∞-Cosmoi for Lean

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Chapter 1

∞-Cosmoi

1.1 Overview

Following [RV22], from which this document was excerpted, we aim to develop the basic theory of ∞-categories in a model independent fashion using a common axiomatic framework that is satisfied by a variety of models. In contrast with prior "analytic" treatments of the theory of ∞-categories — in which the central categorical notions are defined in reference to the coordinates of a particular model — our approach is "synthetic," proceeding from definitions that can be interpreted simultaneously in many models to which our proofs then apply.

To achieve this, our strategy is not to axiomatize what infinite-dimensional categories *are*, but rather to axiomatize the categorical "universe" in which they *live*. This motivates the notion of an ∞-*cosmos*, which axiomatizes the universe in which ∞-categories live as objects.[1] So that theorem statement about ∞-cosmoi suggest their natural interpretation, we recast ∞-*category* as a technical term, to mean an object in some (typically fixed) ∞-cosmos. Several common models of $(\infty, 1)$-categories[2] are ∞-categories in this sense, but our ∞-categories also include certain models of (∞, n)-categories[3] as well as fibered versions of all of the above. Thus each of these objects are ∞-categories in our sense and our theorems apply to all of them.[4] This usage of the term "∞-categories" is meant to interpolate between the classical one, which refers to any variety of weak infinite-dimensional categories, and the common one, which is often taken to mean quasi-categories or complete Segal spaces.

Much of the development of the theory of ∞-categories takes place not in the full ∞-cosmos but in a quotient that we call the *homotopy 2-category*, the name chosen because an ∞-cosmos is something like a category of fibrant objects in an enriched model category and the homotopy 2-category is then a categorification of its homotopy category. The homotopy 2-category is a strict

[1] Metaphorical allusions aside, our ∞-cosmoi resemble the fibrational cosmoi of Street [Str74].

[2] Quasi-categories, complete Segal spaces, Segal categories, and 1-complicial sets (naturally marked quasi-categories) all define the ∞-categories in an ∞-cosmos.

[3] n-quasi-categories, Θ_n-spaces, iterated complete Segal spaces, and n-complicial sets also define the ∞-categories in an ∞-cosmos, as do saturated (née weak) complicial sets, a model for (∞, ∞)-categories.

[4] There is a sense, however, in which many of our definitions are optimized for those ∞-cosmoi whose objects are $(\infty, 1)$-categories. A good illustration is provided by the notion of *discrete ∞-category*. In the ∞-cosmoi of $(\infty, 1)$-categories, the discrete ∞-categories are the ∞-groupoids, but this is not true for the ∞-cosmoi of (∞, n)-categories.

2-category — like the 2-category of categories, functors, and natural transformations[5] — and in this way the foundational proofs in the theory of ∞-categories closely resemble the classical foundations of ordinary category theory except that the universal properties they characterize, e.g., when a functor between ∞-categories defines a cartesian fibration, are slightly weaker than in the familiar case of strict 1-categories.

There are many alternate choices we could have made in selecting the axioms of an ∞-cosmos. One of our guiding principles, admittedly somewhat contrary to the setting of homotopical higher category theory, was to allow us to work as strictly as possible, with the aim of shortening and simplifying proofs. As a consequence of these choices, the ∞-categories in an ∞-cosmos and the functors and natural transformations between them assemble into a 2-category rather than a bicategory. To help us achieve this counterintuitive strictness, each ∞-cosmos comes with a specified class of maps between ∞-categories called *isofibrations*. The isofibrations have no homotopy-theoretic meaning, as any functor between ∞-categories is equivalent to an isofibration with the same codomain. However, isofibrations permit us to consider strictly commutative diagrams between ∞-categories and allow us to require that the limits of such diagrams satisfy a universal property up to simplicially enriched isomorphism. Neither feature is essential for the development of ∞-category theory. Similar proofs carry through to a weaker setting, at the cost of more time spent considering coherence of higher cells.

An ∞-cosmos is a particular sort of *simplicially enriched category* with certain *simplicially enriched limits*. While the notion of simplicially enriched category currently exists in Mathlib, simplicially enriched limits do not, so in §**??** we first introduce the prerequisite notions of simplicially enriched limits that will be required to state the definition of an ∞-cosmos in §**??**.

The authors of this blueprint are particularly indebted to:

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1.2 Simplicial sets

Before introducing an axiomatic framework that allows us to develop ∞-category theory in general, we first consider one model in particular: *quasi-categories*, which were introduced in 1973 by Boardman and Vogt [BV73] in their study of homotopy coherent diagrams. Ordinary 1-categories give examples of quasi-categories via the construction of Definition 1.2.5. Joyal first undertook the task of extending 1-category theory to quasi-category theory in [Joy02] and [Joy08] and in several unpublished draft book manuscripts. The majority of the results in this section are due to him.

[5] In fact this is another special case: there is an ∞-cosmos whose objects are ordinary categories and its homotopy 2-category is the usual category of categories, functors, and natural transformations. This 2-category is as old as category theory itself, introduced in Eilenberg and Mac Lane's foundational paper [EML45].

Definition 1.2.1 (the simplex category)**.** Let $\mathbf{\Delta}$ denote the **simplex category** of finite nonempty ordinals $[n] = \{0 < 1 < \cdots < n\}$ and order-preserving maps. These include in particular the

$$\begin{array}{lcc} \text{elementary face operators} & [n-1] \xrightarrow{\delta^i} [n] & 0 \leq i \leq n \\ \text{elementary degeneracy operators} & [n+1] \xrightarrow{\sigma^i} [n] & 0 \leq i \leq n \end{array}$$

whose images, respectively, omit and double up on the element $i \in [n]$. Every morphism in $\mathbf{\Delta}$ factors uniquely as an epimorphism followed by a monomorphism; these epimorphisms, the **degeneracy operators**, decompose as composites of elementary degeneracy operators, while the monomorphisms, the **face operators**, decompose as composites of elementary face operators.

Definition 1.2.2. The category of **simplicial sets** is the category $s\mathcal{S}et := \mathcal{S}et^{\mathbf{\Delta}^{\mathrm{op}}}$ of presheaves on the simplex category.

We write $\Delta[n]$ for the **standard** n**-simplex** the simplicial set represented by $[n] \in \mathbf{\Delta}$, and $\Lambda^k[n] \subset \partial\Delta[n] \subset \Delta[n]$ for its k**-horn** and **boundary sphere**, respectively. The sphere $\partial\Delta[n]$ is the simplicial subset generated by the codimension-one faces of the n-simplex, while the horn $\Lambda^k[n]$ is the further simplicial subset that omits the face opposite the vertex k.

Given a simplicial set X, it is conventional to write X_n for the set of n**-simplices**, defined by evaluating at $[n] \in \mathbf{\Delta}$. By the Yoneda lemma, each n-simplex $x \in X_n$ corresponds to a map of simplicial sets $x \colon \Delta[n] \to X$. Accordingly, we write $x \cdot \delta^i$ for the ith face of the n-simplex, an $(n-1)$-simplex classified by the composite map

$$\Delta[n-1] \xrightarrow{\delta^i} \Delta[n] \xrightarrow{x} X.$$

The right action of the face operator defines a map $X_n \xrightarrow{\cdot\delta^i} X_{n-1}$. Geometrically, $x \cdot \delta^i$ is the "face opposite the vertex i" in the n-simplex x.

The category of simplicial sets, as a presheaf category, is very well-behaved: it is cartesian closed and has all small limits and colimits. Instances of these facts currently appear in Mathlib. The definition of a quasi-category can be found there as well.

Definition 1.2.3 (quasi-category)**.** A **quasi-category** is a simplicial set A in which any **inner horn** can be extended to a simplex, solving the displayed lifting problem:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & A \\ \Delta[n] \arrow[ur, dashed] \end{tikzcd} \qquad \text{for } \; n \geq 2, \; 0 < k < n. \tag{1.2.4}$$

Quasi-categories were first introduced by Boardman and Vogt [BV73] under the name "weak Kan complexes," a **Kan complex** being a simplicial set admitting extensions as in (1.2.4) along all horn inclusions $n \geq 1, 0 \leq k \leq n$. Since any topological space can be encoded as a Kan complex,[6] in this way spaces provide examples of quasi-categories.

Categories also provide examples of quasi-categories via the nerve construction.

[6]The total singular complex construction defines a functor from topological spaces to simplicial sets that is an equivalence on their respective homotopy categories — weak homotopy types of spaces correspond to homotopy equivalence classes of Kan complexes [Qui67, §II.2]. The left adjoint "geometrically realizes" a simplicial set as a topological space.

Definition 1.2.5 (nerve)**.** The category $\mathcal{C}at$ of 1-categories embeds fully faithfully into the category of simplicial sets via the **nerve** functor. An n-simplex in the nerve of a 1-category C is a sequence of n composable arrows in C, or equally a functor $\mathbb{n}+\mathbb{1} \to C$ from the ordinal category $\mathbb{n}+\mathbb{1} := [n]$ with objects $0, \ldots, n$ and a unique arrow $i \to j$ just when $i \leq j$.

The map $[n] \mapsto \mathbb{n}+\mathbb{1}$ defines a fully faithful embedding $\mathbf{\Delta} \hookrightarrow \mathcal{C}at$. From this point of view, the nerve functor can be described as a "restricted Yoneda embedding" which carries a category C to the restriction of the representable functor $\hom(-, C)$ to the image of this inclusion. This is an instance of a more general family of "nerve-type constructions."

Remark 1.2.6. The nerve of a category C is **2-coskeletal** as a simplicial set, meaning that every sphere $\partial\Delta[n] \to C$ with $n \geq 3$ is filled uniquely by an n-simplex in C, or equivalently that the nerve is canonically isomorphic to the right Kan extension of its restriction to 2-truncated simplicial sets.[7]Note a sphere $\partial\Delta[2] \to C$ extends to a 2-simplex if and only if that arrow along its diagonal edge is the composite of the arrows along the edges in the inner horn $\Lambda^1[2] \subset \partial\Delta[2] \to C$. The simplices in dimension 3 and above witness the associativity of the composition of the path of composable arrows found along their **spine**, the 1-skeletal simplicial subset formed by the edges connecting adjacent vertices. In fact, as suggested by the proof of Proposition 1.2.7, any simplicial set in which inner horns admit *unique* fillers is isomorphic to the nerve of a 1-category.

We decline to introduce explicit notation for the nerve functor, preferring instead to identify 1-categories with their nerves. As we shall discover the theory of 1-categories extends to ∞-categories modeled as quasi-categories in such a way that the restriction of each ∞-categorical concept along the nerve embedding recovers the corresponding 1-categorical concept. For instance, the standard simplex $\Delta[n]$ is isomorphic to the nerve of the ordinal category $\mathbb{n}+1$, and we frequently adopt the latter notation — writing $\mathbb{1} := \Delta[0]$, $\mathbb{2} := \Delta[1]$, $\mathbb{3} := \Delta[2]$, and so on — to suggest the correct categorical intuition.

To begin down this path, we must first verify the implicit assertion that has just been made. A proof of the following result will appear in Mathlib soon.

Proposition 1.2.7 (nerves are quasi-categories)**.** *Nerves of categories are quasi-categories.*

Proof. Via the isomorphism $C \cong \mathrm{cosk}_2 C$ from Remark 1.2.6 and the associated adjunction $\mathrm{sk}_2 \dashv \mathrm{cosk}_2$ of, the required lifting problem displayed below-left transposes to the one displayed below-right:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & C \cong \mathrm{cosk}_2 C \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \mathrm{sk}_2\Lambda^k[n] \arrow[r] \arrow[d, hook] & C \\ \mathrm{sk}_2\Delta[n] \arrow[ur, dashed] & \end{tikzcd}$$

The functor sk_2 replaces a simplicial set by its **2-skeleton**, the simplicial subset generated by the simplices of dimension at most two. For $n \geq 4$, the inclusion $\mathrm{sk}_2\Lambda^k[n] \hookrightarrow \mathrm{sk}_2\Delta[n]$ is an isomorphism, in which case the lifting problems on the right admit (unique) solutions. So it remains only to solve the lifting problems on the left in the cases $n = 2$ and $n = 3$.

[7]The equivalence between these two perspectives is non-obvious and makes use of Reedy category theory (see [RV22, §C.4-5]), which does not currently exist in Mathlib.

To that end consider

$$
\begin{tikzcd}
\Lambda^1[2] \arrow[r] \arrow[d, hook] & C \\
\Delta[2] \arrow[ur, dashed] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda^1[3] \arrow[r] \arrow[d, hook] & C \\
\Delta[3] \arrow[ur, dashed] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda^2[3] \arrow[r] \arrow[d, hook] & C \\
\Delta[3] \arrow[ur, dashed] &
\end{tikzcd}
$$

An inner horn $\Lambda^1[2] \to C$ defines a composable pair of arrows in C; an extension to a 2-simplex exists precisely because any composable pair of arrows admits a (unique) composite.

An inner horn $\Lambda^1[3] \to C$ specifies the data of three composable arrows in C, as displayed in the following diagram, together with the composites gf, hg, and $(hg)f$.

$$
\begin{tikzcd}
& c_1 \arrow[dd, "g"' near end] \arrow[dr, "hg"] & \\
c_0 \arrow[ur, "f"] \arrow[rr, "(hg)f" near end] \arrow[dr, "gf"'] & & c_3 \\
& c_2 \arrow[ur, "h"'] &
\end{tikzcd}
$$

Because composition is associative, the arrow $(hg)f$ is also the composite of gf followed by h, which proves that the 2-simplex opposite the vertex c_1 is present in C; by 2-coskeletality, the 3-simplex filling this boundary sphere is also present in C. The filler for a horn $\Lambda^2[3] \to C$ is constructed similarly. $\square$

We now turn to the homotopy category functor. The following definitions and results are not currently in Mathlib but will appear there soon.

Definition 1.2.8 (homotopy relation on 1-simplices)**.** A parallel pair of 1-simplices f, g in a simplicial set X are **homotopic** if there exists a 2-simplex whose boundary takes either of the following forms[8]

$$
\begin{tikzcd}
& y \arrow[dr, equal] & \\
x \arrow[ur, "f"] \arrow[rr, "g"'] & & y
\end{tikzcd}
\qquad
\begin{tikzcd}
& x \arrow[dr, "f"] & \\
x \arrow[ur, equal] \arrow[rr, "g"'] & & y
\end{tikzcd}
\tag{1.2.9}
$$

or if f and g are in the same equivalence class generated by this relation.

In a quasi-category, the relation witnessed by either of the types of 2-simplex on display in (1.2.9) is an equivalence relation and these equivalence relations coincide.

Lemma 1.2.10 (homotopic 1-simplices in a quasi-category)**.** *Parallel 1-simplices f and g in a quasi-category are homotopic if and only if there exists a 2-simplex of any or equivalently all of the forms displayed in* (1.2.9)*.*

Definition 1.2.11 (the homotopy category [GZ67, §2.4])**.** By 1-truncating, any simplicial set X has an underlying reflexive directed graph with the 0-simplices of X defining the objects and the 1-simplices defining the arrows:

$$
\begin{tikzcd}
X_1 \arrow[r, shift left=2, "\cdot\delta^1"] \arrow[r, shift right=2, "\cdot\delta^0"'] & X_0 \arrow[l, "\cdot\sigma^0" description]
\end{tikzcd},
$$

[8] The symbol " $=$ " is used in diagrams to denote a degenerate simplex or an identity arrow.

By convention, the source of an arrow $f \in X_1$ is its 0th face $f \cdot \delta^1$ (the face opposite 1) while the target is its 1st face $f \cdot \delta^0$ (the face opposite 0). The **free category** on this reflexive directed graph has X_0 as its object set, degenerate 1-simplices serving as identity morphisms, and nonidentity morphisms defined to be finite directed paths of nondegenerate 1-simplices. The **homotopy category** $\mathsf{h}X$ of X is the quotient of the free category on its underlying reflexive directed graph by the congruence[9] generated by imposing a composition relation $h = g \circ f$ witnessed by 2-simplices

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & x_2 \end{tikzcd}$$

This relation implies in particular that homotopic 1-simplices represent the same arrow in the homotopy category.

The homotopy category of the nerve of a 1-category is isomorphic to the original category, as the 2-simplices in the nerve witness all of the composition relations satisfied by the arrows in the underlying reflexive directed graph. Indeed, the natural isomorphism $\mathsf{h}C \cong C$ forms the counit of an adjunction, embedding $\mathcal{C}at$ as a reflective subcategory of $s\mathcal{S}et$.

Proposition 1.2.12. *The nerve embedding admits a left adjoint, namely the functor which sends a simplicial set to its homotopy category:*

$$\begin{tikzcd} \mathcal{C}at \arrow[r, hook, bend right] \arrow[r, phantom, "\perp"] & s\mathcal{S}et \arrow[l, bend right, "\mathsf{h}"'] \end{tikzcd}$$

The adjunction of Proposition 1.2.12 exists for formal reasons, via results which have already been formalized in Mathlib, once the category $\mathcal{C}at$ is known to be cocomplete. A proof of this fact does not currently exist in Mathlib, however, and in fact the adjunction between the homotopy category and the nerve can be used to construct colimits of categories, as it embeds $\mathcal{C}at$ as a reflective subcategory of a cocomplete category (see [Rie16, 4.5.16]). Thus, the approach that is currently being formalized instead gives a direct proof.

Proof. For any simplicial set X, there is a natural map from X to the nerve of its homotopy category $\mathsf{h}X$; since nerves are 2-coskeletal, it suffices to define the map $\mathrm{sk}_2 X \to \mathsf{h}X$, and this is given immediately by the construction of Definition 1.2.11. Note that the quotient map $X \to \mathsf{h}X$ becomes an isomorphism upon applying the homotopy category functor and is already an isomorphism whenever X is the nerve of a category. Thus the adjointness follows by direct verification of the triangle equalities. ☐

The homotopy category of a quasi-category admits a simplified description.

Lemma 1.2.13 (the homotopy category of a quasi-category)**.** *If A is a quasi-category then its* ***homotopy category*** $\mathsf{h}A$ *has*

- *the set of 0-simplices A_0 as its objects*
- *the set of homotopy classes of 1-simplices A_1 as its arrows*

[9] A binary relation $\sim$ on parallel arrows of a 1-category is a **congruence** if it is an equivalence relation that is closed under pre- and post-composition: if $f \sim g$ then $hfk \sim hgk$.

- *the identity arrow at $a \in A_0$ represented by the degenerate 1-simplex $a \cdot \sigma^0 \in A_1$*
- *a composition relation $h = g \circ f$ in $\mathsf{h}A$ between the homotopy classes of arrows represented by any given 1-simplices $f, g, h \in A_1$ if and only if there exists a 2-simplex with boundary*

$$\begin{tikzcd} & a_1 \arrow[dr, "g"] & \\ a_0 \arrow[ur, "f"] \arrow[rr, "h"'] & & a_2 \end{tikzcd}$$

Definition 1.2.14 (isomorphism in a quasi-category)**.** A 1-simplex in a quasi-category is an **isomorphism**[10] just when it represents an isomorphism in the homotopy category. By Lemma 1.2.13 this means that $f\colon a \to b$ is an isomorphism if and only if there exists a 1-simplex $f^{-1}\colon b \to a$ together with a pair of 2-simplices

$$\begin{tikzcd} & b \arrow[dr, dashed, "f^{-1}"] & \\ a \arrow[ur, "f"] \arrow[rr, equal] & & a \end{tikzcd} \qquad \begin{tikzcd} & a \arrow[dr, "f"] & \\ b \arrow[ur, dashed, "f^{-1}"] \arrow[rr, equal] & & b \end{tikzcd}$$

The properties of the isomorphisms in a quasi-category are somewhat technical to prove and will likely be a pain to formalize (see [RV22, §D]). Here we focus on a few essential results, which are more easily obtainable.

Just as the arrows in a quasi-category A are represented by simplicial maps $\mathbb{2} \to A$ whose domain is the nerve of the free-living arrow, the isomorphisms in a quasi-category can be represented by diagrams $\mathbb{I} \to A$ whose domain, called the **homotopy coherent isomorphism**, is the nerve of the free-living isomorphism:

Corollary 1.2.15. *An arrow f in a quasi-category A is an isomorphism if and only if it extends to a homotopy coherent isomorphism*

$$\begin{tikzcd} \mathbb{2} \arrow[r, "f"] \arrow[d, hook] & A \\ \mathbb{I} \arrow[ur, dashed] & \end{tikzcd}$$

Remark 1.2.16. If this result proves too annoying to formalize without the general theory of "special-outer horn filling," we might instead substitute a finite model of the homotopy coherent isomorphism for $\mathbb{I}$.

Quasi-categories define the fibrant objects in a model structure due to Joyal. We use the term *isofibration* to refer to the fibrations between fibrant objects in this model structure, which admit the following concrete description.

Definition 1.2.17 (isofibration)**.** A simplicial map $f\colon A \to B$ between quasi-categories is an **isofibration** if it lifts against the inner horn inclusions, as displayed below-left, and also against

[10] Joyal refers to these maps as "isomorphisms" while Lurie refers to them as "equivalences." We prefer, wherever possible, to use the same term for ∞-categorical concepts as for the analogous 1-categorical ones.

the inclusion of either vertex into the free-living isomorphism $\mathbb{I}$.[11]

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} \qquad \begin{tikzcd} \mathbb{1} \arrow[r] \arrow[d, hook] & A \arrow[d, two heads, "f"] \\ \mathbb{I} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

To notationally distinguish the isofibrations, we depict them as arrows "$\twoheadrightarrow$" with two heads.

We now introduce the weak equivalences and trivial fibrations between fibrant objects in the Joyal model structure.

Definition 1.2.18 (equivalences of quasi-categories)**.** A map $f\colon A \to B$ between quasi-categories is an **equivalence** if it extends to the data of a "homotopy equivalence" with the free-living isomorphism $\mathbb{I}$ serving as the interval: that is, if there exist maps $g\colon B \to A$,

$$\begin{tikzcd} & A \\ A \arrow[ur, equals] \arrow[r, "\alpha"] \arrow[dr, "gf"'] & A^{\mathbb{I}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & B \\ B \arrow[ur, "fg"] \arrow[r, "\beta"] \arrow[dr, equals] & B^{\mathbb{I}} \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & B \end{tikzcd}$$

We write "$\xrightarrow{\sim}$" to decorate equivalences and $A \simeq B$ to indicate the presence of an equivalence $A \xrightarrow{\sim} B$.

Remark 1.2.19. If $f\colon A \to B$ is an equivalence of quasi-categories, then the functor $\mathsf{h}f\colon \mathsf{h}A \to \mathsf{h}B$ is an equivalence of categories, where the data displayed above defines an equivalence inverse $\mathsf{h}g\colon \mathsf{h}B \to \mathsf{h}A$ and natural isomorphisms encoded by the composite[12] functors

$$\mathsf{h}A \xrightarrow{\mathsf{h}\alpha} \mathsf{h}(A^{\mathbb{I}}) \longrightarrow (\mathsf{h}A)^{\mathbb{I}} \qquad \mathsf{h}B \xrightarrow{\mathsf{h}\beta} \mathsf{h}(B^{\mathbb{I}}) \longrightarrow (\mathsf{h}B)^{\mathbb{I}}$$

Definition 1.2.20. A map $f\colon X \to Y$ between simplicial sets is a **trivial fibration** if it admits lifts against the boundary inclusions for all simplices

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, hook] & X \arrow[d, two heads, "\sim"', "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \text{for} \quad n \geq 0 \tag{1.2.21}$$

We write "$\overset{\sim}{\twoheadrightarrow}$" to decorate trivial fibrations.

The notation "$\overset{\sim}{\twoheadrightarrow}$" is suggestive: the trivial fibrations between quasi-categories are exactly those maps that are both isofibrations and equivalences. This can be proven by a relatively standard although rather technical argument in simplicial homotopy theory [RV22, D.5.6].

[11]The free-living isomorphism is the 0-coskeletal simplicial set on two vertices, or equally the nerve of the free-living isomorphism of categories.

[12]Note that $\mathsf{h}(A^{\mathbb{I}}) \not\cong (\mathsf{h}A)^{\mathbb{I}}$ in general. Objects in the latter are homotopy classes of isomorphisms in A, while objects in the former are homotopy coherent isomorphisms, given by a specified 1-simplex in A, a specified inverse 1-simplex, together with an infinite tower of coherence data indexed by the nondegenerate simplices in $\mathbb{I}$.

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