5 Probability Distributions (Discrete Variables)

5.2 Probability Distributions of a Discrete Random Variable

Probability Distributions of a Discrete Random Variable

Consider a coin-tossing experiment where two coins are tossed and no heads, one head, or two heads are observed.

If we define the random variable *x* to be the number of heads observed when two coins are tossed, *x* can take on the value 0, 1, or 2. The probability of each of these three events can be calculated using techniques from Chapter 4:

$$P(x = 0) = P(0\text{H}) = P(\text{TT}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = 0.25$$

$$P(x = 1) = P(1\text{H}) = P(\text{HT or TH}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2} = 0.50$$

$$P(x = 2) = P(2\text{H}) = P(\text{HH}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = 0.25$$

Probability Distributions of a Discrete Random Variable

These probabilities can be listed in any number of ways. One of the most convenient is a table format known as a *probability distribution* (see Table 5.1).

x	$P(x)$
0	0.25
1	0.50
2	0.25

Probability Distribution: Tossing Two Coins

Table 5.1

Probability distribution A distribution of the probabilities associated with each of the values of a random variable. The probability distribution is a theoretical distribution; it is used to represent populations.

Probability Distributions of a Discrete Random Variable

In an experiment in which a single die is rolled and the number of dots on the top surface is observed, the random variable is the number observed.

The probability distribution for this random variable is shown in Table 5.2.

x	1	2	3	4	5	6
$P(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Probability Distribution: Rolling a Die

Table 5.2

Sometimes it is convenient to write a rule that algebraically expresses the probability of an event in terms of the value of the random variable.

Probability Distributions of a Discrete Random Variable

This expression is typically written in formula form and is called a *probability function*.

Probability function A rule that assigns probabilities to the values of the random variables.

A probability function can be as simple as a list that pairs the values of a random variable with their probabilities.

Tables 5.1 and 5.2 show two such listings.

x	$P(x)$
0	0.25
1	0.50
2	0.25

Probability Distribution: Tossing Two Coins

Table 5.1

Probability Distributions of a Discrete Random Variable

However, a probability function is most often expressed in formula form.

Consider a die that has been modified so that it has one face with one dot, two faces with two dots, and three faces with three dots.

Let *x* be the number of dots observed when this die is rolled.

The probability distribution for this experiment is presented in Table 5.3.

x	$P(x)$
1	$\frac{1}{6}$
2	$\frac{2}{6}$
3	$\frac{3}{6}$

Probability Distribution: Rolling the Modified Die

Table 5.3

Probability Distributions of a Discrete Random Variable

Each of the probabilities can be represented by the value of *x* divided by 6;

that is, each $P(x)$ is equal to the value of *x* divided by 6, where $x = 1, 2$, or 3.

Thus,

$$P(x) = \frac{x}{6} \qquad \text{for} \qquad x = 1, 2, 3$$

is the formula for the probability function of this experiment.

Probability Distributions of a Discrete Random Variable

The probability function for the experiment of rolling one ordinary die is

$$P(x) = \frac{1}{6} \qquad \text{for} \qquad x = 1, 2, 3, 4, 5, 6$$

This particular function is called a **constant function** because the value of $P(x)$ does not change as x changes.

Probability Distributions of a Discrete Random Variable

Every probability function must display the two basic properties of probability.

These two properties are

(1) the probability assigned to each value of the random variable must be between zero and one, inclusive, and

(2) the sum of the probabilities assigned to all the values of the random variable must equal one—that is,

Property 1 $0 \leq \text{each } P(x) \leq 1$

Property 2 $\sum_{\text{all } x} P(x) = 1$

Example 2 – *Determining a Probability Function*

Is $P(x) = \frac{x}{10}$ for x = 1, 2, 3, 4 a probability function?

Solution:

To answer this question we need only test the function in terms of the two basic properties.

The probability distribution is shown in Table 5.4.

x	$P(x)$
1	$\frac{1}{10} = 0.1$ ✓
2	$\frac{2}{10} = 0.2$ ✓
3	$\frac{3}{10} = 0.3$ ✓
4	$\frac{4}{10} = 0.4$ ✓
	$\frac{10}{10} = 1.0$

Probability Distribution for $P(x) = \frac{x}{10}$ for x = 1, 2, 3, 4

Table 5.4

Example 2 – *Solution*

Property 1 is satisfied because 0.1, 0.2, 0.3, and 0.4 are all numerical values between zero and one.

(See the ✓ indicating each value was checked.) Property 2 is also satisfied because the sum of all four probabilities is exactly one.

Since both properties are satisfied, we can conclude that $P(x) = \frac{x}{10}$ for x = 1, 2, 3, 4 is a probability function.

Example 2 – *Solution*

cont'd

What about $P(x = 5)$ (or any value other than $x = 1, 2, 3$, or 4) for the function $P(x) = \frac{x}{10}$ for $x = 1, 2, 3, 4$?

$P(x = 5)$ is considered to be zero.

That is, the probability function provides a probability of zero for all values of x other than the values specified as part of the domain.

Probability distributions can be presented graphically.

Example 2 – *Solution*

cont'd

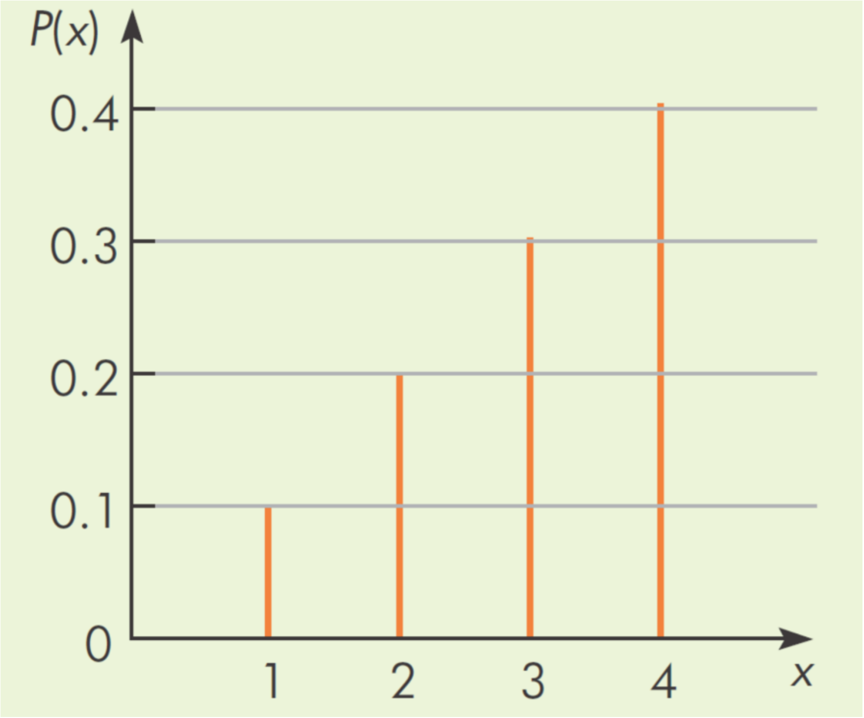
Regardless of the specific graphic representation used, the values of the random variable are plotted on the horizontal scale, and the probability associated with each value of the random variable is plotted on the vertical scale.

The probability distribution of a discrete random variable could be presented by a set of line segments drawn at the values of x with lengths that represent the probability of each x.

Example 2 – *Solution*

cont'd

Figure 5.1 shows the probability distribution of $P(x) = \frac{x}{10}$ for x = 1, 2, 3, 4



Line Representation: Probability Distribution for $P(x) = \frac{x}{10}$ for x = 1, 2, 3, 4

Figure 5.1

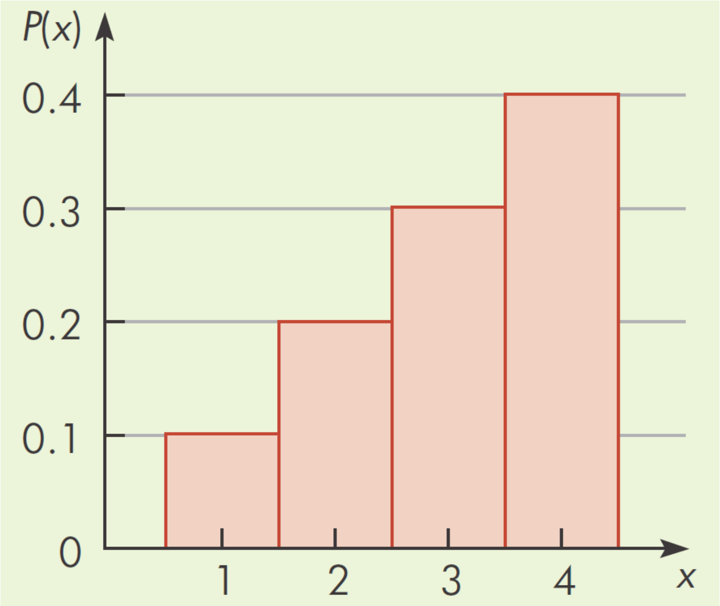
Example 2 – *Solution*

cont'd

A regular histogram is used more frequently to present probability distributions.

Figure 5.2 shows the probability distribution of Figure 5.1 as a **probability histogram.**

The histogram of a probability distribution uses the physical area of each bar to represent its assigned probability.



Histogram: Probability Distribution for $P(x) = \frac{x}{10}$ for x = 1, 2, 3, 4

Figure 5.2

Example 2 – *Solution*

cont'd

The bar for $x = 2$ is 1 unit wide (from 1.5 to 2.5) and 0.2 unit high.

Therefore, its area (length × width) is (0.2)(1) = 0.2, the probability assigned to $x = 2$.

The areas of the other bars can be determined in similar fashion.

This area representation will be an important concept in Chapter 6 when we begin to work with continuous random variables.

Mean and Variance of a Discrete Probability Distribution

Mean and Variance of a Discrete Probability Distribution

Probability distributions may be used to represent theoretical populations, the counterpart to samples.

We use **population parameters** (mean, variance, and standard deviation) to describe these probability distributions just as we use **sample statistics** to describe samples.

Notes

1. $\bar{x}$ is the mean of the sample.

2. s^2 and s are the variance and standard deviation of the sample, respectively.

Mean and Variance of a Discrete Probability Distribution

3. $\bar{x}$, s^2, and s are called *sample statistics*.

4. μ (lowercase Greek letter mu) is the mean of the population.

5. σ^2 (sigma squared) is the variance of the population.

6. σ (lowercase Greek letter sigma) is the standard deviation of the population.

Mean and Variance of a Discrete Probability Distribution

7. μ, σ^2 and σ are called *population parameters*. (A parameter is a constant; σ, σ^2, and σ are typically unknown values in real statistics problems. About the only time they are known is in a textbook problem setting for the purposes of learning and understanding.)

Mean and Variance of a Discrete Probability Distribution

The *mean of the probability distribution* of a discrete random variable, or the *mean of a discrete random variable*, is found in a manner that takes full advantage of the table format of a discrete probability distribution.

The mean of a discrete random variable is often referred to as its *expected value*.

Mean of a discrete random variable (expected value)

The mean, μ, of a discrete random variable x is found by multiplying each possible value of x by its own probability and then adding all the products together:

Mean and Variance of a Discrete Probability Distribution

mean of x:

mu = sum of (each x multiplied by its own probability)

$$\mu = \Sigma[xP(x)] \tag{5.1}$$

The variance of a discrete random variable is defined in much the same way as the variance of sample data, the mean of the squared deviations from the mean.

Mean and Variance of a Discrete Probability Distribution

Variance of a discrete random variable The variance, σ^2, of a discrete random variable x is found by multiplying each possible value of the squared deviation from the mean, $(x - \mu)^2$, by its own probability and then adding all the products together:

variance:

sigma squared = sum of (squared deviation times probability)

$$\sigma^2 = \Sigma[(x - \mu)^2 P(x)] \quad \textbf{(5.2)}$$

Mean and Variance of a Discrete Probability Distribution

Formula (5.2) is often inconvenient to use; it can be reworked into the following form(s):

variance:

sigma squared = *sum of* (x^2 *times probability*) – [*sum of* (x *times probability*)]^2

$$\sigma^2 = \Sigma[x^2 P(x)] - \{ \Sigma[xP(x)]\}^2 \qquad \textbf{(5.3a)}$$

or

$$\sigma^2 = \Sigma[x^2 P(x)] - \mu^2 \qquad \textbf{(5.3b)}$$

Mean and Variance of a Discrete Probability Distribution

Likewise, standard deviation of a random variable is calculated in the same manner as is the standard deviation of sample data.

Standard deviation of a discrete random variable

The positive square root of variance.

$$\text{standard deviation: } \sigma = \sqrt{\sigma^2} \qquad \textbf{(5.4)}$$

Example 5 – *Mean, Variance, and Standard Deviation of a Discrete Random Variable*

A coin is tossed three times. Let the "number of heads" that occur in those three tosses be the random variable, x.

Find the mean, variance, and standard deviation of x.

Solution:

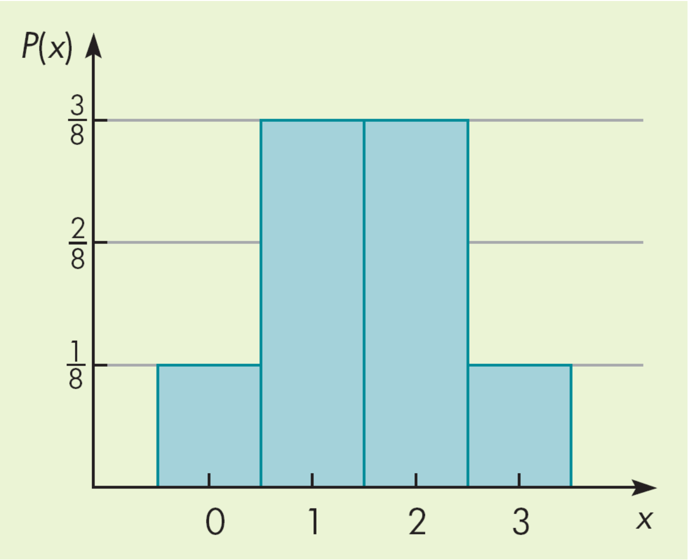
There are eight possible outcomes (all equally likely) to this experiment: {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

One outcome results in $x = 0$, three in $x = 1$, three in $x = 2$, and one in $x = 3$. Therefore, the probabilities for this random variable are $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, and $\frac{1}{8}$.

Example 5 – *Solution*

cont'd

The probability distribution associated with this experiment is shown in Figure 5.3 and in Table 5.6.



Probability Distribution: Number of Heads in Three Tosses of Coin

Figure 5.3

Example 5 – *Solution*

cont'd

The necessary extensions and summations for the calculation of the mean, variance, and standard deviation are also shown in Table 5.6.

x	$P(x)$	$xP(x)$	x^2	$x^2P(x)$
0	$\frac{1}{8}$ ✓	$\frac{0}{8}$	0	$\frac{0}{8}$
1	$\frac{3}{8}$ ✓	$\frac{3}{8}$	1	$\frac{3}{8}$
2	$\frac{3}{8}$ ✓	$\frac{6}{8}$	4	$\frac{12}{8}$
3	$\frac{1}{8}$ ✓	$\frac{3}{8}$	9	$\frac{9}{8}$
	$\Sigma[P(x)] = \frac{8}{8} = 1.0$ ck	$\Sigma[xP(x)] = \frac{12}{8} = 1.5$		$\Sigma[x^2P(x)] = \frac{24}{8} = 3.0$

Extensions Table of Probability Distribution of Number of Heads in Three Coin Tosses

Table 5.6

Example 5 – *Solution*

The mean is found using formula (5.1):

$$\mu = \Sigma[xP(x)] = \mathbf{1.5}$$

This result, 1.5, is the mean of the theoretical distribution for the random variable "number of heads" observed per set of three coin tosses.

It is expected that the mean for many observed values of the random variable will also be approximately equal to this value. The variance is found using formula (5.3a):

$$\sigma^2 = \Sigma[\mathrm{x}^2\, P(x)] - \{\, \Sigma[xP(x)]\}^2$$

$$= 3.0 - (1.5)^2$$

Example 5 – *Solution*

cont'd

$$= 3.0 - 2.25$$

$$= \mathbf{0.75}$$

The standard deviation is found using formula (5.4):

$$\sigma = \sqrt{\sigma^2}$$

$$= \sqrt{0.75}$$

$$= 0.866$$

$$= \mathbf{0.87}$$

Example 5 – *Solution*

cont'd

That is, 0.87 is the standard deviation of the theoretical distribution for the random variable "number of heads" observed per set of three coins tosses.

It is expected that the standard deviation for many observed values of the random variable will also be approximately equal to this value.