National Technical university of Athens

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Thesis

Inflation in String-induced cosmologies with Torsion

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Spring semester 2024-2025

Contents

1 Introduction

The current cosmological model, based on the Friedmann-Lemaître-Robertson-Walker (FLRW) solution of General Relativity (GR) with a positive cosmological constant (> 0) and Cold Dark Matter (CDM), accurately describes the universe's large-scale structure and evolution, with satisfying precision to observational data. This model, known as ΛCDM, relies on a de Sitter equation of state $w = -1$, but despite its successes, both theoretical and experimental challenges need an extension or revision of the framework. Issues such as the nature and origin of the dark sector (including dark matter and dark energy), black hole solutions, as well as cosmological tensions like the Hubble constant (H_0) tension and the σ_8 tension, require a more generalized or alternative cosmological theory. In this context, various modifications of GR have been proposed (modified gravity theories) and growing interest in string-inspired effective theories.

While introducing torsion into spacetime alone is insufficient to address these issues, low-energy effective string theory offers a more promising approach. In this framework, the antisymmetric Kalb-Ramond (KR) field, part of the massless bosonic multiplet in string theory, can act as a source of torsion. It has been shown that after compactification of extra dimensions in string theory, axion-like degrees of freedom emerge, along with their coupling terms, which govern their behavior in the early universe. Furthermore, in this context, vielbeins and the spin connection are treated as independent fields, leading to non-trivial torsion in the equations of motion.

In this assignment, we will present the mathematical tools necessary to explore Einstein-Cartan theory in a contorted spacetime, as outlined in[1]. We will demonstrate how axion-like currents arise naturally from the incorporation of torsion into the manifold and discuss the physical implications of these phenomena. Additionally, we will present an alternative solution inspired by string theory, where torsion emerges naturally in the low-energy limit due to the variations of the KR field.

2 Torsion in Einstein-Cartan-Sciama-Kibble (ECSK)

Before proceeding to the calculation and explanation of torsion and its properties on a spacetime, the mathematical framework must be provided. The main parts that compose the framework is the tensor definition, the operations and the basis i.e. metric and coordinate system.

2.1 Mathematics

2.1.1 Spacetime properties

It is important to mention and explain the notation used in this assignment. Let $\mathcal{M}$ be a (3+1)-dimensional manifold parametrized by coordinates x^μ, with $\mu = 0, 1, 2, 3$, described by $g_{\mu\nu}$ metric. $T_p\mathcal{M}$ is the corresponding tangent space at point p described by the corresponding η_{ab} Minkowski metric. Using the vielbein notation (analysed belolw), one can express the metric as:

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab} \tag{2.1.1}$$

For the vielbein components, the set of vectors comprising an orthonormal basis, we take into consideration the relations $E^\mu{}_a e_\nu{}^a = \delta^\mu_\nu$ and $E_\mu{}^a e^\nu{}_b = \delta^a_b$.

In this assignment we use objects that have Greek and/or Latin indices. Latin indices are locally flat indices that are used on objects that "live" on Lorentzian spacetime or else on a Tangent space described by Minkowski metric ϵ_{ab}. They are independent of the coordinate system used. Greek indices are used on objects that "live" on a specific chart of our manifold which is described by the GR metric tensor $g_{\mu}\nu$. The main purpose during the calculations is to eliminate the Greek indices in order to obtain a more simplified and global notation, easy to handle. The transformation of a vector or a tensor is something like a change of basis:

$$\text{Vector: } V^a = e^a_\mu V^\mu \tag{2.1.2}$$

$$\text{Tensor: } T^a_b = e^a_\mu T^\mu_b = e^a_\mu e^\nu_b T^\mu_\nu \tag{2.1.3}$$

This notation, is useful for two reasons. First, we have the ability to describe spinor fields (which in general "live" in a Lorentzian space) and calculate their covariant derivatives adding an extra feature in the Theory. Second, we can think of tensors as tensor-valued differential forms more specifically one can think of a p-form "living" on a chart of the manifold that its components are tensors.

2.1.2 Wedge product

Denoted as $\wedge$, the wedge product, or else exterior product is defined as the anti-symmetric tensor product of the cotangent space basis elements. The mathematical definition is shown below:

$$dx^\mu \wedge dx^\nu = \frac{1}{2}(dx^\mu \times dx^\nu - dx^\nu \times dx^\mu) = -dx^\nu \wedge dx^\mu \tag{2.1.4}$$

One general property of the wedge product for a q-form α_q and a p-form b_p, is:

$$\alpha_q \wedge b_p = (-1)^{pq} b_p \wedge \alpha_q \tag{2.1.5}$$

2.1.3 Covariant derivative

The covariant derivative definition is already known from the GR course. Here, we will provide an alternative definition of an operation of the covariant derivative acting on a q-form $Q^{\alpha,\cdots}_{b,\cdots}$:

$$D(\omega)Q^{\alpha,\cdots}_{b,\cdots} = dQ^{\alpha,\cdots}_{b,\cdots} + \omega^\alpha_c \wedge Q^{c,\cdots}_{b,\cdots} + ... - (-1)^q Q^{\alpha,\cdots}_{d,\cdots} \wedge \omega^d_b \tag{2.1.6}$$

The covariant derivative provides a connection between tangent spaces on a manifold, through the definition of the parallel transport property.

One very important property of the tangent space at a point p on the manifold, is the metric compatibility. The Minkowski metric describes the tangent space, so the expression for the metric compatibility is:

$$\nabla \eta_{ab} = 0 \tag{2.1.7}$$

Metric compatible spaces preserve the inner product of parallely transported vectors or other objects.

2.1.4 Mathematical objects used

Based on vielbein notation, we will introduce the main objects we need for our purpose.
First, noted as $\boldsymbol{\omega}^a_b = \omega^a_{b\mu} dx^\mu$, the affine Lorentz connection otherwise called **affine spin connection**, is an one-form with the Greek index suppressed, that corresponds to spinors or to general tensors with Lorentz indices. The spin connection introduced in Lorentz group, plays the same role as Christoffel symbols in

gravity, used for parallel transports and covariant derivatives definitions (next subsection). Using the **metric compatibility** property in our spacetime we can prove the antisymmetrization of this one-form:

$$D(\omega)\eta_{ab}=0\Rightarrow \cancelto{0}{d\eta_{ab}}-\eta_{cb}\wedge\omega^{c}{}_{a}-\eta_{ac}\wedge\omega^{c}{}_{b}=0\Rightarrow \eta_{cb}(\omega_\mu)^{c}{}_{a}=-\eta_{ac}(\omega_\mu)^{c}_{b}\Rightarrow \boxed{\boldsymbol{\omega}_{ab}=-\boldsymbol{\omega}_{ba}} \tag{2.1.8}$$

Furthermore, let us introduce the torsion tensor $\boldsymbol{T}^a$ (again with the Greek indices suppressed), the geometric derivation of which will be explained later. For now we need the algebraic definition in order to illustrate some influences on the rest of the objects used for the spacetime description. The tensor in vielbein notation is written:

$$\boldsymbol{T}^a=\frac{1}{2}T^{a}{}_{\mu\nu}dx^\mu\wedge dx^\nu\equiv de^a+\omega^{a}{}_{b}\wedge e^b \tag{2.1.9}$$

Considering a spacetime with Torsion (something we will next explain geometrically), we can split this tensor into a torsion-free one-form, $\mathring{\omega}$ and contorsion one-form, $K^a_{b\mu}$. So, the spin connection can be written as:

$$\omega^{a}{}_{b\mu}=\mathring{\omega}^{a}{}_{b\mu}+K^{a}{}_{b\mu} \tag{2.1.10}$$

We can relate the contorsion tensor K_{abc} with the torsion tensor to make the calculations easier in the future. It has been proved that:

$$K_{abc}=\frac{1}{2}(T_{cab}-T_{abc}-T_{bca})\Rightarrow T_{[abc]}=-2K_{[abc]} \tag{2.1.11}$$

In this case we chose the notation in which K_{abc}, is antisymmetric in the 1st and 2nd index $K_{abc}=-K_{bac}$ (or else $K_{\mu\nu\lambda}=-K_{\nu\mu\lambda}$). To change the notation from minkowski to amnifold coordinates we can simply do $K^{a}{}_{bc}=K^{a}{}_{b\mu}E^{\mu}{}_{c}$.
The next object we need to introduce, is the two-form of the **generalised Riemann curvature** again with the entire tensor form (with the Greek indices included in bolt) $\boldsymbol{R}^a_b$. It is defined as:

$$\boldsymbol{R}^{a}{}_{b}=d\omega^{a}{}_{b}+\omega^{a}{}_{c}\wedge\omega^{c}{}_{b} \tag{2.1.12}$$

Using the covariant derivative definition, we will provide an alternative formula for the generalised Riemann curvature:

$$\begin{aligned}\boldsymbol{R}^{a}{}_{b}&=d\omega^{a}{}_{b}+\omega^{a}{}_{c}\wedge\omega^{c}{}_{b}\\&=d(\mathring{\omega}^{a}{}_{b}+K^{a}{}_{b})+(\mathring{\omega}^{a}{}_{c}+K^{a}{}_{c})\wedge(\mathring{\omega}^{c}{}_{b}+K^{c}{}_{b})\\&=\underbrace{d\mathring{\omega}^{a}{}_{b}+\mathring{\omega}^{a}{}_{c}\wedge\mathring{\omega}^{c}{}_{b}}_{\mathring{R}^{a}{}_{b}}+\underbrace{dK^{a}{}_{b}+\mathring{\omega}^{a}{}_{c}\wedge K^{c}{}_{b}+K^{a}{}_{c}\wedge\mathring{\omega}^{c}{}_{b}}_{D(\mathring{\omega})K^{a}{}_{b}=dK^{a}{}_{b}+\mathring{\omega}^{a}{}_{c}\wedge K^{c}{}_{b}-(-1)^{1}K^{a}{}_{c}\wedge\mathring{\omega}^{c}{}_{b}}+K^{a}{}_{c}\wedge K^{c}{}_{b}\\&=\boxed{\mathring{R}^{a}{}_{b}+D(\mathring{\omega})K^{a}{}_{b}+K^{a}{}_{c}\wedge K^{c}{}_{b}}\end{aligned} \tag{2.1.13}$$

The equations 2.1.9 and 2.1.12 are known as *Cartan structure equations.*

Let us derive the affine connection $\Gamma^{\lambda}_{\mu\nu}$, from the torsion tensor and explore at the same time their relation in a contorted spacetime. Using the equations below for the infinitesimal elements:

$$de^a=\frac{1}{2}\partial_\mu e^{a}{}_{\nu}(dx^\mu\wedge dx^\nu)+\frac{1}{2}e^{a}{}_{\mu}\underbrace{(dx^\nu\wedge dx^\mu)}_{-(dx^\mu\wedge dx^\nu)} \tag{2.1.14}$$

$$\omega \wedge e^b = \frac{1}{2}\omega^a{}_{b\mu}e^b{}_\nu(dx^\mu \wedge dx^\nu) - \frac{1}{2}\omega^a{}_{b\nu}e^b{}_\mu(dx^\nu \wedge dx^\mu) \tag{2.1.15}$$

The torsion tensor, using again differential-form notation, appears to be:

$$\begin{aligned} \boldsymbol{T}^a &= [\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu](dx^\mu \wedge dx^\nu) + [\omega^a{}_{b\mu}e^b{}_\mu - \omega^a{}_{b\nu}e^b{}_\nu](dx^\mu \wedge dx^\nu) \\ &= [\partial_\mu e^a{}_\nu + \omega^a{}_{b\mu}e^b{}_\mu - (\partial_\nu e^a{}_\mu + \omega^a{}_{b\nu}e^b{}_\nu)](dx^\mu \wedge dx^\nu) \\ &= [\Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}](dx^\mu \wedge dx^\nu) \\ &= \Gamma^\lambda{}_{[\mu\nu]}(dx^\mu \wedge dx^\nu) \end{aligned} \tag{2.1.16}$$

where $[\mu\nu]$ shows the antisymmetrization in the indices of the affine connection. We calculated the abstract one-form object suppressing the Greek indices, so the components of each object defined by the Greek indices will be:

$$T^a_{\mu\nu} = \Gamma^\lambda{}_{[\mu\nu]} \tag{2.1.17}$$

2.1.5 Geometrical explanation

Now, we will explore the concept of Lie brackets of two covariant derivatives. The lie brackets of two vector fields illustrate how one vector field changes with respect to the other. More specifically, in the expression $[\nabla_\mu, \nabla_\nu]f = \nabla_\mu\nabla_\nu f - \nabla_\nu\nabla_\mu f$, the first term represents how ∇_ν changes in the direction of ∇_μ and vice versa for the second term.
The space where those two covariant derivatives are independent from each other, this means that they don't change while parallely transported across each others flow curves, is torsion free. On the contrary, in a space that is twisted, there is torsion, each parallel transported vector across the flow lines changes. This change, is the failure to close a standard parallelogram in the case where the space has no torsion and it is given from this exact Lie bracket operator.

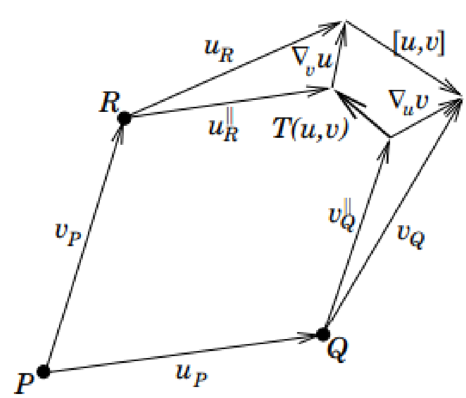

Figure 1: Torsion representation[2].

Let us explain what happens in 1. v_P, a vector at point P, is parallel transported to the point Q, and is denoted as $v_Q^\parallel$, and the same thing happens with the parallel transport of u_P, which is a vector at point P, to the point R, denoted as $u_R^\parallel$. The actual vectors in the vector field look like v_Q and v_R correspondingly. We can measure the difference between $v_Q^\parallel$ and v_Q by taking the covariant derivative of v with respect to u, and

$u_R^{\parallel}$ and u_R by taking the covariant derivative of u with respect to v. Those, two covariant derivatives shows us how much the vectors u, v deviate from the parallel transported ones. Now, we can measure the difference between those two covariant derivatives by taking the Lie bracket. However, T which is the so called Torsion tensor, and represents the difference of the parallel transported vectors $v_Q^{\parallel}$ and $u_R^{\parallel}$. It is found to be:

$$T(\vec{u},\vec{v}) = \nabla_{\vec{u}}\vec{v} - [\vec{u},\vec{v}] - \nabla_{\vec{v}}\vec{u} \tag{2.1.18}$$

This is the formula of the Torsion tensor acting in two vector fields u and v. In case the result of this equation is equal to zero, there is no torsion and we get a closed loop.

Now, it is possible to expand the covariant derivative using the fully diffeomorphic formula:

$$D_\mu(\Gamma)e^a_\mu = \partial_\mu e^a{}_\mu - \Gamma^\lambda{}_{\nu\mu}e^a{}_\lambda = -\omega_\mu{}^a{}_b \tag{2.1.19}$$

The term $\Gamma^\lambda_{\nu\mu}$, is the affine connection which in a case of a torsionless and metric compatible spacetime are unique, and called the *Christoffel symbols*.
Having considered the metric $g_\mu\nu$ and the metric compatibility property, one can rewrite the total linear connection in Riemann-Cartan space for our case:

$$\begin{aligned}\Gamma^\mu{}_{\nu\rho} &= \begin{Bmatrix}\mu\\ \nu\rho\end{Bmatrix} + K^\mu{}_{\nu\rho}\\ &= \frac{1}{2}g^{\mu\rho}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} + \partial_\sigma g_{\rho\nu}) + K^\mu{}_{\nu\rho}\end{aligned} \tag{2.1.20}$$

We have considered that local distances do not change under parallel transport.

2.2 Physics of gravitation with Torsion

Starting with the standard Einstein-Hilbert gravitational action, modifications will be applied based on the new torsion tensor that was arbitrarily introduced to study the impacts of the torsion component in General Relativity. We shall bring the action to a form, convenient to analyse its impacts on cosmology.

2.2.1 Derivation of the gravitational action

It is quite easy to start with the classical Einstein-Hilbert action and replace the Ricci scalar with the contorsion tensor we have derived in the previous chapter in order to take into account the torsion tensor.

$$\begin{aligned}S_{\text{EH}} &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\boldsymbol{R}\\ &\overset{R=R^{ab}_{ab}}{=} \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R^{ab}_{[\mu\nu]}e^\mu_a e^\nu_b\\ &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\frac{1}{2}R^{ab}_{\mu\nu}(e^\mu_a e^\nu_b - e^\nu_a e^\mu_b)\\ &= \frac{1}{4\kappa^2}\int d^4x\sqrt{-g}R^{ab}_{\mu\nu}(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\nu_\rho\delta^\mu_\sigma)e^\rho_a e^\sigma_b\end{aligned}$$

also, we used references[3] to calculate the below useful quantities:

- $$\epsilon^{\mu\nu\kappa\lambda} d^4x \sqrt{-g} = dx^\mu \wedge dx^\nu \wedge dx^\kappa \wedge dx^\lambda \tag{2.2.1}$$

- $$(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma) = \frac{1}{2} \epsilon_{\rho\sigma\kappa\lambda} \epsilon^{\mu\nu\kappa\lambda} \tag{2.2.2}$$

- $$\star e^a \wedge e^b = \frac{1}{2} \epsilon_{\rho\sigma\kappa\lambda} e^{a\rho} e^{b\sigma} dx^\kappa \wedge dx^\lambda \tag{2.2.3}$$

so using 2.2.1 and 2.2.2 at the same time we get:

$$\begin{aligned}
S_{\text{EH}} &= \frac{1}{8\kappa^2} \int (dx^\mu \wedge dx^\nu \wedge dx^\kappa \wedge dx^\lambda) R^{ab}_{\mu\nu} e^\rho_a e^\sigma_b \epsilon_{\rho\sigma\kappa\lambda} \\
&= \frac{1}{2\kappa^2} \int \left(\frac{1}{2} R^{ab}_{\mu\nu} dx^\mu \wedge dx^\nu \right) \wedge \left(\frac{1}{2} \epsilon_{\rho\sigma\kappa\lambda} e^{\rho a'} e^{\sigma b'} dx^\kappa \wedge dx^\lambda \right) \eta_{aa'} \eta_{bb'} \\
&\overset{2.2.3}{=} \frac{1}{2\kappa^2} \int \left(\boldsymbol{R}^{ab} \eta_{aa'} \eta_{bb'} \right) \wedge \star \left(e^{a'} \wedge e^{b'} \right) \\
&= \frac{1}{2\kappa^2} \int (\boldsymbol{R}_{a'b'}) \wedge \star \left(e^{a'} \wedge e^{b'} \right)
\end{aligned}$$

And the final result using the equation 2.1.13, will be eventually:

$$S_{\text{EH}} = \frac{1}{2\kappa^2} \int (\boldsymbol{R}_{ab}) \wedge \star \left(e^a \wedge e^b \right) = \frac{1}{2\kappa^2} \int (\mathring{\boldsymbol{R}}_{ab} + D(\mathring{\omega}) \boldsymbol{K}_{ab} + \boldsymbol{K}_{ac} \wedge \boldsymbol{K}^c{}_b) \wedge \star \left(e^a \wedge e^b \right) \tag{2.2.4}$$

where R is the Ricci scalar, $\kappa^2 = 8\pi G$, G is Newton's gravitational constant.

From the classical general relativity, it is known that we can use the Stokes theorem for the boundary condition in a volume V, in order to simplify the formula:

$$\int_V D(\mathring{\omega}) \boldsymbol{K}_{ab} \wedge \star (e^a \wedge e^b) = 0 \tag{2.2.5}$$

The Einstein-Hilbert action becomes Einstein-Cartan action after adding the torsion property on the spacetime. This means that it describes the gravitational force in the Einstein Cartan spacetime gravity action can be expressed as:

$$\begin{aligned}
S_{\text{grav}} &= \frac{1}{2\kappa^2} \int (\mathring{\boldsymbol{R}}_{ab} + \boldsymbol{K}_{ac} \wedge \boldsymbol{K}^c{}_b) \wedge \star \left(e^a \wedge e^b \right) \\
&= \frac{1}{2\kappa^2} (C_1 + C_2)
\end{aligned} \tag{2.2.6}$$

In the above equation torsion seems to be separated. We notice that, torsion is purely a topological feature, which allows us to treat the action equations similarly to how we would in classical General Relativity. However, if torsion were to act as a physical source or have dynamical effects, we would need to adopt a different approach.

The C_1, C_2 coefficients are used to simplify the action denoted:

- to calculate C_1 one must follow the reverse procedure that was already showed above. Finally we get: $C_1 = \int d^4x\sqrt{-g}\mathring{\boldsymbol{R}}_{ab}$

- and to calculate C_2:

$$\begin{aligned}
C_2 &= \int (\boldsymbol{K}_{ac} \wedge \boldsymbol{K}^c{}_b) \wedge \star \left(e^a \wedge e^b\right) \overset{2.2.3}{=} \int (\boldsymbol{K}_{ac} \wedge \boldsymbol{K}^c{}_b) \wedge \left(\frac{1}{2}\epsilon_{\rho\sigma\kappa\lambda} e^{a\rho} e^{b\sigma} dx^\kappa \wedge dx^\lambda\right) \\
&= \int (K_{ac\mu} K^c{}_{b\nu}) \left(\frac{1}{2}\epsilon_{\rho\sigma\kappa\lambda} e^{a\rho} e^{b\sigma} dx^\mu \wedge dx^\nu \wedge dx^\kappa \wedge dx^\lambda\right) \\
&\overset{2.2.1}{=} \int d^4x\sqrt{-g}(K_{ac\mu} K^c{}_{b\nu}) \left(\frac{1}{2}\epsilon_{\rho\sigma\kappa\lambda}\epsilon^{\mu\nu\kappa\lambda}\right) e^{a\rho} e^{b\sigma} \\
&\overset{2.2.2}{=} \int d^4x\sqrt{-g}(K_{ac\mu} K^c{}_{b\nu}) \left(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\nu_\rho \delta^\mu_\sigma\right) e^{a\rho} e^{b\sigma} \\
&= \int d^4x\sqrt{-g}\left(K_{ac\rho} K^c{}_{b\sigma} - K_{ac\sigma} K^c{}_{b\rho}\right) e^{a\rho} e^{b\sigma} \\
&= \int d^4x\sqrt{-g}\left(K_{ac}{}^a K^c{}_b{}^b - K_{ac}{}^b K^c{}_b{}^a\right)
\end{aligned}$$

Now, we can replace the Latin indices by the Greek ones since they seem to cancel out: $K_{ac}{}^a K^c{}_b{}^b = K_{\lambda\mu}{}^\lambda K^\mu{}_\nu{}^\nu E^\lambda_a e^a_\lambda E^\nu_b e^b_\nu E^\mu_c e^c_\mu = K_{\lambda\mu}{}^\lambda K^\mu{}_\nu{}^\nu$. So we have:

$$C_2 = \int d^4x\sqrt{-g}\left(K_{\lambda\mu}{}^\lambda K^\mu{}_\nu{}^\nu - K_{\lambda\mu}{}^\nu K^\mu{}_\nu{}^\lambda\right) = \int d^4x\sqrt{-g}\left(K_{\lambda\mu}{}^\lambda K^{\mu\nu}{}_\nu - K^\lambda{}_{\mu\nu} K^{\mu\nu}{}_\lambda)\right)$$

Using the antisymmetrization $K^{\mu\nu}{}_\lambda = -K^{\nu\mu}{}_\lambda$ and $K_{\lambda\mu}{}^\lambda = -K_{\mu\lambda}{}^\lambda$ we get:

$$C_2 = \int d^4x\sqrt{-g}\left(K^\lambda{}_{\mu\nu} K^{\nu\mu}{}_\lambda - K_{\mu\lambda}{}^\lambda K^{\mu\nu}{}_\nu\right) \tag{2.2.7}$$

Next we set the C_2 term (it is a scalar value) :

$$\Delta \equiv K^\lambda{}_{\mu\nu} K^{\nu\mu}{}_\lambda - K^{\mu\nu}{}_\nu K_{\mu\lambda}{}^\lambda \overset{2.1.11}{=} T^\nu{}_{\nu\lambda} T^\lambda{}_\lambda{}^\mu - \frac{1}{2} T^\mu{}_{\nu\lambda} T^\nu{}_{\lambda\mu} + \frac{1}{4} T_{\mu\nu\lambda} T^{\mu\nu\lambda} \tag{2.2.8}$$

And finally, the action equation can be simplified into a torsionless and a contorted part:

$$\boxed{S_{\text{grav}} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(\mathring{\boldsymbol{R}}_{ab} + \Delta)} \tag{2.2.9}$$

2.2.2 Torsion tensor decomposition

After some group theory calculations, the torsion tensor can be decomposed in its irreducible parts:

$$T_{\mu\nu\rho} = \frac{1}{3}(T_\nu g_{\mu\rho} - T_\rho g_{\mu\nu}) - \frac{1}{6}\epsilon_{\mu\nu\rho\sigma} S^\sigma + q_{\mu\nu\rho} \tag{2.2.10}$$

where we defined the:

- **torsion trace vector**:

$$T_\mu \equiv T^\nu{}_{\mu\nu} \tag{2.2.11}$$

- **pseudo-scalar axial vector**:

$$S_\mu \equiv \epsilon_{\mu\nu\rho\sigma} T^{\nu\rho\sigma} \tag{2.2.12}$$

And we note the below properties:

- for the antisymmetric tensor

$$q^{\nu}{}_{\rho\nu} = 0 = \epsilon^{\sigma\mu\nu\rho} q_{\mu\nu\rho} \tag{2.2.13}$$

- Rewrite the contorsion tensor as:

$$\boldsymbol{K_{abc}} = \frac{1}{2}\epsilon_{abcd} S^d + \hat{\boldsymbol{K}}_{abc} \tag{2.2.14}$$

- Rewrite the scalar as:

$$\Delta = \frac{3}{2} S_d S^d + \hat{\Delta} \tag{2.2.15}$$

So by the end of this subsection, we have managed to derive the action for gravity for a spacetime with torsion.

2.3 QED in a contorted spacetime

The S_{grav} term that was derived in the previous section describes the gravity in a contorted spacetime, however to study further the physical phenomena a good start is done by adding matter in the universe. In order to add matter we start by adding QED. Before proceeding to more complicated equations, we need to consider first the classical QED action for a massless Dirac fermion.

2.3.1 QED action in a contorted spacetime

Let there be a 3+1 dimensional QED with torsion. The action equation is:

$$S_{Torsion-QED} = \frac{i}{2} \int d^4x \sqrt{-g} \left[\bar{\psi}(x) \gamma^\mu D_\mu(\omega, A) \psi(x) - \overline{D_\mu(\omega, A)\psi(x)} \gamma^\mu \psi(x) \right] \tag{2.3.1}$$

In order to take torsion into account in the above equation it is necessary to replace the covariant derivative to match our theory, thus an additional torsion is required. We use:

$$D_\mu(\omega, A) = D(\omega) - ieA_\mu = \partial_\mu + i\omega^a{}_{b\mu} \sigma^b_a - ieA_\mu \tag{2.3.2}$$

and $\sigma^{ab} = \frac{i}{4} = [\gamma^a, \gamma^b]$ the 4x4 Dirac matrices on the tangent space T_pM of the manifold.

Eventually, one can rewrite this action using some differential geometry as:

$$S_{Torsion-QED} = S_{Classical-QED} + \underbrace{\frac{1}{8} \int d^4x \sqrt{-g} \bar{\psi}(x) \{\gamma^c, \sigma^{ab}\} K_{abc} \psi(x)}_{\text{Torsion Term}} \tag{2.3.3}$$

The final term is the torsion tensor acting on a fermion. The whole expression between $\bar{psi}$ and ψ can be considered as an operator:

$$
\begin{aligned}
\text{Torsion Term} &= \frac{1}{8}\int d^4x\sqrt{-g}\bar{\psi}(x)\left(2\epsilon^{abc}{}_d\gamma^d\gamma^5\right)K_{abc}\psi(x)\\
&= \frac{1}{8}\int d^4x\sqrt{-g}\bar{\psi}(x)\left(2\epsilon^{abc}{}_d\gamma^d\gamma^5\right)\left(\frac{1}{2}(T_{cab}-T_{abc}-T_{bca})\right)\psi(x)\\
&= -\frac{1}{8}\int d^4x\sqrt{-g}\bar{\psi}(x)\left(\gamma^d\gamma^5\right)\left(\epsilon^{abc}{}_dT_{cab}-\epsilon^{abc}{}_dT_{abc}-\epsilon^{abc}{}_dT_{bca}\right)\psi(x)\\
&= -\frac{1}{8}\int d^4x\sqrt{-g}\bar{\psi}(x)\left(\gamma^d\gamma^5\right)\left(\epsilon^{abc}{}_dT_{abc}\right)\psi(x)\\
&= -\frac{1}{8}\int d^4x\sqrt{-g}\bar{\psi}(x)\left(\gamma^d\gamma^5\right)(6S_d)\psi(x)\\
&= -\frac{3}{4}\int d^4x\sqrt{-g}\bar{\psi}(x)S_d\gamma^d\gamma^5\psi(x)
\end{aligned}
\tag{2.3.4}
$$

We used equations 2.1.11 and 2.2.12.

Using the $g_{\mu\nu} = diag(-1,1,1,1)$ Minkowski metric, we maintained the minus sign in the term, from the calculation: $\epsilon^{abcd} = \epsilon^{abc}{}_d \cdot g^{dd}$. So the final expression is:

$$
S_{Torsion-QED} = S_{Classical-QED} - \frac{3}{4}\int d^4x\sqrt{-g}S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi \tag{2.3.5}
$$

In the second term we notice that we have a vector field S_μ coupling to the fermion's axial current.

2.3.2 Current conservation correction & Torsion-induced axions

Including the Maxwell tensor for QED we get the final expression for the action:

$$
\begin{aligned}
S_{Torsion-QED} &= S_{grav} + S_{Torsion-QED} + S_{Maxwell}\\
&= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(\mathring{\boldsymbol{R}}_{ab}+\Delta) + S_{Classical-QED} - \frac{3}{4}\int d^4x\sqrt{-g}S_\mu\bar{\psi}\gamma^\mu\gamma^5\psi\\
&- \frac{1}{4}\int d^4x\sqrt{-g}F^{\mu\nu}F_{\mu\nu}
\end{aligned}
\tag{2.3.6}
$$

This action takes into consideration the gravity and the QED that represents the matter content in our universe. Having the full action we can use it to calculate the stress-energy tensor and after that the equations of motion. The stress-energy tensor can be decomposed as follows:

$$
\begin{aligned}
T_{\mu\nu} &= T^A_{\mu\nu} + T^\psi_{\mu\nu} + T^S_{\mu\nu}\\
&= F_{\mu\lambda}F^\lambda_\nu - \frac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - \left(\frac{i}{2}\bar{\psi}\gamma_{(\mu}D_{\nu)} - \gamma_{(\mu}D_{\nu)}\psi\right) + \frac{3}{4}S_{(\mu}\bar{\psi}\gamma_{\nu)}\gamma^5\psi\\
&- \frac{3}{2\kappa^2}\left(S_\mu S_\nu - \frac{1}{2}g_{\mu\nu}S_\alpha S^\alpha\right)
\end{aligned}
\tag{2.3.7}
$$

The equations of motion after taking the variation of the full-developed action 2.3.6, are proved to be:

$$
T_\mu = 0 \tag{2.3.8}
$$

$$
q_{\mu\nu\rho} = 0 \tag{2.3.9}
$$

$$S = \frac{\kappa^2}{2} j^5 \tag{2.3.10}$$

Now the final expression for the spin-connection in this contorted universe becomes:

$$\begin{aligned} \omega^{\alpha\beta}{}_{\mu} &= \mathring{\omega}^{\alpha\beta}{}_{\mu} + K^{\alpha\beta}_{\mu} \\ &= \mathring{\omega}^{\alpha\beta}{}_{\mu} + \frac{\kappa^2}{4} \epsilon^{ab}{}_{cd} e^{c}_{\mu} j^5 \end{aligned} \tag{2.3.11}$$

For massless fermion, we can denote the equation of motion, by taking the $S_{\text{Torsion-QED}}$ and deriving the Dirac equation:

$$\frac{\partial S}{\partial \bar{\psi}} = 0 \Rightarrow i\gamma^{\mu} D_{\mu}(\omega, A)\psi = \frac{3}{4} S_{\mu} \gamma^{\mu} \gamma^5 \psi \tag{2.3.12}$$

We notice that the pseudo-vector S_{μ} seems to play the role of an axial source. We also notice conservation laws to be true not only for the fermionic current, but also for the torsion pseudo-vector S:

$$d \star j^5 = 0 \overset{2.3.10}{\Rightarrow} d \star S = 0 \tag{2.3.13}$$

Now, we notice that the action is proportional to the S_{μ} squared, one could integrate it using a path integral and get the repulsive four fermion interaction from the torsion term[3]:

$$-\frac{3}{16} \int j^5 \wedge \star j^5 \tag{2.3.14}$$

However, we notice that the conservation of the axial current of a fermion is violated at a quantum level. Specifically, the one-loop path integral of the conservation gives a non zero value, even in the torsionless limit[3]:

$$d \star j^5 = \frac{e^2}{8\pi} F \wedge F - \frac{1}{96\pi^2} R^a_b \wedge R^a_b \equiv \mathcal{G}(\omega, A) \tag{2.3.15}$$

The violation of torsion conservation due to anomalies can be corrected by performing perturbation theory within the path integral framework. To achieve this, one should introduce a δ-functional constraint in the path integral and sum up to all possible torsion fields S_{μ} on the path. Using this δ-functional it is ensured that the conservation law $d \star S = 0$ holds at every point of the spacetime. The pseudo-scalar field Φ added to the expression, acts as a Lagrangian multiplier ensuring that only the torsion terms that satisfy this condition will contribute eventually to the path integral. In this way, Φ guarantees the conservation of both, the torsion field S_{μ} and the axial current j^5_{μ} even in the presence of quantum anomalies. By summing over all possible configurations of the torsion field, we maintain the consistency of the quantum theory.

Let's break down the steps for the path integral calculation.
The δ-functional used is defined:

$$\delta(d \star S) = \int D\Phi \, \exp\left(i \int \Phi d \star S\right) \tag{2.3.16}$$

By integrating 2.3.16 in the path integral the initial expression is:

$$\begin{aligned}\mathcal{Z} &\propto \int DS\ D\Phi\ \delta(d\star S)\ \exp\left(i\int\left[\frac{3}{4\kappa^2}S\wedge\star S-\frac{3}{4}S\wedge\star j^5\right]\right)\\ &\propto \int DS\ D\Phi\ \ \exp\left(i\int\left[\frac{3}{4\kappa^2}S\wedge\star S-\frac{3}{4}S\wedge\star j^5+\Phi d\star S\right]\right)\end{aligned} \tag{2.3.17}$$

It is easier to work with Greek notation in this case so we are going to replace all the wedge products in the same way we did before (using 2.2.1, 2.2.2 and 2.2.3):

- $\int S\wedge\star S=\int d^4x\sqrt{-g}S^\mu S_\mu$
- $\int S\wedge\star j^5=\int d^4x\sqrt{-g}S^\mu j^5_\mu=\int d^4x\sqrt{-g}S_\mu j^{5\mu}$
- and because Φ and S_μ are pseudo-scalars: $\int \Phi d\star S=\int d^4x\sqrt{-g}\ \Phi\ \partial_\mu S^\mu$

So, the initial expression 2.3.17 transformation is shown below, and it is then simplified using the square completion technique[3]:

$$\begin{aligned}\mathcal{Z} &\propto \int DS^\mu\ D\Phi\ \ \exp\left(i\int d^4x\sqrt{-g}\left[\frac{3}{4\kappa^2}S^\mu S_\mu-\frac{3}{4}S^\mu j^5_\mu+\Phi\ \partial_\mu S^\mu\right]\right)\\
&\propto \int DS^\mu\ D\Phi\ \ \exp\left(i\int d^4x\sqrt{-g}\left[\frac{3}{4\kappa^2}S^\mu S_\mu+S^\mu\underbrace{\left(-\frac{3}{4}j^5_\mu+\Phi\ \partial_\mu\right)}_{J_\mu}\right]\right)\\
&\propto \int DS^\mu\ D\Phi\ \ \exp\left(i\int d^4x\sqrt{-g}\left[\frac{3}{4\kappa^2}S^\mu S_\mu+S^\mu J_\mu\right]\right)\\
&\propto \int DS^\mu\ D\Phi\ \ \exp\left(i\int d^4x\sqrt{-g}\left[\frac{\sqrt{3}}{2^2\kappa^2}S^\mu S_\mu+2\frac{\sqrt{3}}{2\kappa}\frac{\kappa}{\sqrt{3}}S^\mu J_\mu+\frac{\kappa^2}{3}J_\mu J^\mu-\frac{\kappa^2}{3}J_\mu J^\mu\right]\right)\\
&\propto \int DS^\mu\ D\Phi\ \ \exp\left(i\int d^4x\sqrt{-g}\left[\frac{3}{4\kappa^2}\left(S^\mu+\frac{2\kappa^2}{3}J^\mu\right)\left(S_\mu+\frac{2\kappa^2}{3}J_\mu\right)-\frac{\kappa^2}{3}J_\mu J^\mu\right]\right)\\
&\propto \int DS^\mu\ D\Phi\ \ \underbrace{\exp\left(i\int d^4x\sqrt{-g}\left[\frac{3}{4\kappa^2}\left(S^\mu+\frac{2\kappa^2}{3}J^\mu\right)\left(S_\mu+\frac{2\kappa^2}{3}J_\mu\right)\right]\right)}_{\text{of the form }\int e^{it^2}dt\text{ Gaussian integral over S}}\exp\left(-i\int d^4x\sqrt{-g}\frac{\kappa^2}{3}J_\mu J^\mu\right)\end{aligned}$$

The first exponential contracts to a Gaussian integral during the integration over S_μ, so it is a constant which can be absorbed in the norm of S. The value of this constant is calculated from the Gaussian integral over S seems to be $\frac{\sqrt{6\pi}}{3}\kappa(i+1)$. The second exponential has no dependence on S^μ, thus it remains intact. So we end up with the path integral depending only on the $J^\mu J_\mu$ term:

$$\begin{aligned}\mathcal{Z} &\propto \int D\Phi\ \exp\left(-i\int d^4x\sqrt{-g}\frac{\kappa^2}{3}J_\mu J^\mu\right)\\
\mathcal{Z} &\propto \int D\Phi\ \exp\left(-i\int d^4x\sqrt{-g}\frac{\kappa^2}{3}\left(-\frac{3}{4}j^5_\mu+\Phi\ \partial_\mu\right)\left(-\frac{3}{4}j^{5\mu}+\Phi\ \partial^\mu\right)\right)\\
\mathcal{Z} &\propto \int D\Phi\ \exp\left(-i\int d^4x\sqrt{-g}\frac{\kappa^2}{3}\left(\frac{9}{16}j^5_\mu j^{5\mu}-\frac{3}{2}j^{5\mu}\partial_\mu\Phi+\Phi^2\ \partial^2_\mu\right)\right)\\
\mathcal{Z} &\propto \int D\Phi\ \exp\left(i\int d^4x\sqrt{-g}\left(-\frac{1}{2}\frac{3\kappa^2}{8}j^5_\mu j^{5\mu}+\frac{\kappa^2}{2}j^{5\mu}\partial_\mu\Phi-\frac{\kappa^2}{3}\ (\partial_\mu\Phi)^2\right)\right)\end{aligned}$$

Rescaling the Φ pseudo-scalar field as $\Phi = \sqrt{\frac{3}{2\kappa^2}}b$ we get:

$$\mathcal{Z} \propto \int D\Phi \exp\left(i\int d^4x\sqrt{-g}\left(-\frac{1}{2}\frac{3\kappa^2}{8}j^5_\mu j^{5\mu} + \sqrt{\frac{3\kappa^2}{8}}j^{5\mu}\partial_\mu b - \frac{1}{2}\left(\partial_\mu b\right)^2\right)\right)$$

Finally, we set $\frac{1}{f_\phi} = \sqrt{\frac{3\kappa^2}{8}}$:

$$\mathcal{Z} \propto \int D\Phi \exp\left(i\int d^4x\sqrt{-g}\left(-\frac{1}{2f_\phi}j^5_\mu j^{5\mu} + \frac{1}{f_\phi}j^{5\mu}\partial_\mu b - \frac{1}{2}\left(\partial_\mu b\right)^2\right)\right) \tag{2.3.18}$$

The final form 2.3.18 of the path integral 2.3.17, reminds us of a kinetic term of a massless axion-like degree of freedom $b(x)$ coupled to fermions, which emerges from torsion through quantum anomalies. The constant f_b is the axion decay that determines the interaction strength between the axion and fermions. One can write:

$$\mathcal{Z} \propto \int D\Phi \exp\left(i\int d^4x\sqrt{-g}\left(-\frac{1}{2f_\phi}j^5_\mu j^{5\mu} + \frac{1}{f_\phi}\mathcal{G}(\omega, A) - \frac{1}{2}\left(\partial_\mu b\right)^2\right)\right) \tag{2.3.19}$$

Where the $\mathcal{G}(\omega, A)$, is the same anomalous term mentioned in 2.3.15. This redefinition shows that QED on a torsionful space is equivalent to QED on a torsionless space coupled to an axion. In this way, torsion becomes dynamical due to quantum anomalies and the current that is conserved in this case, is the $J^5 = j^5 + f_\phi d\phi$ from the axion equations of motion[3].

Since the effective field theory approach we made above, guarantees the conservation law for the current J^5, there should be a conserved "Torsion Charge":

$$Q_S = \int \star S \tag{2.3.20}$$

The model described in this section, includes only fermion interactions with Torsion for QED. Torsion is gravitational in nature which means interactions are allowed with all fermion species. Expanded to other groups e.g.non-Abelian SU(3) QCD group, one must sum over all possible interactions to find the total axial current. During the QCD cosmological era, chiral anomalies of the axial fermion current induce a breaking of the action shift-symmetry by introducing a potential of the form[4]:

$$V(b) = \int d^4x\sqrt{-g}\Lambda^4_{\mathrm{QCD}}\left[1 - cos\left(\frac{b}{f_b}\right)\right] \tag{2.3.21}$$

From the potential equation it appears that the action mass-term is introduced $m_b = \frac{\Lambda^2_{\mathrm{QCD}}}{f_b}$ and it can play the role of the dark matter component in the universe.

3 Torsion in string-inspired cosmology

3.1 String-inspired contorted cosmology and derivation of axion field

Before discussing the effects of torsion in cosmology, it is important to introduce some key concepts and the theoretical background. Currently, it is believed we are living in a dark matter-dominated era. Many models attempt to investigate this era and the most famous one is the ΛCDM model (Λ is the positive cosmological constant and CDM stands for Cold Dark Matter). Although this model works well on large cosmological scales, tensions arise when it is applied to smaller scales. For example, discrepancies exist between observed data and theoretical predictions of the Hubble parameter and the σ_8 parameter, which characterizes the growth of galactic structures.

One potential solution to these issues is to consider the vacuum in the ΛCDM model as a metastable de Sitter vacuum, where torsion is present. In this case, Dark Matter and Dark Energy could have pure geometrical origins, rather than being independent fields or particles.

3.1.1 String-induced gravitational action of the early bosonic universe

Starting from the early universe phase, where a bosonic gravitational theory arises from String theory, three primary fields compose the theory: the spin-0 dilaton, the spin-2 graviton and the spin-1 KR tensor field $B_{\mu\nu} = -B_{\nu\mu}$ (it arises naturally in the low-energy limit of the String theory). In this context, one can find out that torsion is induced naturally from string theory as a tensor field, without invoking the geometrical tensor we described above.
Due to the Abelian-gauge transformation of the Kalb-Ramond (KR) field $B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu}\theta_{\nu]}$, we can introduce the field strength $H_{\mu\nu\rho}$ in the same way as the electromagnetic tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This field strength $H_{\mu\nu\rho}$ is defined[5]:

$$\begin{aligned} H_{\mu\nu\rho} &= \frac{1}{2}(\partial_\mu B_{\nu\rho} - \partial_\mu B_{\rho\nu} + \partial_\nu B_{\rho\mu} - \partial_\nu B_{\mu\rho} + \partial_\rho B_{\mu\nu} - \partial_\rho B_{\nu\mu}) \\ &\overset{B_{\mu\nu}=-B_{\nu\mu}}{=} \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} = \partial_{[\mu} B_{\nu\rho]} \end{aligned} \tag{3.1.1}$$

and it reminds us of the typical contorsion tensor definition 2.1.11. Considering graviton, dilaton and the KR-field the general form of the action in 26-dimensional space is given[5]:

$$S_B = \int d^{26}x \frac{1}{2\kappa^2}\sqrt{-g}e^{-2\Phi}\left(R - \frac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} + ...\right) \tag{3.1.2}$$

By freezing out the dilaton field, set $\Phi = \Phi_0$ =constant, the action is simplified and a low-energy effective theory can be explored. After string compactification, (3+1)-dimensional particle phenomenology is not affected, so one can use the effective gravitational action expansion to the lowest $a' = M_s^2$ order ($\mathcal{O}((\alpha')^0)$). Ignoring the dilaton term, action can be re-expressed:

$$S_B = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R - \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu} + ...\right) \tag{3.1.3}$$

Where we normalised $\mathcal{H}_{\lambda\mu\nu} = \kappa^{-1}H_{\lambda\mu\nu}$ and ... represent the higher derivative terms. As mentioned above, comparing the actions from the previous analysis 2.2.9 one observes that $\mathcal{H}_{\lambda\mu\nu}$, is the corresponding contorsion tensor. This means that, the affine connection-Christoffel symbols- have a contributing term $\mathcal{H}_{\lambda\mu\nu}$ in the same way as in the previous analysis.:

$$\Gamma^\rho_{\mu\nu} = \mathring{\Gamma}^\rho_{\mu\nu} + \frac{\kappa}{\sqrt{3}}H^\rho_{\mu\nu} \tag{3.1.4}$$

In the same way we confronted quantum anomalies in the GR-torsion analysis, in this case we need to confront anomalies that appear due to gauge versus gravitational anomalies. Green and Schwartz added appropriate counterterms in the effective action by making a few modifications in the defined field strength of the KR-field. The field strength has now an extra modification with a Lorentz Ω_{3L} and a Yang-Mills Ω_{3Y}

gauge Chern-Simons terms

$$\mathcal{H} = dB + \frac{\alpha'}{8\kappa}\left(\Omega_{3L} - \Omega_{3Y}\right) \tag{3.1.5}$$

$$\Omega_{3L} = \omega^a{}_c \wedge d\omega^c{}_a + \frac{2}{3}\omega^a{}_c \wedge \omega^c{}_d \wedge \omega^d{}_a, \quad \Omega_{3Y} = A \wedge dA + A \wedge A \wedge A \tag{3.1.6}$$

This has as a consequence for the Bianchi identity to be expressed as:

$$d\mathcal{H} = d\left(dB + \frac{\alpha'}{8\kappa}\left(\Omega_{3L} - \Omega_{3Y}\right)\right) \overset{d^2=0}{=} d\left(\frac{\alpha'}{8\kappa}\left(\Omega_{3L} - \Omega_{3Y}\right)\right) \tag{3.1.7}$$

We notice that:

$$d(\Omega_{3L}) \overset{d^2=0}{=} d\omega^a{}_c \wedge d\omega^c{}_a + \frac{2}{3}(d\omega^a{}_c \wedge \omega^c{}_d \wedge \omega^d{}_a + \omega^a{}_c \wedge d\omega^c{}_d \wedge \omega^d{}_a + \omega^a{}_c \wedge \omega^c{}_d \wedge d\omega^d{}_a)$$

We know that: $d\omega^a{}_c = R^a{}_c - \omega^a{}_c \wedge \omega^c{}_a$. If we replace this equation in the $d(\Omega_{3L})$ and make indices rearrangements (detailed calculation are showed in personal notes), we get a term:

$$d(\Omega_{3L}) = R^a{}_c \wedge R^c{}_a = Tr(R \wedge R)$$

We follow the same procedure for $d\Omega_{3Y} = -Tr(F \wedge F)$, where $F = dA + A \wedge A$. Finally, the form of the derivative of the field strength can be written:

$$d\mathcal{H} = \frac{\alpha'}{8\kappa} Tr\left(R \wedge R - F \wedge F\right) \tag{3.1.8}$$

The non-zero term on the right-hand side of equation 3.1.8 represents a mixed quantum anomaly, arising from both gauge fields (F) and gravitational curvature (R). This type of anomaly indicates a violation of classical symmetries at the quantum level, involving both gauge and gravitational interactions. To address this anomaly, we will employ a similar approach to the one used for resolving quantum anomalies in the case of General Relativity (GR) with torsion, where additional counterterms or constraints are introduced to restore consistency at the quantum level.

In the effective (3+1)-dimensional spacetime of the early low-energy universe, where bosons exist as external fields of the theory, quantum anomalies are still present. Fermions do not emerge until the end of inflation, when they arise from the decay of the false vacuum. Under these conditions, the electromagnetic tensor A (gauge field) can be set to zero (since fermions do not exist yet), allowing the Bianchi identity to be reformulated. This sets the stage for addressing the quantum anomalies through the path integral of the action. Using the Bianchi identity from[3], we simplify its form and finally we get:

$$\varepsilon_{abc}{}^{\mu}\partial_\mu \mathcal{H}^{abc} = \frac{\alpha'}{16\kappa}\sqrt{-g}(R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} - F^{\mu\nu}F_{\mu\nu}) \overset{A=0}{=} -\sqrt{-g}\mathcal{G}(\omega) = \frac{\alpha'}{16\kappa}\partial_\mu\left(\sqrt{-g}\,\mathcal{K}^\mu(\omega)\right) \tag{3.1.9}$$

Where $\varepsilon = \sqrt{-g}\epsilon$, $\epsilon^{\mu\nu\rho\sigma} = \frac{sgn(g)}{\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}$ and $\tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}R^{\alpha\beta}{}_{\rho\sigma}$ the dual tensor of the curvature. The $\mathcal{K}^\mu$ tensor expresses the total derivative of the gravitational Chern-Simons anomalous terms.

Having expressed the mixed quantum anomaly given by the Bianchi identity 3.1.9, the next step is to implement it as a δ-functional constraint, exactly as we did in the contorted QED case to resolve the issue of the current conservation. The δ-functional that we are using is defined:

$$\Pi_x \delta\left(\varepsilon_{abc}{}^{\mu}\partial_\mu \mathcal{H}^{abc} - \frac{\alpha'}{16\kappa}\sqrt{-g}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right) \Rightarrow \Pi_x \delta\left(\varepsilon^{\mu\nu\rho\sigma}\partial_\mu \mathcal{H}_{\nu\rho\sigma} - \frac{\alpha'}{16\kappa}\sqrt{-g}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right)$$
$$= \int Db \, \exp\left(i\int d^4x\, \frac{b(x)}{\sqrt{3}}\left(\varepsilon^{\mu\nu\rho\sigma}\partial_\mu \mathcal{H}_{\nu\rho\sigma} - \frac{\alpha'}{16\kappa}\sqrt{-g}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right)\right) \tag{3.1.10}$$
$$= \int Db \, \exp\left(-i\int d^4x \,\sqrt{-g}\left(\frac{\partial_\mu b(x)}{\sqrt{3}}\epsilon_{\mu\nu\rho\sigma}\mathcal{H}^{\nu\rho\sigma} + \frac{b(x)}{\sqrt{3}}\frac{\alpha'}{16\kappa}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right)\right)$$

The path integral of the action can be expressed:

$$\mathcal{Z} = \int Dg\ DH\ \delta(\varepsilon^{\mu\nu\rho\sigma}\partial_\mu \mathcal{H}_{\nu\rho\sigma} - \frac{\alpha'}{16\kappa}\sqrt{-g}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma})\ exp\left(i\int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R - \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu}\right]\right)$$
$$= \int Dg\ DH\ Db\ exp\left(i\int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R - \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu} - \frac{\partial_\mu b(x)}{\sqrt{3}}\epsilon_{\mu\nu\rho\sigma}\mathcal{H}^{\nu\rho\sigma} - \frac{b(x)}{\sqrt{3}}\frac{\alpha'}{16\kappa}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right]\right)$$

We use the same mathematical trick as before-complete the square to form a gaussian integral. But before this we need to change the indices on the third term:

$$\partial_\mu b\, \epsilon_{\mu\nu\rho\sigma}\mathcal{H}^{\nu\rho\sigma} \xrightarrow[\sigma\to\nu,\ \mu\to\rho]{\nu\to\lambda,\ \rho\to\mu} \partial_\rho b\, \epsilon_{\rho\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu}$$

Square completion:

$$\mathcal{Z} = \int Dg\ DH\ Db\ \exp\left(i\int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R - \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu} - \frac{\partial_\rho b}{\sqrt{3}}\ \epsilon_{\rho\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu} - \frac{b}{\sqrt{3}}\frac{\alpha'}{16\kappa}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\right]\right)$$
$$= \int Dg\ DH\ Db\ \exp\left(i\int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R - \frac{b}{\sqrt{3}}\mathcal{G}(\omega) - \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu} - 2\left(\frac{1}{\sqrt{6}}\right)\left(\frac{\sqrt{6}}{2\sqrt{3}}\right)\partial_\rho b\ \epsilon_{\rho\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu}\right]\right)$$
$$= \int Dg\ DH\ Db\ \exp\left(i\int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R - \frac{b}{\sqrt{3}}\mathcal{G}(\omega) - \frac{1}{6}\mathcal{H}_{\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu}\right.\right.$$
$$\left.\left.-2\left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{2}}\right)\partial_\rho b\ \epsilon_{\rho\lambda\mu\nu}\mathcal{H}^{\lambda\mu\nu} - \frac{1}{2}\epsilon_{\rho\lambda\mu\nu}\epsilon^{\rho\lambda\mu\nu}\partial_\rho b\,\partial^\rho b + \frac{1}{2}\epsilon_{\rho\lambda\mu\nu}\epsilon^{\rho\lambda\mu\nu}\partial_\rho b\,\partial^\rho b\right]\right)$$

Remember $\epsilon_{\rho\lambda\mu\nu}\epsilon^{\rho\lambda\mu\nu} = 24$:

$$\mathcal{Z} = \int Dg\ DH\ Db\ \exp\left(i\int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R - \frac{b}{\sqrt{3}}\mathcal{G}(\omega) - \left[\frac{1}{\sqrt{6}}\mathcal{H}_{\lambda\mu\nu} + \frac{1}{\sqrt{2}}\partial_\rho b\ \epsilon^{\rho\lambda\mu\nu}\right]\right.\right.$$
$$\left.\left.\left[\frac{1}{\sqrt{6}}\mathcal{H}^{\lambda\mu\nu} + \frac{1}{\sqrt{2}}\partial^\rho b\ \epsilon_{\rho\lambda\mu\nu}\right] + 12\,\partial_\rho b\,\partial^\rho b\right]\right)$$
$$= \int Dg\ DH\ Db\ \exp\left(i\int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R - \frac{b}{\sqrt{3}}\mathcal{G}(\omega) - \frac{1}{6}\left[\mathcal{H}_{\lambda\mu\nu} + \sqrt{3}\partial_\rho b\ \epsilon^{\rho\lambda\mu\nu}\right]^2 + 12\,\partial_\rho b\,\partial^\rho b\right]\right)$$

The second term can be calculated during the path integration over DH, as a gaussian integral and give a constant value which will be multiplied with the rest of the exponential, and this means it can be absorbed in the norm of H and neglected. The rest of the factors do not have any dependence whatsoever on the H, torsion field and by rescaling the b field to be $b' = \sqrt{24}b$, the path integral is simplified to:

$$\mathcal{Z} = \int Dg\ Db\ \exp\left(i\int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R - \frac{b'}{6\sqrt{2}}\mathcal{G}(\omega) + \frac{1}{2}\,\partial_\rho b'\,\partial^\rho b'\right]\right)$$

Now integrating over the rest of the variables, Dg for graviton and Db for the axion-like pseudoscalar, we get the action:

$$S_B^{eff} = \int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R + \frac{\sqrt{2}\alpha'}{192\kappa}b(x)(R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma} - F^{\mu\nu}F_{\mu\nu}) + \frac{1}{2}\,\partial_\rho b\,\partial^\rho b + ...\right] \tag{3.1.11}$$

As noticed from the process above, $b(x)$ is a naturally induced from the string theory axion(one of the axions from string theory) that couples to gravitational Chern-Simons terms making torsion H, a dynamical parameter in the theory $\epsilon_{\mu\nu\rho\sigma}H^{\nu\rho\sigma} = \partial_\mu b(x)$. Since action is purely bosonic, 4-fermion interactions are not observed and the anomalies considered, are purely gravitational and are not related to fermion anomalies as we previously saw in QED. This gravitational model is a Chern-Simons modified gravity model.

3.1.2 Chern-Simons terms and Hirzebruch-Pontryagin topological density

Before rewriting the previous result 3.1.11 as a composition of three main action terms, it is essential to provide an explanation why we introduced the Chern-Simons gravitational terms.
Chern-Simons terms first introduced in 3-dimensional gauge field (Maxwell's theory) to study the impacts on what??. Based on[6], the additional CS-term in Maxwell's theory derives a topological Pontryagin density through its divergence which contributes to the final form of the Lagrangian. Due to a massive term appearing in the equations of motion, the photon acquires two polarizations with different dispersion relations for the frequency $\omega = \sqrt{|k|^2 \pm \mu|k|}$. This modification leads to physical consequences such as birefringence of the vacuum, however such phenomena have not been observed in nature based on measurements from distant galaxies, it is possible though to be produced in lab under conditions of high magnetic fields.
In this case, the form of the CS-terms is given in 3.1.6. In a similar way, a 4-dimensional topological current can be related to those terms, as already shown in 3.1.9, and provide an expression for the corresponding Pontryagin density, which will be then used to simplify the effective action. The related 4-current is given:

$$K^\mu = 2\epsilon^{\mu\alpha\beta\gamma}\left[\frac{1}{2}\Gamma^\sigma{}_{\alpha\tau}\partial_\beta\Gamma^\tau{}_{\gamma\sigma} + \frac{1}{3}\Gamma^\sigma{}_{\alpha\tau}\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma}\right] \tag{3.1.12}$$

Let us now calculate analytically the divergence of the four-dimensional topological current and prove the result noted in Bianchi's identity in 3.1.9. First, we calculate and simplify the partial derivative of 3.1.12:

$$\begin{aligned}
\partial_\mu K^\mu &= 2\partial_\mu\left(\epsilon^{\mu\alpha\beta\gamma}\left[\frac{1}{2}\Gamma^\sigma{}_{\alpha\tau}\partial_\beta\Gamma^\tau{}_{\gamma\sigma} + \frac{1}{3}\Gamma^\sigma{}_{\alpha\tau}\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma}\right]\right) \\
&= 2\epsilon^{\mu\alpha\beta\gamma}\left[\frac{1}{2}\partial_\mu\Gamma^\sigma{}_{\alpha\tau}\partial_\beta\Gamma^\tau{}_{\gamma\sigma} + \frac{1}{3}\partial_\mu\Gamma^\sigma{}_{\alpha\tau}\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma} + \frac{1}{3}\Gamma^\sigma{}_{\alpha\tau}\partial_\mu\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma} + \frac{1}{3}\Gamma^\sigma{}_{\alpha\tau}\Gamma^\tau{}_{\beta\eta}\partial_\mu\Gamma^\eta{}_{\gamma\sigma}\right] \\
&= \epsilon^{\mu\alpha\beta\gamma}\left[\partial_\mu\Gamma^\sigma{}_{\alpha\tau}\partial_\beta\Gamma^\tau{}_{\gamma\sigma} + 2\partial_\mu\Gamma^\sigma{}_{\alpha\tau}\Gamma^\tau{}_{\beta\eta}\Gamma^\eta{}_{\gamma\sigma}\right] \xrightarrow[\gamma\rightarrow\beta]{\nu\rightarrow\alpha,\ \alpha\rightarrow\beta} \\
&= \epsilon^{\mu\nu\alpha\beta}\left[\partial_\mu\Gamma^\sigma{}_{\nu\tau}\partial_\alpha\Gamma^\tau{}_{\beta\sigma} + 2\partial_\mu\Gamma^\sigma{}_{\nu\tau}\Gamma^\tau{}_{\alpha\eta}\Gamma^\eta{}_{\beta\sigma}\right]
\end{aligned} \tag{3.1.13}$$

We will now prove that the above expression is equal to $\frac{1}{2}\tilde{R}^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$. We can rename the indices and make some calculations as follows:

$$
\begin{aligned}
\frac{1}{2}\tilde{R}^{\sigma\kappa\mu\nu}R_{\sigma\kappa\mu\nu} &= \frac{1}{2}\tilde{R}^{\sigma}{}_{\tau}{}^{\mu\nu}R_{\sigma\kappa\mu\nu}g^{\kappa\tau} = \frac{1}{2}\tilde{R}^{\sigma}{}_{\tau}{}^{\mu\nu}R_{\sigma}{}^{\tau}{}_{\mu\nu} = \frac{1}{2}\tilde{R}^{\tau}{}_{\sigma}{}^{\mu\nu}R^{\tau}{}_{\sigma\mu\nu} = \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}R^{\tau}{}_{\sigma\alpha\beta}R^{\tau}{}_{\sigma\mu\nu} \\
&= \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\left(\partial_\nu\Gamma^{\tau}{}_{\mu\sigma} - \partial_\mu\Gamma^{\tau}{}_{\nu\sigma} + \Gamma^{\tau}{}_{\nu\eta}\Gamma^{\eta}{}_{\mu\sigma} - \Gamma^{\tau}{}_{\mu\eta}\Gamma^{\eta}{}_{\nu\sigma}\right)\left(\partial_\beta\Gamma^{\tau}{}_{\alpha\sigma} - \partial_\alpha\Gamma^{\tau}{}_{\beta\sigma} + \Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\alpha\sigma} - \Gamma^{\tau}{}_{\alpha\eta}\Gamma^{\eta}{}_{\beta\sigma}\right) \\
&= \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\Big(\partial_\nu\Gamma^{\tau}{}_{\mu\sigma}\partial_\beta\Gamma^{\tau}{}_{\alpha\sigma} - \partial_\nu\Gamma^{\tau}{}_{\mu\sigma}\partial_\alpha\Gamma^{\tau}{}_{\beta\sigma} + \partial_\nu\Gamma^{\tau}{}_{\mu\sigma}\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\alpha\sigma} - \partial_\nu\Gamma^{\tau}{}_{\mu\sigma}\Gamma^{\tau}{}_{\alpha\eta}\Gamma^{\eta}{}_{\beta\sigma} \\
&\quad -\partial_\mu\Gamma^{\tau}{}_{\nu\sigma}\partial_\beta\Gamma^{\tau}{}_{\alpha\sigma} + \partial_\mu\Gamma^{\tau}{}_{\nu\sigma}\partial_\alpha\Gamma^{\tau}{}_{\beta\sigma} - \partial_\mu\Gamma^{\tau}{}_{\nu\sigma}\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\alpha\sigma} + \partial_\mu\Gamma^{\tau}{}_{\nu\sigma}\Gamma^{\tau}{}_{\alpha\eta}\Gamma^{\eta}{}_{\beta\sigma} \\
&\quad \Gamma^{\tau}{}_{\nu\eta}\Gamma^{\eta}{}_{\mu\sigma}\partial_\beta\Gamma^{\tau}{}_{\alpha\sigma} - \Gamma^{\tau}{}_{\nu\eta}\Gamma^{\eta}{}_{\mu\sigma}\partial_\alpha\Gamma^{\tau}{}_{\beta\sigma} + \Gamma^{\tau}{}_{\nu\eta}\Gamma^{\eta}{}_{\mu\sigma}\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\alpha\sigma} - \Gamma^{\tau}{}_{\nu\eta}\Gamma^{\eta}{}_{\mu\sigma}\Gamma^{\tau}{}_{\alpha\eta}\Gamma^{\eta}{}_{\beta\sigma} \\
&\quad -\Gamma^{\tau}{}_{\mu\eta}\Gamma^{\eta}{}_{\nu\sigma}\partial_\beta\Gamma^{\tau}{}_{\alpha\sigma} + \Gamma^{\tau}{}_{\mu\eta}\Gamma^{\eta}{}_{\nu\sigma}\partial_\alpha\Gamma^{\tau}{}_{\beta\sigma} - \Gamma^{\tau}{}_{\mu\eta}\Gamma^{\eta}{}_{\nu\sigma}\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\alpha\sigma} + \Gamma^{\tau}{}_{\mu\eta}\Gamma^{\eta}{}_{\nu\sigma}\Gamma^{\tau}{}_{\alpha\eta}\Gamma^{\eta}{}_{\beta\sigma}\Big) \\
&\overset{\text{Levi-Civita}}{=} \frac{1}{4}\epsilon^{\mu\nu\alpha\beta}\left(4\partial_\mu\Gamma^{\tau}{}_{\nu\sigma}\partial_\alpha\Gamma^{\tau}{}_{\beta\sigma} + 4\partial_\mu\Gamma^{\tau}{}_{\nu\sigma}\Gamma^{\tau}{}_{\alpha\eta}\Gamma^{\eta}{}_{\beta\sigma} + 4\Gamma^{\tau}{}_{\mu\eta}\Gamma^{\eta}{}_{\nu\sigma}\partial_\alpha\Gamma^{\tau}{}_{\beta\sigma} + 4\Gamma^{\tau}{}_{\mu\eta}\Gamma^{\eta}{}_{\nu\sigma}\Gamma^{\tau}{}_{\alpha\eta}\Gamma^{\eta}{}_{\beta\sigma}\right)
\end{aligned}
$$

The last term is totally antisymmetric when it is multiplied by the Levi-Civita symbol, thus it vanishes completely from the expression. Finally, we get:

$$
\begin{aligned}
\frac{1}{2}\tilde{R}^{\sigma\kappa\mu\nu}R_{\sigma\kappa\mu\nu} &= \epsilon^{\mu\nu\alpha\beta}\left(\partial_\mu\Gamma^{\tau}{}_{\nu\sigma}\partial_\alpha\Gamma^{\tau}{}_{\beta\sigma} + \partial_\mu\Gamma^{\tau}{}_{\nu\sigma}\Gamma^{\tau}{}_{\alpha\eta}\Gamma^{\eta}{}_{\beta\sigma} + \partial_\alpha\Gamma^{\tau}{}_{\mu\eta}\Gamma^{\eta}{}_{\nu\sigma}\Gamma^{\tau}{}_{\beta\sigma}\right) \\
&\overset{\text{Levi-Civita}}{=} \epsilon^{\mu\nu\alpha\beta}\left(\partial_\mu\Gamma^{\tau}{}_{\nu\sigma}\partial_\alpha\Gamma^{\tau}{}_{\beta\sigma} + 2\partial_\mu\Gamma^{\tau}{}_{\nu\sigma}\Gamma^{\tau}{}_{\alpha\eta}\Gamma^{\eta}{}_{\beta\sigma}\right) \overset{3.1.12}{=} \partial_\mu K^\mu
\end{aligned}
\tag{3.1.14}
$$

Based on the above result, the effective action of the early bosonic universe with the pseudoscalar degrees of freedom 3.1.11 can be expressed using the topological current K^μ:

$$
\begin{aligned}
S_B^{eff} &= \int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R + \frac{\sqrt{2}\alpha'}{192\kappa}b(x)(R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}) + \frac{1}{2}\,\partial_\rho b\,\partial^\rho b + ...\right] \\
&= \int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}R + \frac{\sqrt{2}\alpha'}{96\kappa}b(x)(u_\mu K^\mu) + \frac{1}{2}\,\partial_\rho b\,\partial^\rho b + ...\right] \\
&= S_{grav} + S_{b-grav} + S_b
\end{aligned}
\tag{3.1.15}
$$

Where $u_\mu = \nabla_\mu b(x)$.

3.1.3 Derivation of the modified Einstein's equations

Using the alternative definition of the effective action one can derive the modified Einstein equations, by a metric variation application.
Some useful expressions for the analysis are:

- the metric determinant variation is:

$$\delta g = g g^{\mu\nu}\delta g_{\mu\nu} \tag{3.1.16}$$

- the square root of the metric determinant is:

$$\delta(\sqrt{-g}) = -\frac{\delta g}{2\sqrt{-g}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g_{\mu\nu} \tag{3.1.17}$$

- Christoffel symbol variation with respect to the metric:

$$\delta\Gamma^{\sigma}_{\alpha\tau} = \frac{g^{\sigma\nu}}{2}(\nabla_{\alpha}\delta g_{\nu\tau} + \nabla_{\tau}\delta g_{\nu\alpha} - \nabla_{\nu}\delta g_{\alpha\tau}) \tag{3.1.18}$$

Starting with the gravitational term, it is known from the standard theory that:

$$\begin{aligned}
\delta S_{grav} &= \delta\left(\int d^4x\sqrt{-g}\frac{R}{2\kappa^2}\right) = \delta\left(\int d^4x\sqrt{-g}\frac{g^{\mu\nu}R_{\mu\nu}}{2\kappa^2}\right) \\
&= \frac{1}{2\kappa^2}\int d^4x\left(\sqrt{-g}\ \delta g^{\mu\nu}R_{\mu\nu} - \delta(\sqrt{-g})R\right) \\
&\overset{3.1.17}{=} \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\underbrace{\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)}_{G_{\mu\nu}}\delta g^{\mu\nu}
\end{aligned}$$

$$\delta S_{grav} = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu} \tag{3.1.19}$$

Afterwards, metric variations over S_b will be calculated:

$$\begin{aligned}
\delta S_b &= \frac{1}{2}\delta\left(\int d^4x\sqrt{-g}\partial_{\mu}b\ \partial^{\mu}b\right) = \frac{1}{2}\delta\left(\int d^4x\sqrt{-g}g^{\mu\nu}\partial_{\mu}b\ \partial_{\nu}b\right) \\
&= \frac{1}{2}\int d^4x\left(\sqrt{-g}\ \delta g^{\mu\nu}\partial_{\mu}b\ \partial_{\nu}b - \delta(\sqrt{-g})\partial_{\rho}b\ \partial^{\rho}b\right)
\end{aligned}$$

Through 3.1.17:

$$\delta S_b = \int d^4x\sqrt{-g}\left(\frac{1}{2}\partial_{\mu}b\ \partial_{\nu}b - \frac{1}{4}g_{\mu\nu}\partial_{\rho}b\ \partial^{\rho}b\right)\delta g^{\mu\nu} \tag{3.1.20}$$

Finally, the gravity-pseudoscalar coupling action term will be computed:

$$\begin{aligned}
\delta S_{b-grav} &= \frac{\sqrt{2}a'}{192}\delta\left(\int d^4x\sqrt{-g}\ 2\ b(x)\partial_{\mu}K^{\mu}\right) = \frac{\sqrt{2}a'}{96}\delta\left(\int d^4x\sqrt{-g}b(x)\partial_{\mu}K^{\mu}\right) \\
&= \frac{\sqrt{2}a'}{96}\int d^4x\left(\delta(\sqrt{-g})b(x)\partial_{\mu}K^{\mu} + \sqrt{-g}\ b(x)\ \delta(\partial_{\mu}K^{\mu})\right)
\end{aligned}$$

We will calculate the quantity δK^{μ}, taking into account the properties of the Levi-Civita tensor and the symmetrization of the Christoffel symbols:

$$\begin{aligned}
\delta K^{\mu} &= \delta\left(2\epsilon^{\mu\alpha\beta\gamma}\left[\frac{1}{2}\Gamma^{\sigma}{}_{\alpha\tau}\partial_{\beta}\Gamma^{\tau}{}_{\gamma\sigma} + \frac{1}{3}\Gamma^{\sigma}{}_{\alpha\tau}\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\gamma\sigma}\right]\right) \\
&= 2\epsilon^{\mu\alpha\beta\gamma}\left[\frac{1}{2}\delta\Gamma^{\sigma}{}_{\alpha\tau}\partial_{\beta}\Gamma^{\tau}{}_{\gamma\sigma} + \frac{1}{2}\Gamma^{\sigma}{}_{\alpha\tau}\partial_{\beta}\delta\Gamma^{\tau}{}_{\gamma\sigma} + \frac{1}{3}\delta\Gamma^{\sigma}{}_{\alpha\tau}\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\gamma\sigma} + \frac{1}{3}\Gamma^{\sigma}{}_{\alpha\tau}\delta\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\gamma\sigma} + \frac{1}{3}\Gamma^{\sigma}{}_{\alpha\tau}\Gamma^{\tau}{}_{\beta\eta}\delta\Gamma^{\eta}{}_{\gamma\sigma}\right]
\end{aligned}$$

- The 2$^{\text{nd}}$ term becomes:

$$\epsilon^{\mu\alpha\beta\gamma}\Gamma^{\sigma}{}_{\alpha\tau}\partial_{\beta}\delta\Gamma^{\tau}{}_{\gamma\sigma} = \epsilon^{\mu\beta\gamma\alpha}\partial_{\beta}\Gamma^{\sigma}{}_{\gamma\tau}\delta\Gamma^{\tau}{}_{\alpha\sigma} \overset{\sigma\leftrightarrow\tau}{=} \epsilon^{\mu\beta\gamma\alpha}\partial_{\beta}\Gamma^{\tau}{}_{\gamma\sigma}\delta\Gamma^{\sigma}{}_{\alpha\tau} \tag{3.1.21}$$

- The 4$^{\text{th}}$ term becomes:

$$\frac{1}{3}\epsilon^{\mu\alpha\beta\gamma}\Gamma^{\sigma}{}_{\alpha\tau}\delta\Gamma^{\tau}_{\beta\eta}\Gamma^{\eta}{}_{\gamma\sigma} \overset{\alpha\leftrightarrow\beta}{=} -\frac{1}{3}\epsilon^{\mu\beta\alpha\gamma}\Gamma^{\sigma}{}_{\beta\tau}\delta\Gamma^{\tau}_{\alpha\eta}\Gamma^{\eta}{}_{\gamma\sigma} \overset{\text{permute}}{=} -\frac{1}{3}\epsilon^{\mu\alpha\beta\gamma}\Gamma^{\eta}{}_{\beta\sigma}\delta\Gamma^{\sigma}_{\alpha\tau}\Gamma^{\tau}{}_{\gamma\eta} \tag{3.1.22}$$

- The 5$^{\text{th}}$ term becomes:

$$\frac{1}{3}\epsilon^{\mu\alpha\beta\gamma}\Gamma^{\sigma}{}_{\alpha\tau}\Gamma^{\tau}_{\beta\eta}\delta\Gamma^{\eta}{}_{\gamma\sigma} \overset{\alpha\leftrightarrow\gamma}{=} -\frac{1}{3}\epsilon^{\mu\gamma\beta\alpha}\Gamma^{\sigma}{}_{\gamma\tau}\delta\Gamma^{\tau}_{\beta\eta}\Gamma^{\eta}{}_{\alpha\sigma} \overset{\text{permute}}{=} -\frac{1}{3}\epsilon^{\mu\alpha\beta\gamma}\Gamma^{\tau}{}_{\gamma\eta}\Gamma^{\eta}_{\beta\sigma}\delta\Gamma^{\sigma}{}_{\alpha\tau} \tag{3.1.23}$$

δK^{μ} becomes:

$$\begin{aligned}
\delta K^{\mu} &= 2\epsilon^{\mu\alpha\beta\gamma}\left[\frac{1}{2}\partial_{\beta}\Gamma^{\tau}{}_{\gamma\sigma} - \frac{1}{2}\partial_{\gamma}\Gamma^{\tau}{}_{\beta\sigma} + \frac{1}{3}\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\gamma\sigma} - \frac{1}{3}\Gamma^{\eta}{}_{\beta\sigma}\Gamma^{\tau}{}_{\gamma\eta} - \frac{1}{3}\Gamma^{\tau}{}_{\gamma\eta}\Gamma^{\eta}{}_{\beta\sigma}\right]\delta\Gamma^{\sigma}{}_{\alpha\tau} \\
&= \epsilon^{\mu\alpha\beta\gamma}\left[\partial_{\beta}\Gamma^{\tau}{}_{\gamma\sigma} - \partial_{\gamma}\Gamma^{\tau}{}_{\beta\sigma} + \frac{2}{3}\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\gamma\sigma} - \frac{4}{3}\Gamma^{\tau}{}_{\gamma\eta}\Gamma^{\eta}{}_{\beta\sigma}\right]\delta\Gamma^{\sigma}{}_{\alpha\tau} \\
&= \epsilon^{\mu\alpha\beta\gamma}\underbrace{[\partial_{\beta}\Gamma^{\tau}{}_{\gamma\sigma} - \partial_{\gamma}\Gamma^{\tau}{}_{\beta\sigma} + \frac{2}{3}\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\gamma\sigma} + \frac{1}{3}\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\gamma\sigma} - \Gamma^{\tau}{}_{\gamma\eta}\Gamma^{\eta}{}_{\beta\sigma}}_{R^{\tau}{}_{\sigma\gamma\beta}} \\
&\quad - \frac{1}{3}\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\gamma\sigma} + \Gamma^{\tau}{}_{\gamma\eta}\Gamma^{\eta}{}_{\beta\sigma} - \frac{4}{3}\Gamma^{\tau}{}_{\gamma\eta}\Gamma^{\eta}{}_{\beta\sigma}]\delta\Gamma^{\sigma}{}_{\alpha\tau} \\
&= \epsilon^{\mu\alpha\beta\gamma}\left[R^{\tau}{}_{\sigma\gamma\beta} - \frac{1}{3}\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\gamma\sigma} - \frac{1}{3}\Gamma^{\tau}{}_{\gamma\eta}\Gamma^{\eta}{}_{\beta\sigma}\right]\delta\Gamma^{\sigma}{}_{\alpha\tau}
\end{aligned}$$

The last two terms cancel out under Levi-Civita permutations:

$$\epsilon^{\mu\alpha\beta\gamma}\left[\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\gamma\sigma} + \Gamma^{\tau}{}_{\gamma\eta}\Gamma^{\eta}{}_{\beta\sigma}\right] = \epsilon^{\mu\alpha\beta\gamma}\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\gamma\sigma} + \epsilon^{\mu\alpha\gamma\beta}\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\gamma\sigma} = \epsilon^{\mu\alpha\beta\gamma}\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\gamma\sigma} - \epsilon^{\mu\alpha\beta\gamma}\Gamma^{\tau}{}_{\beta\eta}\Gamma^{\eta}{}_{\gamma\sigma} = 0$$

Thus, the final result for the topological current variation will be:

$$\delta K^{\mu} = \epsilon^{\mu\alpha\beta\gamma}R^{\tau}{}_{\sigma\gamma\beta}\delta\Gamma^{\sigma}{}_{\alpha\tau} \tag{3.1.24}$$

Going back to the b-gravity coupling variation:

$$\begin{aligned}
\delta S_{b-grav} &= \frac{\sqrt{2}a'}{96}\int d^4x\sqrt{-g}\ \nabla_{\mu}b(x)\ \epsilon^{\mu\alpha\beta\gamma}R^{\tau}{}_{\sigma\gamma\beta}\delta\Gamma^{\sigma}{}_{\alpha\tau} \\
&\overset{3.1.18}{=} \frac{\sqrt{2}a'}{96}\int d^4x\sqrt{-g}\ \nabla_{\mu}b(x)\ \epsilon^{\mu\alpha\beta\gamma}R^{\tau}{}_{\sigma\gamma\beta}\frac{g^{\sigma\nu}}{2}(\nabla_{\alpha}\delta g_{\nu\tau} + \nabla_{\tau}\delta g_{\nu\alpha} - \nabla_{\nu}\delta g_{\alpha\tau}) \\
&= \frac{\sqrt{2}a'}{192}\int d^4x\sqrt{-g}\ \nabla_{\mu}b(x)\ \epsilon^{\mu\alpha\beta\gamma}R^{\tau\nu}{}_{\gamma\beta}(\nabla_{\alpha}\delta g_{\nu\tau} + \nabla_{\tau}\delta g_{\nu\alpha} - \nabla_{\nu}\delta g_{\alpha\tau}) \\
&= \frac{\sqrt{2}a'}{192}\int d^4x\sqrt{-g}\ \nabla_{\mu}b(x)\ \epsilon^{\mu\alpha\beta\gamma}(\underbrace{R^{\tau\nu}{}_{\gamma\beta}\nabla_{\alpha}\delta g_{\nu\tau}}_{\text{does not contribute}} + R^{\tau\nu}{}_{\gamma\beta}\nabla_{\tau}\delta g_{\nu\alpha} - \underbrace{R^{\tau\nu}{}_{\gamma\beta}\nabla_{\nu}\delta g_{\alpha\tau}}_{\tau\leftrightarrow\nu}) \\
&= \frac{\sqrt{2}a'}{96}\int d^4x\sqrt{-g}\ \nabla_{\mu}b(x)\ \epsilon^{\mu\alpha\beta\gamma}R^{\tau\nu}{}_{\gamma\beta}\nabla_{\tau}\delta g_{\nu\alpha}
\end{aligned}$$

With partial integration, we get:

$$= \frac{\sqrt{2}a'}{96}\int d^4x\sqrt{-g}\ \left(\nabla_{\tau}\left[\nabla_{\mu}b(x)\ \epsilon^{\mu\alpha\beta\gamma}R^{\tau\nu}{}_{\gamma\beta}\delta g_{\nu\alpha}\right] + \nabla_{\mu}b(x)\ \epsilon^{\mu\alpha\beta\gamma}\nabla_{\tau}R^{\tau\nu}{}_{\gamma\beta}\delta g_{\nu\alpha}\right)$$

Using Stoke's theorem on the second term, it vanishes over the boundary. So the first term only survives:

$$\begin{aligned}\delta S_{b-grav} &= \frac{\sqrt{2}a'}{96}\int d^4x\sqrt{-g}\ \nabla_\tau\left[\nabla_\mu b(x)\ \epsilon^{\mu\alpha\beta\gamma}R^{\tau\nu}{}_{\gamma\beta}\right]\delta g_{\nu\alpha} \\ &= \frac{\sqrt{2}a'}{96}\int d^4x\sqrt{-g}\ \left[\nabla_\tau\nabla_\mu b(x)\ \underbrace{\epsilon^{\mu\alpha\beta\gamma}R^{\tau\nu}{}_{\gamma\beta}}_{-2R^{\tau\nu\mu\alpha}}+\nabla_\mu b(x)\ \epsilon^{\mu\alpha\beta\gamma}\nabla_\tau R^{\tau\nu}{}_{\gamma\beta}\right]\delta g_{\nu\alpha} \\ &= \frac{\sqrt{2}a'}{96}\int d^4x\sqrt{-g}\ \left[-2\nabla_\tau\nabla_\mu b(x)\ R^{\tau\nu\mu\alpha}+\nabla_\mu b(x)\ \epsilon^{\mu\alpha\beta\gamma}\nabla_\tau R^{\tau\nu}{}_{\gamma\beta}\right]\delta g_{\nu\alpha}\end{aligned}$$

Now using Bianchi identity we can write:

$$\begin{aligned}&\nabla_\tau R^{\tau\nu}{}_{\gamma\beta} = \nabla_\gamma R^\nu{}_\beta - \nabla_\beta R^\nu{}_\gamma \Rightarrow \\ &\qquad \epsilon^{\mu\alpha\beta\gamma}\nabla_\tau R^{\tau\nu}{}_{\gamma\beta} = \epsilon^{\mu\alpha\beta\gamma}\nabla_\gamma R^\nu{}_\beta - \epsilon^{\mu\alpha\beta\gamma}\nabla_\beta R^\nu{}_\gamma = \epsilon^{\mu\alpha\beta\gamma}\nabla_\gamma R^\nu{}_\beta - \epsilon^{\mu\alpha\gamma\beta}\nabla_\gamma R^\nu{}_\beta \\ &\qquad = 2\epsilon^{\mu\alpha\beta\gamma}\nabla_\gamma R^\nu{}_\beta\end{aligned} \tag{3.1.25}$$

Through 3.1.25:

$$\delta S_{b-grav} = \frac{\sqrt{2}a'}{96}\int d^4x\sqrt{-g}\ \left[-2\nabla_\tau\nabla_\mu b(x)\ R^{\tau\nu\mu\alpha}+2\nabla_\mu b(x)\nabla_\tau\epsilon^{\mu\alpha\beta\gamma}\nabla_\gamma R^\nu{}_\beta\right]\delta g_{\nu\alpha} \xrightarrow{\alpha\to\mu}$$

$$\delta S_{b-grav} = \frac{\sqrt{2}a'}{192}\ 4\int d^4x\sqrt{-g}\ \underbrace{\left[\nabla_\mu b(x)\nabla_\tau\epsilon^{\mu\alpha\beta\gamma}\nabla_\gamma R^\nu{}_\beta - \nabla_\tau\nabla_\mu b(x)R^{\tau\nu\mu\alpha}\right]}_{C^{\mu\nu}}\delta g_{\nu\alpha} \tag{3.1.26}$$

3.1.4 Running Vacuum Model (RVM)

3.2 Cosmological evolution of the String-induced model

3.3 Discussion about impacts on cosmology

In the model under consideration, the KR-axion field plays a crucial role in driving inflation. The axion is part of the massless gravitational multiplet in string theory, and its coupling to the gravitational Chern-Simons (CS) term leads to important physical effects, including inflation. Assuming the existence of sources for primordial Gravitational Waves (GW) on an FLRW-inflationary phase of the universe, the Hubble parameter H is approximately a constant (slowly varying). In this way, it has been proved that the gravitational anomaly integrated over GW perturbations, with spatial momenta k up to a cutoff value μ at Ultra-Violet, is analogous to $H^3\mu^4$[7]:

$$\langle R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\rangle \propto H^3\mu^4 n_* \tag{3.3.1}$$

Where n_* is the number density of the sources of GW. In the early universe, the density becomes time-independent and thus, there is dependence only on Hubble constant. This condensation of primordial GW contributes significantly to the total energy density of the universe during inflation and provides a dynamic mechanism for driving inflation.

In the same early universe now,[7] there is a metastable de Sitter spacetime in which the KR axion field $b(x)$ evolves in time during inflation, and its derivative is proportional to the approximately constant Hubble parameter:

$$\dot{b} \approx \sqrt{2\epsilon}HM_{Pl} \Rightarrow b(t) \approx \sqrt{2\epsilon}HM_{Pl}t + b(0) \tag{3.3.2}$$

where $\epsilon = \mathcal{O}(10^{-2})$ a phenomenological parameter so that it satisfies Planck data. The axion field evolves linearly with cosmic time, since its derivative is constant, leading to an axion-driven inflationary phase.
Using the Running-Vacuum model scenario, in a de sitter space ($P_{rvm} = -\rho_{rvm}$), the energy density is described:

$$\rho_{total} = \rho_b + \rho_{\mathrm{gCS}} + \rho_{\mathrm{condensate}} \propto H^4 \tag{3.3.3}$$

where $\rho_b, \rho_{\mathrm{gCS}}$ and $\rho_{\mathrm{condensate}}$ are the energy density contributions from the b field, the gravitational CS term $R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}$ and the condensation of primordial GW. This analogy, indicates that inflation does not need an inflaton-like external field for the inflation. Instead inflation is self-driven by the dependence on a non-linear term of H^4, of the running vacuum energy density.

4 Conclusion

In conclusion, the study of torsion within both geometrical General Relativity (GR) and string-inspired models offers a deep understanding of the potential extensions of classical gravity and cosmology.

In the geometrical framework of GR, based on the Einstein-Cartan-Sciama-Kibble (ECSK) theory, spacetime is equipped with both curvature and torsion. In this setup, torsion becomes a dynamical variable directly related to the spin density of matter (fermions) and introduces new degrees of freedom. Focusing on QED, it has been shown that torsion interactions with fermionic currents are non-trivial and quite important at high spin densities. More specifically, studies have shown that the torsion-spinor coupling, lead to the replacing of Big Bang singularities with a Big Bounce scenario, providing right after, a non-singular extremely rapid expansion of the universe without requiring inflation[8]. This approach is quite different from the string-induced torsion that we explored in the second part of the assignment.

On the other hand, the string-inspired model, provided a more dynamic and far-reaching role for torsion. Here, torsion arises from the Kalb-Ramond (KR) field, a component of the massless bosonic sector of string theory(Early bosonic universe effective theory). This torsion is associated with a totally antisymmetric field strength, which, after compactification, leads to the appearance of a dynamical axion-like pseudoscalar field. Unlike the geometric torsion of GR, string-induced torsion couples to gravitational anomalies via the Chern-Simons term, playing an important role in the early universe's evolution. In cosmology, string-induced torsion offers significant insights into several issues. It can drive inflation through gravitational wave condensates, contributing to the vacuum energy in the early universe. This leads to a string running vacuum model (RVM), where the Hubble parameter evolves with the vacuum energy, potentially providing a self-consistent inflationary scenario without the need for an external inflaton field. Additionally, the KR axion, arising from torsion, provides a potential candidate for dark matter, offering a geometric origin to the dark sector of the universe.

Thus, while the torsion in ECSK theory follows a different approach in explaining the beginning of the universe, string-induced torsion seems to explain better the new possibilities in understanding the early universe's dynamics, inflation, and the dark sector. These insights provide a promising direction for future investigations into the role of torsion in fundamental physics and cosmology.

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