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Decision Support

# Interior analysis of the green product mix solution

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ABSTRACT

When modeling optimal product mix under emission restrictions produces a solution with unacceptable level of profit, analyst is moved to investigate the cause(s). Interior analysis (IA) is proposed for this purpose. With IA, analyst can investigate the impact of accommodating emission controls in step-by-step one-at-a-time manner and in doing so track how profit and other important features of product mix degrade and to which emission control enforcements its diminution may be attributed. In this way, analyst can assist manager in identifying implementation strategies. Although IA is presented within context of a linear programming formulation of the green product mix problem, its methodology may be applied to other modeling frameworks. Quantity dependent penalty rates and transformations of emissions to forms with or without economic value are included in the modeling and illustrations of IA.


## 1. Introduction

Let greening product mix refer to the manner in which the environmental impact (e.g., carbon footprint or volume of $CO_2$ emissions) of production is incorporated within the mathematical modeling of what, how, and how much to produce. It should reflect the organization's intention to (i) restrict or eliminate the impact quantities and (ii) account for their economic consequences. In the modeling, the greening may take form in a variety of ways. It may appear as restrictions on the volume of certain emissions and other by-products of production; restrictions that relate to energy consumption or a preferred mix of energy sources, e.g., coal-fired, hydro, nuclear, etc.; charges that reflect penalties associated with emission generation and disposal; costs associated with emission treatment; and a representation that reflects the organization's participation in the trading of unused amounts of a regulated emission allowance. The exchange is known as cap-and-trade and in some situations a trading forum exists, see Galbraith (2013), Jaehn and Letmathe (2010), and Letmathe and Balakrishnan (2005). Modeling green product mix may also include the transformation of the impact quantities to environmentally less harmful and possibly marketable commodities and products. The transformations may occur within the operations environment in which the mix products are produced and may compete for use of the resources therein.

Modeling product mix under emission restrictions and a single measure or objective of what is best is generally straightforward. However, the optimal product mix that accommodates all sought after emission controls may exclude flagship product(s); result in product volumes that may challenge maintenance of market share or be so disparate that it creates problems in moving product through the various fabrication processes; upset product/process fit; call for underutilization of expensive resources; or most importantly result in a prohibitive reduction in the optimizing objective. Clearly, the value of the objective under emission restrictions cannot be better than the value of the unrestricted product mix solution.

When the modeling produces a green product mix solution with prohibitive results, the following is proposed for assessing cause. Suppose information was available that would allow the analyst to identify the results of successive one-at-a-time accommodations of the emission restrictions within the green product mix model of interest. By stepping through the greening in this manner, the analyst can track and attribute cause and effect to the diminution in the objective (e.g., profitability) with each accommodation. At some point, a prohibitive reduction in the objective will emerge and the causative emission restriction identified. This is referred to as the tipping point. Assuming the restrictions do not uniformly impact the objective, a reasonable way to begin the investigation is the identification of the single least detrimental emission restriction to incorporate in the model, followed by the least detrimental

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pair that includes the least detrimental single emission restriction, and to proceed similarly until all emission restrictions are accounted for. The necessary information for proceeding in this manner is found among the solutions to all subproblems of the green product mix problem in which the emission restrictions are accommodated one at a time, in pairs, triples, etc. Each subproblem represents a scenario of specified inclusions and exclusions among the set of emission restrictions under study, a subset scenario. The generation and use of the subproblem scenarios constitute interior analysis (IA), the subject of this paper. In the literature of green product mix modeling, this method of investigation is not well developed.

The results of IA may be useful in a variety of ways. For example, emergence of the tipping point may move the decision maker to consider alternate implementation scenarios. The merit of scenarios without the offending emission restriction and its combination with restrictions that compound the loss in the objective may be considered in formulating a strategy of partial implementation. Although some emission controls would not be explicitly enforced in these scenarios, their proximity to desired targets may be acceptable. Or the identified emission restriction(s) may be voluntary not mandated and as such postponable and eliminated from current consideration. The identification of the tipping point may also help the analyst assess the magnitude of needed offsets to the loss in the objective due to imposition of the emission restrictions.

In other ways, examination of all subsets of emission controls using IA has analytical value. They may reveal features of the green solution set that would not otherwise be evident. For example, the subproblem results provide insight to the manner in which the product mix quantities vary with the emission restrictions. Enforcement of some emission controls individually or in combination with others may eliminate some products from the mix and drive others to their upper bounds. In other situations, the values of certain product mix quantities may be invariant in any scenario of emission control. They are robust with respect to the greening.

In this paper, IA is illustrated within the framework of a linear programming (LP) formulation of a green product mix problem. Although the LP form of a green product mix model is not the totality of representations, it is the form of many mix models discussed in the literature. Its form is easily understood and amenable to the editing that is required for streaming all possible scenarios of emission control. A procedure for streaming the scenarios and organizing the results for IA is presented in this article.

Without loss of generality, emissions will refer collectively to the gases, solid wastes, effluents, scrap, etc. that result from production and whose return to the environment in untreated state would be harmful, illegal, or perceived as poor citizenship. Transformed emissions include recycled materials, recovered compounds, treated effluents, composited by-products, and other forms.

The rest of the paper is organized in the following manner. Literature relating to greening, its place in product mix determination, and strategies for implementation are reviewed in the next section. The formal presentation of the methodology of IA appears in Section 3 and illustrated with examples in Section 4. The paper concludes with summary and remarks in Section 5.

## 2. Literature review

The following literature discussion is intended to provide context for the modeling framework in which IA is illustrated. The context is found in the literature of the product mix problem and its green form as well as the literature of the mitigation of greenhouse gas emissions and other by-products of production. Other contributions related to the subject of this paper are found in the literature of waste disposal and post-optimality analysis of the product mix solution.

### 2.1. Literature of the product mix problem

Early in its history, the product mix problem was formulated as an LP problem and applied in a wide variety of settings. Over time, formulation innovations emerged to address special features of the decision making environment. They included fuzzy features such as the decision maker's ability/inability to rationalize the tradeoffs in product mix decision making. Recent contributions to accommodate fuzzy modeling aspects included Bhattacharya and Vasant (2007), Bhattacharya, Vasant, Sarkar, and Mukherjee (2008), Hasuike and Ishii (2009), Kunsch, Springael, and Brans (2004), Kunsch and Springael (2008), and Tsai and Hung (2009). Susanto and Bhattacharya (2011) accommodated fuzzy features under multiple objectives. Bhattacharya, Sarkar, and Mukherjee (2006) utilized an analytical hierarchy process (AHP) in modeling product mix determination and Chaharsooghi and Jafari (2007) used simulated annealing. Other contributions included accommodation of activity-based costing (ABC) aspects and theory of constraints (TOC) approach to the determination of optimal product mix. For the latter, see Plenert (1993). Kee (1995) incorporated both. The integration was intended to capture the interaction between costs (direct and indirect) of production and resource capacity (utilization and expansion) in determining optimal mix. Kee and Schmidt (2000) presented a model in which ABC and TOC solutions were special cases. Malik and Sullivan (1995) also addressed ABC aspects in their modeling. Onwubolu (2001) combined tabu search and TOC and Onwubolu and Mutingi (2001a, 2001b) investigated use of genetic algorithms and TOC in modeling product mix determination.

### 2.2. Literature of green product mix determination

Contributions to the greening of product mix determination are found in a variety of sources and consist of mathematical models, modeling methodologies, and formulation innovations for the abatement of a variety of emissions that accompany the production of the mix products. The contributions of Kunsch et al. (2004), Kunsch and Springael (2008), Mirzaesmaeeli, Elkamel, Douglas, Croiset, and Gupta (2010), Mollersten, Yan, and Westermark (2003), and Tsai et al. (2012) addressed $CO_2$ reduction; Dvorak, Chlapek, Jecha, Puchyr, and Stehlik (2010), the mitigation of dioxins and $NO_x$ emissions; and Lu, Huang, Liu, and He (2008), the abatement of greenhouse gas emissions (e.g. $CO_2$, methane) resulting from the treatment/disposal of solid waste. Jaehn and Letmathe (2010), Letmathe and Balakrishnan (2005), and Rong and Lahdelma (2007) formulated for modeling purposes the trading of unused emission allowances. Rong and Lahdelma (2007) also contributed a multi-period stochastic optimization model for the combined production of heat and power among multiple installations. Their model incorporated emission penalties and the trading of unused $CO_2$ emission allowances. Lu et al. (2008) addressed uncertainty among environmental parameters using interval-parameter programming methodology in their mixed integer programming model of solid waste disposal. Kunsch and Springael (2008) utilized fuzzy reasoning methodology to address parameter uncertainties in their modeling of electricity production for residential distribution. In the modeling, they included a carbon tax scheme based on the fossil fuel used to produce electricity. Mirzaesmaeeli et al. (2010) provided a deterministic multi-period mixed integer linear programming model for the determination of the optimal mix (sourcing) of electric energy subject to demand restrictions and mitigation of $CO_2$ emission. Tsai et al. (2012) modeled green product mix determination under flexible capacity expansion and taxing of $CO_2$ emission at volume-dependent rates. Lu and Chen (2013) presented a two-product mix formulation under a single emission constraint (carbon) and objective function (profit). Other treatments of greening production are found in Boons (2002), Klemes, Varbanov, Pierucci, and Huisingh (2010), and Sheng and Srinivasan (1995).

The post-optimality investigations of the solutions to these product mix models are generally conventional. Malik and Sullivan (1995) investigated the sensitivity of their product mix solution to certain model parameters and conducted shadow price studies of capacitated resources and other mix restrictions. Najm, El-Fadel, Ayoub, El-Taha, and Al-Awar (2002) examined the sensitivity of solid waste disposition (recycling, composting, incineration, and landfilling) to revenues from the sale of treated and recovered materials and operational costs using simulation methods. Kee (1995) used a form of sensitivity analysis to investigate the consequences of relieving a binding/bottleneck constraint in a mixed integer product mix model that incorporated ABC and TOC features. According to Kee (1995), this form of sensitivity analysis "may be used to estimate the benefits that may accrue from relieving a constraint and identifying the subsequent set of activities that will become a bottleneck…". The same may be said for accommodating emission restrictions one at a time, in pairs, in triples, etc. and observing the impact on outcome measures of importance such as profitability. This is the methodology of IA.

A related literature addressed converting emissions to products in the form of recycled and processed by-products. According to Atasu, Guide, and Van Wassenhove (2008), some conversions have economic value that depending "on the economics of a particular situation, recovery processes may reuse the entire product, selected modules, components, and/or parts". Many successful conversions of emissions have been reported. Some are significant in created value and others serve as illustration of the organization's commitment to greening. For the former, see Anonymous (2012a). Other conversions include carbon dioxide ($CO_2$) to polymeric feedstock, Petrie (2008); starch and food waste to bio-degradable plastic, Anonymous (2007); and organic waste to renewable biodiesel fuel, Anonymous (2012b).

Of particular interest in this paper are the green product mix models of Letmathe and Balakrishnan (2005) and Tsai et al. (2012). Letmathe and Balakrishnan (2005) modeled a real situation consisting of twelve mix products and five emission controls; cost/charges associated with emission generation; and costs (revenues) resulting from the procurement (sale) of emission allowance when emission generation exceeded (fell short of) the compliance quantity. Tsai et al. (2012) showed how penalties that varied with the quantity of emissions could be accommodated with use of piecewise linear functions. Their modeling innovations and the proposed incorporation of emission conversions/treatments in product mix determination provided a representative framework for the illustration of IA in this paper.

These product mix investigations produced useful results. Interior analysis goes further using the outcomes of accommodating emission restrictions individually, in pairs, triples, etc. to identify the tipping point in emission controls, observe the interactions between emission controls and product mix quantities, and examine the merit of alternate implementation scenarios consisting of subsets of emission controls. This approach is not well developed in the product mix literature and is nascent to the analysis of the green product mix solution; see Lu and Chen (2013). Similar to TOC, IA may be looked upon as an investigation of the impact of relieving constraints.

### *2.3. Green strategy and product mix determination*

Much like green supply chain management, greening product mix is an integral part of the overall environmental strategy of an organization, Ambec and Lanoie (2008), Daniel, Diakoulaki, and Pappis (1997), King and Lenox (2001), Letmathe and Balakrishnan (2005), Penkuhn, Spengler, Puchert, and Rentz (1997), Radulescu, Radulescu, and Radulescu (2009), and Tsai et al. (2012) among others.

## 3. The methodology of interior analysis

Interior analysis is based upon the results of incorporating emission controls one at a time, in pairs, triples, etc. within a product mix problem of interest. The enumeration of each scenario of control, composing the corresponding green product mix subproblem, solving it, and evaluating the outcome feature(s) of the solutions constitute the methodology. The post-optimality analysis of the solutions allows the analyst to track and attribute cause and effect to the diminution in profitability and/or other meaningful outcome features resulting from the greening of a product mix model of interest. Given a set of $M$ number of emission controls for consideration, each scenario of control may be looked upon as a subset of 1, 2, …, $M$ emission controls. There are $2^M$ possibilities. The terms subset and emission control scenario are used interchangeably in the discussion.

Within a green product mix model, an emission control can be operationalized in a variety of ways. It may occur through the addition of constraint(s) that represent restrictions on the volume of the emissions that result from production of the mix products; through term(s) in the objective function that reflect costs or penalties associated with emission production; through modeling features that relate to the organization's participation in a cap-and-trade exchange; or through additions to the objective function, the constraint set, and the repertoire of decision variables to account for conversion of emissions to more benign forms or to commodities and products with market value.

The scheme for enumerating all subsets of $M$ emission controls appears in the Supplementary Material for this article in the document titled Subset Enumeration Scheme. Through successive labeling operations on the string of positive integers 1, …, $M$, the scheme generates all subsets/scenarios of emission controls. The labeling scheme is a simpler version of the backtracking schemes of Balas (1965) and Narula and Wellington (1985).

Given the outcomes for all scenarios of emission control, IA proceeds in the following way. Let scenario $\upsilon(=1,\ldots,2^M)$ be denoted as $s_\upsilon(r) = \{S_1,\ldots,S_M\}$, where $S_m = 0(1)$ indicates that emission control $m(=1,\ldots,M)$ was excluded (included) in the scenario and $r = S_1 + \cdots + S_M$. Further, associate with each scenario $s_\upsilon(r)$ the corresponding outcome value $o_\upsilon(r)$, e.g., the value of the optimizing objective for scenario $\upsilon$. To facilitate IA, the $o_\upsilon(r)$ values are reordered in the following way. Let $o^i(r)$ denote a reordered outcome value and $s^i(r)$ the associated scenario. For $r = 0$, $o^1(0)$ is the outcome value for the null scenario $s^1(0) = \{0,\ldots,0\}$ with no emission controls. For each $r = 1, \ldots, M-1$, the reordering is such that $o^i(r) \geqslant o^{i+1}(r)$ for any two adjacent values, i.e. for each $r$, the outcome values $o^i(r)$ are in descending order. The $s^i(r)$ are correspondingly reordered. For $r = 0,1, \ldots, M$ and $n = 2^M$, let $s^j(r)$, $j = 1, \ldots, n$ denote the reordered scenarios and $o^j(r)$, $j = 1, \ldots, n$ denote the associated outcome values. Then $o^2(1)$ is the best outcome value for any scenario with one emission control and $s^2(1)$ indicates the associated scenario. In $s^2(1)$, the $m$ for which $S_m = 1$ identifies the best individual emission control to incorporate in the product mix model. Set $\rho_1$ to the $m$ so identified and $f_1$ to the $o^2(1)$ value. The best emission control to combine with $\rho_1$ is found among scenarios $s^j(2)$ with $S\rho_1 = 1$. Among them, identify the scenario associated with the best $o^j(2)$ value. In addition to $S\rho_1 = 1$, that scenario has one other $S_m = 1$. Set $\rho_2$ to the $m$ so identified and $f_2$ to the best $o_j(2)$ value so found. At this point, the best two-step accommodation ($\rho_1$ then $\rho_2$) is identified. Continue in this manner with step $r(=3,\ldots,M)$ to identify the best $\rho_r$ to combine with $\rho_1, \rho_2, \ldots, \rho_{r-1}$. At each step $r$, search for the best outcome value $o^j(r)$ associated with $s^j(r)$ in which $S_m = 1$, $m = \rho_1, \rho_2, \ldots, \rho_{r-1}$. That scenario has one other $S_m = 1$. Set $\rho_r = m$ so identified and $f_r$ to the best outcome value $o^j(r)$ so described. Then $\rho_1, \ldots, \rho_r$ is the best $r$-step accommodation. The procedure concludes at step $r = M$ with the sequence $\rho_1, \ldots, \rho_M$ and the corresponding outcome values $f_1, \ldots, f_M$. Through this stepwise investigation of one-at-a-time accommodation of emissions controls, cause $\rho_1, \ldots, \rho_M$ is attributed to the corresponding diminution $f_1, \ldots, f_M$ in the optimizing objective. In doing so, a prohibitive value of the optimizing objective may emerge at some step signaling a tipping point in the accommodation of emission controls.

## 4. Numerical illustrations

Two illustrations of IA are presented. Illustration 1 is straightforward IA of the Letmathe and Balakrishnan (2005) model. Illustration 2 is IA of their model with adaptations that included quantity dependent piecewise linear emission penalty rates and emission-to-product transformations. The form of the quantity dependent emission penalties scheme is adapted from Tsai and Lin (1990) and Tsai et al. (2012). The illustrations are intended to demonstrate the capacity of IA to produce useful and insightful results under different product mix formulations. They are representative of the product mix modeling discussed in Section 2.

### 4.1. Numerical illustration 1

Letmathe and Balakrishnan (2005) modeled determination of optimal mix among 12 (=$I$) products whose quantities were denoted $X_1, \ldots, X_{12}$; whose production required use of five (=$J$) resources in the amounts $R_1, \ldots, R_5$; and whose processing resulted in five (=$M$) emissions in the amounts $E_1, \ldots, E_5$. For the purpose of IA, the adapted form of their model is:

Maximize

$$\begin{aligned}&800X_1 + 700X_2 + 500X_3 + 1000X_4 + 2200X_5 + 2300X_6\\&\quad + 1600X_7 + 2600X_8 + 700X_9 + 1000X_{10} + 1300X_{11}\\&\quad + 1800X_{12} - 50R_1 - 100R_2 - 3R_3 - 5R_4 - 2R_5 - S_4E_4\\&\quad - 5S_5E_5^+ + 4S_5E_5^- \end{aligned} \tag{1.1}$$

subject to:

$$\begin{aligned}R_1 &= 3X_1 + 3X_2 + 3X_3 + 4X_4 + 10X_5 + 9X_6 + 8X_7 + 10X_8\\&\quad + 3X_9 + 3X_{10} + 5X_{11} + 7X_{12}\end{aligned} \tag{1.2}$$

$$\begin{aligned}R_2 &= X_1 + X_2 + X_3 + 2X_4 + 6X_5 + 10X_6 + 4X_7 + 10X_8 + X_9\\&\quad + 2X_{10} + 3X_{11} + 6X_{12}\end{aligned} \tag{1.3}$$

$$\begin{aligned}R_3 &= 20X_1 + 20X_2 + 10X_3 + 50X_4 + 80X_5 + 60X_6 + 60X_7\\&\quad + 100X_8 + 30X_9 + 60X_{10} + 60X_{11} + 70X_{12}\end{aligned} \tag{1.4}$$

$$\begin{aligned}R_4 &= 30X_1 + 30X_2 + 30X_3 + 40X_4 + 100X_5 + 80X_6 + 70X_7\\&\quad + 100X_8 + 30X_9 + 40X_{10} + 50X_{11} + 90X_{12}\end{aligned} \tag{1.5}$$

$$\begin{aligned}R_5 &= 50X_1 + 50X_2 + 50X_3 + 50X_4 + 50X_5 + 50X_6 + 50X_7\\&\quad + 50X_8 + 50X_9 + 50X_{10} + 50X_{11} + 50X_{12}\end{aligned} \tag{1.6}$$

$$\begin{aligned}E_1 &= 4X_1 + 3X_2 + 2X_3 + 5X_4 + 3X_5 + 8X_6 + 10X_7 + 20X_8\\&\quad + 4X_9 + 5X_{10} + 12X_{11} + 10X_{12}\end{aligned} \tag{1.7}$$

$$\begin{aligned}E_2 &= 10X_1 + 20X_2 + X_3 + 15X_4 + 50X_5 + 100X_6 + 50X_7\\&\quad + 280X_8 + 40X_9 + 40X_{10} + 50X_{11} + 40X_{12}\end{aligned} \tag{1.8}$$

$$\begin{aligned}E_3 &= 6X_1 + 7X_2 + X_3 + 5X_4 + 4X_5 + 4X_6 + 5X_7 + 7X_8 + 5X_9\\&\quad + 6X_{10} + 6X_{11} + 6X_{12}\end{aligned} \tag{1.9}$$

$$\begin{aligned}E_4 &= 4X_1 + 3X_2 + X_3 + 5X_4 + 16X_5 + 7X_6 + 7X_7 + 15X_8\\&\quad + X_9 + 7X_{10} + 2X_{11} + 9X_{12}\end{aligned} \tag{1.10}$$

$$\begin{aligned}E_5 &= 2X_1 + X_2 + X_3 + X_4 + 4X_5 + 2X_6 + 3X_7 + 4X_8 + 2X_9\\&\quad + 2X_{10} + 2X_{11} + 6X_{12}\end{aligned} \tag{1.11}$$

$$E_1 \leqslant S_1 60{,}000 + (1 - S_1)B \tag{1.12}$$

$$E_2 \leqslant S_2 8R_2 + (1 - S_2)B \tag{1.13}$$

$$\begin{aligned}E_3 &\leqslant S_3 6(X_1 + X_2 + X_3 + X_4 + S_2X_5 + X_6 + X_7 + X_8 + X_9\\&\quad + X_{10} + X_{11} + X_{12}) + (1 - S_3)B\end{aligned} \tag{1.14}$$

$$E_5 = S_5(50{,}000 + E_5^+ - E_5^-) \tag{1.15}$$

$$X_1 \leqslant 1600 \tag{1.16}$$

$$X_2 \leqslant 2400 \tag{1.17}$$

$$X_3 \leqslant 7500 \tag{1.18}$$

$$X_4 \leqslant 3000 \tag{1.19}$$

$$X_5 \leqslant 2000 \tag{1.20}$$

$$X_6 \leqslant 4000 \tag{1.21}$$

$$X_7 \leqslant 5000 \tag{1.22}$$

$$X_8 \leqslant 6500 \tag{1.23}$$

$$X_9 \leqslant 1000 \tag{1.24}$$

$$X_{10} \leqslant 2200 \tag{1.25}$$

$$X_{11} \leqslant 1000 \tag{1.26}$$

$$X_{12} \leqslant 1800 \tag{1.27}$$

$$\begin{aligned}&E_1,\ E_2,\ E_3,\ E_4,\ E_5,\ E_5^+,\ E_5^-,\ R_1,\ R_2,\ R_3,\ R_4,\ R_5,\ X_1,\ X_2,\ X_3,\\&\quad X_4,\ X_5,\ X_6,\ X_7,\ X_8,\ X_9,\ X_{10},\ X_{11},\ X_{12} \geqslant 0\end{aligned} \tag{1.28}$$

$$S_1,\ S_2,\ S_3,\ S_4,\ S_5 = 0, 1. \tag{1.29}$$

Note that the resource quantities $R_1, \ldots, R_5$ were not restricted. However, market considerations bounded the product mix quantities, see (1.16)–(1.27). The $B$ in (1.12)–(1.14) is a large positive number of sufficient magnitude to indicate no real bound. Let (1) refer inclusively to (1.1)–(1.29).

In (1), the greening was addressed through production restrictions (1.12)–(1.14) for emission quantities $E_1$, $E_2$, and $E_3$; through the per unit cost/penalty of $-1$ for emission quantity $E_4$, see (1.1); and in (1.1) with the cap-and-trade benefit $(+4E_5^-)$ and penalty $(-5E_5^+)$ of under-achieving $(E_5^-)$ or over-achieving $(E_5^+)$ emission 5's compliance quantity of 50,000 as specified in (1.15). For each scenario $\nu(=1,\ldots,32)$ of emission control, (1) was edited according to $s_\nu(r) = \{S_1,S_2,S_3,S_4,S_5\}$ and solved where $S_m = 0(1)$ indicated exclusion (inclusion) of control for emission m $(=1,\ldots,5)$. Execution of the Subset Enumeration Scheme described in the

Supplementary Material for this article produced the streaming of the $s_{t}(r)$, $j = 1, \ldots, 32$. Solving (1) in this manner and reordering the outcomes as described in Section 3 produced the results that populate Table 1. Details of the solutions appear in the document titled Illustration 1 among the Supplementary Material that accompany this article.

The solutions to (1) without and with the five emission controls appear respectively in rows 1 and 32 of Table 1. From the linear programming properties of (1) under any emission restriction, the objective should be no greater than $5,386,000 (row 1 of Table 1). The consequences of accommodating all five emission controls included large reduction in the optimizing objective from $5,386,000 to $1,781,188 (row 32 of Table 1); significant changes in the constituency of the optimal mix with elimination of five products ($X_i = 0$, $i = 6$–$10$); and large reductions in resource usage. These results indicated the inability of the processes as modeled to accommodate the five emission controls with reasonable operating consequences.

In Table 1, note the response of the mix quantities to the emission controls. Among the thirty-two scenarios including the null scenario (row 1) of no emission control, the values of the mix quantities $X_1$, $X_2$, $X_4$, and $X_5$ are invariant and equal to their respective upper bound values; the mix quantities $X_8$, $X_9$, $X_{10}$, and $X_{12}$ are at either their lower or upper bound values and at the latter for all subset scenarios of emission controls 3, 4, and 5; mix quantities $X_8$, $X_9$, and $X_{10}$ have a tendency to their lower bound value of zero for scenarios with emission control 2; $X_{11}$ has three distinct values including its lower and upper bound values; $X_6$ and $X_7$, four values; and $X_3$, five values. Most (24) scenarios have five zero-value mix quantities. Among all scenarios, over one-third of the product mix values are zero. Further, the inclusion of emission control 3 in any scenario did not change the value of the objective. It is not a binding constraint in any scenario. The addition of emission 4 to any scenario did not improve the objective. However, in twelve of the fifteen scenarios in which emission 5 is combined with other emissions in pairs, triples, etc., the values of the objective are greater with its inclusion. The results for the null scenario in row 1 serve as a useful benchmark in assessing the impact of the greening.

In order to attribute cause to the diminution in the value of the optimizing objective, the step-by-step procedure described in Section 3 was applied to the solutions to (1). Given the $o^j(r)$ and $s^j(r)$ shown in Table 1, the analysis began with identification of the single ($r = 1$) best emission control to accommodate in (1), followed by the best pair ($r = 2$) that included the single best emission control, then the best triple ($r = 3$) that included the best pair of controls, and continued in this manner until $r = 5$. The details are as follows. In the discussion, note the correspondence between the j of $o^j(r)$ and $s^j(r)$ and the contents of row j in Table 1. Among outcome values $o^j(1)$, the largest is $o^2(1) = 5{,}386{,}000$ for scenario $s^2(1) = \{0,0,1,0,0\}$. Therefore enforcement of emission 3's target is the single best emission control. Set $\rho_1 = 3$ and $f_1 = 5{,}386{,}000$. For $r = 2$, among outcome values $o^j(2)$ with $s^j(2) = \{*,*,1,*,*\}$, the largest is $o^7(2) = 5{,}212{,}000$ with $s^7(2)=\{0,0,1,0,1\}$. Then emission 3 is best paired with emission 5. Set $\rho_2 = 5$ and $f_2 = 5{,}212{,}000$. The asterisk in $\{*,*,*,*,*\}$ indicates that the value in the position so noted is free to be 0 or 1. However the sum of the numerical elements of the set must equal $r$. For $r = 3$, among outcome values $o^j(3)$ with $s^j(3) = \{*,*,1,*,1\}$, $o^{17}(3) = 4{,}956{,}000$ is the largest with

**Table 1**
Results of interior analysis for numerical illustration 1.

| Row $j$ | Included emission controls[a] | No. of included emissions $r$ | Scenario of included emission controls $s^j(r)$ | Objective function value $o^j(r)$ | Product mix quantities[b](in thousands) | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $X_3$ | $X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10}$ | $X_{11}$ | $X_{12}$ |
| 1 | – | 0 | {0,0,0,0,0} | 5,386,000 | 0 | 4 | 5 | 6.5 | 1 | 2.2 | 1 | 1.8 |
| 2 | 3 | 1 | {0,0,1,0,0} | 5,386,000 | 0 | 4 | 5 | 6.5 | 1 | 2.2 | 1 | 1.8 |
| 3 | 5 | 1 | {0,0,0,0,1} | 5,212,000 | 0 | 4 | 5 | 6.5 | 1 | 2.2 | 1 | 1.8 |
| 4 | 4 | 1 | {0,0,0,1,0} | 5,130,300 | 0 | 4 | 5 | 6.5 | 1 | 2.2 | 1 | 1.8 |
| 5 | 1 | 1 | {1,0,0,0,0} | 2,395,000 | 0 | 1.3 | 0 | 0 | 1 | 2.2 | 0 | 0 |
| 6 | 2 | 1 | {0,1,0,0,0} | 1,947,167 | 7.500 | 0 | 1.883 | 0 | 0 | 0 | 0 | 1.8 |
| 7 | 3, 5 | 2 | {0,0,1,0,1} | 5,212,000 | 0 | 4 | 5 | 6.5 | 1 | 2.2 | 1 | 1.8 |
| 8 | 3, 4 | 2 | {0,0,1,1,0} | 5,130,300 | 0 | 4 | 5 | 6.5 | 1 | 2.2 | 1 | 1.8 |
| 9 | 4, 5 | 2 | {0,0,0,1,1} | 4,956,300 | 0 | 4 | 5 | 6.5 | 1 | 2.2 | 1 | 1.8 |
| 10 | 1, 5 | 2 | {1,0,0,0,1} | 2,492,600 | 0 | 1.3 | 0 | 0 | 1 | 2.2 | 0 | 0 |
| 11 | 1, 3 | 2 | {1,0,1,0,0} | 2,395,000 | 0 | 1.3 | 0 | 0 | 1 | 2.2 | 0 | 0 |
| 12 | 1, 4 | 2 | {1,0,0,1,0} | 2,308,900 | 0 | 1.3 | 0 | 0 | 1 | 2.2 | 0 | 0 |
| 13 | 2, 5 | 2 | {0,1,0,0,1} | 1,984,967 | 7.500 | 0 | 1.883 | 0 | 0 | 0 | 0 | 1.8 |
| 14 | 2, 3 | 2 | {0,1,1,0,0} | 1,947,167 | 7.500 | 0 | 1.883 | 0 | 0 | 0 | 0 | 1.8 |
| 15 | 2, 4 | 2 | {0,1,0,1,0} | 1,849,683 | 7.500 | 0 | 1.883 | 0 | 0 | 0 | 0 | 1.8 |
| 16 | 1, 2 | 2 | {1,1,0,0,0} | 1,785,104 | 3.092 | 0.152 | 0 | 0 | 0 | 0 | 0 | 1.8 |
| 17 | 3, 4, 5 | 3 | {0,0,1,1,1} | 4,956,300 | 0 | 4 | 5 | 6.5 | 1 | 2.2 | 1 | 1.8 |
| 18 | 1, 3, 5 | 3 | {1,0,1,0,1} | 2,492,600 | 0 | 1.3 | 0 | 0 | 1 | 2.2 | 0 | 0 |
| 19 | 1, 4, 5 | 3 | {1,0,0,1,1} | 2,406,500 | 0 | 1.3 | 0 | 0 | 1 | 2.2 | 0 | 0 |
| 20 | 1, 3, 4 | 3 | {1,0,1,1,0} | 2,308,900 | 0 | 1.3 | 0 | 0 | 1 | 2.2 | 0 | 0 |
| 21 | 2, 3, 5 | 3 | {0,1,1,0,1} | 1,984,967 | 7.500 | 0 | 1.883 | 0 | 0 | 0 | 0 | 1.8 |
| 22 | 2, 4, 5 | 3 | {0,1,0,1,1} | 1,887,483 | 7.500 | 0 | 1.883 | 0 | 0 | 0 | 0 | 1.8 |
| 23 | 1, 2, 5 | 3 | {1,1,0,0,1} | 1,861,921 | 3.092 | 0.152 | 0 | 0 | 0 | 0 | 0 | 1.8 |
| 24 | 2, 3, 4 | 3 | {0,1,1,1,0} | 1,849,683 | 7.500 | 0 | 1.883 | 0 | 0 | 0 | 0 | 1.8 |
| 25 | 1, 2, 3 | 3 | {1,1,1,0,0} | 1,785,104 | 3.092 | 0.152 | 0 | 0 | 0 | 0 | 0 | 1.8 |
| 26 | 1, 2, 4 | 3 | {1,1,0,1,0} | 1,704,300 | 3.011 | 0 | 0.138 | 0 | 0 | 0 | 0 | 1.8 |
| 27 | 1, 3, 4, 5 | 4 | {1,0,1,1,1} | 2,406,500 | 0 | 1.3 | 0 | 0 | 1 | 2.2 | 0 | 0 |
| 28 | 2, 3, 4, 5 | 4 | {0,1,1,1,1} | 1,887,483 | 7.500 | 0 | 1.883 | 0 | 0 | 0 | 0 | 1.8 |
| 29 | 1, 2, 3, 5 | 4 | {1,1,1,0,1} | 1,861,921 | 3.092 | 0.152 | 0 | 0 | 0 | 0 | 0 | 1.8 |
| 30 | 1, 2, 4, 5 | 4 | {1,1,0,1,1} | 1,781,188 | 3.056 | 0 | 0 | 0 | 0 | 0 | 0.107 | 1.8 |
| 31 | 1, 2, 3, 4 | 4 | {1,1,1,1,0} | 1,704,300 | 3.011 | 0 | 0.138 | 0 | 0 | 0 | 0 | 1.8 |
| 32 | 1, 2, 3, 4, 5[c] | 5 | {1,1,1,1,1} | 1,781,188 | 3.056 | 0 | 0 | 0 | 0 | 0 | 0.107 | 1.8 |

[a] 1, Represents inclusion of constraint (1.24) in (2); 2, inclusion of (1.25); 3, inclusion of (1.26); 4, inclusion of – $E_4$ in the objective function (1.1); and 5, inclusion of constraint (1.27) and $-5E_5^+ + 4E_5^-$ in objective function (1.1).

[b] Not shown are the following product mix quantities that are the same in each scenario: $X_1 = 1.6$, $X_2 = 2.4$, $X_4 = 3.0$, and $X_5 = 2.0$.

[c] Solution of Letmathe and Balakrishnan (2005).

$s^{17}(3) = \{0, 0, 1, 1, 1\}$. Therefore emissions 3 and 5 are best combined with emission 4. Set $\rho_3 = 4$ and $f_3 = 4{,}956{,}000$. At this point, note that the values of the mix quantities $X_1, \ldots, X_{12}$ for scenario $s^{17}(3)$ and the null scenario $s^1(0)$ are identical but the objective function value $o^{17}(3)$ is 7.98% smaller than the $o^1(0)$ value. For $r = 4$, among outcome values $o^j(4)$ with $s^j(4) = \{*, *, 1, 1, 1\}$, $o^{27}(4) = 2{,}406{,}000$ is the largest with $s^{27}(4) = \{1, 0, 1, 1, 1\}$. Therefore emission 1 is best combined with emissions 3, 4, and 5. Set $\rho_4 = 1$ and $f_4 = 2{,}406{,}000$. The procedure concluded at step 5 (=$r$) with addition of emission 2 and $\rho_5 = 2$ and $f_5 = 1{,}781{,}188$. Note the 51.45% reduction in the outcome values between steps 3 and 4. The addition of emission 1 (=$\rho_4$) at step 4 may be looked upon as the tipping point in the stepwise accommodation of the five emission controls. Also observe that the addition of emission 2 at step 5 resulted in further reduction of 25.97% in the outcome value. The accommodation of emissions 1 and 2 alone or in combinations with the other controls resulted in at least a 55.62% reduction in use of resource 1; 71.22% reduction in the use of resource 2; and reductions of 63.49% for resource 3, 54.12% for resource 4, and 33.83% for resource 5. The largest changes in the values of the mix quantities relative to the benchmark quantities (row 1 of Table 1) occurred with scenarios that included emissions 1 and 2. Stepping through the greening of (1) with one-at-a-time additions revealed these aspects of the solution space that may not otherwise be evident to the analyst.

The above results may move the decision maker to consider an implementation strategy of 'satisficing' the emission targets not enforced in a scenario of partial accommodation. For the partial accommodation scenario $s^{17}(3) = \{0, 0, 1, 1, 1\}$, the unrestricted quantities $E_1$ (=291,600) and $E_2$ (=2,929,00) are respectively 4.86 (=$E_1$/60,000) and 2.20 (=$E_2/(8 * 166{,}200)$) times their target values, see (1.12) and (1.13). Although satisficing may not appear attractive for this partial accommodation, in other applications the multiples for the unrestricted emissions may be acceptable, i.e. satisficing. If so, a partial accommodation scenario may serve as a first strategy of accommodation that in time may allow the market value (e.g. product price, branding recognition, etc.) of the greening to grow or the development of offsets to the reduction in the objective that accompany enforcement of emission controls 1 and 2. In addition, since the values of the mix quantities $X_1, \ldots, X_{12}$ for scenarios $s^{17}(3) = \{0, 0, 1, 1, 1\}$ and the null scenario $s^1(0) = \{0, 0, 0, 0, 0\}$ are identical, implementing controls for just emissions 3, 4, and 5 may be attractive for reason of process fit, i.e. none of the controls would alter the existing compatibility of product, process, and locus of production. Under the assumption that the null scenario of no emission controls has good product/process fit, controlling just emissions 3, 4, and 5 may be a good first implementation of controls that maintains the fit. See the document titled Illustration 1 among the Supplementary Material for more details of the solutions to (1).

### 4.2. Numerical illustration 2

Consider the following scenario that is reflective of the product mix modeling reviewed in Section 2, adapted within model (1), and analyzed using IA. Suppose the analyst is interested in investigating the impact of a certain quantity-dependent emission charge scheme and the benefit, if any, of concurrently transforming certain emissions-to-products within the product mix production environment modeled in (1). Because the technology for producing real products 1–12 modeled in (1) was not available, for the sake of discussion consider the twelve subject products to be food related and the scenario of interest to consist of quantity dependent penalties for emissions 1 and 4 and three transformations of emissions-to-products. Further, consider emission 1 to have three segments (ranges) of quantity dependent penalty rates and emission 4 to have four as described in Table 2. In addition, let $XE_y$ denote the quantity of emissions-to-product y (=1,2,3); $RSXE_j$, the amount of standard resource $j$ (=1,...,5) required to process the three emissions-to-product quantities; and $RDXE_w$, the quantity of resource $w$ (=1,2) dedicated to the processing of the same quantities. The scenario is described in (2.1)–(2.64) and collectively referred to as (2).

**Table 2**
The assumed piecewise linear emission penalties of illustration 2.

| Emission | Segment t | Quantity lower bound | Quantity upper bound | Penalty rate per unit | Upper bound of emission penalty charge |
|---|---|---|---|---|---|
| 1 | 1 | >0 | 50,000 | 4 | 200,000 |
| | 2 | >50,000 | 55,000 | 5 | 225,000[a] |
| | 3 | >55,000 | 60,000 | 6 | 255,000[b] |
| 4 | 1 | >0 | 40,000 | 1.00 | 40,000 |
| | 2 | >40,000 | 60,000 | 1.50 | 70,000[c] |
| | 3 | >60,000 | 80,000 | 2.00 | 110,000[d] |
| | 4 | >80,000 | 100,000 | 2.50 | 160,000[e] |

[a] =200,000 + 5 * 5000.
[b] =225,000 + 6 * 5000.
[c] =40,000 + 1.50 * 20,000.
[d] 70,000 + 2.00 * 20,000.
[e] 110,000 + 2.50 * 20,000.

The modeling of the five standard resource requirements in (1) was adopted in (2). For the assumed setting of this illustration, they could consist of resources consumed in ingredient preparation, blending of the same, heating/cooking, shaping, and packaging. Some could be used for processing food products 1–12 and emissions-to-products 1–3, see (2.52)–(2.56). The quantity of standard resource $j$ (=1,...,5) required for both purposes was denoted as $RT_j$ (=$R_j + RSXE_j$) in (2.59)–(2.63). Two specialized resources/processings dedicated to transforming the emissions to products were assumed, see (2.57) and (2.58). They could represent precipitation of certain compounds from the untreated emission quantities; blending of emissions with other substances to create biodegradable materials; or shaping and packaging the end products in forms different from the food products. Specifications (1.2)–(1.29) were maintained in (2) and referenced as (2.2)–(2.29) respectively.

For the conversion of emissions-to-products, simple transformations were assumed that related the quantities of the emissions-to-products to the untreated emission volumes by weight, see (2.49)–(2.51). When elected, each transformation may be looked upon as commitment to treat the emissions in the indicated manner with positive or otherwise impact on the objective. Because the emissions-to-products were not core products, their per unit standard resource requirements were assigned modest values ($\leqslant$10) for less expensive resources 3 and 5 of Illustration 1 and zero for the other resources, see (2.52)–(2.56). The per unit dedicated resource requirements of the three emissions-to-products were given arbitrary values ($\leqslant$15), see (2.55) and (2.56). For simplicity in illustration, no additional emission quantities were assumed to result from the transformations.

In the objective function (2.1), the coefficient of $RT_j$ is the same as that specified for $R_j$ in (1.1), $j = 1, \ldots, 5$, and reflects the assumption that the unit cost of using standard resource j for processing standard products 1–12 or emissions-to-products 1–3 is the same. The unit cost of $RDXE_1$ and $RDXE_2$ was set to 2, the smallest unit cost of the standard resources. Given that the transformed emissions are not core products and should not command commensurate market value, the per unit revenue contribution of each was assigned an arbitrary value (=75) equal to fifteen percent of smallest food product per unit revenue (=500), see (2.1).

Quantity dependent penalty rates for emissions 1 and 4 were accommodated using indicator variables $\alpha_{10}$, $\alpha_{11}$, $\alpha_{12}$, $\alpha_{13}$, $\theta_{11}$, $\theta_{12}$, and $\theta_{13}$ for emission 1 and $\alpha_{40}, \alpha_{41}, \ldots, \alpha_{44}, \theta_{41}, \ldots, \theta_{44}$ for emission 4. The per unit quantity dependent penalty charges for emission 1 were assigned the values 1, 1.50, 2, 2.50 and larger values (4, 5, 6) were given for emission 4 to account for variation in the disincentives to produce emissions. The quantity dependent charge scheme of Table 2 was assumed for the setting of (2). See document titled Qty depend rates.pdf in the Supplementary Material that accompany this paper. The adaptations to (1) resulted in the following:

Maximize

$$\begin{aligned}&800X_1+700X_2+500X_3+1000X_4+2200X_5+2300X_6\\&\quad+1600X_7+2600X_8+700X_9+1000X_{10}\\&\quad+1300X_{11}+1800X_{12}-50RT_1-100RT_2-3RT_3-5RT_4\\&\quad-2RT_5-(1-S_2)E_4-5E_5^{+}+4E_5^{-}\\&\quad+75S_3XE_1+75S_4XE_2+75S_5XE_3-2RDXE_1-2RDXE_2\\&\quad-S_1(200{,}000\alpha_{11}+225{,}000\alpha_{12}+255{,}000\alpha_{13})\\&\quad-S_2(40{,}000\alpha_{41}+70{,}000\alpha_{42}+110{,}000\alpha_{43}+160{,}000\alpha_{44})\end{aligned} \tag{2.1}$$

subject to
restrictions (1.2)–(1.29)
that are referenced respectively as (2.2)–(2.29) here. (2.2–2.29)

When $S_1 = 1$, (2.30)–(2.37) are operational; otherwise not. (2.30)

$$\alpha_{10}-\theta_{11}\leqslant 0 \tag{2.31}$$
$$\alpha_{11}-\theta_{11}-\theta_{12}\leqslant 0 \tag{2.32}$$
$$\alpha_{12}-\theta_{12}-\theta_{13}\leqslant 0 \tag{2.33}$$
$$\alpha_{13}-\theta_{13}\leqslant 0 \tag{2.34}$$
$$\alpha_{10}+\alpha_{11}+\alpha_{12}+\alpha_{13}=1 \tag{2.35}$$
$$\theta_{11}+\theta_{12}+\theta_{13}=1 \tag{2.36}$$
$$0\leqslant \alpha_{10},\alpha_{11},\alpha_{12},\alpha_{13}\leqslant 1 \tag{2.37}$$
$$E_1=50{,}000\alpha_{11}+55{,}000\alpha_{12}+60{,}000\alpha_{13} \tag{2.38}$$

When $S_2 = 1$, (2.38)–(2.46) are operational; otherwise not. (2.39)

$$\alpha_{40}-\theta_{41}\leqslant 0 \tag{2.40}$$
$$\alpha_{41}-\theta_{41}-\theta_{42}\leqslant 0 \tag{2.41}$$
$$\alpha_{42}-\theta_{42}-\theta_{43}\leqslant 0 \tag{2.42}$$
$$\alpha_{43}-\theta_{43}-\theta_{44}\leqslant 0 \tag{2.43}$$
$$\alpha_{44}-\theta_{44}\leqslant 0 \tag{2.44}$$
$$\alpha_{40}+\alpha_{41}+\alpha_{42}+\alpha_{43}+\alpha_{44}=1 \tag{2.45}$$
$$\theta_{41}+\theta_{42}+\theta_{43}+\theta_{44}=1 \tag{2.46}$$
$$0\leqslant \alpha_{40},\alpha_{41},\alpha_{42},\alpha_{43},\alpha_{44}\leqslant 1 \tag{2.47}$$
$$E_4=40{,}000\alpha_{41}+60{,}000\alpha_{42}+80{,}000\alpha_{43}+100{,}000\alpha_{44} \tag{2.48}$$

$$XE_1=S_3(0.1E_1+0.02E_2+0E_3+0E_4+0E_5) \tag{2.49}$$
$$XE_2=S_4(0.2E_1+0E_2+0.03E_3+0E_4+0E_5) \tag{2.50}$$
$$XE_3=S_5(0.1E_1+0E_2+0E_3+0.25E_4+0E_5) \tag{2.51}$$

$$RSXE_1=0 \tag{2.52}$$
$$RSXE_2=0 \tag{2.53}$$
$$RSXE_3=10S_3XE_1+2XS_3E_2+7S_5XE_3 \tag{2.54}$$
$$RSXE_4=0 \tag{2.55}$$
$$RSXE_5=5S_3XE_1+5S_4XE_2+5S_5XE_3 \tag{2.56}$$

$$RDXE_1=10S_3XE_1+6S_4XE_2+11S_5XE_3 \tag{2.57}$$
$$RDXE_2=8S_3XE_1+15S_4XE_2+7S_5XE_3 \tag{2.58}$$

$$RT_1=R_1+RSXE_1 \tag{2.59}$$
$$RT_2=R_2+RSXE_2 \tag{2.60}$$
$$RT_3=R_3+RSXE_3 \tag{2.61}$$
$$RT_4=R_4+RSXE_4 \tag{2.62}$$
$$RT_5=R_5+RSXE_5 \tag{2.63}$$

$$S_3XE_1+S_4XE_2+S_5XE_3\leqslant 0.10(E_1+E_2+E_3+E_4+E_5) \tag{2.64}$$

$$\begin{aligned}&XE_1,\ XE_2,\ XE_3,\ RSXE_1,\ RSXE_2,\ RSXE_3,\ RSXE_4,\ RSXE_5,\ RT_1,\ RT_2,\\&RT_3,\ RT_4,\ RT_5,\ RDXE_1,\ RDXE_2\geqslant 0\end{aligned} \tag{2.65}$$

$$\theta_{11},\ \theta_{12},\ \theta_{13},\ \theta_{41},\ \theta_{42},\ \theta_{43},\ \theta_{44}=0,\ 1. \tag{2.66}$$

The following convention was adopted in performing IA of (2): 1 denoted accommodation of quantity dependent charges for emission 1 in (2); 2, the same for emission 4; 3, 4, and 5, indicated emissions-to-product 1, 2, and 3 in the amounts $XE_1$, $XE_2$, and $XE_3$, respectively. For these five representations, the streaming sequences $s_v(r)$, $v = 1, \ldots, 32$ of Illustration 1 were used in solving (2).

The results of solving (2) under all combinations of accommodations 1–5 appear in Table 3. The mix quantities $X_1$, $X_2$, $X_4$, $X_5$, $X_8$–$X_{10}$, and $X_{12}$ were invariant among the thirty-two scenarios including the null scenario (row 1). Among the results of Table 3, fifteen scenarios have objective function values smaller than the null scenario's (1,781,188) with at least one of accommodations 1,2,3 is included in each.

Given the solutions to (2) presented in Table 3, IA proceeded in the following manner. Since $o^2(1) = 2{,}017{,}777$ for scenario $s^2(1) = \{0,0,0,1,0\}$ is maximal among the $o^j(1)$, the single best accommodation is 4. Set $\rho_1 = 4$ and $f_1 = 2{,}017{,}777$. For $r = 2$, because the maximal $o^j(2)$ with $s^j(2) = \{*,*,*,1,*\}$ is $o^7(2) = 2{,}227{,}466$ for scenario $s^7(2) = \{0,0,0,1,1\}$, 4 is best paired with 5. Set $\rho_2 = 5$ and $f_2 = 2{,}227{,}466$. For $r = 3$, the maximal $o^j(3)$ for scenarios with $s^j(3) = \{*,*,*,1,1\}$ is $o^{17}(3) = 2{,}215{,}480$ for $s^{17}(3) = \{0,0,1,1,1\}$, accommodations 4 and 5 are best in combination with 3. Set $\rho_3 = 3$ and $f_3 = 2{,}215{,}480$. For $r = 4$, the maximal $o^j(4)$ for scenarios with $s^j(4) = \{*,*,1,1,1\}$ is $o^{27}(4) = 2{,}184{,}161$ with $s^{27}(4) = \{0,1,1,1,1\}$. Then 3,4,5 are best combined with 2. Set $\rho_4 = 2$ and $f_4 = 2{,}184{,}166$. Accommodation 1 is the last addition with $o^{32}(5) = 1{,}929{,}161$ and $s^{32}(5) = \{1,1,1,1,1\}$. Set $\rho_5 = 1$ and $f_5 = 1{,}929{,}161$. In summary, accommodations 4(=$\rho_1$), 5 (=$\rho_2$), 3(=$\rho_3$), 2(=$\rho_4$), and 1(=$\rho_5$) resulted respectively in outcomes values of 2,017,777, 2,227,466, 2,215,480, 2,184,161, and 1,929,161.

The step-by-step analysis revealed that the deterioration in the objective set in with the introduction of accommodation 3 (emissions treatment 1) at step 3 and aggravated thereafter with the stepwise incorporations of accommodations 2 and 1 (quantity dependent emission charges for emissions 4 and 1) in that order. Consequently, step 3 with the addition of accommodation 3 may be looked upon as the tipping point in the step-by-step accommodations. In fact, the addition of accommodation 3 to any scenario did not have a net positive contribution to the objective. Accommodation 1 diminished the objective more than accommodation 2.

If the results of IA for (2) are intended as an assessment of the manner in which quantity dependent emission charges will impact product mix, the following is clear. The new emission charge scheme will significantly reduce profitability. Compare the objective function value $o^1(0) = 1{,}781{,}188$ for the null scenario $s^1(0) = \{0,0,0,0,0\}$ in row 1 of Table 3 to the value $o^{16}(2) = 1{,}500{,}657$ for the scenario $s^{16}(2) = \{1,1,0,0,0\}$ in row 16 that accommodated just the new charge schemes for emissions 1 and 4 (accommodations 1 and 2). However, when the new emission charge scheme was combined with emission treatments 2 and 3 (accommodations 4, 5), profitability improved. If accommodations 1 and 2 are unavoidable and accommodations 3–5 are

**Table 3**
Results of interior analysis for numerical illustration 2.

| Row $j$ | Included accommodations[a] | No. of included accomo-dations $r$ | Scenario $s^j(r)$ | Objective function value $o_j(r)$ | Product mix quantities[b] (in thousands) | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | $X_3$ | $X_6$ | $X_7$ | $X_{11}$ | $XE_1$ | $XE_2$ | $XE_3$ |
| 1 | – | 0 | {0,0,0,0,0} | 1,781,188 | 3.056 | 0 | 0 | 0.107 | 0 | 0 | 0 |
| 2 | 4 | 1 | {0,0,0,1,0} | 2,017,777 | 3.056 | 0 | 0 | 0.107 | 0 | 13.917 | 0 |
| 3 | 5 | 1 | {0,0,0,0,1} | 1,990,877 | 3.092 | 0.152 | 0 | 0 | 0 | 0 | 26.239 |
| 4 | 3 | 1 | {0,0,1,0,0} | 1,769,400 | 3.056 | 0 | 0 | 0.107 | 11.788 | 0 | 0 |
| 5 | 2 | 1 | {0,1,0,0,0} | 1,751,082 | 3.056 | 0 | 0 | 0.107 | 0 | 0 | 0 |
| 6 | 1 | 1 | {1,0,0,0,0} | 1,530,114 | 2.657 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7 | 4, 5 | 2 | {0,0,0,1,1} | 2,227,466 | 3.092 | 0.152 | 0 | 0 | 0 | 13.917 | 26.239 |
| 8 | 3, 4 | 2 | {0,0,1,1,0} | 2,005,989 | 3.056 | 0 | 0 | 0.107 | 11.788 | 13.917 | 0 |
| 9 | 2, 4 | 2 | {0,1,0,1,0} | 1,987,671 | 3.056 | 0 | 0 | 0.107 | 0 | 13.917 | 0 |
| 10 | 3, 5 | 2 | {0,0,1,0,1} | 1,978,891 | 3.092 | 0.152 | 0 | 0 | 11.986 | 0 | 26.239 |
| 11 | 2, 5 | 2 | {0,1,0,0,1} | 1,959,443 | 3.092 | 0.152 | 0 | 0 | 0 | 0 | 26.239 |
| 12 | 1, 4 | 2 | {1,0,0,1,0} | 1,762,777 | 3.056 | 0 | 0 | 0.107 | 0 | 13.917 | 0 |
| 13 | 2, 3 | 2 | {0,1,1,0,0} | 1,739,294 | 3.056 | 0 | 0 | 0.107 | 11.788 | 0 | 0 |
| 14 | 1, 5 | 2 | {1,0,0,0,1} | 1,735,877 | 3.092 | 0.152 | 0 | 0 | 0 | 0 | 26.239 |
| 15 | 1, 3 | 2 | {1,0,1,0,0} | 1,518,650 | 2.657 | 0 | 0 | 0 | 11.465 | 0 | 0 |
| 16 | 1, 2 | 2 | {1,1,0,0,0} | 1,500,657 | 2.657 | 0 | 0 | 0 | 0 | 0 | 0 |
| 17 | 3, 4, 5 | 3 | {0,0,1,1,1} | 2,215,480 | 3.092 | 0.152 | 0 | 0 | 11.986 | 13.917 | 26.239 |
| 18 | 2, 4, 5 | 3 | {0,1,0,1,1} | 2,196,032 | 3.092 | 0.152 | 0 | 0 | 0 | 13.917 | 26.239 |
| 19 | 2, 3, 4 | 3 | {0,1,1,1,0} | 1,975,883 | 3.056 | 0 | 0 | 0.107 | 11.788 | 13.917 | 0 |
| 20 | 1, 4, 5 | 3 | {1,0,0,1,1} | 1,972,466 | 3.092 | 0.152 | 0 | 0 | 0 | 13.917 | 26.239 |
| 21 | 2, 3, 5 | 3 | {0,1,1,0,1} | 1,947,572 | 3.011 | 0 | 138 | 0 | 11.818 | 0 | 26.194 |
| 22 | 1, 3, 4 | 3 | {1,0,1,1,0} | 1,750,959 | 3.056 | 0 | 0 | 0.107 | 11.818 | 13.917 | 0 |
| 23 | 1, 2, 4 | 3 | {1,1,0,1,0} | 1,732,671 | 3.056 | 0 | 0 | 0.107 | 0 | 13.917 | 0 |
| 24 | 1, 3, 5 | 3 | {1,0,1,0,1} | 1,723,895 | 2.657 | 0 | 0 | 0 | 11.465 | 0 | 25.656 |
| 25 | 1, 2, 5 | 3 | {1,1,0,0,1} | 1,705,903 | 2.657 | 0 | 0 | 0 | 0 | 0 | 25.656 |
| 26 | 1, 2, 3 | 3 | {1,1,1,0,0} | 1,489,193 | 2.657 | 0 | 0 | 0 | 11.465 | 0 | 0 |
| 27 | 2, 3, 4, 5 | 4 | {0,1,1,1,1} | 2,184,161 | 3.011 | 0 | 138 | 0 | 11.818 | 13.917 | 26.194 |
| 28 | 1, 3, 4, 5 | 4 | {1,0,1,1,1} | 1,960,480 | 3.092 | 0.152 | 0 | 0 | 11.986 | 13.917 | 26.239 |
| 29 | 1, 2, 4, 5 | 4 | {1,1,0,1,1} | 1,941,032 | 3.092 | 0.152 | 0 | 0 | 0 | 13.917 | 26.239 |
| 30 | 1, 2, 3, 4 | 4 | {1,1,1,1,0} | 1,720,883 | 3.056 | 0 | 0 | 0.107 | 11.788 | 13.917 | 0 |
| 31 | 1, 2, 3.5 | 4 | {1,1,1,0,1} | 1,694,438 | 2.657 | 0 | 0 | 0 | 11.465 | 0 | 25.656 |
| 32 | 1, 2, 3, 4, 5 | 5 | {1,1,1,1,1} | 1,929,161 | 3.011 | 0 | 138 | 0 | 11.818 | 13.917 | 26.194 |

[a] 1 Represents inclusion of quantity dependent rates for emission 1 in (3); 2, inclusion of quantity dependent rates for emission 4 in (3); 3, 4, 5 respectively indicate inclusion of $XE_1$, $XE_2$, $XE_3$ in (3).
[b] Not shown are the following product mix quantities that are the same in each scenario: $X_1 = 1.6$, $X_2 = 2.4$, $X_4 = 3.0$, $X_5 = 2.0$, $X_8 = X_9 = X_{10} = 0$, $X_{12} = 1.8$.

elective, scenario $s^{29}(4) = \{1,1,0,1,1\}$ of row 29 in Table 3 with objective function value $o^{29}(4) = 1{,}941{,}032$ is superior in profitability to the all-inclusive scenario $s^{32}(5) = \{1,1,1,1,1\}$ with objective value $o^{32}(5) = 1{,}929{,}161$. Scenaro $s^{29}(4)$ may serve as a first stage partial implementation of accommodations 1–5.

In summary, much like IA of (1), the results for this illustration revealed the implementation options between the null scenario and the all inclusive scenario in row 32 of Table 3. Among the scenarios in between the two, accommodation 3 (emission treatment 1) unlike accommodations 4 and 5 (emission treatments 2, 3) had a uniform negative impact on the outcome values; the product mix quantities showed small variation; and accommodations 4–5 as modeled had some capacity to offset the impact on the objective of the quantity dependent emission charge scheme. Without accommodations 4 and 5, the consequence (row 16) of the emission charge scheme was a large reduction in the objective. See the document titled Illustration 2 among the Supplementary Material for more details of the solutions to (2).

## 5. Conclusions

When the green product mix solution produces adverse outcomes, interior analysis provides means for investigating the cause. Interior analysis was presented as a tool of analysis and illustrated with green product mix models that incorporated several contemporary formulation innovations. Through the illustrations, demonstration was given to the manner in which the step-by-step one-at-a-time accommodation of emission controls allows the analyst to track how the product mix feature(s) of interest degrades and to which emission control enforcements it may be attributed. Based on the results of IA for the examples, suggested implementation strategies were discussed. For ease of illustration and without loss of applicability, IA was illustrated with LP formulations of a real green product mix problem.

Given the environment of emission regulation, to have one restriction such as (1.12) represent emission control is too limiting. As shown with (1) and (2), product mix modeling has the capacity to accommodate emission control in a variety of forms in addition to volume restrictions. Among others, they include a scheme for selling and procuring emission allowance in an emission trading forum, incorporation of penalties and costs based on volume, as well as converting/treating emissions within the product mix production environment. Whatever the forms of control incorporated in the modeling, IA provides means for investigating their individual impact on mix determination as well as the effect of their interactions in pairs, triples, etc. This approach to post-optimality investigation of product mix modeling has not been well developed in the literature. Interior analysis is proposed as a tool of analysis to complement the methodology of product mix determination.

Further research should be devoted to identifying the feasibility of implicitly eliminating inferior scenarios in the enumeration scheme; use of IA for green product mix models with multiple objectives and fuzzy modeling features; and the applicability of IA to other allocation models.

## Appendix A. Supplementary material

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.ejor.2014.02.029.

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