Big O Notation

This section contains some math

Don't worry, we'll survive

Objectives

- Motivate the need for something like Big O Notation
- Describe what Big O Notation is
- Simplify Big O Expressions
- Define "time complexity" and "space complexity"
- Evaluate the time complexity and space complexity of different algorithms using Big O Notation
- Describe what a logarithm is

I know

What's the idea here?

Imagine we have multiple implementations of the same function.

How can we determine which one is the "best?"

"Write a function that accepts a string input and returns a reversed copy"


Great!
Pretty Good
Only OK
Ehhhhh
Awful

Who Cares?

- It's important to have a precise vocabulary to talk about how our code performs
- Useful for discussing trade-offs between different approaches
- When your code slows down or crashes, identifying parts of the code that are inefficient can help us find pain points in our applications
- Less important: it comes up in interviews!

An Example

Suppose we want to write a function that calculates the sum of all numbers from 1 up to (and including) some number *n*.

```
function addUpTo(n) {
  let total = 0;
  for (let i = 1; i <= n; i++) {
    total += i;
  }
  return total;
}
```

```
function addUpTo(n) {
  return n * (n + 1) / 2;
}
```

Which one is better?

ZOMG WUT

```
  addUpTo(n) =     1     +     2     +     3     + ... + (n - 1) +     n
+ addUpTo(n) =     n     + (n - 1) + (n - 2) + ... +     2     +     1
-------------------------------------------------------------------------
2 addUpTo(n) = (n + 1) + (n + 1) + (n + 1) + ... + (n + 1) + (n + 1)
```

n copies

```
2addUpTo(n) = n * (n + 1)

addUpTo(n) = n(n + 1) / 2
```

OMG YES MATHZ

What does better mean?

- Faster?
- Less memory-intensive?
- More readable?

Let's focus here first

Why not use timers?

```
function addUpTo(n) {
  let total = 0;
  for (let i = 1; i <= n; i++) {
    total += i;
  }
  return total;
}

let t1 = performance.now();
addUpTo(1000000000);
let t2 = performance.now();
console.log(`Time Elapsed: ${(t2 - t1) / 1000} seconds.`)
```

Let's visualize this!

The Problem with Time

- Different machines will record different times
- The *same* machine will record different times!
- For fast algorithms, speed measurements may not be precise enough?

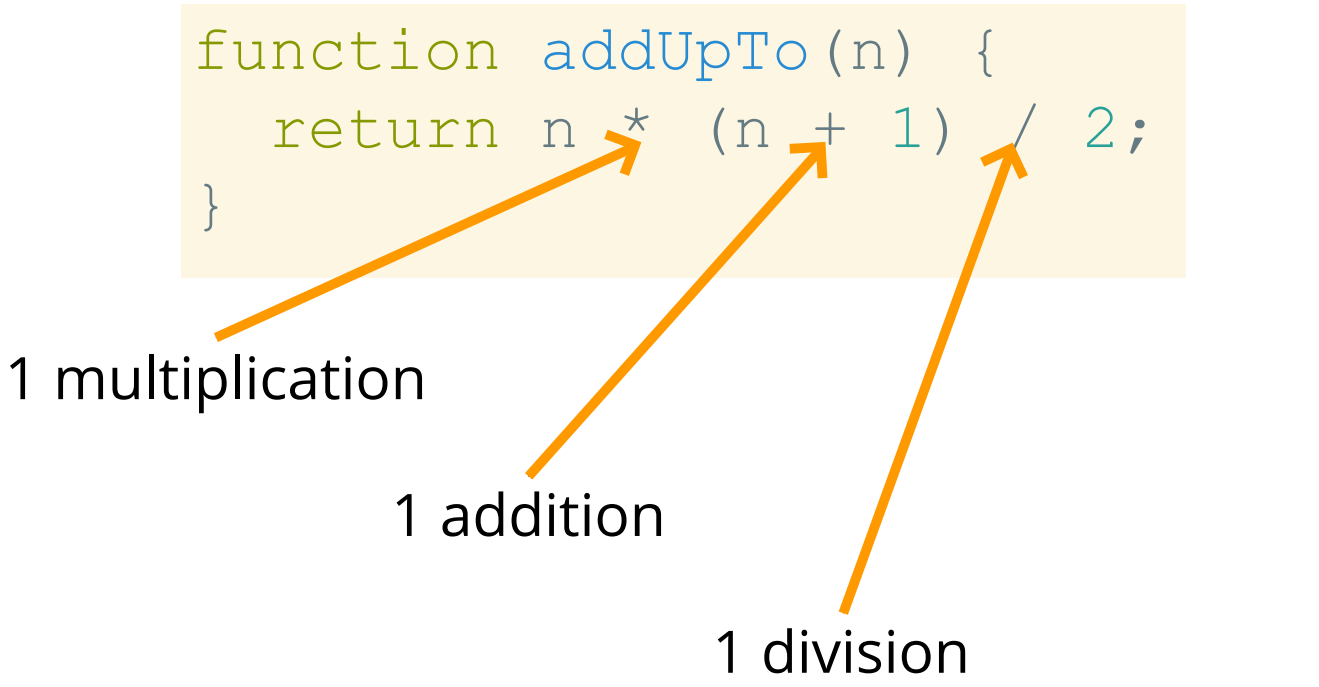
If not time, then what?

Rather than counting *seconds*,
which are so variable...

Let's count the *number* of
simple operations the
computer has to perform!

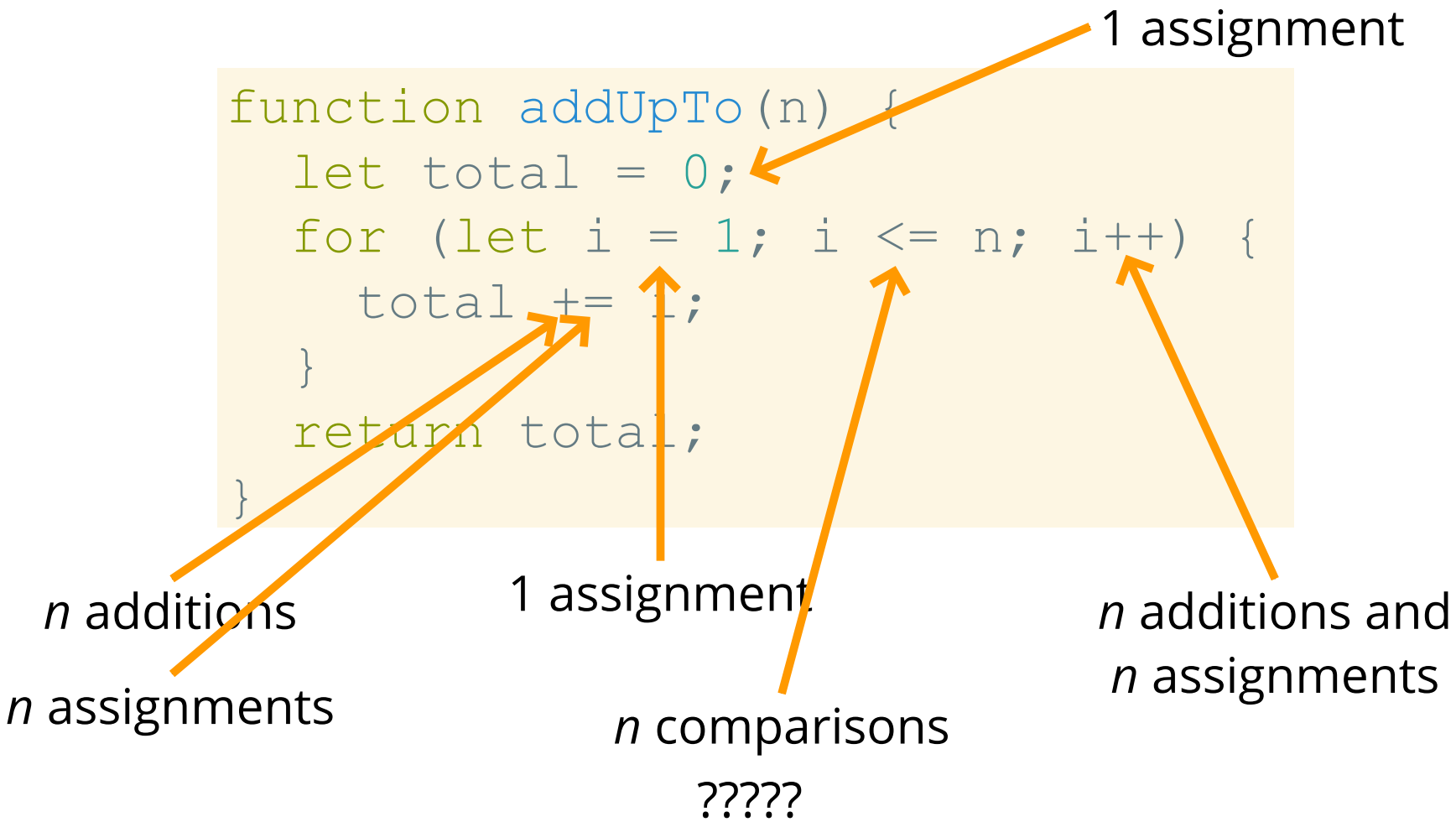
Counting Operations

```
function addUpTo(n) {
  return n * (n + 1) / 2;
}
```



3 simple operations, regardless of the size of *n*

Counting Operations



Counting is hard!

Depending on what we count, the number of operations can be as low as $2n$ or as high as $5n + 2$

But regardless of the exact number, the number of operations grows roughly *proportionally with n*

If n doubles, the number of operations will also roughly double

Introducing....Big O

Big O Notation is a way to
formalize fuzzy counting

It allows us to talk formally about
how the runtime of an algorithm
grows as the inputs grow

We won't care about the details,
only the trends

Big O Definition

We say that an algorithm is **O(f(n))** if the number of simple operations the computer has to do is eventually less than a constant times **f(n)**, as **n** increases

- f(n) could be linear (f(n) = n)
- f(n) could be quadratic (f(n) = n^2)
- f(n) could be constant (f(n) = 1)
- f(n) could be something entirely different!

Example

```
function addUpTo(n) {
  return n * (n + 1) / 2;
}
```

Always 3 operations

O(1)

```
function addUpTo(n) {
  let total = 0;
  for (let i = 1; i <= n; i++) {
    total += i;
  }
  return total;
}
```

Number of operations is (eventually) bounded by a multiple of *n* (say, 10*n*)

O(*n*)

Another Example

O(*n*)

O(*n*)

```
function countUpAndDown(n) {
  console.log("Going up!");
  for (let i = 0; i < n; i++) {
    console.log(i);
  }
  console.log("At the top!\nGoing down...");
  for (let j = n - 1; j >= 0; j--) {
    console.log(j);
  }
  console.log("Back down. Bye!");
}
```

Number of operations is (eventually) bounded by a multiple of *n* (say, 10*n*)

O(*n*)

OMG MOAR EXAMPLEZ

```
function printAllPairs(n) {
  for (var i = 0; i < n; i++) {
    for (var j = 0; j < n; j++) {
      console.log(i, j);
    }
  }
}
```

O(*n*) O(*n*)

O(n) operation inside of an O(n) operation.

O(n^2)

YOUR TURN

Simplifying Big O Expressions

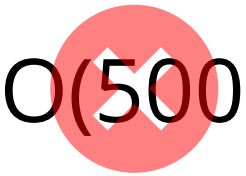
When determining the time complexity of an algorithm, there are some helpful rule of thumbs for big O expressions.

These rules of thumb are consequences of the definition of big O notation.

Constants Don't Matter

$O(2n)$

$O(n)$



$O(500)$

$O(1)$

$O(13n^2)$

$O(n^2)$

Smaller Terms Don't Matter

$O(n + 10)$ $O(n)$

$O(1000n + 50)$ $O(n)$

$O(n^2 + 5n + 8)$ $O(n^2)$

Big O Shorthands

- Analyzing complexity with big O can get complicated
- There are several rules of thumb that can help
- These rules won't **ALWAYS** work, but are a helpful starting point

Big O Shorthands

1. Arithmetic operations are constant
2. Variable assignment is constant
3. Accessing elements in an array (by index) or object (by key) is constant
4. In a loop, the the complexity is the length of the loop times the complexity of whatever happens inside of the loop

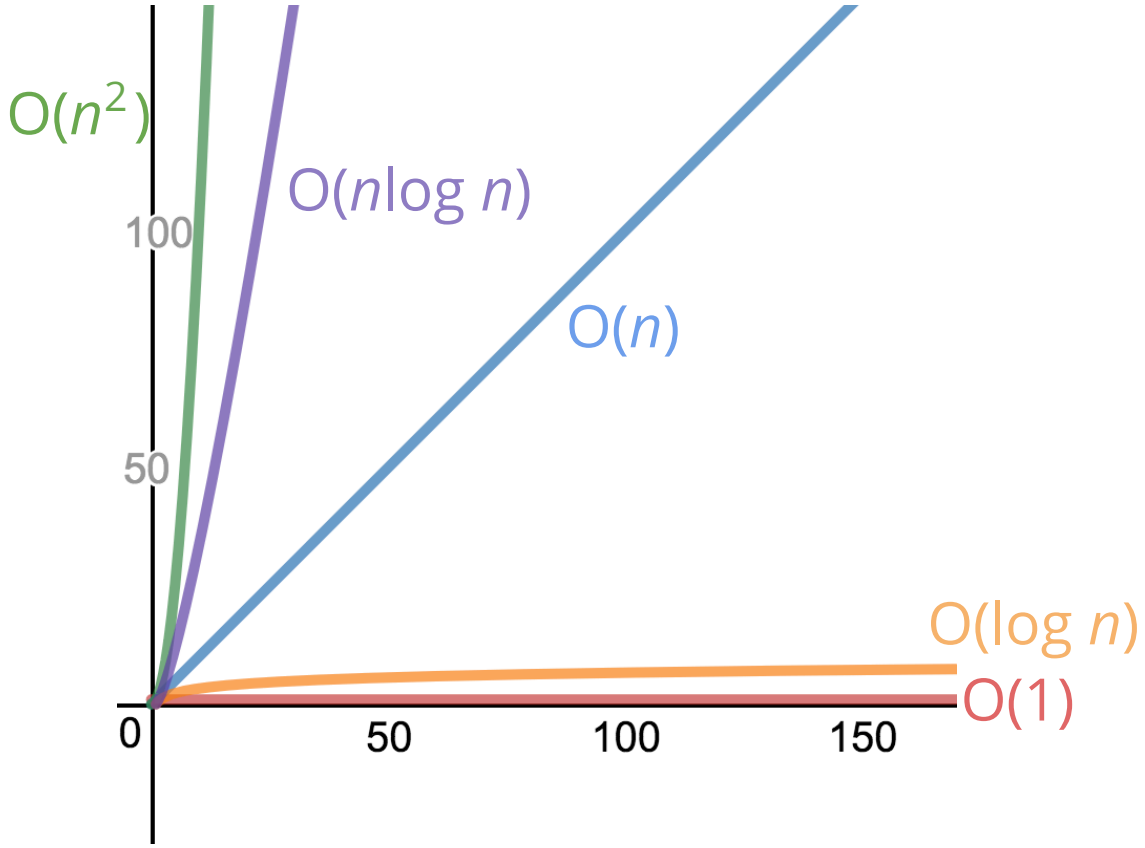
A Couple More Examples

```
function logAtLeast5(n) {
  for (var i = 1; i <= Math.max(5, n); i++) {
    console.log(i);
  }
}
```

O(*n*)

```
function logAtMost5(n) {
  for (var i = 1; i <= Math.min(5, n); i++) {
    console.log(i);
  }
}
```

O(1)


O(n^2)
O($n$log n)
O(n)
O(log n)
O(1)
100
50
0
50
100
150

YOUR TURN

Space Complexity

So far, we've been focusing on **time complexity**:
how can we analyze the *runtime* of an algorithm as
the size of the inputs increases?

We can also use big O notation to analyze **space complexity**:
how much additional memory do we need to allocate in order
to run the code in our algorithm?

What about the inputs?

Sometimes you'll hear the term **auxiliary space complexity** to refer to space required by the algorithm, not including space taken up by the inputs.

Unless otherwise noted, when we talk about space complexity, technically we'll be talking about auxiliary space complexity.

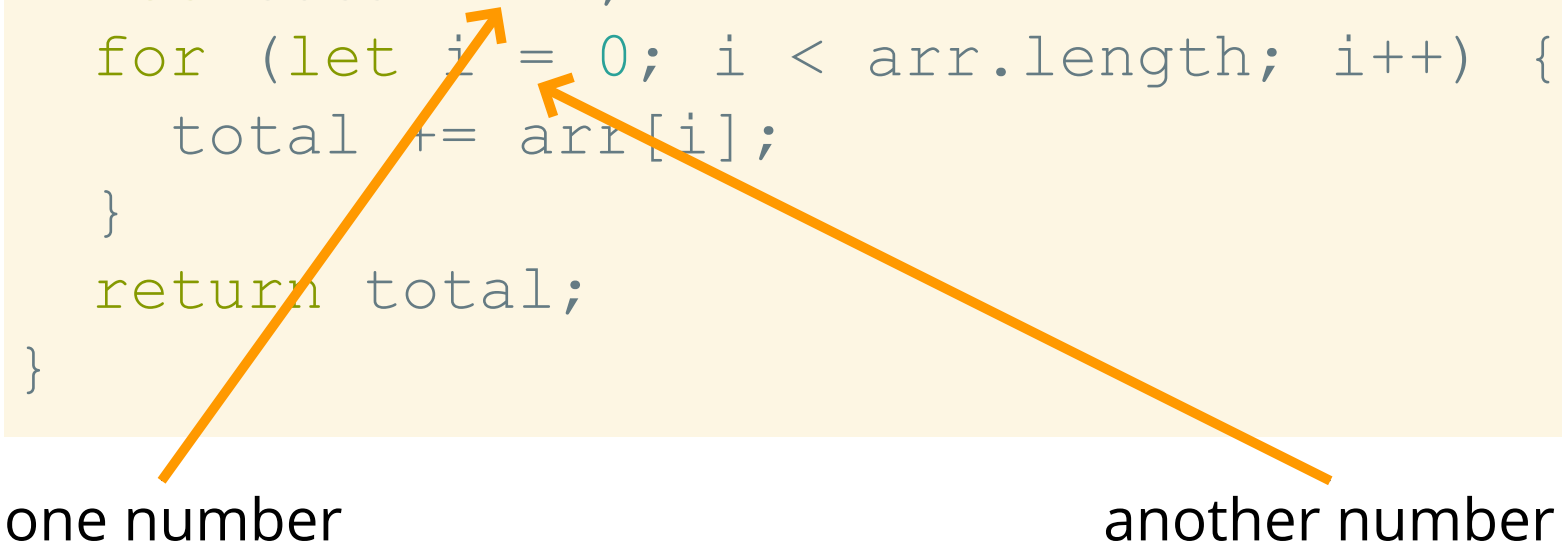
Space Complexity in JS

Rules of Thumb

- Most primitives (booleans, numbers, undefined, null) are constant space
- Strings require O(*n*) space (where *n* is the string length)
- Reference types are generally O(*n*), where n is the length (for arrays) or the number of keys (for objects)

An Example

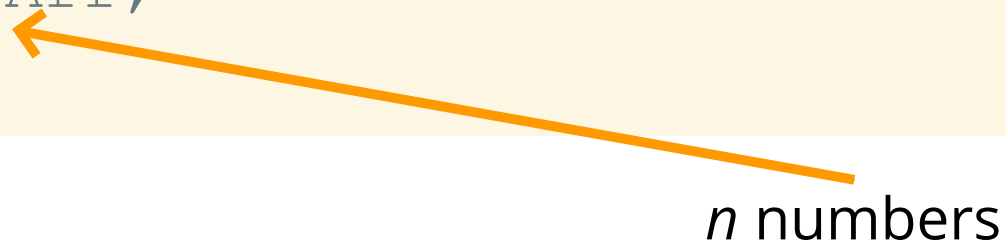
```
function sum(arr) {
  let total = 0;
  for (let i = 0; i < arr.length; i++) {
    total += arr[i];
  }
  return total;
}
```



O(1) space!

Another Example

```
function double(arr) {
  let newArr = [];
  for (let i = 0; i < arr.length; i++) {
    newArr.push(2 * arr[i]);
  }
  return newArr;
}
```



O(*n*) space!

YOUR TURN

Logarithms

We've encountered some of the most common complexities: O(1), O(n), O(n^2)

Sometimes big O expressions involve more complex mathematical expressions

One that appears more often than you might like is the logarithm!

Wait, what's a log again?

$\log_2(8) = 3 \longrightarrow 2^3 = 8$

$\log_2(\textit{value}) = \textit{exponent} \longrightarrow 2^{\textit{exponent}} = \textit{value}$

We'll omit the 2.

$\log === \log_2$

Wut.

This isn't a math course, so here's a rule of thumb.

The logarithm of a number roughly measures the number of times you can divide that number by 2 **before you get a value that's less than or equal to one**.

Logarithm Examples

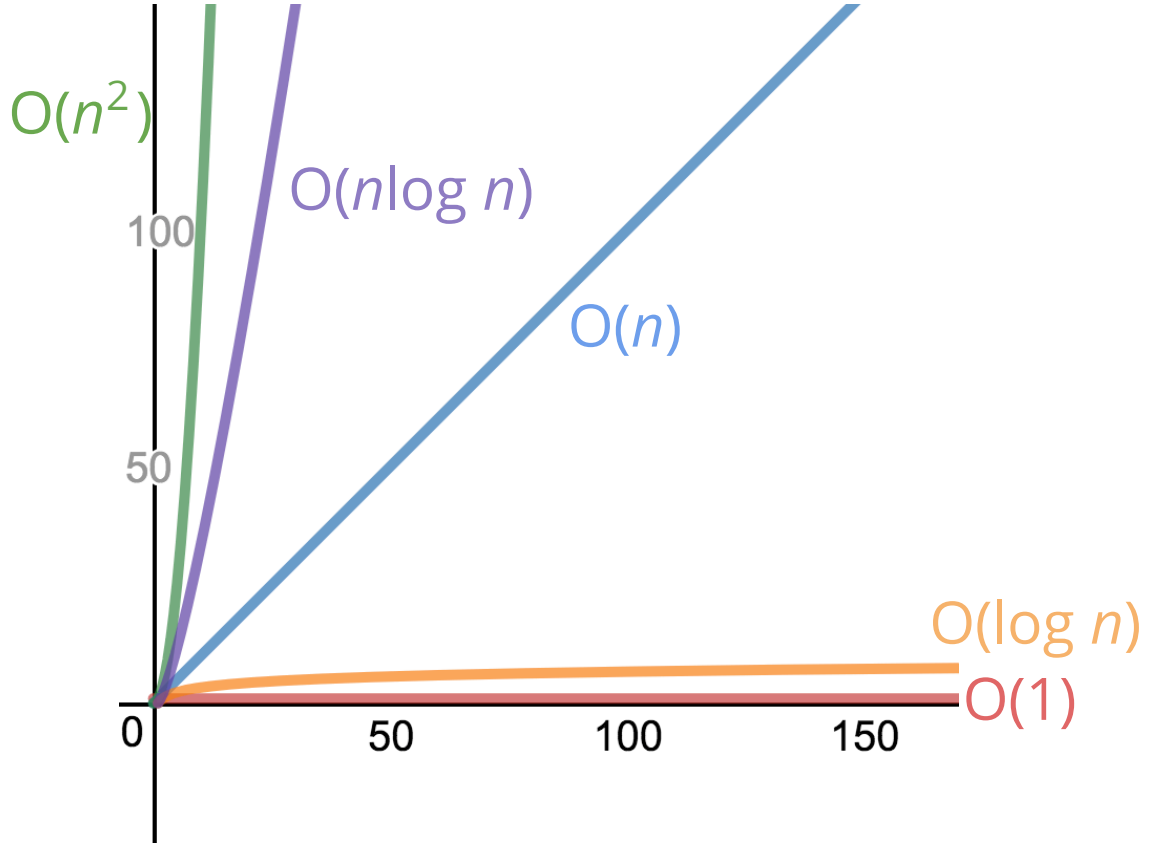
	8	25
÷ 2		÷ 2
	4	12.5
÷ 2		÷ 2
	2	6.25
÷ 2		÷ 2
	1	3.125
		÷ 2
		1.5625
		÷ 2
		0.78125

$\log(8) = 3$

$\log(25) \approx 4.64$

Logarithm Complexity

Logarithmic time complexity is great!



Who Cares?

Certain searching algorithms have logarithmic time complexity.

Efficient sorting algorithms involve logarithms.

Recursion sometimes involves logarithmic space complexity.

...and more!

Recap

- To analyze the performance of an algorithm, we use Big O Notation
- Big O Notation can give us a high level understanding of the time or space complexity of an algorithm
- Big O Notation doesn't care about precision, only about general trends (linear? quadratic? constant?)
- The time or space complexity (as measured by Big O) depends only on the algorithm, not the hardware used to run the algorithm
- Big O Notation is everywhere, so get lots of practice!