Mathematical Modelling of Social Change

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This paper describes a feasibility study into the use of mathematical models in the social sciences, particularly aimed at problems normally tackled using statistical methods. For initial simplicity, we describe higher education entry in the UK and compare the results of the model to UCAS data. The results are very encouraging and suggest that this is a useful method which potentially has much wider benefits for the social sciences than currently appreciated or understood. We describe a model in non-technical terms, discuss the possible roles of mathematical modelling in the social sciences and suggest how they might complement existing statistical methods.

1. Introduction

This paper describes a feasibility study into the application of mathematical modelling techniques to social science, paying particular attention to the sort of results required in the formation of social policy. We propose using mathematical models resembling those utilized successfully in recent years in population modelling in mathematical biology. These result in non-linear partial differential equations which describe the time-dependent behaviour of several different groups of the UK population (as a function of age) including some terms which represent social interactions. These equations are solved using computational methods, and importantly, results are compared to social data in order to assess the accuracy of the model.

The work undertaken constitutes a feasibility study, and as such we have started with limited aspirations. In this initial project we have developed a mathematical model of a straight forward, uncontroversial social process. We test the results obtained from this model on data in order to demonstrate that our approach shows promise. We concentrate on a simple model describing changing trends in higher education with the view to eventually describe some of the associated changes in employment patterns in the UK.

2. Why mathematics?

Quantitative social researchers regularly undertake secondary analyses of microdata, including datasets such as the General Household Survey, Labour Force Survey and Samples of Anonymised Records from the Census. Analyses of these, as well as primary sources, are principally undertaken using statistical methods. Research of this type generally aims to produce generalisable descriptions of the social phenomena and to understand the underlying relationships between variables in the data for the purposes of explanation, in either theoretical or applied contexts.

One important application of social research is the identification and description of social change. This area is of particularly important to those individuals and organisations whose principal interest is in future social patterns for either theoretical or policy purposes. For these research consumers, the ability to make projections and/or predictions is of considerable interest.

Standard statistical methods can be problematic when used to make projections. Regression models based on data for a specified range may not be valid for data points outside that range (as the approximate (possibly linear) relationship assumed in these models may not be applicable to data outside that range) and extrapolation is likely to produce greater error than interpolation. Time series decomposition methods are also highly inductive and assume that observed trends for a studied social characteristic will continue into the future.

Mathematical models based on differential equations are increasingly being used in a range of disciplines, for example in engineering, biology and economics. The modelling of population dynamics and epidemics (eg. Dietz 1982, Murray 1989) is of particular relevance to social sciences. It is unsurprising therefore that an application of these methods within social science is in socio-epidemiology (see for example Blower et. al. 1991). However, it is only now that we are starting to investigate the utility of these methods for what might be considered more standard social science research problems (for example Waugh and Lacey's 1999 work on council house allocations).

The current focus on mathematics is not the first time that mathematics has come to the attention of social researchers. There was a lot of activity in the 1960's and the 1970's (see for example Coleman 1964, Doreian 1970, Leik and Meeker 1975, Mapes 1971 and Montroll and Badger 1974). These attempts to use mathematical models for sociological and social policy questions were hampered by the restrictions of computing power at the time and preceded our understanding that deterministic systems could result in chaos. The first of these limitations severely restricted what was possible. The second meant that mathematical models necessarily appeared to be more deterministic than social reality. In an anti-reductionist social science culture this undoubtedly did little to help their reputation.

Interesting work continues, in a limited range of areas such as the analysis of social networks (see Doreian et. al. 1996, Eve et. al 1997, Fararo(ed.) 1984, Heckathon 1989, or Helbing et. al. 2000 for various different examples of modern mathematical sociology).

It is not our intention to suggest that mathematical models are universally appropriate, nor that statistical methods should not be used. Our intention is to investigate the utility of mathematical models to assess whether the technique has potential as an additional tool available to the social researcher in a broader range of fields than they are currently used. Mathematical models have particular strengths which constitute specific advantages over statistical approaches. These are;

- their form is more specifically tailored to the process being modelled
- they are well suited to making predictions, and allow us to explore the limits of predictability

Mathematical models are most suited to macro-analysis of change over time. In analyses at this scale the unit of analysis is a society and individual change contributes only a small amount towards overall social change. They tend to describe feedback systems where the circumstances at a certain time impact on those at a later time.

3. What are mathematical models?

Mathematical models have been characterised as 'thought experiments'(Blower et.al). This means that they have potential for producing simulations where it is not possible to obtain real-life data (for an example of a social science application of simulation see Gilbert 1997).

This characterisation also suggests the extent to which mathematical models are specified in isolation from real data. The model is a formal mathematical expression of the theoretised process being described. Constraints and qualifications are also formalised mathematically. This produces a set of related mathematical expressions which can be solved without recourse to data. (However, data may be necessary for providing the initial state from which to start the simulation.)

The concept of 'a model' within mathematics is quite different from that within statistics. In the latter a model typically involves a single equation whose form is taken off the peg, from a standard toolbox of model types (e.g. logistic regression, OLS, loglinear), into which variables are plugged before being parameterized.

The models we describe here are based on differential equations, which describe change (in some variable of interest) over time. They do not assume linearity and can produce chaotic solutions. This however, does not mean that the solutions are necessarily chaotic. The recognition that some models can produce solutions which are predictable for only a limited period caused quite a sensation in the late 1980s and early 1990s within social science as within other disciplines. This has led some authors to talk about a paradigmatic 'chaotic world' in which nothing is predictable (e.g. Byrne 1997). Such accounts generally overstate the applicability of chaos and misrepresent the world view of modelers. However the insights provided by chaos and the related understanding of non-linear equations enables us to do two things:

- Produce a model which could anticipate qualitative discontinuity
- Establish the limits of prediction

4. A Model of Higher Education Entry

We chose a simple and well understood process to model for the sake of establishing the feasibility of the method.

Our model describes the process of entry into Higher Education (H.E.) as a function of age, using a simple model borrowed from mathematical biology. We aim to use it to preduct future values of the entry rates into HE as a function of age in the short to medium term. For the moment we ignore differences in gender and ethnicity for simplicity.

The features of the process we seek to incorporate into the model are:

1. People are born and die at particular rates
2. During their lives they either enter higher education and obtain higher qualifications or do not
3. The rate at which people enter higher education will be related on the proportion of people in the population who either have, or are studying for higher educational qualifications - as more people obtain higher qualifications, the more desirable they become
4. People are increasingly likely to obtain higher education later in life

We will base our model upon the age-structured models discussed in Charlesworth (1980) and Murray (1989), and extend them to describe up take of higher education. Let u represent the density of the population who (i) have a degree, or (ii) are currently at university studying for one, or (iii) have been accepted onto such a course. Let v represent the density of the rest of the population. We consider the distribution of u and v with age x. Therefore, $u = u(x,t)$ and $v = v(x,t)$, where t is time. Consequently, the number of people aged between x and $x+a$ who have a degree, are currently at university, or have been accepted on such a course is equal to $\int_x^{x+a} u(y,t)\,dy$ (where y is a dummy variable). We assume that x and t are measured in identical units (in years). We use the standard age structured population models commonly used in mathematical biology, but include extra terms to represent social factors (in this case, the proportion of the population entering higher education). This is similar in some ways to the approach adopted in the modelling of epidemics (see for example Dietz 1982).

We use a continuous-time model (see Charlesworth 1980, Murray 1989). This gives

$$u_t + u_x = f(u,v,x,t) - c(x)\,u \tag{1}$$

and

$$v_t + v_x = -f(u,v,x,t) - c(x)\,v \tag{2}$$

where subscripts denote differentiation with respect to that subscript, $c(x)$ is a function which describes the likelihood of a person of age x dying and f is the rate at which people enter higher education. (For simplicity, we neglect people who drop out of their courses, but this can easily be incorporated.) The initial conditions are $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$, where u_0 and v_0 are given functions. The boundary condition for v at $x=0$ expresses the birth rate of the population. This will be given by some function of $u+v$ integrated over x between $x=0$ and $x=\infty$, weighted appropriately towards those in the population involved in reproduction. For this description of the birth rate, we will choose a so called density dependent model (see Charlesworth 1980 for a detailed discussion of the various options for this boundary condition). One such example is when the resulting total population satisfies logistic growth or decay (see Murray 1989). In this case the total population $N = \int_0^\infty [u(x,t) + v(x,t)]\,dx$ would satisfy the differential equation $N_t = rN - rN^2/K$, where r and K are constants. The boundary condition on u is $u(0,t) = 0$, since nobody is in higher education at birth.

We model f by $f = \gamma < u > v$, where $\gamma(x,t)$ weights the distribution of people accepted onto higher education courses towards certain age groups (in particular, those in their late teens), and $< u >$ is the sum of u over all people at a certain time, so that

$$< u > = \int_0^\infty u dx.$$

We choose γ so that $\int_0^\infty \gamma(x,t)\,dx$ is independent of time, so that changes in γ with respect to t do not cause any increase in $< u >$.[1] The reasons for this choice of f are as follows. Firstly, the more people there are of age x without a degree, then the greater the number of higher education applicants of age x there potentially will be at any given time. Hence the relationship between f and v. Secondly, the presence of undergraduates and graduates in society tends to enhance the perceived value of a degree (by for example peer group pressure), and hence f is proportional to the total number of people in category u (so f is proportional to $< u >$).

As the proportion of mature students has increased we need γ to depend on time, and for γ to slowly spread out with respect to x as time t increases. We achieve this by describing γ by

[1] Changes in γ describe any changes that may occur in the shape of the distribution of the age profile of UCAS applicants.

the linear diffusion equation, namely $\gamma_t = D\gamma_{xx}$ where D is a positive constant.[2] This equation for γ should be solved for $x > x_c$ together with the boundary condition $\gamma_x = 0$ at $x = x_c$, where x_c is an age below which people never enter higher education, and an initial condition $\gamma = \gamma_0(x)$ at $t = 0$ where γ_0 is some function giving the initial distribution of γ. These equations for γ can be solved using Laplace transforms. For $0 \leq x < x_c$, γ is taken to be equal to zero. This idea of using diffusion to model a spread of influence or ideas is not new. (See Leik and Meeker 1975, for example.) The diffusion equation in this case can be derived formally by considering a conservation equation for γ and a Fickian-type flux of γ given by $-D\gamma_x$.[3]

Of course, applications to universities mainly occur in the autumn, and students predominantly start on new courses in September and October. However, in this simple example we imagine applications and acceptances occuring continuously throughout the year. This keeps the model as simple as possible. The model can easily be made more realistic in this respect by incorporating appropriate Heaviside step functions into f to allow most applications to occur only during a certain period of the year, and time delay terms to represent successful applicants emerging as new undergraduates in September/October of each year. This would result in the solutions for u containing step like structures.

The above model is only one of several possibilities which we could consider. Another possibility is to differentiate between those currently in higher education $a(x,t)$ and those people who already have a degree $b(x,t)$.

Of course our model does not describe any sudden changes in government policy relating to higher education. These must be regarded for the purposes of the model as unpredictable, and any results which we produce may be valid only up to any change in policy by government. But the same limitation will apply to any method.

This set of equations can be solved using standard analytical and computational techniques. The solution method is discussed separately in a more detailed technical paper (Decent and Wathan, in preparation). The results are shown in the following figures. Figure 1 shows the total number of entrants into HE as a cumulative count. The graph shows the increase in the total value of u measured in millions of people since 1993 (i.e $\int_0^\infty (u - u_0)\, dx$) plotted against the year. The continuous line is the solution to the model equations and the stars are data points for comparison. (Data obtained from the UCAS website.[4]) Very encouraging agreement can be seen. (Note that, as already described, the model could be improved by allowing for the fact that applications occur only in certain months of the year. This would result in steps in the solution shown in Figure 1.)

Figures 2 to 6 show the total number of people entering HE in 1994, 1995, 1996, 1997 and 1998 respectively. The vertical axis shows the number of entrants. The horizontal axis shows age in years. Stars show UCAS data points. Squares show results from our model. In each figure, the leftmost points show the total number of people aged under 18 entering HE. Going left to right, the points then show ages in the following age groupings: 18-19, 19-20, 20-21, 21-22, 22-23, 23-24, 24-25, 25-30, 30-40 and finally over 40. (For example, in each figure, the square showing the number of people entering HE aged between 20 and 21 years old is based on an integral of the theoretical solution between the limits $x = 20$ and $x = 21$.) The data points are plotted in the centre of each range, except for the range under 18, which is plotted at the age 17.5, and the range over 40, which is plotted at 50. Once again, very encouraging agreement is

[2] This is analogous to the diffusion of chemicals or heat. See for example Billingham & King (2000).
[3] See Murray (1989) for an example of this in the diffusion of chemicals.
[4] address www.ucas.ac.uk/figures/archive/summary/index.html

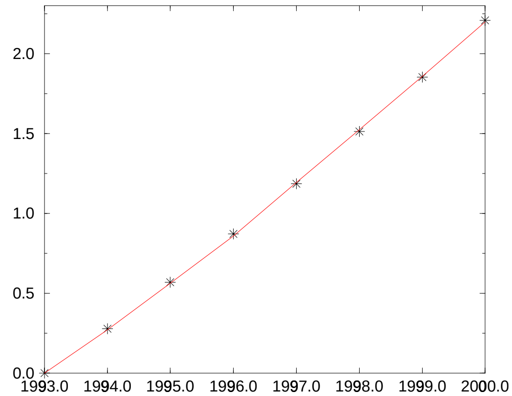

Figure 1: Total number of people entering HE since 1993 as a cumulative count. The model predicts the solid line. The UCAS data points are shown as stars.

obtained between theory and data.

4. Discussion

We have seen that the results of our feasibility study are encouraging. However our results are preliminary and further work is planned.

Several questions arise. If our method can be used successfully to predict future values of HE variables then how accurate is our method compared to existing statistical regression and time-series methods, and how far into the future can our method be used for prediction? As with many non-linear mathematical models, sensitivity on the initial data may result in a theoretical limit to the second of these questions. (For example, this is also the case with predicting the weather from mathematical models.) These questions need to be addressed by a more detailed comparison with data and further numerical simulations.

We also plan to extend our model by considering the entry into HE by different ethnic groups for men and women separately. Also, the effects of this upon employment participation is a further important extension. By considering these this will allow us to fully appreciate the scope and usefulness of our method.

We conclude by making the important observation that mathematical models will enable social scientists working in these fields to examine different questions from those currently answerable by statistical methods. In the long term future, we expect mathematical models to be used routinely by social scientists in addition to currently used statistical methods. Mathematical models allow prediction of future values of quantitative variables. Currently social scientists may spot a trend of interest in society, and mathematical models would allow for the evaluation of whether this trend will in future become more or less prominent. Another important benefit of mathematical models over statistical models is when considering qualitative changes in the nature of quantitative variables. For example, it is of interest to know whether a quantitative variable is increasing or decreasing monotonically, or whether it is oscillating with time, and whether the system under consideration is close to a qualitative switch between these regimes.

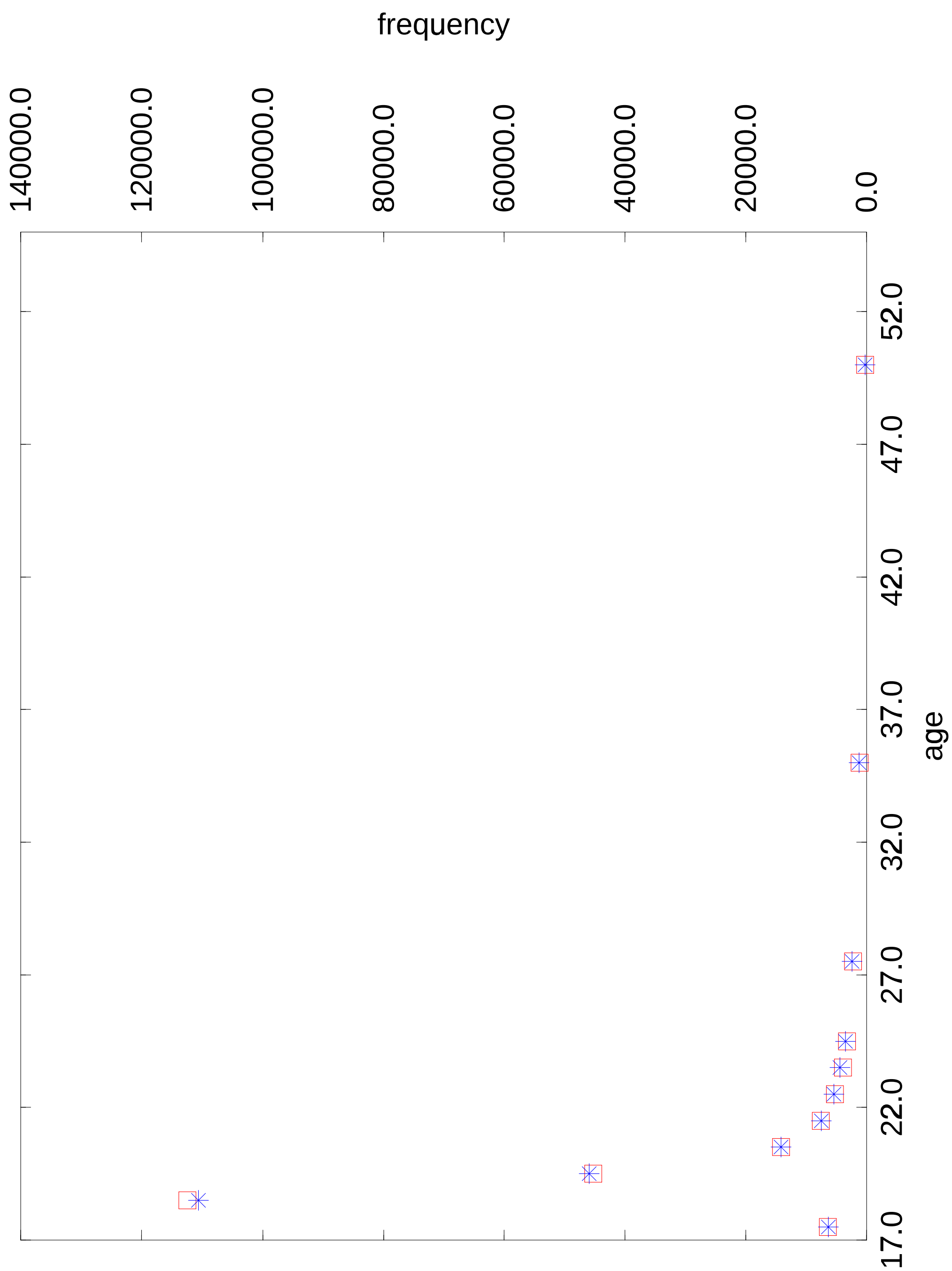

Figure 2: The total number of people entering HE in 1994 plotted against age.

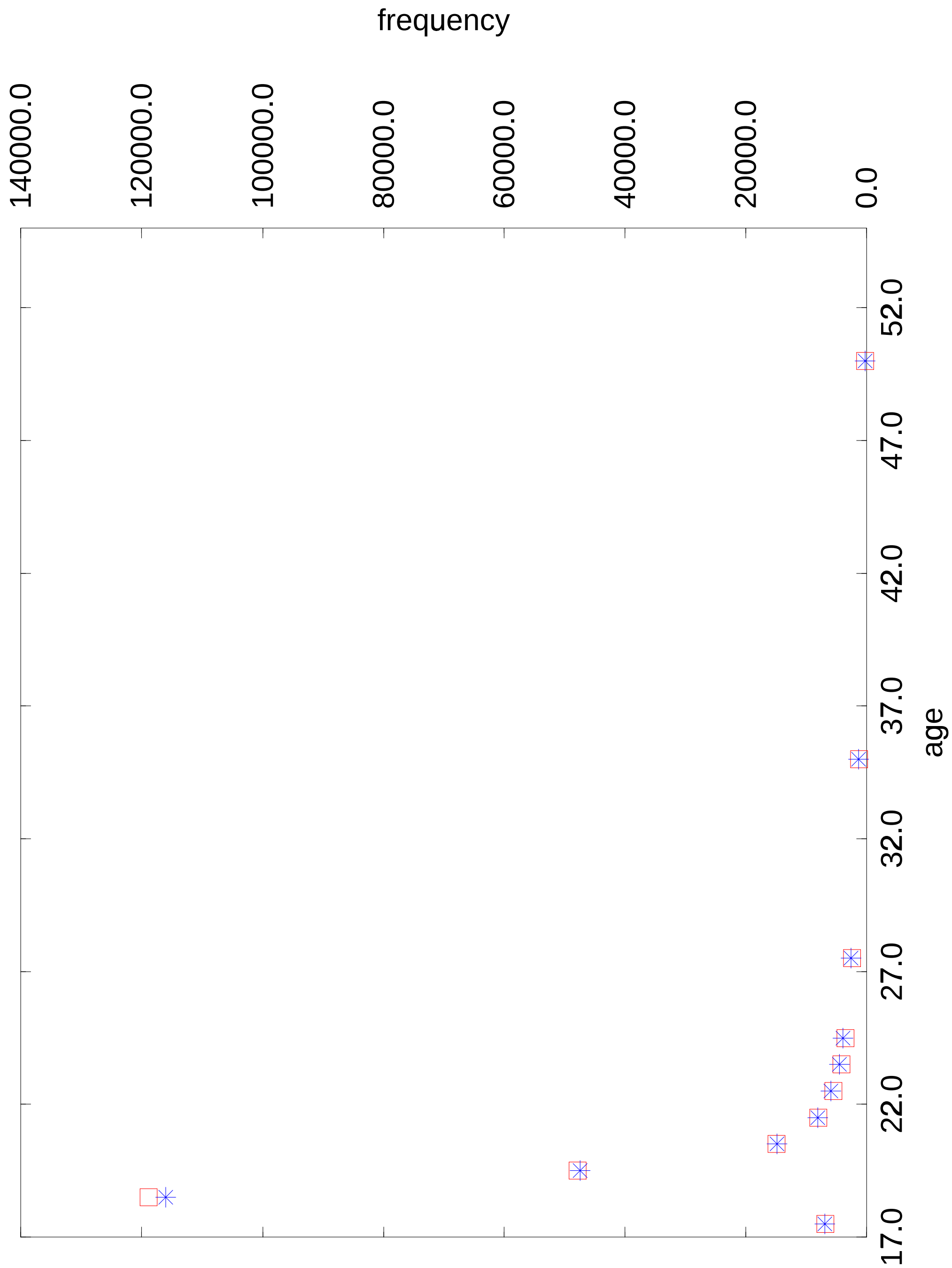

Figure 3: Total number of people entering HE in 1995 plotted against age.

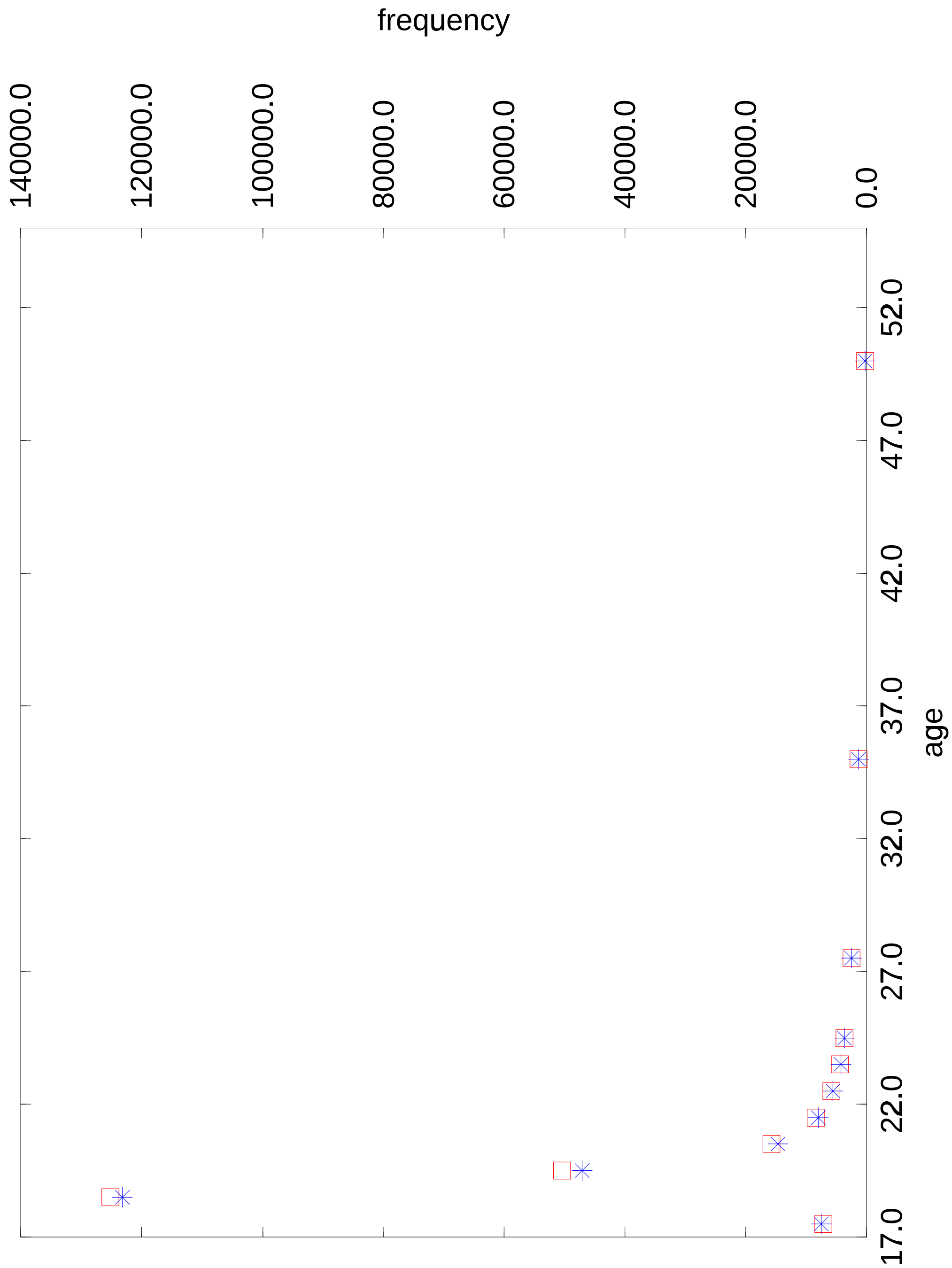

Figure 4: Total number of people entering HE in 1996 plotted against age.

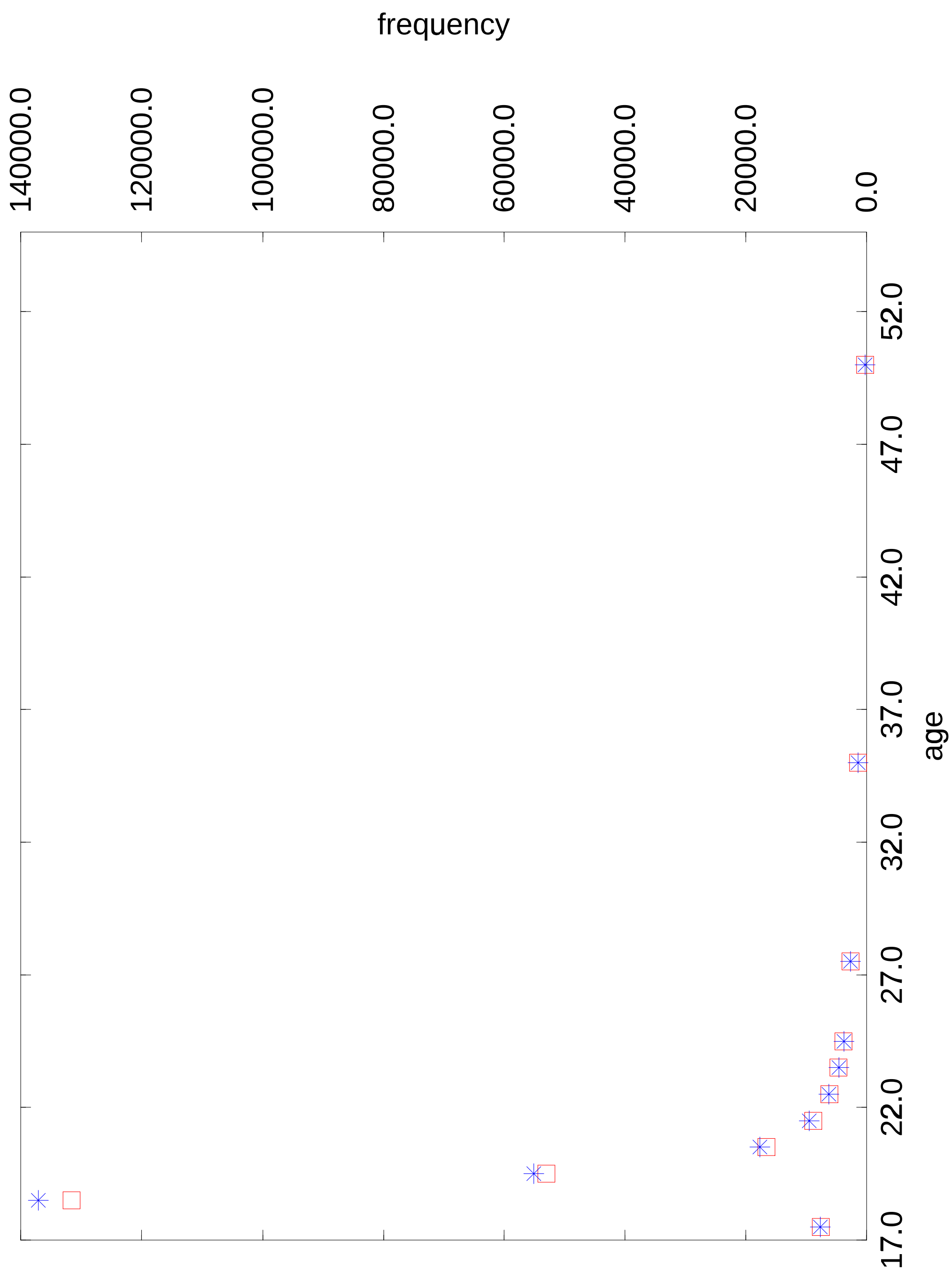

Figure 5: Total number of people entering HE in 1997 plotted against age.

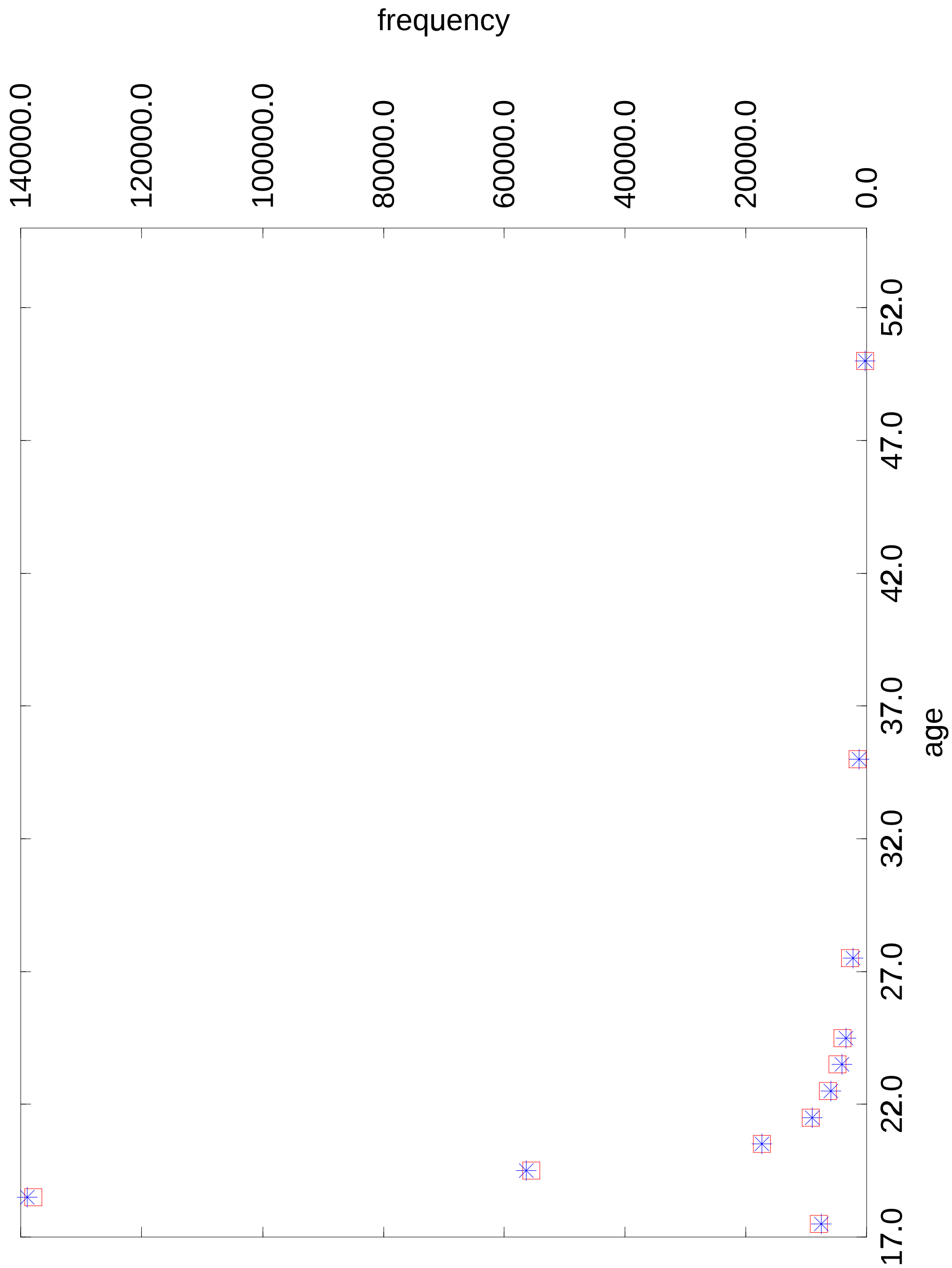

Figure 6: Total number of people entering HE in 1998 plotted against age.

One last possible benefit is for the comparison of differing sociological theories. If two competing theories were incorporated into mathematical models then the models would allow the comparison of the results of the theories against data.

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