## Reduction to Entropy Equation

GRMHD codes occasionally have difficulties handling the energy equation. One then uses entropy evolution as a backup. When we have only gas and magnetic fields, the entropy equation is straightforward. The gas entropy per unit mass is

$$s_{\rm gas} = \frac{k}{(\gamma-1)m}\ln\left(\frac{p}{\rho^\gamma}\right) = \frac{k}{m}\ln\left(\frac{p^n}{\rho^{n+1}}\right), \qquad \gamma = 1+\frac{1}{n}, \tag{1}$$

and the entropy per unit volume is

$$S_{\rm gas} = \rho s_{\rm gas}, \tag{2}$$

where $p$ and $\rho$ are measured in the gas rest frame. For entropy evolution, we say that the gas entropy per unit mass is conserved (temporarily ignoring shocks and reconnection), hence

$$\frac{ds_{\rm gas}}{d\tau} = 0, \quad \text{i.e.} \quad (\rho s_{\rm gas} u^\mu)_{;\mu} = 0, \tag{3}$$

where the second equation is equivalent to the first, but written in a flux-conservative form.

In the case of radiation hydrodynamics or radiation MHD, the gas entropy is no longer conserved since there is energy transfer between gas and radiation. In the fluid rest frame, let $\hat{E}_{\rm rad}$ be the radiation energy density and $\hat{B}_{\rm gas}$ be the frequency integrated blackbody intensity corresponding to the temperature $T_{\rm gas}$ of the gas. Let $\kappa$ be the frequency-integrated absorption opacity coefficient, which has units of cm$^{-1}$. (This is different from the standard opacity which has units of cm$^2$g$^{-1}$; the two are related by a factor of $\rho$.) The evolution of the gas entropy per unit mass in the fluid frame is described by the standard relation $Tds_{\rm gas} = dq$, where $dq$ is the amount of heat energy (heating minus cooling) added to the gas per unit mass. This gives

$$T_{\rm gas}\frac{ds_{\rm gas}}{d\tau} = \frac{dq_{\rm gas}}{d\tau} = \frac{\kappa}{\rho}(c\hat{E}_{\rm rad} - 4\pi\hat{B}_{\rm gas}). \tag{4}$$

The extra $\rho$ in the denominator on the right hand side is to convert this quantity to heating rate per unit mass (erg g$^{-1}$s$^{-1}$).

We would like to write equation (4) in the form of an evolution equation for the entropy per unit volume, $S_{\rm gas} = \rho s_{\rm gas}$. To do this, start with the following relation in flat space,

$$\frac{\partial}{\partial t}(\rho s_{\rm gas}) + \vec{\nabla}\cdot(\rho s_{\rm gas}\vec{v}) = s_{\rm gas}\left[\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v})\right] + \rho\left[\frac{\partial s_{\rm gas}}{\partial t} + \vec{v}\cdot\vec{\nabla}s_{\rm gas}\right], \tag{5}$$

where $\vec{v}$ is the 3-velocity of the gas. The first square brackets on the right vanishes by mass conservation, while the second term in the second square brackets vanishes in the fluid frame (where $\vec{v} = 0$). Furthermore, in the fluid frame, $\partial s_{\rm gas}/\partial t$ is equal to the total (or Lagrangian) time derivative $ds_{\rm gas}/dt = ds_{\rm gas}/d\tau$ of equation (4). Thus, equation (5) becomes

$$\frac{\partial}{\partial t}(\rho s_{\rm gas}) + \vec{\nabla} \cdot (\rho s_{\rm gas}\vec{v}) = \frac{\kappa}{T_{\rm gas}}(c\hat{E}_{\rm rad} - 4\pi\hat{B}_{\rm gas}). \tag{6}$$

The quantity on the left hand side of equation (6) is clearly the 4-divergence of the entropy 4-flux $\rho s_{\rm gas} u^\mu$. Also, the quantity on the right is the time-component of the fluid frame radiation force density $\hat{G}^\mu$ divided by $T_{\rm gas}$. Thus, we can rewrite equation (6) in the following covariant form:

$$(\rho s_{\rm gas} u^\mu)_{;\mu} = -\frac{1}{T_{\rm gas}} G^\mu u_\mu. \tag{7}$$

Hopefully, this equation has all the units and signs right (**Olek, please check**). The equation can be used to replace the time component of the usual conservation law $(T^\mu_\nu)_{;\mu} = G_\nu$ whenever the latter leads to numerical difficulties.

The entropy equation for the radiation is slightly more complicated since the equation equivalent to equation (4) will have two terms on the right hand side: (i) energy transfer from the gas to the radiation (right hand side of eq. 4 but with the opposite sign), and (ii) energy gain due to radiation flow: $-\vec{\nabla} \cdot \hat{\vec{F}}_{\rm rad}$. However, the entropy equation of radiation is probably not needed, since the energy equation is already quite simple. In fact, very likely, the entropy equation will simply give back the energy equation.