An introduction to quantum information and computation

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Lecture 6: The theory of entanglement

1) Recap on entanglement

We said early on in these lecture notes that given a composite quantum system, for instance a two-qubit system, composed of the qubit of Alice and of the qubit of Bob, and the state

$$|\Psi\rangle \in \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$|\Psi\rangle$ is entangled if it is not a product state.

Product state: $|\Psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$.

In the two-qubit setting, an example of entangled state that we saw are the four Bell states:

$$\frac{|01\rangle \pm |10\rangle}{\sqrt{2}} \qquad \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$$

Yet, there are reasons to be unhappy with this definition.

First, it is a negative definition. Entangled is whatever is not a product state. It would be better to have a positive definition.

Second, it is not quantitative at all. Let us consider these two entangled states:

$$\frac{|01\rangle + |10\rangle}{\sqrt{2}} \qquad \sqrt{0.99}\,|01\rangle + \sqrt{0.01}\,|10\rangle.$$

In the first case we have a maximally mixed state.

In the second case we have an entangled state that is basically a product state, because its almost $\sim |01\rangle$.
It would be very important to have a quantitative measure of entanglement, something that could say that some states are more entangled than others.

2) A view on entanglement through measurement correlations.

Let us consider a two-qubit system. Alice and Bob have one qubit each. We consider the joint observable

$$\hat{A} \otimes \hat{B}$$

Acts on Alice's qubit

Acts on Bob's qubit.

The 2-qubit system is described by $|\psi\rangle$ and the expectation value of AB is:

$$\langle AB \rangle = \langle \psi | \hat{A} \otimes \hat{B} | \psi \rangle$$

$|\psi\rangle$ is a product state $\Longrightarrow$ $\langle AB \rangle = \langle A \rangle \cdot \langle B \rangle$.

PROOF. Simple implication. $|\psi\rangle$ is of the form $|\phi_A\rangle|\phi_B\rangle$.
Hence,

$$\langle \psi | \hat{A}\hat{B} | \psi \rangle = \langle \phi_A | \hat{A} | \phi_A \rangle \cdot \langle \phi_B | \hat{B} | \phi_B \rangle .$$

$$\langle A \rangle = \langle \psi | A \otimes \mathbb{1}_B | \psi \rangle = \langle \phi_A | \hat{A} | \phi_A \rangle \underbrace{\langle \phi_B | \phi_B \rangle}_{=1}$$

$$\langle B \rangle = \langle \psi | \mathbb{1}_A \otimes B | \psi \rangle = \langle \phi_A | \phi_A \rangle \langle \phi_B | \hat{B} | \phi_B \rangle .$$

The implication follows. $\blacksquare$

A product state is a state without measurement correlations between the two parties.

This means that if Alice wants to measure an observable A on her qubit she can simply forget about the existence of Bob's qubit.

3) A view on entanglement via the ensemble interpretation of quantum mechanics.

Let us consider a 2-qubit system, one for Alice and one for Bob. The initial state is a pure state of this form:

$$|\Psi\rangle = |\phi_A\rangle \otimes |+_B\rangle$$

where $|\phi_A\rangle$ does not need to be specified and instead $|+\rangle = \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$ for Bob's qubit.

According to the ensemble interpretation of quantum mechanics, to speak of $|\Psi\rangle$ means that we have a large number of systems that are identically prepared in the 2-qubit state $|\Psi\rangle$.

$\boxed{\lvert\Psi\rangle}$	$\boxed{\lvert\Psi\rangle}$	$\boxed{\lvert\Psi\rangle}$	$\boxed{\lvert\Psi\rangle}$	$\boxed{\lvert\Psi\rangle}$	...	$\boxed{\lvert\Psi\rangle}$
1	2	3	4		...	N

On each of these experiments we perform a measurement of Z_B ONLY of Bob's qubit. Alice does not do any measurement. What happens?

Half of the systems is projected to the state $|\phi_A\rangle \otimes |0_B\rangle$.

Half of the systems is projected to the state $|\phi_A\rangle \otimes |1_B\rangle$

$\boxed{|\phi_A\rangle|1_B\rangle}$ $\boxed{|\phi_A\rangle|0_B\rangle}$ $\boxed{|\phi_A\rangle|0_B\rangle}$ $\boxed{|\phi_A\rangle|1_B\rangle}$ $\cdots$ $\boxed{|\phi_A\rangle|0_B\rangle}$

In principle our ensemble of systems is a statistical mixture but we soon realize that if we discard Bob's qubit and we only focus on Alice's qubit we actually have a pure state!

$\boxed{|\phi_A\rangle}$ $\boxed{|\phi_A\rangle}$ $\boxed{|\phi_A\rangle}$ $\boxed{|\phi_A\rangle}$ $\boxed{|\phi_A\rangle}$ $\cdots$ $\boxed{|\phi_A\rangle}$

This fact is a direct consequence of the fact that we assumed a product state. Very generally, for the state

$$|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$$

and considering the generic observable $\hat{B}$ for Bob's qubit with eigenvalues $\{b_1, b_2\}$ and eigenvectors $\{|b_1\rangle, |b_2\rangle\}$, after a measurement of $\hat{B}$

$N_1 = |\langle b_1|\phi_B\rangle|^2 \times N$ setups are projected to $|\phi_A\rangle|b_1\rangle$

$N_2 = |\langle b_2|\phi_B\rangle|^2 \times N$ setups are projected to $|\phi_A\rangle|b_2\rangle$

If we discard Bob's qubit, Alice's qubit is described by a pure state, $|\phi_A\rangle$.

This is in general not true if we consider a two qubit system that is entangled, for instance:

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

According to the ensemble interpretation of quantum mechanics:

$$\boxed{|\psi\rangle} \quad \boxed{|\psi\rangle} \quad \boxed{|\psi\rangle} \quad \boxed{|\psi\rangle} \quad \boxed{|\psi\rangle} \quad \cdots \quad \boxed{|\psi\rangle}$$

We now perform a measurement of Z_B on Bob's qubit.

Half of the systems get projected to $|00\rangle$

Half of the systems get projected to $|11\rangle$

The N systems are now described by a statistical mixture:

$$\left\{ \frac{1}{2}, |00\rangle \; ; \; \frac{1}{2}, |11\rangle \right\}$$

Differently from what we discussed before, this remains true also when we discard Bob's qubit. Alice qubit is described by a statistical mixture

$$\left\{ \frac{1}{2}, |0_A\rangle \; ; \; \frac{1}{2}, |1_A\rangle \right\}$$

This fact that after discarding Bob's qubit we get a statistical mixture is at the heart of the theory of entanglement.
In simple words it says that an entangled state contain correlations between Alice and Bob qubits. If Bob's qubit is discarded, then we lose information and what was a pure state becomes a statistical mixture.

4) Partial trace

We now discuss a mathematical technique that can be used to formalize this idea of discarding Bob's qubit.

Let us consider ρ, the density matrix the describes the two-qubit system composed of Alice's and Bob's qubits.

We define the reduced density matrix ρ_A the density matrix that is obtained by performing a partial trace on Bob's qubit:

$$\rho_A = \sum_i \langle i_B | \rho | i_B \rangle = Tr_B[\rho]$$

ρ_A is a density matrix only for Alice's qubit.

- Example. Take $|\psi_{AB}\rangle = |00\rangle = |0_A\rangle \otimes |0_B\rangle$

$$\rho_A = tr_B\left[|00\rangle\langle 00| \right] = \langle 0_B| \; |0\rangle_A |0\rangle_B \langle 0|_A \langle 0|_B \; |0_B\rangle$$
$$+ \langle 1_B | \; |0\rangle_A |0\rangle_B \langle 0|_A \langle 0|_B \; |1_B\rangle$$
$$= |0_A\rangle\langle 0_A|$$

- Example. Take $|\psi_{AB}\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$ and thus

$$\rho = \frac{1}{2}\left(|00\rangle + |11\rangle \right)\left(\langle 00| + \langle 11| \right)$$
$$= \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11| \right)$$

Let us now compute ρ_A

$$\rho_A = \langle 0_B|\rho|0_B\rangle + \langle 1_B|\rho|1_B\rangle$$

$$= \frac{1}{2}|0_A\rangle\langle 0_A| + \frac{1}{2}|1_A\rangle\langle 1_A| = \frac{1}{2}\mathbb{1}.$$

Example: $\rho = \frac{1}{3}\left(|00\rangle + |01\rangle + |11\rangle\right)\left(\langle 00| + \langle 01| + \langle 11|\right)$

$$\rho_A = \langle 0_B|\rho|0_B\rangle + \langle 1_B|\rho|1_B\rangle$$

$$= \frac{1}{3}|0_A\rangle\langle 0_A| + \frac{1}{3}\left(|0_A\rangle + |1_A\rangle\right)\left(\langle 0_A| + \langle 1_A|\right)$$

$$= \frac{2}{3}|0_A\rangle\langle 0_A| + \frac{1}{3}|1_A\rangle\langle 1_A| + \frac{1}{3}\left(|0_A\rangle\langle 1_A| + |1_A\rangle\langle 0_A|\right)$$

Can we prove that in general ρ_A is a density matrix?

Let us start from the definition $\rho_A = \langle 0_B|\rho|0_B\rangle + \langle 1_B|\rho|1_B\rangle$, where ρ is a well-defined 2-qubit density matrix.

- Hermiticity. $\rho_A^\dagger = \langle 0_B|\rho^\dagger|0_B\rangle + \langle 1_B|\rho^\dagger|1_B\rangle \underset{\rho=\rho^\dagger}{=} \rho_A$

- $\mathrm{tr}\,\rho_A = 1$. $\rightarrow$ $\mathrm{tr}\,\rho_A = \langle 0_A|\rho_A|0_A\rangle + \langle 1_A|\rho_A|1_A\rangle =$

$$= \langle 0_A 0_B|\rho|0_A 0_B\rangle + \langle 0_A 1_B|\rho|0_A 1_B\rangle +$$

$$+ \langle 1_A 0_B|\rho|1_A 0_B\rangle + \langle 1_A 1_B|\rho|1_A 1_B\rangle$$

We are infact taking the trace over a basis of the 2-qubit Hilbert space!

Hence: $\mathrm{tr}\,\rho_A = \mathrm{tr}\,\rho = 1$. Good!

- Positivity. $\langle \phi_A | \rho_A | \phi_A \rangle \geq 0 \quad \forall |\phi_A\rangle$.

We show by taking the explicit expression:

$$\langle \phi_A | \rho_A | \phi_A \rangle = \langle \phi_A 0_B | \rho | \phi_A 0_B \rangle + \langle \phi_A 1_B | \rho | \phi_A 1_B \rangle$$

Using the fact that ρ is positive, both terms of the sum are ≥ 0 and hence:

$$\langle \phi_A | \rho_A | \phi_A \rangle \geq 0 \quad \forall |\phi_A\rangle.$$

5) Rényi-2 entanglement entropy.

If we get back to the examples we discussed above:

$$|\Psi_{AB}\rangle = |00\rangle \longrightarrow \rho_A = |0\rangle\langle 0|$$

$$|\Psi_{AB}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \longrightarrow \rho_A = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

As we had intuitively expected, when $|\Psi_{AB}\rangle$ is a product state, ρ_A is a pure state. For $|\Psi_{AB}\rangle$ which is entangled, we find that ρ_A is a statistical mixture (mixed state).

The idea is to base our quantitative theory of entanglement on the purity of the reduced density matrix.

$$\mathrm{Tr}\left[\rho_A^2\right]$$

We thus define the second Rényi entropy:

$$S_2(\rho_A) = -\log\left(\mathrm{tr}[\rho_A^2]\right)$$

Theorem Given a 2-qubit pure state $|\Psi_A\rangle$, the reduced density matrix ρ_A is pure IF AND ONLY IF $|\Psi_{AB}\rangle$ is a product state.

Corollary

- $S_2(\rho_A) = 0$ IF AND ONLY IF $|\Psi_{AB}\rangle$ is a product state.
- $S_2(\rho_A) > 0$ IF AND ONLY IF $|\Psi_{AB}\rangle$ is entangled.
- $S_2(\rho_A)$ has maximal value when ρ_A has lowest possible purity. For a single-qubit density matrix, the minimal value of the purity is $1/2$. Hence the maximal value of $S_2(\rho_A)$ is $\log 2$.
- $S_2(\rho_A)$ is a quantitative measure of entanglement for pure states $|\Psi_{AB}\rangle$.

PROOF OF THE THEOREM

PRODUCT STATE $\Longrightarrow$ ρ_A is pure.

$$|\Psi_{AB}\rangle = |\phi_A\rangle|\phi_B\rangle \longrightarrow \rho = |\phi_A\rangle|\phi_B\rangle\langle\phi_A|\langle\phi_B|$$

$\rho_A = \mathrm{Tr}_B[\rho]$. In order to perform the trace, we take as orthonormal basis of $\mathcal{H}_B$ the states $\{|\phi_B\rangle, |\phi_B'\rangle\}$, with $\langle\phi_B|\phi_B'\rangle = 0$.

Hence:

$$\rho_A = \langle \phi_B | \rho | \phi_B \rangle + \langle \phi_B' | \rho | \phi_B' \rangle = |\phi_A\rangle\langle\phi_A| \text{ which is pure.}$$

ρ_A is pure $\Longrightarrow$ $|\psi_{AB}\rangle$ is a product state.

Consider $\rho_A = |\phi_A\rangle\langle\phi_A|$

We get back to $|\psi_{AB}\rangle$ and in general I can put in evidence a term with $|\phi_A\rangle$:

$$|\psi_{AB}\rangle = \alpha |\phi_A\rangle|\phi_B\rangle + \beta |\phi_A'\rangle|\psi_B\rangle$$

I know that $\langle\phi_A|\phi_A\rangle = \langle\phi_A'|\phi_A'\rangle = 1$ and $|\phi_A'\rangle$, that is, they form an orthonormal basis of $\mathcal{H}_A$.

If I want that $\rho_A = |\phi_A\rangle\langle\phi_A|$ then necessarily $\beta = 0$ because if $\beta \neq 0$ then I will for sure have other terms proportional to

- $|\phi_A'\rangle\langle\phi_A'|$
- $|\phi_A\rangle\langle\phi_A'|$
- $|\phi_A'\rangle\langle\phi_A|$

But then, if $\beta = 0$, then: $|\psi_{AB}\rangle = |\phi_A\rangle|\phi_B\rangle$, which is a product state.

∎

Remark: This proof could have been carried out in a most clear and elegant form by considering the

Schmidt decomposition of the state $|\Psi_{AB}\rangle$. We will not discuss this in the context of these lectures.

Why a log? Why not just the purity? This is related to the notion of additivity, and of entropy.

I require that for a product state $|\Psi_{AB}\rangle$, the entanglement entropy of the total system matches the sum of the entanglement entropies of the two subparts:

For $|\Psi_{AB}\rangle$ a product state

$$\underbrace{S(\rho_{AB})}_{\substack{\text{Pure state}\\ =0}} = \underbrace{S(\rho_A)}_{\substack{\text{Pure state}\\ =0}} + \underbrace{S(\rho_B)}_{\substack{\text{Pure state}\\ =0}}$$

The result is more general and holds in general for an uncorrelated state

$$\rho_{AB} = \rho_A \otimes \rho_B$$

Be careful: this state can be mixed!

ρ_A is the reduced density matrix: $= \mathrm{Tr}_B[\rho_{AB}]$

ρ_B is the reduced density matrix: $= \mathrm{Tr}_A[\rho_{AB}]$

Let us now compute the second Rényi entropy of ρ_{AB}:

$$-\log \mathrm{tr}[\rho_{AB}^2] = -\log \mathrm{tr}[\rho_A^2 \otimes \rho_B^2]$$

We now use the fact that: $\mathrm{tr}[O] = \mathrm{Tr}_A\left[\mathrm{tr}_B[O]\right]$. (We prove it below).

$$\cdot \; \mathrm{Tr}_B[\rho_A^2 \otimes \rho_B^2] = \rho_A^2 \cdot \mathrm{tr}_B[\rho_B^2]$$

- $\mathrm{tr}_A\left[\mathrm{tr}_B\left[\rho_A^2 \otimes \rho_B^2\right]\right] = \mathrm{tr}_A\left[\rho_A^2\right] \cdot \mathrm{tr}_B\left[\rho_B^2\right]$

Hence: $-\log \mathrm{tr}\left[\rho_{AB}^2\right] = -\log \mathrm{tr}_A\left[\rho_A^2\right] \mathrm{tr}_B\left[\rho_B^2\right] = -\log\left(\mathrm{tr}_A\left[\rho_A^2\right]\right) - \log\left(\mathrm{tr}_B\left[\rho_B^2\right]\right)$

From which we derive

$$S_2(\rho_A \otimes \rho_B) = S_2(\rho_A) + S_2(\rho_B)$$

6) Rényi and von-Neumann entanglement entropies

The Rényi-2 entanglement entropy is one of the many entanglement entropies that can be defined for a quantum system that can be bipartite.
We have seen that given $|\psi_{AB}\rangle$ a two-qubit system, we can define

$$S_2(\rho_A) = -\log\left(\mathrm{tr}[\rho_A^2]\right)$$

with ρ_A the reduced density matrix.
In general, we define the Rényi entropy of order n

$$S_n(\rho_A) = \frac{1}{1-n} \log\left(\mathrm{tr}[\rho_A^n]\right) \qquad n \geq 2 \text{ and integer.}$$

There is another quantity, the von Neumann entanglement entropy

$$S_{VN}(\rho_A) = -\mathrm{tr}\left[\rho_A \log(\rho_A)\right]$$

All these quantities share the property that $S_n(\rho_A) = 0$ and $S_{VN}(\rho_A) = 0$ when $|\psi_{AB}\rangle$ is a product state. When the state $|\psi_{AB}\rangle$ is entangled, then $S_n(\rho_A) > 0$ and $S_{VN}(\rho_A) > 0$.

Note that all the entanglement entropies that we introduced are non-linear function of ρ_A. We now that for a single-qubit observable, its expectation value can be written as:

$$\langle \hat{O} \rangle = \mathrm{tr}[\rho_A \hat{O}].$$

The entanglement entropies have a definition that at first sight is very similar to that. In fact is different because is not linear in ρ_A.

$$\mathrm{Tr}[\rho_A^n] \text{ is qualitatively different from } \mathrm{Tr}[\rho_A O].$$

In summary, entanglement entropies are not associated to observables.

Measuring an entanglement entropy is possible but requires tricks!

7) Why speaking of entropies?

Let us consider ρ_A $\to$ It is an Hermitian operator, hence I can put it in diagonal form:

$$\rho_A = \sum_{\alpha=1}^{2} \lambda_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$$

(for a single qubit I only have two eigenvalues).

The entropies we just wrote are non-linear function of the eigenvalues

- $S_2(\rho_A) = -\log(\lambda_1^2 + \lambda_2^2)$
- $S_n(\rho_A) = \dfrac{1}{1-n} \log(\lambda_1^n + \lambda_2^n)$
- $S_{VN}(\rho_A) = -(\lambda_1 \log(\lambda_1) + \lambda_2 \log(\lambda_2))$

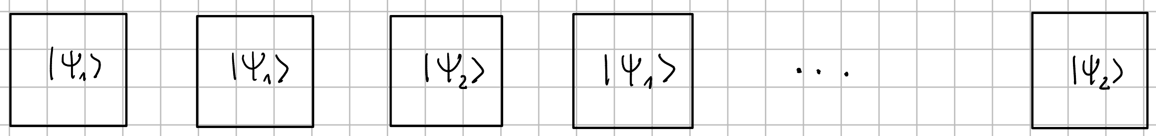
Remember that $\{|\psi_1\rangle, |\psi_2\rangle\}$ form an orthonormal basis: $\langle\psi_i|\psi_j\rangle = \delta_{ij}$.

In this basis, ρ_A has a very simple matrix form:

$$\rho_A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

We said that the diagonal matrix elements can be interpreted as probabilities. ρ_A thus represents the following single-qubit statistical mixture:

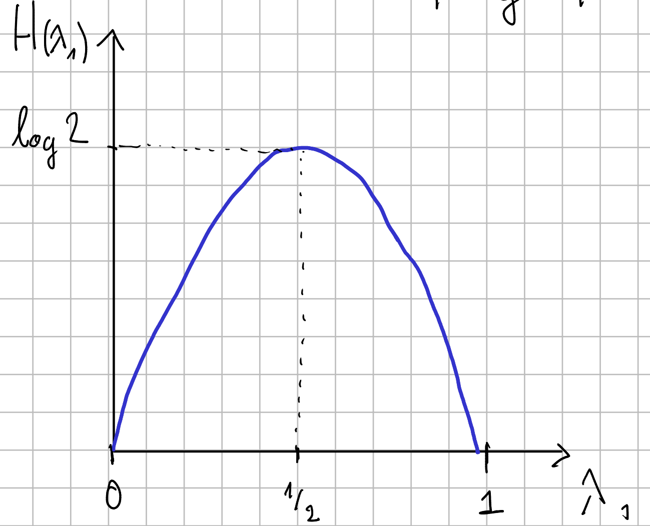
$$\left\{ |\Psi_1\rangle, \lambda_1 \; ; \; |\Psi_2\rangle, \lambda_2 \right\}$$



If we measure ρ_A, with probability λ_1 I will get information on $|\Psi_1\rangle$. With probability λ_2 I will get information on $|\Psi_2\rangle$.

Idea: associate an entropy to this stochastic process.
We define the Shannon entropy:

$$H(\lambda_1) = -\lambda_1 \log \lambda_1 - \lambda_2 \log \lambda_2$$
$$= -\lambda_1 \log \lambda_1 - (1-\lambda_1) \log (1-\lambda_1)$$



$$H(\lambda_1) = -\mathrm{tr}\left[\rho_A \log \rho_A\right] = S_{VN}(\rho_A)$$

The Shannon entropy is a generalization of the Boltzmann entropy, which is defined when all probabilities are the same.

- Boltzmann: I have d equiprobable microstates, $p = 1/d$

$$S_B = + \log d = - \log p = - \sum_i p \log p$$

- Shannon: $S_S = - \sum_i p_i \log p_i$ works in general for configurations that are not equiprobable.

Shannon entropy is considered to be a measure of surprise of a random variable.

- Large Shannon entropy: it is very difficult to say in advance the outcome of the random variable.
  High surprise.
  In our case, it corresponds to $\lambda_1 = \lambda_2 = \frac{1}{2}$
  $S = \log 2$

- Small Shannon entropy: it is very easy to say in advance the outcome of the random variable.
  Little surprise.
  In our case, it corresponds to $\lambda_1 \sim 1$ and $\lambda_2 \sim 0$.

Let us now reinterprete from this standpoint the result on the additivity of the entanglement entropy for an uncorrelated state

$$S_2(\rho_A \otimes \rho_B) = S_2(\rho_A) + S_2(\rho_B)$$

It is as difficult to predict the properties of the global system as to separately predict the properties of A and B.

In the presence of entanglement between A and B this is not true. For $|\psi_{AB}\rangle$ pure

$$0 = S_2(\rho_{AB}) \leq S_2(\rho_A) + S_2(\rho_B)$$

This means that predicting the properties of the total system is easier than predicting separately the local properties of A and B.

Take for instance $|\psi_{AB}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

Globally, we know very well with which state we are dealing with. Locally, this is not true, as it looks as a fully mixed state:

$$\rho_A = \frac{1}{2}\mathbb{1} \qquad \rho_B = \frac{1}{2}\mathbb{1}$$

That is why we say that entanglement is a form of non-local quantum correlations among the different parties that compose a quantum system.