Approximating Nash Equilibria Using Small-Support Strategies

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Finding a Nash equilibrium of a game and in particular a bimatrix game is one of the most central problems in algorithmic game theory. Recently it was proved that it is not possible to solve this problem or give a fully polynomial time approximation scheme for it unless PPAD is in P [5, 2, 3].

I the light of the above results, the question of finding approximate equilibria emerges as an important open problem. Is there a polynomial time approximation scheme for computing an equilibria? What is the best achievable approximation? An α-approximate equilibria is a pair of mixed strategies (X, Y) with payoffs (x, y) such that if the best response to Y has payoff x' and the best response to X has payoff y' then $x \geq \alpha x'$ and $y \geq \alpha y'$. Like [4], one can adopt an additive notion for the approximation, but it will not affect our result here.

Surprisingly, the best approximation algorithm known for this problem is a linear-time algorithm that finds a $\frac{1}{2}$-approximate equilibria by examining all strategies with support of size at most 2 [4]. The support of a strategy is the set of pure strategies used to construct it.

A natural question is whether it is possible to improve the factor $\frac{1}{2}$ by searching over strategies with larger support. Althöfer [1] (for zero-sum games) and Lipton *et al.* [6] showed that there is always an ϵ-approximate Nash equilibria with support of size $O(\frac{\log n}{\epsilon^2})$.

We will show that the above two results are *asymptotically optimum*. We prove that it is not possible to improve the factor $\frac{1}{2}$, if the support of both players are less than or equal to $\log_2 n - 2\log_2 \log n$ improving the factor $\frac{1}{4}$ given in [1, 4].

Furthermore, we prove that for any $0 < \epsilon < 1$, it is not possible to find a $\frac{1}{1+\epsilon}$-approximate equilibria using strategies of support $O(\frac{\log n}{\epsilon^2})$. We also show that it is not possible to find a $(\frac{1}{2} + \epsilon)$-equilibria even if we limit the support of one of the players to $\frac{\log m}{1-\log \epsilon}$. All our negative results apply to both zero-sum and $0 - 1$ bimatrix games as well.

On the flip side, we give an algorithm that finds an α-approximate Nash equilibria for α slightly bigger than $\frac{1}{2}$. In the solution of our algorithm, one of the players play a pure strategy but the support of the strategy of the other player may be arbitrarily large.

Theorem 1 *For any large enough m and $0 \leq \epsilon \leq 1$, there exists a constant-sum $0/1$ game of size m such that if we limit the row player to strategies with support of size less than $\frac{\log m}{1-\log \epsilon}$, it is not possible to achieve an α-approximate Nash equilibrium for $\alpha \leq \frac{1}{2}(1+\epsilon)$.*

Proof : Let $s = \lfloor \frac{\log m}{1-\log \epsilon} \rfloor$ and $n = \lfloor \frac{s}{\epsilon} \rfloor$. For such values for s and n we have $\binom{n}{s} < (\frac{en}{s})^s \leq m$. Hence, we can generate an $n \times m$ matrix C having all possible columns consisting of s 1's and $n - s$

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0's. Obtain R from C by exchanging 0's and 1's. Suppose the row player plays at most s rows. Then some column has only 1's in these rows in C. Thus, the best response for column player has payoff 1. Furthermore, row player can get payoff $1-\frac{s}{n}$ by playing uniformly on all of the n rows, independent of the strategy of the column player. So the sum of the payoffs of the best responses of the players is $2-\frac{s}{n}$, while sum of their payoffs is 1. Therefore, $\frac{1}{2}(1+\frac{s}{n})$ approximation is not possible. $\square$

Theorem 2 *Consider a zero-sum game where R is an n by n matrix with entries chosen uniformly at random from $\{0,1\}$ and C is the matrix obtained from R by exchanging 0 and 1. Then, with high probability, for any $\alpha > \frac{1}{2}$, no pair of strategies with supports of size smaller than $\log_2 n - 2\log_2 \log n$ has an α-approximate Nash equilibrium. Furthermore, for any $0<\epsilon<1$, with high probability, no pair of strategies with supports of size smaller than $O(\frac{\log n}{\epsilon^2})$ has a $\frac{1}{1+\epsilon}$-approximate Nash equilibrium.*

Proof : Let $k = \log_2 n - 2\log_2 \log n$. We show with high probability, for any choice of k columns of R, there is a row of R which has all 1's in these k columns. Similarly, for any choice of k rows of C, some row of C has all 1's in these k rows. Therefore with high probability, the best response of each player has payoff 1. But the payoffs of the two players sum up to 1, so one of the players will have payoff at most $\frac{1}{2}$. Hence, there is no α-approximate Nash equilibrium with $\alpha > \frac{1}{2}$.

The probability that in some row of R, some k chosen columns will not be all 1 in that row is $1-2^{-k}$, so the probability that this will be the case in all n rows of R is $(1-2^{-k})^n$. Since $n2^{-k} > k\log n$, the probability that this will happen in some choice of k columns is $(1-2^{-k})^n n^k << 1$. This proves the first part of the Theorem.

For the second part, let S be the set of the rows in the support of row player. We first prove the theorem for the case that the distributions of the mixed strategies of the players are uniform over their supports and the size of the supports is k. For column j, let s_j be the sum of the entries of column j that are in A_R, e.g. $s_j = \sum_{i\in S} C_{ij}$. We claim with high probability, there exists a column j such that $s_j \geq (1+\epsilon)\frac{k}{2}$. Hence, the payoff of column player by choosing column j is at least $\frac{(1+\epsilon)}{2}$. Similarly for row player, with high probability, the best response has payoff at least $\frac{1+\epsilon}{2}$. But the sum of the payoffs of both players is 1. Therefore, there is no α-approximate Nash equilibrium.

To prove the claim, consider an arbitrary column j. Let $s = \lceil (1+\epsilon)k/2 \rceil$. By Sterling's formula, there exist constants c_1 and c_2 such that:

$$\begin{aligned}
P[s_j \geq s] &\geq P[s_j = s] = \frac{1}{2^k}\binom{k}{s} \\
&\geq \frac{c_1}{2^k\sqrt{k}} \frac{(k)^k}{((1+\epsilon)k/2)^{(1+\epsilon)k/2}((1-\epsilon)k/2)^{(1-\epsilon)k/2}} \\
&= \frac{c_1}{\sqrt{k}} e^{-\frac{k}{2}((1+\epsilon)\log(1+\epsilon)+(1-\epsilon)\log(1-\epsilon))} \qquad (1)\\
&\geq \frac{1}{\sqrt{k}} e^{-c_2 k\epsilon^2} \qquad (2)
\end{aligned}$$

Now we can choose a constant c such that $k = \frac{\log n}{c\epsilon^2}$, and $P[s_j \geq s] \geq n^{-1/2}$. So the probability that none of the columns has at least $(1+\epsilon)\frac{k}{2}$ 1's in these k rows is less than $(1-n^{-1/2})^n$. The probability that this happens in some choice of k columns is less than $(1-n^{-1/2})^n n^k << 1$. This proves the result for uniform strategies over the support.

For nonuniform strategies, we limit the strategy space to the strategies that the probability of playing any rows or columns is equal to $\frac{j}{k^2}$, for some integer $0 \leq j \leq k^2$. This gives an additive error of $\frac{1}{k}$ in approximation of s_j's, which is negligible for large n. The number of such strategies can be closely approximated by $(k^2)^k$. Without loss of generality, assume that rows in S are the rows 1 to k. Let $r = (r_1, \ldots, r_k)$ be the distribution of the mixed strategy. For $1 \leq i \leq k$, the i'th *cyclic permutation* of r is the mixed strategy with distribution $(r_{1+i}, \ldots, r_{k+i})$ where addition is in module k. Therefore, the payoff of column player by playing column j, when the row player strategy is the i'th cyclic permutation, is $\sum_{l=1}^{k} r_{l+i} c_{lj}$. Because the average payoffs of all of the k cyclic permutations is equal to s_j, at least one of these permutations guarantees payoff s_j for the column player. The entries of C are chosen independently at randomly, so by (2), $P[s_j \geq s] \geq \frac{1}{k\sqrt{k}} e^{-c_2 k \epsilon^2}$. So we can choose constant c' such that $k = \frac{\log n}{c'\epsilon^2}$, and $P[s_j \geq s] \geq n^{-1/2}$. The probability that the total payoff of the best responses of the players, for all pair of the strategies be less than $1+\epsilon$, is at most $2(1 - P[s_j \geq s])^n (k^2)^k n^k = o(1)$ which completes the proof. □

Theorem 3 *Let R and C be arbitrary matrices of size at most n. There exists a function $f(n) = (2 + o(1))^n$ such that for any $0 < \epsilon < \frac{1}{4nf(n)}$ there is a pure row strategy and a mixed column strategy that gives an α-approximate Nash equilibrium with $\alpha = \frac{1}{2}(1+\epsilon)$. This approximate equilibrium can be computed in polynomial time.*

Proof : Without loss of generality we can assume that $r_{11} = 1$ is the biggest element in R and the rest of the elements are non-negative. Let multi-set $S_1 = 1$. Assume that row player chooses row 1 and column player plays the column in S_1 with probability $\frac{1}{2}(1-\epsilon)$ and plays column y_1 with probability $\frac{1}{2}(1+\epsilon)$, where c_{1y_1} is the highest entry in the first row. This guarantees α for column player with respect to the best response. If this pair of strategies also gives a similar guarantee to row player then we are done. Otherwise there is another row, let us say row 2, which gives a payoff greater than $\frac{1}{2}(1+\epsilon)$ times the current payoff. Then row player chooses this row. Now column player chooses uniformly among the columns in multi-set $S_2 = S_1 \cup y_1$ with probability $\frac{1}{2}(1-\epsilon)$ and plays column y_2 with probability $\frac{1}{2}(1+\epsilon)$ where c_{2y_2} is the highest entry in row 2. While there exists a row that by playing it row player is better off by at least α, she chooses this row and column player changes her strategy accordingly.

Without loss of generality assume that row players chooses the rows in the increasing order from 1 to n. Let μ_i be the average of the entries of row i in S_i, i.e. $\mu_i = \frac{1}{i}\sum_{j \in S_i} r_{ij}$. By induction, we show $\mu_i \geq 1 - f(i)\epsilon$. The basis clearly holds. Because row player preferred row $i+1$ to row i we have $\frac{(1-\epsilon)\mu_i}{(1-\epsilon)\mu_{i+1} + (1+\epsilon)y_{i+1}} < \frac{1}{2}(1+\epsilon)$. y_{i+1} is at most 1 which implies $\mu_{i+1} > \frac{2}{1+\epsilon}\mu_i - \frac{1+\epsilon}{1-\epsilon} > 1 - (2+o(1))f(i)\epsilon$. Also, row player never chooses a row that she has chosen before. Note that if row player gives up playing row i, then μ_i is less than $\frac{1}{2}(1+\epsilon)$. If she is currently playing row k then

$$\frac{(1-\epsilon)\mu_k}{(1-\epsilon)\frac{i\mu_i + k - i}{k} + (1+\epsilon)} \geq \frac{(1 - f(k)\epsilon)}{1 - \frac{i(1-\epsilon)}{2k} + \frac{(1+\epsilon)}{(1-\epsilon)}} \geq \frac{1 - \frac{1}{4n}}{2 - \frac{1-\epsilon}{n} + \frac{\epsilon^2}{1-\epsilon}} \geq \frac{1}{2}(1+\epsilon)$$

With this observation row player will finally stay at one row and the players reach an α-approximate equilibrium. □

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