4 Probability

4.6 Are Mutual Exclusiveness and Independence Related?

Are Mutual Exclusiveness and Independence Related?

Mutually exclusive events and independent events are two very different concepts based on definitions that start from very different orientations.

The two concepts can easily be confused because they interact with each other and are intertwined by the probability statements we use in describing these concepts.

To describe these two concepts and eventually understand the distinction between them as well as the relationship between them, we need to agree that the events being considered are two nonempty events defined on the same sample space and therefore each has nonzero probabilities.

Are Mutual Exclusiveness and Independence Related?

Note

Students often have a hard time realizing that when we say, “Event A is a nonempty event” and write, “$P(A) > 0$,” we are describing the same situation.

The words and the algebra often do not seem to have the same meaning. In this case the words and the probability statement both tell us that event A exists within the sample space.

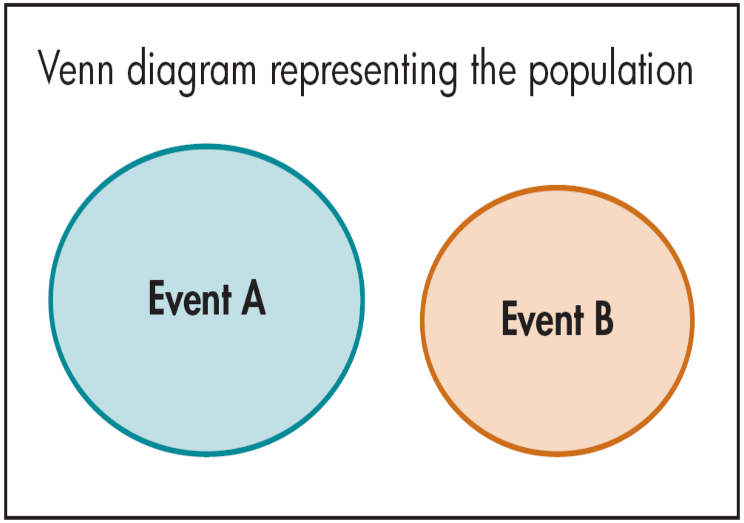
Mutually Exclusive

Mutually Exclusive

Mutually exclusive events are two nonempty events defined on the same sample space and share no common elements.

This means:

1. In words: In this Venn diagram, the closed areas representing each event "do not intersect"; in other words, they are disjoint sets, or no intersection occurs between their respective sets.



Mutually Exclusive

2. In algebra: ,P(A and B) = 0, which says, “The intersection of the two events is an empty set”; in other words, there is no intersection between their respective sets.

Notice that the concept of mutually exclusive is based on the relationship of the elements that satisfy the events.

Mutually exclusive is not a probability concept by definition—it just happens to be easy to express the concept using a probability statement.

Independence

Independence

Independent events are two nonempty events defined on the same sample space that are related in such a way that the occurrence of either event does not affect the probability of the other event.

This means that:

1. In words: If event A has (or is known to have) already occurred, the probability of event B is unaffected (that is, the probability of B after knowing event A had occurred remains the same as it was before knowing event A had occurred).

Independence

In addition, it is also the case when *A* and *B* interchange roles that if event *B* has (or is known to have) already occurred, the probability of event *A* is unaffected (i.e., the probability of *A* is still the same after knowing event *B* had occurred as it was before).

This is a "mutual relationship"; it works both ways.

2. In Algebra: $P(\text{B} \mid \text{A}) = P(\text{B} \mid \text{not A}) = P(\text{B})$ and

$$P(\text{A} \mid \text{B}) = P(\text{A} \mid \text{not B}) = P(\text{A})$$

Or with a few words to help interpret the algebra P(B, knowing A has occurred) = *P*(B, knowing A has not occurred

Independence

$= P(\text{B})$ and $P(\text{A, knowing B has occurred})$

$= P(\text{A, knowing B has not occurred})$

$= P(\text{A})$.

Notice that the concept of independence is based on the effect one event (in this case, the lack of effect) has on the probability of the other event.

Independence

Let's look at the following four demonstrations relating to mutually exclusive and independent events:

Demonstration I

Given: $P(\mathrm{A}) = 0.4$, $P(\mathrm{B}) = 0.5$, and A and B are mutually exclusive; are they independent?

Answer: If A and B are mutually exclusive events, $P(\mathrm{A} \mid \mathrm{B}) = 0.0$, and because we are given $P(\mathrm{A}) = 0.4$, we see that the occurrence of B has an effect on the probability of A. Therefore A and B are not independent events.

Independence

Conclusion I: If the events are mutually exclusive, they are NOT independent.

Demonstration II

Given: P(A) = 0.4, P(B) = 0.5, and A and B are independent; are events A and B mutually exclusive?

Answer: If A and B are independent events, then the P(A and B) = P(A) • P(B) = 0.4 • 0.5 = 0.20 and because P(A and B) is greater than zero, events A and B must intersect, meaning the events are not mutually exclusive.

Independence

Conclusion II: If the events are independent, they are NOT mutually exclusive.

Demonstration III

Given: $P(\mathrm{A}) = 0.4$, $P(\mathrm{B}) = 0.5$, and A and B are not mutually exclusive; are events A and B independent?

Answer: Because A and B are not mutually exclusive events, it must be that $P(\mathrm{A\ and\ B})$ is greater than zero. Now, if $P(\mathrm{A\ and\ B})$ happens to be exactly 0.20, then events A and B are independent

$$[P(\mathrm{A}) \cdot P(\mathrm{B}) = 0.4 \cdot 0.5 = 0.20],$$

Independence

but if P(A and B) is any other positive value, say 0.1, then events A and B are not independent.

Therefore, events A and B could be either independent or dependent; some other information is needed to make that determination.

Conclusion III: If the events are not mutually exclusive, they MAY be either independent or dependent; additional information is needed to determine which.

Independence

Demonstration IV

Given: P(A) = 0.4, P(B) = 0.5, and A and B are not independent; are events A and B mutually exclusive?

Answer: Because A and B are not independent events, it must be that P(A and B) is different from 0.20, the value it would be if they were independent

$$[P(A) \cdot P(B) = 0.4 \cdot 0.5 = 0.20]$$

Now, if P(A and B) happens to be exactly 0.00, then events A and B are mutually exclusive, but if P(A and B) is any other positive value, say 0.1, then events A and B are not mutually exclusive.

Independence

Therefore, events *A* and *B* could be either mutually exclusive or not; some other information is needed to make that determination.

Conclusion IV: If the events are NOT independent, they MAY be either mutually exclusive or not mutually exclusive; additional information is needed to determine which.

Independence

Advice

Work very carefully, starting with the information you are given and the definitions of the concepts involved.

What not to do

Do not rely on the first “off-the-top” example you can think of to lead you to the correct answer. It typically will not!

The following examples give further practice with these probability concepts.

Example 27 – *Using Several Probability Rules*

A production process produces thousands of items. On the average, 20% of all items produced are defective. Each item is inspected before it is shipped. The inspector misclassifies an item 10% of the time;
that is,

$$P(\text{classified good} \mid \text{defective item})$$
$$= P(\text{classified defective} \mid \text{good item})$$
$$= 0.10$$

What proportion of the items will be "classified good"?

Example 27 – *Solution*

What do we mean by the event "classified good"?

G: The item is good.

D: The item is defective.

CG: The item is classified good by the inspector.

CD: The item is classified defective by the inspector.

CG consists of two possibilities: "the item is good and is correctly classified good" and "the item is defective and is misclassified good." Thus,

$$P(\text{CG}) = P[(\text{CG and G}) \text{ or } (\text{CG and D})]$$

Example 27 – *Solution*

cont'd

Since the two possibilities are mutually exclusive, we can start by using the addition rule:

$$P(\text{CG}) = P[(\text{CG and G}) + (\text{CG and D})]$$

The condition of an item and its classification by the inspector are not independent. The multiplication rule for dependent events must be used.

Therefore,

$$P(\text{CG}) = P[(\text{G}) \cdot P(\text{CG} \mid \text{G})] + [P(\text{D}) \cdot P(\text{CG} \mid \text{D})]$$

Example 27 – *Solution*

cont'd

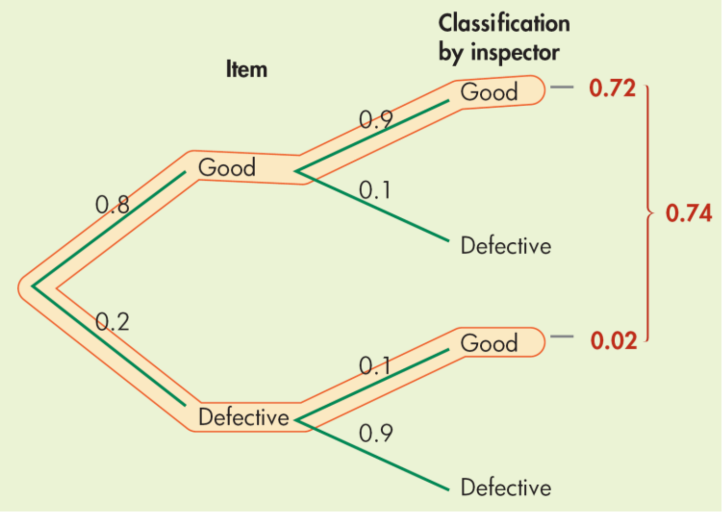
Substituting the known probabilities in Figure 4.7, we get

$$P(\text{CG}) = [(0.8)(0.9)] + [(0.2)(0.1)]$$

$$= 0.72 + 0.02$$

$$= \mathbf{0.74}$$

That is, 74% of the items are classified good.



Using Several Probability Rules

Figure 4.7