NAND programming language

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1 The NAND Programming Language

Version: 0.2

The NAND programming language was designed to accompany the upcoming book "Introduction to Theoretical Computer Science". This is an appendix to this book, which is also available online as a Jupyter notebook.

The NAND programming language is part of a family of languages:

- **NAND:** The NAND programming language is equivalent in expressive power to *Boolean circuits*. Specifically NAND programs are straightline programs with the *NAND* operation, which are equivalent to Boolean circuits with *NAND* gates.
- **NAND++:** The NAND++ programming language is equivalent in expressive power to *Turing Machines*. NAND++ programs are obtained by adding *loops* and *unbounded arrays* to NAND programs. Specifically, NAND++ programs can be thought of as capturing *oblivious single tape Turing Machines*, that are polynomially equivalent to all other standard variants of Turing Machines and Turing-equivalent models of computation.
- **NAND<<:** The NAND<< programing language is a formalization of *RAM Machines*. NAND<< programs are obtained by adding integer-valued variables and arrays, and the standard "C type" operations on them, as well as *indirect indexing* of arrays via integer variables. NAND<< programs are equivalent up to polylogarithmic terms to standard models of RAM machines, and up to polynomial terms with NAND++ and all other standard Turing-equivalent models.
- **QNAND:** QNAND is only used in a single chapter of the book, and is meant to capture *Quantum Boolean Circuits* and so can be used to define the class **BQP** of polyonmial time quantum computaiton.

1.1 Syntax of NAND programs

This notebook/appendix is concerned with the first and simplest of these languages: the *NAND Programming Languages*. A NAND program consists of a sequence of lines, each of the following form:

```
foo := NAND(bar,blah)
```

where `foo`, `bar` and `blah` are variable identifiers.

We have two special types of variables: *input variables* have the form `X[` i `]` where i is a natural number, and *output variables* have the form `Y[` j `]` where j is a natural number. When a NAND program is *executed* on input $x \in \{0,1\}^n$, the variable `X[`i`]` is assigned the value x_i for all $i \in [n]$. The *output* of the program is the list of m values `Y[0]`... `Y[`$m-1$`]`, where $m-1$ is the largest index for which the variable `Y[`$m-1$`]` is assigned a value in the program.

Here is an example of a NAND program:

```
In [76]: xor = r'''
         u = NAND(X[0],X[1])
         v = NAND(X[0],u)
         w = NAND(X[1],u)
         Y[0] = NAND(v,w)
         '''
```

To evaluate this program on inputs $x_0, x_1 \in \{0,1\}$, we can use a simple *Python* function that would keep track of the values assigned to variables.

```
In [102]: def NAND(x,y):
              """Compute the NAND of two 0/1 valued variables."""
              return 1 - x*y
```

```
In [79]: import re
         def numinout(prog):
             '''Compute the number of inputs and outputs of a NAND program, given as a string
             n = max([int(s[2:-1]) for s in re.findall(r'X\[\d+\]',prog)])+1
             m = max([int(s[2:-1]) for s in re.findall(r'Y\[\d+\]',prog)])+1
             return n,m

         numinout(xor)

Out[79]: (2, 1)
```

```
In [80]: from collections import defaultdict

         def EVAL(prog,x):
             """Evaluate NAND program prog on input x."""
             (n,m) = numinout(prog)
             vartable = defaultdict(int) # dictionary where missing entries are automatically
             for i in range(n): vartable[f'X[{i}]']=x[i]
             for line in filter(None,re.split('\n+',prog)): # split into lines
                 foo,bar,blah, = filter(None,re.split('\s*=\s*NAND\s*\(|\,|\)\s*|\s+',line)) #
                 vartable[foo] =  NAND(vartable[bar],vartable[blah])
             return [vartable[f'Y[{j}]'] for j in range(m)]
```

```
In [81]: EVAL(xor,[1,1])

Out[81]: [0]
```

While we could easily write a "NAND interpreter" in any programming language, by design a NAND program uses valid Python code. So we can also write NAND programs as python functions:

```
In [83]: def XOR(a,b):
             t1 = NAND(a,b)
             t2 = NAND(a,t1)
             t3 = NAND(b,t1)
             return NAND(t2,t3)

         XOR(0,1)
```

```
Out[83]: 1
```

We can translate such a function into standard NAND code using a simple trick. We will "override" the NAND function to take a pair of strings `bar`, `blah` instead of integers as input and generate the line of code `foo = NAND(bar,blah)` instead of computing the NAND function.

```
In [106]: from inspect import signature
          def numarguments(f):
              """Number of arguments a Python function takes."""
              return len(signature(f).parameters)

          def nandcode(f):
              n = numarguments(f)
              counter = 0 # to ensure unique temporary variables.
              code = ''
              global NAND
              def tempNAND(bar,blah):
                  nonlocal code, counter
                  var = f'Temp[{counter}]'
                  counter += 1
                  code += f'{var} = NAND({bar},{blah})\n'
                  return var

              NAND , tempNAND = tempNAND, NAND # Override NAND with its temporary version
              # (this is a hack and won't play nicely with exceptions or multithreading)
              outputs = f(*[f'X[{i}]' for i in range(n)]) # execute f on the strings "X[0]", "
              NAND , tempNAND = tempNAND, NAND # Restore original NAND

              if type(outputs)==str: outputs = [outputs] # make single output into singleton l

              for j in range(len(outputs)):
                  code = code.replace(outputs[j],f'Y[{j}]')
              return code

          print(nandcode(XOR))
```

```
Temp[0] = NAND(X[0],X[1])
Temp[1] = NAND(X[0],Temp[0])
Temp[2] = NAND(X[1],Temp[0])
Y[0] = NAND(Temp[1],Temp[2])
```

```
In [105]: XOR(1,1)
```

```
Out[105]: 0
```

1.2 Syntactic sugar

NAND is pretty bare bones, and writing NAND code directly gets real old real fast. However, we can use "syntactic sugar" to make it a little less tedious. For example we can use function definitions to avoid copying again and again repetitive code.

We will use the Python-like syntax of `def func(...):` to define functions and so we can write the XOR on 4 bits function as follows:

```
In [107]: def XOR4(a,b,c,d):
              return XOR(XOR(a,b),XOR(c,d))

          XOR4(0,1,0,1)
```

```
Out[107]: 0
```

To verify that this is indeed merely "syntactic sugar" and this can be translated to pure NAND we can use our `nandcode` function:

```
In [108]: print(nandcode(XOR4))
```

```
Temp[0] = NAND(X[0],X[1])
Temp[1] = NAND(X[0],Temp[0])
Temp[2] = NAND(X[1],Temp[0])
Temp[3] = NAND(Temp[1],Temp[2])
Temp[4] = NAND(X[2],X[3])
Temp[5] = NAND(X[2],Temp[4])
Temp[6] = NAND(X[3],Temp[4])
Temp[7] = NAND(Temp[5],Temp[6])
Temp[8] = NAND(Temp[3],Temp[7])
Temp[9] = NAND(Temp[3],Temp[8])
Temp[10] = NAND(Temp[7],Temp[8])
Y[0] = NAND(Temp[9],Temp[10])
```

Once we have this, we can also define other functions such as AND, OR, NOT, and also the IF function that on input `cond`, `a` and `b` returns `a` if `cond` equals 1 and `b` otherwise.

```
In [134]: def NOT(a):
              return NAND(a,a)

          def AND(a,b):
```

```
    return NOT(NAND(a,b))

def OR(a,b):
    return NAND(NOT(a),NOT(b))

def IF(cond,first,sec):
    temp1 = AND(cond,first) # zero if cond=0, otherwise first
    temp2 = AND(NOT(cond),sec) # zero if cond=1, otherwise second
    return OR(temp1,temp2)

def one(a):
    return NAND(a,NOT(a))

def zero(a):
    return NOT(one(a))

def COPY(a):
    return NOT(NOT(a))

IF(0,1,0)
```

```
Out[134]: 0
```

We can use more python-inspired syntactic sugar:

```
In [139]: def increment(X): # increment integer given in binary representation
              n = len(X)
              Y = ["*"]*(n+1) # will be overwritten anyway
              carry = one(X[0])
              for i in range(n):
                  Y[i] = XOR(X[i],carry)
                  carry = AND(X[i],carry)
              Y[n] = COPY(carry)
              return Y

          def inc5(a,b,c,d,e):
              return increment([a,b,c,d,e])

In [140]: inc5(1,1,0,0,0)

Out[140]: [0, 0, 1, 0, 0, 0]

In [141]: print(nandcode(inc5))

Temp[0] = NAND(X[0],X[0])
Temp[1] = NAND(X[0],Temp[0])
Temp[2] = NAND(X[0],Temp[1])
Temp[3] = NAND(X[0],Temp[2])
Temp[4] = NAND(Temp[1],Temp[2])
```

```
Y[0] = NAND(Temp[3],Temp[4])
Temp[6] = NAND(X[0],Temp[1])
Temp[7] = NAND(Temp[6],Temp[6])
Temp[8] = NAND(X[1],Temp[7])
Temp[9] = NAND(X[1],Temp[8])
Temp[10] = NAND(Temp[7],Temp[8])
Y[1] = NAND(Temp[9],Temp[10])
Temp[12] = NAND(X[1],Temp[7])
Temp[13] = NAND(Temp[12],Temp[12])
Temp[14] = NAND(X[2],Temp[13])
Temp[15] = NAND(X[2],Temp[14])
Temp[16] = NAND(Temp[13],Temp[14])
Y[2] = NAND(Temp[15],Temp[16])
Temp[18] = NAND(X[2],Temp[13])
Temp[19] = NAND(Temp[18],Temp[18])
Temp[20] = NAND(X[3],Temp[19])
Temp[21] = NAND(X[3],Temp[20])
Temp[22] = NAND(Temp[19],Temp[20])
Y[3] = NAND(Temp[21],Temp[22])
Temp[24] = NAND(X[3],Temp[19])
Temp[25] = NAND(Temp[24],Temp[24])
Temp[26] = NAND(X[4],Temp[25])
Temp[27] = NAND(X[4],Temp[26])
Temp[28] = NAND(Temp[25],Temp[26])
Y[4] = NAND(Temp[27],Temp[28])
Temp[30] = NAND(X[4],Temp[25])
Temp[31] = NAND(Temp[30],Temp[30])
Temp[32] = NAND(Temp[31],Temp[31])
Y[5] = NAND(Temp[32],Temp[32])
```

```
In [144]: EVAL(nandcode(inc5),[1,0,1,0,0])

Out[144]: [0, 1, 1, 0, 0, 0]
```

We can create functions such as inc5 for every n via a little Python trickery:

```
In [231]: def restrict(f,n):
              """Create function that restricts the function f to exactly n inputs"""
              args = ", ".join(f'arg{i}' for i in range(n))
              exec(rf'''
          def _temp({args}):
              return {f.__name__}([{args}])
              ''',globals())
              return _temp

In [234]: inc7 = restrict(increment,7)

          inc7(1,1,1,0,0,0,1)
```

```
Out[234]: [0, 0, 0, 1, 0, 0, 1, 0]
```

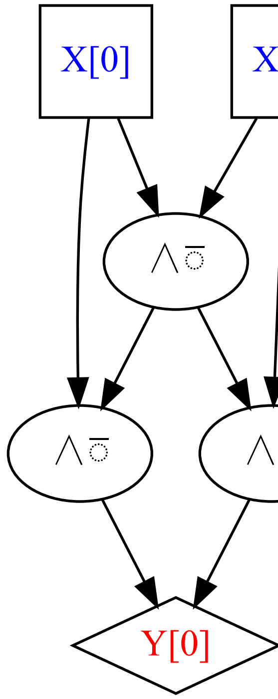
```
In [170]: print(nandcode(inc7))

Temp[0] = NAND(X[0],X[0])
Temp[1] = NAND(X[0],Temp[0])
Temp[2] = NAND(X[0],Temp[1])
Temp[3] = NAND(X[0],Temp[2])
Temp[4] = NAND(Temp[1],Temp[2])
Y[0] = NAND(Temp[3],Temp[4])
Temp[6] = NAND(X[0],Temp[1])
Temp[7] = NAND(Temp[6],Temp[6])
Temp[8] = NAND(X[1],Temp[7])
Temp[9] = NAND(X[1],Temp[8])
Temp[10] = NAND(Temp[7],Temp[8])
Y[1] = NAND(Temp[9],Temp[10])
Temp[12] = NAND(X[1],Temp[7])
Temp[13] = NAND(Temp[12],Temp[12])
Temp[14] = NAND(X[2],Temp[13])
Temp[15] = NAND(X[2],Temp[14])
Temp[16] = NAND(Temp[13],Temp[14])
Y[2] = NAND(Temp[15],Temp[16])
Temp[18] = NAND(X[2],Temp[13])
Temp[19] = NAND(Temp[18],Temp[18])
Temp[20] = NAND(X[3],Temp[19])
Temp[21] = NAND(X[3],Temp[20])
Temp[22] = NAND(Temp[19],Temp[20])
Y[3] = NAND(Temp[21],Temp[22])
Temp[24] = NAND(X[3],Temp[19])
Temp[25] = NAND(Temp[24],Temp[24])
Temp[26] = NAND(X[4],Temp[25])
Temp[27] = NAND(X[4],Temp[26])
Temp[28] = NAND(Temp[25],Temp[26])
Y[4] = NAND(Temp[27],Temp[28])
Temp[30] = NAND(X[4],Temp[25])
Temp[31] = NAND(Temp[30],Temp[30])
Temp[32] = NAND(X[5],Temp[31])
Temp[33] = NAND(X[5],Temp[32])
Temp[34] = NAND(Temp[31],Temp[32])
Y[5] = NAND(Temp[33],Temp[34])
Temp[36] = NAND(X[5],Temp[31])
Temp[37] = NAND(Temp[36],Temp[36])
Temp[38] = NAND(X[6],Temp[37])
Temp[39] = NAND(X[6],Temp[38])
Temp[40] = NAND(Temp[37],Temp[38])
Y[6] = NAND(Temp[39],Temp[40])
Temp[42] = NAND(X[6],Temp[37])
Temp[43] = NAND(Temp[42],Temp[42])
```

```
Temp[44] = NAND(Temp[43],Temp[43])
Y[7] = NAND(Temp[44],Temp[44])
```

1.3 NAND Programs and circuits

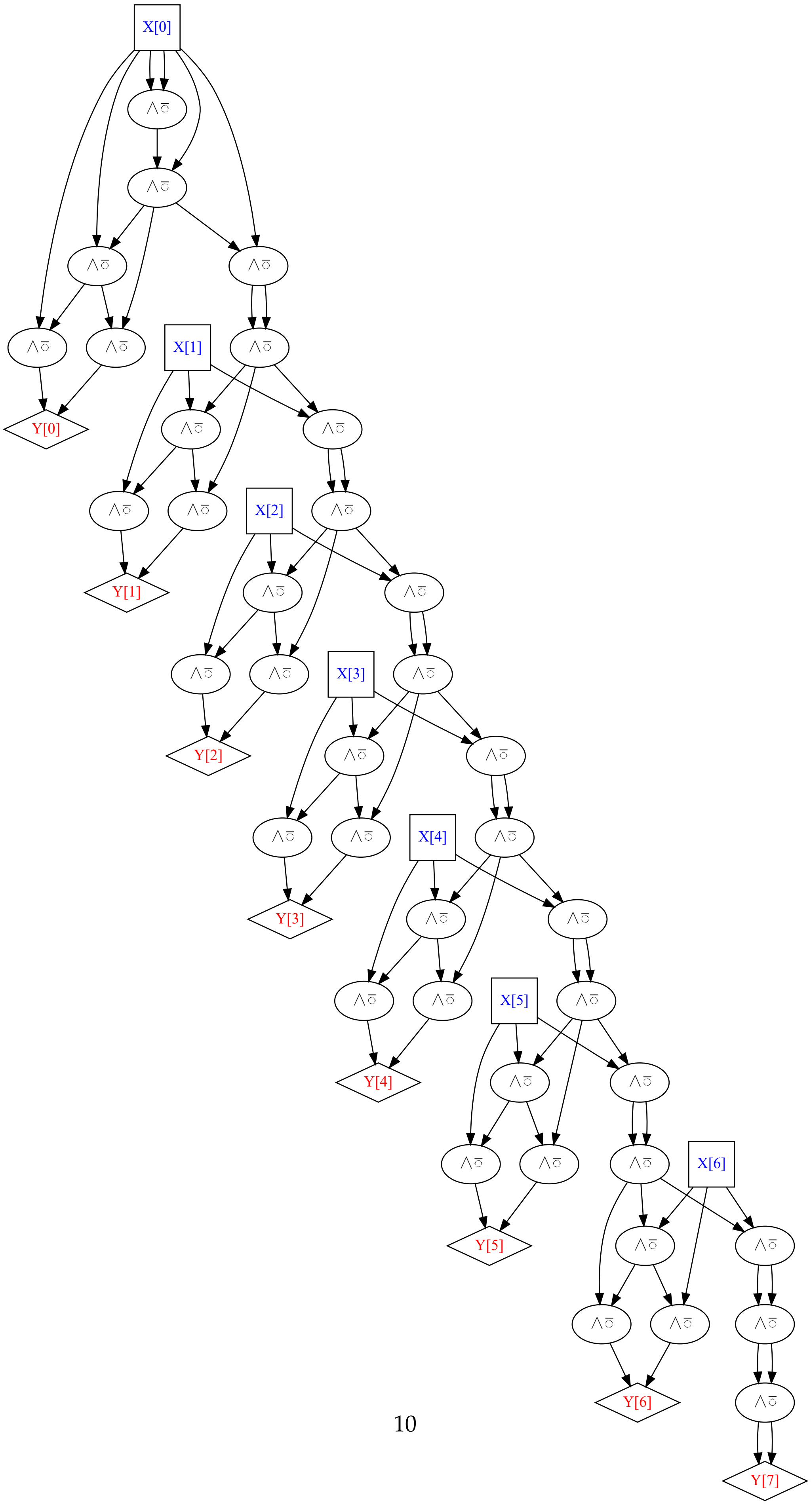
NAND programs are equivalent to the model of Boolean circuits. We can present the graph corresponding to a NAND function using ideas similar to those we used to print the code

```
In [194]: import graphviz
          from graphviz import Graph
          from graphviz import Digraph

          def nandcircuit(f):
              """Compute the graph representating a NAND circuit for a NAND program, given as
              n = numarguments(f)
              counter = 0 # to ensure unique temporary variables.
              G = Digraph()


              global NAND
              def tempNAND(bar,blah):
                  nonlocal G, counter
                  var = f'Temp[{counter}]'
                  counter += 1
                  G.node(var,label="\u0305")
                  G.edge(bar,var)
                  G.edge(blah,var)
                  return var

              for i in range(n):
                  G.node(f'X[{i}]',label=f'X[{i}]', fontcolor='blue',shape='square')

              NAND , tempNAND = tempNAND, NAND # Override NAND with its temporary version
              # (this is a hack and won't play nicely with exceptions or multithreading)
              outputs = f(*[f'X[{i}]' for i in range(n)]) # execute f on the strings "X[0]", "
              NAND , tempNAND = tempNAND, NAND # Restore original NAND

              if type(outputs)==str: outputs = [outputs] # make single output into singleton l

              for j in range(len(outputs)):
                  G.node(outputs[j],label=f'Y[{j}]',fontcolor='red',shape='diamond')
              return G

In [195]: nandcircuit(XOR)

Out[195]:
```



In [196]: `nandcircuit(inc7)`

Out[196]:


X[0]
X[1]
X[2]
X[3]
X[4]
X[5]
X[6]
Y[0]
Y[1]
Y[2]
Y[3]
Y[4]
Y[5]
Y[6]
Y[7]

1.4 Computing every function

It turns out that we can compute *every* function $f : \{0,1\}^n \to \{0,1\}$ by some NAND program.

The crucial element for that is the function `LOOKUP` that on input an index $i \in [n]$ (represented as a string of length $\log n$) and a table $T \in \{0,1\}^n$, outputs t_i.

```
In [226]: def LOOKUP(i,T):
              l = len(i)
              if l==1:
                  return IF(i[0],T[1],T[0])
              return IF(i[l-1],LOOKUP(i[:-1],T[2**(l-1):]),LOOKUP(i[:-1],T[:2**(l-1)]))

          LOOKUP([1,1,1],[0,1,1,0,1,1,0,1])
```

```
Out[226]: 1
```

We can extract the NAND code of LOOKUP using the usual tricks.

```
In [239]: # generalize restrict to handle functions that take more than one array
          def restrict(f,*numinputs):
              """Create function that restricts the function f to exactly given input lengths 
              k = len(numinputs)
              args = [""]*k

              for i in range(k):
                  args[i] = ", ".join(f'arg_{i}_{j}' for j in range(numinputs[i]))
              sig = ", ".join(args)
              call = ", ".join(f"[{args[i]}]" for i in range(k))
              exec(rf'''
          def _temp({sig}):
              return {f.__name__}({call})
              ''',globals())
              return _temp
```

```
In [240]: def funclookup(l):
              return restrict(LOOKUP,l,2**l)
```

```
In [241]: funclookup(3)(1,1,1,0,1,1,0,1,1,0,1)
```

```
Out[241]: 1
```

```
In [242]: print(nandcode(funclookup(3)))
```

```
Temp[0] = NAND(X[0],X[10])
Temp[1] = NAND(Temp[0],Temp[0])
Temp[2] = NAND(X[0],X[0])
```

```
Temp[3] = NAND(Temp[2],X[9])
Temp[4] = NAND(Temp[3],Temp[3])
Temp[5] = NAND(Temp[1],Temp[1])
Temp[6] = NAND(Temp[4],Temp[4])
Temp[7] = NAND(Temp[5],Temp[6])
Temp[8] = NAND(X[0],X[8])
Temp[9] = NAND(Temp[8],Temp[8])
Temp[10] = NAND(X[0],X[0])
Temp[11] = NAND(Temp[10],X[7])
Temp[12] = NAND(Temp[11],Temp[11])
Temp[13] = NAND(Temp[9],Temp[9])
Temp[14] = NAND(Temp[12],Temp[12])
Temp[15] = NAND(Temp[13],Temp[14])
Temp[16] = NAND(X[1],Temp[7])
Temp[17] = NAND(Temp[16],Temp[16])
Temp[18] = NAND(X[1],X[1])
Temp[19] = NAND(Temp[18],Temp[15])
Temp[20] = NAND(Temp[19],Temp[19])
Temp[21] = NAND(Temp[17],Temp[17])
Temp[22] = NAND(Temp[20],Temp[20])
Temp[23] = NAND(Temp[21],Temp[22])
Temp[24] = NAND(X[0],X[6])
Temp[25] = NAND(Temp[24],Temp[24])
Temp[26] = NAND(X[0],X[0])
Temp[27] = NAND(Temp[26],X[5])
Temp[28] = NAND(Temp[27],Temp[27])
Temp[29] = NAND(Temp[25],Temp[25])
Temp[30] = NAND(Temp[28],Temp[28])
Temp[31] = NAND(Temp[29],Temp[30])
Temp[32] = NAND(X[0],X[4])
Temp[33] = NAND(Temp[32],Temp[32])
Temp[34] = NAND(X[0],X[0])
Temp[35] = NAND(Temp[34],X[3])
Temp[36] = NAND(Temp[35],Temp[35])
Temp[37] = NAND(Temp[33],Temp[33])
Temp[38] = NAND(Temp[36],Temp[36])
Temp[39] = NAND(Temp[37],Temp[38])
Temp[40] = NAND(X[1],Temp[31])
Temp[41] = NAND(Temp[40],Temp[40])
Temp[42] = NAND(X[1],X[1])
Temp[43] = NAND(Temp[42],Temp[39])
Temp[44] = NAND(Temp[43],Temp[43])
Temp[45] = NAND(Temp[41],Temp[41])
Temp[46] = NAND(Temp[44],Temp[44])
Temp[47] = NAND(Temp[45],Temp[46])
Temp[48] = NAND(X[2],Temp[23])
Temp[49] = NAND(Temp[48],Temp[48])
Temp[50] = NAND(X[2],X[2])
```

```
Temp[51] = NAND(Temp[50],Temp[47])
Temp[52] = NAND(Temp[51],Temp[51])
Temp[53] = NAND(Temp[49],Temp[49])
Temp[54] = NAND(Temp[52],Temp[52])
Y[0] = NAND(Temp[53],Temp[54])
```

1.5 Representing NAND programs

We can represent a NAND program in many ways including the string of its source code, as the graph corresponding to its circuit. One simple representation of a NAND program we will use is as the following:

We represent a NAND program of t intermediate variables, s lines, n input variables, and m input variables as a triple (n, m, L) where L is a list of s triples of the form (a, b, c) of numbers in $[n + t + m]$.

A triple (a, b, c) corresponds to the line assigning to the variable corresponding a the NAND of the variables corresponding to b and c. We identify the first n variables with the input and the last m variables with the outputs.

We can again compute this representation using Python:

```
In [243]: def nandrepresent(f):
              """Compute the list of triple representation for a NAND program, given by a Pyth
              n = numarguments(f)
              counter = n # to ensure unique temporary variables.
              L = [] # list of tuples


              global NAND
              def tempNAND(bar,blah):
                  nonlocal L, counter
                  var = counter
                  counter += 1
                  L += [(var,bar,blah)]
                  return var

              NAND , tempNAND = tempNAND, NAND # Override NAND with its temporary version
              # (this is a hack and won't play nicely with exceptions or multithreading)
              outputs = f(*range(n)) # execute f on the strings "X[0]", "X[1]", ...
              NAND , tempNAND = tempNAND, NAND # Restore original NAND

              if type(outputs)==int: outputs = [outputs] # make single output into singleton l
              m = len(outputs)

              # make sure outputs are last m variables
              for j in range(m):
                  def flip(a):
                      if a==outputs[j]:  return counter-m+j
```

```
                if a==counter-m+j: return outputs[j]
                return a
            L = [(flip(a),flip(b),flip(c)) for (a,b,c) in L]
        return (n,m,L)

    nandrepresent(XOR)
```

Out[243]: (2, 1, [(2, 0, 1), (3, 0, 2), (4, 1, 2), (5, 3, 4)])

We can directly evaluate a NAND program based on its list of triples representation:

In [244]:
```
def EVALnand(prog,X):
    """Evaluate a NAND program from its list of triple representation."""
    n,m,L = prog
    vartable = X+[0]*(max(max(a,b,c) for (a,b,c) in L)-n+1)
    for (a,b,c) in L:
        vartable[a] = NAND(vartable[b],vartable[c])

    return [vartable[-m+j] for j in range(m)]
```

In [245]:
```
EVALnand(nandrepresent(XOR),[1,0])
```

Out[245]: [1]

1.6 Universal circuit evaluation or NAND interpreter in NAND

We can construct a NAND program P that given the representation of a NAND program Q and an input x, outputs $Q(x)$. We can obviously compute such a function since every finite function is computable by a NAND program, but it turns out we can do so in a program that is polynomial in the size of P (maybe even quasiliinear).

The implementation of this program not included at this time in this appendix.

In [ ]: