# Questão 2 - (a)

Resolução

26 de fevereiro de 2024

2)

(a) Calcule:

$$\Delta = \sum_{n=0}^{\infty} \langle 1|x^2|n\rangle\langle n|x^2|1\rangle$$

**RESPOSTA:** Sabemos que:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}x + \frac{ip}{\sqrt{2m\omega\hbar}}$$

e

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}x - \frac{ip}{\sqrt{2m\omega\hbar}}$$

Portanto, podemos obter $x$ somando $\hat{a}$ com $\hat{a}^\dagger$ e o isolando:

$$\begin{aligned} \hat{a} + \hat{a}^\dagger &= \sqrt{\frac{2m\omega}{\hbar}}x \\ x &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \end{aligned}$$

Sabemos pela relação de Dirac que:

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = 1$$

Logo, $\Delta$ se resume a:

$$\Delta = \langle 1|x^2 x^2|1\rangle$$

$$\Delta = \langle 1|x^4|1\rangle$$

Agora, devemos calcular $x^4$ para calcular o valor de $\Delta$:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger)$$

$$x^4 = \frac{\hbar^2}{4m^2\omega^2}(\hat{a} + \hat{a}^\dagger)^4$$

Agora é só substituir em $\Delta$ e aplicar os operadores $\hat{a}$ e $\hat{a}^\dagger$, lembrando que para aplicá-los devemos seguir:

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$
$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

Então, bora lá:

$$\begin{aligned}
\Delta &= \langle 1|x^4|1\rangle \\
\Delta &= \langle 1|\frac{\hbar^2}{4m^2\omega^2}(\hat{a}+\hat{a}^\dagger)^4|1\rangle \\
\Delta &= \frac{\hbar^2}{4m^2\omega^2}\langle 1|(\hat{a}+\hat{a}^\dagger)^4|1\rangle \\
\Delta &= \frac{\hbar^2}{4m^2\omega^2}\langle 1|\hat{a}^4 + \hat{a}^{\dagger 4} + \hat{a}^3\hat{a}^\dagger + \hat{a}^2\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\hat{a}^2 + \hat{a}^\dagger\hat{a}^3 + \hat{a}^{\dagger 3}\hat{a} + \hat{a}^{\dagger 2}\hat{a}\hat{a}^\dagger + \\
&\quad + \hat{a}^\dagger\hat{a}\hat{a}^{\dagger 2} + \hat{a}\hat{a}^{\dagger 3} + \hat{a}^2\hat{a}^{\dagger 2} + \hat{a}^{\dagger 2}\hat{a}^2 + \hat{a}\hat{a}^{\dagger 2}\hat{a} + \hat{a}^\dagger\hat{a}^2\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger|1\rangle \\
\Delta &= \frac{\hbar^2}{4m^2\omega^2}\Big(\langle 1|\hat{a}^4|1\rangle + \langle 1|\hat{a}^{\dagger 4}|1\rangle + \langle 1|\hat{a}^3\hat{a}^\dagger|1\rangle + \langle 1|\hat{a}^2\hat{a}^\dagger\hat{a}|1\rangle + \\
&\quad + \langle 1|\hat{a}\hat{a}^\dagger\hat{a}^2|1\rangle + \langle 1|\hat{a}^\dagger\hat{a}^3|1\rangle + \langle 1|\hat{a}^{\dagger 3}\hat{a}|1\rangle + \langle 1|\hat{a}^{\dagger 2}\hat{a}\hat{a}^\dagger|1\rangle + \\
&\quad + \langle 1|\hat{a}^\dagger\hat{a}\hat{a}^{\dagger 2}|1\rangle + \langle 1|\hat{a}\hat{a}^{\dagger 3}|1\rangle + \langle 1|\hat{a}^2\hat{a}^{\dagger 2}|1\rangle + \langle 1|\hat{a}^{\dagger 2}\hat{a}^2|1\rangle + \\
&\quad + \langle 1|\hat{a}\hat{a}^{\dagger 2}\hat{a}|1\rangle + \langle 1|\hat{a}^\dagger\hat{a}^2\hat{a}^\dagger|1\rangle + \langle 1|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}|1\rangle + \langle 1|\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger|1\rangle\Big) \\
\Delta &= \frac{\hbar^2}{4m^2\omega^2}\Big(\langle 1|\hat{a}^3\sqrt{1}|0\rangle + \langle 1|\hat{a}^{\dagger 3}\sqrt{2}|2\rangle + \langle 1|\hat{a}^3\sqrt{2}|2\rangle + \langle 1|\hat{a}^2\hat{a}^\dagger\sqrt{1}|0\rangle + \\
&\quad + \langle 1|\hat{a}\hat{a}^\dagger\hat{a}\sqrt{1}|0\rangle + \langle 1|\hat{a}^\dagger\hat{a}^2\sqrt{1}|0\rangle + \langle 1|\hat{a}^{\dagger 3}\sqrt{1}|0\rangle + \langle 1|\hat{a}^{\dagger 2}\hat{a}\sqrt{2}|2\rangle + \\
&\quad + \langle 1|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\sqrt{2}|2\rangle + \langle 1|\hat{a}\hat{a}^{\dagger 2}\sqrt{2}|2\rangle + \langle 1|\hat{a}^2\hat{a}^\dagger\sqrt{2}|2\rangle + \langle 1|\hat{a}^{\dagger 2}\hat{a}\sqrt{1}|0\rangle + \\
&\quad + \langle 1|\hat{a}\hat{a}^{\dagger 2}\sqrt{1}|0\rangle + \langle 1|\hat{a}^\dagger\hat{a}^2\sqrt{2}|2\rangle + \langle 1|\hat{a}^\dagger\hat{a}\hat{a}^\dagger\sqrt{1}|0\rangle + \langle 1|\hat{a}\hat{a}^\dagger\hat{a}\sqrt{2}|2\rangle\Big) \\
\Delta &= \frac{\hbar^2}{4m^2\omega^2}\Big(\cancelto{0}{\langle 1|\hat{a}^2\sqrt{0}\sqrt{1}|-1\rangle} + \langle 1|\hat{a}^{\dagger 2}\sqrt{3}\sqrt{2}|3\rangle + \langle 1|\hat{a}^2\sqrt{2}\sqrt{2}|1\rangle + \langle 1|\hat{a}^2\sqrt{1}\sqrt{1}|1\rangle + \\
&\quad + \cancelto{0}{\langle 1|\hat{a}\hat{a}^\dagger\sqrt{0}\sqrt{1}|-1\rangle} + \cancelto{0}{\langle 1|\hat{a}^\dagger\hat{a}\sqrt{0}\sqrt{1}|-1\rangle} + \langle 1|\hat{a}^{\dagger 2}\sqrt{1}\sqrt{1}|1\rangle + \langle 1|\hat{a}^{\dagger 2}\sqrt{2}\sqrt{2}|1\rangle + \\
&\quad + \langle 1|\hat{a}^\dagger\hat{a}\sqrt{3}\sqrt{2}|3\rangle + \langle 1|\hat{a}\hat{a}^\dagger\sqrt{3}\sqrt{2}|3\rangle + \langle 1|\hat{a}^2\sqrt{3}\sqrt{2}|3\rangle + \cancelto{0}{\langle 1|\hat{a}^{\dagger 2}\sqrt{0}\sqrt{1}|-1\rangle} + \\
&\quad + \langle 1|\hat{a}\hat{a}^\dagger\sqrt{1}\sqrt{1}|1\rangle + \langle 1|\hat{a}^\dagger\hat{a}\sqrt{2}\sqrt{2}|1\rangle + \langle 1|\hat{a}^\dagger\hat{a}\sqrt{1}\sqrt{1}|1\rangle + \langle 1|\hat{a}\hat{a}^\dagger\sqrt{2}\sqrt{2}|1\rangle\Big) \\
\Delta &= \frac{\hbar^2}{4m^2\omega^2}\Big(\langle 1|\hat{a}^\dagger\sqrt{4}\sqrt{3}\sqrt{2}|4\rangle + \langle 1|\hat{a}\sqrt{1}\sqrt{2}\sqrt{2}|0\rangle + \langle 1|\hat{a}\sqrt{1}\sqrt{1}\sqrt{1}|0\rangle + \langle 1|\hat{a}^\dagger\sqrt{2}\sqrt{1}\sqrt{1}|2\rangle \\
&\quad + \langle 1|\hat{a}^\dagger\sqrt{2}\sqrt{2}\sqrt{2}|2\rangle + \langle 1|\hat{a}^\dagger\sqrt{3}\sqrt{3}\sqrt{2}|2\rangle + \langle 1|\hat{a}\sqrt{4}\sqrt{3}\sqrt{2}|4\rangle + \langle 1|\hat{a}\sqrt{3}\sqrt{3}\sqrt{2}|2\rangle + \\
&\quad + \langle 1|\hat{a}\sqrt{2}\sqrt{1}\sqrt{1}|2\rangle + \langle 1|\hat{a}^\dagger\sqrt{1}\sqrt{2}\sqrt{2}|0\rangle + \langle 1|\hat{a}^\dagger\sqrt{1}\sqrt{1}\sqrt{1}|0\rangle + \langle 1|\hat{a}\sqrt{2}\sqrt{2}\sqrt{2}|2\rangle\Big) \\
\Delta &= \frac{\hbar^2}{4m^2\omega^2}\Big(\langle 1|\sqrt{5}\sqrt{4}\sqrt{3}\sqrt{2}|5\rangle + \cancelto{0}{\langle 1|\sqrt{0}\sqrt{1}\sqrt{2}\sqrt{2}|-1\rangle} + \cancelto{0}{\langle 1|\sqrt{0}\sqrt{1}\sqrt{1}\sqrt{1}|-1\rangle} + \langle 1|\sqrt{3}\sqrt{2}\sqrt{1}\sqrt{1}|3\rangle \\
&\quad + \langle 1|\sqrt{3}\sqrt{2}\sqrt{2}\sqrt{2}|3\rangle + \langle 1|\sqrt{3}\sqrt{3}\sqrt{3}\sqrt{2}|3\rangle + \langle 1|\sqrt{4}\sqrt{4}\sqrt{3}\sqrt{2}|3\rangle + \langle 1|\sqrt{2}\sqrt{3}\sqrt{3}\sqrt{2}|1\rangle + \\
&\quad + \langle 1|\sqrt{2}\sqrt{2}\sqrt{1}\sqrt{1}|1\rangle + \langle 1|\sqrt{1}\sqrt{1}\sqrt{2}\sqrt{2}|1\rangle + \langle 1|\sqrt{1}\sqrt{1}\sqrt{1}\sqrt{1}|1\rangle + \langle 1|\sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2}|1\rangle\Big)
\end{aligned}$$

$$
\begin{aligned}
\Delta &= \frac{\hbar^2}{4m^2\omega^2}\Big[\Big(\sqrt{5}\sqrt{4}\sqrt{3}\sqrt{2}\Big)\cancelto{0}{\langle 1|5\rangle} + \Big(\sqrt{3}\sqrt{2}\sqrt{1}\sqrt{1}\Big)\cancelto{0}{\langle 1|3\rangle} + \Big(\sqrt{3}\sqrt{2}\sqrt{2}\sqrt{2}\Big)\cancelto{0}{\langle 1|3\rangle} + \\
&\quad + \Big(\sqrt{3}\sqrt{3}\sqrt{3}\sqrt{2}\Big)\cancelto{0}{\langle 1|3\rangle} + \Big(\sqrt{4}\sqrt{4}\sqrt{3}\sqrt{2}\Big)\cancelto{0}{\langle 1|3\rangle} + \Big(\sqrt{2}\sqrt{3}\sqrt{3}\sqrt{2}\Big)\cancelto{1}{\langle 1|1\rangle} + \Big(\sqrt{2}\sqrt{2}\sqrt{1}\sqrt{1}\Big)\cancelto{1}{\langle 1|1\rangle} + \\
&\quad + \Big(\sqrt{1}\sqrt{1}\sqrt{2}\sqrt{2}\Big)\cancelto{1}{\langle 1|1\rangle} + \Big(\sqrt{1}\sqrt{1}\sqrt{1}\sqrt{1}\Big)\cancelto{1}{\langle 1|1\rangle} + \Big(\sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2}\Big)\cancelto{1}{\langle 1|1\rangle}\Big] \\
\Delta &= \frac{\hbar^2}{4m^2\omega^2}\Big[\Big(\sqrt{2}\sqrt{3}\sqrt{3}\sqrt{2}\Big) + \Big(\sqrt{2}\sqrt{2}\sqrt{1}\sqrt{1}\Big) + \Big(\sqrt{1}\sqrt{1}\sqrt{2}\sqrt{2}\Big) + \Big(\sqrt{1}\sqrt{1}\sqrt{1}\sqrt{1}\Big) + \Big(\sqrt{2}\sqrt{2}\sqrt{2}\sqrt{2}\Big)\Big] \\
\Delta &= \frac{\hbar^2}{4m^2\omega^2}\Big[6 + 2 + 2 + 1 + 4\Big] \\
\Delta &= \frac{15\hbar^2}{4m^2\omega^2}
\end{aligned}
$$

Portanto, a resposta é:

$$
\Delta = \frac{15\hbar^2}{4m^2\omega^2}
$$