g'^{\mu\nu} = \frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} g^{\rho\sigma} \tag{6.3.4}$$

Based on 6.3.2, 6.3.4, and 6.3.3, the transformation properties of the metric will provide a relation between the perturbations before and after the transformation. Since the flat Minkowski spacetime remains invariant under such diffeomorphisms, the covariant transformation law for the perturbation is given by:

$$g'^{\mu\nu} = \frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} g^{\rho\sigma} \Rightarrow (\eta^{\mu\nu} - h'^{\mu\nu}) = \left[\frac{\partial x^{\rho}}{\partial x^{\mu}} - \frac{\partial \xi^{\rho}(x)}{\partial x^{\mu}}\right]\left[\frac{\partial x^{\sigma}}{\partial x^{\nu}} - \frac{\partial \xi^{\sigma}(x)}{\partial x^{\nu}}\right](\eta^{\rho\sigma} - h^{\rho\sigma}) \Rightarrow$$
$$h'^{\mu\nu} = h^{\mu\nu} - (\eta^{\mu\sigma}\partial_{\nu}\xi^{\sigma} + \eta^{\rho\nu}\partial_{\nu}\xi^{\rho}) \Rightarrow$$

$$h'^{\rho\sigma} = h^{\mu\nu} - (\partial^{\nu}\xi^{\mu} + \partial^{\mu}\xi^{\nu}) \tag{6.3.5}$$

Terms involving second-order derivatives or combinations of $|\partial_{\mu}\xi^{\rho}| \ll 1$ and $|h_{\mu\nu}| \ll 1$ are neglected as they are negligibly small. Since the derivative $|\partial\xi| \ll 1$ is at most of the same order as $|h_{\mu\nu}|$, it follows that the perturbation remains small under any coordinate transformation. This confirms the local (gauge) invariance of the linearized theory.

Based on the metric perturbation expansion 6.3.1, we can recalculate the Christoffel symbols, the Riemann and Ricci tensors, and the Ricci scalar to express the Einstein equation based on this perturbative expansion:

- $\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}(\nabla_{\mu}g_{\nu\lambda} + \nabla_{\nu}g_{\lambda\mu} + \nabla_{\lambda}g_{\mu\nu}) \xrightarrow[|h\partial h| \ll 1]{\partial_{\rho}\eta_{\mu\nu}=0} \Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\lambda}(\nabla_{\mu}h_{\nu\lambda} + \nabla_{\nu}h_{\lambda\mu} + \nabla_{\lambda}h_{\mu\nu})$
- $R_{\mu\nu\rho\sigma} = \frac{1}{2}(\nabla_{\mu}\nabla_{\sigma}h_{\nu\rho} + \nabla_{\nu}\nabla_{\mu}h_{\rho\sigma} + \nabla_{\nu}\nabla_{\rho}h_{\mu\sigma} - \nabla_{\mu}\nabla_{\nu}h_{\rho\sigma} - \nabla_{\mu}\nabla_{\rho}h_{\nu\sigma} - \nabla_{\nu}\nabla_{\sigma}h_{\mu\rho})$
- $R_{\mu\nu} = \frac{1}{2}(-\Box h_{\mu\nu} + \nabla_{\alpha}\nabla_{\mu}h_{\nu}{}^{\alpha} + \nabla_{\alpha}\nabla_{\nu}h_{\mu}{}^{\alpha} - \nabla_{\mu}\nabla_{\nu}h^{\alpha}{}_{\alpha})$
- $R = \nabla_{\mu}\nabla_{\nu}h^{\mu\nu} - \Box h$

Where $\Box = \nabla_{\mu}\nabla^{\mu} = \nabla^{\mu}\nabla_{\mu}$, the d' Alembertian, and ∇_{μ}, the covariant derivative of the metric $g_{\mu\nu}$. As previously stated, we get rid of all terms that involve the perturbation $h_{\mu\nu}$ or its derivative beyond first order, as well as any mixed terms combining the perturbation and its derivatives, such as $|h\partial h|$, considering them negligibly small.

Below, we perform a calculation of the first part of the Einstein equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}\left(-\Box h_{\mu\nu} + \nabla^{\alpha}\nabla_{\mu}h_{\nu\alpha} + \nabla^{\alpha}\nabla_{\nu}h_{\mu\alpha} - \nabla_{\mu}\nabla_{\nu}h - \eta_{\mu\nu}\nabla^{\rho}\nabla^{\sigma}h_{\rho\sigma} + \eta_{\mu\nu}\Box h\right)$$

Changing the variables:

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \tag{6.3.6}$$

We get:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{2}\Big(&-\Box\bar{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box\bar{h} + \nabla^{\alpha}\nabla_{\mu}\bar{h}_{\nu\alpha} - \frac{1}{2}\nabla^{\alpha}\nabla_{\mu}\eta_{\alpha\nu}\bar{h} + \nabla^{\alpha}\nabla_{\nu}\bar{h}_{\mu\alpha} - \frac{1}{2}\nabla^{\alpha}\nabla_{\nu}\eta_{\alpha\mu}\bar{h} \\ &-\eta_{\mu\nu}\nabla^{\rho}\nabla^{\sigma}\bar{h}_{\rho\sigma} + \frac{1}{2}\eta_{\mu\nu}\nabla^{\rho}\nabla^{\sigma}\eta_{\rho\sigma}\bar{h} + \nabla_{\mu}\nabla_{\nu}\bar{h} - \eta_{\mu\nu}\nabla_{\rho}\nabla^{\rho}\bar{h}\Big) \\ = \frac{1}{2}&\left(-\Box\bar{h}_{\mu\nu} + \nabla^{\alpha}\nabla_{\mu}\bar{h}_{\nu\alpha} + \nabla^{\alpha}\nabla_{\nu}\bar{h}_{\mu\alpha} - \eta_{\mu\nu}\nabla^{\rho}\nabla^{\sigma}\bar{h}_{\rho\sigma}\right) \end{aligned}$$

Many terms cancel out and eventually Einstein's equation is expressed:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu} \Rightarrow$$

$$-\Box\bar{h}_{\mu\nu} + \nabla^{\alpha}\nabla_{\mu}\bar{h}_{\nu\alpha} + \nabla^{\alpha}\nabla_{\nu}\bar{h}_{\mu\alpha} - \eta_{\mu\nu}\nabla^{\rho}\nabla^{\sigma}\bar{h}_{\rho\sigma} = \frac{16\pi G}{c^4}T_{\mu\nu} \tag{6.3.7}$$

By analogy to the electromagnetic field gauge invariance, we can fix our gauge transformation into the most convenient expression. Expressed in equation 6.3.5, we can make a choice of the vector field ξ_{μ} so that in the new coordinates, $\nabla^{\beta}\bar{h}'_{\alpha\beta} = 0$. If we achieve this, then the last three terms of the left-hand side of the equation 6.3.7.

Let us consider a coordinate transformation 6.3.3. The transformation for the modified perturbation $\bar{h}$ is:

$$\bar{h} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \quad \rightarrow \quad \bar{h}' = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h' \tag{6.3.8}$$

So in this case:

$$\begin{aligned}\bar{h}' = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h' &= h_{\mu\nu} - (\partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu}) - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\left(h_{\alpha\beta} - (\partial_{\alpha}\xi_{\beta} - \partial_{\beta}\xi_{\alpha})\right)\\ &= h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} - \frac{1}{2}\eta_{\mu\nu}h + \eta_{\mu\nu}\partial^{\alpha}\xi_{\alpha} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} + \eta_{\mu\nu}\partial^{\alpha}\xi_{\alpha}\end{aligned}$$

$$\boxed{\bar{h}' = \bar{h}_{\mu\nu} - (\partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu}) + \eta_{\mu\nu}\partial^{\alpha}\xi_{\alpha}} \tag{6.3.9}$$

Now that the perturbation $h_{\mu\nu}$ has been defined in the new coordinate system x'^{μ}, we impose the condition $\nabla^{\beta}\bar{h}'_{\alpha\beta} = 0$ to determine the appropriate ξ^{μ} such that the three last terms in the wave equation 6.3.7 vanish.

$$\nabla^{\beta}\bar{h}'_{\alpha\beta} = 0 \;\Rightarrow\; \nabla^{\beta}[\bar{h}_{\alpha\beta} - (\partial_{\alpha}\xi_{\beta} + \partial_{\beta}\xi_{\alpha}) + \eta_{\alpha\beta}\partial^{\mu}\xi_{\mu}] = 0 \;\Rightarrow\; \partial^{\beta}\bar{h}_{\alpha\beta} - \partial^{\beta}\partial_{\alpha}\xi_{\beta} + \partial^{\beta}\partial_{\beta}\xi_{\alpha} + \partial_{\alpha}\partial^{\mu}\xi_{\mu} = 0$$

$$\boxed{\nabla^{\beta}\bar{h}_{\alpha\beta} = \partial^{\beta}\partial_{\beta}\xi_{\alpha} = \Box\xi} \tag{6.3.10}$$

By applying the condition 6.3.10, we ensure the existence of gravitational waves in vacuum since the vanishing divergence of the stress-energy tensor leads to a coordinate system where a wave equation is clear:

$$\nabla_{\mu}T^{\mu\nu} = 0 \;\Rightarrow\; \Box\bar{h}'_{\mu\nu} = 0 \tag{6.3.11}$$

This gravitational wave equation implies that GWs travel at the speed of light in a vacuum.

In our case, we analyze metric perturbations of the spatially flat part of the FLRW metric, considering weak gravitational waves in the presence of the Chern-Simons term. This allows us to follow the methodology above, and work in the transverse-traceless basis to simplify the wave solutions and investigate the final contribution from the CS-term.

6.3.2 Transverse-Traceless representation

One straightforward solution to the equation 6.3.11 is the plane wave solution. Considering that the wave equation is linear, any generic wave solution can be written as a linear combination of plane waves. Let us assume a plane wave solution of the form:

$$\bar{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\sigma x^\sigma} \tag{6.3.12}$$

Where k_σ are the components of the wave covector $\vec{k} = (\frac{\omega}{c}, k_x, k_y, k_z)$. By replacing the solution to the initial differential equation, we get:

$$\Box \bar{h}_{\mu\nu} = 0 \;\Rightarrow\; g^{\alpha\beta}\; \partial_\alpha \partial_\beta (A_{\mu\nu} e^{ik_\sigma x^\sigma}) = 0 \Rightarrow\; A_{\mu\nu} g^{\alpha\beta}\; \partial_\alpha (e^{ik_\sigma x^\sigma} (ik_\sigma) \delta^\sigma{}_\beta) = 0 \Rightarrow$$
$$A_{\mu\nu} g^{\alpha\beta}\; \partial_\alpha (e^{ik_\sigma x^\sigma} (ik_\beta)) = 0 \Rightarrow\; A_{\mu\nu} g^{\alpha\beta}\; e^{ik_\sigma x^\sigma} (ik_\beta)(ik_\sigma \delta^\sigma{}_\alpha) = 0 \Rightarrow\; A_{\mu\nu} (ik^\alpha)(ik_\alpha) = 0 \Rightarrow$$

$$k^\alpha k_\alpha = 0 \;\; \text{(null/lightlike)} \tag{6.3.13}$$

Now, the steps we will follow to reduce the $A_{\mu\nu}$ components are below:

- Since the metric perturbation tensor is symmetric, we should note that $A_{\mu\nu} = A_{\nu\mu}$. Because of this condition, the independent components of $A_{\mu\nu}$ reduce from 16 to 10.

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix} \Rightarrow A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{12} & A_{22} & A_{23} & A_{24} \\ A_{13} & A_{23} & A_{33} & A_{34} \\ A_{14} & A_{24} & A_{34} & A_{44} \end{bmatrix}$$

- Four more constraints can be obtained through the gauge invariance condition. Using a gauge fixing so that $\partial^\beta \bar{h}_{\alpha\beta} = \Box\xi = 0 \Rightarrow k^\beta A_{\alpha\beta} = 0$, the four constraints reduce the independent components from 10 to 6.

- Since the Lorentz gauge is a class of coordinate systems, I can choose a specific direction. Let us consider an observer with a 4-velocity $U = (U_t, U_x, U_y, U_z)$. We may choose a specific direction of propagation of the GWs with respect to the observer so that we get 3 more constraints. We want the amplitude to be transverse with respect to the observer, and if we choose $U = (U_t, 0, 0, 0)$, the equation that satisfies those two conditions is:

$$U^\mu A_{\mu\nu} = 0 \Rightarrow A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{22} & A_{23} & A_{24} \\ 0 & A_{23} & A_{33} & A_{34} \\ 0 & A_{24} & A_{34} & A_{44} \end{bmatrix}$$

- Finally, by considering the traceless constraint (fourth constraint) we get the equation:

$$A^\mu_\mu = 0 \Rightarrow \bar{h}^\mu_\mu = 0 \Rightarrow h^\mu_\nu - \frac{1}{2}\delta^\mu_\mu h = 0 \Rightarrow h = 0$$

Eventually, we have concluded that $\bar{h}_{\mu\nu} = h_{\mu\nu}$ and the final form of the amplitude 4x4 matrix for a wave travelling in the z-direction where $U = (U_t, 0, 0, 0)$ and $\vec{k} = (\frac{\omega}{c}, 0, 0, \frac{\omega}{c})$ is:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{22} & A_{23} & 0 \\ 0 & A_{23} & A_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{22} & A_{23} & 0 \\ 0 & A_{23} & -A_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{\text{rename}}{\Rightarrow} A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{+} & A_{\times} & 0 \\ 0 & A_{\times} & -A_{+} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

6.4 Appendix D - Quantization of fields in curved spacetime

6.4.1 Quantization of fields in FLRW universe

In this appendix, we will present the methodology that we followed for the quantization of the ϕ complex scalar field in a curved background. Let us consider an FLRW universe:

$$ds^2 = dt^2 - \alpha^2(t)dx^2 = \alpha^2(\eta)(d\eta^2 - dx^2) \tag{6.4.1}$$

where $d\eta = \alpha(t)dt \Rightarrow \eta = \int_{t_0}^{\tau} \frac{dt}{a(t)}$ is the conformal time. Therefore, the metric is $g_{\mu\nu} = \alpha^2(\eta)\eta_{\mu\nu}$ and $\sqrt{-g} = \alpha^4(\eta)$. At this point, we will analyze the behavior of a complex scalar field as it matches our theory. The action is expressed:

$$S = \int d^4x\sqrt{-g}[g^{\mu\nu}(\partial_\mu\phi)^\dagger\partial_\nu\phi - V(\phi^\dagger\phi)]$$

Using the Euler-Lagrange, we derive the equations of motion for the ϕ and $\phi^\dagger$:

$$\frac{\partial L}{\partial\phi^\dagger} = \partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi^\dagger)} \Rightarrow -\sqrt{-g}V_{,\phi} = \partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) \overset{\text{FLRW}}{\Rightarrow} \alpha^4(\eta)V_{,\phi} = \partial(\alpha^2(\eta)\eta^{\mu\nu}\partial_\nu\phi) \Rightarrow$$
$$\frac{1}{\alpha^4(\eta)}\partial(\alpha^2(\eta)\eta^{\mu\nu}\partial_\nu\phi) + V_{,\phi} = 0 \Rightarrow \frac{2\alpha'\alpha}{\alpha^4}\phi' + \frac{1}{\alpha^2}(\phi'' - \nabla^2\phi) + V_{,\phi} = 0$$

Now considering zero potential $V_{,\phi} = 0$, since the GWs contain massless terms, and rescaling ϕ, accordingly:

- $\chi = \alpha(\eta)\phi \Rightarrow \phi = \frac{\chi}{\alpha(\eta)}$
- $\phi' = \frac{\chi'}{\alpha} - \frac{\chi\alpha'}{\alpha^2}$
- $\phi' = \frac{\chi''}{\alpha} - \frac{2\chi'\alpha'}{\alpha^2} - \frac{\chi\alpha''}{\alpha^2} + -\frac{2\chi(\alpha')^2}{\alpha^3}$

The equation of motion becomes:

$$\chi'' - \nabla^2\chi - \frac{\alpha''}{\alpha}\chi = 0 \tag{6.4.2}$$

Expanding in Fourier modes, we could simplify the system into a harmonic oscillator problem.

$$\chi(\vec{x}, \eta) = \int \frac{d^3k}{(2\pi)^{3/2}}\chi(k)e^{i\vec{k}\vec{x}} \Rightarrow \nabla^2\chi(\vec{x}, \eta) = -k^2\chi(\vec{x}, \eta) \tag{6.4.3}$$

Thus, the equation 6.4.4 becomes:

$$\chi_k'' + \left(k^2 - \frac{\alpha''}{\alpha}\right)\chi_k = 0 \tag{6.4.4}$$

Due to the Gravitational Chern-Simons contribution term, the expansion of the universe is affected. This means that instead of a simple $\alpha(\eta)$, we choose a $z(\eta) = \alpha(\eta)\sqrt{1-L_{CS}}$ expansion variable.

6.5 Appendix E - Dimensional analysis

Having considered natural units $c = 1$ and $\hbar = 1$, it comes as a result of special relativity that: $[m] = [s] = [M]^{-1} = \frac{1}{E}$, where M and E are the mass and energy units correspondingly.

For the integration, we need:

$$[d^4x] = [m^4] = \frac{1}{E^4} \qquad \text{and} \qquad [\kappa] = [m] = \frac{1}{E}$$

The Torsion-QED action is:

$$\begin{aligned} S_{Torsion-QED} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(\mathring{\boldsymbol{R}}_{ab} + \Delta) + S_{Classical-QED} - \frac{3}{4}\int d^4x\sqrt{-g}S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi \\ -\frac{1}{4}\int d^4x\sqrt{-g}F^{\mu\nu}F_{\mu\nu} \end{aligned}$$

Unit check:

- $[\mathring{\boldsymbol{R}}_{ab}] = [m^{-2}] = E^2$
- $[\Delta] = [m^{-2}] = E^2$
- $[\psi] = [\bar{\psi}] = [m^{-1}] = E$
- $[S_\mu] = [m^{-2}] = E^2$

The effective action is:

$$S_B = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu} + ...\right)$$

Unit check:

- $[R] = [m^{-2}] = E^2$
- $[\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu}] = [\kappa^{-2}H_{\lambda\mu\nu}H^{\lambda\mu\nu}] = [\partial_{[\mu}B_{\nu\rho]}\partial^{[\mu}B^{\nu\rho]}] = [m^{-4}] = E^4$

Thus, the action is indeed dimensionless.

6.6 Appendix F - Mathematica Scripts

Some useful Mathematica scripts can be found on this GitHub repository. Contents:

- Variations of the Einstein-Cartan contorted action
- Metric Perturbations in Minkowski Metric
- Transition phase in the String inspired cosmology (RVM toy model)
- Calculation of the R_{CS} in stiff matter era
- Calculation of the R_{CS} in inflationary era

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