Decision Analysis 097328

Lecture #13

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1 Convergence to Nash Equilibrium in Games

In this lecture we shall consider general games (i.e., not necessarily zero-sum). We shall overview the basic notion of Nash Equilibrium and show that under certain conditions, if the decisions of each player in the game follow some low-regret algorithm, the dynamics of the game converge (in some sense) to such equilibrium.

1.1 Basic notions

We shall first introduce some basic notions of Game Theory.

Definition 1 (Game)**.** A game is defined by the tuple

$$\Gamma = (n, \{S_i \,:\, i \in [n]\}, \{u_i : S \to \mathbb{R} \,:\, i \in [n]\})$$

where n is the number of players, S_i is the set of actions of player i and $u_i : S \to \mathbb{R}$ is the utility function of player i. The set $S = S_1 \times S_2 \times \cdots \times S_n$ is the set of joint actions.

For $s \in S$, we shall denote by s_i the action of player i and by $s_{-i} \in S_{-i}$ the actions of all players except of player i, where $S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$.

Example 2 (Matching Pennies)**.** The game is defined by the following game matrix:

	0	1
0	+1 , -1	-1 , +1
1	-1 , +1	+1 , -1

Each cell of the matrix contains the payoffs of both players. This game is a zero-sum game.

Example 3 (Prisoner's Dilemma)**.** In this game, the players are two prisoners that are interrogated by the police (while being separated from each other). Each have the option of either betray his partner or remain silent. If one of them betrays the other while his partner remain silent, he goes free and the silent partner receives the full 10-year sentence. If both remain silent, both prisoners are sentenced to only 1 year in jail for a minor charge. If each betrays the other, each receives a 5-year sentence.

This game (which is clearly *not* zero-sum) can be described by the following matrix:

	betray	not betray
betray	1 , 1	0 , 10
not betray	10 , 0	5 , 5

Definition 4 (Best Response)**.** Let $s \in S$. An action $x_i \in S_i$ of player i is a *best response* to s_{-i} if

$$\forall x \in S_i \ . \ u_i(x_i, s_{-i}) \geq u_i(x, s_{-i}).$$

We shall denote the best response to s_{-i} by $\mathrm{BR}(s_{-i})$.

Definition 5 (Nash Equilibrium)**.** A vector $s \in S$ is a Nash Equilibrium (NE) if for any player i,

$$u_i(s_i, s_{-i}) \geq u_i(\mathrm{BR}(s_{-i}), s_{-i}).$$

In other words, a set of strategies is an NE if no player can benefit from changing his action, assuming that the other players do not. The NE is perhaps the most fundamental notion of Game Theory, and its importance is seen from the following result proved by Nash.

Theorem 6 (Nash, 51')**.** *Any game with mixed strategies has at least one Nash Equilibrium.*

Nash Equilibrium is generally hard to compute. The hardness of the problem is not captured by the class NP (it is not NP-hard, since a NE always exists) but rather by a different complexity class called PPAD.

1.2 Concave and Socially-Concave games

We shall next consider some more specific classes of games: concave games and socially-concave games. Concavity is natural in economics, and many practical instances of games in this field are concave.

Definition 7 (Concave Game)**.** A game Γ is concave if the utility function $u_i(s) = u_i(s_i, s_{-i})$ of each player i is concave in s_i.

In certain cases, concave games have a unique Nash Equilibrium.

Theorem 8 (Rosen, 65')**.** *A concave game has a Nash Equilibrium. In addition, if the utility functions u_i are strictly convex, then the game has a* ***unique*** *Nash Equilibrium.*

Next, we turn to define socially-concave games.

Definition 9 (Socially-Concave Game)**.** A game Γ is socially-concave if:

(a) There exist some $\lambda_1, \ldots, \lambda_n > 0$ for which the function $\sum_i \lambda_i u_i$ is concave.

(b) For all $i \in [n]$, the function u_i is convex in the actions of the other players. That is, for any fixed $s_i \in S_i$ the function $u_i(s_i, x_{-i})$ is convex in $x_{-i} \in S_{-i}$.

As we shall prove near the end of the lecture, a socially-concave game is also a concave game.

Lemma 10. *If in a socially-concave game Γ the utility functions u_i are twice differentiable, then Γ is also a concave game.*

1.3 Main result: Convergence to Nash Equilibrium

We now arrive to the main result of this lecture. This result shows that in the case of socially-concave games, a Nash Equilibrium can be computed efficiently using regret minimization algorithms.

Theorem 11 (EMN, 2009)**.** *If in a (repeated) socially-concave game each player plays according to a low-regret algorithm, then the average strategy vector converges to a Nash Equilibrium.*

Proof. Let us denote by $x^t \in S$ the action vector at time t, and by $x_i^t \in S_i$ the action of the player i at time t. In addition, let $u_i^t = u_i(x^t)$ and define

$$\hat{x}^t = \frac{1}{t}\sum_{\tau=1}^{t} x^\tau \quad ; \quad \hat{u}^t = \frac{1}{t}\sum_{\tau=1}^{t} u^\tau.$$

Since the players play low-regret, we have for all players i,

$$\max_{y_i \in S_i} \sum_{t=1}^{T} u_i(y_i, x_{-i}^t) - \sum_{t=1}^{T} u_i(x_i^t, x_{-i}^t) \leq R_i^T$$

with $R_i^T = o(T)$, and we would like to show that for all i,

$$u_i(\hat{x}_i^T) \geq u_i(\mathrm{BR}(\hat{x}_{-i}^T), \hat{x}_{-i}^T) - \varepsilon_T \qquad \text{such that} \qquad \varepsilon_T \to 0. \tag{1}$$

First, using the properties of socially-concave games, we have

$$\begin{aligned}
\sum_i \lambda_i \hat{u}_i^t &= \sum_i \lambda_i \left(\frac{1}{t}\sum_{\tau=1}^{t} u_i(x^\tau) \right) \\
&\geq \sum_i \lambda_i \left(\max_{y_i \in S_i} \frac{1}{t} \sum_{\tau=1}^{t} u_i(y_i, x_{-i}^\tau) - \frac{R_i^t}{t} \right) && \text{(low-regret of each player)} \\
&\geq \sum_i \lambda_i \left(\frac{1}{t}\sum_{\tau=1}^{t} u_i(\mathrm{BR}(\hat{x}_{-i}^t), x_{-i}^\tau) - \frac{R_i^t}{t} \right) \\
&\geq \sum_i \lambda_i u_i \left(\mathrm{BR}(\hat{x}_{-i}^t), \frac{1}{t}\sum_{\tau=1}^{t} x_{-i}^\tau \right) - \sum_i \lambda_i \frac{R_i^t}{t} && \text{(convexity of } u_i \text{in } x_{-i}) \\
&= \sum_i \lambda_i u_i(\mathrm{BR}(\hat{x}_{-i}^t), \hat{x}_{-i}^t) - \sum_i \lambda_i \frac{R_i^t}{t} \\
&\geq \sum_i \lambda_i u_i(\hat{x}^t) - \sum_i \lambda_i \frac{R_i^t}{t} && \text{(definition of BR)} \\
&\geq \sum_i \lambda_i \left(\frac{1}{t}\sum_{\tau=1}^{t} u_i(x^\tau) \right) - \sum_i \lambda_i \frac{R_i^t}{t}. && \text{(property (a) of SC game)}
\end{aligned}$$

This implies that

$$\sum_i \lambda_i u_i(\hat{x}^t) \geq \sum_i \lambda_i u_i(\mathrm{BR}(\hat{x}_{-i}^t), \hat{x}_{-i}^t) - \sum_i \lambda_i \frac{R_i^t}{t}. \tag{2}$$

However, by the definition of best-response we have for any $j \in [n]$

$$\sum_{i \neq j} \lambda_i u_i(\hat{x}^t) \leq \sum_{i \neq j} \lambda_i u_i(\mathrm{BR}(\hat{x}^t_{-i}), \hat{x}^t_{-i}) \tag{3}$$

and by subtracting (3) from (2) we conclude that for any $j \in [n]$,

$$u_j(\hat{x}^t) \geq u_j(\mathrm{BR}(\hat{x}^t_{-j}), \hat{x}^t_{-i}) - \frac{1}{\lambda_j} \sum_i \lambda_i \frac{R_i^t}{t}.$$

That is, since $R_i^T = o(T)$ we got (1) with

$$\varepsilon_T = \frac{1}{\lambda_j} \sum_i \lambda_i \frac{R_i^T}{T} \longrightarrow 0.$$

□

1.4 Proof of Lemma 10

Finally, we prove Lemma 10.

Proof of Lemma 10. For any player $j \in [n]$, we have $\partial^2_{x_j} \sum_i \lambda_i u_i \leq 0$ since $\sum_i \lambda_i u_i$ is concave. However, for $i \neq j$ the function u_i is convex in x_j, so that $\partial^2_{x_j} u_i \geq 0$. Hence, it must hold that $\partial^2_{x_j} u_j \leq 0$, that is, the utility function u_j of player j is concave. □