Truncation errors

Accuracy of finite difference approximations

$$T_1 \quad \Rightarrow \quad \left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1} - u_i}{\Delta x} - \frac{\Delta x}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{(\Delta x)^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \ldots$$

forward difference truncation error $\mathcal{O}(\Delta x)$

$$T_2 \quad \Rightarrow \quad \left(\frac{\partial u}{\partial x}\right)_i = \frac{u_i - u_{i-1}}{\Delta x} + \frac{\Delta x}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)_i - \frac{(\Delta x)^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \ldots$$

backward difference truncation error $\mathcal{O}(\Delta x)$

$$T_1 - T_2 \quad \Rightarrow \quad \left(\frac{\partial u}{\partial x}\right)_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x} - \frac{(\Delta x)^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_i + \ldots$$

central difference truncation error $\mathcal{O}(\Delta x)^2$

Leading truncation error

$$\epsilon_\tau = \alpha_m(\Delta x)^m + \alpha_{m+1}(\Delta x)^{m+1} + \ldots \approx \alpha_m(\Delta x)^m$$

Approximation of second-order derivatives

Central difference scheme

$$T_1 + T_2 \quad \Rightarrow \quad \left(\frac{\partial^2 u}{\partial x^2}\right)_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} + \mathcal{O}(\Delta x)^2$$

Alternative derivation

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i = \left[\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right]_i = \lim_{\Delta x \to 0} \frac{\left(\frac{\partial u}{\partial x}\right)_{i+1/2} - \left(\frac{\partial u}{\partial x}\right)_{i-1/2}}{\Delta x}$$

$$\approx \frac{\frac{u_{i+1}-u_i}{\Delta x} - \frac{u_i - u_{i-1}}{\Delta x}}{\Delta x} = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}$$

Variable coefficients $\qquad f(x) = d(x)\frac{\partial u}{\partial x} \qquad$ *diffusive flux*

$$\left(\frac{\partial f}{\partial x}\right)_i \approx \frac{f_{i+1/2} - f_{i-1/2}}{\Delta x} = \frac{d_{i+1/2}\frac{u_{i+1}-u_i}{\Delta x} - d_{i-1/2}\frac{u_i - u_{i-1}}{\Delta x}}{\Delta x}$$

$$= \frac{d_{i+1/2}u_{i+1} - (d_{i+1/2} + d_{i-1/2})u_i + d_{i-1/2}u_{i-1}}{(\Delta x)^2}$$