Probability and Statistics Review

Sample Space and Events

The outcome of a random experiment cannot
be predicted with 100% accuracy. We need to assign probabilities to the possible outcomes of the experiment

Sample Space

The sample space of a random experiment is a set that contains every possible outcome of the experiment.

Event

An event is a subset of the sample space

Example

Suppose that we flip an unbiased coin twice. The sample space will be {HH, TT, HT, TH}. An event can be “obtaining heads on both flips” , which is {HH} .

Probability

For a sample space S, the probability measure **P** is a function that assigns each event E ⊂ S a number **P(**E) satisfying the following axioms:

1. 0≤ P(E) ≤ 1
2. P(S) = 1
3. For any sequence of mutually exclusive events E1, E2, ... (that is, events for which $E_i \cap E_j$ = Ø when i ≠ j), we have

$$\boldsymbol{P}\left[\bigcup_{i \geq 1}\left(E_i\right)\right] = \sum_{\mathrm{i} \geq 1} P\left[E_i\right]$$

Axioms of Probability

Let **P** be a probability measure.

(1) (Monotonicity.) If A ⊂ B, then **P**[A] ≤ **P**[B].

(2) (Complement rule.) For every event E, one has **P**[E^c] = 1 − **P**[E].

(3) (Empty event.) **P**[∅] = 0

(4) (Inclusion-exclusion formula.) For every events A, B ⊂ S, one has
P[A ∪ B] = **P**[A] + **P**[B] − **P**[A ∩ B]

Conditional Probability

Let S be a sample space.
Suppose a random event B is drawn

The conditional probability of another event A given B is the probability that A happened given that B happened.

Quantitatively, it is $\mathbf{P}(A \mid B) = \dfrac{\mathbf{P}(A \cap B)}{\boldsymbol{P}(B)}$

This is known as “the probability of A conditioned on B ”

Independent events

We say that two events A and B such that $\mathbf{P}(A), \mathbf{P}(B) > 0$ are independent if
$\mathbf{P}(A|B) = \mathbf{P}(A)$ and $\mathbf{P}(B|A) = \mathbf{P}(B)$.   or, equivalently, if

$\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$

Intuitively: knowing A occurred does not change the likelihood that B occurred, and vice versa.

Example: Flipping a coin

Random Variable

- A **random variable**, usually denoted by X, is a rule that assigns a numerical value to each outcome in a sample space. Random variables may be either discrete or continuous.
- We use random variables (r.v.) to model data that are uncertain, e.g.

Number of heads in ten coin tosses

Share of votes for a candidate in an election

Average # of hours spent on homework each week

Household income in the U.S.

Indicator Functions

An Important application of random variable is an indicator function:

$$I\{E\} = \begin{cases} 1 \ if \ E \ occurs \\ 0 \ otherwise \end{cases}$$

Expected Value

Expected value, or **expectation**/population mean, is the weighted average of the possible values that the variable can take, weighed on the probability of each value occurring.

Example:

If a random variable X takes on values of -1, 0, and 2, with probabilities 0.3, 0.3, and 0.4 respectively, then the expectation of X equals

$$E[X] = (-1)(0.3) + (0)(0.3) + (2)(0.4) = 0.5$$

Linearity of Expectation

Let X and Y be two discrete random variables, and let a ∈ ℝ be a nonrandom constant. Then,

$\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y]$

and

$\mathbf{E}[aX] = a\mathbf{E}[X]$

Conditional Expectation

The expectation of a random variable X conditional on Y=y is denoted by E[X|Y=y]

$E[X|A] = \sum \;\; x \, P[X = x \;|A]$

Variance

Suppose that X has expectation E[X]. Its variance is
Var(X) = E[(X − E[X $]^2$)] = E[X^2] – E[X $]^2$

The variance is a measure of far X will be, on average, from its expected value E[X]. Stated another way, the variance measures how random a random variable is.

Note: Standard Deviation SD[X] = $\sqrt{Var[X]}$

Covariance

- The covariance measures the amount of linear dependence between two random variables
- Covariance between X and Y is $Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$

Correlation

Given a pair of random variables (X,Y) , the correlation coefficient ρ is defined as

$$\rho_{X,Y} = \frac{cov(X,Y)}{SD[X]SD[Y]}$$