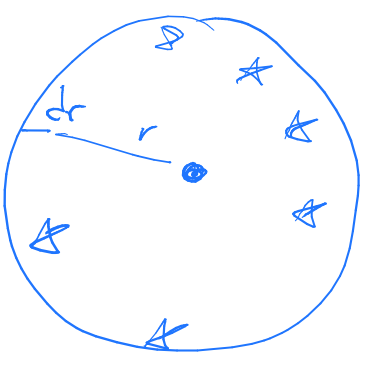
Chapter 1: Introduction

Parsec (pc): distance at which 1AU subtends an angle of 1 arcsecond

Chapter 2: Fundamental observations

The night is Dark



$$F(r) = \frac{L}{4\pi r^2}$$

$$dJ(r) = \frac{L}{4\pi r^2} \cdot n r^2 dr$$

$$= \frac{nL}{4\pi} dr$$

$$\Rightarrow J(r) = \int_0^\infty \frac{nL}{4\pi} dr = \infty$$

sky is infinitely bright

$\Rightarrow$ • universe has finite age

$\rightarrow$ light from some stars hasn't had time to reach us yet

(Edgar Allan Poe 1848)

(1st)

The universe is isotropic and homogeneous

- no prefered directions
- no prefered locations

at large scales (>100 Mpc)

average density of the universe:

$$\rho_0 \sim 3\times10^{-27}\ kg/m^3$$

- cosmological principle $\rightarrow$ there is nothing special about our location in the universe

Galaxies show redshift proportional to their distance

$$z \equiv \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}}$$

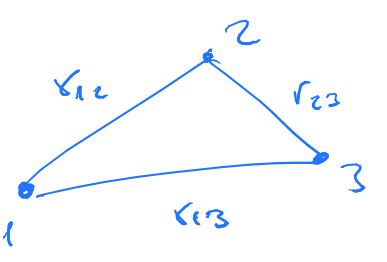
$$z = \frac{H_0}{c} r \quad \text{Hubble's Law}$$

$$z \ll 1 \Rightarrow z = \frac{v}{c}$$

$$\Rightarrow \frac{H_0 r}{\cancel{c}} = \frac{v}{\cancel{c}} \Rightarrow v = H_0 r$$

$$H_0 = 500 \frac{km}{s} Mpc^{-1}$$

Hubble underestimated the distances to galaxies



$$r_{12} = a(t)\, r_{12}(t_0)$$

$a(t) \rightarrow$ how the expansion of the universe depends on time

$$v_{12}(t) = \frac{d}{dt} r_{12} = \dot{a}\, r_{12}(t_0) = \underbrace{\frac{\dot{a}}{a}}_{H_0} r_{12}(t)$$

$$\Rightarrow v = H_0 r \ ; \ H = \frac{\dot{a}}{a}$$

$$v = \frac{r}{t_0} \Rightarrow t_0 = \frac{r}{v} = \frac{r}{H_0 r} = H_0^{-1}$$

$$H_0^{-1} = \frac{1}{70} km^{-1} s\, Mpc \approx 14 \pm 1{,}4\ Gyr$$

all the galaxies in the universe were together at $t_0 = 14$ Gyr

The universe contains different types of Particles

baryonic matter $\rightarrow$ only protons and neutrons contribute to the mass density

total energy density

$$\varepsilon_\gamma = \alpha T^4$$

The universe is filled with cosmic Microwave Background

CMB, $T_0 = 2{,}7$ K

if mass is conserved in an expanding universe, in the past, it was denser

$\Rightarrow T \Rightarrow$ baryonic matter ionized $\rightarrow$ opaque

$$\cancel{dQ} = dE + PdV$$

$$\Rightarrow dE = -PdV$$

$$\Rightarrow d(\alpha T^4 V) = -\alpha T^4/3$$

Chapter 3

Newton vs Einstein

equivalence principle

gravitational mass = inertial mass

$\Rightarrow$

* gravity affects photons, even though they have no mass

Fermat principle $\rightarrow$ light travels between two points along a path which minimizes the travel time

euclidean space $\rightarrow$ straight line

$\Rightarrow$ light bends $\rightarrow$ the space is not euclidean

Describing curvature

$$ds^2 = dx^2 + dy^2$$

or $ds^2 = dr^2 + r^2 d\phi^2$

sphere surface $\rightarrow$ $ds = R\,d\theta = R\,d\frac{x}{R}$ (arc / radius)

$$\left.\begin{aligned} ds^2 &= dr^2 + R^2 \sin^2\left(\tfrac{r}{R}\right) d\phi^2 \\ ds^2 &= dr^2 + R^2 \cos^2\left(\tfrac{r}{R}\right) d\phi^2 \end{aligned}\right\} \text{metrics}$$

geometry

K: curvature constant

0: flat

+1: positively curved space

-1: negatively curved

R: radius of curvature

3D space with K=+1:

$$ds^2 = dr^2 + r^2 \sin^2\left(\tfrac{r}{R}\right)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

3 possible metrics for a homogeneous, isotropic 3D space:

$$ds^2 = dr^2 + S_K^2(r)\, d\Omega^2$$

$$d\Omega = d\theta^2 + \sin^2(\theta)\, d\phi^2$$

$$S_K(r) = \begin{cases} R\sin\left(\frac{r}{R}\right) & (K=+1) \\ r & (K=0) \\ R\sinh\left(\frac{r}{R}\right) & (K=-1) \end{cases}$$

The Robertson-Walker metric

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2 \rightarrow \text{Minkowski}$$

photon trajectory:

$$ds^2 = 0 = -c^2 dt^2 + dr^2 + r^2 d\Omega^2$$

if $\theta = \phi = \text{const}$ (fixed direction)

$$\Rightarrow c^2 dt^2 = dr^2$$

$$\Rightarrow \frac{dr}{dt} = \pm c$$

Robertson-Walker metric:

$$ds^2 = -c^2 dt^2 + a^2(t)\underbrace{\left(\frac{dr^2}{1-\frac{kr^2}{R_0^2}} + r^2 d\Omega^2\right)}_{\text{spatial metric for uniformly curved space at radius } R_0}$$

or

$$ds^2 = -c^2 dt^2 + a^2(t)\left[dr^2 + S_K^2(r)\, d\Omega^2\right]$$

$\rightarrow a(t_0) = 1$

$a(t) \rightarrow$ how distances grow with time

$\rightarrow K,\ a(t_0),\ R_0$

observations:

the universe is not isotropic and homogeneous on small scales

but the universe is "lumpy"

$\hookrightarrow$ dense regions, held together with their own gravity

- solar systems
- galaxies
- clusters of galaxies

> 100 Mpc $\rightarrow$ isotropic, homogeneous distribution

described by $a(t)$

Proper distance

$$ds^2 = c^2 dt^2 \left(dr^2 + S_K^2(r)\, d\Omega^2\right)$$

$\theta, \phi = \text{const}$

$$\Rightarrow ds = a(t)\, dr$$

Proper distance $\rightarrow$ length of the spatial geodesic between two events at a fixed scale factor $a(t)$ fixed at the value $a(t)$

$$d_p(t) = a(t) \int_0^r dr = a(t)\, r$$

$$\Rightarrow \dot{d}_p(t) = \dot{a} r = \frac{\dot{a}}{a} d_p \qquad \left(r = \frac{d_p}{a}\right)$$

$$\Rightarrow v_p(t) = H_0\, d_p(t_0)$$

$$\Rightarrow H_0 = \left(\frac{\dot{a}}{a}\right)_{t=t_0}$$

$$R(t) = R_0 a(t)$$

$$\Rightarrow d_H(t_0) \equiv \frac{c}{H_0} \quad \text{Hubble distance}$$

≈ 4300 Mpc

galaxies farther than 4300 Mpc are moving away from us at speeds greater than the speed of light

$$\Rightarrow c\, dt = a(t)\, dr$$

$$\Rightarrow \frac{c\, dt}{a(t)} = dr \quad / \int$$

$$c \int_{t_0}^{t_0} \frac{dt}{a(t)} = \int_0^r dr = r$$

$$\Rightarrow c \int_{t_0 + \frac{1}{H_0}}^{t_0} \frac{dt}{a(t)} = r$$

$$\Rightarrow \int_{t_0 + \frac{1}{H_0}}^{t_0} \frac{dt}{a(t)} = \int_{t_0}^{t_0} \frac{dt}{a(t)}$$

$$\Rightarrow \quad / - \int_{t_0 + \frac{1}{H_0}}^{t_0} \frac{dt}{a(t)} = \int_{t_0 + \frac{1}{H_0}}^{t_0} + \int_{t_0}^{t_0}$$

$$\Rightarrow \cancel{\int_{t_0}^{t_0}} + \cancel{\int_{t_0 + \frac{1}{H_0}}^{t_0}} + \int_{t_0 + \frac{1}{H_0}}^{t_0} = \int_{t_0}^{t_0 + \frac{1}{H_0}}$$

$$\Rightarrow \int_{t_0}^{t_0 + \frac{1}{H_0}} \frac{dt}{a(t)} = \int_{t_0 - \frac{1}{H_0}}^{t_0} \frac{dt}{a(t)}$$

$$a(t) \sim t^{\alpha}$$

$$\Rightarrow \frac{1}{a(t_0)} \lambda_0 = \frac{1}{a(t_e)} \lambda_e$$

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{\lambda_0}{\lambda_e} - 1$$

$$\Rightarrow \frac{\lambda_0}{\lambda_e} = \frac{a(t_0)}{a(t_e)}$$

$$\Rightarrow z = \frac{a(t_0)}{a(t_e)} - 1 = \frac{1}{a(t_e)} - 1$$

ex:

$$z + 1 = \frac{1}{a(t_e)} \Rightarrow a(t_e) = \frac{1}{3}$$

redshift depends only on the relative scale factors $a(t)$

we observe the galaxy ($z \sim 2$) when the universe had a scale factor $a(t_e) = 1/3$

it doesn't depend on the transition $a(t_e) \rightarrow a(t_0)$

Lecture 9 Fluid Dynamics

If the current is partially ionized then

$\vec{B}, \vec{v} \parallel \nabla \rho$

If not we that equations

The Navier equations

Euler's fluid equation and Poisson eq

$$\boxed{\nabla P = \rho \vec{a}_{grav}}$$

Newton's Gravity

$\vec{a}$ gravitational potential

ρ mass density

Pick values $\langle \rho \rangle$, $\langle P \rho \rho \rho

Chapter 5 : Single-Component Universe

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{kc^2}{R_0^2 a^2}$$

$$\dot\varepsilon + \frac{3\dot a}{a}(\varepsilon + P) = 0$$

$$P = w\varepsilon$$

Evolution of energy density

$$P = \sum_i w_i \varepsilon_i$$

$$\Rightarrow \quad \dot\varepsilon_w + \frac{3\dot a}{a}(1+w)\varepsilon_w = 0$$

$$\frac{d\,\varepsilon_w}{\varepsilon_w} = -\frac{3\,da}{a}(1+w)$$

$$\ln(\varepsilon_w) = -3(1+w)\ln(a)$$

$$\varepsilon_w = \varepsilon_{w0}\, a^{-3(1+w)} \qquad \varepsilon_{w0} = \varepsilon_w\big|_{a=1}$$

non-relativistic matter ($w=0$)

$$\varepsilon_m(a) = \varepsilon_{m0}\, a^{-3}$$

$$\varepsilon_m(a) = \frac{\varepsilon_{m0}}{a^3}$$

radiation ($w = \frac{1}{3}$) relativistic

$$\varepsilon_r(a) = \varepsilon_{r0}\, a^{-3(1+\frac{1}{3})}$$

$$= \varepsilon_{r0}\, a^{-4}$$

average density starlight is currently only 3% of the energy density of CMB

$\Rightarrow$ density radiation $>$ density matter

$$\Omega_{r0} = \Omega_{m0} a^{-1}\Omega_{r0} = 3.4\cdot10^{-5}$$

$$\boxed{\begin{aligned}\Omega_{m0} &\approx 0.3\\ \Omega_{\Lambda 0} &\approx 0.7\end{aligned}} \Rightarrow \Omega_{r0} = 1 - \Omega_{\Lambda 0} - \Omega_{m0}$$

$$\frac{\varepsilon_a(a)}{\varepsilon_m(a)} = \frac{\varepsilon_{m0}\, a^3}{\varepsilon_{r0}}$$

$a \to 0 \Rightarrow$ horizon ∞

$\Rightarrow$ $a \to \infty$ (future) $\to w = 0$ "matter"

$$a = \frac{1}{1+z}$$

Many component :

$$\dot a^2 = \frac{8\pi G}{3c^2}\sum_i \varepsilon_{i0}\, a^{-1-3w_i} - \frac{kc^2}{R_0^2}$$

Curvature only

in an empty expanding universe

$$a(t) = \frac{1}{1+z} = \frac{t_0}{t} \qquad \left(a(t) = \frac{t}{t_0}\right) \text{ empty universe}$$

$$\Rightarrow \quad t_0 = \frac{2\,a_0}{1+z} = \frac{H_0^{-1}}{1+z}$$

$$\cdots \quad d_L(z) = \frac{c}{H_0}\ln(1+z)$$

$$d_p(z) = \frac{c}{H_0}\frac{\ln(1+z)}{1+z}$$

Spatially flat universes

spatially flat $\to k=0$

single component w

$$\dot a^2 = \frac{8\pi G}{3c^2}\varepsilon_{w0}\,(1+z)^{-3(1+w)}$$

guess : $a \propto t^q \Rightarrow \dot a = q\,t^{q-1}$

$$q^2 t^{2q-2} = \frac{8\pi G}{3c^2}\varepsilon_{w0}\, t^{-q(1+3w)}$$

$$\Rightarrow \quad 2q - 2 = -q - 3wq$$

$$\Rightarrow \quad q(2+3w) = 2$$

$$q = \frac{2}{2+3w}$$

$$\Rightarrow \quad a(t) = \left(\frac{t}{t_0}\right)^q = \left(\frac{t}{t_0}\right)^{\frac{2}{2+3w}}$$

$$\varepsilon(t) = \varepsilon_{w0}\, a^{-3(1+w)}$$

$$= \varepsilon_{w0}\left(\frac{t}{t_0}\right)^{-\frac{3(1+w)}{2+3w}}$$

$$\varepsilon_{w0} = \varepsilon_{w0} = \frac{3c^2}{8\pi G}H_0^2$$

$$1+z = \frac{a(t_0)}{a(t_e)} = \left(\frac{t_0}{t_e}\right)^{\frac{2}{2+3w}}$$

$$\Rightarrow \quad t_e = \frac{t_0}{(1+z)^{\frac{2+3w}{2}}} = \frac{2}{3(1+w)H_0}\frac{1}{(1+z)^{\frac{2+3w}{2}}}$$

$$d_p(t_e) = c\int_{t_e}^{t_0}\frac{dt}{a(t)} = c\left(\frac{t}{t_0}\right)^{-\frac{2}{2+3w}}dt$$

$$= \left(\frac{t}{t_0}\right)^{\frac{2}{2+3w}}\frac{t_0}{\frac{1+w}{1+3w}}\Bigg|_{t_e}^{t_0}$$

$$= c\,t_0\,\frac{3(1+w)}{1+w}\left(1 - \left(\frac{t_e}{t_0}\right)^{\frac{1+3w}{2}}\right) \quad //$$

$$\Rightarrow \quad d_p(t_e) = \frac{c}{H_0}\frac{2}{1+3w}\left(1 - (1+z)^{-\frac{(1+3w)}{2}}\right)$$

$$d_{em}(t_e) = c\int_{t_e}^{t_0}\frac{dt}{a(t)}$$

$$= c\,t_0\,\frac{3(1+w)}{1+3w} = \frac{c}{H_0}\frac{2}{1+3w}$$

Matter only

only non-relativistic matter : $w=0$

$$t_0 = \frac{2}{3H_0} \quad ; \quad d_{em}(t_e) = 3t_0 = \frac{2c}{H_0}$$

$$a_m(t) = \left(\frac{t}{t_0}\right)^{2/3}$$

$$d_p(t_e) = c\int_{t_e}^{t_0}\frac{dt}{\left(\frac{t}{t_0}\right)^{2/3}} = c\int_{t_e}^{t_0}\left(\frac{t_0}{t_e}\right)^{2/3}dt$$

$$= 3\left(\frac{t_0}{t_e}\right)^{\frac{2}{3}}t_e^{\frac{1}{3}}\Bigg|_{t_e}^{t_0} = 3t_0\left(1 - \frac{1}{t_0^{1/3}}\right)$$

$$t_0 = \frac{2}{3H_0} \quad \Rightarrow \quad 1+z = \left(\frac{t_0}{t_e}\right)^{\frac{2}{3}}$$

$$\Rightarrow \quad 1+z = \left(\frac{t_0}{t_e}\right)^{\frac{1}{3}}$$

$$\Rightarrow \quad \sqrt{1+z} = \left(\frac{t_0}{t_e}\right)^{\frac{1}{2}}$$

$$\Rightarrow \quad \left(\frac{t_0}{t_e}\right)^{\frac{1}{3}} = \frac{1}{\sqrt{1+z}}$$

$$\Rightarrow \quad d_p(t_e) = \frac{2c}{H_0}(1+z)\left(1 - \frac{1}{\sqrt{1+z}}\right)$$

Radiation only

$(w = \frac{1}{3})$ only time $\to$ pure adiabatic $\to$ no matter factor important

$$t_0 = \frac{1}{2H_0} \quad \Rightarrow \quad d_{em}(t_e) = 2t_0 = \frac{c}{H_0}$$

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2}$$

$$d_p(t_e) = c\int_{t_e}^{t_0}\frac{dt}{a(t)} = c\int_{t_e}^{t_0}\frac{dt}{\left(\frac{t}{t_0}\right)^{1/2}}$$

$$= 2\,c\,t_0\left(\frac{t}{t_0}\right)^{1/2}\Bigg|_{t_e}^{t_0} = 2\,c\,t_0\left(1 - \left(\frac{t_e}{t_0}\right)^{1/2}\right)$$

$$= \frac{c}{H_0}\left(1 - \frac{1}{1+z}\right)$$

$$d_p(t_e) = \frac{c}{H_0}\frac{z}{(1+z)}$$

$$\varepsilon_r(t) = \varepsilon_{r0}\left(\frac{t}{t_0}\right)^{-2} \quad t \to 0 \;,\; \varepsilon_r \to \infty$$

Lambda Only

$(w = -1)$ cosmological constant component

$$\Rightarrow \quad \dot a^2 = \frac{8\pi G\varepsilon_\Lambda}{3c^2}a^2$$

$$\Rightarrow \quad a(t) = e^{H_\Lambda(t-t_0)} \quad ; \quad H_\Lambda = \left(\frac{8\pi G\varepsilon_\Lambda}{3c^2}\right)^{1/2}$$

$$d_p(t_e) = c\int_{t_e}^{t_0}\frac{dt}{a(t)} = c\int_{t_e}^{t_0} e^{-H_\Lambda(t-t_0)}dt$$

$$= \frac{c\,e^{-H_\Lambda(t-t_0)}}{-H_\Lambda}\Bigg|_{t_e}^{t_0} = \frac{-c}{H_\Lambda}\left(e^{-H_\Lambda(t_0-t_e)} - 1\right) \quad ?$$

$$= \frac{c}{H_0}z$$

Chapter 6 : Multiple component universes

$$H^2(t) = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2 a(t)}$$

$$\frac{\kappa}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_0 - 1)$$

$$\Rightarrow \quad H^2(t) = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{H_0^2}{a^2(t)}(\Omega_0 - 1)$$

$$\Rightarrow \quad \frac{H^2}{H_0^2} = \frac{\varepsilon(t)}{\varepsilon_{c,0}} + \frac{1-\Omega_0}{a^2(t)}$$

$$\varepsilon_{c,0} \equiv \frac{3c^2 H_0^2}{8\pi G}$$

$$\varepsilon_m = \frac{\varepsilon_{m,0}}{a^3} \quad ; \quad \varepsilon_r = \frac{\varepsilon_{r,0}}{a^4}$$

current evidence $\Rightarrow$ $\varepsilon_{\Lambda,0} = \varepsilon_\Lambda = $ cte

quintessence $\rightarrow$ component of the universe whose eq of state parameter $-1 < w < 1/3$
$\Rightarrow \ddot{a} > 0$

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{1-\Omega_0}{a^2}$$

$\Omega_0 = 1 \rightarrow$ spatially flat

$$\frac{\dot{a}}{H_0} = \left(\frac{\Omega_{r,0}}{a^2} + \frac{\Omega_{m,0}}{a} + \Omega_{\Lambda,0}a^2 + (1-\Omega_0)\right)^{1/2}$$

$$\Rightarrow \quad H_0 t = \int_0^a \frac{da}{(\text{o/o})^{1/2}}$$

$\hookrightarrow$ can be integrated numerically

early stages $\rightarrow$ radiation dominates expansion

$$\Rightarrow \quad H_0 t \approx \int_0^a \frac{a\,da}{\sqrt{\Omega_{r,0}}} \approx \frac{a^2}{2\sqrt{\Omega_{r,0}}}$$

$$\Leftrightarrow \quad a(t) \approx \left(2\sqrt{\Omega_{r,0}}\,H_0 t\right)^{1/2}$$

Benchmark Model $\sim$ radiation-matter equality
$a_{rm} \sim 2{,}8 \times 10^{-4}$
$a_{m\Lambda} \sim 0{,}75$

Matter + Curvature

$w = 0$
flat $\quad a(t) = \left(\frac{t}{t_0}\right)^{2/3}$

matter + curved: $(\Omega_{m,0} = \Omega_0)$

$$\frac{H^2}{H_0^2} = \frac{\Omega_0}{a^3} + \frac{1-\Omega_0}{a^2}$$

matter universe will cease to expand if $\Omega_0 > 1$ and $\kappa = +1$

a matter-dominated universe with $\Omega_0 > 1$
$\rightarrow$ expand to a_{max} $\rightarrow$ Big crunch

density	curvature	ultimate fate
$\Omega_0 < 1$	$\kappa = -1$	Big Chill at t
$\Omega_0 = 1$	$\kappa = 0$	Big Chill at $t^{2/3}$
$\Omega > 1$	$\kappa = +1$	Big Crunch

$$\frac{\dot{a}}{H_0^2} = \frac{\Omega_0}{a} + (1-\Omega_0)$$

$$H_0 t = \int_0^a \frac{da}{\left(\frac{\Omega_0}{a} + (1-\Omega_0)\right)^{1/2}}$$

$$a(\theta) = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}(1 - \cos\theta)$$

$$t(\theta) = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}(\theta - \sin\theta)$$

Matter + Lambda

$\sim$

Matter + Curvature + Lambda

$\sim$

Radiation + Matter

$\sim$

Benchmark Model $\rightarrow$ best fit to the current data

$\kappa = 0$ flat
radiation + matter + Λ

radiation = CMB $\sim 2{,}725$ K $\sim \Omega_{\gamma,0} = 5{,}0 \times 10^{-5}$

$\Omega_{bary,0} \approx 0{,}04$
$\Omega_{dm,0} \approx 0{,}26$
$\Omega_\Lambda = 1 - \Omega_{m,0} - \Omega_{r,0} \approx 0{,}7$

$t_{rm} = 4{,}7 \times 10^4$ yr
$t_{m\Lambda} = 9{,}8$ Gyr
$t_0 = 13{,}5$ Gyr

Chapter 7: Measuring Cosmological Parameters

A search for two numbers

Taylor series $a(t)$:

$$a(t) = a_0 + \dot{a}\Big|_{t_0}(t-t_0) + \frac{1}{2}\ddot{a}\Big|_{t_0}(t-t_0)^2$$

$$\frac{a(t)}{a_0} \approx 1 + \frac{\dot{a}}{a}\Big|_{t_0}(t-t_0) + \frac{1}{2}\frac{\ddot{a}}{a}\Big|_{t_0=0}(t-t_0)^2$$

$$a(t) \approx 1 + H_0(t-t_0) - \frac{1}{2}q_0 H_0^2 (t-t_0)^2$$

$q_0 \to$ deceleration parameter

$$q_0 \equiv \left(\frac{-\ddot{a}a}{\dot{a}^2}\right)_{t=t_0} = -\left(\frac{\ddot{a}}{aH^2}\right)_{t=t_0}$$

multiple component

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3c^2}\sum_w \varepsilon_w(1+3w)$$

$$\frac{-\ddot{a}}{aH^2} = \frac{1}{2}\left[\frac{8\pi G}{3c^2H^2}\right]\sum_w \varepsilon_w(1+3w) \qquad \frac{\varepsilon_w}{\varepsilon_c} = \Omega_w$$

$$= \frac{1}{2}\sum_w \Omega_w(1+3w)$$

$$q_0 = \frac{1}{2}\sum_w \Omega_{w,0}(1+3w)$$

$$q_0 = \frac{1}{2}\left(\Omega_{m,0} + 2\Omega_{r,0} - 2\Omega_{\Lambda,0}\right)$$

$$= \Omega_{r,0} + \frac{1}{2}\Omega_{m,0} - \Omega_{\Lambda,0}$$

Luminosity distance

$$\lambda_0 = \frac{1}{a(t_e)}\lambda_e = (1+z)\lambda_e$$

- the expansion of the universe causes the energy of each photon to decrease

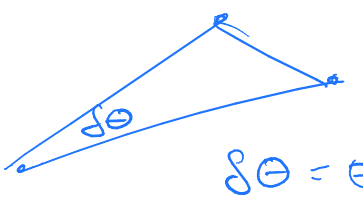
$$E_0 = \frac{E_e}{1+z}$$

expanding, spatially curved universe:

$$f = \frac{L}{4\pi S_k^2(r)(1+z)^2}$$

$$d_L = S_k(r)(1+z)$$

Angular diameter distance



$\delta\theta = \theta_2 - \theta_1$

$$d_A = \frac{l}{\delta\theta}$$

$$d_s = a(t_e)S_k(r)\delta\theta$$

$$d_s = l$$

$$l = a(t_e)S_k(r)\delta\theta = \frac{S_k(r)\delta\theta}{1+z}$$

$$d_A \equiv \frac{S_k(r)}{1+z} = \frac{l}{\delta\theta}$$

$$d_A = \frac{d_L}{(1+z^2)}$$

Standard candles & the Hubble constant

- s.c. with Luminosity L
- measure z and f
- $d_L = \frac{L}{(4\pi f)^{1/2}}$
- plot cz v/s d_L
- slope $\to H_0$ $(z \ll 1)$

standard candles $\to$ Cepheid variable stars

$\bar{L} = 400 \to 40.000\,L_\odot$

$\bar{P} = 1{,}5 \to 60$ days

Standard candles & the accelerating universe

$\hookrightarrow$ type Ia supernovae

- the rise and fall time of a type Ia SN tells you its peak luminosity

Chapter 8 : Dark Matter

Visible Matter

$\Omega_{*,0} \approx 0.004$

80% of the stellar mass comes from stars less luminous than the sun

In a galaxy, the mass of interstellar gas is $\approx 10\%$ of the mass of the stars

stellar mass is small compared to the mass of the intercluster gas between galaxies

$$M_{gas} \sim 6 M_{stars}$$

Dark Matter in Galaxies

majority of the matter in the universe is non baryonic dark matter

→ doesn't absorb, emit or scatter light

→ look its gravitational influence

→ looking at the orbital speeds of stars in spiral galaxies

$$a = \frac{v^2}{R} = \frac{GM(R)}{R^2} \Rightarrow v = \sqrt{\frac{GM(R)}{R}}$$

surface brightness

$$I(R) = I(0)\, e^{\frac{-R}{R_s}}$$

if stars contributed all the mass

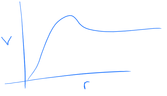
$$v \propto \frac{1}{\sqrt{R}} \quad \text{Keplerian rotation}$$

$$\frac{b}{a} = \cos i$$

redshift → radial velocity $v_r(R) = cz(R)$

$$v_r(R) = v_{gal} + v(R)\sin i$$

$$\Rightarrow v(R) = \frac{v_r(R) - v_{gal}}{\sin i} = \frac{v_r(R) - v_{gal}}{\sqrt{1 - \frac{b^2}{a^2}}}$$



since the orbital speed > than stars + gas

⇒ dark matter halo

dark halo $\sim 10\, M_{galaxy}$

$$\Omega_{gal,0} \approx 0.016$$

Dark Matter in Clusters

Potential energy cluster $W = -\alpha \frac{GM^2}{r_n}$

$M = \sum_i m_i$, r_n : half mass radius

$$K = \frac{1}{2} M \langle v \rangle^2 \quad ; \quad \langle v \rangle = \frac{1}{M} \sum m_i \dot{x}_i^2$$

$$I(\text{inertia}) \equiv \sum m_i x_i^2$$

$$\ddot{I} = 2 \sum m_i (\dot{x}_i \cdot \ddot{x}_i) + 4K$$

$$= 2W + 4K \quad \text{Virial theorem}$$

steady state

$$K = \frac{-W}{2}$$

$$\Rightarrow \frac{1}{2} M \langle v \rangle^2 = \frac{\alpha}{2} \frac{GM^2}{r_n}$$

$$\Rightarrow M = \frac{\langle v \rangle^2 r_n}{\alpha G}$$

intracluster gas

$$\frac{dP}{dr} = \frac{-GM(r)\rho(r)}{r^2}$$

$$P = \frac{\rho k T}{\mu m_p} \quad \text{perfect gas law}$$

$$\Rightarrow M(r) = \frac{kT(r)}{G\mu m_p} r \left[-\frac{d \ln \rho}{d \ln r} - \frac{d \ln T}{d \ln r} \right]$$

$$\Omega_{clus,0} \approx 0.2$$

Gravitational lensing

dark matter affect trajectories of light

dark matter → MACHOs : Massive Compact Halo Objects

→ brown dwarfs, white dwarfs, neutron stars, black holes

$$\alpha = \frac{4GM}{c^2 b}$$

$$\theta_E = \left(\frac{4GM}{c^2 d} \left(\frac{1-x}{x} \right) \right)^{1/2}$$

lensing event → star become brighter as the angular distance between the star and MACHO decreases, and dimmer as the angular distance increases again

lensing events → 20% halo mass MACHO

What's the Matter?

neutrinos ~ Black holes

Particles which interact via the weak nuclear force, but which are much more massive than the upper limit on the neutrino mass

→ Weakly Interacting Massive Particles → WIMP's

Chapter 9 The Cosmic Microwave Background

$$q = \frac{m_{proc}}{n_\gamma} \approx \frac{0.25\,m^3}{4\pi(1.9\,m)^3} \approx 5\times10^{-3}$$

Baryons are subordinated by photons

Observing the CMB

If the universe started in a hot dense state → should now be filled with microwave radiation

- isotropic, unpolarized Temperature
- Microwaves with wavelengths shorter than 1 cm are strongly absorbed by water molecules
- At any angular position (θ,ϕ) the spectrum of the CMB is very close to a Blackbody
- CMB has the dipole deviation $\propto T$
  → Doppler shift because of COBE (Cosmic Background Explorer) motion

Velocities around:

Earth	370 km/s
Sun	20 km/s
Galaxy center	220 km/s
Local Group	80 km/s

- After the dipole deviation is subtracted the remaining temperature fluctuations are small

$$\frac{\delta T}{T} \approx 10^{-5} \rightarrow \text{isotropy}$$

→ Hot big bang Model

Recombination and Decoupling

When the temperature matter sees from being an ionized plasma to a gas of neutral atoms

opaque → transparent

recombination $\hat{=}$ baryons ionized → neutral
→ time at Period $\hat{=}$ Epoch

Photon decoupling $\hat{=}$ time at which the rate at which photons scatter from electrons becomes smaller than the Hubble parameter
→ electrons cease to interact with e^-
→ universe become transparent

Last scattering $\hat{=}$ time at which a photon undergoes its last scattering from e^- (with σ_T)

early universe ~ pure H-plasma

neutrino charge neutrality → $n_e = n_p$

$$X = \frac{n_p}{n_p + n_H} = \frac{n_p}{n_{baryons}} = \frac{n_p}{n_{nucl}}$$

X ≪ baryons ionized
X ≫ neutral atoms

$$H + \gamma \rightleftharpoons p + e^-$$

before recombination $kT_{rec} \approx 0.3\,eV$
→ H about half → $X \sim 1$

Thomson Scattering:

$$\gamma + e^- \rightarrow \gamma + e^-$$

$$\lambda(\text{free path}) = \frac{1}{n_e \sigma_T}$$

$$\tau = \frac{c}{3}\, n_e \sigma_T c \quad (\text{scattering rate})$$

$$n_e = n_{baryons} X = \frac{n_{baryons}\, X}{a^3}$$

$$\Rightarrow \tau = \frac{n_{baryons}\, \sigma_T c}{a^3}$$

photons, e^-, p → thermal equilibrium

when τ drops below $H = \frac{\dot{a}}{a}$:
e^- are being diluted by expansion more rapidly than the photons can interact with them
→ γ decouples from e^- → transparent

IF H had remained constant
→ decoupling at Tatom (low energetic keep H ionized)

The physics of recombination

$$H + \gamma \rightleftharpoons p + e^-$$

Saha-Boltzmann equation:

$$n_x = g_x \left(\frac{m_x kT}{2\pi\hbar^2}\right)^{3/2} \exp\left(\frac{m_x \mu}{kT}\right)$$

non-relativistic $kT \ll m_x c^2$

$$\frac{n_H}{n_p n_e} = \frac{g_H}{g_p g_e}\left(\frac{m_H}{m_p m_e}\right)^{3/2}\left(\frac{kT}{2\pi\hbar^2}\right)^{-3/2} \exp\left(\frac{(m_p c^2 + m_e c^2 - m_H c^2)}{kT}\right)$$

$$m_H - m_p \quad Q = 13.6\,eV = (m_p c^2 + m_e c^2 - m_H c^2)$$

$$\Rightarrow \frac{n_H}{n_p n_e} = \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2} \exp\left(\frac{Q}{kT}\right) \quad \text{Saha equation}$$

recombination T also $x = 1/2$

$$kT_{rec} \approx 0.323\,eV$$

$$n(z) = n_p(z)\,\sigma_T c = X(z)\,(1+z)^3\, n_{baryon}\,\sigma_T c$$

scatterings

$$\tau(z) = \int_z^\infty \Gamma(z)\,dt$$

optical depth

$t \rightarrow$ time of last scattering

$$\tau(z) = \int_0^\infty \frac{\Gamma(z)\,dz}{H(z)(1+z)} = 0.0055 \int_0^\infty X(z)\,(1+z)^{\frac{1}{2}}\,dz$$

recombination, photon decoupling, last scattering
→ matter dominated era

Temperature Fluctuations

$\Delta_\theta = \frac{1}{60}$, 1 degree angular size of T° fluctuation in the CMB

last scattering surface

fluctuations in CMB are now $l(l+1\,\sigma_l) > 1\,\text{Gpc}\,\sigma_l$
$> 0.1\,\text{Gpc}$

$$\frac{\delta T}{T}(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta,\phi)$$

$C(\theta)$: correlation function

$$C(\theta) = \left\langle \frac{\delta T}{T}(\hat{z}) \frac{\delta T}{T}(\hat{z}') \right\rangle_{\hat{z}\cdot\hat{z}' = \cos\theta}$$

$C(\theta, l(l+1))$: complete statistical description of the T° fluctuations

$$C(\theta) = \frac{1}{4\pi}\sum_l (2l+1)\, C_l\, P_l(\cos\theta)$$

multipole moments

→ measure of T° fluctuations on the angular scale $\frac{\theta \sim 180}{l}$

$l = 0$ (monopole)
last dipole → Doppler shift the lower motion

$l \geq 2$ → tells us about the fluctuations present at the time of last scattering

$$\Delta T = \left(\frac{l(l+1)}{2\pi}\, C_l\right)^{\frac{1}{2}} \langle T \rangle$$

T° fluctuations has a peak in $l = 200 \rightarrow l^*$

What causes the fluctuations?

$$\frac{c}{H(z_{dec})} \approx 0.1\,\text{Gpc}$$

$$\theta_H = \frac{\sqrt{\Omega_m(z_{dec})}}{1+z} \approx l^* \quad \text{(angular size of last scattering)}$$

at the time of last scattering $\Omega_m \sim \Omega_b > \Omega_{rad}$

T° fluctuations arise from density fluctuations in the distribution of dm.

$$\boxed{\Delta\rho = \delta\rho}$$

$$\rho_m(x) = \bar{\rho}_m + \delta\bar{\rho}_m$$

$$\nabla^2(\delta\phi) = \frac{4\pi G}{c^2}\delta\rho$$

$$\frac{\delta T}{T} = \frac{1}{3}\frac{\delta\phi}{c^2}$$

- photon-baryon fluid → oscillates because of potential wells
  → compressed - gravity
  → expands because of pressure and so on
  → acoustic oscillations

at the time of decoupling
→ γb may be compressed → bulge
or expanded → valley

→ $l \approx 200$, $\sigma \approx l^*$ → potential wells photons reach max compression

The observed peak → K → or Dal

$$C_s = \sqrt{1/3}\, c$$

$$\Rightarrow \theta_{Dal} = 0.04\,\text{rad}$$

Chapter 10 : Nucleosynthesis and the early universe

Nuclear physics and Cosmology

Big Bang Nucleosynthesis (BBN) → first three minutes

$\frac{3}{4}\Omega_{baryonic}$ ~ only $^1H \equiv$ protons

$$Y_p \equiv \frac{\rho(^4He)}{\rho_{bary}} \sim 0{,}24 \text{ in stars}$$

Neutrons and protons

free neutron → instable

$$n \to p + e^- + \bar{\nu}_e$$

at $t = 0{,}1$ sec

$$\gamma + \gamma \rightleftarrows e^- + e^+$$

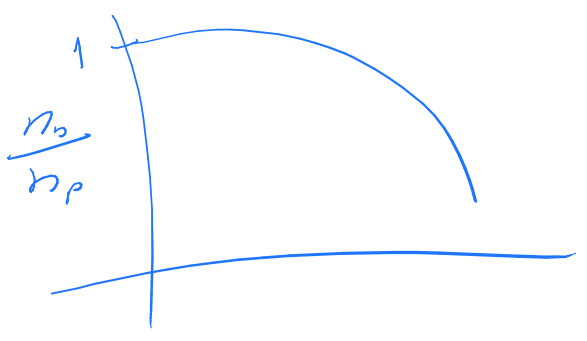
$$n + \nu_e \rightleftarrows p + e^-$$

$$n + e^+ \rightleftarrows p + \bar{\nu}_e$$

neutron, proton density → Maxwell-Boltzmann eq

$$\Rightarrow \frac{n_n}{n_p} = \left(\frac{m_n}{m_p}\right)^{3/2} \exp\left(-\frac{(m_n - m_p)c^2}{kT}\right)$$

$$\approx \exp\left(-\frac{Q_n}{kT}\right)$$



$n \to p$

- a neutron will fuse with a proton more readily (facilmente) than a $p + p$

$$p + p \rightleftarrows D + e^+ + \nu_e \quad \text{weak interaction}$$

$$p + n \rightleftarrows D + \gamma \quad \text{strong interaction}$$

$p + p$ rate < $p + n$

- $p + p$ coulomb potential
- σ_{pp} cross section < σ_{np}

- after ≈ 1 hour → T^0, ρ too low for fusion

BBN proceeds until every free neutron is bounded

→ theoretically $Y_p \approx \frac{1}{3}$

but $Y_p \approx 0{,}24$ //

Deuterium synthesis

$$p + n \rightleftarrows D + \gamma$$

photoionization: $p + e^- \to H + \gamma$

$t_{nuc} \approx 200\,s \to n_D = n_n$

Beyond Deuterium

$$D + p \rightleftarrows {}^3He + \gamma$$

$$D + n \rightleftarrows {}^3H + \gamma$$

$$D + D \rightleftarrows {}^4He + \gamma$$

$$D + D \rightleftarrows {}^3H + p$$

$$D + D \rightleftarrows {}^3He + n$$

There is never a large amount of ^3H or ^3He

once BBN begins, $D, {}^3H, {}^3He \to {}^4He$

end with all baryons in form of free p or ^4He.

quasars → iluminate intergalactic gas

→ if intergalactic gas has no stars

$\sim \frac{D}{H} \sim$ primordial value

- when we look at the quasar, we se an absorption line at $\lambda_H(1 + z_{ul})$

Baryon - Antibaryon asymmetry

Chapter 11: Inflation and the very early universe

Flatness problem:

- the universe is nearly flat today, and was even flatter in the past

horizon problem:

- the universe is nearly isotropic and homogeneous today, and was even more so in the past

monopole problem:

- the universe is free of magnetic monopoles

Flatness Problem

$\Omega = 1$ or $k = 0$

fine-tuning:

$|1-\Omega_0| \lesssim 10^{-60}$

$\Omega = 1 \rightarrow$ no physical explanation

$k = 0 \rightarrow$ satisfactory to find mechanism for flattening

Horizon Problem

How can 2 points that never had time to swap information be so nearly identical properties?

- size of last scattering $\rightarrow$ smaller - dominated

$d_H(t_0) \sim 13\,Mpc \sim 2°$

$\frac{\delta T}{T} \sim 10^{-5}$ on scales $\theta > 2°$ or $l < 100$

Monopole Problem

phase transition GUT $\rightarrow$ magnetic monopoles

$\downarrow$

loss of symmetry $\rightarrow$ topological defects

- no evidence of monopoles
- no isolated matter/antimatter pairs

The inflation solution

attractive idea $\ddot{a} > 0$

$\frac{\ddot{a}}{a} = \frac{\dot{\rho}_T}{3} > 0 \quad (w = -1)$

$a \propto e^{Ht}$

$$a(t) = \begin{cases} a(t_i/t_f)^{1/2} & t < t_i \\ a \cdot e^{H(t-t_i)} & t_i < t < t_f \\ a e^{H(t_f - t_i)} \left(\frac{t}{t_f}\right)^{1/2} & t > t_f \end{cases}$$

$$\frac{a(t_f)}{a(t_i)} = e^{N}$$

$N \equiv H \cdot (t_f - t_i)$

GUT:

$\frac{a(t_f)}{a(t_i)} \sim 10^{13}$

$|1-\Omega| \propto t^{1/(N+1)/(N+3)}$

if $w \neq -1$:

$|1-\Omega(t)| \propto e^{-2Ht}$

$\Rightarrow |1-\Omega(t_f)| = e^{-2N}|1-\Omega(t_i)|$

if $k = 1 \Rightarrow |1-\Omega| \sim 1 \quad (\Omega = 0)$

$\Rightarrow |1-\Omega(t_f)| \sim e^{-2N} \sim 10^{-87} \sim e^{-200}$

even if the universe at t_i was not flat

only if $N \geq 100$ $\rightarrow$ flattening the universe

e-foldings

if $|1-\Omega_0| \lesssim 10^{-62} \rightarrow N \geq 60$

$d_{hor}(t_i) = e^{N}(2t_i + H_i^{-1})$

$\Rightarrow d_{hor}(t_i) \sim 6m10^{-18}m$

$d_{hor}(t_f) \sim 7 \times 10^{16}m \sim 0.8\,pc$

inflation $\rightarrow$ increase the horizon length by $\sim e^{N}$ over which would have been without inflation

if $N = 100$: $d_{hor}(t_f) \sim 10^{17}\,Mpc$

$\rightarrow$ enough for the LS surface to be in causal contact

---

if magnetic monopoles were created before or during inflation

$\rightarrow$ density of monopoles slow

$n_M \propto e^{-3Ht}$

after $N = 100 \rightarrow n_M(t_f) \sim 5 \times 10^{-44}\,m^{-3}$

$t_f \sim t_0 \rightarrow 10^{-4}\,Mpc$

- the probability of finding a monopole is small

The physics of inflation

- What triggers inflation $t \sim t_f$?
- Why does it end at $t = t_f$?
- Why keep it slow-roll not the last curvature the fluctuations in the energy density?

- There is no consensus about the physics of inflation

$\phi(\vec{r}, t)$: scalar field $\rightarrow V(\phi)$

energy density

$\rho_\phi = \frac{1}{2}\frac{1}{\hbar c^2}\dot{\phi}^2 + V(\phi)$

if ϕ is homogeneous

$P_\phi = \frac{1}{2}\frac{1}{\hbar c^2}\dot{\phi}^2 - V(\phi)$ not the contact eqn.

if $\dot{\phi}^2 \ll \hbar c^2 V(\phi)$ $\quad (\dot{\phi} \sim \Lambda)$

$\Rightarrow \rho_\phi \approx -P_\phi \approx V(\phi)$

inflation $\Rightarrow \phi$ small $\sim V(\phi)$ large

Field eq:

$\ddot{\phi} + 3H(t)\dot{\phi} + \frac{dV}{d\phi} = 0$

$\Rightarrow \ddot{\phi} + 3H(t)\dot{\phi} + \hbar c^2 \frac{dV}{d\phi} = 0$

"slow-roll"

inflation field reach "terminal velocity"

$\ddot{\phi} = 0 \Rightarrow 3H\dot{\phi} = -\hbar c^2 \frac{dV}{d\phi}$

$\dot{\phi}^2 \ll \hbar c^2 V \Rightarrow \left(\frac{dV}{d\phi}\right)^2 \ll 9H^2 V / \hbar c^2$

$\Rightarrow \left(\frac{M_{Pl}}{V}\frac{dV}{d\phi}\right)^2 \ll 1$ $\quad$ Planck mass

with $H = \left(\frac{8\pi G V}{3c^2}\right)^{1/2}$ "metric"

around ϕ_0, V is slow-roller, damped by "Hubble friction" $(3H\dot{\phi})$

energy of ϕ interacts with γ or relativistic particles $\Rightarrow$ γ reheat the universe

- the transition from false to true vacuum releases an energy V_0 which goes to reheat the universe

also, after $N = 100$ e-folding $\frac{\delta\rho}{\rho} \sim 10^{-5}$

BUT

- inflation exists initially sub-microscopic scales and blowing them up to macroscopic scales
- microscopic scale $\rightarrow$ vacuum $\rightarrow$ quantum fluctuations (virtual particles popping in and out existence)
- inflation takes QF $\rightarrow$ macroscopic scale "Great superclusters from tiny quantum fluctuations grown"

Chapter 11: The Theory of Distortion

- Large Scale Structure (LSS) of the universe
- LSS gives statistical survey of cosmological properties, or density field $\delta(\vec{x}, \tau)$
- Galaxies collect under influence of overall tidal forces

Gaussian Probability

Average overdensity:

$$\delta(\vec{x}) = \frac{1}{V}\int_V d^3x\, \delta(\vec{x})$$

Probability Density Distribution

$$P(\delta) = \frac{P(\delta)\, d\delta}{d\delta}$$

$\delta \to$ overdense region

$\delta \to$ underdense region

A Halo is a low probability theory

$$P = \frac{1}{\sqrt{2\pi}}\int_{\delta_c}^{\infty} d\delta \left(\frac{\delta^2}{2\sigma^2}\right)$$

$$\Rightarrow \int_{\delta_c}^{\infty} \frac{-\delta}{\sigma^2} d\delta$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \frac{-\delta}{\sigma^2} \exp\left(\frac{-\delta^2}{2\sigma^2}\right)$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \frac{

Sunday October 13th

Q1

Sub question 1

$$\frac{1}{x+1} + \frac{1}{x-1} = \frac{2}{x^2-1}$$

$$\frac{1}{x+1} + \frac{1}{x-1} = \frac{2}{(x+1)(x-1)}$$

$$\frac{(x-1)+(x+1)}{(x+1)(x-1)} = \frac{2}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2}{(x+1)(x-1)}$$

$$2x = 2$$

$$x = 1$$

Sub question 2

$$\frac{x+2}{(x+1)(x-1)} + \frac{x-2}{(x+1)(x-1)} = \frac{2x}{(x+1)(x-1)}$$

$$\frac{(x+2)+(x-2)}{(x+1)(x-1)} = \frac{2x}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x}{(x+1)(x-1)}$$

$$2x = 2x$$

$$x = x$$

Sub question 3

$$\frac{x+3}{(x+1)(x-1)} + \frac{x-3}{(x+1)(x-1)} = \frac{2x^2}{(x+1)(x-1)}$$

$$\frac{(x+3)+(x-3)}{(x+1)(x-1)} = \frac{2x^2}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^2}{(x+1)(x-1)}$$

$$2x = 2x^2$$

$$x = x^2$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 4

$$\frac{x+4}{(x+1)(x-1)} + \frac{x-4}{(x+1)(x-1)} = \frac{2x^3}{(x+1)(x-1)}$$

$$\frac{(x+4)+(x-4)}{(x+1)(x-1)} = \frac{2x^3}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^3}{(x+1)(x-1)}$$

$$2x = 2x^3$$

$$x = x^3$$

$$x^3 - x = 0$$

$$x(x^2-1) = 0$$

$$x(x-1)(x+1) = 0$$

$$x = 0 \quad x = 1 \quad x = -1$$

Sub question 5

$$\frac{x+5}{(x+1)(x-1)} + \frac{x-5}{(x+1)(x-1)} = \frac{2x^4}{(x+1)(x-1)}$$

$$\frac{(x+5)+(x-5)}{(x+1)(x-1)} = \frac{2x^4}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^4}{(x+1)(x-1)}$$

$$2x = 2x^4$$

$$x = x^4$$

$$x^4 - x = 0$$

$$x(x^3-1) = 0$$

$$x(x-1)(x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 6

$$\frac{x+6}{(x+1)(x-1)} + \frac{x-6}{(x+1)(x-1)} = \frac{2x^5}{(x+1)(x-1)}$$

$$\frac{(x+6)+(x-6)}{(x+1)(x-1)} = \frac{2x^5}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^5}{(x+1)(x-1)}$$

$$2x = 2x^5$$

$$x = x^5$$

$$x^5 - x = 0$$

$$x(x^4-1) = 0$$

$$x(x-1)(x^2+1)(x+1) = 0$$

$$x = 0 \quad x = 1 \quad x = -1$$

Sub question 7

$$\frac{x+7}{(x+1)(x-1)} + \frac{x-7}{(x+1)(x-1)} = \frac{2x^6}{(x+1)(x-1)}$$

$$\frac{(x+7)+(x-7)}{(x+1)(x-1)} = \frac{2x^6}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^6}{(x+1)(x-1)}$$

$$2x = 2x^6$$

$$x = x^6$$

$$x^6 - x = 0$$

$$x(x^5-1) = 0$$

$$x(x-1)(x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 8

$$\frac{x+8}{(x+1)(x-1)} + \frac{x-8}{(x+1)(x-1)} = \frac{2x^7}{(x+1)(x-1)}$$

$$\frac{(x+8)+(x-8)}{(x+1)(x-1)} = \frac{2x^7}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^7}{(x+1)(x-1)}$$

$$2x = 2x^7$$

$$x = x^7$$

$$x^7 - x = 0$$

$$x(x^6-1) = 0$$

$$x(x-1)(x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 9

$$\frac{x+9}{(x+1)(x-1)} + \frac{x-9}{(x+1)(x-1)} = \frac{2x^8}{(x+1)(x-1)}$$

$$\frac{(x+9)+(x-9)}{(x+1)(x-1)} = \frac{2x^8}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^8}{(x+1)(x-1)}$$

$$2x = 2x^8$$

$$x = x^8$$

$$x^8 - x = 0$$

$$x(x^7-1) = 0$$

$$x(x-1)(x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 10

$$\frac{x+10}{(x+1)(x-1)} + \frac{x-10}{(x+1)(x-1)} = \frac{2x^9}{(x+1)(x-1)}$$

$$\frac{(x+10)+(x-10)}{(x+1)(x-1)} = \frac{2x^9}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^9}{(x+1)(x-1)}$$

$$2x = 2x^9$$

$$x = x^9$$

$$x^9 - x = 0$$

$$x(x^8-1) = 0$$

$$x(x-1)(x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 11

$$\frac{x+11}{(x+1)(x-1)} + \frac{x-11}{(x+1)(x-1)} = \frac{2x^{10}}{(x+1)(x-1)}$$

$$\frac{(x+11)+(x-11)}{(x+1)(x-1)} = \frac{2x^{10}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{10}}{(x+1)(x-1)}$$

$$2x = 2x^{10}$$

$$x = x^{10}$$

$$x^{10} - x = 0$$

$$x(x^9-1) = 0$$

$$x(x-1)(x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 12

$$\frac{x+12}{(x+1)(x-1)} + \frac{x-12}{(x+1)(x-1)} = \frac{2x^{11}}{(x+1)(x-1)}$$

$$\frac{(x+12)+(x-12)}{(x+1)(x-1)} = \frac{2x^{11}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{11}}{(x+1)(x-1)}$$

$$2x = 2x^{11}$$

$$x = x^{11}$$

$$x^{11} - x = 0$$

$$x(x^{10}-1) = 0$$

$$x(x-1)(x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 13

$$\frac{x+13}{(x+1)(x-1)} + \frac{x-13}{(x+1)(x-1)} = \frac{2x^{12}}{(x+1)(x-1)}$$

$$\frac{(x+13)+(x-13)}{(x+1)(x-1)} = \frac{2x^{12}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{12}}{(x+1)(x-1)}$$

$$2x = 2x^{12}$$

$$x = x^{12}$$

$$x^{12} - x = 0$$

$$x(x^{11}-1) = 0$$

$$x(x-1)(x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 14

$$\frac{x+14}{(x+1)(x-1)} + \frac{x-14}{(x+1)(x-1)} = \frac{2x^{13}}{(x+1)(x-1)}$$

$$\frac{(x+14)+(x-14)}{(x+1)(x-1)} = \frac{2x^{13}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{13}}{(x+1)(x-1)}$$

$$2x = 2x^{13}$$

$$x = x^{13}$$

$$x^{13} - x = 0$$

$$x(x^{12}-1) = 0$$

$$x(x-1)(x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 15

$$\frac{x+15}{(x+1)(x-1)} + \frac{x-15}{(x+1)(x-1)} = \frac{2x^{14}}{(x+1)(x-1)}$$

$$\frac{(x+15)+(x-15)}{(x+1)(x-1)} = \frac{2x^{14}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{14}}{(x+1)(x-1)}$$

$$2x = 2x^{14}$$

$$x = x^{14}$$

$$x^{14} - x = 0$$

$$x(x^{13}-1) = 0$$

$$x(x-1)(x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 16

$$\frac{x+16}{(x+1)(x-1)} + \frac{x-16}{(x+1)(x-1)} = \frac{2x^{15}}{(x+1)(x-1)}$$

$$\frac{(x+16)+(x-16)}{(x+1)(x-1)} = \frac{2x^{15}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{15}}{(x+1)(x-1)}$$

$$2x = 2x^{15}$$

$$x = x^{15}$$

$$x^{15} - x = 0$$

$$x(x^{14}-1) = 0$$

$$x(x-1)(x^{13}+x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 17

$$\frac{x+17}{(x+1)(x-1)} + \frac{x-17}{(x+1)(x-1)} = \frac{2x^{16}}{(x+1)(x-1)}$$

$$\frac{(x+17)+(x-17)}{(x+1)(x-1)} = \frac{2x^{16}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{16}}{(x+1)(x-1)}$$

$$2x = 2x^{16}$$

$$x = x^{16}$$

$$x^{16} - x = 0$$

$$x(x^{15}-1) = 0$$

$$x(x-1)(x^{14}+x^{13}+x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 18

$$\frac{x+18}{(x+1)(x-1)} + \frac{x-18}{(x+1)(x-1)} = \frac{2x^{17}}{(x+1)(x-1)}$$

$$\frac{(x+18)+(x-18)}{(x+1)(x-1)} = \frac{2x^{17}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{17}}{(x+1)(x-1)}$$

$$2x = 2x^{17}$$

$$x = x^{17}$$

$$x^{17} - x = 0$$

$$x(x^{16}-1) = 0$$

$$x(x-1)(x^{15}+x^{14}+x^{13}+x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 19

$$\frac{x+19}{(x+1)(x-1)} + \frac{x-19}{(x+1)(x-1)} = \frac{2x^{18}}{(x+1)(x-1)}$$

$$\frac{(x+19)+(x-19)}{(x+1)(x-1)} = \frac{2x^{18}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{18}}{(x+1)(x-1)}$$

$$2x = 2x^{18}$$

$$x = x^{18}$$

$$x^{18} - x = 0$$

$$x(x^{17}-1) = 0$$

$$x(x-1)(x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 20

$$\frac{x+20}{(x+1)(x-1)} + \frac{x-20}{(x+1)(x-1)} = \frac{2x^{19}}{(x+1)(x-1)}$$

$$\frac{(x+20)+(x-20)}{(x+1)(x-1)} = \frac{2x^{19}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{19}}{(x+1)(x-1)}$$

$$2x = 2x^{19}$$

$$x = x^{19}$$

$$x^{19} - x = 0$$

$$x(x^{18}-1) = 0$$

$$x(x-1)(x^{17}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Q2

Sub question 1

$$\frac{1}{x+1} + \frac{1}{x-1} = \frac{2}{x^2-1}$$

$$\frac{1}{x+1} + \frac{1}{x-1} = \frac{2}{(x+1)(x-1)}$$

$$\frac{(x-1)+(x+1)}{(x+1)(x-1)} = \frac{2}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2}{(x+1)(x-1)}$$

$$2x = 2$$

$$x = 1$$

Sub question 2

$$\frac{x+2}{(x+1)(x-1)} + \frac{x-2}{(x+1)(x-1)} = \frac{2x}{(x+1)(x-1)}$$

$$\frac{(x+2)+(x-2)}{(x+1)(x-1)} = \frac{2x}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x}{(x+1)(x-1)}$$

$$2x = 2x$$

$$x = x$$

Sub question 3

$$\frac{x+3}{(x+1)(x-1)} + \frac{x-3}{(x+1)(x-1)} = \frac{2x^2}{(x+1)(x-1)}$$

$$\frac{(x+3)+(x-3)}{(x+1)(x-1)} = \frac{2x^2}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^2}{(x+1)(x-1)}$$

$$2x = 2x^2$$

$$x = x^2$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 4

$$\frac{x+4}{(x+1)(x-1)} + \frac{x-4}{(x+1)(x-1)} = \frac{2x^3}{(x+1)(x-1)}$$

$$\frac{(x+4)+(x-4)}{(x+1)(x-1)} = \frac{2x^3}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^3}{(x+1)(x-1)}$$

$$2x = 2x^3$$

$$x = x^3$$

$$x^3 - x = 0$$

$$x(x^2-1) = 0$$

$$x(x-1)(x+1) = 0$$

$$x = 0 \quad x = 1 \quad x = -1$$

Sub question 5

$$\frac{x+5}{(x+1)(x-1)} + \frac{x-5}{(x+1)(x-1)} = \frac{2x^4}{(x+1)(x-1)}$$

$$\frac{(x+5)+(x-5)}{(x+1)(x-1)} = \frac{2x^4}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^4}{(x+1)(x-1)}$$

$$2x = 2x^4$$

$$x = x^4$$

$$x^4 - x = 0$$

$$x(x^3-1) = 0$$

$$x(x-1)(x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 6

$$\frac{x+6}{(x+1)(x-1)} + \frac{x-6}{(x+1)(x-1)} = \frac{2x^5}{(x+1)(x-1)}$$

$$\frac{(x+6)+(x-6)}{(x+1)(x-1)} = \frac{2x^5}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^5}{(x+1)(x-1)}$$

$$2x = 2x^5$$

$$x = x^5$$

$$x^5 - x = 0$$

$$x(x^4-1) = 0$$

$$x(x-1)(x^2+1)(x+1) = 0$$

$$x = 0 \quad x = 1 \quad x = -1$$

Sub question 7

$$\frac{x+7}{(x+1)(x-1)} + \frac{x-7}{(x+1)(x-1)} = \frac{2x^6}{(x+1)(x-1)}$$

$$\frac{(x+7)+(x-7)}{(x+1)(x-1)} = \frac{2x^6}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^6}{(x+1)(x-1)}$$

$$2x = 2x^6$$

$$x = x^6$$

$$x^6 - x = 0$$

$$x(x^5-1) = 0$$

$$x(x-1)(x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 8

$$\frac{x+8}{(x+1)(x-1)} + \frac{x-8}{(x+1)(x-1)} = \frac{2x^7}{(x+1)(x-1)}$$

$$\frac{(x+8)+(x-8)}{(x+1)(x-1)} = \frac{2x^7}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^7}{(x+1)(x-1)}$$

$$2x = 2x^7$$

$$x = x^7$$

$$x^7 - x = 0$$

$$x(x^6-1) = 0$$

$$x(x-1)(x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 9

$$\frac{x+9}{(x+1)(x-1)} + \frac{x-9}{(x+1)(x-1)} = \frac{2x^8}{(x+1)(x-1)}$$

$$\frac{(x+9)+(x-9)}{(x+1)(x-1)} = \frac{2x^8}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^8}{(x+1)(x-1)}$$

$$2x = 2x^8$$

$$x = x^8$$

$$x^8 - x = 0$$

$$x(x^7-1) = 0$$

$$x(x-1)(x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 10

$$\frac{x+10}{(x+1)(x-1)} + \frac{x-10}{(x+1)(x-1)} = \frac{2x^9}{(x+1)(x-1)}$$

$$\frac{(x+10)+(x-10)}{(x+1)(x-1)} = \frac{2x^9}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^9}{(x+1)(x-1)}$$

$$2x = 2x^9$$

$$x = x^9$$

$$x^9 - x = 0$$

$$x(x^8-1) = 0$$

$$x(x-1)(x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 11

$$\frac{x+11}{(x+1)(x-1)} + \frac{x-11}{(x+1)(x-1)} = \frac{2x^{10}}{(x+1)(x-1)}$$

$$\frac{(x+11)+(x-11)}{(x+1)(x-1)} = \frac{2x^{10}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{10}}{(x+1)(x-1)}$$

$$2x = 2x^{10}$$

$$x = x^{10}$$

$$x^{10} - x = 0$$

$$x(x^9-1) = 0$$

$$x(x-1)(x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 12

$$\frac{x+12}{(x+1)(x-1)} + \frac{x-12}{(x+1)(x-1)} = \frac{2x^{11}}{(x+1)(x-1)}$$

$$\frac{(x+12)+(x-12)}{(x+1)(x-1)} = \frac{2x^{11}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{11}}{(x+1)(x-1)}$$

$$2x = 2x^{11}$$

$$x = x^{11}$$

$$x^{11} - x = 0$$

$$x(x^{10}-1) = 0$$

$$x(x-1)(x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 13

$$\frac{x+13}{(x+1)(x-1)} + \frac{x-13}{(x+1)(x-1)} = \frac{2x^{12}}{(x+1)(x-1)}$$

$$\frac{(x+13)+(x-13)}{(x+1)(x-1)} = \frac{2x^{12}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{12}}{(x+1)(x-1)}$$

$$2x = 2x^{12}$$

$$x = x^{12}$$

$$x^{12} - x = 0$$

$$x(x^{11}-1) = 0$$

$$x(x-1)(x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 14

$$\frac{x+14}{(x+1)(x-1)} + \frac{x-14}{(x+1)(x-1)} = \frac{2x^{13}}{(x+1)(x-1)}$$

$$\frac{(x+14)+(x-14)}{(x+1)(x-1)} = \frac{2x^{13}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{13}}{(x+1)(x-1)}$$

$$2x = 2x^{13}$$

$$x = x^{13}$$

$$x^{13} - x = 0$$

$$x(x^{12}-1) = 0$$

$$x(x-1)(x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 15

$$\frac{x+15}{(x+1)(x-1)} + \frac{x-15}{(x+1)(x-1)} = \frac{2x^{14}}{(x+1)(x-1)}$$

$$\frac{(x+15)+(x-15)}{(x+1)(x-1)} = \frac{2x^{14}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{14}}{(x+1)(x-1)}$$

$$2x = 2x^{14}$$

$$x = x^{14}$$

$$x^{14} - x = 0$$

$$x(x^{13}-1) = 0$$

$$x(x-1)(x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 16

$$\frac{x+16}{(x+1)(x-1)} + \frac{x-16}{(x+1)(x-1)} = \frac{2x^{15}}{(x+1)(x-1)}$$

$$\frac{(x+16)+(x-16)}{(x+1)(x-1)} = \frac{2x^{15}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{15}}{(x+1)(x-1)}$$

$$2x = 2x^{15}$$

$$x = x^{15}$$

$$x^{15} - x = 0$$

$$x(x^{14}-1) = 0$$

$$x(x-1)(x^{13}+x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 17

$$\frac{x+17}{(x+1)(x-1)} + \frac{x-17}{(x+1)(x-1)} = \frac{2x^{16}}{(x+1)(x-1)}$$

$$\frac{(x+17)+(x-17)}{(x+1)(x-1)} = \frac{2x^{16}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{16}}{(x+1)(x-1)}$$

$$2x = 2x^{16}$$

$$x = x^{16}$$

$$x^{16} - x = 0$$

$$x(x^{15}-1) = 0$$

$$x(x-1)(x^{14}+x^{13}+x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 18

$$\frac{x+18}{(x+1)(x-1)} + \frac{x-18}{(x+1)(x-1)} = \frac{2x^{17}}{(x+1)(x-1)}$$

$$\frac{(x+18)+(x-18)}{(x+1)(x-1)} = \frac{2x^{17}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{17}}{(x+1)(x-1)}$$

$$2x = 2x^{17}$$

$$x = x^{17}$$

$$x^{17} - x = 0$$

$$x(x^{16}-1) = 0$$

$$x(x-1)(x^{15}+x^{14}+x^{13}+x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 19

$$\frac{x+19}{(x+1)(x-1)} + \frac{x-19}{(x+1)(x-1)} = \frac{2x^{18}}{(x+1)(x-1)}$$

$$\frac{(x+19)+(x-19)}{(x+1)(x-1)} = \frac{2x^{18}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{18}}{(x+1)(x-1)}$$

$$2x = 2x^{18}$$

$$x = x^{18}$$

$$x^{18} - x = 0$$

$$x(x^{17}-1) = 0$$

$$x(x-1)(x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 20

$$\frac{x+20}{(x+1)(x-1)} + \frac{x-20}{(x+1)(x-1)} = \frac{2x^{19}}{(x+1)(x-1)}$$

$$\frac{(x+20)+(x-20)}{(x+1)(x-1)} = \frac{2x^{19}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{19}}{(x+1)(x-1)}$$

$$2x = 2x^{19}$$

$$x = x^{19}$$

$$x^{19} - x = 0$$

$$x(x^{18}-1) = 0$$

$$x(x-1)(x^{17}+x^{16}+x^{15}+x^{14}+x^{13}+x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Q3

Sub question 1

$$\frac{1}{x+1} + \frac{1}{x-1} = \frac{2}{x^2-1}$$

$$\frac{1}{x+1} + \frac{1}{x-1} = \frac{2}{(x+1)(x-1)}$$

$$\frac{(x-1)+(x+1)}{(x+1)(x-1)} = \frac{2}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2}{(x+1)(x-1)}$$

$$2x = 2$$

$$x = 1$$

Sub question 2

$$\frac{x+2}{(x+1)(x-1)} + \frac{x-2}{(x+1)(x-1)} = \frac{2x}{(x+1)(x-1)}$$

$$\frac{(x+2)+(x-2)}{(x+1)(x-1)} = \frac{2x}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x}{(x+1)(x-1)}$$

$$2x = 2x$$

$$x = x$$

Sub question 3

$$\frac{x+3}{(x+1)(x-1)} + \frac{x-3}{(x+1)(x-1)} = \frac{2x^2}{(x+1)(x-1)}$$

$$\frac{(x+3)+(x-3)}{(x+1)(x-1)} = \frac{2x^2}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^2}{(x+1)(x-1)}$$

$$2x = 2x^2$$

$$x = x^2$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 4

$$\frac{x+4}{(x+1)(x-1)} + \frac{x-4}{(x+1)(x-1)} = \frac{2x^3}{(x+1)(x-1)}$$

$$\frac{(x+4)+(x-4)}{(x+1)(x-1)} = \frac{2x^3}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^3}{(x+1)(x-1)}$$

$$2x = 2x^3$$

$$x = x^3$$

$$x^3 - x = 0$$

$$x(x^2-1) = 0$$

$$x(x-1)(x+1) = 0$$

$$x = 0 \quad x = 1 \quad x = -1$$

Sub question 5

$$\frac{x+5}{(x+1)(x-1)} + \frac{x-5}{(x+1)(x-1)} = \frac{2x^4}{(x+1)(x-1)}$$

$$\frac{(x+5)+(x-5)}{(x+1)(x-1)} = \frac{2x^4}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^4}{(x+1)(x-1)}$$

$$2x = 2x^4$$

$$x = x^4$$

$$x^4 - x = 0$$

$$x(x^3-1) = 0$$

$$x(x-1)(x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 6

$$\frac{x+6}{(x+1)(x-1)} + \frac{x-6}{(x+1)(x-1)} = \frac{2x^5}{(x+1)(x-1)}$$

$$\frac{(x+6)+(x-6)}{(x+1)(x-1)} = \frac{2x^5}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^5}{(x+1)(x-1)}$$

$$2x = 2x^5$$

$$x = x^5$$

$$x^5 - x = 0$$

$$x(x^4-1) = 0$$

$$x(x-1)(x^2+1)(x+1) = 0$$

$$x = 0 \quad x = 1 \quad x = -1$$

Sub question 7

$$\frac{x+7}{(x+1)(x-1)} + \frac{x-7}{(x+1)(x-1)} = \frac{2x^6}{(x+1)(x-1)}$$

$$\frac{(x+7)+(x-7)}{(x+1)(x-1)} = \frac{2x^6}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^6}{(x+1)(x-1)}$$

$$2x = 2x^6$$

$$x = x^6$$

$$x^6 - x = 0$$

$$x(x^5-1) = 0$$

$$x(x-1)(x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 8

$$\frac{x+8}{(x+1)(x-1)} + \frac{x-8}{(x+1)(x-1)} = \frac{2x^7}{(x+1)(x-1)}$$

$$\frac{(x+8)+(x-8)}{(x+1)(x-1)} = \frac{2x^7}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^7}{(x+1)(x-1)}$$

$$2x = 2x^7$$

$$x = x^7$$

$$x^7 - x = 0$$

$$x(x^6-1) = 0$$

$$x(x-1)(x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 9

$$\frac{x+9}{(x+1)(x-1)} + \frac{x-9}{(x+1)(x-1)} = \frac{2x^8}{(x+1)(x-1)}$$

$$\frac{(x+9)+(x-9)}{(x+1)(x-1)} = \frac{2x^8}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^8}{(x+1)(x-1)}$$

$$2x = 2x^8$$

$$x = x^8$$

$$x^8 - x = 0$$

$$x(x^7-1) = 0$$

$$x(x-1)(x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 10

$$\frac{x+10}{(x+1)(x-1)} + \frac{x-10}{(x+1)(x-1)} = \frac{2x^9}{(x+1)(x-1)}$$

$$\frac{(x+10)+(x-10)}{(x+1)(x-1)} = \frac{2x^9}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^9}{(x+1)(x-1)}$$

$$2x = 2x^9$$

$$x = x^9$$

$$x^9 - x = 0$$

$$x(x^8-1) = 0$$

$$x(x-1)(x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 11

$$\frac{x+11}{(x+1)(x-1)} + \frac{x-11}{(x+1)(x-1)} = \frac{2x^{10}}{(x+1)(x-1)}$$

$$\frac{(x+11)+(x-11)}{(x+1)(x-1)} = \frac{2x^{10}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{10}}{(x+1)(x-1)}$$

$$2x = 2x^{10}$$

$$x = x^{10}$$

$$x^{10} - x = 0$$

$$x(x^9-1) = 0$$

$$x(x-1)(x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 12

$$\frac{x+12}{(x+1)(x-1)} + \frac{x-12}{(x+1)(x-1)} = \frac{2x^{11}}{(x+1)(x-1)}$$

$$\frac{(x+12)+(x-12)}{(x+1)(x-1)} = \frac{2x^{11}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{11}}{(x+1)(x-1)}$$

$$2x = 2x^{11}$$

$$x = x^{11}$$

$$x^{11} - x = 0$$

$$x(x^{10}-1) = 0$$

$$x(x-1)(x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 13

$$\frac{x+13}{(x+1)(x-1)} + \frac{x-13}{(x+1)(x-1)} = \frac{2x^{12}}{(x+1)(x-1)}$$

$$\frac{(x+13)+(x-13)}{(x+1)(x-1)} = \frac{2x^{12}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{12}}{(x+1)(x-1)}$$

$$2x = 2x^{12}$$

$$x = x^{12}$$

$$x^{12} - x = 0$$

$$x(x^{11}-1) = 0$$

$$x(x-1)(x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 14

$$\frac{x+14}{(x+1)(x-1)} + \frac{x-14}{(x+1)(x-1)} = \frac{2x^{13}}{(x+1)(x-1)}$$

$$\frac{(x+14)+(x-14)}{(x+1)(x-1)} = \frac{2x^{13}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{13}}{(x+1)(x-1)}$$

$$2x = 2x^{13}$$

$$x = x^{13}$$

$$x^{13} - x = 0$$

$$x(x^{12}-1) = 0$$

$$x(x-1)(x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1) = 0$$

$$x = 0 \quad x = 1$$

Sub question 15

$$\frac{x+15}{(x+1)(x-1)} + \frac{x-15}{(x+1)(x-1)} = \frac{2x^{14}}{(x+1)(x-1)}$$

$$\frac{(x+15)+(x-15)}{(x+1)(x-1)} = \frac{2x^{14}}{(x+1)(x-1)}$$

$$\frac{2x}{(x+1)(x-1)} = \frac{2x^{14}}{(x+1)(x-1)}$$

$$2x = 2x^{14}$$

$$x = x^{14}$$

$$x^{14} - x = 0$$

$$x(x^{13}-1) = 0$$

$$x(x-1)(x^{12}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+x^2+x+1