Periods of join algebraic cycles

Jorge Duque Franco[1] and Roberto Villaflor Loyola [2]

Abstract

We determine the cycle class of join algebraic cycles inside smooth hypersurfaces by means of their periods. We show that being a join algebraic cycle is equivalent to have its associated Artin Gorenstein algebra isomorphic to the tensor product of the Artin Gorenstein algebras of each generating cycle. We apply this result to study several Hodge loci of join algebraic cycles, in particular we study when such Hodge loci are smooth and reduced at second order approximation. Moreover, in terms of the Hilbert function of the associated Artin Gorenstein algebra we describe 0-dimensional fake linear cycles, and use them to produce fake linear (and some non-linear) cycles in any dimension and degree.

1 Introduction

Let $\mathbb{P}^{n+1}$ be an odd dimensional projective space (i.e. n even), and consider two odd dimensional (i.e. k is also even) linear subspaces $\mathbb{P}^{k+1}, \mathbb{P}^{n-k-1} \subseteq \mathbb{P}^{n+1}$ such that $\mathbb{P}^{k+1} \cap \mathbb{P}^{n-k-1} = \varnothing$. Assume $X_1 := \{f(x) = 0\} \subseteq \mathbb{P}^{k+1}$ and $X_2 := \{g(y) = 0\} \subseteq \mathbb{P}^{n-k-1}$ are two degree d smooth hypersurfaces, then

$$X := \{f(x) + g(y) = 0\} \subseteq \mathbb{P}^{n+1}$$

is also a smooth degree d hypersurface. Given two half dimensional algebraic subvarieties $Z_1 \subseteq X_1$ and $Z_2 \subseteq X_2$, their *join* $J(Z_1, Z_2) \subseteq X$ is the closure of the union of all lines connecting one point of Z_1 with one point of Z_2 inside $\mathbb{P}^{n+1}$. In other words, in terms of their homogeneous coordinate rings

$$S(J(Z_1, Z_2)) = \frac{\mathbb{C}[x, y]}{I(J(Z_1, Z_2))} = \frac{\mathbb{C}[x]}{I(Z_1)} \otimes \frac{\mathbb{C}[y]}{I(Z_2)} = S(Z_1) \otimes S(Z_2),$$

or equivalently in terms of their affine cones

$$C(J(Z_1, Z_2)) = C(Z_1) \times C(Z_2).$$

The above definition is compatible with rational equivalence and so, it can be extended bilinearly to a map

$$J : \mathrm{CH}^{\frac{k}{2}}(X_1) \otimes \mathrm{CH}^{\frac{n-k-2}{2}}(X_2) \to \mathrm{CH}^{\frac{n}{2}}(X).$$

This is a classical construction in algebraic geometry. Our main result relates the periods of a join of two algebraic cycles with the periods of each generating cycle. We state it in terms of residue forms introduced by Griffiths in his classical work [Gri69].

[1] Universidad de Chile, Facultad de Ciencias, Departamento de Matemáticas, Las Palmeras 3425, Santiago, Chile, georgy11235@gmail.com

[2] Pontificia Universidad Católica de Chile, Facultad de Matemáticas, Campus San Joaquín, Avenida Vicuña Mackenna 4860, Santiago, Chile roberto.villaflor@mat.uc.cl

Theorem 1.1. Let $Z_1 \in \mathrm{CH}^{\frac{k}{2}}(X_1)$ and $Z_2 \in \mathrm{CH}^{\frac{n-k-2}{2}}(X_2)$. For any homogeneous polynomials $P(x) \in \mathbb{C}[x]$ and $Q(y) \in \mathbb{C}[y]$ such that $\deg(P(x) \cdot Q(y)) = (d-2)(\frac{n}{2}+1)$ we have

$$\text{(1)}\quad \frac{\frac{n}{2}!}{\frac{k}{2}! \cdot \frac{n-k-2}{2}!} \int_{J(Z_1,Z_2)} \mathrm{res}\left(\frac{P(x)Q(y)\Omega}{(f(x)+g(y))^{\frac{n}{2}+1}}\right) = -2\pi i \cdot \int_{Z_1} \mathrm{res}\left(\frac{P\Omega'}{f^{\frac{k}{2}+1}}\right) \cdot \int_{Z_2} \mathrm{res}\left(\frac{Q\Omega''}{g^{\frac{n-k}{2}}}\right)$$

if $\deg(P) = (d-2)(\frac{k}{2}+1)$ and $\deg(Q) = (d-2)(\frac{n-k}{2})$, and is zero otherwise. Where Ω, Ω', and Ω'' are the standard top forms of $\mathbb{P}^{n+1}$, $\mathbb{P}^{k+1}$ and $\mathbb{P}^{n-k-1}$ respectively.

The main idea in the proof is a toric birational modification of the ambient space which reduces the computation to a smooth hypersurface of a projective simplicial toric variety, and then use tools recently developed by the second author in [VL23] to describe residue forms along hypersurfaces in toric ambient.

As a consequence of the main theorem we can relate the cycle class of the join of two such algebraic cycles with the cycle class of each of them as follows: Recall first that by Griffiths' theorem [Gri69] we can always write the primitive part of the cycle class of an algebraic cycle $Z \in \mathrm{CH}^{\frac{n}{2}}(X)$ inside a smooth degree d hypersurface $X = \{F = 0\} \subseteq \mathbb{P}^{n+1}$ of even dimension n as a residue form

$$\text{(2)}\qquad [Z]_{\mathrm{prim}} = \frac{(-1)^{\frac{n}{2}+1}\frac{n}{2}!}{d}\mathrm{res}\left(\frac{P_Z\Omega}{F^{\frac{n}{2}+1}}\right)^{\frac{n}{2},\frac{n}{2}}$$

for a unique $P_Z \in R^F_{(d-2)(\frac{n}{2}+1)}$. In consistency with the main result of [VL22a] we say P_Z is the *polynomial associated* to the algebraic cycle Z. Using this notation we can state our second main result.

Theorem 1.2. In the context of Theorem 1.1 we have

$$\text{(3)}\qquad P_{J(Z_1,Z_2)} = P_{Z_1} \cdot P_{Z_2}.$$

Furthermore, if $\delta \in H^{\frac{n}{2},\frac{n}{2}}(X,\mathbb{Q})_{\mathrm{prim}}$ then

$$\text{(4)}\qquad R^{f+g,\delta} = R^{f,[Z_1]} \otimes R^{g,[Z_2]} \iff \delta = c \cdot [J(Z_1,Z_2)]_{\mathrm{prim}} \quad \text{for some } c \in \mathbb{Q}^\times.$$

After understanding the cycle class of a join of two algebraic cycles in terms of the cycle classes of each of them, it is natural to ask if this allows us to relate their corresponding Hodge loci. Recall that given a Hodge cycle $\delta \in H^{\frac{n}{2},\frac{n}{2}}(X,\mathbb{Q})$ we can consider $X = X_{t_0}$ as the central element of the family $\pi : \mathcal{X} \to T$ of all smooth degree d hypersurfaces of even dimension n of $\mathbb{P}^{n+1}$. Then in any small simply connected analytic neighbourhood of t_0 we can do a parallel transport of $\delta = \delta_{t_0}$ to $\delta_t \in H^n(X_t,\mathbb{Q})$ for $t \in (T,t_0)$ and consider the *Hodge locus* to be the germ of analytic subvariety

$$\text{(5)}\qquad V_\delta := \{t \in (T,t_0) : \delta_t \in H^{\frac{n}{2},\frac{n}{2}}(X_t,\mathbb{Q})\}.$$

In fact, the Hodge locus comes with a natural analytic scheme structure which might be non-reduced. This non-reduceness might be detected for instance using the quadratic fundamental form introduced by Maclean (see [Mac05]). As a first application of Theorem 1.2 we relate the quadratic fundamental form of the Hodge loci $V_{[Z_1]}$, $V_{[Z_2]}$ and $V_{[J(Z_1,Z_2)]}$, this is done in Corollary 4.1.

As secondary applications we illustrate how the join description can be used to determine the Artinian Gorenstein ideal associated to some combinations of linear cycles in Fermat varieties and their quadratic fundamental forms. We mainly focus on combinations of two linear cycles of the form $r\mathbb{P}^{\frac{n}{2}} + \check{r}\check{\mathbb{P}}^{\frac{n}{2}} \in \mathrm{CH}^{\frac{n}{2}}(X)$ for $X = \{x_0^d + \cdots + x_{n+1}^d = 0\}$ such that $\dim \mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = m \geq \frac{n}{2} - \frac{d}{d-2}$. This kind of combinations have already been studied by Movasati and the second author [MV18, VL22a, Mov22] as a non-trivial case to study the Variational Hodge Conjecture for reducible algebraic cycles. After [VL22a, Theorem 1.3] it is known that for $m < \frac{n}{2} - \frac{d}{d-2}$ the Hodge locus $V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ is smooth and corresponds to $V_{[\mathbb{P}^{\frac{n}{2}}]} \cap V_{[\check{\mathbb{P}}^{\frac{n}{2}}]}$. The remaining cases which we study in this article were studied first by Movasati [Mov21, Chapter 18] who raised some conjectures about their smoothness based on computational evidence. Later in [Mov22] Movasati conjectured the smoothness of the Hodge locus in the case $(d, m) = (3, \frac{n}{2} - 3)$. This conjecture was recently disproved by Kloosterman for $n \geq 10$ in [Klo23]. In that work Kloosterman also studies the smoothness of the cases $(d, m) = (3, \frac{n}{2} - 2)$ and $(4, \frac{n}{2} - 2)$. Kloosterman's methods rely only on the first order approximation to the Hodge loci, namely the Infinitesimal Variations of Hodge Structure. We complement their results by describing the Artinian Gorenstein ideal of all such algebraic cycles at Fermat, and by showing the vanishing of the quadratic fundamental form. In the remaining case where $m = \frac{n}{2} - 1$ and $r \neq \check{r}$ we show (using the quadratic fundamental form) that the Hodge locus $V_{r[\mathbb{P}^{\frac{n}{2}}]+\check{r}[\check{\mathbb{P}}^{\frac{n}{2}}]}$ is not smooth (see Theorem 5.1). We remark this was already pointed out by Movasati in [Mov21, Theorem 18.3] for a finite number of examples.

As a third application of the join construction we describe algebraic representatives of fake linear cycles in Fermat varieties of degree 3, 4 and 6, which were discovered by the authors in the previous article [DFVL23]. In fact, using the Hilbert function associated to a Hodge cycle (see Definition 6.1) we introduce the notion of fake version of any $\frac{n}{2}$-dimensional algebraic subvariety of any n-dimensional smooth hypersurface (see Definition 7.1). Under this new notion, all fake linear cycles (inside any hypersurface) have codimension of the Zariski tangent space of their associated Hodge loci equal to $\binom{\frac{n}{2}+d}{d} - (\frac{n}{2}+1)^2$, which is conjecturally the smallest possible codimension of a Hodge locus. This bound is known to be attained by linear cycles, and is also known to be lower bound for $d >> n$ (by the work of Otwinowska [Otw02]) and for the Hodge loci passing through the Fermat variety (by the work of Movasati [Mov17]). This led Movasati to conjecture, originally only at the Fermat variety [Mov21, Conjecture 18.8], that the Hodge loci attaining the mentioned bound at the level of the Zariski tangent space are only the Hodge loci of linear cycles. The second author [VL22b] proved this conjecture for all Fermat varieties of degree $d \neq 3, 4, 6$ and later the authors [DFVL23] disproved it for all Fermat varieties of degree $d = 3, 4, 6$ by showing the existence of fake linear cycles. Using the join construction we are able to show the following result.

Theorem 1.3. *For any degree d and even dimension n, there are infinitely many smooth degree d hypersurfaces X of dimension n in $\mathbb{P}^{n+1}$ containing infinitely many $\frac{n}{2}$-dimensional fake linear cycles in $\mathbb{P}(H^{\frac{n}{2},\frac{n}{2}}(X,\mathbb{Q})_{\mathrm{prim}})$.*

The article is organized as follows: in Section 2 we recall some preliminaries about Artinian Gorenstein ideals and the quadratic fundamental form of a Hodge locus. Section 3 is devoted to the proof of Theorem 1.1. In Section 4 we prove Theorem 1.2 and use it to relate the quadratic fundamental forms of the Hodge loci of $V_{[Z_1]}$, $V_{[Z_2]}$ and $V_{[J(Z_1,Z_2)]}$ (this is Corollary 4.1). Section 5 is devoted to the computation of the Artinian Gorenstein ideal and their associated quadratic fundamental form for all combinations of two linear cycles not covered by [VL22a, Theorem 1.3].

In Section 6 we introduce the Hilbert function associated to a Hodge cycle, and use this notion in Section 7 to introduce the concept of fake algebraic cycles. This section also contains the proof of Theorem 1.3.

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2 Preliminaries

2.1 Artinian Gorenstein ideal associated to a Hodge cycle

For the sake of completeness we will briefly recall some known facts about Artinian Gorenstein ideals associated to Hodge cycles in smooth hypersurfaces of the projective space. For a more complete exposition see [VL22b].

Definition 2.1. A graded $\mathbb{C}$-algebra R is *Artinian Gorenstein* if there exist $\sigma \in \mathbb{N}$ such that

(i) $R_e = 0$ for all $e > \sigma$,

(ii) $\dim_{\mathbb{C}} R_\sigma = 1$,

(iii) the multiplication map $R_i \times R_{\sigma-i} \to R_\sigma$ is a perfect pairing for all $i = 0, \dots, \sigma$.

The number $\sigma =: \mathrm{soc}(R)$ is the *socle of* R. We say that an ideal $I \subseteq \mathbb{C}[x_0, \dots, x_{n+1}]$ is *Artinian Gorenstein of socle* $\sigma =: \mathrm{soc}(I)$ if the quotient ring $R = \mathbb{C}[x_0, \dots, x_{n+1}]/I$ is Artinian Gorenstein of socle σ.

The definition of the following ideal appeared first in the work of Voisin [Voi89] for surfaces, and later in the work of Otwinowska [Otw03] for higher dimensional varieties.

Definition 2.2. Let $X = \{F = 0\} \subseteq \mathbb{P}^{n+1}$ be a smooth degree d hypersurface of even dimension n, and $\lambda \in H^{\frac{n}{2},\frac{n}{2}}(X, \mathbb{Z})$ be a non-trivial Hodge cycle. Consider $J^F := \langle \frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_{n+1}} \rangle$ to be the Jacobian ideal, we define the *Artinian Gorenstein ideal associated to* λ as

$$J^{F,\lambda} := (J^F : P_\lambda), \tag{6}$$

where $P_\lambda \in \mathbb{C}[x_0, \dots, x_{n+1}]_{(d-2)(\frac{n}{2}+1)}$ is such that $\lambda_{\mathrm{prim}} = \mathrm{res}\left(\frac{P_\lambda \Omega}{F^{\frac{n}{2}+1}}\right)^{\frac{n}{2},\frac{n}{2}}$. This ideal is Artinian Gorenstein of $\mathrm{soc}(J^{F,\lambda}) = (d-2)(\frac{n}{2}+1) = \frac{1}{2}\mathrm{soc}(J^F)$.

Remark 2.1. The importance of this ideal is that it determines the cycle and its Hodge locus, that is (see for instance [VL22b, Corollary 2.3, Remark 2.3])

$$J^{F,\lambda_1} = J^{F,\lambda_2} \iff \exists c \in \mathbb{Q}^\times : (\lambda_1 - c\cdot\lambda_2)_{\text{prim}} = 0 \iff V_{\lambda_1} = V_{\lambda_2}.$$

This ideal also encodes the information of the first-order approximation of the Hodge loci in a simple way. More precisely, let $T \subseteq \mathbb{C}[x_0,\dots,x_{n+1}]_d$ be the parameter space of smooth degree d hypersurfaces of $\mathbb{P}^{n+1}$, of even dimension n. For $t \in T$, let $X_t = \{F = 0\} \subseteq \mathbb{P}^{n+1}$ be the corresponding hypersurface. For every Hodge cycle $\lambda \in H^{\frac{n}{2},\frac{n}{2}}(X_t,\mathbb{Z})$, we can compute the Zariski tangent space of its associated Hodge locus V_λ as

$$T_tV_\lambda = J_d^{F,\lambda}. \tag{7}$$

Where we have identified $T_tT \simeq \mathbb{C}[x_0,\dots,x_{n+1}]_d$.

2.2 Quadratic fundamental form

In this section we will explore the second order invariant of the IVHS associated to the Hodge locus $V_{[Z]}$ described by Maclean [Mac05]. This invariant allows us to derive geometric information about the Hodge locus, namely the Hodge locus is either singular or non-reduced. For this type of application see [DFVL23].

The quadratic fundamental form was described in the context of surfaces for the classical Noether-Lefschetz loci by Maclean [Mac05]. However in higher dimensions it also gives a partial description of the quadratic fundamental form.

Definition 2.3. Let M be a smooth m-dimensional analytic scheme, V a vector bundle on M and σ a section of V. Let W be the zero locus of σ and let $x \in W$. The *quadratic fundamental form of σ at x* is

$$q_{\sigma,x} : T_xW \otimes T_xW \to V_x/\mathrm{Im}(d\sigma_x)$$

given in local coordinates $(z_1,\dots,z_m)$ around x by

$$q_{\sigma,x}\left(\sum_{i=1}^m \alpha_i \frac{\partial}{\partial z_i}, \sum_{j=1}^m \beta_j \frac{\partial}{\partial z_j}\right) = \sum_{i=1}^m \alpha_i \frac{\partial}{\partial z_i}\left(\sum_{j=1}^m \beta_j \frac{\partial}{\partial z_j}(\sigma)\right).$$

In our context we will take $M = (T,0)$, $V = \bigoplus_{p=0}^{\frac{n}{2}-1} \mathcal{F}^p/\mathcal{F}^{p+1}$ and $x = 0$. Where $T \subseteq H^0(\mathcal{O}_{\mathbb{P}^{n+1}}(d))$ is the parameter space of smooth degree d hypersurfaces of $\mathbb{P}^{n+1}$, $\pi : X \to T$ is the corresponding family, $\mathcal{F}^p = R^n\pi_*\Omega^{\bullet\geq p}_{X/T}$, and $0 \in T$ corresponds to the Fermat variety. In order to construct a section σ of V around x, let $\lambda \in H^{\frac{n}{2},\frac{n}{2}}(X_d^n)_{\text{prim}} \cap H^n(X_d^n,\mathbb{Z})$ be a Hodge cycle, and consider $\overline{\lambda}$ its induced flat section in $\mathcal{F}^0/\mathcal{F}^{\frac{n}{2}}$. If we fix a holomorphic splitting $\mathcal{F}^0/\mathcal{F}^{\frac{n}{2}} \simeq V$ and we take σ as the image of $\overline{\lambda}$ under this splitting, then $W = V_\lambda$. In this context we can identify $T_xW = J_d^{F,\lambda}$ (7), $V_x = \bigoplus_{q=\frac{n}{2}+1}^n R^F_{d(q+1)-n-2}$ and $d\sigma_x = \cdot P_\lambda$. The computation of the degree $d + (d-2)(\frac{n}{2}+1)$ piece of $q = q_{\sigma,x}$ under these identifications was done by Maclean [Mac05, Theorem 7] as follows.

Theorem 2.1 (Maclean)**.** *The degree $r := d+(d-2)(\frac{n}{2}+1)$ piece of the fundamental quadratic form is $q|_{\mathrm{Sym}^2(J_d^{F,\lambda})}$ where*

$$q : \mathrm{Sym}^2(J^{F,\lambda}) \to R^F/\langle P_\lambda\rangle$$

is the bilinear form given by

$$q(G,H)=\sum_{i=0}^{n+1}\left(H\frac{\partial Q_i}{\partial x_i}-R_i\frac{\partial G}{\partial x_i}\right) \tag{8}$$

where

$$G\cdot P_\lambda=\sum_{i=0}^{n+1}Q_i\frac{\partial F}{\partial x_i}\quad\text{ and }\quad H\cdot P_\lambda=\sum_{i=0}^{n+1}R_i\frac{\partial F}{\partial x_i}.$$

Remark 2.2. In particular $q(\cdot,H)=0$ for any $H\in J^F$.

3 Periods of join of algebraic cycles

In this section we compute the periods of joins of algebraic cycles. Then we use this information to relate the cycle class and Artinian Gorenstein ideals of them. In particular, we prove Theorem 1.1. Let us recall the context we are working in.

We start with $\mathbb{P}^{n+1}$ odd dimensional (i.e. n even), and two odd dimensional (i.e. k is also even) linear subspaces $\mathbb{P}^{k+1},\mathbb{P}^{n-k-1}\subseteq\mathbb{P}^{n+1}$ such that $\mathbb{P}^{k+1}\cap\mathbb{P}^{n-k-1}=\varnothing$. Inside them we have the smooth degree d hypersurfaces $X_1:=\{f(x)=0\}\subseteq\mathbb{P}^{k+1}$, $X_2:=\{g(y)=0\}\subseteq\mathbb{P}^{n-k-1}$ and

$$X:=\{f(x)+g(y)=0\}\subseteq\mathbb{P}^{n+1}.$$

Each hypersurface contains a half dimensional algebraic cycle $Z_1\in\mathrm{CH}^{\frac{k}{2}}(X_1)$, $Z_2\in\mathrm{CH}^{\frac{n-k-2}{2}}(X_2)$ and their join $J(Z_1,Z_2)\in\mathrm{CH}^{\frac{n}{2}}(X)$.

Proof of Theorem 1.1 In order to avoid confusion let $u=(u_0:\cdots:u_{k+1})$ be the coordinates of $\mathbb{P}^{k+1}$, $v=(v_0:\cdots:v_{n-k-1})$ be the coordinates of $\mathbb{P}^{n-k-1}$ and $(x:y)=(x_0:\cdots:x_{k+1}:y_0:\cdots:y_{n-k-1})$ be the coordinates of $\mathbb{P}^{n+1}$. Since (1) is independent of the choice of coordinates for $\mathbb{P}^{k+1}$ and $\mathbb{P}^{n-k-1}$, we can assume by Bertini's theorem that $X_1\cap\{u_0=0\}$ and $X_2\cap\{v_0=0\}$ are smooth hyperplane sections. By the bilinearity of (1) we can reduce ourselves to the case of monomials $P(x)=x^\alpha$ and $Q(y)=y^\beta$. Let us treat first the case where $\deg(x^\alpha)=(d-2)(\frac{k}{2}+1)$ and $\deg(y^\beta)=(d-2)(\frac{n-k}{2})$. Let us denote

$$\omega_{\alpha\beta}:=\mathrm{res}\left(\frac{x^\alpha y^\beta\Omega}{(f(x)+g(y))^{\frac{n}{2}+1}}\right)^{\frac{n}{2},\frac{n}{2}}\in H^{\frac{n}{2}}(X,\Omega_X^{\frac{n}{2}}),$$

$$\omega_{\alpha}:=\mathrm{res}\left(\frac{u^\alpha\Omega'}{f(u)^{\frac{k}{2}+1}}\right)^{\frac{k}{2},\frac{k}{2}}\in H^{\frac{k}{2}}(X_1,\Omega_{X_1}^{\frac{k}{2}}),$$

$$\omega_{\beta}:=\mathrm{res}\left(\frac{v^\beta\Omega''}{g(v)^{\frac{n-k}{2}}}\right)^{\frac{n-k-2}{2},\frac{n-k-2}{2}}\in H^{\frac{n-k-2}{2}}(X_2,\Omega_{X_2}^{\frac{n-k-2}{2}}).$$

Consider the birational map

$$\varphi:\mathbb{P}^{k+1}\times\mathbb{P}^{n-k-1}\times\mathbb{P}^1\dashrightarrow\mathbb{P}^{n+1}$$

$$\varphi(u,v,t) = (t_0 v_0 u : t_1 u_0 v)$$

whose indeterminacy locus is given by $C_0 \cup C_1 \cup C_2$ for

$$C_0 = \{u_0 = v_0 = 0\}, \quad C_1 = \{u_0 = t_0 = 0\}, \quad C_2 = \{v_0 = t_1 = 0\}.$$

Let $\mathbb{P}_\Sigma$ be the projective simplicial toric variety obtained by successively blowing-up C_0, C_1 and C_2

$$\begin{tikzcd} \mathbb{P}_\Sigma \arrow[r, "\pi"] \arrow[dr, "\widetilde{\varphi}"'] & \mathbb{P}^{k+1} \times \mathbb{P}^{n-k-1} \times \mathbb{P}^1 \arrow[d, dashed, "\varphi"] \\ & \mathbb{P}^{n+1} \end{tikzcd}.$$

Let us denote their Cox rings by

$$\begin{aligned} S(\mathbb{P}^{n+1}) &= \mathbb{C}[x_0,\dots,x_{k+1},y_0,\dots,y_{n-k-1}], \\ S(\mathbb{P}^{k+1}\times\mathbb{P}^{n-k-1}\times\mathbb{P}^1) &= \mathbb{C}[u_0,\dots,u_{k+1},v_0,\dots,v_{n-k-1},t_0,t_1], \\ S(\mathbb{P}_\Sigma) &= \mathbb{C}[a_0,\dots,a_{k+1},b_0,\dots,b_{n-k-1},s_0,s_1,e_0,e_1,e_2], \end{aligned}$$

hence we have the identifications induced by φ and π

$$\begin{gathered} x_0 = t_0 u_0 v_0, \ \dots\ , \quad x_{k+1} = t_0 u_{k+1} v_0, \\ y_0 = t_1 u_0 v_0, \ \dots\ , \ y_{n-k-1} = t_1 u_0 v_{n-k-1}, \\ u_0 = a_0 e_0 e_1, \ u_1 = a_1, \ \dots\ , \quad u_{k+1} = a_{k+1}, \\ v_0 = b_0 e_0 e_2, \quad v_1 = b_1, \ \dots\ , v_{n-k-1} = b_{n-k-1}, \\ t_0 = s_0 e_1, \ t_1 = s_1 e_2. \end{gathered}$$

In order to understand the fan of $\mathbb{P}_\Sigma$ let us write first the primitive generators of the rays corresponding to each variable. Let M_1, M_2 and M_3 be the character lattices of $\mathbb{P}^{k+1}$, $\mathbb{P}^{n-k-1}$ and $\mathbb{P}^1$ respectively. Let $N_i := M_i^\vee$ be the dual lattice. Then, the primitive generators of the rays of Σ belong to $N := N_1 \oplus N_2 \oplus N_3$. Let us denote by $\{r_j^{(i)}\}_j$ the canonical basis of N_i, then the primitive generators of the rays of $\Sigma(1)$ correspond to

$$\rho_{a_i} = (r_i^{(1)}, 0, 0), \quad \rho_{a_0} = -\sum_{i=1}^{k+1} \rho_{a_i}, \quad \rho_{b_j} = (0, r_j^{(2)}, 0), \quad \rho_{b_0} = -\sum_{j=1}^{n-k-1} \rho_{b_j},$$

$$\rho_{s_1} = (0,0,r_1^{(3)}), \quad \rho_{s_0} = -\rho_{s_1}, \quad \rho_{e_0} = \rho_{a_0} + \rho_{b_0}, \quad \rho_{e_1} = \rho_{a_0} + \rho_{s_0}, \quad \rho_{e_2} = \rho_{b_0} + \rho_{s_1},$$

for $i = 1,\dots,k+1$ and $j = 1,\dots,n-k-1$. In order to describe the (maximal) cones of $\Sigma(n+1)$, we write the generators of its irrelevant ideal as follows

$$\begin{aligned} B(\Sigma) = \Big\langle \big\{ & a_0 a_i b_0 b_j s_0 e_1, \ a_0 a_i b_j s_0 s_1 e_1, \ a_0 a_i b_j s_1 e_1 e_2, \ a_i b_0 b_j s_0 e_1 e_2, \ a_0 a_i b_0 b_j s_1 e_2, \\ & a_i b_0 b_j s_0 s_1 e_2, \ a_0 b_0 b_j s_0 e_0 e_1, \ a_0 b_j s_0 s_1 e_0 e_1, \ a_0 b_j s_1 e_0 e_1 e_2, \ a_i b_0 s_0 e_0 e_1 e_2, \\ & a_0 a_i b_0 s_1 e_0 e_2, \ a_i b_0 s_0 s_1 e_0 e_2, \ a_0 b_0 s_0 e_0 e_1 e_2, \ a_0 b_0 s_1 e_0 e_1 e_2 \big\}_{\substack{1 \le i \le k+1 \\ 1 \le j \le n-k-1}} \Big\rangle. \end{aligned}$$

Let $Y \subseteq \mathbb{P}_\Sigma$ be the strict transform of $X \subseteq \mathbb{P}^{n+1}$ under the birational morphism $\widetilde{\varphi}$. In particular

$$Y = \{F := (s_0 b_0)^d f(u) + (s_1 a_0)^d g(v) = 0\} \subseteq \mathbb{P}_\Sigma$$

is a smooth hypersurface (here we use that $X_1 \cap \{u_0 = 0\}$ and $X_2 \cap \{v_0 = 0\}$ are smooth). Let $W \in \mathrm{CH}^{\frac{n}{2}}(Y)$ be the strict transform of $J(Z_1, Z_2) \in \mathrm{CH}^{\frac{n}{2}}(X)$. Since $\pi_*(W) = Z_1 \times Z_2 \times \mathbb{P}^1$, in order to obtain (1) it is enough to check that

$$\frac{\frac{n}{2}!}{\frac{k}{2}! \cdot \frac{n-k-2}{2}!} \cdot \int_W \widetilde{\varphi}^* \omega_{\alpha\beta} = - \int_W \pi^*(\mathrm{pr}_1^* \omega_\alpha \cup \mathrm{pr}_2^* \omega_\beta \cup \mathrm{pr}_3^* \theta) \tag{9}$$

for $\theta \in H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1})$ the polarization (whose period is $2\pi i$). Let

$$X_{1,2} := \{f(x) = g(y) = 0\} \subseteq X \subseteq \mathbb{P}^{n+1}$$

which is a smooth complete intersection of bi-degree (d, d), and $J(Z_1, Z_2) \in \mathrm{CH}^{\frac{n}{2}}(X_{1,2})$. Since the open sets $V_j := \{x_j \frac{\partial f(x)}{\partial x_j} \neq 0\}$ and $V'_\ell := \{y_\ell \frac{\partial g(y)}{\partial y_\ell} \neq 0\}$ cover $X_{1,2}$ for $j = 0, \dots, k$ and $\ell = 0, \dots, n-k-2$, we can assume by the moving lemma that $J(Z_1, Z_2)$ is supported in a collection of smooth subvarieties of $X_{1,2}$ contained in $\bigcup_{j=0}^{\frac{k}{2}} U_j \cup \bigcup_{\ell=0}^{\frac{n-k-2}{2}} V_\ell$. Let us denote by $W_0 \subseteq Y$ the strict transform of any of such subvarieties. Thus, in order to prove (9) it is enough to show that

$$\frac{\frac{n}{2}!}{\frac{k}{2}! \cdot \frac{n-k-2}{2}!} \cdot \widetilde{\varphi}^* \omega_{\alpha\beta}|_{W_0} = -\pi^*(\mathrm{pr}_1^* \omega_\alpha \cup \mathrm{pr}_2^* \omega_\beta \cup \mathrm{pr}_3^* \theta)|_{W_0} \tag{10}$$

in $H^n_{\mathrm{dR}}(W_0, \mathbb{C}) \simeq H^{\frac{n}{2}}(W_0, \Omega^{\frac{n}{2}}_{W_0})$. We can compute the left hand side of (10) using a toric version of a theorem due to Carlson and Griffiths [VL23, Theorem 8.1] which computes the residue map in Cech cohomology relative to the Jacobian cover $\mathcal{U} = \{U_i\}_{i=0}^{n+6}$ of Y, where $U_i = \{F_i \neq 0\}$ and F_i are the partial derivatives of F with respect to the homogeneous coordinates of $\mathbb{P}_\Sigma$. Let us denote by Ω''' the standard top form of $\mathbb{P}_\Sigma$. Since $\widetilde{\varphi}^*\Omega = -e_0 e_1 e_2 u_0^{n-k} v_0^{k+2} t_0^{k+1} t_1^{n-k-1} \Omega'''$, we get

$$\begin{aligned} \widetilde{\varphi}^* \omega_{\alpha\beta} &= \mathrm{res}\left(\frac{(t_0 v_0)^{(d-2)(\frac{k}{2}+1)} (t_1 u_0)^{(d-2)(\frac{n-k}{2})} u^\alpha v^\beta \widetilde{\varphi}^* \Omega}{(e_0 e_1 e_2)^{n+2} F^{\frac{n}{2}+1}} \right)^{\frac{n}{2}, \frac{n}{2}} \\ &= -\mathrm{res}\left(\frac{s_0^{d(\frac{k}{2}+1)-1} s_1^{d(\frac{n-k}{2})-1} a_0^{d(\frac{n-k}{2})} b_0^{d(\frac{k}{2}+1)} e_0 u^\alpha v^\beta \Omega'''}{F^{\frac{n}{2}+1}} \right)^{\frac{n}{2}, \frac{n}{2}} \\ &= \frac{-1}{\frac{n}{2}!} \left\{ \frac{s_0^{d(\frac{k}{2}+1)-1} s_1^{d(\frac{n-k}{2})-1} a_0^{d(\frac{n-k}{2})} b_0^{d(\frac{k}{2}+1)} e_0 u^\alpha v^\beta \Omega'''_J}{F_J} \right\}_{|J| = \frac{n}{2}+1} \in H^{\frac{n}{2}}(\mathcal{U}, \Omega^{\frac{n}{2}}_Y), \end{aligned} \tag{11}$$

where we are using the notation from [VL23]. On the other hand we have

$$\pi^* \mathrm{pr}_1^* \omega_\alpha|_{W_0} = \frac{1}{\frac{k}{2}!} \left\{ \frac{u^\alpha \Omega'_K}{f_K} \right\}_{|K| = \frac{k}{2}+1} \in H^{\frac{k}{2}}(\pi^{-1} \mathrm{pr}_1^{-1} \mathcal{U}_1, \Omega^{\frac{k}{2}}_{W_0}),$$

$$\pi^* \mathrm{pr}_2^* \omega_\beta|_{W_0} = \frac{1}{\frac{n-k-2}{2}!} \left\{ \frac{v^\beta \Omega''_L}{g_L} \right\}_{|L| = \frac{n-k}{2}} \in H^{\frac{n-k-2}{2}}(\pi^{-1} \mathrm{pr}_2^{-1} \mathcal{U}_2, \Omega^{\frac{n-k-2}{2}}_{W_0}),$$

$$\pi^* \mathrm{pr}_3^* \theta|_{W_0} = \frac{t_0 dt_1 - t_1 dt_0}{t_0 t_1} \in H^1(\pi^{-1} \mathrm{pr}_3^{-1} \mathcal{U}_3, \Omega^1_{W_0}),$$

where $\mathcal{U}_1$ and $\mathcal{U}_2$ are the Jacobian covers of X_1 and X_2 respectively, while $\mathcal{U}_3$ is the standard open cover of $\mathbb{P}^1$. When restricted to W_0, the coverings $\pi^{-1}\mathrm{pr}_1^{-1}\mathcal{U}_1$, $\pi^{-1}\mathrm{pr}_2^{-1}\mathcal{U}_2$ and $\pi^{-1}\mathrm{pr}_3^{-1}\mathcal{U}_3$ admit a common refinement $\mathcal{V} = \{V_{(j,\ell,r)}\}_{(j,\ell,r)} = \{V_{(j,0,0)}\}_{j=0}^{\frac{k}{2}} \cup \{V_{(0,\ell,1)}\}_{\ell=0}^{\frac{n-k-2}{2}}$ where

$$V_{(j,0,0)} = \{u_j f_j(u) v_0 t_0 \neq 0\} = \widetilde{\varphi}^{-1} V_j \quad \text{and} \quad V_{(0,\ell,1)} = \{u_0 v_\ell g_\ell(v) t_1 \neq 0\} = \widetilde{\varphi}^{-1} V'_\ell.$$

Hence

$$\frac{k}{2}! \cdot \frac{n-k-2}{2}! \cdot (\pi^*\mathrm{pr}_1^*\omega_\alpha|_{W_0} \cup \pi^*\mathrm{pr}_2^*\omega_\beta|_{W_0} \cup \pi^*\mathrm{pr}_3^*\theta|_{W_0})_{(j_1,\ell_1,r_1),\ldots,(j_{\frac{n}{2}+1},\ell_{\frac{n}{2}+1},r_{\frac{n}{2}+1})} =$$
$$(-1)^{\frac{nk}{4}+\frac{n}{2}+1} \cdot \frac{u^\alpha v^\beta \Omega'_{(j_1,\ldots,j_{\frac{k}{2}+1})} \wedge \Omega''_{(\ell_{\frac{k}{2}+1},\ldots,\ell_{\frac{n}{2}})} \wedge (t_{r_{\frac{n}{2}}} dt_{r_{\frac{n}{2}+1}} - t_{r_{\frac{n}{2}+1}} dt_{r_{\frac{n}{2}}})}{f_{(j_1,\ldots,j_{\frac{k}{2}+1})}(u) g_{(\ell_{\frac{k}{2}+1},\ldots,\ell_{\frac{n}{2}})}(v) t_{r_{\frac{n}{2}}} t_{r_{\frac{n}{2}+1}}}$$

in Cech cohomology relative to the cover $\mathcal{V}$. We remark that the above formula for the cup product in Cech cohomology is well defined only for ordered tuples of indexes (the other tuples are defined by skew-symmetric extension), and the cohomology class is independent of this choice of the ordering. We will order the tuples (j,ℓ,r) lexicographically but with decreasing order in each entry. Then for an ordered set of tuples

$$\frac{k}{2}! \cdot \frac{n-k-2}{2}! \cdot (\pi^*(\mathrm{pr}_1^*\omega_\alpha \cup \mathrm{pr}_2^*\omega_\beta \cup \mathrm{pr}_3^*\theta)|_{W_0})_{(j_1,\ell_1,r_1),\ldots,(j_{\frac{n}{2}+1},\ell_{\frac{n}{2}+1},r_{\frac{n}{2}+1})} = \tag{12}$$
$$(-1)^{\frac{nk}{4}+\frac{n}{2}+1} \cdot \frac{u^\alpha v^\beta \pi^*(\Omega'_{(\frac{k}{2},\ldots,0)}) \wedge \pi^*(\Omega''_{(\frac{n-k-2}{2},\ldots,0)}) \wedge (s_1 e_2 d(s_0 e_1) - s_0 e_1 d(s_1 e_2))}{f_{(0,\ldots,\frac{k}{2})}(u) g_{(0,\ldots,\frac{n-k-2}{2})}(v) s_0 s_1 e_1 e_2}$$

if $(j_1,\ldots,j_{\frac{k}{2}+1}) = (\frac{k}{2},\ldots,0)$, $(\ell_{\frac{k}{2}+1},\ldots,\ell_{\frac{n}{2}}) = (\frac{n-k-2}{2},\ldots,0)$ and $(r_{\frac{n}{2}}, r_{\frac{n}{2}+1}) = (1,0)$, and is zero otherwise. Now it is routine to verify (10) in the open covering $\mathcal{V}$ (which is a sub-covering of $\mathcal{U}|_{W_0}$) using (11) and (12).

For the case where $\deg(x^\alpha) \neq (d-2)(\frac{k}{2}+1)$, let $r := \deg(x^\alpha) - (d-2)(\frac{k}{2}+1)$. By the same argument as above, it is enough for us to show that

$$\widetilde{\varphi}^*\omega_{\alpha\beta}|_{W_0} = 0 \in H^{\frac{n}{2}}(\mathcal{V}, \Omega^{\frac{n}{2}}_{W_0}).$$

Using (11) in this covering we can write

$$\widetilde{\varphi}^*\omega_{\alpha\beta}|_{W_0} = \pi^*(\mathrm{pr}_{12}^*\eta \cup \mathrm{pr}_3^*\widetilde{\theta})|_{W_0},$$

for $\eta \in H^{\frac{n}{2}-1}(\mathrm{pr}_{12}(\mathcal{V}), \Omega^{\frac{n}{2}-1}_{X_1\times X_2}|_{\mathrm{pr}_{12}(W_0)})$ given by

$$\eta_{(j_1,\ell_1,r_1),\ldots,(j_{\frac{n}{2}},\ell_{\frac{n}{2}},r_{\frac{n}{2}})} = \left(\frac{v_0}{u_0}\right)^r \cdot \frac{u^\alpha v^\beta \Omega'_{(j_1,\ldots,j_{\frac{k}{2}+1})} \wedge \Omega''_{(\ell_{\frac{k}{2}+1},\ldots,\ell_{\frac{n}{2}})}}{f_{(j_1,\ldots,j_{\frac{k}{2}+1})}(u) g_{(\ell_{\frac{k}{2}+1},\ldots,\ell_{\frac{n}{2}})}(v)},$$

where each open set of the covering $\mathrm{pr}_{12}(\mathcal{V}) = \{T_{(j,\ell,r)}\} = \{T_{(j,0,0)}\}_{j=0}^{\frac{k}{2}} \cup \{T_{(0,\ell,1)}\}_{\ell=0}^{\frac{n-k-2}{2}}$ is of the form $T_{(j,0,0)} = \{u_j v_0 f_j(u) \neq 0\}$ or $T_{(0,\ell,1)} = \{u_0 v_\ell g_\ell(v) \neq 0\}$. And where

$$\widetilde{\theta} = \left(\frac{t_0}{t_1}\right)^r \theta \in H^1(\mathcal{U}_3, \Omega^1_{\mathbb{P}^1}).$$

The result follows since $\widetilde{\theta} = 0$ for $r \neq 0$. ∎

4 Cycle class and Hodge loci of join algebraic cycles

In this section we translate the periods relation of Theorem 1.1 into relations of the corresponding cycle classes and Hodge loci in the context of join algebraic cycles. The first relation is the content of Theorem 1.2 which we prove in the following.

Proof of Theorem 1.2 Applying [VL22a, Proposition 6.1] to Theorem 1.1 we obtain

$$c = \frac{\frac{n}{2}! \cdot d}{\frac{k}{2}! \cdot \frac{n-k-2}{2}!} c_1 c_2 \tag{13}$$

where $c, c_1, c_2 \in \mathbb{C}^\times$ are the unique complex numbers such that

$$\frac{(-1)^{\frac{n}{2}+1}\frac{n}{2}!}{d} PQP_{J(Z_1,Z_2)} \equiv c \cdot \det(\mathrm{Hess}(f+g)) \pmod{J^{f+g}}$$
$$\frac{(-1)^{\frac{k}{2}+1}\frac{k}{2}!}{d} PP_{Z_1} \equiv c_1 \cdot \det(\mathrm{Hess}(f)) \pmod{J^{f}}$$
$$\frac{(-1)^{\frac{n-k}{2}}\frac{n-k-2}{2}!}{d} QP_{Z_2} \equiv c_2 \cdot \det(\mathrm{Hess}(g)) \pmod{J^{g}}$$

for $P \in \mathbb{C}[x]_{(d-2)(\frac{k}{2}+1)}$ and $Q \in \mathbb{C}[y]_{(d-2)(\frac{n-k}{2})}$. Since $R^{f+g} = R^f \otimes R^g$ and $\det(\mathrm{Hess}(f+g)) = \det(\mathrm{Hess}(f)) \cdot \det(\mathrm{Hess}(g))$ it follows that

$$PQP_{J(Z_1,Z_2)} \equiv PQP_{Z_1}P_{Z_2} \pmod{J^{f+g}}$$

for all $P \in \mathbb{C}[x]_{(d-2)(\frac{k}{2}+1)}$ and $Q \in \mathbb{C}[y]_{(d-2)(\frac{n-k}{2})}$. In particular,

$$x^\alpha y^\beta (P_{J(Z_1,Z_2)} - P_{Z_1}P_{Z_2}) = 0 \in R^{f+g} \tag{14}$$

for all monomials such that $\deg(x^\alpha) = (d-2)(\frac{k}{2}+1)$ and $\deg(y^\beta) = (d-2)(\frac{n-k}{2})$. On the other hand if $\deg(x^\alpha) > (d-2)(\frac{k}{2}+1)$ then $x^\alpha P_{Z_1} = 0 \in R^{f+g}$ and similarly if $\deg(y^\beta) > (d-2)(\frac{n-k}{2})$ then $y^\beta P_{Z_2} = 0 \in R^{f+g}$. Hence, it follows from the second part of Theorem 1.1 that (14) holds for any monomial of degree $\deg(x^\alpha y^\beta) = (d-2)(\frac{n}{2}+1)$. Since R^{f+g} is Artinian Gorenstein of socle in degree $(d-2)(n+2)$ we obtain (3).

Now if an element $T \in R^{f+g}_e$ is zero in $R^{f+g,\delta} = R^{f,[Z_1]} \otimes R^{g,[Z_2]}$ then

$$T = \sum_{i=0}^{e} T_i(x) \cdot \check{T}_{e-i}(y)$$

where $T_i(x) \in R^f_i$, $\check{T}_{e-i}(y) \in R^g_{e-i}$ and for each $i = 0, \dots, e$, we have $T_i \in (J^f : P_{Z_1})$ or $\check{T}_{e-i} \in (J^g : P_{Z_2})$. Hence such a T satisfies that $T \cdot P_{Z_1} \cdot P_{Z_2} = 0 \in R^{f+g}$ and so

$$J^{f+g,\delta} \subseteq (J^{f+g}, P_{Z_1} \cdot P_{Z_2}) = J^{f+g,[J(Z_1,Z_2)]}.$$

Since both are Artinian Gorenstein ideals of socle in degree $(d-2)(\frac{n}{2}+1)$, they are equal and (4) follows from Remark 2.1. ■

In Section 2.2 we recalled the quadratic fundamental form, which is a second order invariant of the Hodge loci that vanishes when the corresponding Hodge locus is smooth and reduced. As a consequence of Theorem 1.2 we can relate the quadratic fundamental form of $V_{[J(Z_1,Z_2)]}$ with those of $V_{[Z_1]}$ and $V_{[Z_2]}$ as follows.

Corollary 4.1. In the same context of Theorem 1.1 let us denote by

$$q : \mathrm{Sym}^2(J^{f+g,[J(Z_1,Z_2)]}) \to R^{f+g}/\langle P_{Z_1} \cdot P_{Z_2} \rangle,$$
$$q_1 : \mathrm{Sym}^2(J^{f,[Z_1]}) \to R^f/\langle P_{Z_1} \rangle,$$
$$q_2 : \mathrm{Sym}^2(J^{g,[Z_2]}) \to R^g/\langle P_{Z_2} \rangle,$$

the bilinear forms introduced in (8) associated to $J(Z_1, Z_2)$, Z_1 and Z_2 respectively. Consider

$$G = A_1(x,y)G_1(x) + A_2(x,y)G_2(y) \in J^{f+g,[J(Z_1,Z_2)]}$$
$$H = B_1(x,y)H_1(x) + B_2(x,y)H_2(y) \in J^{f+g,[J(Z_1,Z_2)]}$$

with $G_1, H_1 \in J^{f,[Z_1]}$, $G_2, H_2 \in J^{g,[Z_2]}$. Then

$$q(G,H) = A_1 B_1 P_{Z_2} q_1(G_1, H_1) + A_2 B_2 P_{Z_1} q_2(G_2, H_2). \tag{15}$$

In consequence for any degree $e \geq 0$ we have the following:

(i) If q_1 and q_2 vanish in all degrees $\ell \leq e$, then q vanishes in degree e.

(ii) If q vanishes in degree e, then q_1 vanishes in all degrees $\ell \leq e$ such that $2(e-\ell) \leq (d-2)(\frac{n-k}{2})$ and q_2 vanishes in all degrees $\ell \leq e$ such that $2(e-\ell) \leq (d-2)(\frac{k}{2}+1)$.

Proof Write

$$G_1 \cdot P_{Z_1} = \sum_{i=0}^{k+1} Q_i(x) \frac{\partial f}{\partial x_i} \quad , \quad G_2 \cdot P_{Z_2} = \sum_{j=0}^{n-k-1} R_j(y) \frac{\partial g}{\partial y_j} \ ,$$
$$H_1 \cdot P_{Z_1} = \sum_{i=0}^{k+1} S_i(x) \frac{\partial f}{\partial x_i} \quad , \quad H_2 \cdot P_{Z_2} = \sum_{j=0}^{n-k-1} T_j(y) \frac{\partial g}{\partial y_j} \ .$$

then it follows by (8) that

$$q(G,H) = A_1 B_1 P_{Z_2} q_1(G_1, H_1) + A_2 B_2 P_{Z_1} q_2(G_2, H_2)$$
$$+ B_1 P_{Z_2} \sum_{i=0}^{k+1} (H_1 Q_i - G_1 S_i) \frac{\partial A_1}{\partial x_i} + B_2 P_{Z_1} \sum_{j=0}^{n-k-1} (H_2 R_j - G_2 T_j) \frac{\partial A_2}{\partial y_j}.$$

Note that $\sum_{i=0}^{k+1}(H_1 Q_i - G_1 S_i)\frac{\partial f}{\partial x_i} = 0$ and $\sum_{j=0}^{n-k-1}(H_2 R_j - G_2 T_j)\frac{\partial g}{\partial y_j} = 0$. Since $(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_{k+1}})$ and $(\frac{\partial g}{\partial y_0}, \dots, \frac{\partial g}{\partial y_{n-k-1}})$ are regular sequences, it follows by the exactness of the Koszul complex that $H_1 Q_i - G_1 S_i \in J^f$ and $H_2 R_j - G_2 T_j \in J^g$, and so we obtain (15). From (15) we obtain (i) by a direct computation in the generators of $J^{f+g,[J(Z_1,Z_2)]}$ which are generators of either $J^{f,[Z_1]}$ or $J^{g,[Z_2]}$ (by (4)).

In order to show (ii) consider $G = H = A(y)G_1(x)$ for any $\ell \leq e$ and any $G_1 \in J_\ell^{f,[Z_1]}$. Then by (15)

$$A^2 \cdot P_{Z_2} \cdot q_1(G_1, G_1) = 0 \in R^{f+g}/\langle P_{Z_1} \cdot P_{Z_2} \rangle$$

for all $A \in \mathbb{C}[y]_{e-\ell}$. In particular, for any monomial $y^\beta \in \mathbb{C}[y]_{2(e-\ell)}$ we can write it as $y^\beta = A_1 A_2$ with $A_1, A_2 \in \mathbb{C}[y]_{e-\ell}$ and so

$$y^\beta \cdot P_{Z_2} \cdot q_1(G_1, G_1) = \left(\frac{(A_1 + A_2)^2}{4} - \frac{(A_1 - A_2)^2}{4} \right) P_{Z_2} \cdot q_1(G_1, G_1) = 0 \in R^{f+g}/\langle P_{Z_1} \cdot P_{Z_2} \rangle.$$

From this, it follows in fact that for any polynomial $S(y) \in \mathbb{C}[y]_{2(e-\ell)}$

$$S(y) \cdot P_{Z_2} \cdot q_1(G_1, G_1) = 0 \in R^{f+g}/\langle P_{Z_1} \cdot P_{Z_2} \rangle.$$

As $2(e - \ell) \leq (d - 2)(\frac{n-k}{2})$ we can take some $S(y)$ such that $S(y) \cdot P_{Z_2} \notin J^g$, then there exists some $T(x, y) \in \mathbb{C}[x, y]$ of bi-degree $(\ell, 2(e - \ell))$ such that

$$P_{Z_2}(S(y) \cdot q_1(G_1, G_1) - T(x, y) \cdot P_{Z_1}) \in J^{f+g}. \tag{16}$$

Considering $S_1(y), \dots, S_t(y) \in \mathbb{C}[y]_{2(e-\ell)}$ such that $\{S(y), S_1(y), \dots, S_t(y)\}$ is a basis of $R^g_{2(e-\ell)}$ and $\{S_1(y), \dots, S_p(y)\}$ is a basis of $\ker(R^g_{2(e-\ell)} \xrightarrow{\cdot P_{Z_2}} R^g_{2(e-\ell)+(d-2)(\frac{n-k}{2})})$ we can write

$$T(x, y) = U(x)S(y) + \sum_{h=1}^{t} U_h(x) S_h(y) \in R^{f+g}.$$

Since $\{P_{Z_2}S(y), P_{Z_2}S_{p+1}(y), \dots, P_{Z_2}S_t(y)\}$ is a basis of $R^g_{2(e-\ell)+(d-2)(\frac{n-k}{2})}$ and $R^{f+g} = R^f \otimes R^g$, then (16) is equivalent to have

$$q_1(G_1, G_1) - U(x) \cdot P_{Z_1} = 0 \in R^f$$

and $P_{Z_1} U_h(x) = 0 \in R^f$ for all $h = p+1, \dots, t$. Therefore $q_1(G_1, G_1) = 0 \in R^f/\langle P_{Z_1} \rangle$. Similarly one gets that $q_2(G_2, G_2) = 0$ for $G_2(y) \in J^{g,[Z_2]}_\ell$ and $\ell \leq e$ such that $2(e-\ell) \leq (d-2)(\frac{k}{2}+1)$. ■

5 Examples in Fermat varieties

In this section we give examples of join algebraic cycles inside Fermat varieties, illustrating how we can use the join structure to simplify their study. We focus on combinations of two linear cycles inside low degrees Fermat varieties, whose corresponding Hodge locus is not known to be reduced.

Along this section $X := \{F := x_0^d + \cdots + x_{n+1}^d = 0\}$ is the degree d Fermat variety of even dimension n. Its automorphism group corresponds to $\mathrm{Aut}(X) = G \rtimes \mathfrak{S}_{n+2}$, where $\mathfrak{S}_{n+2}$ acts by permutation on the coordinates and $G = (\mathbb{Z}/d\mathbb{Z})^{n+2}/\mathrm{Im}(a \in \mathbb{Z}/d\mathbb{Z} \mapsto (a, \dots, a) \in (\mathbb{Z}/d\mathbb{Z})^{n+2}) \simeq (\mathbb{Z}/d\mathbb{Z})^{n+1}$ acts diagonally as

$$g \cdot (x_0 : \cdots : x_{n+1}) = (\zeta_d^{g_0} x_0 : \cdots : \zeta_d^{g_{n+1}} x_{n+1}),$$

where for any $k > 0$, ζ_k denotes the k-th primitive root of unity $e^{\frac{2\pi i}{k}}$. The Fermat variety contains several $\frac{n}{2}$-dimensional linear cycles, which are obtained as the orbit under the action of $\mathrm{Aut}(X)$ on the cycle

$$\mathbb{P}^{\frac{n}{2}} := \{x_0 - \zeta_{2d} x_1 = x_2 - \zeta_{2d} x_3 = \cdots = x_n - \zeta_{2d} x_{n+1} = 0\}.$$

Example 5.1. Consider the zero dimensional Fermat variety $X_0 = \{x_0^d + x_1^d = 0\}$, and a point $Z_0 = \{(\zeta_{2d} : 1)\} \subseteq X_0$. Since this is a complete intersection cycle, it follows by [VL22a, Theorem 1.1] that the cycle class of Z_0 has primitive part

$$[Z_0]_{\text{prim}} = \frac{-1}{d}\text{res}\left(\frac{P_{Z_0}(x_0dx_1 - x_1dx_0)}{x_0^d + x_1^d}\right)$$

for the associated degree $(d-2)$ polynomial

$$P_{Z_0} = d\zeta_{2d}\left(\frac{x_0^{d-1} - (\zeta_{2d}x_1)^{d-1}}{x_0 - \zeta_{2d}x_1}\right).$$

Consequently $J^{x_0^d+x_1^d,[Z_0]} = \langle x_0 - \zeta_{2d}x_1, x_1^{d-1}\rangle$ and the quadratic fundamental form q vanishes (by Remark 2.2 this is reduced to check that $q(x_0 - \zeta_{2d}x_1, x_0 - \zeta_{2d}x_1) = 0$).

For higher dimensions, the Fermat polynomial $x_0^d + \cdots + x_{n+1}^d$ can be written as a sum of $\frac{n}{2}+1$ Fermat polynomials in two variables. Let $X_i = \{x_{2i-2}^d + x_{2i-1}^d = 0\}$, and $Z_i = \{(\zeta_{2d} : 1)\} \subseteq X_i$ for each $i = 1, \ldots, \frac{n}{2}+1$, then

$$\mathbb{P}^{\frac{n}{2}} = J(Z_1, \ldots, Z_{\frac{n}{2}+1}).$$

In consequence

$$P_{\mathbb{P}^{\frac{n}{2}}} = d^{\frac{n}{2}+1}\zeta_{2d}^{\frac{n}{2}+1}\prod_{i=1}^{\frac{n}{2}+1}\left(\frac{x_{2i-2}^{d-1} - (\zeta_{2d}x_{2i-1})^{d-1}}{x_{2i-2} - \zeta_{2d}x_{2i-1}}\right)$$

and so $J^{F,[\mathbb{P}^{\frac{n}{2}}]} = \langle x_0 - \zeta_{2d}x_1, x_1^{d-1}, \ldots, x_n - \zeta_{2d}x_{n+1}, x_{n+1}^{d-1}\rangle$. By item (i) of Corollary 4.1 its quadratic fundamental form also vanishes. One can do similar computations for all other linear cycles in the Fermat variety.

Example 5.2. Let $-1 \leq m \leq \frac{n}{2}$ be an integer. Consider inside $\mathbb{P}^{n+1}$ the linear subvarieties

$$\mathbb{P}^{n-m} := \{x_{n-2m} - \zeta_{2d}x_{n-2m+1} = x_{n-2m+2} - \zeta_{2d}x_{n-2m+3} = \cdots = x_n - \zeta_{2d}x_{n+1} = 0\},$$

$$\mathbb{P}^{\frac{n}{2}} := \{x_0 - \zeta_{2d}x_1 = x_2 - \zeta_{2d}x_3 = \cdots = x_{n-2m-2} - \zeta_{2d}x_{n-2m-1} = 0\} \cap \mathbb{P}^{n-m},$$

$$\check{\mathbb{P}}^{\frac{n}{2}} := \{x_0 - \zeta_{2d}^{\alpha_0}x_1 = \cdots = x_{n-2m-2} - \zeta_{2d}^{\alpha_{n-2m-2}}x_{n-2m-1} = 0\} \cap \mathbb{P}^{n-m},$$

where $\alpha_0, \alpha_2, \ldots, \alpha_{n-2m-2} \in \{3, 5, \ldots, 2d-1\}$. Then

$$\mathbb{P}^m := \mathbb{P}^{\frac{n}{2}} \cap \check{\mathbb{P}}^{\frac{n}{2}} = \{x_0 = x_1 = x_2 = x_3 = \cdots = x_{n-2m-1} = 0\} \cap \mathbb{P}^{n-m}.$$

It turns out that the linear combination $Z := r\mathbb{P}^{\frac{n}{2}} + \check{r}\check{\mathbb{P}}^{\frac{n}{2}}$ of these two $\frac{n}{2}$-dimensional linear cycles is a join algebraic cycle, for all $r, \check{r} \in \mathbb{Z} \setminus \{0\}$. In fact, inside each degree d Fermat variety

$$X_1 := \{f := x_0^d + \cdots + x_{n-2m-1}^d = 0\} \subseteq \mathbb{P}^{n-2m-1}$$

and

$$X_2 := \{g := x_{n-2m}^d + \cdots + x_{n+1}^d = 0\} \subseteq \mathbb{P}^{2m+1}$$

we can consider the algebraic cycles $Z_1 \in \mathrm{CH}^{\frac{n}{2}-m-1}(X_1)$ and $Z_2 \in \mathrm{CH}^m(X_2)$ given by

$$Z_1 := rL + \check{r}\check{L},$$

$$Z_2 := \{x_{n-2m} - \zeta_{2d}x_{n-2m+1} = x_{n-2m+2} - \zeta_{2d}x_{n-2m+3} = \cdots = x_n - \zeta_{2d}x_{n+1} = 0\} \subseteq X_2,$$

where

$$L := \{x_0 - \zeta_{2d}x_1 = x_2 - \zeta_{2d}x_3 = \cdots = x_{n-2m-2} - \zeta_{2d}x_{n-2m-1} = 0\} \subseteq X_1,$$
$$\check{L} := \{x_0 - \zeta_{2d}^{\alpha_0}x_1 = x_2 - \zeta_{2d}^{\alpha_2}x_3 = \cdots = x_{n-2m-2} - \zeta_{2d}^{\alpha_{n-2m-2}}x_{n-2m-1} = 0\} \subseteq X_1.$$

Since $\mathbb{P}^{\frac{n}{2}} = J(L, Z_2)$ and $\check{\mathbb{P}}^{\frac{n}{2}} = J(\check{L}, Z_2)$ then

$$Z = r\mathbb{P}^{\frac{n}{2}} + \check{r}\check{\mathbb{P}}^{\frac{n}{2}} = rJ(L, Z_2) + \check{r}J(\check{L}, Z_2) = J(Z_1, Z_2) \in \mathrm{CH}^{\frac{n}{2}}(X),$$

where $X = \{F := f + g = x_0^d + \cdots + x_{n+1}^d = 0\}$ is the n-dimensional Fermat variety. By [VL22a, Theorem 1.3] the Hodge locus $V_{[Z]}$ satisfies

$$V_{[Z]} = V_{[\mathbb{P}^{\frac{n}{2}}]} \cap V_{[\check{\mathbb{P}}^{\frac{n}{2}}]}$$

whenever $d \geq 3$ and $m < \frac{n}{2} - \frac{d}{d-2}$. On the other hand, it follows from [Mov21, Propositions 17.8 and 17.9] that $V_{[\mathbb{P}^{\frac{n}{2}}]} \cap V_{[\check{\mathbb{P}}^{\frac{n}{2}}]}$ is smooth and reduced without restrictions on d and m. In particular, for $d \geq 3$ and $m < \frac{n}{2} - \frac{d}{d-2}$, $V_{[Z]}$ is smooth and reduced. The cases not covered in [VL22a, Theorem 1.3] are: $(d, m) = (3, \frac{n}{2} - 3)$, in which case Movasati conjectured $V_{[Z]}$ is smooth (see [Mov22]), $(d, m) = (3, \frac{n}{2} - 2)$, $(4, \frac{n}{2} - 2)$ and $m = \frac{n}{2} - 1$ with $r \neq \check{r}$. In this last case when $r = \check{r}$ the algebraic cycle Z is a complete intersection and $V_{[Z]}$ parametrizes hypersurfaces containing a complete intersection of type $(1, 1, \ldots, 1, 2)$. In the recent article [Klo23] Kloosterman showed that if $(d, m) = (3, \frac{n}{2} - 3)$, $n \geq 10$ and $r \neq \check{r}$ then $V_{[Z]}$ is not smooth, disproving Movasati's conjecture. Moreover, he showed that when $r = \check{r}$ and $n \geq 4$, $V_{[Z]}$ is smooth. Similar results are obtained by Kloosterman in the cases $(d, m) = (3, \frac{n}{2} - 2)$ and $(4, \frac{n}{2} - 2)$. We will analyze each of the above cases separately, using the join description to determine their associated Artin Gorenstein ideals and corresponding quadratic fundamental forms.

Proposition 5.1. Consider the notation of Example 5.2. For $d = 3,\ n \geq 4,\ m = \frac{n}{2} - 3$, the Artinian Gorenstein ideal $J^{F,[Z]}$ associated to the algebraic cycle Z is

$$\begin{aligned} J^{F,[Z]} = \Big\langle & \{x_j^2\}_{j=0}^{n+1},\ \{x_{2j} - \zeta_6 x_{2j+1}\}_{j=3}^{\frac{n}{2}},\ \{x_{2j}x_{2j+1}\}_{j=0}^{2},\ A_1x_1x_3x_4 + A_2x_1x_3x_5 \\ & x_0x_2 + B_1x_1x_2 + B_2x_1x_3,\ x_0x_3 + C_1x_1x_2 + C_2x_1x_3,\ x_0x_4 + D_1x_1x_4 + D_2x_1x_5 \\ & x_0x_5 + E_1x_1x_4 + E_2x_1x_5,\ x_2x_4 + F_1x_3x_4 + F_2x_3x_5,\ x_2x_5 + G_1x_3x_4 + G_2x_3x_5 \Big\rangle \end{aligned}$$

where $(A_1 : A_2) = (-r + \check{r}\zeta_6^{\alpha_0+\alpha_2+\alpha_4} : r\zeta_6 - \check{r}\zeta_6^{\alpha_0+\alpha_2+2\alpha_4}) \in \mathbb{P}^1$,

$$B_1 = \frac{-(\zeta_6^{\alpha_0+\alpha_2} - \zeta_6)}{\zeta_6^{\alpha_2} - \zeta_6},\ B_2 = \frac{\zeta_6^{\alpha_2+1}(\zeta_6^{\alpha_0} - \zeta_6)}{\zeta_6^{\alpha_2} - \zeta_6},\ C_1 = \frac{-(\zeta_6^{\alpha_0} - \zeta_6)}{\zeta_6^{\alpha_2} - \zeta_6},\ C_2 = \frac{\zeta_6(\zeta_6^{\alpha_0} - \zeta_6^{\alpha_2})}{\zeta_6^{\alpha_2} - \zeta_6},$$

$$D_1 = \frac{-(\zeta_6^{\alpha_0+\alpha_4} - \zeta_6^2)}{\zeta_6^{\alpha_4} - \zeta_6},\ D_2 = \frac{\zeta_6^{\alpha_4+1}(\zeta_6^{\alpha_0} - \zeta_6)}{\zeta_6^{\alpha_4} - \zeta_6},\ E_1 = \frac{-(\zeta_6^{\alpha_0} - \zeta_6)}{\zeta_6^{\alpha_4} - \zeta_6},\ E_2 = \frac{\zeta_6(\zeta_6^{\alpha_0} - \zeta_6^{\alpha_4})}{\zeta_6^{\alpha_4} - \zeta_6},$$

$$F_1 = \frac{-(\zeta_6^{\alpha_2+\alpha_4} - \zeta_6^2)}{\zeta_6^{\alpha_4} - \zeta_6},\ F_2 = \frac{\zeta_6^{\alpha_4+1}(\zeta_6^{\alpha_2} - \zeta_6)}{\zeta_6^{\alpha_4} - \zeta_6},\ G_1 = \frac{-(\zeta_6^{\alpha_2} - \zeta_6)}{\zeta_6^{\alpha_4} - \zeta_6},\ G_2 = \frac{\zeta_6(\zeta_6^{\alpha_2} - \zeta_6^{\alpha_4})}{\zeta_6^{\alpha_4} - \zeta_6}.$$

In particular, the degree $k := \frac{n}{2} + 4$ piece of the quadratic fundamental form vanishes on $\mathrm{Sym}^2(J_3^{F,[Z]})$.

Proof Since $Z = J(Z_1, Z_2)$ is a join algebraic cycle, by Theorem 1.2 it is enough to compute $J^{f,[Z_1]}$ and $J^{g,[Z_2]}$. In Example 5.1 we already computed $J^{g,[Z_2]}$, and so we just need to show that

$$\begin{aligned} J^{f,[Z_1]} =&\Big\langle x_0^2,\ x_1^2,\ x_2^2,\ x_3^2,\ x_4^2,\ x_5^2,\ x_0x_1,\ x_2x_3,\ x_4x_5,\ A_1x_1x_3x_4 + A_2x_1x_3x_5 \\ & x_0x_2 + B_1x_1x_2 + B_2x_1x_3,\ x_0x_3 + C_1x_1x_2 + C_2x_1x_3,\ x_0x_4 + D_1x_1x_4 + D_2x_1x_5 \\ & x_0x_5 + E_1x_1x_4 + E_2x_1x_5,\ x_2x_4 + F_1x_3x_4 + F_2x_3x_5,\ x_2x_5 + G_1x_3x_4 + G_2x_3x_5\Big\rangle. \end{aligned}$$

Note that the right hand side is contained in the left hand side. Assume that $A_1 \neq 0$ (the case where $A_1 = 0$ is analogue), then the ideal generated by the leading terms in the lexicographic monomial ordering is

$$\langle x_0^2, x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_0x_1, x_0x_2, x_0x_3, x_0x_4, x_0x_5, x_2x_3, x_2x_4, x_2x_5, x_4x_5, x_1x_3x_4\rangle \subseteq \mathrm{LT}(J^{f,[Z_1]}).$$

Thus if we show that both monomial ideals have the same Hilbert function we are done (and in fact we conclude that the generators given above are a Gröbner basis of $J^{f,[Z_1]}$). To see this, note that for the left hand side monomial ideal, it is very easy to compute its Hilbert function, and in fact we see that the quotient ring has Hilbert function 1, 6, 6, 1, and 0 for degree bigger than 3. On the other hand the Hilbert function of $R^{f,[Z_1]}$ is of the form 1, ℓ, ℓ, 1 and 0 for degree bigger than 3 (since J^{f,Z_1} is Artinian Gorenstein of socle in degree 3). Thus it is reduced to show that $\ell = 6$. In other words, to show that $J_1^{f,[Z_1]} = 0$. And this can be shown using [VL22a, Proposition 2.1]. The last statement about the quadratic fundamental form follows from Corollary 4.1 item (i) and a routine verification that the quadratic fundamental form $q_1|_{\mathrm{Sym}^2(J^{f,[Z_1]})}$ vanishes in degree ≤ 3. ■

Proposition 5.2. In the context of Example 5.2 consider $d = 3$, $n \geq 2$ and $m = \frac{n}{2} - 2$. The Artinian Gorenstein ideal $J^{F,[Z]}$ associated to algebraic cycle Z is

$$\begin{aligned} J^{F,[Z]} =&\Big\langle \{x_j^2\}_{j=0}^{2n+1},\ \{x_{2j} - \zeta_6 x_{2j+1}\}_{j=2}^{\frac{n}{2}},\ x_0x_1,\ x_2x_3,\ A_1x_1x_2 + A_2x_1x_3, \\ & B_1x_0x_2 + B_2x_1x_2 + B_3x_1x_3,\ C_1x_0x_3 + C_2x_1x_2 + C_3x_1x_3\Big\rangle \end{aligned}$$

where $(A_1 : A_2) = (r\zeta_6^2 + \check{r}\zeta_6^{\alpha_0+\alpha_2} : r - \check{r}\zeta_6^{\alpha_0+2\alpha_2}) \in \mathbb{P}^1$,

$$(B_1 : B_2 : B_3) = \begin{cases} (r\zeta_6^2 + \check{r}\zeta_6^{\alpha_0+\alpha_2} : 0 : r\zeta_6 - \check{r}\zeta_6^{2(\alpha_0+\alpha_2)}) & \text{if } A_1 \neq 0, \\ (r - \check{r}\zeta_6^{\alpha_0+2\alpha_2} : -r\zeta_6 + \check{r}\zeta_6^{2(\alpha_0+\alpha_2)} : 0) & \text{if } A_1 = 0, \end{cases}$$

$$(C_1 : C_2 : C_3) = \begin{cases} (r\zeta_6^2 + \check{r}\zeta_6^{\alpha_0+\alpha_2} : 0 : r - \check{r}\zeta_6^{2\alpha_0+\alpha_2}) & \text{if } A_1 \neq 0, \\ (r - \check{r}\zeta_6^{\alpha_0+2\alpha_2} : -r + \check{r}\zeta_6^{2\alpha_0+\alpha_2} : 0) & \text{if } A_1 = 0. \end{cases}$$

In particular, the degree $k := \frac{n}{2} + 4$ piece of the quadratic fundamental form vanishes.

Proof As in the proof of the previous proposition we just need to show that

$$\begin{aligned} J^{f,[Z_1]} =&\Big\langle x_0^2,\ x_1^2,\ x_2^2,\ x_3^2,\ x_0x_1,\ x_2x_3,\ A_1x_1x_2 + A_2x_1x_3, \\ & B_1x_0x_2 + B_2x_1x_2 + B_3x_1x_3,\ C_1x_0x_3 + C_2x_1x_2 + C_3x_1x_3\Big\rangle. \end{aligned}$$

The right hand side ideal is clearly contained in $J^{f,[Z_1]}$, hence it is enough to show that both ideals have the same Hilbert function. If $A_1 \neq 0$ then the leading terms ideal of the right hand side ideal is

$$\langle x_0^2,\ x_1^2,\ x_2^2,\ x_3^2,\ x_0x_1,\ x_2x_3,\ x_1x_2,\ x_0x_2,\ x_0x_3\rangle$$

whose quotient ring has Hilbert function equal to 1, 4, 1 and 0 for degree bigger than 2. If $A_1 = 0$, the leading terms ideal is

$$\langle x_0^2,\ x_1^2,\ x_2^2,\ x_3^2,\ x_0x_1,\ x_2x_3,\ x_1x_3,\ x_0x_2,\ x_0x_3\rangle$$

whose quotient ring also has Hilbert function equal to 1, 4, 1 and 0 for degree bigger than 2. Thus we are reduced to show that $R^{f,[Z_1]}$ has the same Hilbert function. Since $J^{f,[Z_1]}$ is Artinian Gorenstein of socle in degree 2 we just need to show that $J_1^{f,[Z_1]} = 0$, which follows from [VL22a, Proposition 2.1]. The statement about the quadratic fundamental form follows from Corollary 4.1 item (i) and the fact that $q_1|_{\mathrm{Sym}^2(J^{f,[Z_1]})}$ vanishes in degree ≤ 2. ■

Proposition 5.3. In the context of Example 5.2 let $d = 4$, $n \geq 2$ and $m = \frac{n}{2} - 2$. The Artinian Gorenstein ideal $J^{F,[Z]}$ associated to the algebraic cycle Z is

$$\begin{aligned} J^{F,[Z]} = \Big\langle & \{x_{2j+1}^3\}_{j=0}^{\frac{n}{2}},\ \{x_{2j} - \zeta_8 x_{2j+1}\}_{j=2}^{\frac{n}{2}},\ x_0x_1^2,\ x_2x_3^2,\ A_1x_1^2x_2x_3 + A_2x_1^2x_3^2, \\ & x_0x_2 + B_1x_1x_2 + B_2x_1x_3,\ x_0x_3 + C_1x_1x_2 + C_2x_1x_3, \\ & x_0^2 + D_1x_0x_1 + D_2x_1^2,\ x_2^2 + E_1x_2x_3 + E_2x_3^2 \Big\rangle \end{aligned}$$

where $(A_1 : A_2) = (r\zeta_8^2 + \check{r}\zeta_8^{\alpha_0+\alpha_2} : -(r\zeta_8^3 + \check{r}\zeta_8^{\alpha_0+2\alpha_2})) \in \mathbb{P}^1$,

$$B_1 = \frac{\zeta_8^2 - \zeta_8^{\alpha_0+\alpha_2}}{\zeta_8^{\alpha_2} - \zeta_8},\ B_2 = \frac{\zeta_8(\zeta_8^{\alpha_0+\alpha_2} - \zeta_8^{\alpha_2+1})}{\zeta_8^{\alpha_2} - \zeta_8},\ C_1 = \frac{\zeta_8 - \zeta_8^{\alpha_0}}{\zeta_8^{\alpha_2} - \zeta_8},\ C_2 = \frac{\zeta_8(\zeta_8^{\alpha_0} - \zeta_8^{\alpha_2})}{\zeta_8^{\alpha_2} - \zeta_8},$$

$$D_1 = \frac{-(\zeta_8^{2(\alpha_0+1)} + 1)}{\zeta_8^2(\zeta_8^{\alpha_0} - \zeta_8)},\ D_2 = \frac{\zeta_8^{\alpha_0}(1 + \zeta_8^{\alpha_0+3})}{\zeta_8^2(\zeta_8^{\alpha_0} - \zeta_8)},\ E_1 = \frac{-(\zeta_8^{2(\alpha_2+1)} + 1)}{\zeta_8^2(\zeta_8^{\alpha_2} - \zeta_8)},\ E_2 = \frac{\zeta_8^{\alpha_2}(1 + \zeta_8^{\alpha_2+3})}{\zeta_8^2(\zeta_8^{\alpha_2} - \zeta_8)}.$$

In particular, the degree $k := n + 6$ piece of the quadratic fundamental form vanishes.

Proof As in the other cases, we are reduced to show that

$$\begin{aligned} J^{f,[Z_1]} = \Big\langle & x_1^3,\ x_3^3,\ x_0x_1^2,\ x_2x_3^2,\ A_1x_1^2x_2x_3 + A_2x_1^2x_3^2, \\ & x_0x_2 + B_1x_1x_2 + B_2x_1x_3,\ x_0x_3 + C_1x_1x_2 + C_2x_1x_3, \\ & x_0^2 + D_1x_0x_1 + D_2x_1^2,\ x_2^2 + E_1x_2x_3 + E_2x_3^2 \Big\rangle. \end{aligned}$$

The right hand side ideal is clearly contained in $J^{f,[Z_1]}$. Let us assume that $A_1 \neq 0$ (the case $A_1 = 0$ is analogue), taking the ideal generated by the leading terms in the lexicographical monomial ordering we get

$$\langle x_0^2,\ x_2^2,\ x_0x_2,\ x_0x_3,\ x_1^3,\ x_3^3,\ x_0x_1^2,\ x_2x_3^2,\ x_1^2x_2x_3\rangle \subseteq \mathrm{LT}(J^{f,Z_1}).$$

For the left monomial ideal, the quotient ring has Hilbert function $1, 4, 6, 4, 1$ and 0 for degree bigger than 4. Thus it is enough to show that $J_1^{f,[Z_1]} = 0$ and $\dim\ J_2^{f,[Z_1]} = 4$. For this we use again [VL22a, Proposition 2.1] and check that

$$J_1^{f,[Z_1]} = \langle x_0 - \zeta_8 x_1, x_2 - \zeta_8 x_3\rangle_1 \cap \langle x_0 - \zeta_8^{\alpha_0} x_1, x_2 - \zeta_8^{\alpha_2} x_3\rangle_1 = 0$$

and

$$J_2^{f,[Z_1]} = \langle x_0 - \zeta_8 x_1, x_2 - \zeta_8 x_3\rangle_2 \cap \langle x_0 - \zeta_8^{\alpha_0} x_1, x_2 - \zeta_8^{\alpha_2} x_3\rangle_2$$

$$= \langle (x_0 - \zeta_8 x_1)(x_0 - \zeta_8^{\alpha_0} x_1), (x_2 - \zeta_8 x_3)(x_0 - \zeta_8^{\alpha_0} x_1), (x_0 - \zeta_8 x_1)(x_2 - \zeta_8^{\alpha_2} x_3), (x_2 - \zeta_8 x_3)(x_2 - \zeta_8^{\alpha_2} x_3)\rangle_2.$$

The statement about the quadratic fundamental forms follows from Corollary 4.1 item (i) and the verification that the quadratic fundamental form $q_1|_{\mathrm{Sym}^2(J^{f,[Z_1]})}$ vanishes in degree ≤ 4. ■

Theorem 5.1. With the same notation of Example 5.2, let $d \geq 6$, $m = \frac{n}{2} - 1$ and $r \neq \check{r}$. Then the degree $k := d + (d-2)(\frac{n}{2} + 1)$ piece of quadratic fundamental form does not vanish on $\mathrm{Sym}^2(J_d^{F,[Z]})$. In consequence $V_{[Z]}$ is not smooth if and only if $r \neq \check{r}$.

Proof By Corollary 4.1 item (ii) it is enough to show that $q_1|_{\mathrm{Sym}^2(J^{f,[Z_1]})}$ is non-zero in degree $d-2$. Consider

$$G := x_1^{d-3}((r\zeta_{2d} + \check{r}\zeta_{2d}^{\alpha_0})x_0 + (r\zeta_{2d}^2 + \check{r}\zeta_{2d}^{2\alpha_0})x_1) \in J_{d-2}^{f,[Z_1]}.$$

The quadratic fundamental form at G is

$$q_1(G,G) = r\check{r}\zeta_{2d}^{\alpha_0+1}(\zeta_{2d}^{\alpha_0} - \zeta_{2d})^2 x_0^{d-4} x_1^{d-3}\{(r[\zeta_{2d}^2(d-1) + \zeta_{2d}^{\alpha_0+1}] + \check{r}[\zeta_{2d}^{2\alpha_0}(d-1) + \zeta_{2d}^{\alpha_0+1}])x_0$$

$$+ (r[(d-1)\zeta_{2d}^2(\zeta_{2d}^{\alpha_0} + \zeta_{2d}) + 2\zeta_{2d}^{2\alpha_0+1}] + \check{r}[(d-1)\zeta_{2d}^{2\alpha_0}(\zeta_{2d}^{\alpha_0} + \zeta_{2d}) + 2\zeta_{2d}^{\alpha_0+2}])x_1\}.$$

In order to see that this is non-zero in $R^f/\langle P_{Z_1}\rangle$, let us note first that $\langle P_{Z_1}\rangle_{2d-6}$ is a 2-dimensional subspace of R_{2d-6}^f. In fact, in the basis of $R_{2d-6}^f = \mathbb{C}\cdot x_0^{d-2}x_1^{d-4} \oplus \mathbb{C}\cdot x_0^{d-3}x_1^{d-3} \oplus \mathbb{C}\cdot x_0^{d-4}x_1^{d-2}$

$$\langle P_{Z_1}\rangle_{2d-6} = \mathbb{C}\cdot Q_1 \oplus \mathbb{C}\cdot Q_2$$

where

$$Q_1 = (r\zeta_{2d} + \check{r}\zeta_{2d}^{\alpha_0})x_0^{d-2}x_1^{d-4} + (r\zeta_{2d}^2 + \check{r}\zeta_{2d}^{2\alpha_0})x_0^{d-3}x_1^{d-3} + (r\zeta_{2d}^3 + \check{r}\zeta_{2d}^{3\alpha_0})x_0^{d-4}x_1^{d-2},$$

$$Q_2 = (r\zeta_{2d}^2 + \check{r}\zeta_{2d}^{2\alpha_0})x_0^{d-2}x_1^{d-4} + (r\zeta_{2d}^3 + \check{r}\zeta_{2d}^{3\alpha_0})x_0^{d-3}x_1^{d-3} + (r\zeta_{2d}^4 + \check{r}\zeta_{2d}^{4\alpha_0})x_0^{d-4}x_1^{d-2}.$$

Hence $q_1(G,G)$ vanishes if and only if $q_1(G,G)$, Q_1 and Q_2 are linearly dependent in R_{2d-6}^f. Using the monomial basis of R_{2d-6}^f we can write a 3×3 matrix M whose columns correspond to $q_1(G,G)$, Q_1 and Q_2. Computing its determinant we obtain

$$\det(M) = \pm\zeta_{2d}^{3\alpha_0+3}(\zeta_{2d}^{\alpha_0} - \zeta_{2d})^3 r^2\check{r}^2(r - \check{r})$$

which is non-zero for $r \neq \check{r}$. ■

6 Hilbert function associated to a Hodge cycle

Theorem 1.2 gives us the tensor product structure of the Artinian Gorenstein algebra associated to a join algebraic cycle. This structure helps us to study its associated Hilbert function.

Definition 6.1. Let $X = \{F = 0\} \subseteq \mathbb{P}^{n+1}$ be a smooth degree d hypersurface of even dimension n. For every $\lambda \in H^{\frac{n}{2},\frac{n}{2}}(X,\mathbb{Q})$, its associated *Hilbert function* $\mathrm{HF}_\lambda : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is the Hilbert function of its associated Artinian Gorenstein algebra $R^{F,\lambda} = \mathbb{C}[x_0, \dots, x_{n+1}]/J^{F,\lambda}$.

Corollary 6.1. In the same context of Theorem 1.1 we have $\mathrm{HF}_{[J(Z_1,Z_2)]} = \mathrm{HF}_{[Z_1]} * \mathrm{HF}_{[Z_2]}$, this means that for all $k \geq 0$

$$\mathrm{HF}_{[J(Z_1,Z_2)]}(k) = \sum_{p+q=k} \mathrm{HF}_{[Z_1]}(p) \cdot \mathrm{HF}_{[Z_2]}(q). \tag{17}$$

Proof This follows from Theorem 1.2. ■

Example 6.1. Using the above corollary we can compute the Hilbert function of the examples inside Fermat described in the previous section. As an illustration for one linear cycle in Fermat (see Example 5.1) we get

$$\mathrm{HF}_{[\mathbb{P}^{\frac{n}{2}}]} = \varphi^{*(\frac{n}{2}+1)} \tag{18}$$

where $\varphi : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is the Hilbert function of a point in a 0-dimensional Fermat variety

$$\varphi(k) = \begin{cases} 1 & \text{if } 0 \leq k \leq d-2, \\ 0 & \text{otherwise.} \end{cases}$$

In other words $\mathrm{HF}_{[\mathbb{P}^{\frac{n}{2}}]}(k)$ counts the number of ways of writing k as an ordered sum of $\frac{n}{2}+1$ numbers between 0 and $d-2$.

Remark 6.1. The Hilbert function of (18) is in fact the Hilbert function of a generic linear cycle inside a smooth degree d hypersurface of even dimension n. This follows from the upper semi-continuity of the Hilbert function along the locus of hypersurfaces containing an $\frac{n}{2}$-dimensional linear cycle. In fact, the upper semi-continuity of the Hilbert function holds along the locus of hypersurfaces containing an $\frac{n}{2}$-dimensional complete intersection for any fixed multi-degree. This is a direct consequence of the explicit description of generators of the associated Artinian Gorenstein ideal which can be found in [VL22b, Example 2.1]. In particular we can compute the Hilbert function of a generic complete intersection of type $(1, 1, \dots, 1, k)$ by writing it as a join in Fermat. In general, for other types of algebraic cycles λ we do not know whether the Hilbert function is upper semi-continuous along V_λ or not.

7 Fake algebraic cycles

In the article [DFVL23] the authors found pathological algebraic cycles in all Fermat varieties of degree 3, 4 and 6. They were pathological in the sense that their associated Hodge loci V_λ had the biggest possible Zariski tangent space at the Fermat point without being λ the class of a linear cycle (contradicting a conjecture of Movasati). These cycles were called fake linear cycles and were constructed from an arithmetic viewpoint using the Galois action in the cohomology of the Fermat variety. Using the join construction we can have a better understanding of these cycles as explicit combinations of linear cycles. In this section we will introduce a more general notion of fake algebraic cycles, inside any smooth hypersurface and we will show how one can find hypersurfaces containing fake linear cycles in any degree.

Definition 7.1. Let $X=\{F=0\}\subseteq\mathbb{P}^{n+1}$ be a smooth degree d hypersurface of even dimension n. Let $Z\subseteq X$ be an $\frac{n}{2}$-dimensional algebraic subvariety. A Hodge cycle $\lambda\in H^{\frac{n}{2},\frac{n}{2}}(X,\mathbb{Q})$ is a *fake version of* $[Z]$ if

$$\mathrm{HF}_{\lambda}=\mathrm{HF}_{[Z]}$$

but λ_{prim} is not a scalar multiple of $[Z]_{\mathrm{prim}}$.

Remark 7.1. By [DFVL23, Theorem 1.1] all Fermat varieties of degree $d=3,4,6$ (and only for those degrees) admit fake linear cycles. In fact, in this case the main result shows that $\mathrm{HF}_{\lambda}(d)=\binom{\frac{n}{2}+d}{d}-(\frac{n}{2}+1)^2=\mathrm{HF}_{[\mathbb{P}^{\frac{n}{2}}]}(d)$ implies

$$P_{\lambda}=c_{\lambda}\prod_{j=0}^{\frac{n}{2}}\left(\frac{x_{2j}^{d-1}-(c_jx_{2j+1})^{d-1}}{x_{2j}-c_jx_{2j+1}}\right) \tag{19}$$

for any $c_j\in\zeta_{2d}^{-3}\cdot\{z\in\mathbb{Q}(\zeta_d):\ |z|=1\}$ and some $c_{\lambda}\in\mathbb{Q}(\zeta_{2d})^{\times}$. From this we deduce that

$$J^{F,\lambda}=\langle x_0-c_0x_1,x_2-c_1x_3,\dots,x_n-c_{\frac{n}{2}}x_{n+1},x_0^{d-1},\dots,x_{n+1}^{d-1}\rangle, \tag{20}$$

which in turn implies that $\mathrm{HF}_{\lambda}=\mathrm{HF}_{[\mathbb{P}^{\frac{n}{2}}]}$. In the case where all c_j are d-th roots of -1, λ corresponds to the class of a linear cycle in Fermat. In all other cases λ is a fake linear cycle. The description of the Artinian Gorenstein ideal (20) implies that

$$R^{F,\lambda}=\bigotimes_{j=1}^{\frac{n}{2}+1}R^{F_j,\lambda_j}$$

where $X_j=\{F_j(x_{2j-2},x_{2j-1}):=x_{2j-2}^d+x_{2j-1}^d=0\}\subseteq\mathbb{P}^1$ and λ_j is the class of a 0-cycle such that

$$P_{\lambda_j}=\frac{x_{2j-2}^{d-1}-(c_{2j-2}x_{2j-1})^{d-1}}{x_{2j-2}-c_{2j-2}x_{2j-1}}.$$

In other words, each λ_j is a 0-dimensional fake linear cycle. Since this is a Hodge cycle, there exist $n_{j,1},\dots,n_{j,d}\in\mathbb{Q}$ such that

$$P_{\lambda_j}=\sum_{\ell=1}^{d}n_{j,\ell}\cdot P_{[p_\ell^j]}$$

where $X_j=\{p_1^j,p_2^j,\dots,p_d^j\}\subseteq\mathbb{P}^1$ (note that each $p_\ell^j\in\mathrm{CH}^0(X_j)$ is a linear cycle, and so we know how to compute $P_{[p_\ell^j]}$). It follows from Theorem 1.2 that

$$\lambda_{\mathrm{prim}}=c\cdot J(\lambda_1,\lambda_2,\dots,\lambda_{\frac{n}{2}+1})$$

for some $c \in \mathbb{Q}^\times$. In other words, every fake linear cycle is a linear combination of linear cycles given by

$$\lambda = \sum_{\ell_1,\ell_2,\ldots,\ell_{\frac{n}{2}+1}=1}^{d} (\prod_{j=1}^{\frac{n}{2}+1} n_{j,\ell_j}) \cdot J(p^1_{\ell_1}, p^2_{\ell_2}, \ldots, p^{\frac{n}{2}+1}_{\ell_{\frac{n}{2}+1}}).$$

Remark 7.2. The previous remark shows that the presence of fake linear cycles in degree $d = 3, 4, 6$ Fermat varieties is due to their existence in 0-dimensional Fermat varieties of such degrees. Using this observation one can go further and produce some fake versions of other algebraic cycles obtained as joins. For instance we can produce fake versions of complete intersection cycles of type $(1, 1, \ldots, 1, 2)$ in Fermat varieties of degree $d = 3, 4, 6$ by taking

$$\lambda = J(\lambda_1, [Z_2])$$

where λ_1 is a fake linear cycle in $X_1 = \{x_0^d + \cdots + x_{n-1}^d = 0\}$ and $Z_2 = p_1 + p_2 \in \mathrm{CH}^0(X_2)$ for $X_2 = \{x_n^d + x_{n+1}^d = 0\} = \{p_1, p_2, \ldots, p_d\}$. More generally, for any algebraic cycle given as a cone

$$Z = J(pt, Z_2)$$

we can construct a fake version of Z if we replace the point by a 0-dimensional fake linear cycle. Hence it is natural ask whether there are more 0-dimensional hypersurfaces (of higher degree) containing 0-dimensional fake linear cycles. It turns out that it is not hard to construct hypersurfaces with infinitely many fake linear cycles in any degree.

Theorem 7.1. Let $X = \{F(x_0, x_1) := (x_0 - r_1x_1)(x_0 - r_2x_1) \cdots (x_0 - r_dx_1) = 0\} \subseteq \mathbb{P}^1$ be a smooth degree d hypersurface with $r_i \in \mathbb{Q}$ for all $i = 1, \ldots, d$. Consider for each $c \in \mathbb{Q} \setminus \{r_1, \ldots, r_d\}$ the polynomial

$$P := \frac{a\frac{\partial F}{\partial x_0} - b\frac{\partial F}{\partial x_1}}{x_0 - cx_1} \in R^F_{d-2} \tag{21}$$

for $a = \frac{\partial F}{\partial x_1}(c, 1)$ and $b = \frac{\partial F}{\partial x_0}(c, 1)$. Then

$$\delta := \mathrm{res}\left(\frac{P \cdot (x_0dx_1 - x_1dx_0)}{F}\right) \in H^0(X, \mathbb{Q})_{\mathrm{prim}} \tag{22}$$

is a 0-dimensional fake linear cycle.

Proof For each point $p_i := (r_i : 1) \in X$ we know $[p_i]_{\mathrm{prim}} \in H^0(X, \mathbb{Q})$. Moreover we can write it as a residue applying [VL22a, Theorem 1.1]

$$[p_i]_{\mathrm{prim}} = \frac{-1}{d}\mathrm{res}\left(\frac{P_i \cdot (x_0dx_1 - x_1dx_0)}{F}\right) \in H^0(X, \mathbb{Q})_{\mathrm{prim}}$$

where

$$P_i = \det\begin{pmatrix} 1 & \frac{\frac{\partial F}{\partial x_0}}{x_0 - r_ix_1} - \frac{F}{(x_0-r_ix_1)^2} \\ -r_i & \frac{\frac{\partial F}{\partial x_1}}{x_0 - r_ix_1} + r_i\frac{F}{(x_0-r_ix_1)^2} \end{pmatrix} = \frac{r_i\frac{\partial F}{\partial x_0} + \frac{\partial F}{\partial x_1}}{x_0 - r_ix_1} \in \mathbb{Q}[x_0, x_1]_{d-2}. \tag{23}$$

Since all the points $[p_1]_{\text{prim}}, \dots, [p_d]_{\text{prim}}$ generate the $\mathbb{Q}$-vector space $H^0(X,\mathbb{Q})_{\text{prim}}$ of dimension $d-1$, and the residue map is an isomorphism of $\mathbb{C}[x_0,x_1]_{d-2} = R^F_{d-2} \simeq H^0(X,\mathbb{C})_{\text{prim}}$, it follows that the polynomials $P_1,\dots,P_d$ generate all $\mathbb{Q}[x_0,x_1]_{d-2}$ as $\mathbb{Q}$-vector space. In particular, since $c \in \mathbb{Q}$, then $P \in \mathbb{Q}[x_0,x_1]_{d-2}$ and so we can write it as a $\mathbb{Q}$-linear combination

$$P = q_1 \cdot P_1 + \cdots + q_d \cdot P_d.$$

Hence $\delta = q_1 \cdot [p_1]_{\text{prim}} + \cdots + q_d \cdot [p_d]_{\text{prim}} \in H^0(X,\mathbb{Q})_{\text{prim}}$ is a rational class. To see that it defines a fake linear cycle it is enough to see that

$$J^{F,\delta} = (J^F : P) = \langle x_0 - cx_1, x_0^{d-1}, x_1^{d-1} \rangle$$

and so $\mathrm{HF}_\delta = \mathrm{HF}_{[p_i]}$. ∎

Now, as a corollary of Theorem 7.1 we obtain Theorem 1.3.

Proof of Theorem 1.3 Pick any degree d homogeneous polynomials $F_0,\dots,F_{\frac{n}{2}} \in \mathbb{Q}[x,y]_d$ such that each F_i has only simple rational roots. Define $X := \{F_0(x_0,x_1) + F_1(x_2,x_3) + \cdots + F_{\frac{n}{2}}(x_n,x_{n+1}) = 0\} \subseteq \mathbb{P}^{n+1}$. For each $i = 0,\dots,\frac{n}{2}$ consider $X_i := \{F_i(x_{2i},x_{2i+1}) = 0\} \subseteq \mathbb{P}^1$ and take any fake linear cycle $\delta_i \in H^0(X_i,\mathbb{Q})_{\text{prim}}$. Then by Corollary 6.1

$$\delta := J(\delta_0,\dots,\delta_{\frac{n}{2}}) \in H^{\frac{n}{2},\frac{n}{2}}(X,\mathbb{Q})_{\text{prim}}$$

is a fake linear cycle. ∎

Remark 7.3. A consequence of Theorem 7.1 and [VL22b, Theorem 1.1] is that no automorphism of $\mathbb{P}^1$ transforms all points of the Fermat variety $X = \{x_0^d + x_1^d\} \subseteq \mathbb{P}^1$ into rational points for $d \neq 3,4,6$. On the other hand, it is easy to check that for degrees $d = 3,4,6$ there exists an automorphism of $\mathbb{P}^1$ taking all Fermat points to rational points. This explains the presence of fake linear cycles in Fermat varieties of such degrees.

Example 7.1. Let $X = \{F(x_0,x_1) := (x_0 - r_1x_1)(x_0 - r_2x_1)\cdots(x_0 - r_6x_1) = 0\} \subseteq \mathbb{P}^1$ with $r_1 = 0$, $r_2 = 1$, $r_3 = \frac{1}{2}$, $r_4 = \frac{1}{4}$, $r_5 = \frac{1}{3}$, $r_6 = \frac{2}{5}$. Consider the same notation of Theorem 7.1, and take the fake linear cycle δ of the form (22) where the polynomial P in (21) is defined using the number $c = -1$. Let P_i be the associated polynomial to the point $(r_i : 1) \in X$ for $i = 1,\dots,5$ (this is computed explicitly in (23)). Once we know explicitly all these polynomials, it is an elementary linear algebra problem to find the $\mathbb{Q}$-linear combination of the polynomial P in terms of the polynomials $P_1,\dots,P_5$, which is

$$P = -\frac{207283}{810}P_1 - \frac{68941}{270}P_2 - \frac{507311}{1620}P_3 - \frac{26911}{180}P_4 - \frac{891881}{1620}P_5.$$

For the case of fake linear cycles in Fermat varieties of degree $d = 3,4,6$ one first transforms the Fermat equation to one with only rational roots, and proceeds in the same way as before. In fact, the above example is isomorphic to the Fermat sextic under the composition of the following automorphisms of $\mathbb{P}^1$

$$\phi(x_0 : x_1) = (x_0 : x_0 + x_1)$$

$$\psi(x_0 : x_1) = (x_0 - \zeta_{12}x_1 : (1+\zeta_6^{-1})(x_0 - \zeta_{12}^3 x_1)).$$

We have that $\psi^*\phi^*(F) = -\frac{2\zeta_6^2+1}{40}(x_0^6+x_1^6)$ and the 0-dimensional fake linear cycle $\delta \in H^0(X,\mathbb{Q})_{\text{prim}}$ is transformed to the fake linear cycle $\lambda = \psi^*\phi^*\delta$ inside the Fermat variety $\{x_0^6 + x_1^6 = 0\} \subset \mathbb{P}^1$ given by

$$\lambda = \operatorname{res}\left(\frac{P_\lambda \cdot (x_0 dx_1 - x_1 dx_0)}{x_0^6 + x_1^6}\right)$$

with

$$P_\lambda = c_\lambda \frac{x_0^5 - (c_0 x_1)^5}{x_0 - c_0 x_1},$$

where $c_0 = \zeta_{12}^{-3}\left(\frac{3\zeta_6^2-1}{3-\zeta_6^2}\right) \in \zeta_{12}^{-3} \cdot \mathbb{S}^1_{\mathbb{Q}(\zeta_6)}$ and $c_\lambda = \frac{\zeta_{12}(191\zeta_6+146)}{1-2\zeta_6} \in \mathbb{Q}(\zeta_{12})^\times$.

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