Homework 12

BY 刘家骥 PB20071417

7，解:

三次多项式空间内, 可以得到一组基函数:

$$\varphi_1 = 1, \varphi_2 = x, \varphi_3 = x^2, \varphi_4 = x^3$$

现在在内积定义

$$(f, g) = \int_0^1 \sqrt{x} f(x) g(x)\, \mathrm{d}x$$

下, 使用Gram-Schmidt算法得到

$$e_1 = \varphi_1 = 1$$

$$e_2 = \varphi_2 - \frac{(\varphi_2, e_1)}{(e_1, e_1)} e_1 = x - \frac{3}{5}$$

$$e_3 = \varphi_3 - \frac{(\varphi_3, e_1)}{(e_1, e_1)} e_1 - \frac{(\varphi_3, e_2)}{(e_2, e_2)} e_2 = x^2 - \frac{10}{9} x + \frac{5}{21}$$

$$e_4 = \varphi_4 - \frac{(\varphi_4, e_1)}{(e_1, e_1)} e_1 - \frac{(\varphi_4, e_2)}{(e_2, e_2)} e_2 - \frac{(\varphi_4, e_3)}{(e, e_3)} e_2 = x^3 - \frac{21}{13} x^2 + \frac{945}{1287} x - \frac{35}{429}$$

15, 解:

要让积分值达到极小, 也就是让$c_0 + c_1 x$成为e^x的最佳逼近.

此题中, 内积的定义是

$$(f, g) = \int_0^1 f(x) g(x)\, \mathrm{d}x$$

两个基函数分别为

$$\varphi_0 = 1,\ \varphi_1 = x$$

由最佳逼近的定义, 有法方程组

$$\begin{cases} c_0(\varphi_0, \varphi_0) + c_1(\varphi_1, \varphi_0) = (\mathrm{e}^x, \varphi_0) \\ c_0(\varphi_0, \varphi_1) + c_1(\varphi_1, \varphi_1) = (\mathrm{e}^x, \varphi_1) \end{cases}$$

解得

$$\begin{cases} c_0 = 4\mathrm{e} - 10 \\ c_1 = -6\mathrm{e} + 18 \end{cases}$$

此时, 极小值为

$$\int_0^1 [\mathrm{e}^x + (10 - 4\mathrm{e}) + (6\mathrm{e} - 18)x]^2 \mathrm{d}x = \frac{1}{2}(\text{-}7\mathrm{e}^2 + 40\mathrm{e} - 57) \approx 3.94 \times 10^{-3}$$

16, 解:

权函数是$\rho(x)=\sqrt{x}$, 区间是[0, 1], 记三次最佳平方逼近多项式的形式为

$$p(x)=c_0+c_1x+c_2x^2+c_3x^3$$

法方程组的系数矩阵中,

$$g_{(i+1)(j+1)}=(\varphi_i,\varphi_j)=\int_0^1\sqrt{x}x^ix^j\,\mathrm{d}x=\frac{2}{2(i+j)+3}$$

右侧常数项是

$$(f,\varphi_j)=\int_0^1\sqrt{x}\cos x\,x^j\,\mathrm{d}x$$

由于此积分不便于计算, 我用余弦函数的Taylor展开的前三项作为近似替代, 得到

$$(f,\varphi_j)\approx\int_0^1\sqrt{x}\left(1-\frac{1}{2}x^2+\frac{1}{24}x^4\right)x^j\,\mathrm{d}x=\frac{2}{2j+3}-\frac{1}{2j+7}+\frac{1}{12}\frac{1}{2j+11}$$

即

$$2\begin{pmatrix}\frac{1}{3}&\frac{1}{5}&\frac{1}{7}&\frac{1}{9}\\\frac{1}{5}&\frac{1}{7}&\frac{1}{9}&\frac{1}{11}\\\frac{1}{7}&\frac{1}{9}&\frac{1}{11}&\frac{1}{13}\\\frac{1}{9}&\frac{1}{11}&\frac{1}{13}&\frac{1}{15}\end{pmatrix}\begin{pmatrix}c_0\\c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}\frac{491}{924}\\\frac{691}{2340}\\\frac{2777}{13860}\\\frac{1195}{7956}\end{pmatrix}$$

由此解得

$$\begin{cases}c_0\approx1.228179\\c_1\approx-1.945598\\c_2\approx3.610742\\c_3\approx-2.433610\end{cases}$$

即

$$p(x)\approx1.228179-1.945598x+3.610742x^2-2.433610x^3$$