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On the equivalence of the multistage-insertion and cycle-shrink formulations of the symmetric traveling salesman problem

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Abstract

Arthanari (On the traveling salesman problem, presented at the XI Symposium on Mathematical Programming held at Bonn, West Germany, 1982) proposed a Multistage-insertion (MI)-formulation of the symmetric traveling salesman problem (STSP). This formulation has a polynomial number of constraints. Carr (Polynomial separation procedures and facet determination for inequalities of the traveling salesman polytope, Ph.D. Thesis, Carneige Mellon University, 1995; Separating over classes of TSP inequalities defined by 0 node-lifting in polynomial time, preprint, 1996) proposed the Cycle-shrink relaxation of the STSP. In this paper, we show that there exists a natural transformation which establishes a one-to-one correspondence between the variables of the two formulations, say X and U, such that X is feasible for *MI*-formulation if and only if U is feasible for the other. The size of the Cycle-shrink formulation is larger than that of the MI-formulation, both in the number of variables and constraints. However, both are compact descriptions of the subtour elimination polytope.

Keywords: Traveling salesman problem; Subtour elimination polytope; Problem formulations; Compact descriptions

1. Introduction

Given a complete graph on n vertices, $G=(V,E)$, with edge weights, the symmetric traveling salesman problem (STSP) consists of finding a minimum cost Hamilton cycle in G.

Over the years, researchers have given different formulations to solve the traveling salesman problem as a linear programming problem, with integer/mixed integer variables. We refer to Lawler et al. [9] and Junger et al. [8] for a detailed survey of this problem.

Dantzig et al. [7] posed the STSP as an integer programming problem, wherein subtour elimination constraints were introduced. The corresponding relaxation yields the subtour elimination polytope (SEP). Since

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then, many more formulations have been proposed. Arthanari [1] posed the STSP as the Multistage-insertion (MI) problem. Padberg and Sung [10] provide an approach for an analytical comparison of different formulations of STSP. Arthanari and Usha [3] show that the LP relaxation of MI-formulation is a compact description of the subtour polytope, SEP. Carr [5] shows that the Cycle-shrink relaxation of the STSP is a compact description of the SEP. In this paper, we show that there exists a natural transformation which establishes a one-to-one correspondence between the variables of the two formulations, say X and U, such that X is feasible for MI-formulation if and only if U is feasible for the other. Also, the objective function values differ only by a constant. So, X optimal for MI-formulation means that U is optimal for CS-formulation and vice versa. Also, we establish the correspondence between the constraints of MI-formulation with that of the other relaxation.

In the next section, we describe the SEP$_n$ arising out of the Dantzig et al. [7] formulation when the number of cities equals n. The MI and Cycle-shrink relaxations with some of their properties are described in Sections 3 and 4, respectively. We prove the equivalence of the MI and Cycle-shrink polytopes in Section 5.

2. The subtour elimination polytope SEP$_n$

Let n be the number of cities. Let $G=(V,E)$ be the complete graph on n vertices. Let p_n denote $n(n-1)/2$ for any n. So $|E|=p_n$. Consider any $x\in R^E$. Assume a fixed ordering of the elements of E. Map the edges in E into $\{1,\ldots,p_n\}$. Given a $x\in R^{p_n}$ and a $F\subset E$, we write $x(F)=\sum_{e\in F}x_e$, where x_e is the appropriate coordinate of x corresponding to e. For $S\subseteq V$ define

$$E(S)=\{[u,v]\in E \mid u\in S, v\in S\},$$

$$\delta(S)=\{[u,v]\in E \mid u\in S, v\notin S\},$$

SEP$_n$ is the polytope defined by the set of all $x\in R^{p_n}$ such that (1), (2), and (3) or (4) hold:

$$x_e\geqslant 0,\quad \forall e\in E, \tag{1}$$

$$x(\delta(v))=2,\quad \forall v\in V, \tag{2}$$

$$x(E(S))\leqslant |S|-1,\quad \forall S\subseteq V,\ S\neq\phi \text{ and } S\neq V, \tag{3}$$

$$x(\delta(S))\geqslant 2,\quad \forall S\subseteq V,\ S\neq\phi \text{ and } S\neq V. \tag{4}$$

The dimension of this polytope is p_n-n because the degree constraints flatten the polytope. The number of constraints is exponential in this case.

3. The multistage-insertion relaxation (MI)

In this section, we describe the MI-formulation of the STSP. In local search and heuristic approaches, insertion is a commonly used strategy to solve STSP. Start with the 3-tour $(1,2,3,1)$. We choose one of the edges in $\{(1,2),(1,3),(2,3)\}$ to insert 4 to obtain a 4-*tour*. If our previous decisions yield a $(k-1)$-tour, we select an edge available in the $(k-1)$-tour for inserting k and so on, we proceed until we find an n-tour. Here, we have a multi-stage decision problem.

Define

$$x_{ijk}=\begin{cases}1 & \text{if in stage } (k-3) \text{ the decision is to insert } k \text{ between } i \text{ and } j,\ 1\leqslant i<j\leqslant k-1\\ 0 & \text{otherwise.}\end{cases}$$

Let $X=(x_{124},\dots,x_{n-2,n-1,n})\in B^{\tau_n}=\{0,1\}^{\tau_n}$ where $\tau_n=\sum_{k=4}^{n}(k-1)(k-2)/2$. Let $\mathscr{C}_{ijk}=c_{ik}+c_{jk}-c_{ij}$. Here, $\mathscr{C}_{ijk}$ gives the incremental cost of inserting k between (i,j), where c_{ij} is the cost of visiting city j from city i and $c_{ij}=c_{ji}$. Let $\mathbf{x}_k=(x_{124},\dots,x_{k-2,k-1,k})$. It is clear from the definition that $X=(\mathbf{x}_4,\dots,\mathbf{x}_n)$.

An integer programming formulation of the above MI problem given in [1], is presented below as Problem 1.

Problem 1.

$$\text{minimize} \sum_{k=4}^{n}\sum_{1\leqslant i<j\leqslant k-1}\mathscr{C}_{ijk}x_{ijk}$$

subject to

$$\sum_{1\leqslant i<j\leqslant k-1} x_{ijk}=1,\quad 4\leqslant k\leqslant n, \tag{5}$$

$$\sum_{k=4}^{n}x_{ijk}\leqslant 1,\quad 1\leqslant i<j\leqslant 3, \tag{6}$$

$$-\sum_{r=1}^{i-1}x_{rij}-\sum_{s=i+1}^{j-1}x_{isj}+\sum_{k=j+1}^{n}x_{ijk}\leqslant 0,\quad 4\leqslant j\leqslant n-1;\ 1\leqslant i<j, \tag{7}$$

$$x_{ijk}=0 \text{ or } 1,\quad 1\leqslant i<j\leqslant k-1;\ 4\leqslant k\leqslant n. \tag{8}$$

Relaxing the integer constraints (8) with just non-negativity constraints (as constraints $x_{ijk}\leqslant 1$ are implied by Eq. (5)) and adding the following constraints:

$$-\sum_{r=1}^{i-1}x_{rin}-\sum_{s=i+1}^{n-1}x_{isn}\leqslant 0,\quad i=1,\dots,n-1, \tag{9}$$

we obtain the MI-relaxation of the STSP. Note that constraints (9) are redundant and are added only because their slack variables have special meaning.

Definition 3.1. Solution and feasible solution of Problem 1 are defined as usual. We also say that X is a pre-solution to MI-relaxation if X satisfies inequalities (6), (7) and (9). In addition, if X is non-negative we say X is a feasible pre-solution.

3.1. Theorems about multistage-insertion

We introduce the following notation with an example (from [2]), which facilitates the subsequent discussion.

Definition 3.2. In general, let $E_{[n]}$ denote the matrix corresponding to Eq. (5); let $A_{[n]}$ denote the matrix corresponding to inequalities (6), (7) & (9). Let 1_r denote the row vector of r 1's. Let I_r denote the identity matrix of size $r\times r$. Then we can write recursively.

$$E_{[n]}=\begin{pmatrix}\mathbf{1}_{p3} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & \cdots & \mathbf{1}_{p_{n-2}} & 0\\ 0 & 0 & \cdots & \mathbf{1}_{p_{n-1}}\end{pmatrix}=\begin{pmatrix}E_{[n-1]} & 0\\ \mathbf{0} & \mathbf{1}_{p_{n-1}}\end{pmatrix}.$$

To derive a similar expression for $A_{[n]}$ we first define

$$A^{(n)} = \begin{pmatrix} I_{p_{n-1}} \\ -M_{n-1} \end{pmatrix},$$

where M_i is the $i \times p_i$ node-edge incidence matrix.

Then

$$A_{[n]} = \begin{pmatrix} A^{(4)} & | & A^{(5)} & | & & | & A^{(n)} \\ & | & & | & \ddots & | & \\ \mathbf{0} & | & \mathbf{0} & | & & | & \end{pmatrix} = \begin{pmatrix} A_{[n-1]} & | & A^{(n)} \\ \mathbf{0} & | & \end{pmatrix}.$$

Observe that $A^{(n)}$ is the submatrix of $A_{[n]}$ corresponding to x_n. The number of rows of $0'$s is decreasing from left to right (see Fig. 1).

$$E_{[5]} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$A_{[5]} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \end{pmatrix}$$

Fig. 1 Matrices $E_{[5]}$ and $A_{[5]}$.

Example 3.1. Consider $n=5$, then $p_4=6$, $p_5=10$, and $\tau_n=9$. Let X be given by,

$$X' = (\tfrac{1}{2}\ \ 0\ \ \tfrac{1}{2}; \tfrac{1}{2}\ \ 0\ \ 0\ \ 0\ \ 0\ \ \tfrac{1}{2}) \in R^{\tau_5},$$

where X' denotes transpose of X. X satisfies the equality restriction (5) of the MI-relaxation. Also, the corresponding slack variables u obtained from the rest of the inequalities (6), (7) and (9) are $\geqslant 0$. We have,

$$u' = \begin{pmatrix} e_3 \\ 0 \end{pmatrix} - A_{[5]}X = (0\ \ 1\ \ \tfrac{1}{2}\ \ \tfrac{1}{2}\ \ 1\ \ 0\ \ \tfrac{1}{2}\ \ \tfrac{1}{2}\ \ \tfrac{1}{2}\ \ \tfrac{1}{2}).$$

(Matrices $E_{[5]}$ and $A_{[5]}$ are shown in Fig. 1).

So (X,u) is a feasible solution to the MI-relaxation.

Define

$$\zeta(n) = \{X \mid E_{[n]}X = \mathbf{1}_{n-3},\ X \geqslant 0\}, \tag{10}$$

$$\mathscr{U}(n) = \left\{ u \in R^{p_n},\ u \geqslant 0 \mid \exists X \in \zeta(n) \ni A_{[n]}X + u = \begin{bmatrix} \mathbf{1}_3 \\ 0 \end{bmatrix} \right\}. \tag{11}$$

Remark 3.1. Note that X is a pre-solution to the MI-relaxation is equivalent to saying

$$A_{[n]}X + u = \begin{bmatrix} \mathbf{1}_3 \\ 0 \end{bmatrix}.$$

Remark 3.2. If X is a pre-solution to the MI-relaxation, then pre-multiplying both sides of $A_{[n]}X + u = \begin{bmatrix} \mathbf{1}_3 \\ 0 \end{bmatrix}$ by c and noting that $cA_{[n]}$ is indeed $-\mathscr{C}$, we get

$$cu = c\begin{bmatrix} \mathbf{1}_3 \\ 0 \end{bmatrix} + \mathscr{C}X.$$

So, it is seen that the objective function of MI-formulation is equivalent to cu but for the constant $c_{12}+c_{13}+c_{23}$. (See [3] for details.)

Various properties of MI-formulation are studied in [1–4]. We state two of those results below.

Theorem 3.1. *Any integer feasible solution to Problem* 1 *is a basic solution and corresponds to an n-tour.*

It is interesting to note that the slack variables of the MI-formulation, u, are the edge-tour incidence vectors. Arthanari and Usha [3] obtain the linear description of $\mathscr{U}(n)$ and prove the following theorem.

Theorem 3.2. *The projection $\mathscr{U}(n)$ of the polytope*

$$u \geqslant 0, \quad X \in \zeta(n) \ni A_{[n]}X + u = \begin{bmatrix} \mathbf{1}_3 \\ 0 \end{bmatrix}$$

onto the space of slack variables u is exactly the subtour polytope.

The number of variables not including the slack variables (all non-negative) in this formulation is τ_n and the number of constraints is $(n-3) + p_n$, not counting the non-negativity restrictions. Hence, we have a formulation with polynomial number of constraints, which provides a compact description of the SEP.

4. The Cycle-shrink relaxation (CS)

Cycle-shrink is a polynomial sized linear programming relaxation developed by Carr [5] that implies the validity of all the subtour elimination constraints. The importance of Cycle-shrink lies in the way it is used to separate naturally defined classes of STSP inequalities.

To formulate Cycle-shrink, additional variables are used as follows. First, rename the edge variables, in the standard formulation of the traveling salesman problem, as x_e^0 for each $e \in E$. Arbitrarily label the vertices in V with the integers from 1 through n. Construct the family of graphs $G_k = (V_k, E_k)$ for $i \in V$ such that G_k is the subgraph of the complete graph G which is induced by the set V_k of all those vertices whose labels are greater than k. Then on each graph G_k, create additional variables x_e^k for each $e \in E_k$.

Consider an incidence vector x^0 of a Hamilton cycle $H^0(x^0)$ in G. Let $H^1(x^0)$ be the Hamilton cycle on G_1 formed by removing vertex 1 from $H^0(x^0)$ and linking the neighbors of 1 in $H^0(x^0)$ with an edge. Let $H^k(x^0)$ be the Hamilton cycle on G_k formed by removing vertex k from $H^{k-1}(x^0)$ and linking the neighbors of k in $H^{k-1}(x^0)$ with an edge. The values that we want the additional variables of Cycle-shrink to have, given the values of x^0, can be determined by considering the family of Hamilton cycles as follows:

$(x_e^0 \mid e \in E)$ is the incidence vector of H^0.

$(x_e^k \mid e \in E_k)$ is the incidence vector of H^k for all $k \in \{1, \ldots, n-3\}$.

A complete feasible solution x for Cycle-shrink can thus be represented by

$$x := (x^0, x^1, \ldots, x^{n-3}), \tag{12}$$

where for each $k \in \{0, 1, \ldots, n-3\}$, x^k is a vector having a component for each edge in G_k.

Carr [5] defines the Cycle-shrink relaxation to be the following linear program:

$$\text{minimize} \sum_{e \in E} c_e x_e^0$$

subject to

$$x_e^0 \geqslant 0, \quad e \in E, \tag{13}$$

$$\sum_{e \in \delta(\{j\}) \cap E_k} x_e^k = 2, \quad \forall k \in \{0, \ldots, n-3\}, \ \forall j \in V_k, \tag{14}$$

$$x_e^k \geqslant x_e^{k-1}, \quad \forall k \in \{1, \ldots, n-3\}, \ \forall e \in E_k. \tag{15}$$

Cycle-shrink is a valid relaxation of the STSP [5,6].

4.1. Theorems about Cycle-Shrink

Various properties of the CS-relaxation are studied by Carr [5,6]. He also uses the CS-relaxation to prove polynomial separation over some classes of facet defining inequalities. Here, we state two of those results relevant to our topic of discussion.

Theorem 4.1. *If x is feasible for Cycle-shrink, then x^0 (the vector of original variables) satisfies all the subtour elimination constraints.*

Theorem 4.2. *The projection of the Cycle-shrink polytope onto the space of original variables is exactly the subtour polytope.*

The Cycle-shrink formulation has $\tau_{n+1} + p_{n+1} - 3$ constraints, not counting non-negativity restrictions while the number of variables is τ_{n+1} (all non-negative). Carr [5] shows that the Cycle-shrink relaxation is a compact description of the SEP.

5. Equivalence of multistage-insertion and Cycle-shrink formulations

We now have two formulations of the STSP which have a polynomial number of constraints and both of the formulations are compact descriptions of the SEP. It would be very interesting to compare these two formulations and check for their equivalence. We do this in this section.

Let (X/k) be the restriction of X up to the $(k-3)$rd stage, that is

$$(X/k) = (\mathbf{x}_4, \ldots, \mathbf{x}_k) \quad \text{for } 4 \leqslant k \leqslant n, \ \text{for a given } X.$$

Given X, a pre-solution, let $U^{(k-3)}$ denote the slack variable vector obtained from the MI-relaxation for $n = k$ by substituting (X/k) into the inequalities (6), (7), and (9). Let $U^{(0)} = \mathbf{1}_3$. Also, we assume $U_{ij}^{(k-3)} = 0$ for

$1 \leqslant i < j,\ j > k$ which are not defined at that stage. The following lemma (from [2]) states the connection between $U^{(k-4)}$ and $U^{(k-3)}$.

Lemma 5.1. *Given X, a pre-solution, and $U^{(l)}$ as defined above, we have*

$$U^{(k-4)} - A^{(k)}\mathbf{x}_k = U^{(k-3)} \quad \textit{for all } k,\ 4 \leqslant k \leqslant n. \tag{16}$$

Proof. Immediate from the definition of $U^{(l)}$ and inequalities (6), (7), and (9). These inequalities when rendered in standard form yield, for $n = k,\ k-1$, respectively,

$$A^{(4)}\mathbf{x}_4 + \cdots + A^{(k-1)}\mathbf{x}_{k-1} + A^{(k)}\mathbf{x}_k + U^{(k-3)} = \begin{pmatrix} \mathbf{1}_3 \\ \mathbf{0} \end{pmatrix}, \tag{17}$$

$$A^{(4)}\mathbf{x}_4 + \cdots + A^{(k-1)}\mathbf{x}_{k-1} + U^{(k-4)} = \begin{pmatrix} \mathbf{1}_3 \\ \mathbf{0} \end{pmatrix}. \tag{18}$$

Subtracting (18) from (17) yields the result. □

Note that the converse of this result stated as Lemma 5.2 is also true.

Lemma 5.2. *IF $U^{(l)}$'s satisfy* (16), *with $U^{(0)} = \mathbf{1}_3$, then the corresponding X is a pre-solution.*

Corollary 5.1. *X is a feasible pre-solution if and only if $U_{ij}^{(k-4)} \geqslant U_{ij}^{(k-3)}$, for all (i,j) such that $1 \leqslant i < j \leqslant k-1,\ 4 \leqslant k \leqslant n$.*

Proof. Note that the first p_{k-1} rows of $A^{(k)}$ equal I_{p_k}. As $\mathbf{x}_k \geqslant 0$ for all $k,\ 4 \leqslant k \leqslant n$, Lemma 5.1 implies the result. The other direction is similar. □

Corollary 5.2. *Let $\mathrm{star}(k) = \{(i,k) \mid 1 \leqslant i < k\}$. Given X a pre-solution, we have $\sum_{e \in \mathrm{star}(k)} U_e^{(k-3)} = 2$, for all $k,\ 4 \leqslant k \leqslant n$, if and only if X satisfies constraint* (5).

Proof. From Lemma 5.1, we have $U^{(k-4)} - A^{(k)}\mathbf{x}_k = U^{(k-3)}$. Restrict the attention to the last $k-1$ rows of these equations. Let $U_L^{(k-3)}$ be the subvector of $U^{(k-3)}$ corresponding to the last $k-1$ rows. By definition, $U^{(k-4)}$ has zeros in these rows, and the submatrix corresponding to the last $k-1$ rows of $A^{(k)}$ in fact equals $-M_{k-1}$. Since M_{k-1} is a $((k-1) \times p_{k-1})$ node-edge incidence matrix, every column has exactly two 1's.

We have $U_L^{(k-3)} = M_{k-1}\mathbf{x}_k$. So

$$\begin{aligned} \sum_{e \in \mathrm{star}(k)} U_e^{(k-3)} &= \mathbf{1}'_{k-1} M_{k-1} \mathbf{x}_k \\ &= 2 \sum_{1 \leqslant i < j \leqslant k-1} x_{ijk} \\ &= 2 \quad \text{if and only if } X \text{ satisfies Eq. (5) for } k. \end{aligned}$$

Hence the result. □

Lemma 5.3. *Given X a pre-solution satisfying Eq.* (5),

$$\sum_{r=1}^{i-1} U_{ri}^{(k-3)} + \sum_{s=i+1}^{k} U_{is}^{(k-3)} = 2 \quad \textit{for } 1 \leqslant i \leqslant k-1,\ 3 \leqslant k \leqslant n. \tag{19}$$

Proof. By induction on k.

The result is true for $k=3$, as $U^{(0)}=\mathbf{1}_3$ and the l.h.s for each i is the sum of any two coordinates of $U^{(0)}$.

Assuming that the result is true for $k=l-1$, we have

$$\sum_{r=1}^{i-1} U_{ri}^{(l-4)} + \sum_{s=i+1}^{l-1} U_{is}^{(l-4)} = 2, \quad 1 \leqslant i \leqslant l-2. \tag{20}$$

We shall show that the result is true for $k=l$.

Consider any i, $1 \leqslant i \leqslant l-1$, we shall show that

$$\sum_{r=1}^{i-1} U_{ri}^{(l-3)} + \sum_{s=i+1}^{l} U_{is}^{(l-3)} = 2, \quad 1 \leqslant i \leqslant l-1. \tag{21}$$

We can write l.h.s. of (21) as

$$\sum_{r=1}^{i-1} U_{ri}^{(l-3)} + \sum_{s=i+1}^{l-1} U_{is}^{(l-3)} + U_{il}^{(l-3)}.$$

From Lemma 5.1, we have $U_e^{(l-3)} = U_e^{(l-4)} - x_{e,l}$ for e, an edge in $\{(u,v) | 1 \leqslant u < v \leqslant l-1\}$, and $U_{il}^{(l-3)} = \sum_{r=1}^{i-1} x_{ril} + \sum_{s=i+1}^{l-1} x_{isl}$. Substituting into the above equation and simplifying we get

$$\sum_{r=1}^{i-1} U_{ri}^{(l-3)} + \sum_{s=i+1}^{l} U_{is}^{(l-3)} = \sum_{r=1}^{i-1} U_{ri}^{(l-4)} + \sum_{s=i+1}^{l-1} U_{is}^{(l-4)}. \tag{22}$$

For $1 \leqslant i \leqslant l-2$, we have by hypothesis

$$\sum_{r=1}^{i-1} U_{ri}^{(l-4)} + \sum_{s=i+1}^{l-1} U_{is}^{(l-4)} = 2$$

or we have shown the required result for all i except $i=l-1$.

Now for the remaining case, that is, $i=l-1$

$$\sum_{r=1}^{i-1} U_{ri}^{(l-3)} + \sum_{s=i+1}^{l} U_{is}^{(l-3)} = \sum_{e \in \mathrm{star}(l)} U_e^{(l-3)}$$

and as X satisfies Eq. (5), r.h.s of the above equation equals 2 (by Corollary 5.2).

Hence the result. □

Now we are in a position to show the equivalence of the MI and the Cycle-shrink relaxations, by establishing one-to-one correspondence between the variables that satisfy the constraints of the two relaxations, respectively. While proving this, we also establish the correspondence between the constraints of the two relaxations.

Observe that we can state the constraints of MI-relaxation, using both $X=(\mathbf{x}_4,\dots,\mathbf{x}_n)$ and $U=(U^{(0)},\dots,U^{(n-3)})$ in matrix notation as follows:

$$E_{[n]}X = \mathbf{1}_{n-3},$$

$$U^{(0)} = \mathbf{1}_3,$$

$$U^{(k-4)} - A^{(k)}\mathbf{x}_k = U^{(k-3)} \quad \text{for all } k,\ 4 \leqslant k \leqslant n,$$

$$X \geqslant 0.$$

From this, by eliminating U obviously, we obtain the MI-relaxation. We shall show that eliminating X from this yields the CS-relaxation, but for the numbering of the vertices.

Caution: The numbering of the vertices in the two formulations, MI and CS are not the same, that is, the vertices that are numbered from $1,\dots,n$ are numbered $n,\dots,1$, respectively. Vertex i in MI-formulation refers to vertex $n-i+1$ in CS-formulation. So the set of edges, E_k in CS-formulation corresponds to the set of edges $\{e=(u,v)\mid 1\leqslant u<v\leqslant n-k\}$ in MI-formulation, for $3\leqslant k\leqslant n$. Keeping this in mind, we can rename $U^{(n-k)}$ as x^{k-3}, $3\leqslant k\leqslant n$.

We have the following Lemma.

Lemma 5.4. *Given X feasible for the MI-relaxation we have a unique* $U^{(l)}$, $0\leqslant l\leqslant n-3$ *given by Lemma* 5.1, *which satisfy the constraints of the CS-relaxation.*

Proof. Since X is feasible for the MI-relaxation, we have $x_{ijk}\geqslant 0$ and from Lemma 5.1,

$$x_{ijk}=U_{ij}^{(k-4)}-U_{ij}^{(k-3)}\quad\text{for } 1\leqslant i<j\leqslant k-1,\ 4\leqslant k\leqslant n.$$

So $U_{ij}^{(k-4)}\geqslant U_{ij}^{(k-3)}$, which is equivalent to Eq. (15) of the CS-relaxation. Next, we shall show that constraints (6), that is

$$\sum_{k=4}^{n}x_{ijk}\leqslant 1\quad\text{for } 1\leqslant i<j\leqslant 3$$

imply constraints (13) of the CS-relaxation, for corresponding edges. Substituting for x_{ijk}, we restate constraints (6) as,

$$\sum_{k=4}^{n}(U_{ij}^{(k-4)}-U_{ij}^{(k-3)})\leqslant 1, \tag{23}$$

$$U_{ij}^{(0)}-U_{ij}^{(n-3)}\leqslant 1. \tag{24}$$

With $U_{ij}^{(0)}=1$, we get $U_{ij}^{(n-3)}\geqslant 0$, for all $1\leqslant i<j\leqslant 3$.

Similarly, we can show that constraints (7) and (9), imply the non-negativity of $U_{ij}^{(n-3)}$, for the corresponding edges, respectively.

Hence, we have $U_{ij}^{(n-3)}\geqslant 0$ for $1\leqslant i<j\leqslant n$ which is equivalent to constraints (13) of the CS-Relaxation.

Finally, since X is feasible for the MI-relaxation, by applying Corollary 5.2 and Lemma 5.3 we get equations in U which are equivalent to constraints (14) of the CS-relaxation.

Hence the lemma. □

The other direction is also true and is stated below as Lemma 5.5.

Lemma 5.5. *Given an* x^l, *for* $l=0,\dots,n-3$, *feasible for the CS-relaxation, we have a unique X given by*

$$x_{e,k}=x_e^k-x_e^{k-1},\quad \forall k\in\{1,\dots,n-3\},\ \forall e\in E_k \tag{25}$$

which is feasible for the MI-relaxation.

Proof. Consider the X given by

$$x_{e,k}=x_e^k-x_e^{k-1},\quad \forall k\in\{1,\dots,n-3\},\ \forall e\in E_k.$$

Non-negativity of X follows from the above definition and constraints (15). First we show that X satisfies inequalities (6), that is,

$$\sum_{k=1}^{n-3}x_{e,k}\leqslant 1,\quad \forall e\in E_{n-3}. \tag{26}$$

By substituting for $x_{e,k}$ in the l.h.s. of the above inequality and simplifying we get

$$\sum_{k=1}^{n-3} x_{e,k} = x_e^{n-3} - x_e^0. \tag{27}$$

From Eq. (14), for $k = n-3$, we know that $x_e^{n-3} = 1$, for $e \in E_{n-3}$ and $x_e^{n-3} \geqslant x_e^0$, from inequalities (15). Thus, $0 \leqslant x_e^{n-3} - x_e^0 \leqslant 1$ or $x_{e,k}$ satisfies inequalities (6).

Next we show that X satisfies inequalities (7). That is,

$$-\sum_{e\in\delta(i)\cap E_j} x_{e,j} + \sum_{k=1}^{j-1} x_{[ij],k} \leqslant 0, \quad \text{for } 2 \leqslant j \leqslant n-3,\ i \in V_j. \tag{28}$$

Again, by substitution and simplification we can show that the l.h.s. of the above inequality equals

$$-\sum_{e\in\delta(i)\cap E_j} x_e^j + \sum_{e\in\delta(i)\cap E_{j-1}} x_e^{j-1} - x_{[ij]}^0. \tag{29}$$

From constraints (14), we have $\sum_{e\in\delta(i)\cap E_j} x_e^j = \sum_{e\in\delta(i)\cap E_{j-1}} x_e^{j-1} = 2$, and from constraints (13) we have $x_{[ij]}^0 \geqslant 0$. Thus, X satisfies inequalities (7). Next we show that X satisfies constraints (9), that is,

$$-\sum_{e\in\delta(i)\cap E_1} x_{e,1} \leqslant 0 \quad \text{for } i \in V_1.$$

Using inequalities (15), for $k = 1$ we have

$$x_e^1 - x_e^0 \geqslant 0, \quad e \in E_1.$$

For $i \in V_1$, $\delta(i) \subseteq E_1$, hence we have,

$$\begin{aligned} & x_e^1 - x_e^0 \geqslant 0, \quad e \in \delta(i) \cap E_1,\ i \in V_1, \\ \Rightarrow & \sum_{e\in\delta(i)\cap E_1} x_e^1 - x_e^0 \geqslant 0, \quad i \in V_1, \\ \Rightarrow & \sum_{e\in\delta(i)\cap E_1} x_{e,1} \geqslant 0, \quad i \in V_1. \end{aligned} \tag{30}$$

Finally, we show that X satisfies Eq. (5) by showing

$$\sum_{e\in E_k} x_{e,k} = 1, \quad \forall k \in \{1,\ldots,n-3\}.$$

Note that

$$\sum_{e\in E_k} x_{e,k} = \frac{1}{2}\left[\sum_{i\in V_k} \sum_{e\in\delta(i)\cap E_k} x_{e,k}\right].$$

Now from constraints (14), $\sum_{e\in\delta(i)\cap E_k} x_{e,k}$ can be shown to be equal to $x_{[ik]}^{k-1}$ using

$$\sum_{e\in\delta(i)\cap E_k} x_e^k = \sum_{e\in\delta(i)\cap E_{k-1}} x_e^{k-1} = 2.$$

But $\sum_{i\in V_k} x_{[ik]}^{k-1} = 2$, again from constraints (14).

Thus $\sum_{e\in E_k} x_{e,k} = 1$ as required.

Hence the lemma. □

Thus we have shown the correspondence between the constraints in the two relaxations. The objective functions in the two relaxations differ only by a constant as can be seen from Remark 3.2 and the fact, that the slack variables u in MI-formulation are the x^0 in CS-formulation. This paper establishes the close connection between the MI relaxation and the Cycle-shrink relaxation of STSP.

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