

# Linear Classification & Logistic Regression

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# Recap

## Lecture 2: Linear Regression

- Linear Models overview
- Regression problem statement
- Linear Regression analytical solution
  - Gauss-Markov theorem (BLUE)
  - Instability
- Regularization
  - L2 aka Ridge
    - Analytical solution
  - L1 aka LASSO
    - Weights decay rule
  - Elastic Net
- Metrics in regression
- Model building cycle
  - Train
  - Validation
  - Test

# Outline

- Linear classification
  - margin
  - loss functions
- Logistic regression
  - sigmoid derivation
  - Maximum Likelihood Estimation (MLE)
  - logistic loss
  - probability calibration
- Multiclass aggregation strategies
  - One vs Rest
  - One vs One
- Metrics in classification
  - Accuracy, Balanced accuracy
  - Precision, Recall, F-score
  - ROC curve, PR curve, AUC
  - Confusion matrix

# Linear Classification

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# Classification problem



$$X \in R^{n \times p}$$

$$Y \in C^n$$

$$\text{e.g. } C = \{-1, 1\}$$

$$|C| < +\infty$$

$$c(X) = \hat{Y} \approx Y$$



# Linear classifier

The most simple linear classifier

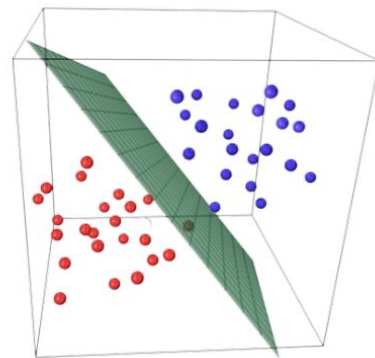
$$c(x) = \begin{cases} 1, & \text{if } f(x) \geq 0 \\ -1, & \text{if } f(x) < 0 \end{cases}$$

or equivalently

$$c(x) = \text{sign}(f(x)) = \text{sign}(x^T w)$$

Geometrical interpretation:  
hyperplane dividing space into two  
subspaces

Why cutoff value is fixed?  
(bias term is implied)





# Margin

Let's define linear model's Margin as

$$M_i = y_i \cdot f(x_i) = y_i \cdot x_i^T w$$

main property:

negative margin reveals misclassification

$$M_i > 0 \Leftrightarrow y_i = c(x_i)$$

$$M_i \leq 0 \Leftrightarrow y_i \neq c(x_i)$$



# Weights choice

Remembering old paradigm

$$\text{Empirical risk} = \sum_{\text{by object}} \text{Loss on object} \longrightarrow \text{Min model params}$$

Essential loss is misclassification

$$\begin{aligned} L_{\text{mis}}(y_i^t, y_i^p) &= [y_i^t \neq y_i^p] = \\ &= [M_i \leq 0] \end{aligned}$$

Disadvantages

- Not differentiable
- Overlooks confidence

Solution:

estimate it with a smooth function

Iverson bracket  $[P] = \begin{cases} 1, & \text{if } P \text{ is true} \\ 0, & \text{otherwise} \end{cases}$



# Square loss

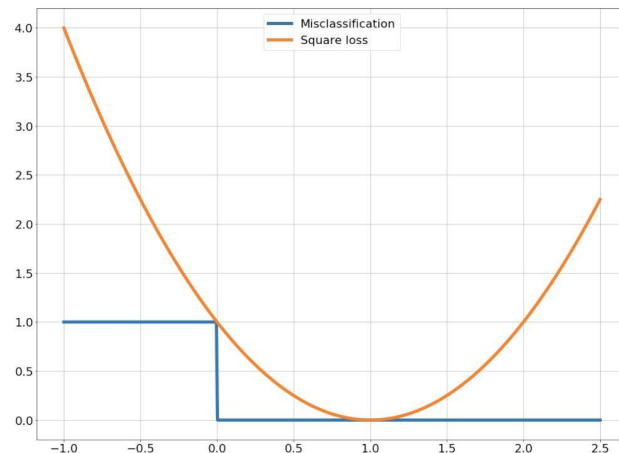


Let's treat classification problem as regression problem:

thus we optimize MSE

$$\begin{aligned} L_{\text{MSE}} &= (y_i - x_i^T w)^2 = \frac{(y_i^2 - y_i \cdot x_i^T w)^2}{y_i^2} = \\ &= (1 - y_i \cdot x_i^T w)^2 = (1 - M_i)^2 \end{aligned}$$

$$Y \in \{-1, 1\} \mapsto Y \in \mathbb{R}$$

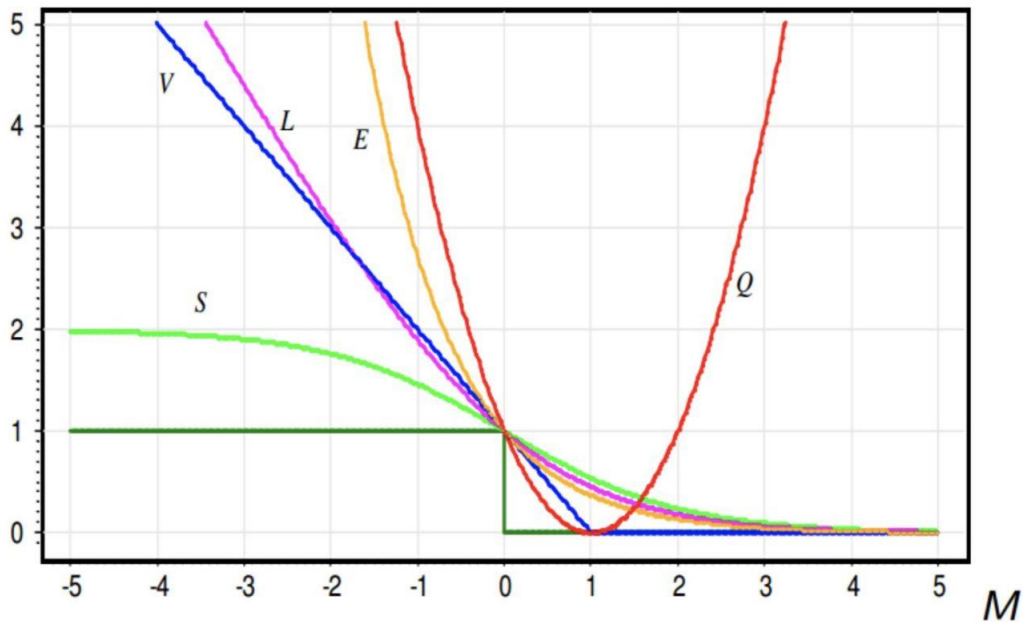


Advantage: already solved

Disadvantage: penalizes for high confidence



# Other losses



$$\begin{aligned}Q(M) &= (1 - M)^2 \\V(M) &= (1 - M)_+ \\S(M) &= 2(1 + e^M)^{-1} \\L(M) &= \log_2(1 + e^{-M}) \\E(M) &= e^{-M}\end{aligned}$$

[Loss functions for classification](#)

# Logistic Regression

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# Intuition



I. Let's try to predict probability of an object to have positive class

$$p_+ = P(y = 1|x) \in [0, 1]$$

II. But all we can predict is a real number!

III. Time for some tricks

$$y = x^T w \in R$$

$$\frac{p_+}{1 - p_+} \in [0, +\infty)$$

$$\log \frac{p_+}{1 - p_+} \in R$$

Here is the match

IV. Reverse to closed form

$$\frac{p_+}{1 - p_+} = \exp(x^T w)$$

$$p_+ = \frac{1}{1 + \exp(-x^T w)} = \sigma(x^T w)$$

# Sigmoid (aka logistic) function

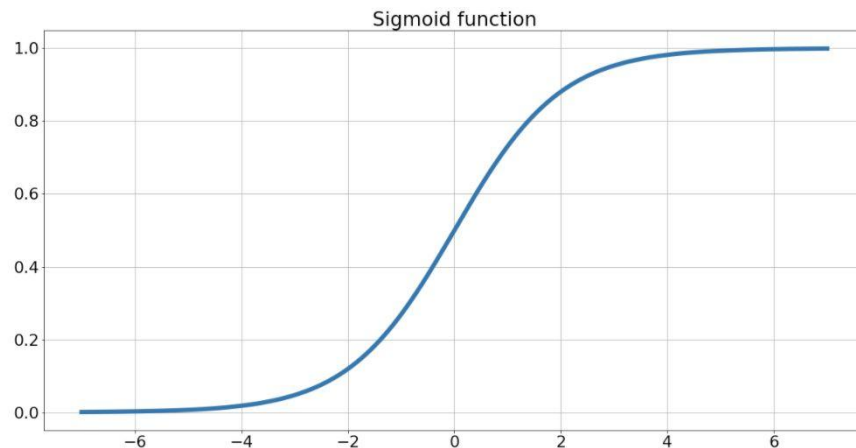


$$\sigma(x) = \frac{1}{1 + \exp(-x)}$$

Sigmoid is odd relative to (0, 0.5) point

Symmetric property:

$$1 - \sigma(x) = \sigma(-x)$$



Derivative:  $\sigma'(x) = \sigma(x) \cdot (1 - \sigma(x))$



# Maximum Likelihood Estimation

Just to remind

$$\log L(w|X, Y) = \log P(X, Y|w) = \log \prod_{i=1}^n P(x_i, y_i|w)$$

Calculating probabilities for objects

$$\text{if } y_i = 1 : P(x_i, 1|w) = \sigma_w(x_i) = \sigma_w(M_i)$$

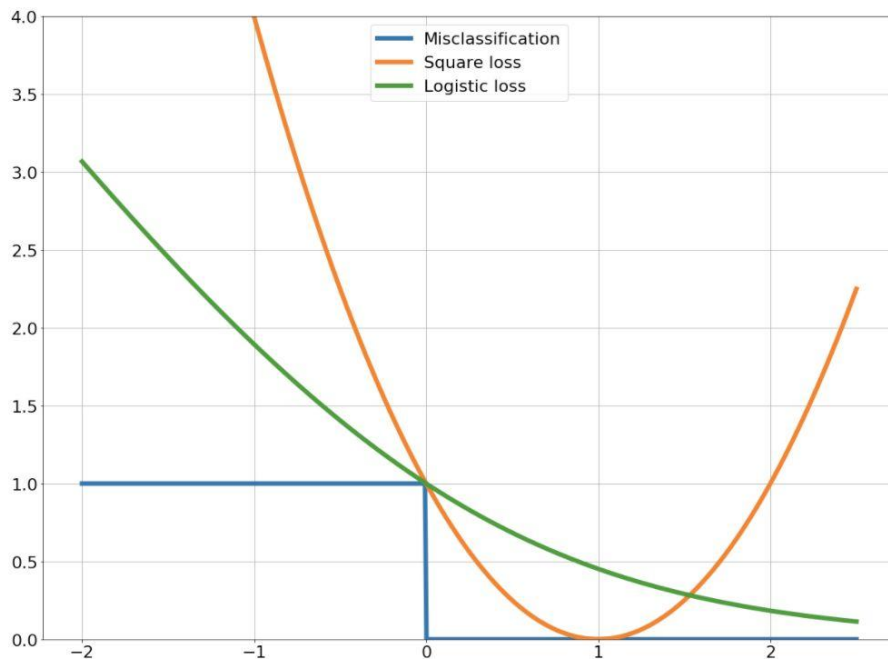
$$\text{if } y_i = -1 : P(x_i, -1|w) = 1 - \sigma_w(x_i) = \sigma_w(-x_i) = \sigma_w(M_i)$$

$$\log L(w|X, Y) = \sum_{i=1}^n \log \sigma_w(M_i) = - \sum_{i=1}^n \log(1 + \exp(-M_i)) \rightarrow \min_w$$



# Logistic loss

$$L_{Logistic} = \log(1 + \exp(-M_i))$$

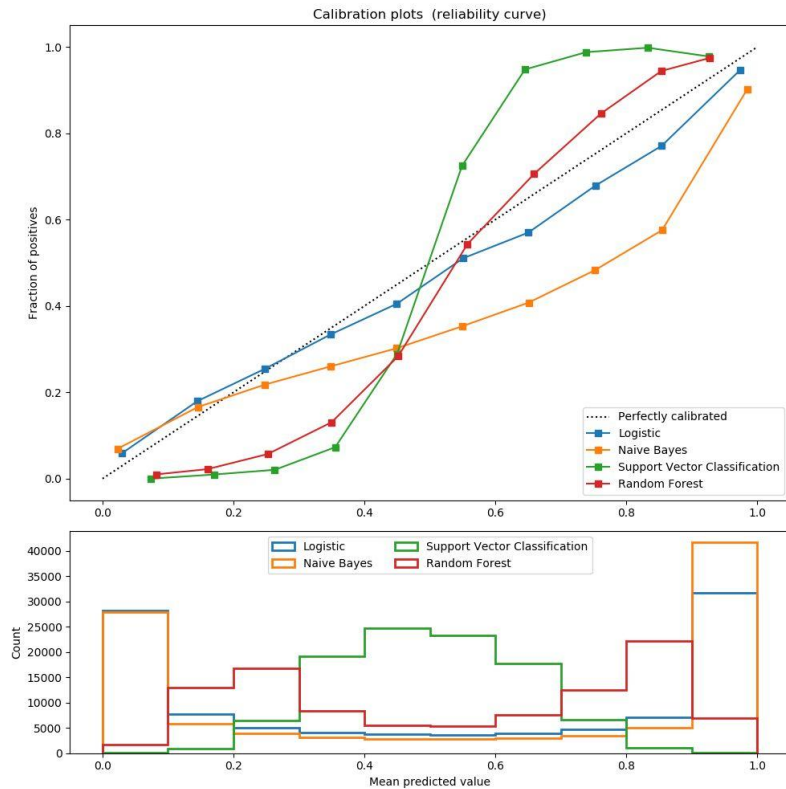




# Probability calibration

By using Logistic Regression  
we generate a Bernoulli distribution  
in each point of space

[Calibration discussion](#)





# Multiclass aggregation strategies

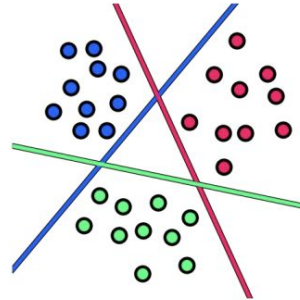
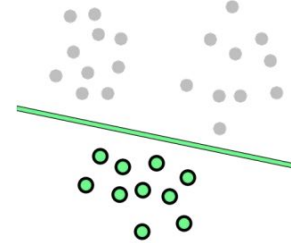
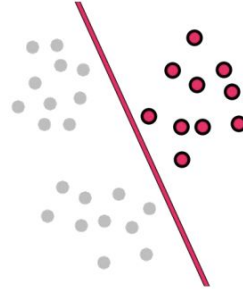
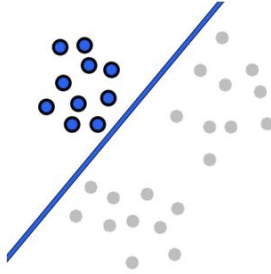
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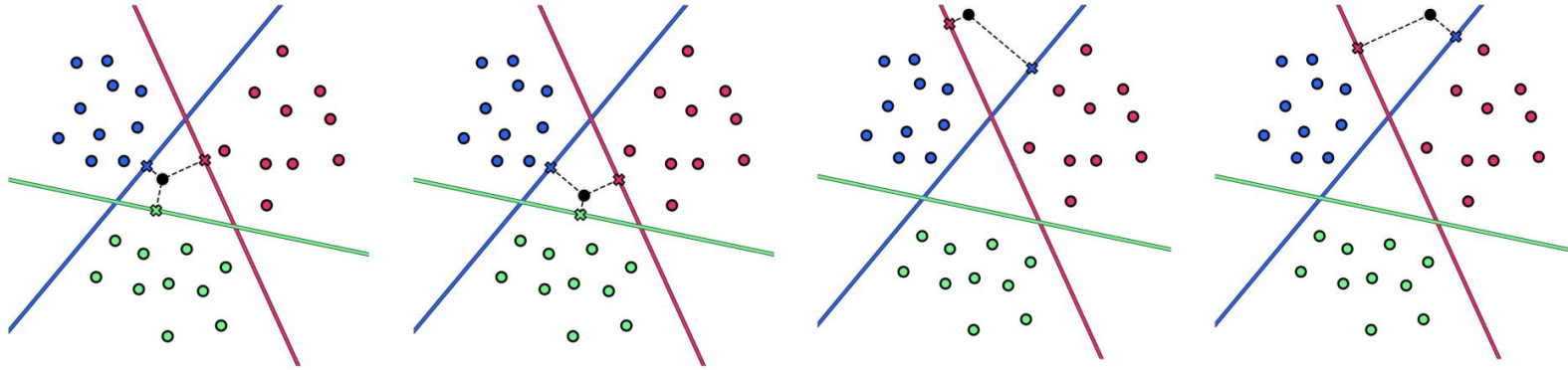


# One vs Rest



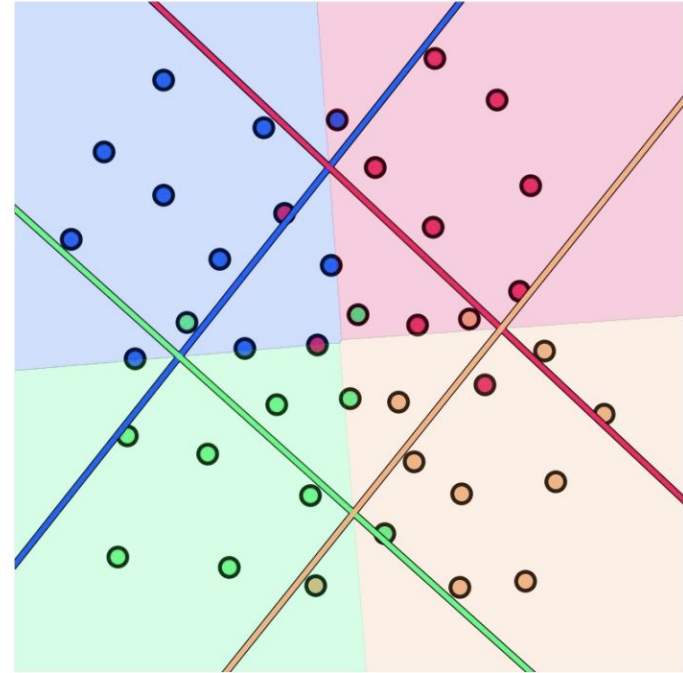
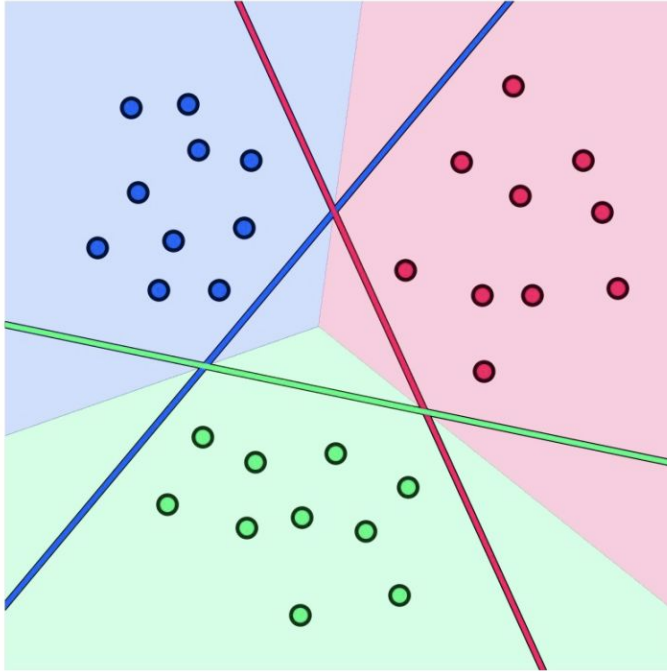
[Images source](#)

# One vs Rest: unclassified regions

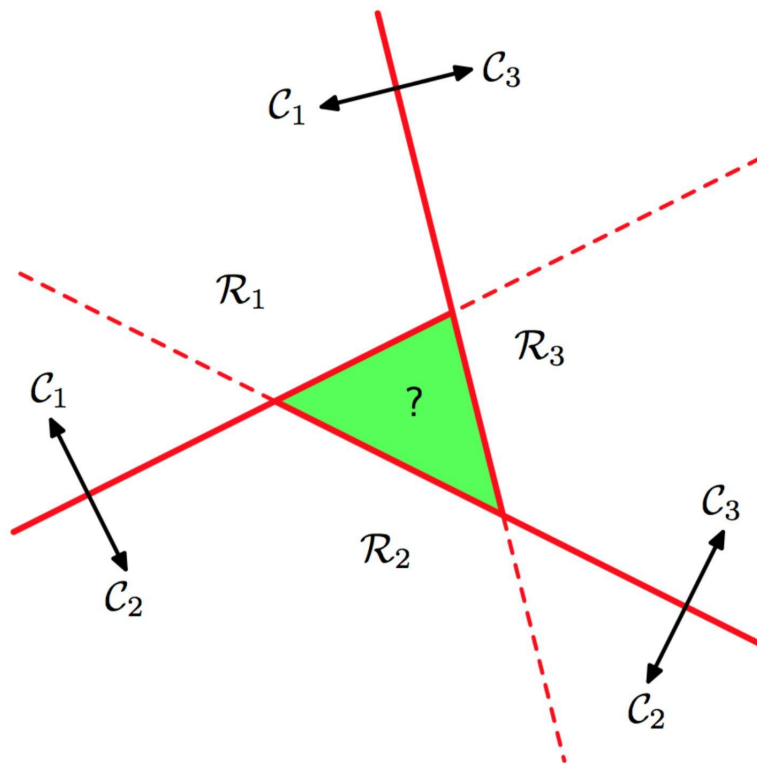




# One vs Rest: final result

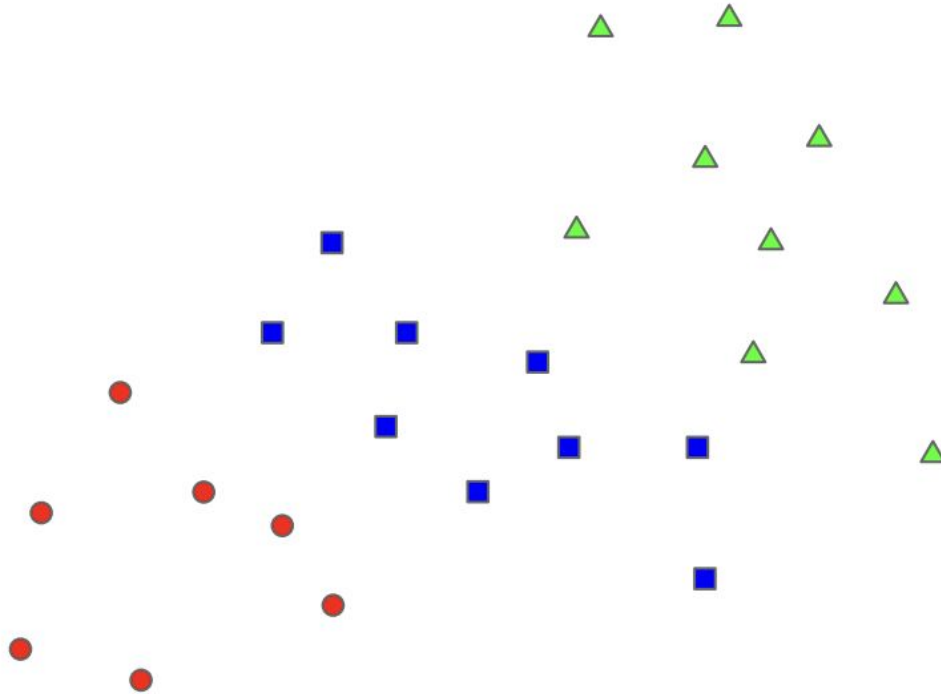


# One vs One





# Failure case?



# Summary



	One vs Rest	One vs One
#classifiers	$k$	$k(k-1)/2$
dataset for each	full	subsampled

# Metrics in classification

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# Metrics

- Accuracy
  - Balanced accuracy
- Precision
- Recall
- F-score
- ROC curve
  - ROC-AUC
- PR curve
  - PR-AUC
- Multiclass generalizations
- Confusion matrix

# Accuracy



Number of right classifications

$$\text{Accuracy} = \frac{1}{n} \sum_{i=1}^n [y_i^t = y_i^p]$$

target: 1 0 1 0 0 0 0 1 0 0

predicted: 0 0 1 0 0 0 0 1 1 0

accuracy = 8/10 = 0.8

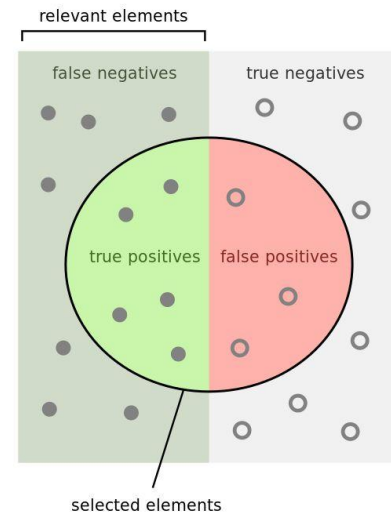
$$\text{Balanced accuracy} = \frac{1}{C} \sum_{k=1}^C \frac{\sum_i [y_i^t = k \text{ and } y_i^t = y_i^p]}{\sum_i [y_i^t = k]}$$



# Precision and Recall

		True condition	
		Condition positive	Condition negative
Predicted condition	Predicted condition positive	True positive	False positive, Type I error
	Predicted condition negative	False negative, Type II error	True negative

$$\text{Precision} = \frac{TP}{TP + FP} \quad \text{Recall} = \frac{TP}{TP + FN}$$

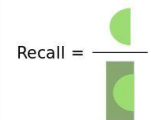


How many selected items are relevant?



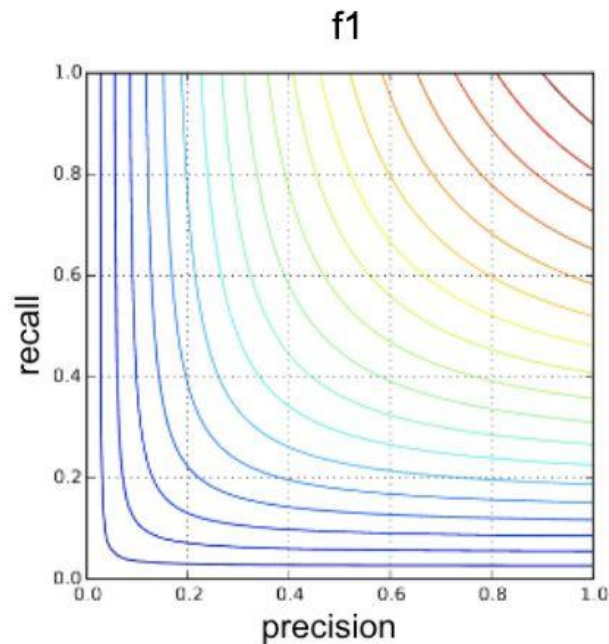
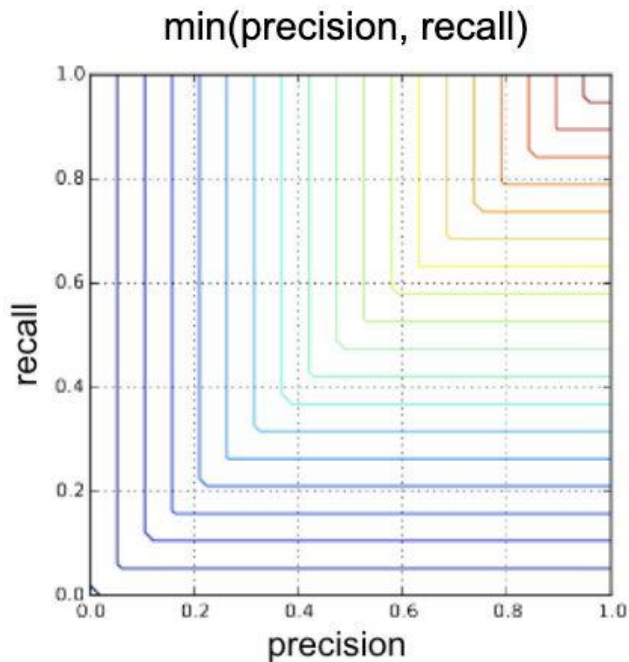
Precision =

How many relevant items are selected?



Recall =

# F-score motivation





# F-score

Harmonic mean of precision and recall

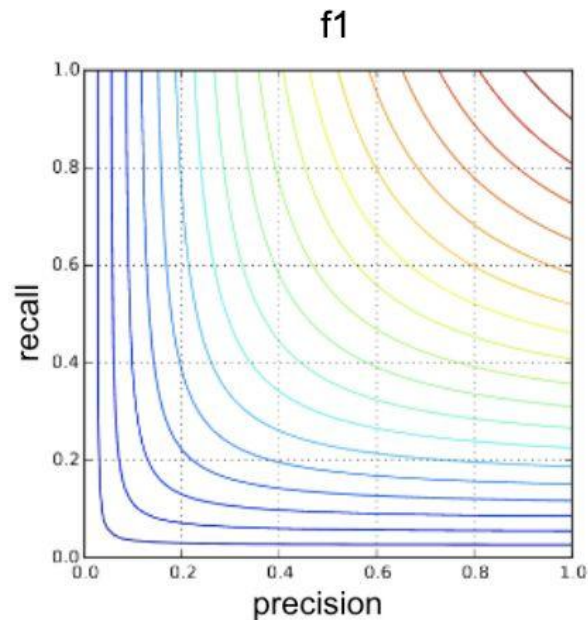
Closer to smaller one

$$F_1 = \frac{2}{\text{precision}^{-1} + \text{recall}^{-1}} = 2 \frac{\text{precision} \cdot \text{recall}}{\text{precision} + \text{recall}}$$

Generalization to different ratio between

Precision and Recall

$$F_\beta = (1 + \beta^2) \frac{\text{precision} \cdot \text{recall}}{\beta^2 \text{precision} + \text{recall}}$$



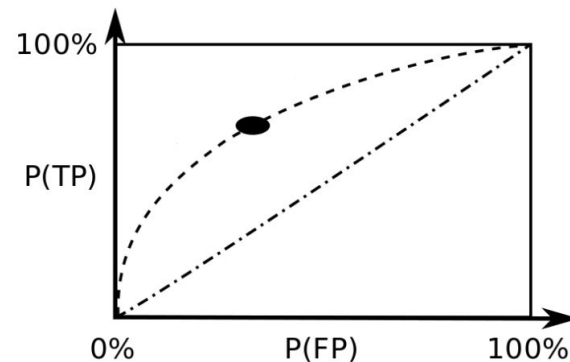


# Receiver Operating Characteristic (ROC)

		True condition	
		Condition positive	Condition negative
Predicted condition	Predicted condition positive	True positive	False positive, Type I error
	Predicted condition negative	False negative, Type II error	True negative

$$FPR = \frac{FP}{FP + TN}$$

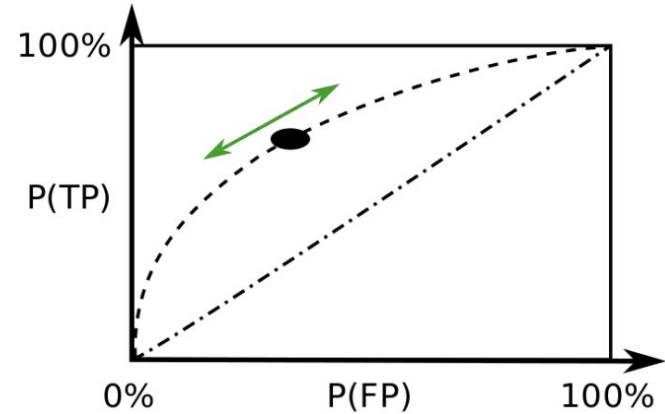
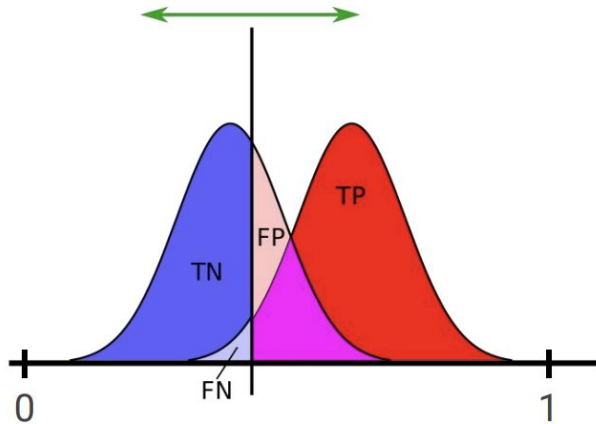
$$TPR = \frac{TP}{TP + FN} (= \text{Recall})$$



# Receiver Operating Characteristic (ROC)



Classifier needs to predict probabilities  
Objects get sorted by positive probability

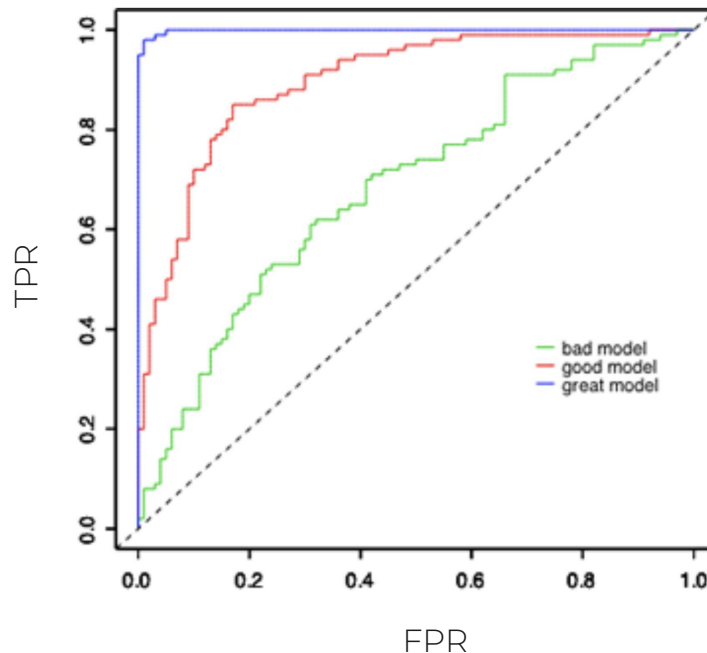


Line is plotted as threshold moves

# Receiver Operating Characteristic (ROC)



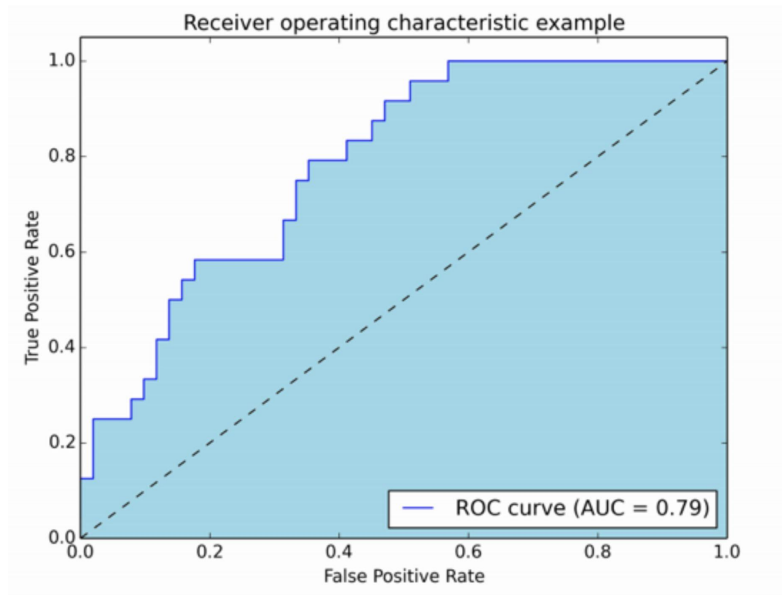
Baseline is random predictions  
Always above diagonal (for reasonable classifier)  
If below - change sign of predictions  
Strictly higher curve means better classifier  
Number of steps (thresholds) not bigger than dataset  
dataset







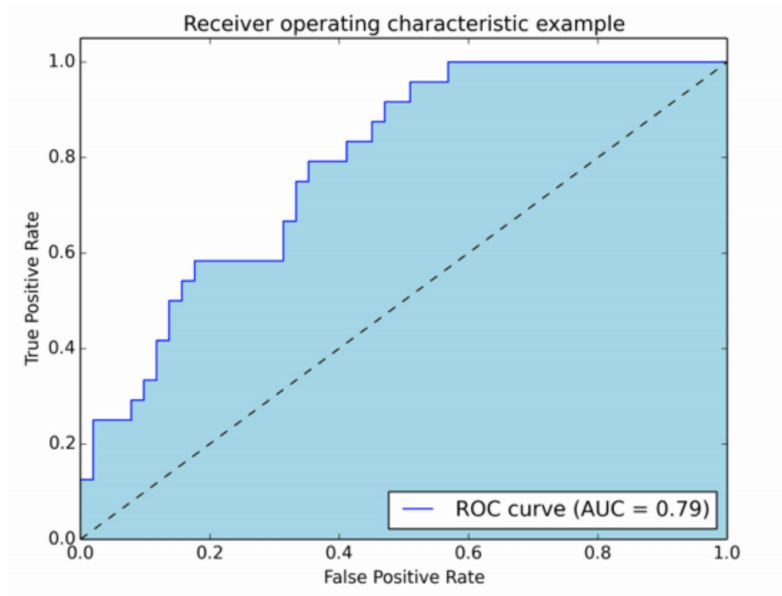
# ROC Area Under Curve (ROC-AUC)



Effectively lays in  $(0.5, 1)$   
Bigger ROC-AUC doesn't imply  
higher curve everywhere  
[More explanations with pictures](#)



# ROC-AUC properties



**Scale-invariant.** It measures how well predictions are ranked, rather than their absolute values.

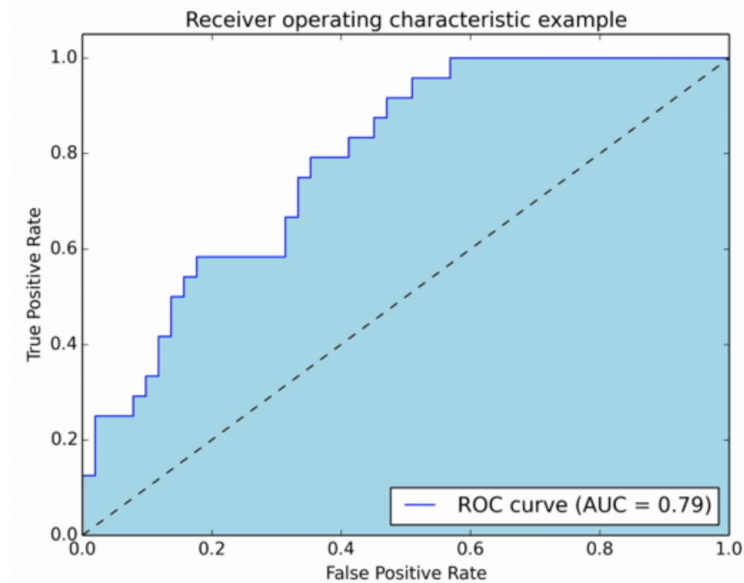
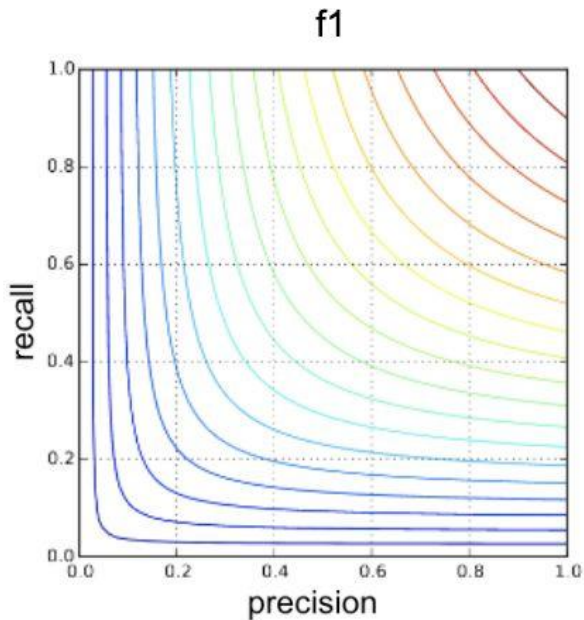
**Classification-threshold-invariant.** It measures the quality of the model's predictions irrespective of what classification threshold is chosen.

[Source](#)



# F-score vs ROC-AUC

Which is better to tune?

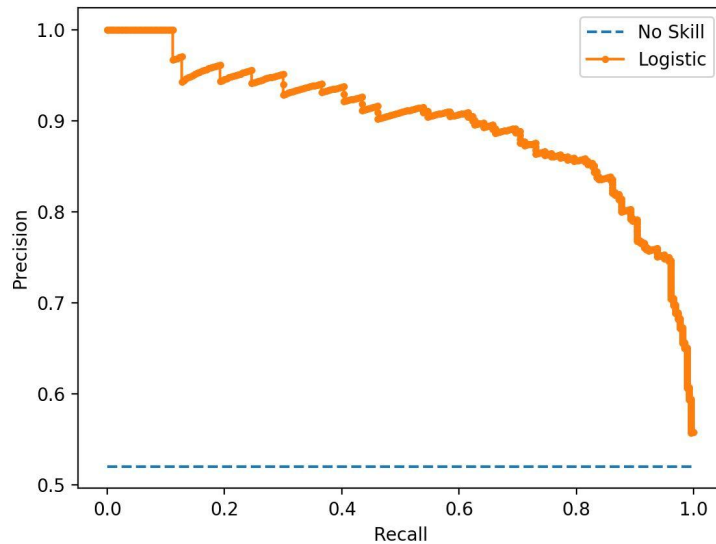




# Precision-Recall Curve

AUC is in  $(0, 1)$   
Source of AP metric  
(important for next semester)

[Nice article](#)





# Multiclass metrics

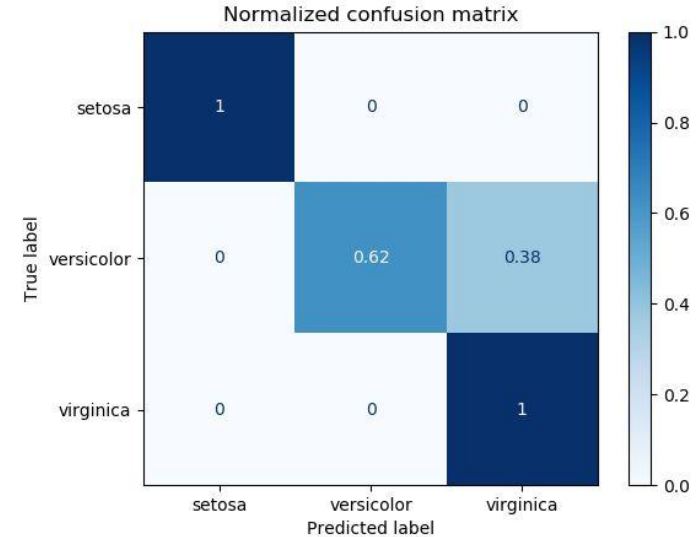
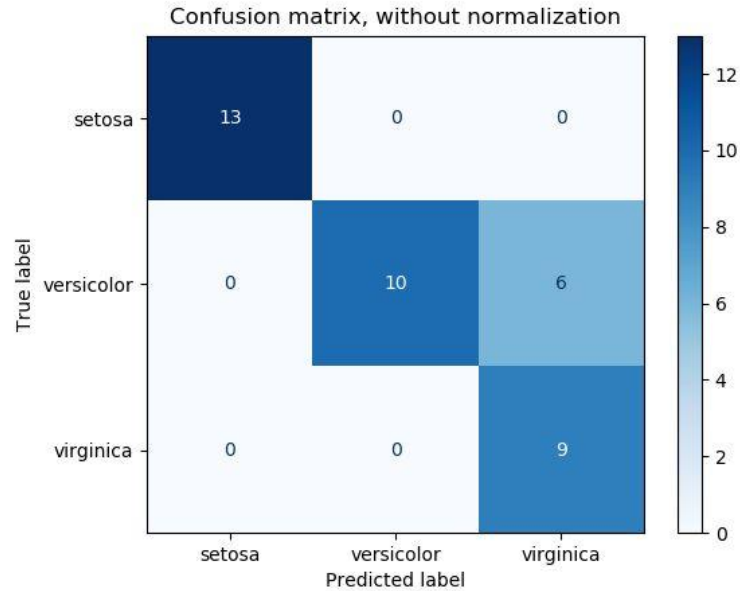
As with linear models we need some magic to measure multiclass problems

Basically it's mean of one or another kind

Detailed info [here](#) and [here](#)

average	Precision	Recall	F_beta
"micro"	$P(y, \hat{y})$	$R(y, \hat{y})$	$F_{\beta}(y, \hat{y})$
"samples"	$\frac{1}{ S } \sum_{s \in S} P(y_s, \hat{y}_s)$	$\frac{1}{ S } \sum_{s \in S} R(y_s, \hat{y}_s)$	$\frac{1}{ S } \sum_{s \in S} F_{\beta}(y_s, \hat{y}_s)$
"macro"	$\frac{1}{ L } \sum_{l \in L} P(y_l, \hat{y}_l)$	$\frac{1}{ L } \sum_{l \in L} R(y_l, \hat{y}_l)$	$\frac{1}{ L } \sum_{l \in L} F_{\beta}(y_l, \hat{y}_l)$
"weighted"	$\frac{1}{\sum_{l \in L}  \hat{y}_l } \sum_{l \in L}  \hat{y}_l  P(y_l, \hat{y}_l)$	$\frac{1}{\sum_{l \in L}  \hat{y}_l } \sum_{l \in L}  \hat{y}_l  R(y_l, \hat{y}_l)$	$\frac{1}{\sum_{l \in L}  \hat{y}_l } \sum_{l \in L}  \hat{y}_l  F_{\beta}(y_l, \hat{y}_l)$

# Confusion matrix



# Revise

- Linear classification
  - margin
  - loss functions
- Logistic regression
  - sigmoid derivation
  - Maximum Likelihood Estimation
  - Logistic loss
  - probability calibration
- Multiclass aggregation strategies
  - One vs Rest
  - One vs One
- Metrics in classification
  - Accuracy, Balanced accuracy
  - Precision, Recall, F-score
  - ROC curve, PR curve, AUC
  - Confusion matrix

# Next time

- Support Vector Machines
- Principal Component Analysis
- Linear Discriminant Analysis



# Thanks for attention!

Questions?

