Data Analytics and Geostatistics:
Spatial Estimation

Lecture outline . . .

- Kriging
- Indicator Kriging

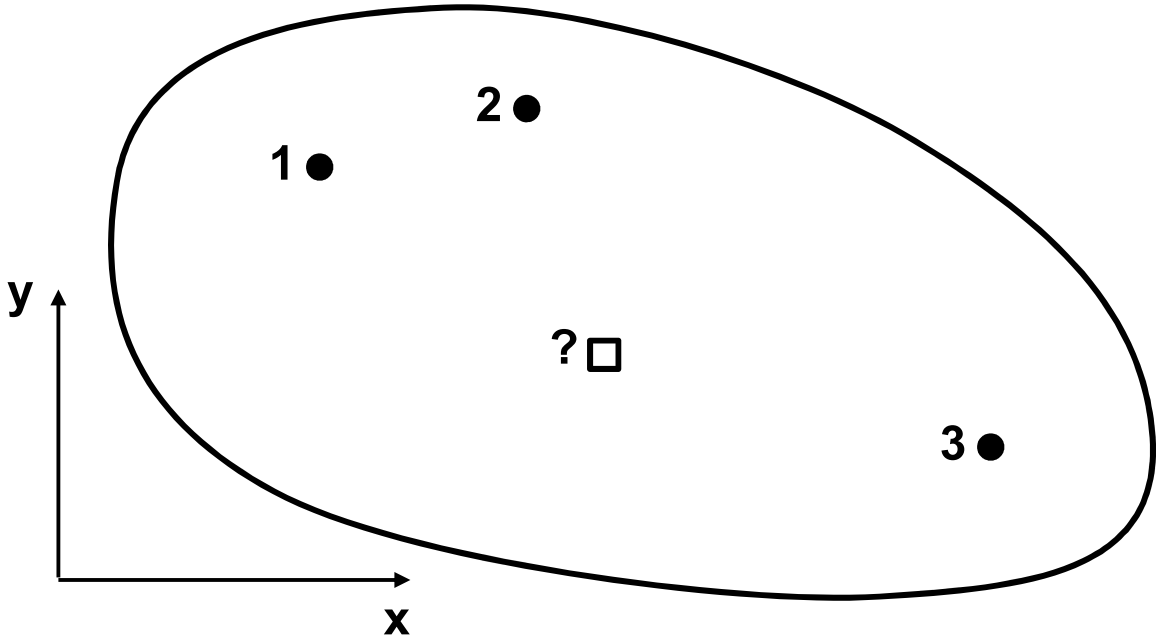
Data Analytics and Geostatistics: Spatial Estimation

Lecture outline . . .

- **Kriging**

Spatial Estimation

- Consider the case of estimating at some unsampled location:

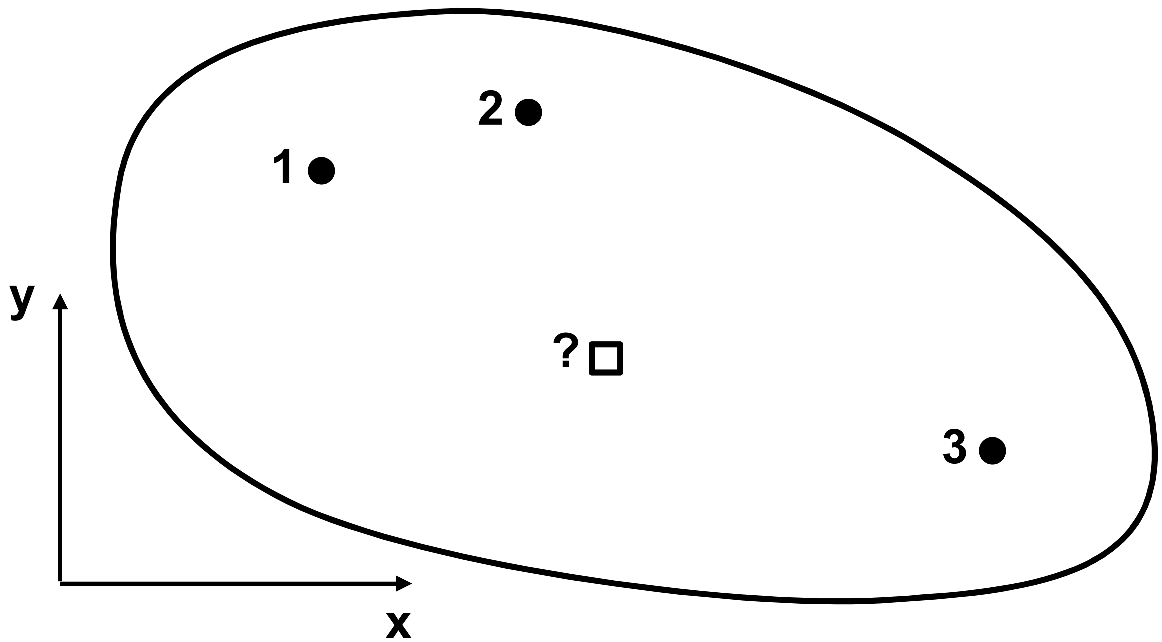


- How would you do this given data, $z(\mathbf{u}_1)$, $z(\mathbf{u}_2)$, and $z(\mathbf{u}_3)$?

- Note: z is the variable of interest (e.g. porosity etc.) and $\mathbf{u}_i$ is the data locations.

Spatial Estimation

- Consider the case of estimating at some unsampled location:



$\mathbf{z}(\mathbf{u}_\alpha)$ **is the data values**

$z^*(\mathbf{u}_0)$ **is an estimate**

λ_α **is the data weights**

m_z **is the global mean**

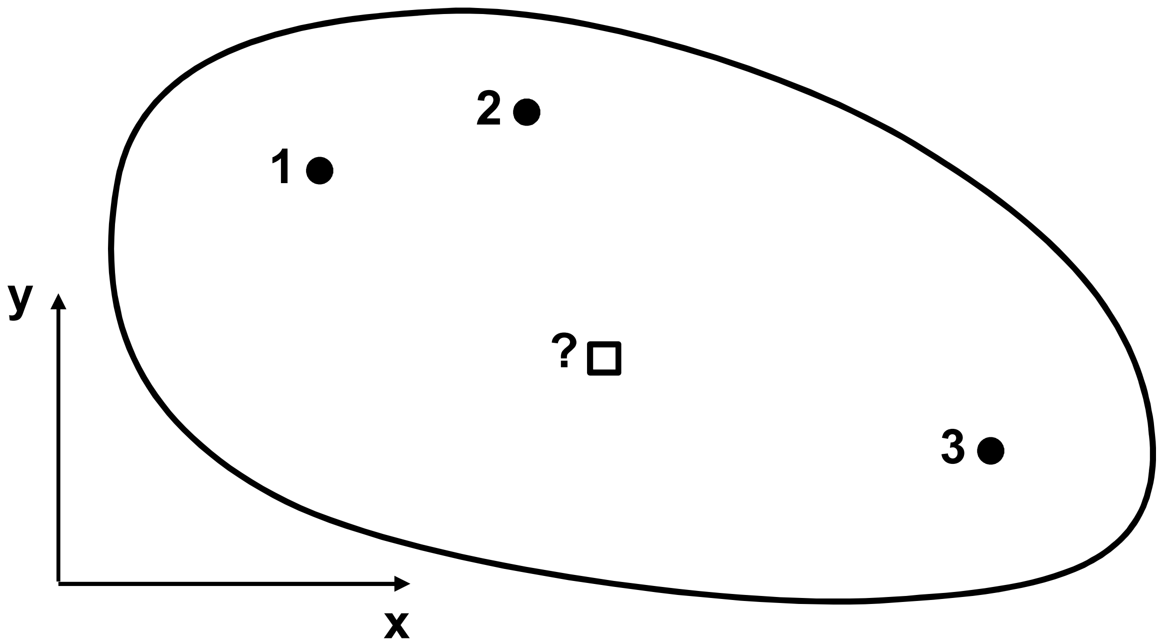
- How would you do this given data, $z(\mathbf{u}_1)$, $z(\mathbf{u}_2)$, and $z(\mathbf{u}_3)$?

$$z^*(\mathbf{u}_0) = \sum_{\alpha=1}^{n} \lambda_\alpha \mathbf{z}(\mathbf{u}_\alpha) + \boxed{\left(1 - \sum_{\alpha=1}^{n} \lambda_\alpha\right) m_z}$$

Unbiasedness Constraint Weights sum to 1.0.

Spatial Estimation

- Consider the case of estimating at some unsampled location:



- How would you do this given data, $z(\mathbf{u}_1)$, $z(\mathbf{u}_2)$, and $z(\mathbf{u}_3)$?

$$\boldsymbol{z^*(\mathbf{u}_0) - m_z(\mathbf{u}_0) = \sum_{\alpha=1}^{n} \lambda_\alpha \left(\mathrm{z}(\mathbf{u}_\alpha) - m_z(\mathbf{u}_\alpha)\right)}$$

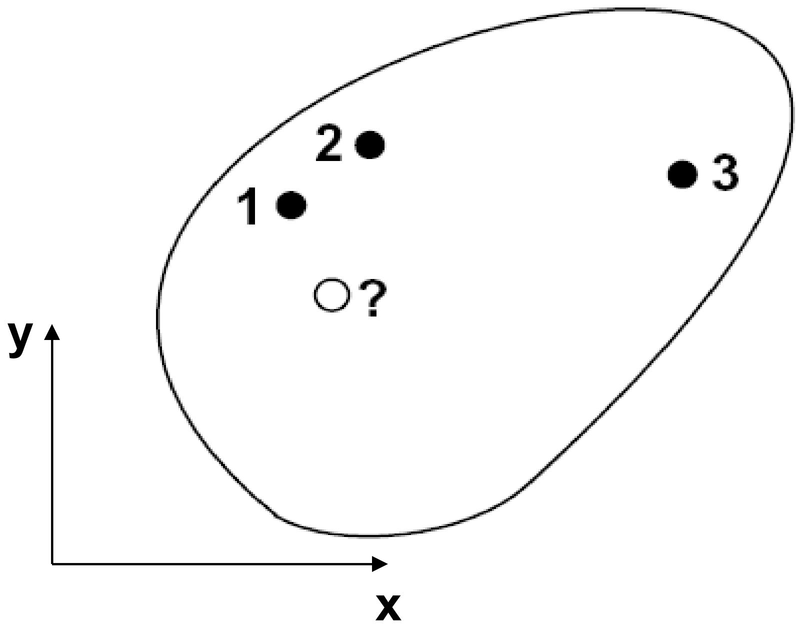
In the case where the mean is non-stationary.

Given y$= \boldsymbol{z - m}$, $\quad \boldsymbol{y^*(\mathbf{u}_0) = \sum_{\alpha=1}^{n} \lambda_\alpha \, \mathrm{y}(\mathbf{u}_\alpha)}$

Simplified with residual, y.

Spatial Estimation

- Consider the case of estimating at some unsampled location:



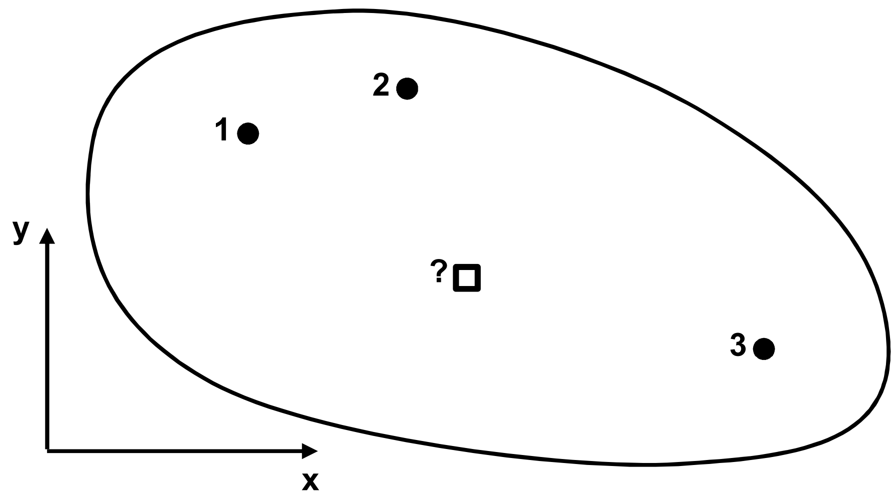
- Linear weighted, sound good. How do we get the weights? $\boldsymbol{\lambda_\alpha, \alpha = 1, \ldots, n}$

$$\boldsymbol{y^*(\mathbf{u}_0) = \sum_{\alpha=1}^{n} \lambda_\alpha \, \mathrm{y}(\mathbf{u}_\alpha)}$$

Simplified with residual, y.

Spatial Estimation

- Consider the case of estimating at some unsampled location:



- Linear weighted, sound good. How do we get the weights? $\boldsymbol{\lambda_{\alpha}, \alpha = 1, \ldots, n}$

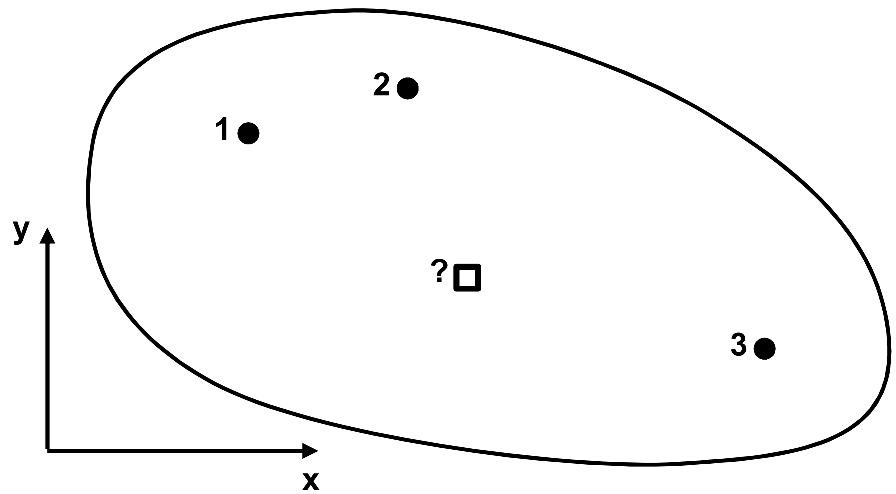
- Equal weighted / average? $\boldsymbol{\lambda_{\alpha} = {}^{1}\!/_{n}}$

Equal weight average of data

- What's wrong with that?

Spatial Estimation

- Consider the case of estimating at some unsampled location:



- How do we get the weights? $\boldsymbol{\lambda_\alpha, \alpha = 1, \ldots, n}$

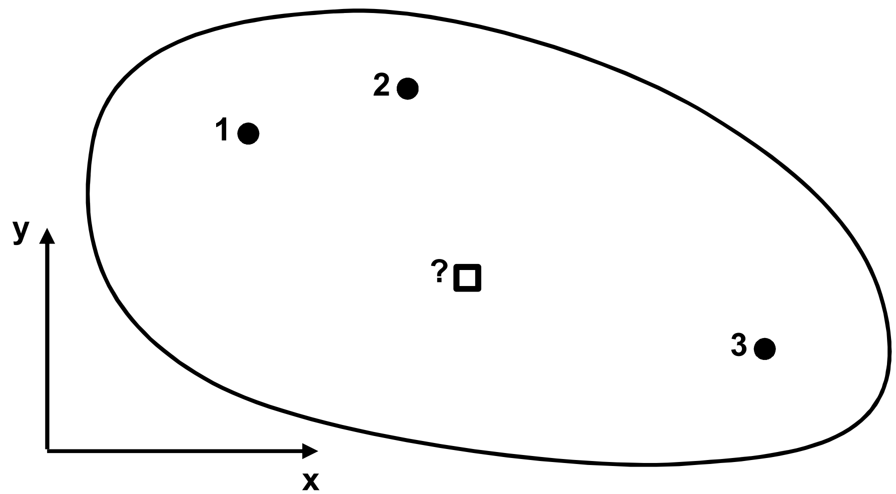
Inverse distance to power standardized so weights sum to 1.0.

- Inverse distance? $$\lambda_\alpha = \frac{1}{dist(\mathbf{u}_0, \mathbf{u}_\alpha)^p} \Big/ \sum_{\alpha=1}^{n} \lambda_\alpha$$

- What’s wrong with that?

Spatial Estimation

- Consider the case of estimating at some unsampled location:



- How do we get the weights? $\boldsymbol{\lambda_\alpha, \alpha = 1, \ldots, n}$

- It would be great to use weight that account for closeness (spatial correlation > distance alone), redundancy (once again with spatial correlation).

- How can we do that?

Derivation of Simple Kriging Equations

- Consider a linear estimator:

$$Y^*(\mathbf{u}) = \sum_{i=1}^{n} \lambda_i \cdot Y(\mathbf{u}_i)$$

where $Y(u_i)$ are the residual data (data values minus the mean) and $Y^*(u_i)$ is the estimate (add the mean back in when we are finished)

Stationary Mean, Variogram

$$E\{Y\} = 0$$

$$2\gamma(\mathbf{h}) = E\left\{\left[Y(\mathbf{u}) - Y(\mathbf{u}+\mathbf{h})\right]^2\right\}$$

- The **estimation variance** is defined as:

$$E\left\{\left[Y^*(u) - Y(u)\right]^2\right\} = \ldots$$

$$= E\left\{\left[Y^*(u)\right]^2\right\} - 2\ E\left\{Y^*(u)\ Y(u)\right\} + E\left\{\left[Y(u)\right]^2\right\}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j E\left\{Y(u_i)\ Y(u_j)\right\} - 2\sum_{i=1}^{n} \lambda_i E\left\{Y(u)\ Y(u_i)\right\} + C(0)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j C(u_i,u_j) - 2\sum_{i=1}^{n} \lambda_i C(u,u_i) + C(0)$$

redundancy **closeness** **variance**

$C(\mathbf{u}_i,\mathbf{u}_j)$ – covariance between data i and j, $C(\mathbf{u}_i,\mathbf{u})$covariance between data and unknown location and $C(0)$ is the variance.

More Derivation

- Optimal weights $\lambda_i, i = 1,\ldots,n$ may be determined by taking partial derivatives of the error variance w.r.t. the weights

$$\frac{\partial[\qquad]}{\partial\lambda_i} = \sum_{j=1}^{n} \lambda_j C(\mathbf{u}_i, \mathbf{u}_i) - 2 \cdot C(\mathbf{u}, \mathbf{u}_i), i = 1, \ldots, n$$

and setting them to zero

$$\sum_{j=1}^{n} \lambda_j C(\mathbf{u}_i, \mathbf{u}_i) = \cdot\, C(\mathbf{u}, \mathbf{u}_i), i = 1, \ldots, n$$

- This system of *n* equations with *n* unknown weights is the simple kriging (SK) system

Kriging Definition

- Estimation approach that relies on linear weights that account for spatial continuity, data closeness and redundancy.
- Weights are unbiased and minimize the estimation variance.

Simple Kriging: Some Details

There are three equations to determine the three weights:

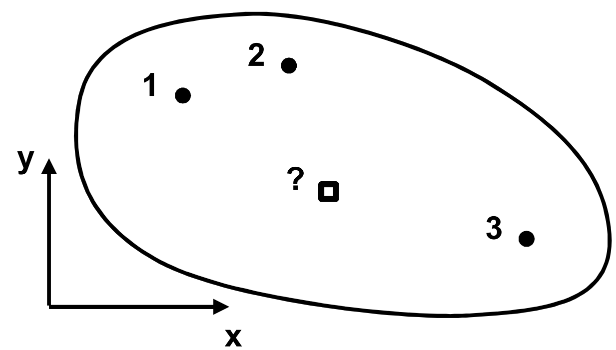
$$\lambda_1 \cdot C(\mathbf{u}_1, \mathbf{u}_1) + \lambda_2 \cdot C(\mathbf{u}_1, \mathbf{u}_2) + \lambda_3 \cdot C(\mathbf{u}_1, \mathbf{u}_3) = C(\mathbf{u}_0, \mathbf{u}_1)$$

$$\lambda_1 \cdot C(\mathbf{u}_2, \mathbf{u}_1) + \lambda_2 \cdot C(\mathbf{u}_2, \mathbf{u}_2) + \lambda_3 \cdot C(\mathbf{u}_2, \mathbf{u}_3) = C(\mathbf{u}_0, \mathbf{u}_2)$$

$$\lambda_1 \cdot C(\mathbf{u}_3, \mathbf{u}_1) + \lambda_2 \cdot C(\mathbf{u}_1, \mathbf{u}_2) + \lambda_3 \cdot C(\mathbf{u}_1, \mathbf{u}_3) = C(\mathbf{u}_0, \mathbf{u}_1)$$

In matrix notation: Recall that $C(\mathbf{h}) = C(0) - \gamma(\mathbf{h})$

$$\underbrace{\begin{bmatrix} C(\mathbf{u}_1, \mathbf{u}_1) & C(\mathbf{u}_1, \mathbf{u}_2) & C(\mathbf{u}_1, \mathbf{u}_3) \\ C(\mathbf{u}_2, \mathbf{u}_1) & C(\mathbf{u}_2, \mathbf{u}_2) & C(\mathbf{u}_2, \mathbf{u}_3) \\ C(\mathbf{u}_3, \mathbf{u}_1) & C(\mathbf{u}_3, \mathbf{u}_2) & C(\mathbf{u}_3, \mathbf{u}_3) \end{bmatrix}}_{\textbf{redundancy}} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \underbrace{\begin{bmatrix} C(\mathbf{u}_o, \mathbf{u}_1) \\ C(\mathbf{u}_o, \mathbf{u}_2) \\ C(\mathbf{u}_o, \mathbf{u}_3) \end{bmatrix}}_{\textbf{closeness}}$$



Properties of Simple Kriging

- Solution exists and is unique of matrix $\left[C(v_i, v_j)\right]$ is positive definite

- Kriging estimator is unbiased: $E\left\{\left[Z - Z^*\right]\right\} = 0$

- Minimum error variance estimator (just try to pick weights, you won't bet it)

- Best Linear Unbiased Estimator

- Provides a measure of the estimation (or kriging) variance (uncertainty in the estimate):

$$\sigma_E^2(\mathbf{u}) = C(0) - \sum_{\alpha=1}^{n} \lambda_\alpha C(\mathbf{u} - \mathbf{u}_\alpha) \qquad \boldsymbol{\sigma_E^2} \rightarrow [\mathbf{0}, \boldsymbol{\sigma_x^2}]$$

More Properties

- Exact interpolator: at data location

- Kriging variance can be calculated before getting the sample information, homoscedastic!

- Kriging takes into account:
  - distance of the information: $C(\mathbf{u},\mathbf{u}_i)$
  - configuration of the data: $C(\mathbf{u}_i,\mathbf{u}_j)$
  - structural continuity of the variable being considered: $C(\mathbf{h})$

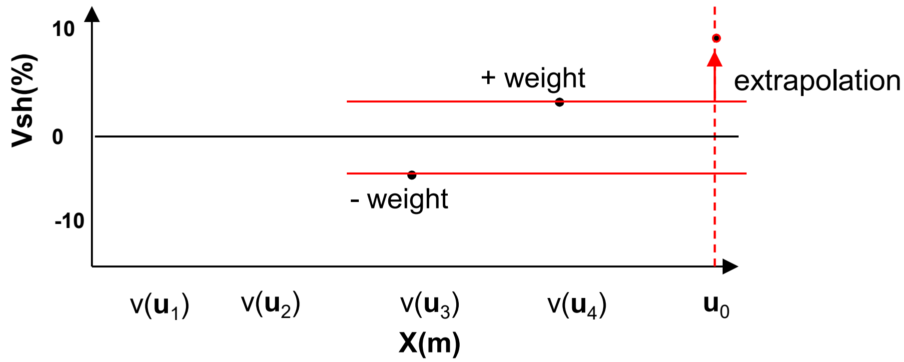
- The smoothing effect of kriging can be forecast – we will return to this with simulation.

- Kriging theory is part of the probabilistic theory of *projectors*: orthogonal projection onto space of linear combinations of the n data (Hilbert space)

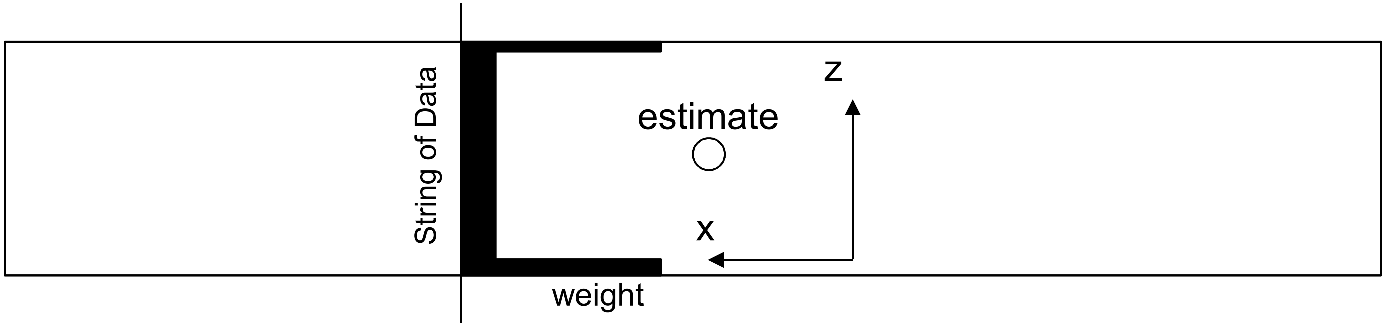
More Properties

- Outside range of the data, simple kriging weights all equal 0.0. The best estimate is the provided mean!

- Screened data will sometimes have negative weights! This allows kriging to extrapolate.



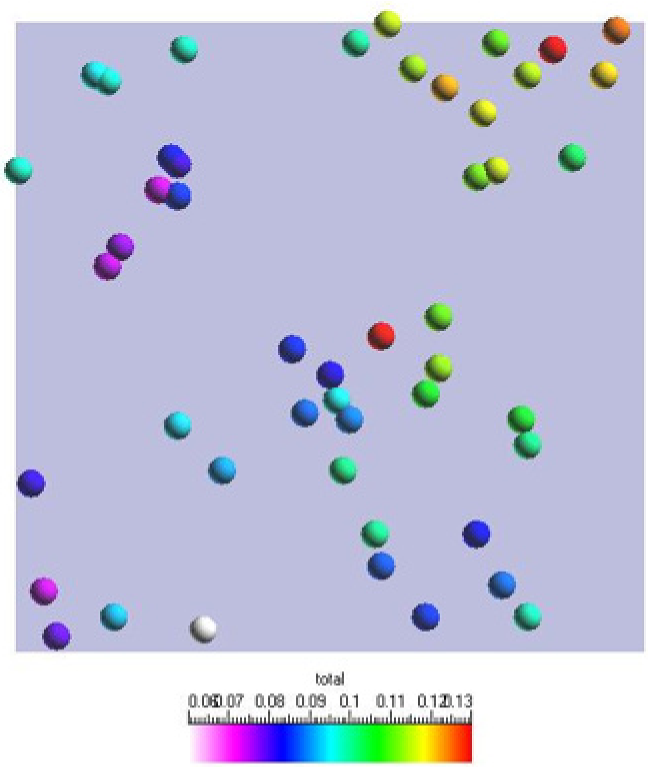
- Strings of data will have an artifact known as the string effect.



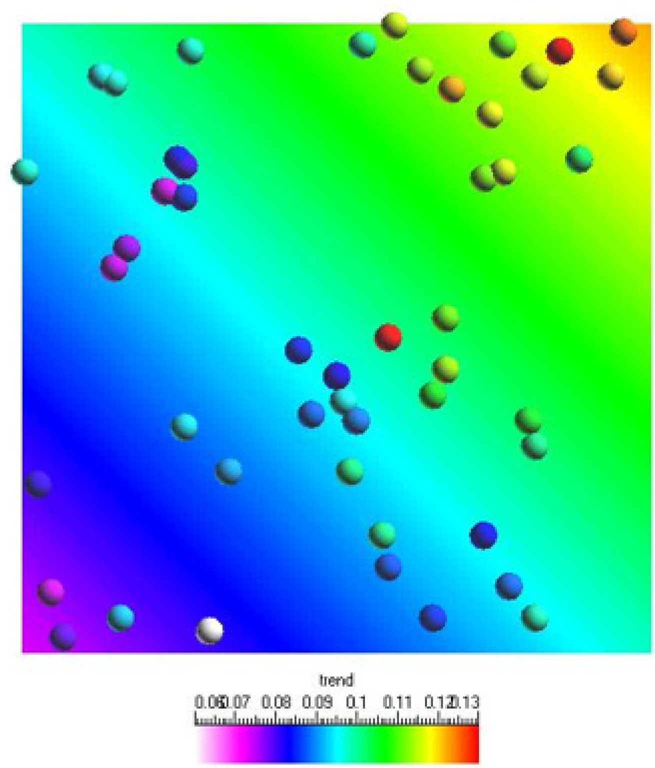
Kriging Estimation Example

- Kriging residual with trend modeling workflow

Distribution of “Hard” Data



Model a Trend Using Hard Data, if Possible

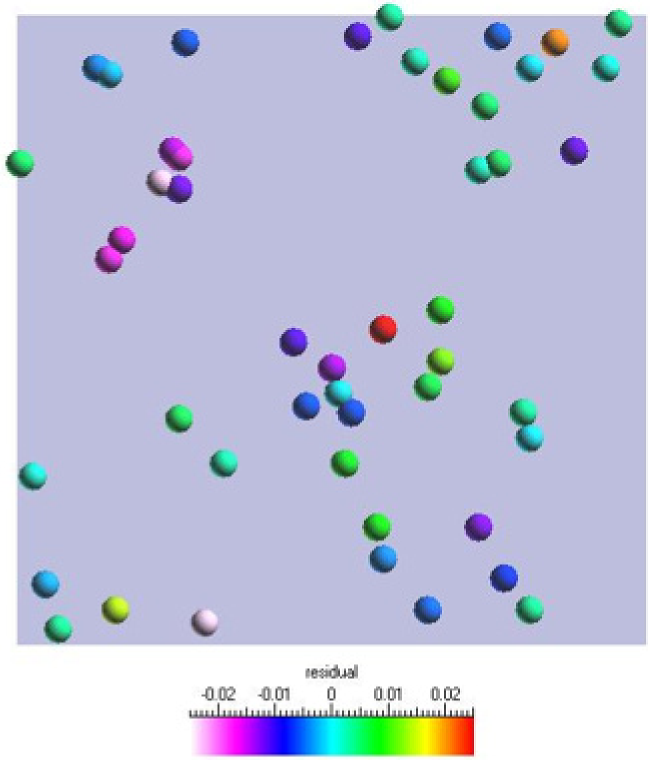


Example by William Meddaugh, Midwestern State University

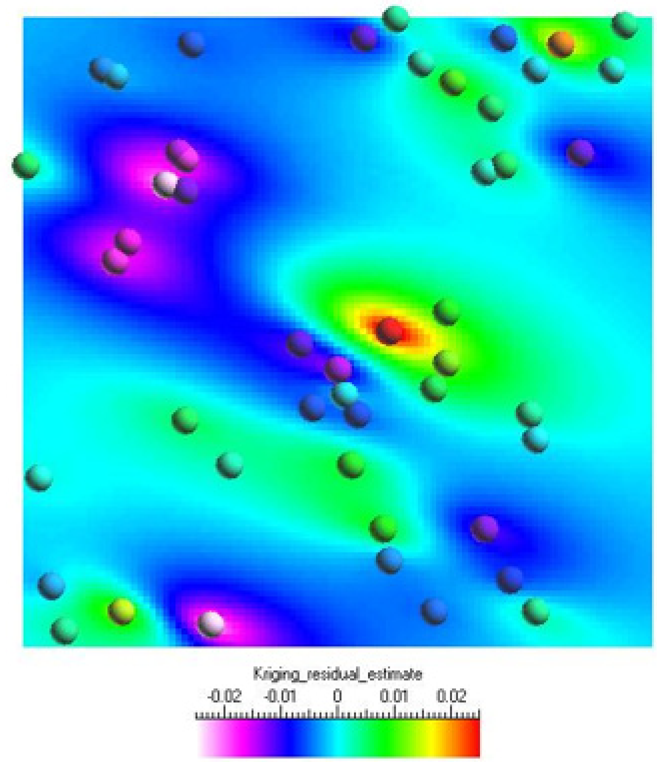
Kriging Estimation Example

- Kriging residual with trend modeling workflow

Calculate Residual at Hard Data Locations:
Residual=Actual Value – Trend Model Value



Perform Variography and Krig the Residuals

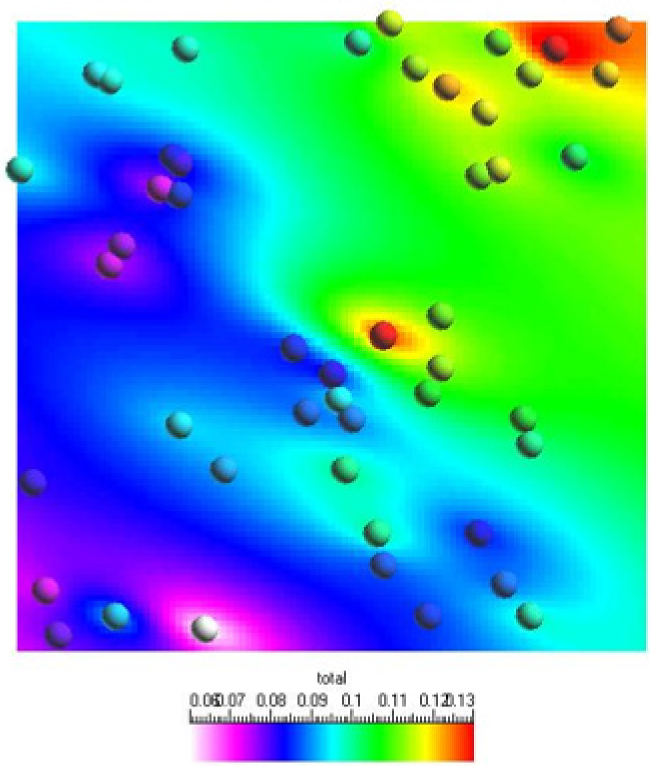


Example by William Meddaugh, Midwestern State University

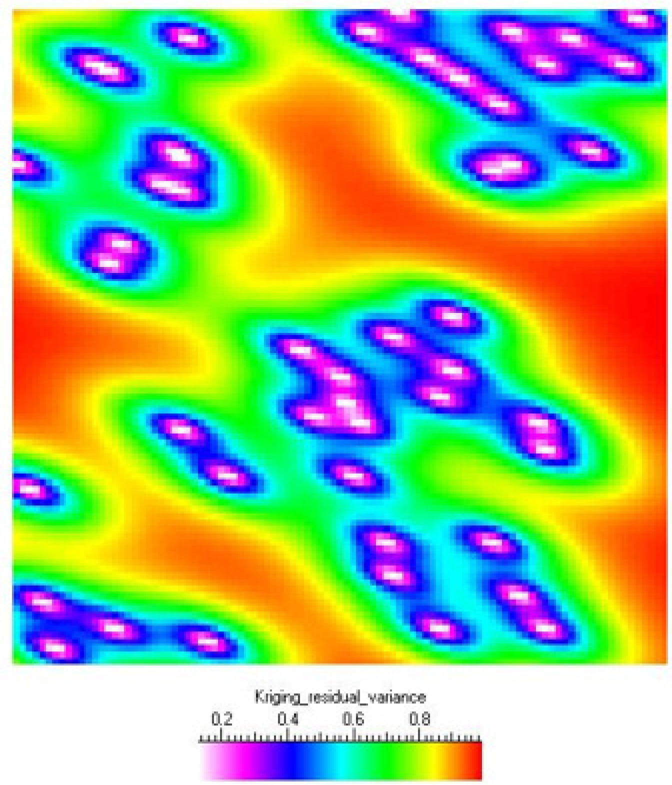
Kriging Estimation Example

- Kriging residual with trend modeling workflow

Add Back the Trend:
Estimate=Kriged Residuals + Trend



Kriging Variance



Example by William Meddaugh, Midwestern State University

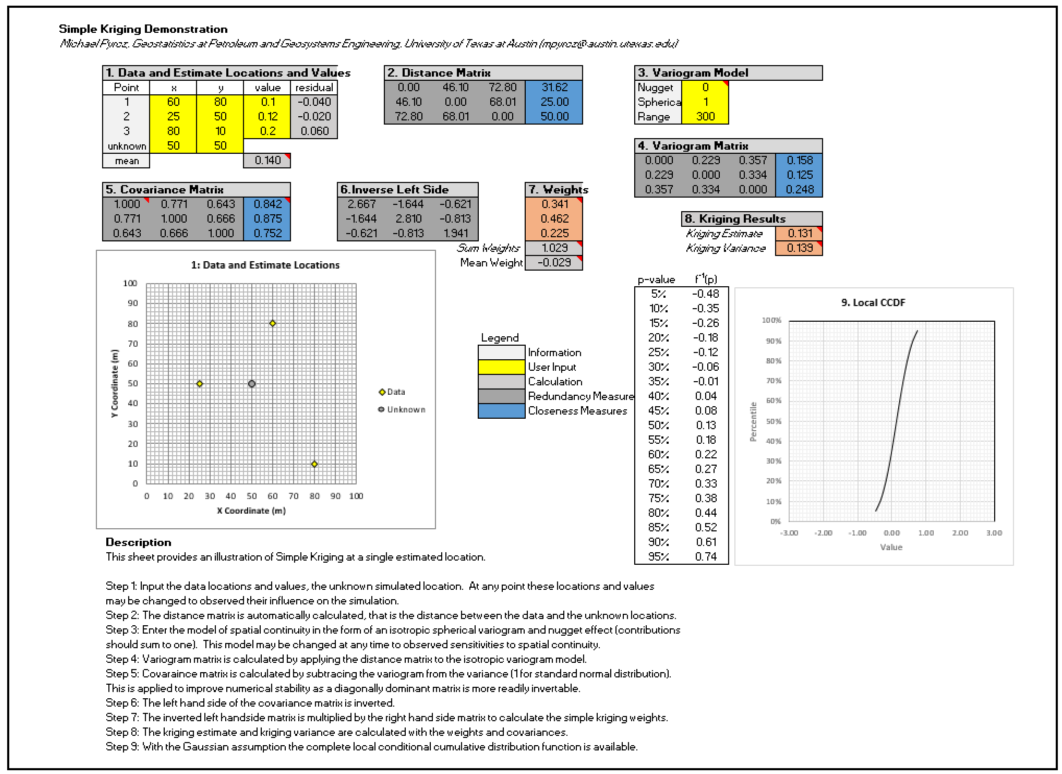
Simple Kriging Hands-on

Here's an opportunity for experiential learning with Simple Kriging.

- **Things to try:**

Pay attention to the kriging weights, kriging estimate and kriging variance while you:

1. Set points 1 and 2 closer together.
2. Put point 1 behind point 2 to create screening.
3. Put all points outside the range.
4. Set the range very large.



File Name: Simple_Kriging_Demo.xlsx   File is at: https://git.io/fNgBK

Spatial Uncertainty Hands-on

Here's an opportunity for experiential learning with Simple Kriging for spatial uncertainty. The kriging estimation variance is very useful.

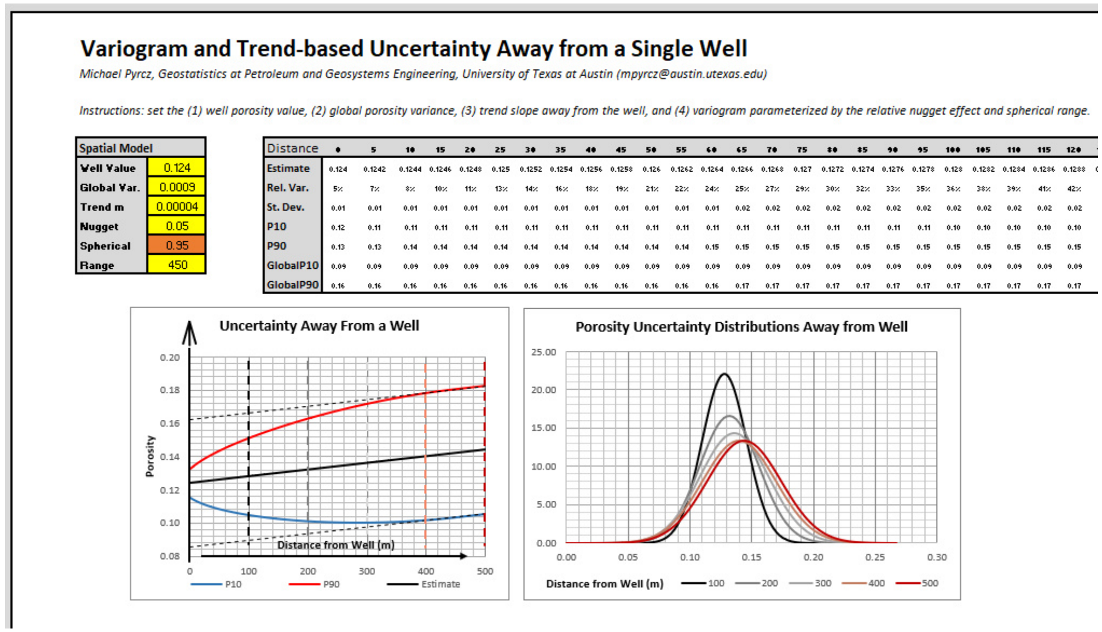
- **Things to try:**

Pay attention to the kriging uncertainty P10, mean and P90 away from the well as you:

1. Change the spatial continuity range.

2. Add and adjust the nugget effect.

3. Modify the trend slope.



Spatial Model	
Well Value	0.124
Global Var.	0.0009
Trend m	0.00004
Nugget	0.05
Spherical	0.95
Range	450

Distance	0	5	10	15	20	25	30	35	40	45	50	55	60
Estimate	0.124	0.1242	0.1244	0.1246	0.1248	0.125	0.1252	0.1254	0.1256	0.1258	0.126	0.1262	0.1264
Rel. Var.	5%	7%	8%	10%	11%	13%	14%	16%	18%	19%	21%	22%	24%
St. Dev.	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
P10	0.12	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.11
P90	0.13	0.13	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.14	0.15
GlobalP10	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09
GlobalP90	0.16	0.16	0.16	0.16	0.16	0.16	0.16	0.16	0.16	0.16	0.16	0.16	0.16

Distance	65	70	75	80	85	90	95	100	105	110	115	120
Estimate	0.1266	0.1268	0.127	0.1272	0.1274	0.1276	0.1278	0.128	0.1282	0.1284	0.1286	0.1288
Rel. Var.	25%	27%	29%	30%	32%	33%	35%	36%	38%	39%	41%	42%
St. Dev.	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
P10	0.11	0.11	0.11	0.11	0.11	0.11	0.11	0.10	0.10	0.10	0.10	0.10
P90	0.15	0.15	0.15	0.15	0.15	0.15	0.15	0.15	0.15	0.15	0.15	0.15
GlobalP10	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09	0.09
GlobalP90	0.17	0.17	0.17	0.17	0.17	0.17	0.17	0.17	0.17	0.17	0.17	0.17

File Name: Uncertainty_Away_from_Well_Demo.xlsx   File is at: https://git.io/fNO2I

Ordinary Kriging: Some Details

Add the constraint of : $\sum_{\alpha=1}^{n} \lambda_\alpha = 1.0$

$$\begin{bmatrix} C(\mathbf{u}_1,\mathbf{u}_1) & C(\mathbf{u}_1,\mathbf{u}_2) & C(\mathbf{u}_1,\mathbf{u}_3) & 1 \\ C(\mathbf{u}_2,\mathbf{u}_1) & C(\mathbf{u}_2,\mathbf{u}_2) & C(\mathbf{u}_2,\mathbf{u}_3) & 1 \\ C(\mathbf{u}_3,\mathbf{u}_1) & C(\mathbf{u}_3,\mathbf{u}_1) & C(\mathbf{u}_3,\mathbf{u}_3) & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \mu \end{bmatrix} = \begin{bmatrix} C(\mathbf{u}_0,\mathbf{u}_1) \\ C(\mathbf{u}_0,\mathbf{u}_2) \\ C(\mathbf{u}_0,\mathbf{u}_3) \\ 1 \end{bmatrix}$$

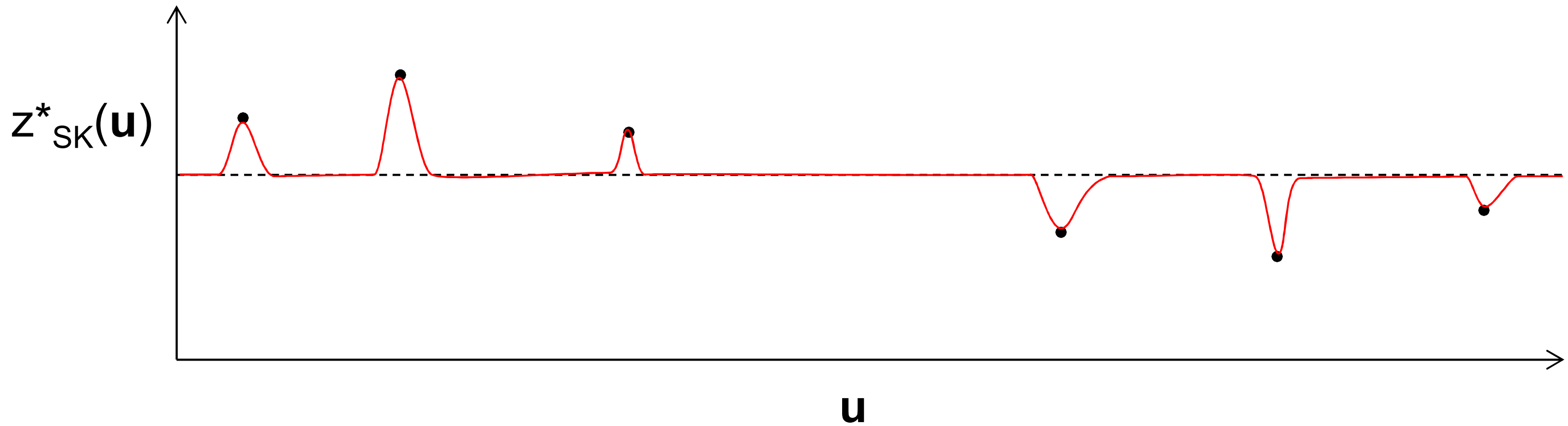
Lagrange Parameter

Recall that $C(\mathbf{h}) = C(0) - \gamma(\mathbf{h})$

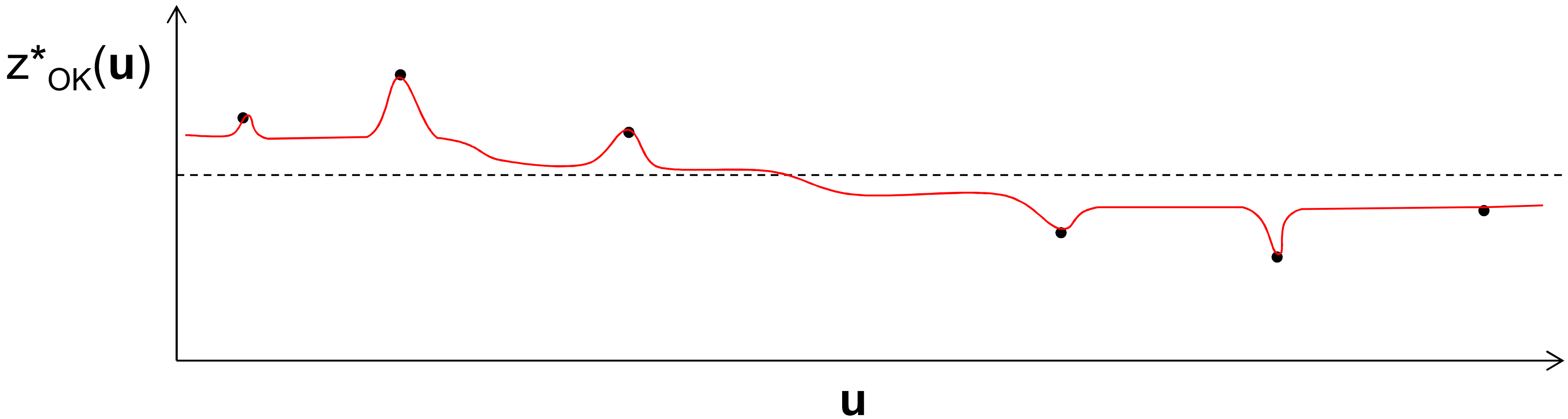
$$z^*(\mathbf{u}_0) = \sum_{\alpha=1}^{n} \lambda_\alpha z(\mathbf{u}_\alpha) + \cancelto{0}{\left(1 - \sum_{\alpha=1}^{n} \lambda_\alpha\right) m_z} \qquad \sigma_E^2(\mathbf{u}) = C(0) - \sum_{\alpha=1}^{n} \lambda_\alpha C(\mathbf{u} - \mathbf{u}_\alpha) + \mu$$

With ordinary kriging the mean does not need to be known. Ordinary kriging estimates the mean locally!

Simple Kriging vs. Ordinary Kriging



Beyond the range of correlation, Simple Kriging estimates the global mean.



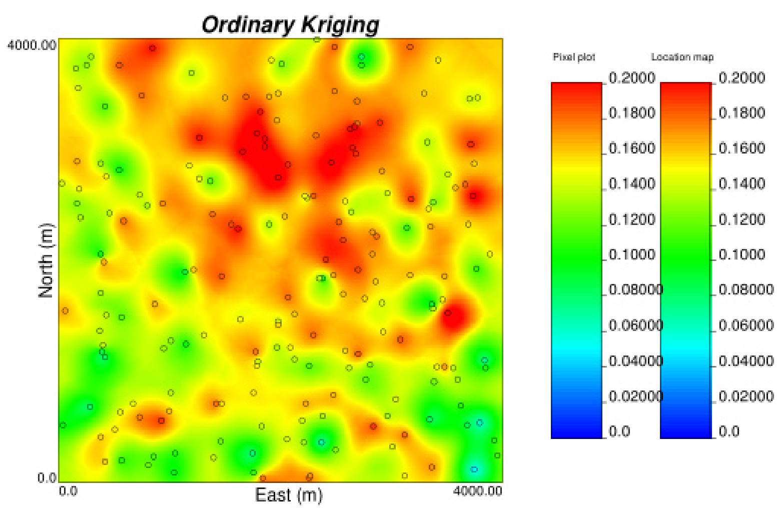
Beyond the range of correlation, Ordinary Kriging estimates with an estimated local mean. Relaxes the stationary mean assumption.

Kriging Demo in GSLIB

- Reformat data and using GSLIB .exes in Command Window

```
Parameters for KB2D
*******************

START OF PARAMETERS:
2D_MV_200wells.dat            -file with data
1   2   4                     -   columns for X, Y, and variable
-1.0e21   1.0e21              -   trimming limits
0                             -debugging level: 0,1,2,3
kb2d.dbg                      -file for debugging output
kb2d_200wells.out             -file for kriged output
400   5.0   10.0              -nx,xmn,xsiz
400   5.0   10.0              -ny,ymn,ysiz
1    1                        -x and y block discretization
1    50                        -min and max data for kriging
2000.0                         -maximum search radius
1   -999.9                    -0=SK, 1=OK,  (mean if SK)
1   0.0                       -nst, nugget effect
1   1.0   0.0 400.0 400.0     -it, c, azm, a_max, a_min
```




```
Parameters for Locpix
*********************

START OF PARAMETERS:
kb2d_200wells.out             -***PIXELPLT***- file with gridded data
1                             -  column number for variable
-1.0e21  1.0e21               -  data trimming limits
1                             -realization number
400    5.0      10.0          -nx,xmn,xsiz
400    5.0      10.0          -ny,ymn,ysiz
0                             -0=arithmetic, 1=log scaling
1                             -0=gray scale, 1=color scale
0                             -0=continuous, 1=categorical
0.0   0.2   0.02              -continuous:  min, max, increm.
4                             -categorical: number of categories
1     3     Code_One          -category(), code(), name()
2     1     Code_Two
3     6     Code_Three
4    10     Code_Four
2D_MV_200wells.dat            -***Locmap***-file with data
1   2   4                     -   columns for X, Y, variable
-1.0    1.0e21                -   trimming limits
0                             -0=arithmetic,  1=log scaling
1                             -0=gray scale,  1=color scale, 3=blackdots
0                             -0=no labels,   1=label each location
0.0   0.2   0.02              -gray/color scale: min, max, increm
0.2                           -label size: 0.1(sml)-1(reg)-10(big)
Ordinary Kriging              -Title
PixLoc_200wells.ps            -Output file
                              -pixtitle
                              -loctitle
East (m)                      -X label
North (m)                     -Y label
1.0                           -Vertical Exageration
```

Go to the Code/GSLIB directory.

Review of Main Points

- Simple kriging (SK) is linear regression with some special properties:
  - Gives the mean and variance of conditional normal distribution
  - Best linear estimate for mean squared error criterion and variogram model
- Estimation variance is expected squared difference between estimate and truth that accounts for:
  - Initial variance if no data are available, the stationary variance of the property
  - The redundancy between the data
  - The closeness of the data to what is being estimated
- We derive simple kriging to minimize the error variance in expected value
- The use of SK estimates directly is somewhat limited, but it is used extensively under a multivariate Gaussian model for inference of conditional means and variances
  - We will discuss more next about simulation.

Data Analytics and Geostatistics: Spatial Estimation

Lecture outline . . .

- **Kriging**
- **Indicator Kriging**

The Indicator Formalism and Indictor Kriging (IK)

Indicator coding is transforming a random variable / function to a probability relative to a category or a threshold.

- If $I\{\mathbf{u}: z_k\}$ is an indicator for a categorical variable,
  - What is the probability of a realization equal to a category?

$$I(\mathbf{u};\boldsymbol{z_k}) = \begin{cases} 1, & \text{if } Z(\mathbf{u}) = \boldsymbol{z_k} \\ 0, & \text{otherwise} \end{cases}$$

  - e.g. given category, $z_2 = 2$, and hard data at $\mathbf{u}_1, z(\mathbf{u}_1) = 2$, then $I\{\mathbf{u}_1; z_2\} = 1$
  - e.g. given category, $z_1 = 1$, and a soft data $\mathbf{u}_2$, $\mathrm{Z}(\mathbf{u}_2)$, then $I\{\mathbf{u}_2; z_1\} = 0.25$

- If $I\{\mathbf{u}: z_k\}$ is an indicator for a continuous variable,
  - What is the probability of a realization less than or equal to a threshold?

$$I(\mathbf{u};\boldsymbol{z_k}) = \begin{cases} 1, & \text{if } Z(\mathbf{u}) \le \boldsymbol{z_k} \\ 0, & \text{otherwise} \end{cases}$$

  - e.g. given threshold, $z_1 = 6\%$, and hard data at $\mathbf{u}_1, z(\mathbf{u}_1) = 8\%$, then $I\{\mathbf{u}_1; z_1\} = 0$
  - e.g. given threshold, $z_4 = 18\%$, and a RV,$\mathrm{Z}(\mathbf{u}_2) = N[16\%, 3\%]$ then $I\{\mathbf{u}_1; z_4\} = 0.75$

Hard and Soft Data Recall

Hard Data

- A local measurement of a property of interest, $z(\mathbf{u}_1)$, for which there is not a significant level of uncertainty and is represented as a constant, z.

$$z(\mathbf{u}_\alpha) = z, \quad \text{e.g. } z(\mathbf{u}_1) = 23.2\%$$

This is commonly assumed for well-based data (well logs and core measures).

Soft Data

- A local measurement of a property of interest, $z(\mathbf{u}_1)$, for which there is a significant level of uncertainty and is represented as a random varialbe, Z.

$$z(\mathbf{u}_\alpha) = Z, \ \text{e.g. } z(\mathbf{u}_1) = F_x(x; \mathbf{u}_1)$$

This is commonly applied for seismic derived local measures.

The Indicator Formalism and Indictor Kriging (IK)

Example of indicator transforms for a categorical variable.

Original Data	$I\{\mathbf{u}_\alpha; z_1 = 1\}$	$I\{\mathbf{u}_\alpha; z_2 = 2\}$	$I\{\mathbf{u}_\alpha; z_3 = 3\}$
$z(\mathbf{u}_1) = 3$	0	0	1
$z(\mathbf{u}_2) = 1$	1	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$z(\mathbf{u}_n) = 2$	0	1	0

Our $z(\mathbf{u}_\alpha)$, $\alpha = 1, \ldots, n$, data become $k = 1, \ldots, K$ sets of n data, new variables that indicates the probability of being each category.

The Indicator Formalism and Indictor Kriging (IK)

Example of indicator transforms for a continuous variable.

Original Data	$I\{\mathbf{u}_\alpha; z_1 = 0.10\}$	$I\{\mathbf{u}_\alpha; z_2 = 0.15\}$	$I\{\mathbf{u}_\alpha; z_3 = 0.20\}$
$z(\mathbf{u}_1) = 0.12$	0	1	1
$z(\mathbf{u}_2) = 0.25$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$z(\mathbf{u}_n) = 0.17$	0	0	1

Our $\mathrm{z}(\mathbf{u}_\alpha)$, $\alpha = 1, \dots, n$, data become $k = 1, \dots, K$ sets of n data, new variables that indicates the probability of being less or equal to each threshold.

Data For Indicator Kriging

- Various constraints that may be applied to indicator coding
- We the indicator approach we can account for souft, uncertainty data

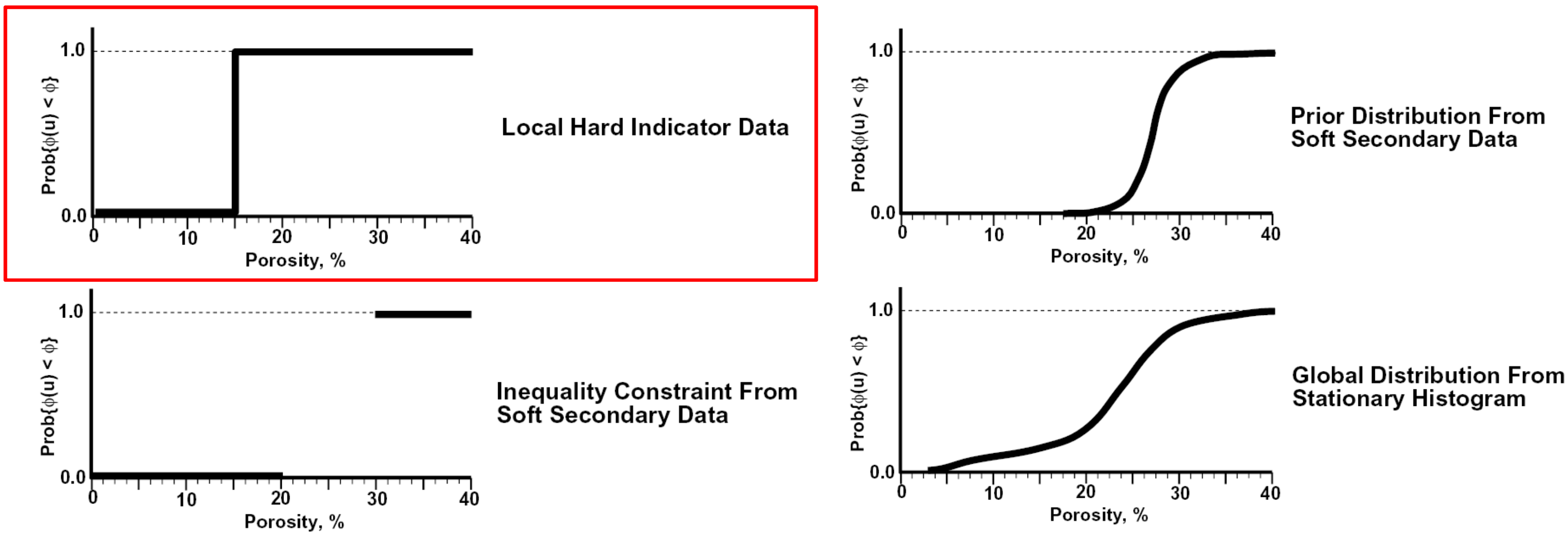


Figure taken from Pyrcz and Deutsch, 2014

Indicator Kriging (IK)

- Assign thresholds for indicator transforms

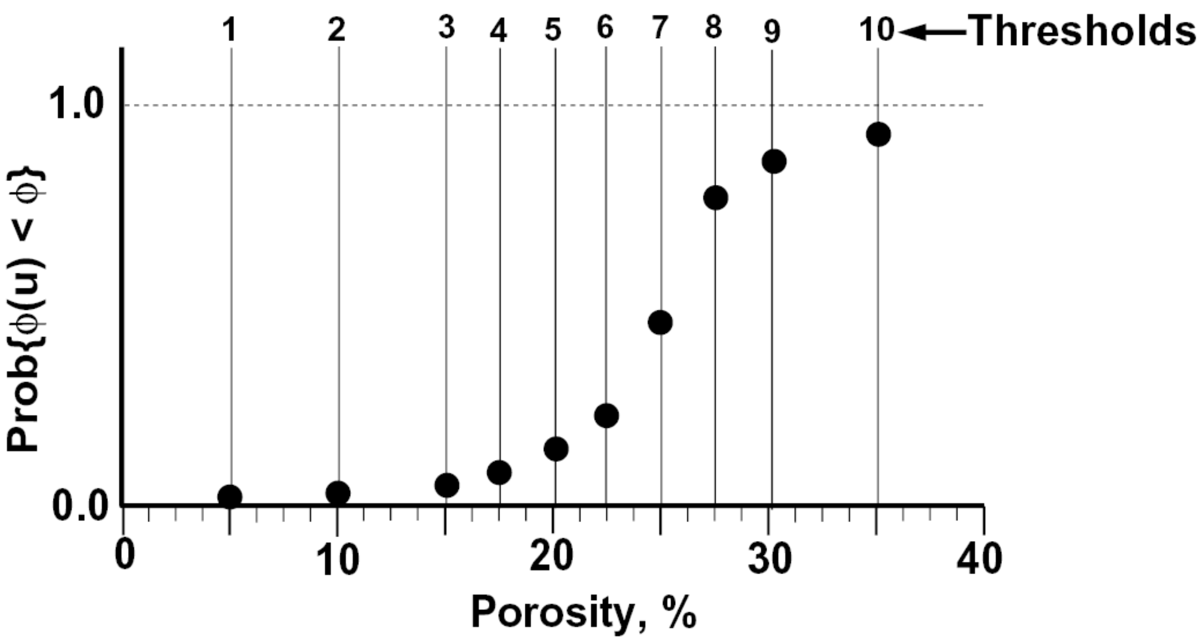


Figure taken from Pyrcz and Deutsch, 2014

- Establish a series of thresholds: $z_k k = 1, \ldots, K$
- May be related to critical thresholds, enough to represent the local distributions of uncertainty

Indicator Variogram Models

	Cumulative Class %	Nugget	Exponential			Spherical		
			3a	sill	anis	range	sill	anis
First Cutoff	6.1	0.17	18.0	0.50	2.3	100.0	0.33	10.0
Second Cutoff	15.5	0.11	47.7	0.54	3.3	150.0	0.35	10.0
Third Cutoff	23.3	0.13	90.0	0.58	5.0	170.0	0.29	10.0
Fourth Cutoff	32.7	0.13	90.0	0.61	6.2	160.0	0.26	10.0
Fifth Cutoff	43.4	0.12	108.0	0.68	6.2	91.0	0.20	7.1
Sixth Cutoff	57.7	0.12	108.0	0.68	7.1	85.0	0.20	7.1
Seventh Cutoff	75.4	0.22	144.0	0.69	6.7	66.0	0.09	5.9

Smoothly Changing Parameters

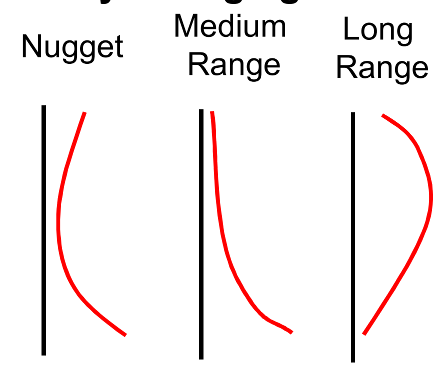


Figure taken from Pyrcz and Deutsch, 2014

- Standardize all models to a unit variance of one, note original sill is $p(1 - p)$

- Model the variograms with smoothly changing parameters for a *consistent* description
  - Transform of the same data, this should impart consistency
  - Reduces order relations issues with indicator estimation and simulation
  - Assists with interpolation of variogram models for new cutoffs

- In this example:
  - **Nugget** is largest for the first and last cutoff
  - **Medium range exponential structure** with consistently increasing range, contribution, and anisotropy
  - ***Long* range spherical model** with decreasing range, decreasing contribution, and increasing anisotropy

Indicator Kriging (IK)

- For all thresholds:
  - find all relevant data: *n*
  - code all data as indicator data at the current threshold:

$$i(\mathbf{u}_\alpha; z_k) = Prob^*\left\{Z(\mathbf{u})_\alpha \le z_k\right\}$$

  - estimate the indicator value at the current threshold at this location:

$$i(\mathbf{u}; z_k)^* = \sum_{\alpha=1}^{n} \lambda_\alpha \cdot i(\mathbf{u}_\alpha; z_k)$$

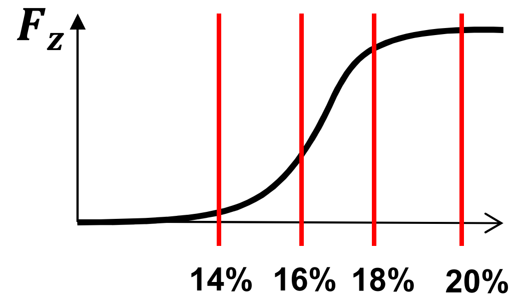
- Correct distribution for order relations

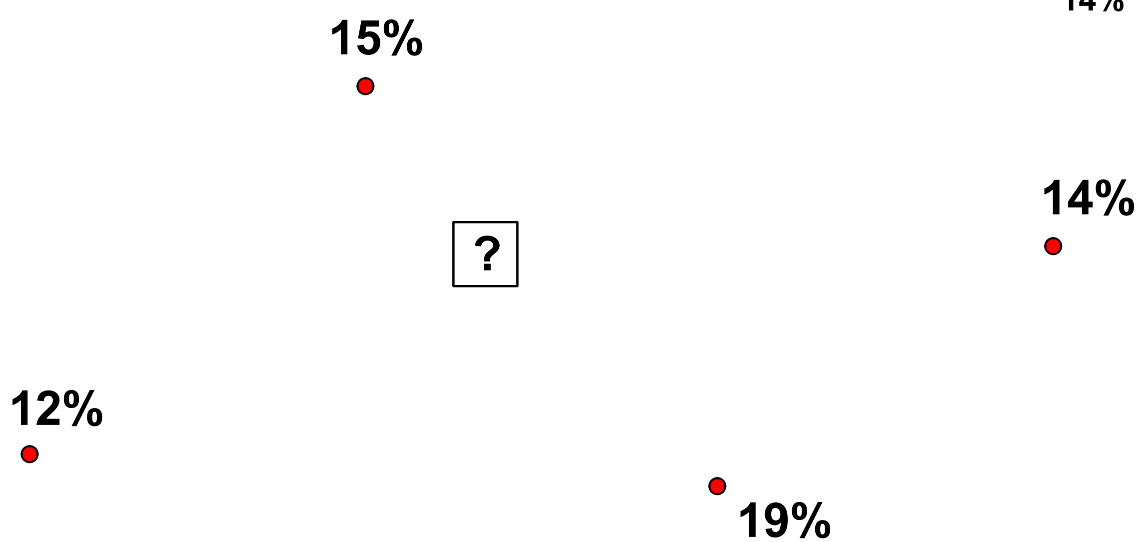
- Use distribution for:
  - measure of uncertainty, probability intervals
  - probability to exceed given thresholds
  - E-type mean estimate, truncated statistics
  - stochastic simulation

Indicator Kriging (IK)

- Demonstration of Indicator Kriging

1. Assign thresholds

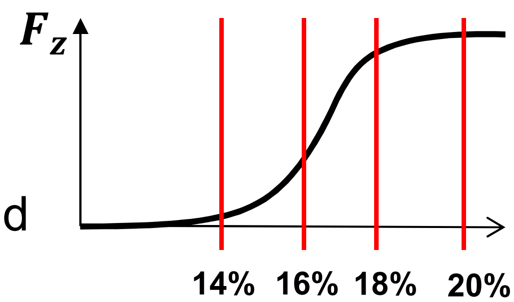




Indicator Kriging (IK)

- Demonstration of Indicator Kriging

1. Assign thresholds
2. Calculate indicator variograms for each threshold
   a) indicator transform
   b) calculate variogram



$i(\mathbf{u}_1; z = z_1)$

0

?

$i(\mathbf{u}_4; z = z_1)$

0

$i(\mathbf{u}_2; z = z_1)$

1

$i(\mathbf{u}_3; z = z_1)$

0

- Indicator Variogram:

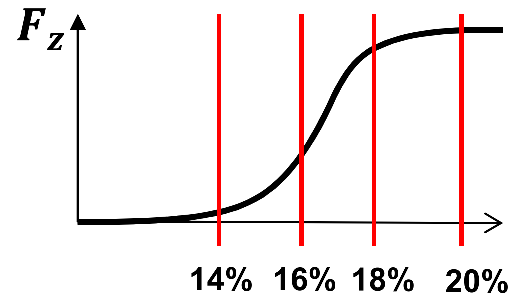
$$\gamma_I(\mathbf{h}; z_k) = \frac{1}{2N(\mathbf{h})} \sum_{i=1}^{N(\mathbf{h})} [i(\mathbf{u}; z_k) - i(\mathbf{u}+\mathbf{h}; z_k)]^2$$

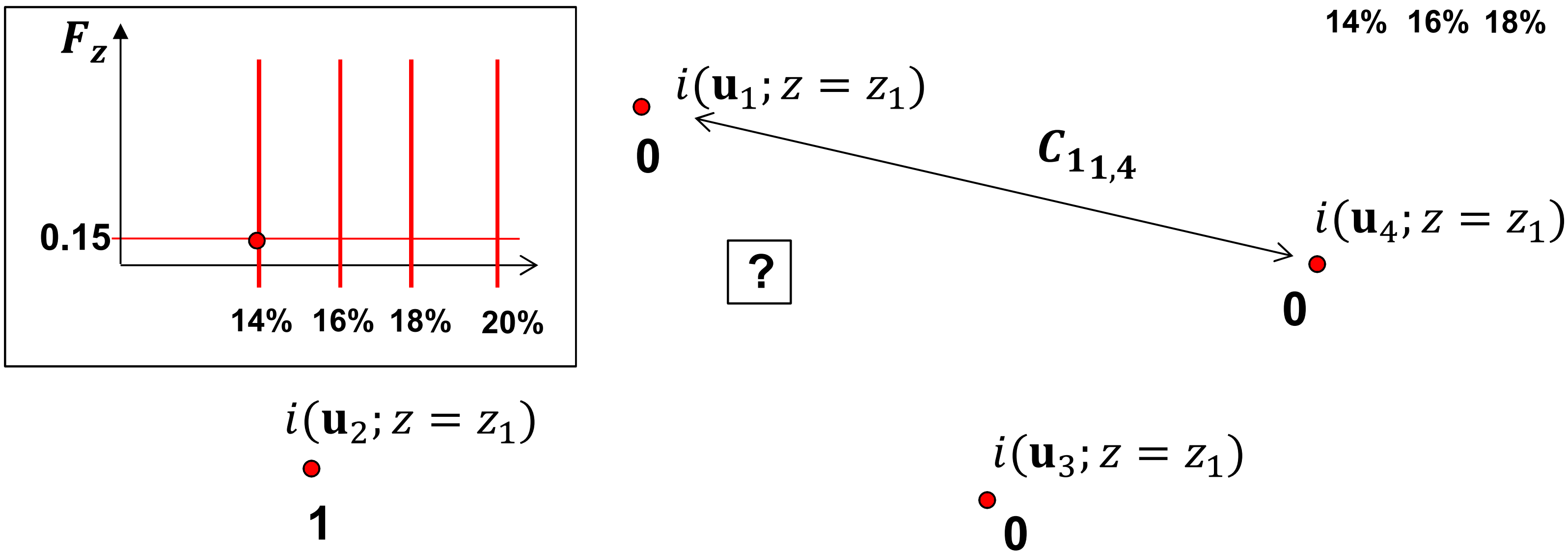
- Interpretation: ½ probability of transition
- Sill = $\sigma_I^2 = p(1-p)$

Indicator Kriging (IK)

- Demonstration of Indicator Kriging

3. For each location:
4. For each threshold: $z = z_1 = 14\%$





- Solve for indicator kriging estimate of probability porosity at location ? is < z_1.

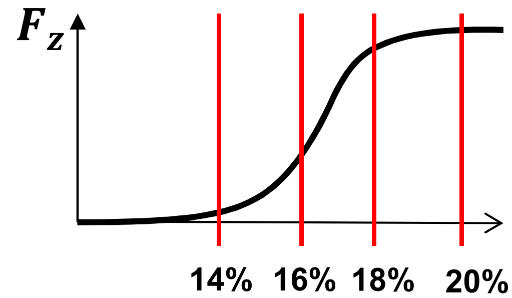
$$\begin{bmatrix} C_{1_{1,1}} & & \\ & \ddots & \\ & & C_{1_{n,n}} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} C_{1_{0,1}} \\ \vdots \\ C_{1_{0,n}} \end{bmatrix}$$

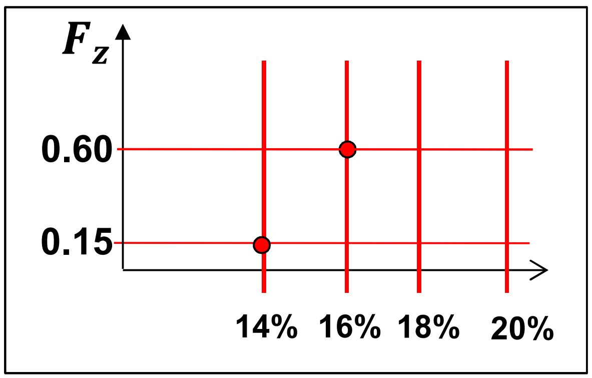
$$z^*(\mathbf{u}_1; z_1) = 0.15$$

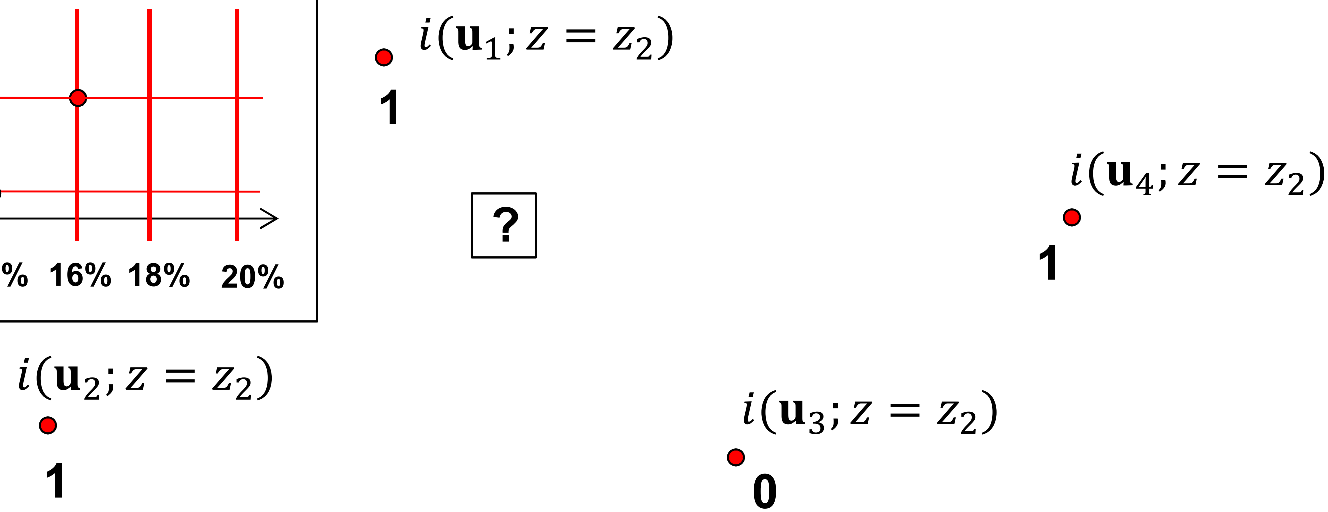
Indicator Kriging (IK)

- Demonstration of Indicator Kriging

3. For each location:
4. For each threshold: $z = z_2 = 16\%$





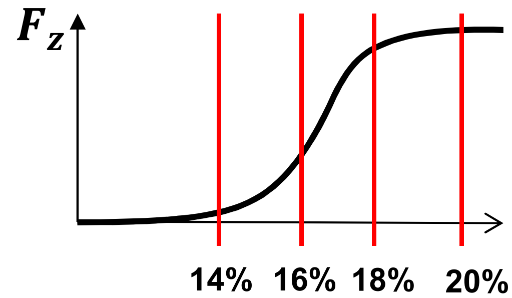


- Solve for indicator kriging estimate of probability porosity at location ? is < z_1.

$$\begin{bmatrix} C_{2_{1,1}} & & \\ & & C_{2_{n,n}} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_n \end{bmatrix} = \begin{bmatrix} C_{2_{0,1}} \\ C_{2_{0,n}} \end{bmatrix}$$

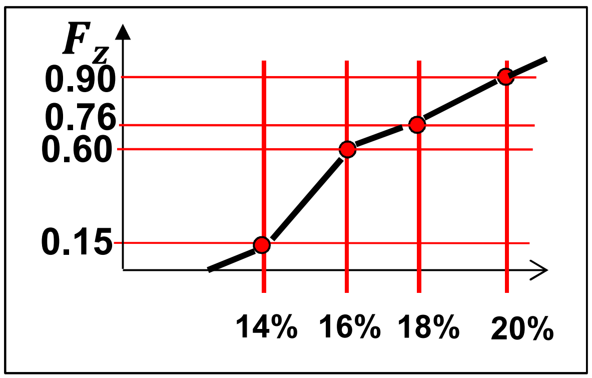
$$z^*(\mathbf{u}_0; z_2) = 0.60$$

Indicator Kriging (IK)



- Demonstration of Indicator Kriging

3. For each location:
4. For each threshold: $z = z_4 = 20\%$



$i(\mathbf{u}_1; z = z_4)$

1

?

$i(\mathbf{u}_4; z = z_4)$

1

$i(\mathbf{u}_2; z = z_4)$

1

$i(\mathbf{u}_3; z = z_4)$

1

- Solve for indicator kriging estimate of probability porosity at location ? is < z_1.
- Indicator kriging directly solves for the local distribution of CDF!

$$\begin{bmatrix} C_{2_{1,1}} & & \\ & & \\ & & C_{2_{n,n}} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \\ \lambda_n \end{bmatrix} = \begin{bmatrix} C_{2_{0,1}} \\ \\ C_{2_{0,n}} \end{bmatrix}$$

$$z^*(\mathbf{u}_0; z_2) = 0.60$$

Order Relations Correction

- Cumulative probability at each threshold was solved with potentially a difference indicator variogram and with a separate kriging.
- There is no direct constraint to impose slope ≥ 0.0.

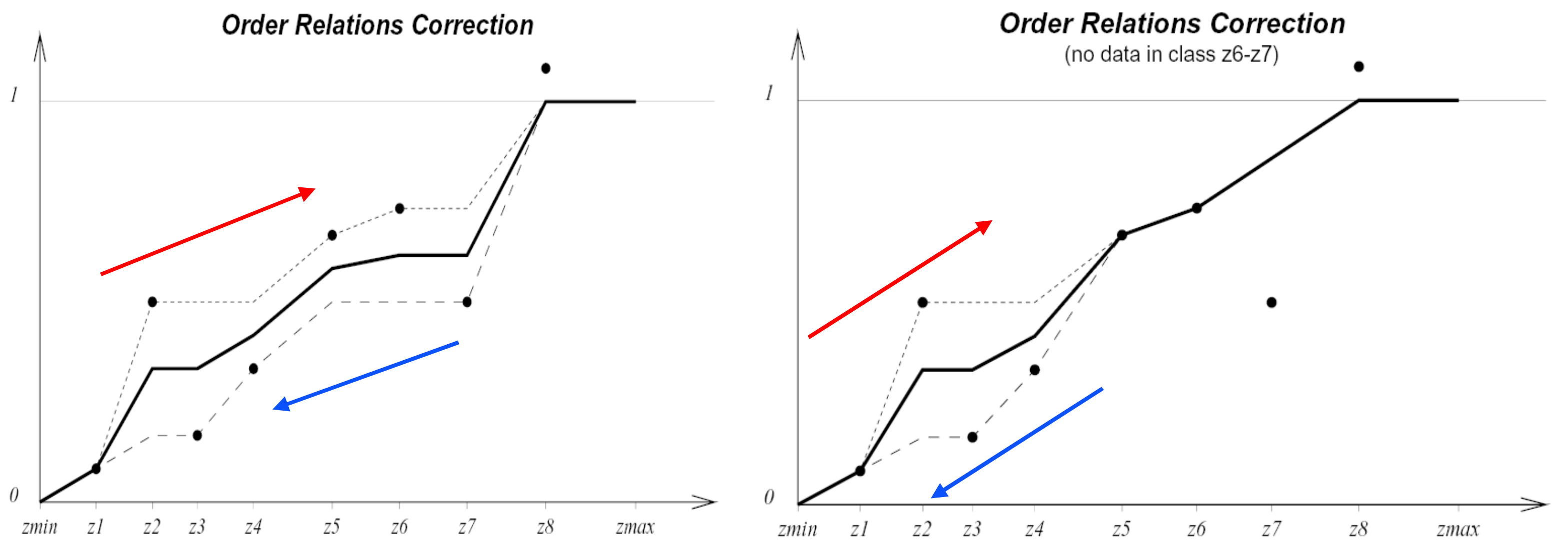


Figure taken from Pyrcz and Deutsch, 2014

- Method:
  - Traverse up along cumulative probabilities, if slope < 0, set slope = 0 (red array)
  - Traverse down along cumulative probabilities, if slope < 0, set slope = 0 (blue arrow)
  - Average result to remove bias

IK: Categorical Variables

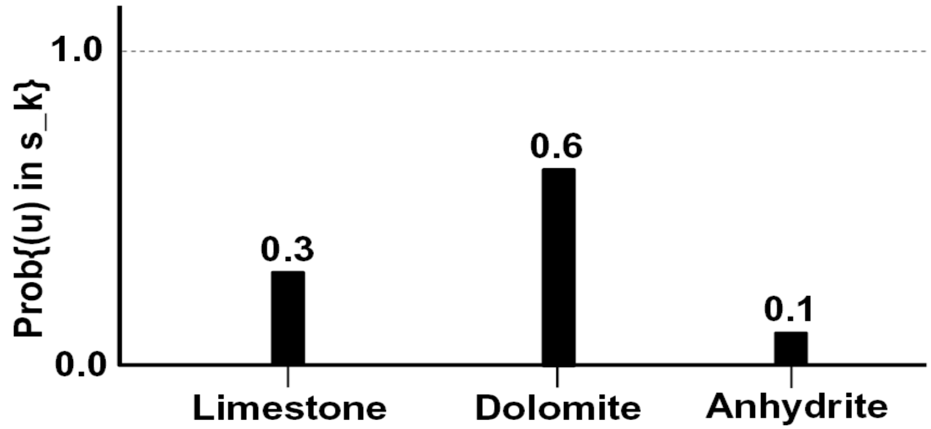


Figure taken from Pyrcz and Deutsch, 2014

- Consider a set of *K* categories: $s_k, k = 1, \ldots, K$
- Indicator variable for each location and each category:

$$i(\mathbf{u}_\alpha; s_k) = \begin{cases} 1 \text{ if category } s_k \text{ prevails at location } \mathbf{u}_\alpha \\ 0 \text{ if not} \end{cases}$$

- Same procedure for indicator kriging
- Order relations:

$$\text{all } i(u; s_k)^*, k = 1, \ldots, K \text{ must be } \geq 0$$

$$\sum_{k=1}^{K} i(\mathbf{u}; s_k)^* = 1.0$$

Ensure Closure

- Results are the probabilities that categories $s_k, k = 1, \ldots, K$ prevail at location u

Indicator Kriging Hands-on

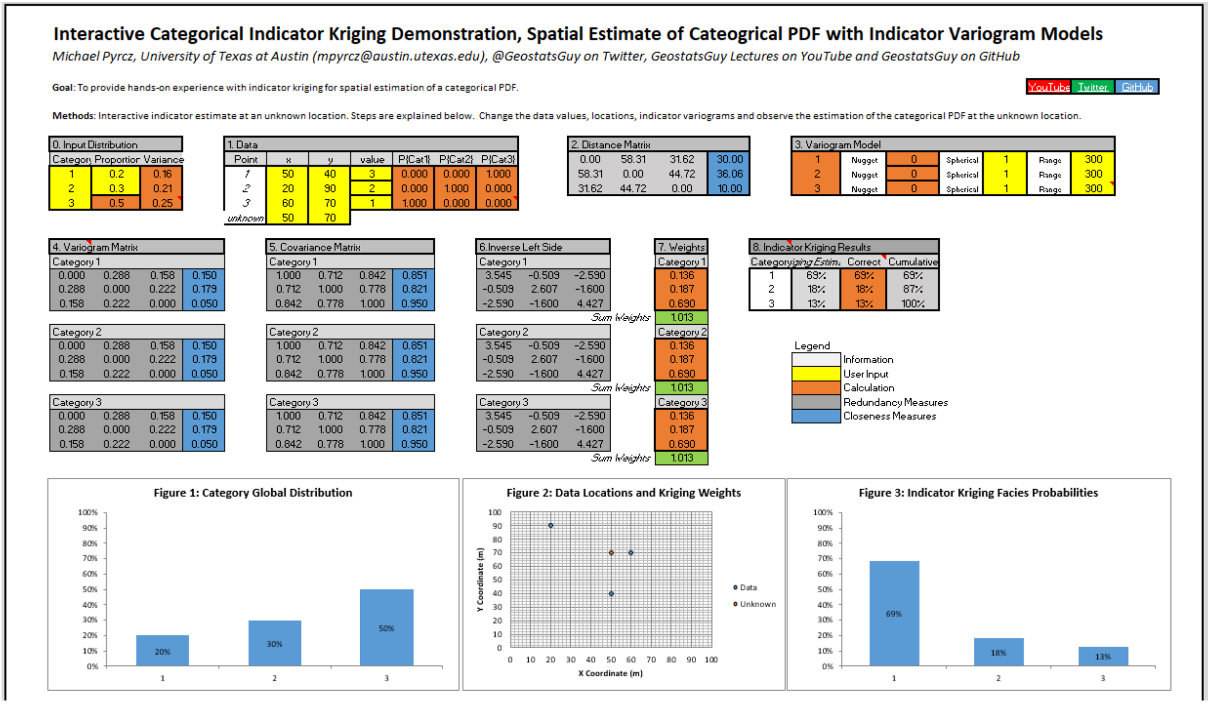
Here's an opportunity for experiential learning with Indicator Kriging.

- **Things to try:**

Pay attention to the kriging weights, kriging estimate and kriging variance while you:

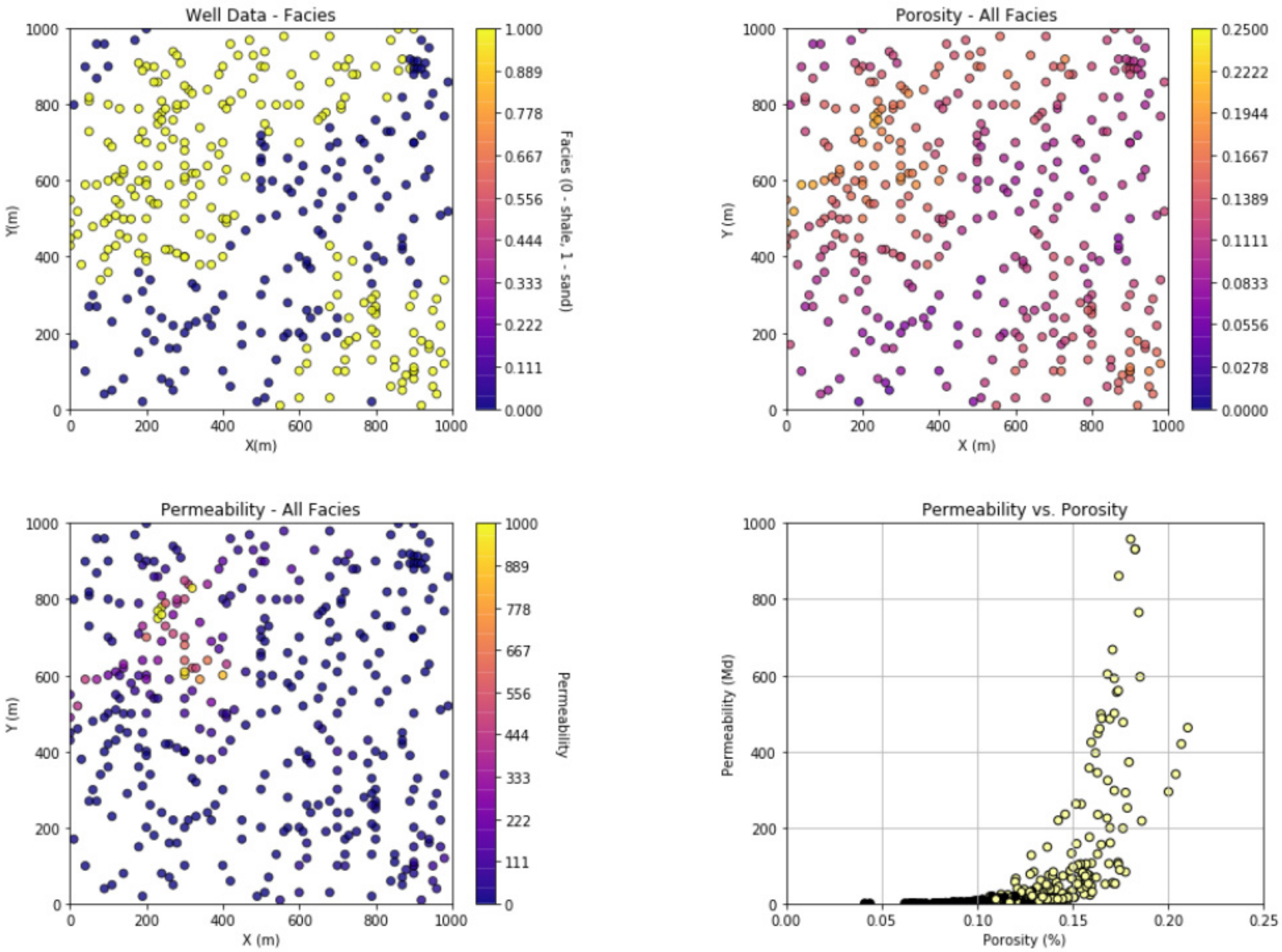
1. Set the ranges all very small.
2. Set points 1 and 2 closer together.
3. Make the indicator variograms very different.

Observed the impact on the estimated PDF.



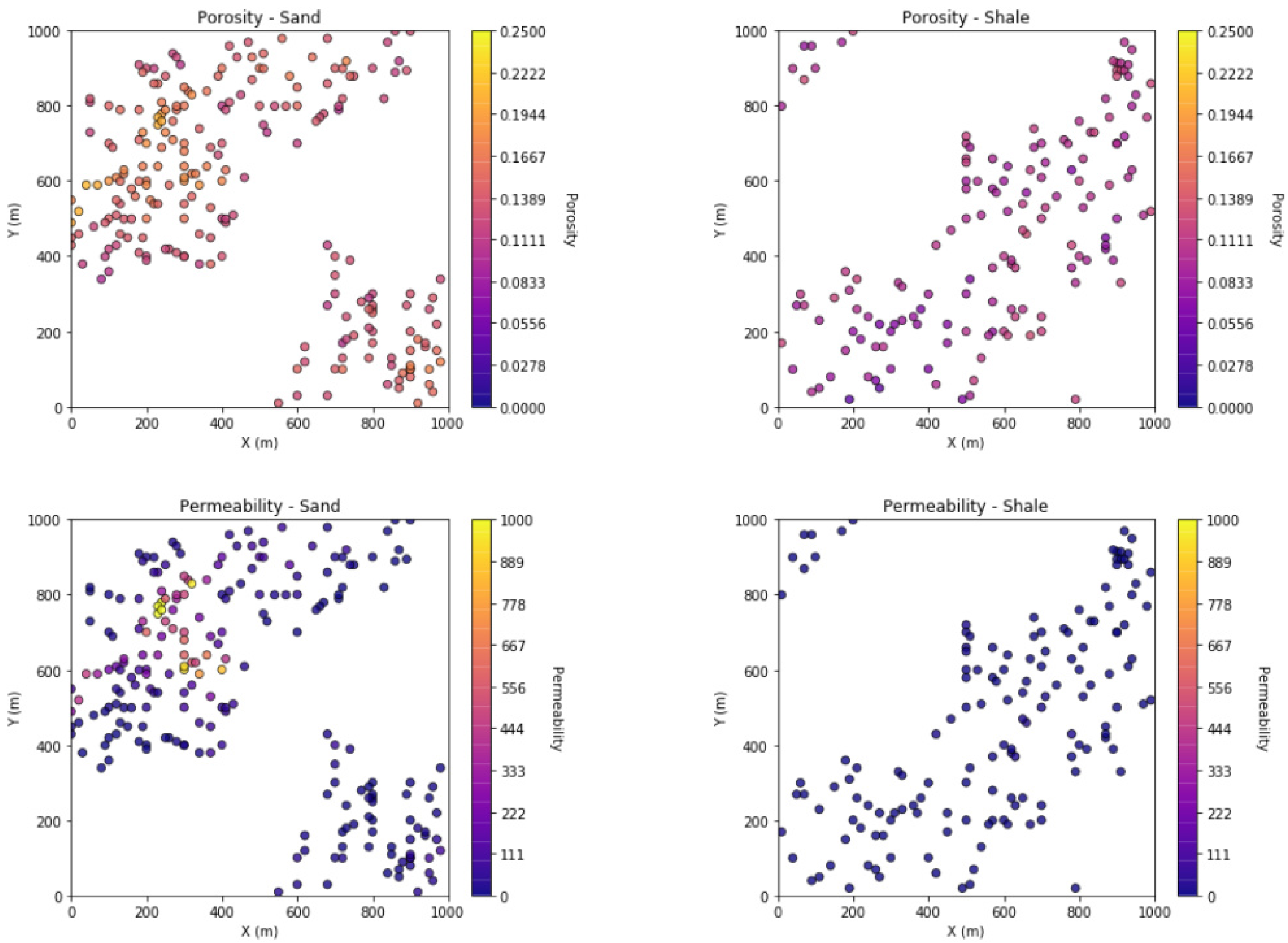
File Name: Indicator_Kriging_Demo.xlsx File is at: https://git.io/fpnY2

Indicator Kriging Demo in Python



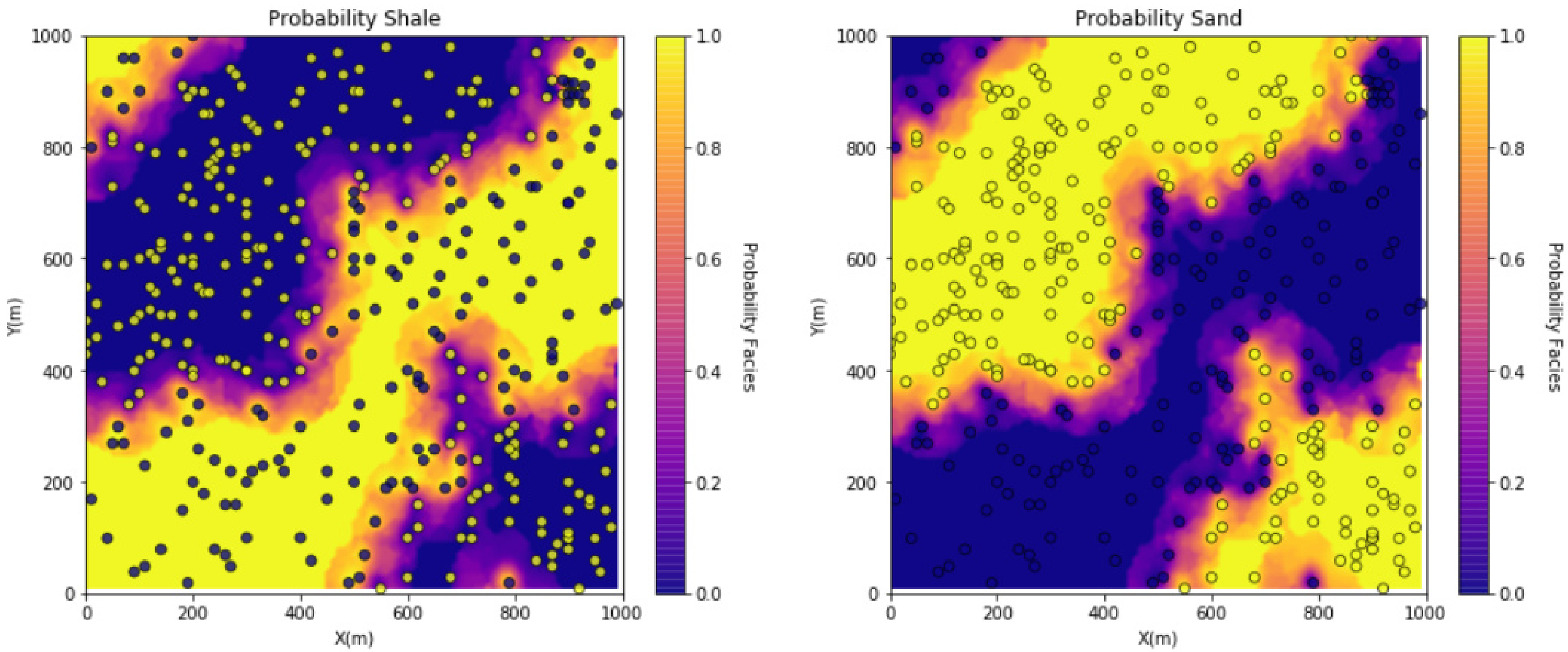
Dataset 12 facies, porosity and permeability.

Indicator Kriging Demo in Python



Dataset 12 porosity and permeability by-facies.

Indicator Kriging Demo in Python



Dataset 12 indicator kriging-based maps of proability of sand and shale.

- With indicator kriging we estimate the local CDF from the data and seconday data.
- No trend used, but with well density and ordinary kriging the result conforms to the reservoir elements.

Indicator Kriging Demo in Python

Here's an opportunity for experiential learning with Kriging and Indicator Kriging.

- **Things to try:**

1. Attempt simple kriging instead of ordinary kriging. What is the result away from data?
2. Change the variogram range and anisotropy and observe the impact.
3. Limit the local search parameters and observe the impact.

GeostatsPy: Spatial Estimation for Subsurface Data Analytics in Python

Michael Pyrcz, Associate Professor, University of Texas at Austin

Twitter | GitHub | Website | GoogleScholar | Book | YouTube | LinkedIn

PGE 383 Exercise: Methods for Spatial Estimation with GeostatsPy

Here's a simple workflow for spatial estimation with kriging and indicator kriging. This step is ciritical for:

1. Prediction away from wells, e.g. pre-drill assessments.
2. Spatial cross validation.
3. Spatial uncertainty modeling.

First let's explain the concept of spatial estimation.

Spatial Estimation

Consider the case of making an estimate at some unsampled location, $z(\mathbf{u_0})$, where z is the property of interest (e.g. porosity etc.) and $\mathbf{u_0}$ is a location vector describing the unsampled location.

How would you do this given data, $z(\mathbf{u_1})$, $z(\mathbf{u_2})$, and $z(\mathbf{u_3})$?

It would be natural to use a set of linear weights to formulate the estimator given the available data.

$$z^*(\mathbf{u}) = \sum_{\alpha=1}^{\mathbf{n}} \lambda_\alpha \mathbf{z}(\mathbf{u}_\alpha)$$

We could add an unbiasedness constraint to impose the sum of the weights equal to one. What we will do is assign the remainder of the weight (one minus the sum of weights) to the global average; therefore, if we have no informative data we will estimate with the global average of the property of interest.

$$z^*(\mathbf{u}) = \sum_{\alpha=1}^{\mathbf{n}} \lambda_\alpha \mathbf{z}(\mathbf{u}_\alpha) + \left(1 - \sum_{\alpha=1}^{\mathbf{n}} \lambda_\alpha\right) \bar{\mathbf{z}}$$

We will make a stationarity assumption, so let's assume that we are working with residuals, y.

$$y^*(\mathbf{u}) = \mathbf{z}^*(\mathbf{u}) - \bar{\mathbf{z}}(\mathbf{u})$$

If we substitute this form into our estimator the estimator simplifies, since the mean of the residual is zero.

$$y^*(\mathbf{u}) = \sum_{\alpha=1}^{\mathbf{n}} \lambda_\alpha \mathbf{y}(\mathbf{u}_\alpha)$$

while satisfying the unbaisedness constraint.

Kriging

Now the next question is what weights should we use?

We could use equal weighting, $\lambda = \frac{1}{n}$, and the estimator would be the average of the local data applied for the spatial estimate. This would not be very informative.

We could assign weights considering the spatial context of the data and the estimate:

File Name: GeostatsPy_kriging File is at: https://git.io/fhjbX

Summary of Indicators

- Indicator transform of continuous variable with thresholds
- Indicator transform of categorical variable, by-category
- Applied to estimate local CDF without assuming Gaussian distribution

- Later - Applied for categorical simulation
  - Replace simple / ordinary kriging with indicator kriging in the sequential context
  - i.e. Monte Carlo from indicator estimated CDF and transform simulated value for each threshold and use as data (sequential approach).

Data Analytics and Geostatistics: Spatial Estimation

Lecture outline . . .

- **Indicator Kriging**