HUGE

Hello

Fuzzy Logic

July 31, 2018

1. Motivation
2. Ideas
3. Fuzzy sets
4. Membership functions
5. Properties

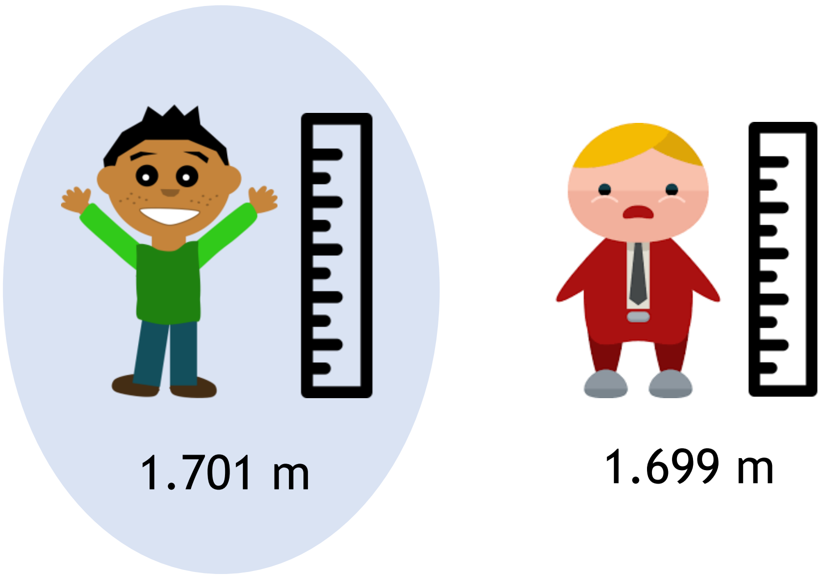
Agenda.

Motivation

Is it not enough with the classic logic?

No, classic logic does not reflects the nature of the concepts and human thoughts.

e.g.
Let's define tall person if he/she is taller than 1.7m



$$Tall\ people = \{\ height \mid height > 1.7\ m\}$$

Basic ideas

Uncertainty: No sure and clear knowledge of any concept.

Modeling and representation: Simplification of a real world situation through abstraction.

Modeling characteristics:

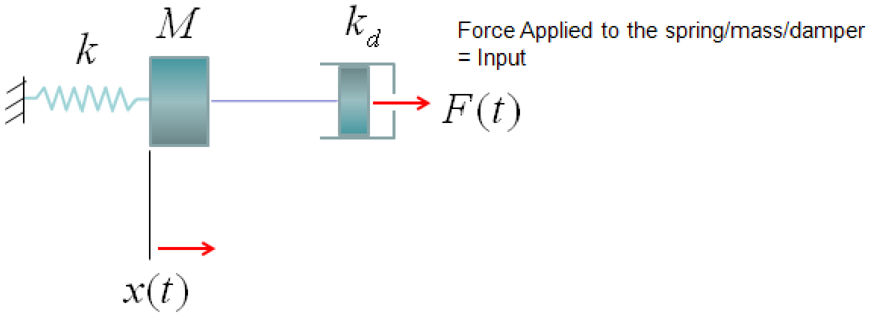
Formality

Optimization

Solvable

Generality

Specificity



Types of uncertainty

Total certainty, certainty

Face or seal?
True or false?

Neural networks are efficient?

The blonde woman is tall or low?

What does A mean? and B? ...
variables not specified

Determinism

Randomness

Ambiguity: More information allows solving the problem

Vagueness: Accuracy in definitions

Confusion

Some types of modeling

Randomness

Risk

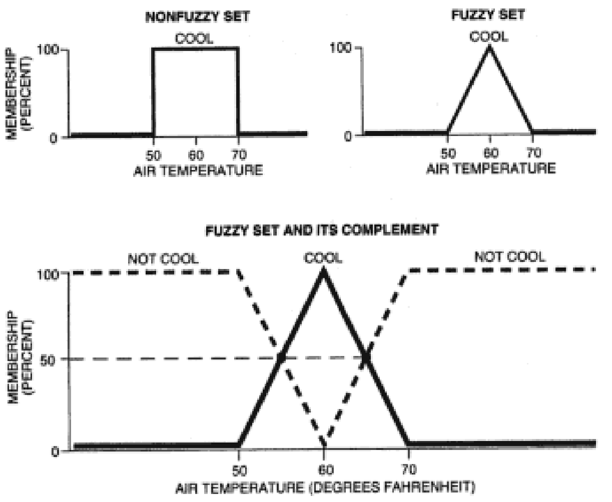
$$P(A) = \frac{N_A}{N}$$

Probability

Ambiguity

More information allows solving the problem

Vagueness-Precision in definitions



Fuzzy sets

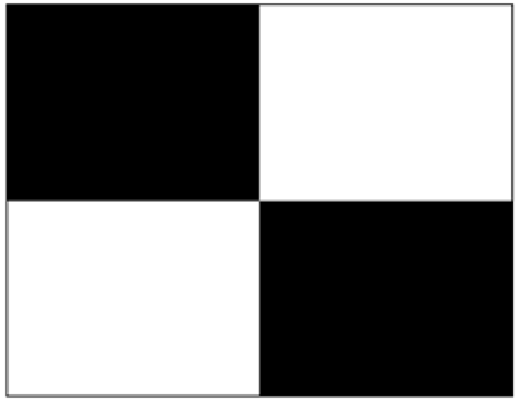
Classic vs. Fuzzy

Classical logic:

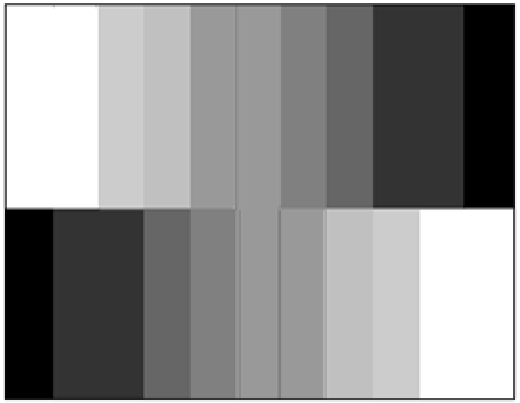
Two values (0,1) are considered to express true / false.

Diffuse logic:

It is a multi-valued type of logic. 'possible'.



Classical logic



Diffuse logic

Fuzzy sets

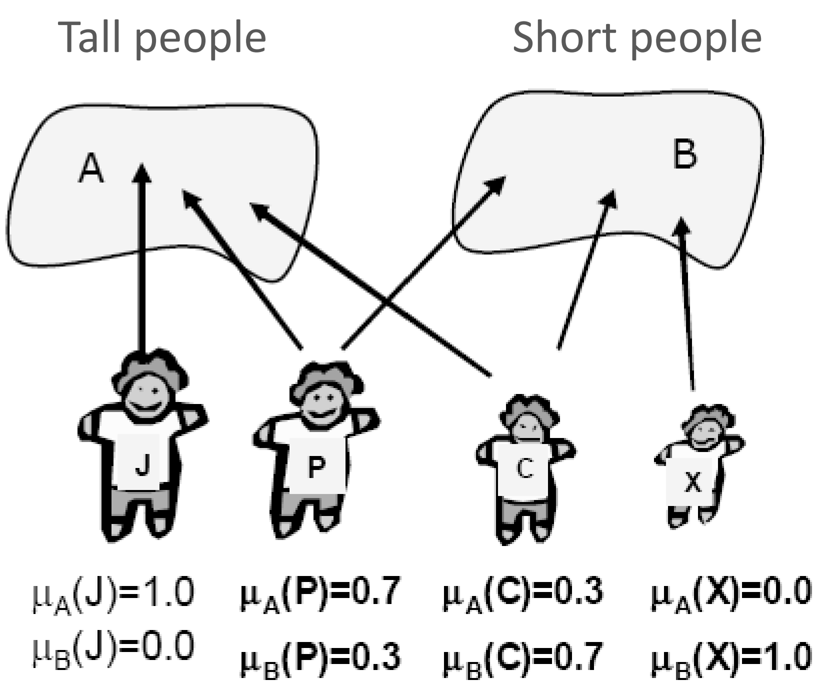
It is a model to describe the meaning of vague or imprecise words.

They are functions that relate a universe of objects.

The membership function that makes this relation for the fuzzy set A is the function

μA (x) gives the degree of belonging of the element x in the diffuse set A.

The fuzzyness describes the vagueness or imprecision of an event, definition or affirmation.



Membership function

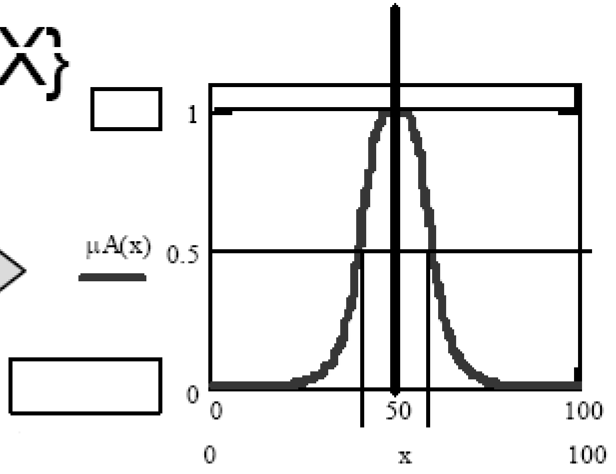
Assign to each element of the set a degree of belonging between 0 and 1. A fuzzy set A in X is defined by the set of ordered pairs

$$x, \mu_a(x)$$

Continuous fuzzy set

$$A = \{ x, \mu_A(x) | x \in X \}$$

$$\mu_A(x) = \frac{1}{1 + \left[\frac{x - 50}{10}\right]^4}$$



Membership function representations

Continuous variables:

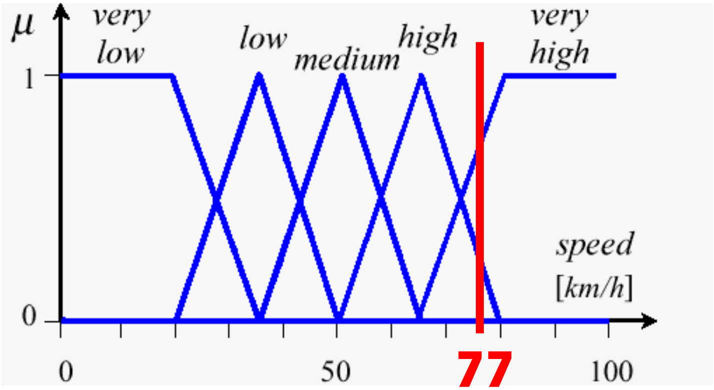
$$A = \int_{x} \frac{\mu_A(x)}{x}$$

Discrete variables:

$$A = \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \ldots \frac{\mu_A(x_n)}{x_n} = \sum_{x \in X} \frac{\mu_A(x_i)}{x_i}$$

The signs of sum and integral do not mean sum or integration but the union of the pairs (x_i , $\mu_A(x_i)$). In both cases, the horizontal line does not mean division. This is a boundary bar.

Representation of fuzzy measurements



Speed at which we are going (77) a vertical is drawn.

Intersection value is taken looking at the vertical axis, there are TWO, the first is at a height of 0.20 and 0.75.

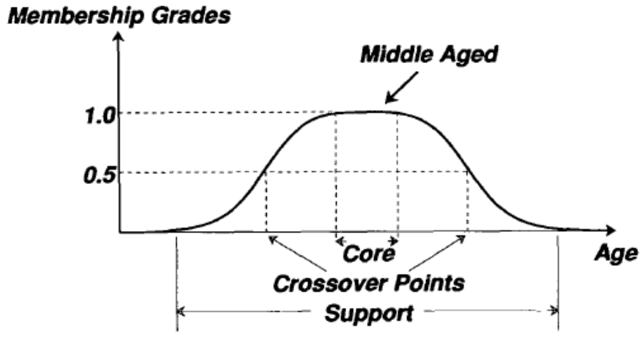
77 Km/h = 0.2 high speed, 0.75 Very high speed. Thus, the system has a “fuzzy” estimate of the current speed.

Membership functions properties

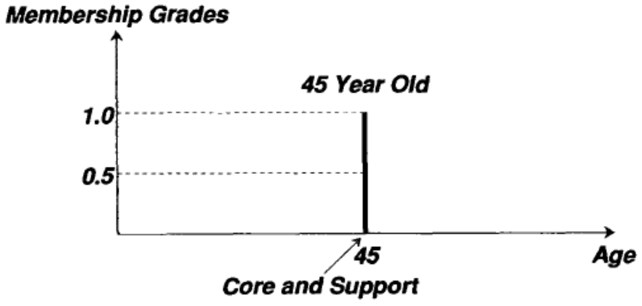
Center: Average of elements $\mu_x = 1$

Fuzzy Singleton: Support is a point F in U with $\mu_F = 1$

Height: Height, Greater membership degree... Normal set = 1



(a)



(b)

Membership functions properties

The cardinality of a classical set is defined as the number of elements in the set.

The cardinality of a diffuse set A is the sum of all the membership degrees of all the elements x in A, that is:

$$|A| = \sum_{x \in U} \mu_A(x)$$

$$|A| = \int_{x \in U} \mu_A(x)dx$$

Valid laws for classical sets but not for fuzzy sets

Classical sets

(E is a classic set and there is nothing in common between E and Ê)

$E \cup \tilde{E} = Universal\ set$

$E \cap \tilde{E} = \emptyset$

Fuzzy sets

(A is a fuzzy set)

$A \cup \tilde{A} \neq Universal\ set$

$A \cap \tilde{A} \neq \emptyset$

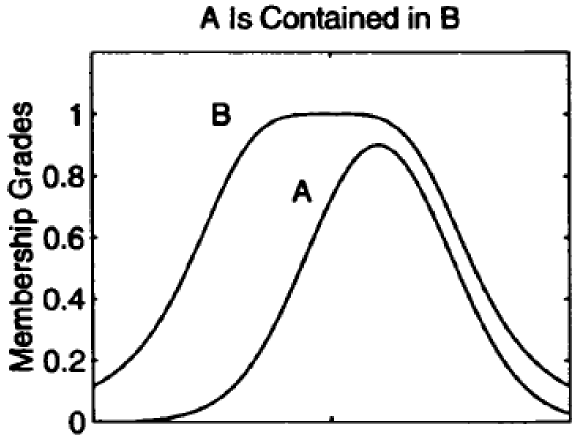
Some classical set rules

Law of contradiction	$A \cap \overline{A} = \emptyset$
Law of the excluded middle	$A \cup \overline{A} = X$
Idempotency	$A \cap A = A,\ A \cup A = A$
Involution	$\overline{\overline{A}} = A$
Commutativity	$A \cap B = B \cap A,\ A \cup B = B \cup A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Absorption	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$
Absorption of complement	$A \cup (\overline{A} \cap B) = A \cup B$ $A \cap (\overline{A} \cup B) = A \cap B$
DeMorgan's laws	$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Fuzzy sets operations: Containment

A is contained in fuzzy set B if and only if $\mu_A(x) \leq \mu_B(x)$ for all x.

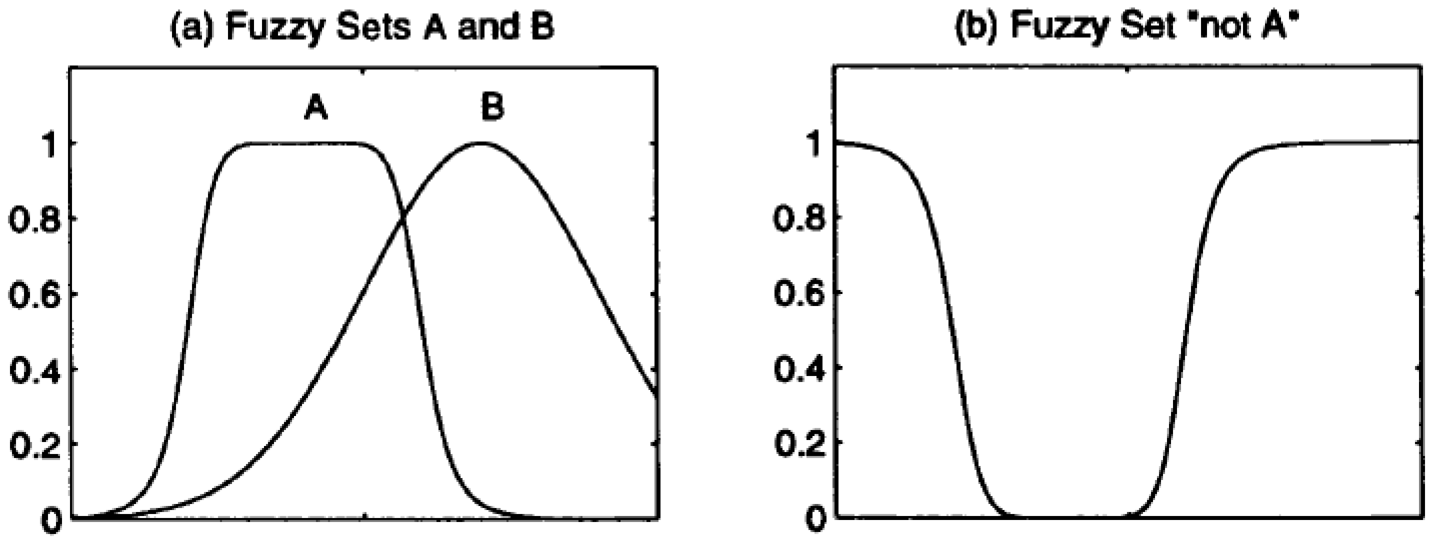
$$A \subset B \iff \mu_A(x) \leq \mu_B(x)$$



Fuzzy sets operations: Complement

The complement of a fuzzy set A, is defined as:

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x)$$



Fuzzy sets operations: Other definitions of the complement

The previous definition is one of many in the literature. In order to be a valid operator, an operator must fulfill the following axiomatic requirements:

1. $c(0) = 1 \qquad c(1) = 0$ **(Boundary)**
2. for all $a, b \in [0,1]$, $if\ \ a < b,\ \ then\ \ c(a) \geq c(b) \qquad a = \mu_A(x) \qquad b = \mu_B(x)$ **(Monotonicity)**
3. $c\big(c(a)\big) = a$ **(No increment)**

Fuzzy sets operations: Other definitions of the complement

- **Sugeno (1977)**

$$c_\lambda = \frac{1 - \alpha}{1 + \lambda\alpha} \qquad \lambda \in (1, \infty)$$

- **Yager (1980)**

$$c_\omega = (1 - \alpha^\omega)^{\frac{1}{\omega}} \qquad \omega \in (0, \infty)$$

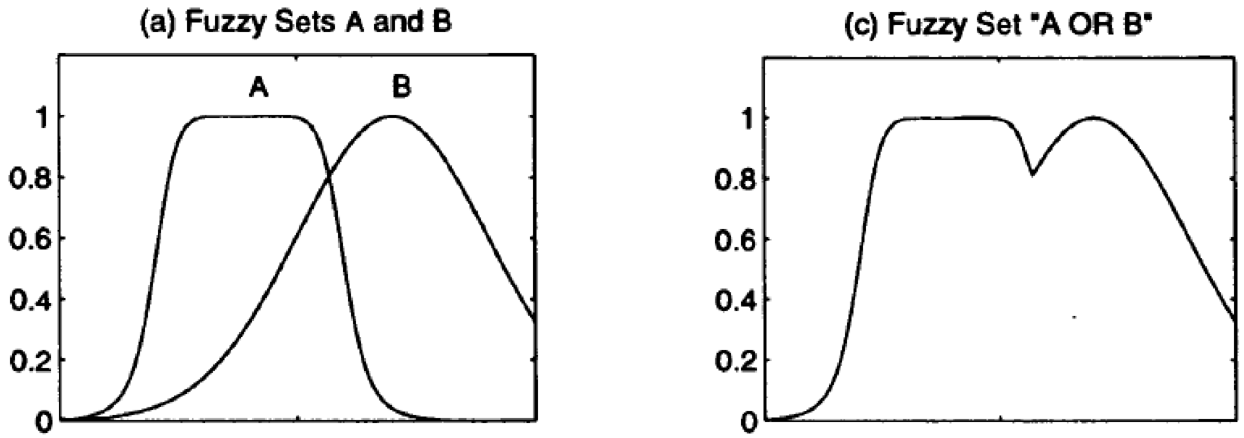
Where α is the fuzzy value to which the complement is applied.

These definitions are actually generalizations of the complement operator, with the possibility of changing parameters.

Fuzzy sets operations: Union

The union of two fuzzy sets A and B, is defined as:

$$A \cup B \iff \max(\mu_A(x), \mu_B(x))$$



In others words, the union is the smallest fuzzy set that contains A and B.

Fuzzy sets operations: Other definitions of the union (S-norm)

The previous definition is one of many in the literature. In order to be a valid operator, an operator must fulfill the following axiomatic requirements:

1. $s(1,1) = 1 \qquad s(0,a) = s(a,0) = a$ **(Boundary)**
2. $s(a,b) = s(b,a)$ **(Commutativity)**
3. $If \quad a \leq a' \quad and \quad b \leq b'$ **then** $s(a,b) \leq s(a', b')$ **(Monotonicity)**
4. $s(s(a,b),c) = s(a, s(b,c))$ **(Associativity)**

Fuzzy sets operations: Other definitions of the union

- **Dombi (1982)**

$$s_\lambda(a,b) = \frac{1}{1 + \left[\left(\frac{1}{a} - 1\right)^{-\lambda} + \left(\frac{1}{a} - 1\right)^{-\lambda}\right]^{-\frac{1}{\lambda}}} \qquad \lambda \in (1,\infty)$$

- **Dubois-Prade (1980)**

$$s_\alpha(a,b) = \frac{a + b \; - ab \; - \min(a, b, 1-\alpha)}{\max(1-a, 1-b, \alpha)} \qquad \alpha \in [0,1]$$

- **Yager (1980)**

$$s_\omega(a,b) = min\left[1, (\alpha^\omega + b^\omega)^{\frac{1}{\omega}}\right] \qquad \omega \in (0,\infty)$$

- **Drastic sum**

$$\mathrm{s}_{\mathrm{ds}}(\mathrm{a},\mathrm{b}) = \begin{cases} a \;\; if \;\; b = 0 \\ b \;\; if \;\; a = 0 \\ 1 \; otherwise \end{cases}$$

Fuzzy sets operations: Other definitions of the union

- **Einstein sum**

$$s_{es}(a,b) = \frac{a+b}{1+ab}$$

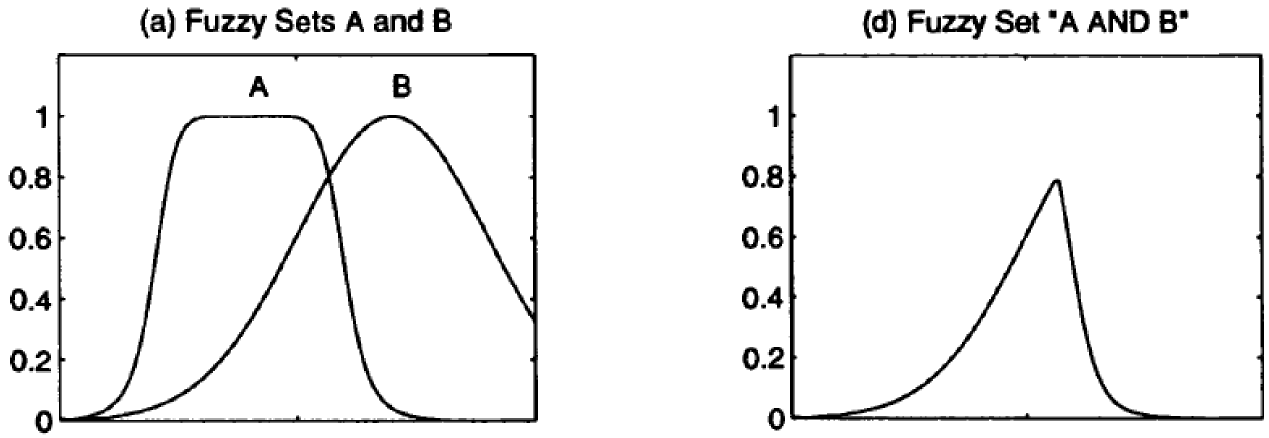
- **Algebraic sum**

$$s_{as}(a,b) = a + b - ab$$

Fuzzy sets operations: Intersection

The intersection of two fuzzy sets A and B, is defined as:

$$A \cap B \iff \min(\mu_A(x), \mu_B(x))$$



In others words, the intersection is the largest fuzzy set which is contained in both A and B.

Fuzzy sets operations: Other definitions of the intersection (T-norm)

The previous definition is one of many in the literature. In order to be a valid operator, an operator must fulfill the following axiomatic requirements:

1. $\mathrm{t}(0{,}0) = 0 \qquad t(1, a) = t(a, 1) = a$ **(Boundary)**
2. $\mathrm{t}(a, b) = t(b, a)$ **(Commutativity)**
3. $If \quad a \leq a' \quad and \quad b \leq b'$ **then** $t(a, b) \leq t(a', b')$ **(Monotonicity)**
4. $t(t(a, b), c) = t(a, t(b, c))$ **(Associativity)**

Fuzzy sets operations: Other definitions of the union

- **Dombi (1982)**

$$t_\lambda(a,b) = \frac{1}{1 + \left[\left(\frac{1}{a} - 1\right)^{\lambda} + \left(\frac{1}{a} - 1\right)^{\lambda}\right]^{\frac{1}{\lambda}}} \qquad \lambda \in (0, \infty)$$

- **Dubois-Prade (1980)**

$$t_\alpha(a,b) = \frac{ab}{\max(a,b,\alpha)} \qquad \alpha \in [0,1]$$

- **Yager (1980)**

$$t_\omega(a,b) = 1 - min\left[1, \left((1-\alpha)^{\omega} + (1-b)^{\omega}\right)^{\frac{1}{\omega}}\right] \qquad \omega \in (0,\infty)$$

- **Drastic product**

$$\mathrm{t_{dp}} = \begin{cases} a \;\; if \;\; b = 1 \\ b \;\; if \;\; a = 1 \\ 0 \; otherwise \end{cases}$$

Fuzzy sets operations: Other definitions of the union

- **Einstein product**

$$t_{es}(a, b) = \frac{ab}{2 - (a + b - ab)}$$

- **Algebraic product**

$$t_{ap}(a, b) = ab$$

T-norms and S-Norms are duals which support the generalization of the DeMorgan's Law

$$t(a,b) = N(s(N(a),N(b)))$$

$$s(a,b) = N(t(N(a),N(b)))$$

Check it:

Algebraic product

$$t_{ap}(a,b) = ab$$

Algebraic sum

$$s_{as}(a,b) = a + b - ab$$

Example

A company is looking for an employee to assign him a new position. They are looking for a person with medium age or high experience but with low salary. The decision must be made from the following database:

Last name	Birth date	Starting date	Salary
Arias	1994	2014	1
Benavides	1990	2009	3
Camargo	1988	2011	2
Díaz	1983	1999	5
Eslava	1995	2013	4

Define the diffuse system that allows taking the decision of the employee that must be promoted. Using the fuzzy system, establish the name of the employee who will have the new position. You should clarify what are the universes of discourse, what are the linguistic values for each variable that is taken into account.

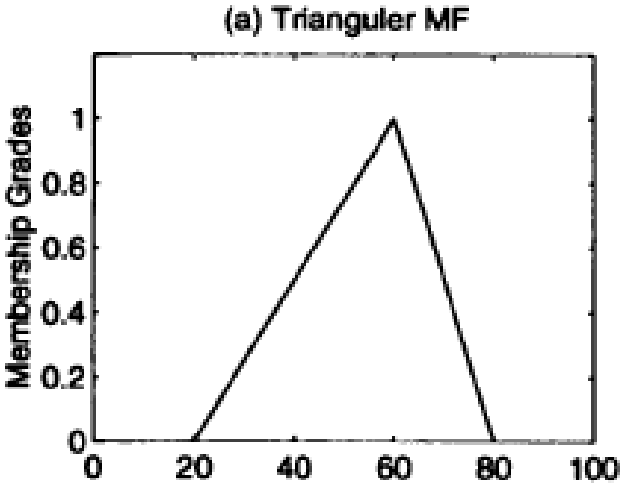
Membership functions

Triangular:

Three parameters {a,b,c}

$$\text{triangle}(x;a,b,c) = \begin{cases} 0, & x \le a. \\ \frac{x-a}{b-a}, & a \le x \le b. \\ \frac{c-x}{c-b}, & b \le x \le c. \\ 0, & c \le x. \end{cases}$$

$triangle(x; 20, 60, 80)$



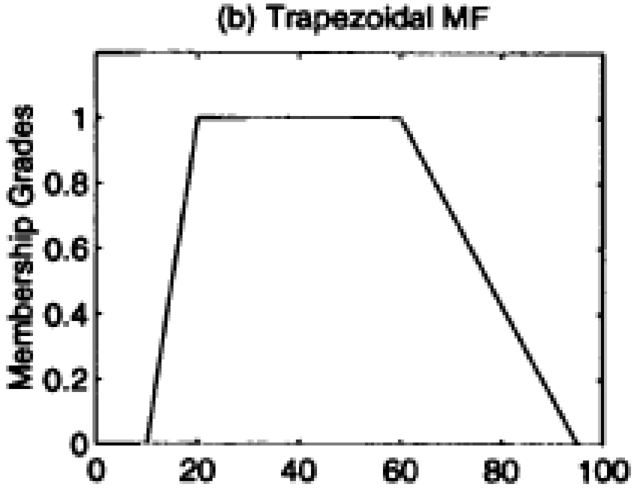
Membership functions

Trapezoidal:

Four parameters {a,b,c,d}

$$\text{trapezoid}(x; a, b, c, d) = \begin{cases} 0, & x \le a. \\ \frac{x-a}{b-a}, & a \le x \le b. \\ 1, & b \le x \le c. \\ \frac{d-x}{d-c}, & c \le x \le d. \\ 0, & d \le x. \end{cases}$$

$trapezoid(x; 10, 20, 60, 95)$



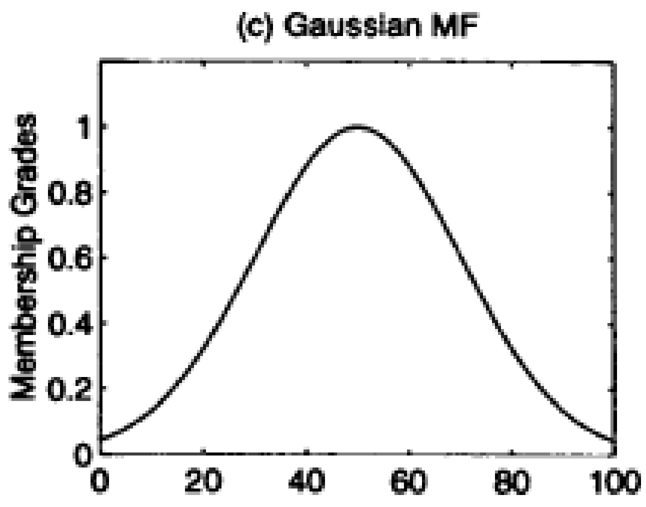
Membership functions

Gaussian:

Two parameters {c,sigma}

$$\text{gaussian}(x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$

$gaussian(x; 50, 20)$



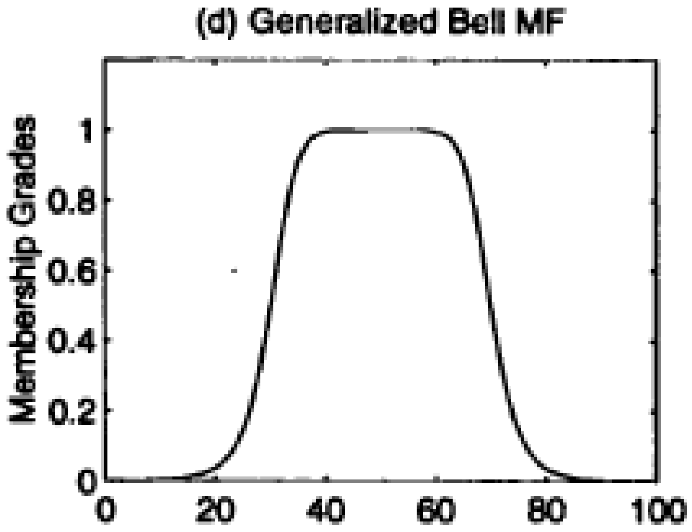
Membership functions

Generalized Bell:

Three parameters {a, b, c}

$$\text{bell}(x;a,b,c)=\frac{1}{1+\left|\frac{x-c}{a}\right|^{2b}}$$

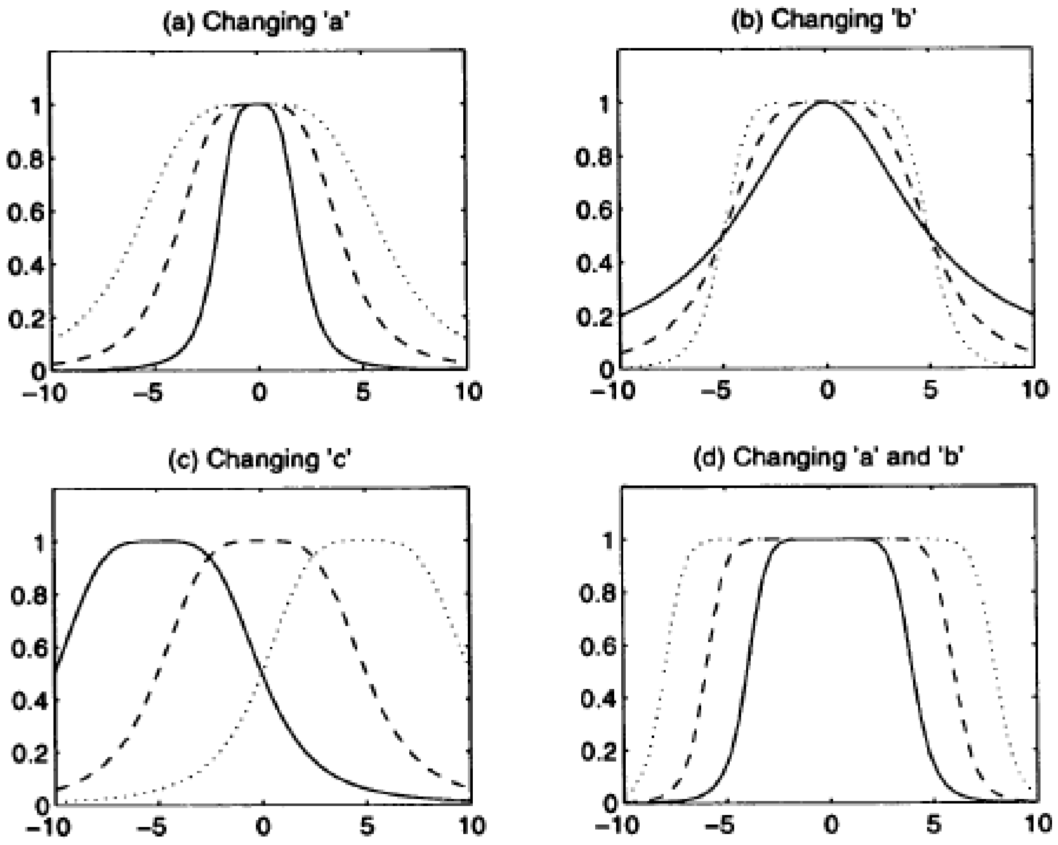
$bell(x; 20, 4, 50)$



The parameter b is usually positive and controls the slope of the crossing points. When it is negative, the form of the membership function becomes an inverted bell. The parameter c controls the center of the bell, while a controls the width.

Membership functions

Generalized Bell:

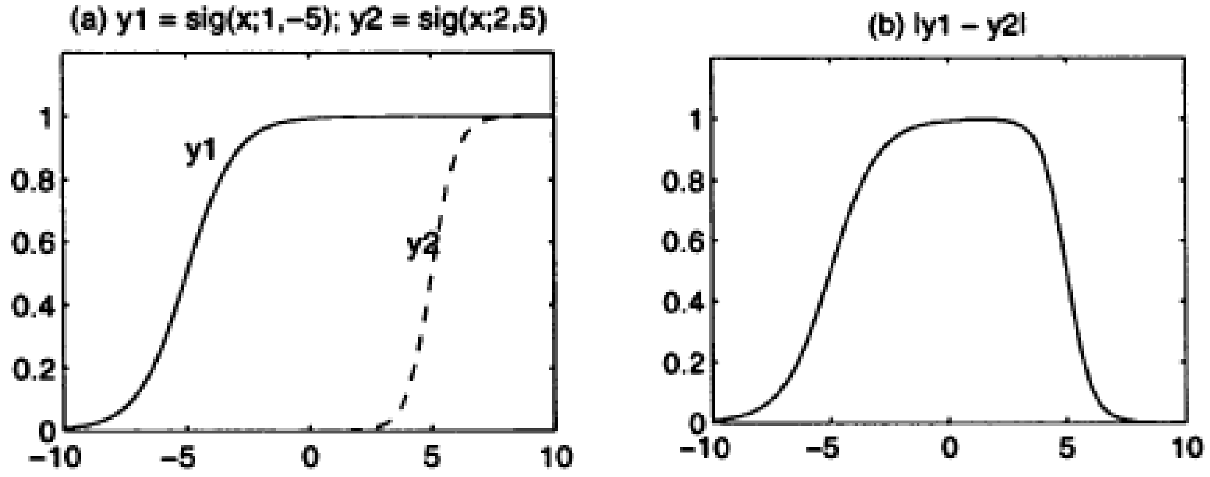


Membership functions

Sigmoid:

Two parameters {a, c}

$$\mathrm{sig}(x; a, c) = \frac{1}{1 + \exp[-a(x - c)]}$$



Ability to generate symmetrical and smooth functions.
Parameter a is responsible for giving direction (left or right) and slope.
The parameter c is the crossing point.

Fuzzy relationships

Binary relationship X x Y:

$$R = \{((x, y), \mu_R(x, y)) | (x, y) \in XxY\}$$

In other words, it is a function of "x" and "y", and is defined for every "x" and "y" in the X x Y plane.

$$\mu_R(x, y) = \begin{cases} \frac{y-x}{y+x+2} & si \quad y \geq x \\ 0 & si \quad y \leq x \end{cases}$$

$$\begin{array}{cc} \mathcal{R} = \begin{bmatrix} 0 & 0.111 & 0.200 & 0.273 & 0.333 \\ 0 & 0 & 0.091 & 0.167 & 0.231 \\ 0 & 0 & 0 & 0.077 & 0.143 \end{bmatrix} & X = \{3, 4, 5\} \end{array}$$

$$Y = \{3, 4, 5, 6, 7\}$$

Fuzzy relationships

Fuzzy relationship examples:

- X is closed to Y (Between numbers)
- X depends on Y (Between events)
- X is similar to Y (Between objects)
- If X is long then Y is short (X is observed and Y is concluded)
- If the level is low then the input flow is high
- If the level is high then the input flow is low

Fuzzy relationships

Controlling the level of the tank:

Implications: Relationship between input and output sets.

- **If the level is low then the input flow is high (R1)**
- **If the level is high then the input flow is low (R2)**

Steps:

1. **Fuzzy sets definition.**
2. **Rules definition.**
3. **Membership functions definition.**
4. **Mamdani implication (?)**

Fuzzy implications

1. In fuzzy systems the human knowledge is represented in term of rules IF – THEN.
2. A fuzzy rule IF – THEN is a conditional expressed as:

Antecedent **Consequence**

If "Fuzzy proposition" then "Fuzzy proposition"

"Atomic fuzzy propositions" :
Unique proposition, simple

x is A
A: Linguistic value of the
Linguistic variable X

"Compound fuzzy propositions" :
Composed proposition

x is A and y is B

$$\mu_{A \cup B}(x, y) = S(\mu_A(x), \mu_B(y))$$

$$\mu_{A \cap B}(x, y) = T(\mu_A(x), \mu_B(y))$$

Fuzzy implications

Classical logic: $p \to q$

Fuzzy logic:

- **Material implication:**

$$R = A \to B = \neg A \cup B.$$

$p \to q \qquad -p \vee q$

$$\mu_{Q_D}(x, y) = max[1 - \mu_{FP_1}(x), \mu_{FP_2}(y)]$$

- **Propositional calculus:**

$$R = A \to B = \neg A \cup (A \cap B)$$

Fuzzy implications

- **Extended propositional calculus:**

$$R = A \to B = (\neg A \cap \neg B) \cup B.$$

And we know now how to do this operations for fuzzy sets (complement, intersection, union)

More fuzzy implications

- **Zadeh implication:**

$$p \rightarrow q \qquad -p \vee (p \wedge q)$$

$$\mu_{Q_Z}(x,y) = max[min(\mu_{FP_1}(x), \mu_{FP_2}(y)), 1 - \mu_{FP_1}(x)]$$

- **Gödel implication:**

$$p \rightarrow q$$

$$\mu_{Q_G}(x,y) = \begin{cases} 1 & if\ \mu_{FP_1}(x) \leq \mu_{FP_2}(y) \\ \mu_{FP_2}(y) & otherwise \end{cases}$$

- **Mamdani implication:**

$$\mu_{Q_{MM}}(x,y) = min[\mu_{FP_1}(x), \mu_{FP_2}(y)]$$

Fuzzy
proposition

$$\mu_{Q_{MP}}(x,y) = \mu_{FP_1}(x)\mu_{FP_2}(y)$$

Fuzzy
relationship

Fuzzy implication

Now:

- If the level is low then the input flow is high (R1)
- If the level is high then the input flow is low (R2)

$R1_{NxF}$: $\min(\mu_A(n), \mu_B(f))$

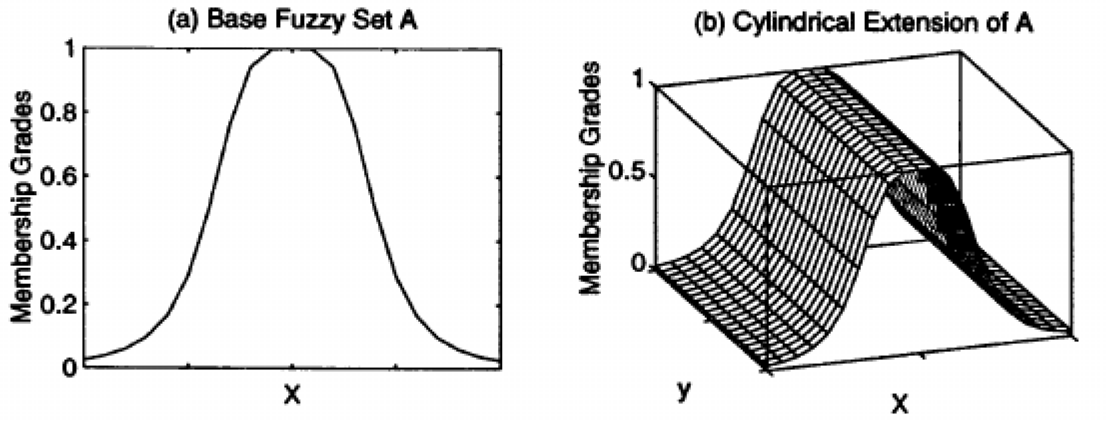
$R2_{NxF}$: $\min(\mu_B(n), \mu_A(f))$

Cylindrical extension

This is the extension of a 1-D fuzzy membership function into a 2-D function.

If A is a fuzzy set in X, then its cylindrical extensión in the plane X x Y is a fuzzy set C(A) defined as:

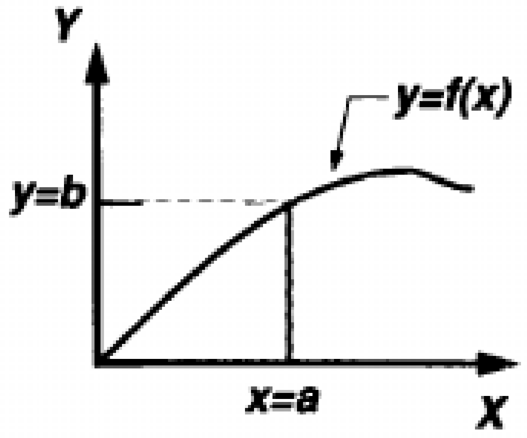
$$C(A) = \int \frac{\mu_A(x)}{(x,y)}$$



Compositional fuzzy rule

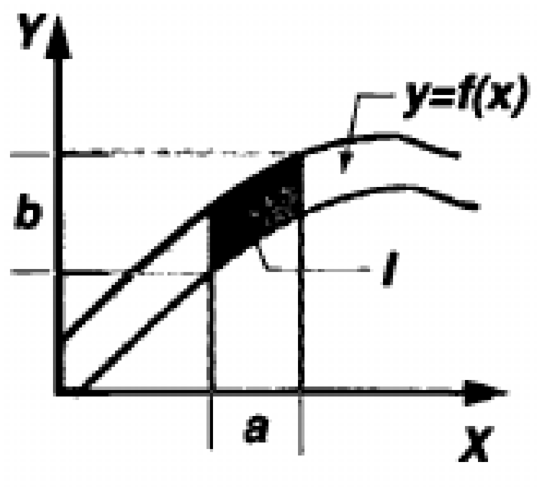
In calculus, one of the first expressions stundents learn is: $f(x) = y$

If $x=a$ is a point, its corresponding y can be obtained using $f(x)$



Compositional fuzzy rule

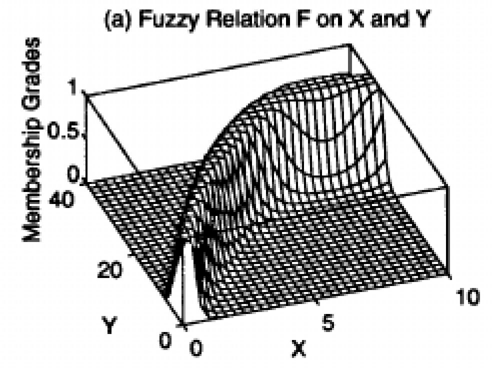
What if x is not a unique value, but a range of values:



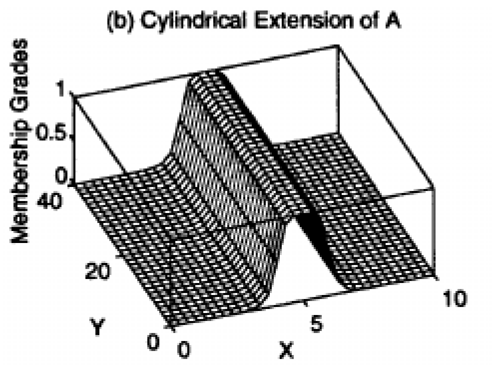
Compositional fuzzy rule

For fuzzy something similar can be done:

Let be F the fuzzy relationship between X and Y.



Let be C(A) the cylindrical extension.



$$\mu_{c(A)}(x, y) = \mu_A(x).$$

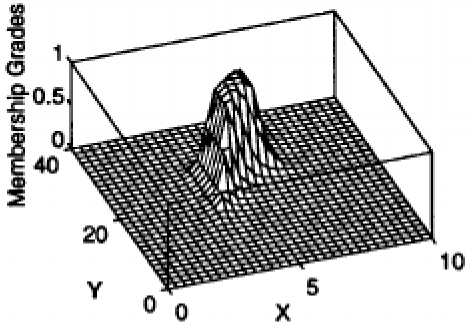
Compositional fuzzy rule

The intersection between F y C(A), is analogous to the shadow region of previous calculus example.
To find the interval we must make the projection the resulting function.

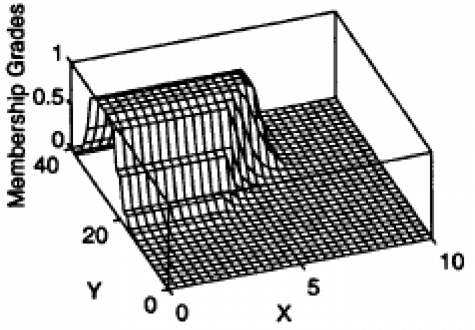
$$\begin{aligned}\mu_{c(A)\cap F}(x,y) &= \min[\mu_{c(A)}(x,y), \mu_F(x,y)] \\ &= \min[\mu_A(x), \mu_F(x,y)].\end{aligned}$$

$$\begin{aligned}\mu_B(y) &= \max_x \min[\mu_A(x), \mu_F(x,y)] \\ &= \vee_x[\mu_A(x) \wedge \mu_F(x,y)].\end{aligned}$$

(c) Minimum of (a) and (b)



(d) Projection of (c) onto Y



Fuzzy reasoning

The basic rule of inference in traditional logic is modus ponens, according to which one can infer the truth of a proposition B from the truth of a proposition A and of the implication A -> B.

$$
\begin{array}{ll}
\text{premise 1 (fact):} & \text{x is } A, \\
\text{premise 2 (rule):} & \text{if x is } A \text{ then y is } B, \\
\hline
\text{consequence (conclusion):} & \text{y is } B.
\end{array}
$$

But human logic is not like that, for example, if we say “if the tomato is red, then it is ripe”, then we know that “if the tomato is half red, then it is more or less mature”

$$
\begin{array}{ll}
\text{premise 1 (fact):} & \text{x is } A', \\
\text{premise 2 (rule):} & \text{if x is } A \text{ then y is } B, \\
\hline
\text{consequence (conclusion):} & \text{y is } B',
\end{array}
$$

Where A 'is close to A, and B' is close to B. Approximate Reasoning.

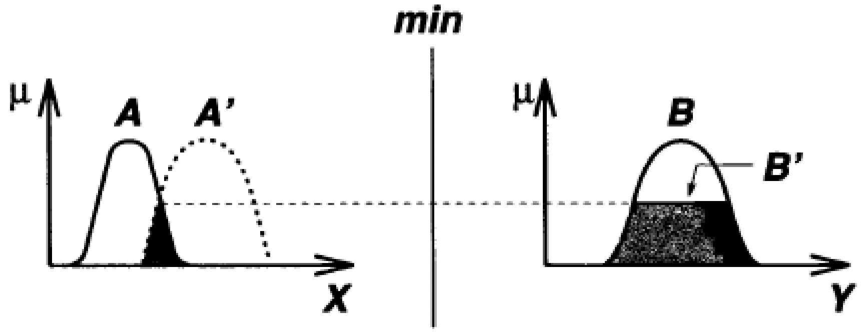
Fuzzy reasoning

The simple scenario:

premise 1 (fact):	x is A,
premise 2 (rule):	if x is A then y is B,
consequence (conclusion):	y is B.

$$\begin{aligned}\mu_{B'}(y) &= [\vee_x(\mu_{A'}(x) \wedge \mu_A(x)] \wedge \mu_B(y) \\ &= w \wedge \mu_B(y).\end{aligned}$$

The maximum degree of similarity of A 'with A, to this portion the cylindrical extension is performed, we make the projection of the extension in Y, and then it intersects with B.



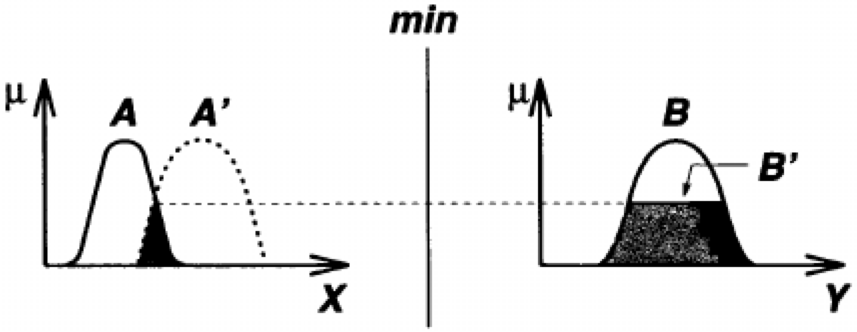
Fuzzy reasoning

The simple scenario:

premise 1 (fact):	x is A,
premise 2 (rule):	if x is A then y is B,
consequence (conclusion):	y is B.

$$\begin{aligned}\mu_{B'}(y) &= [\vee_x(\mu_{A'}(x) \wedge \mu_A(x)] \wedge \mu_B(y) \\ &= w \wedge \mu_B(y).\end{aligned}$$

The maximum degree of similarity of A 'with A, to this portion the cylindrical extension is performed, we make the projection of the extension in Y, and then it intersects with B.

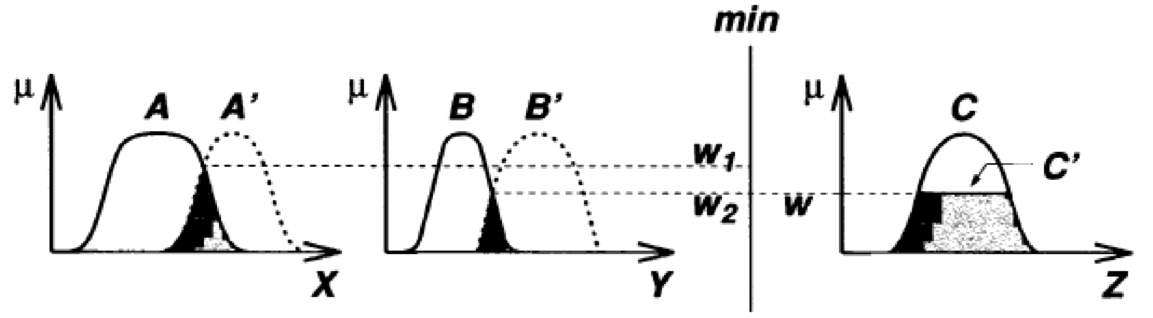


Fuzzy reasoning

Multiple antecedents:

premise 1 (fact):	x is A' and y is B',
premise 2 (rule):	if x is A and y is B then z is C,
consequence (conclusion):	z is C'.

$$
\begin{aligned}
\mu_{C'}(z) &= \vee_{x,y}[\mu_{A'}(x) \wedge \mu_{B'}(y)] \wedge [\mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z)] \\
&= \vee_{x,y}\{[\mu_{A'}(x) \wedge \mu_{B'}(y) \wedge \mu_A(x) \wedge \mu_B(y)]\} \wedge \mu_C(z) \\
&= \underbrace{\{\vee_x[\mu_{A'}(x) \wedge \mu_A(x)]\}}_{w_1} \wedge \underbrace{\{\vee_y[\mu_{B'}(y) \wedge \mu_B(y)]\}}_{w_2} \wedge \mu_C(z) \\
&= (w_1 \wedge w_2) \quad \wedge \mu_C(z),
\end{aligned}
$$

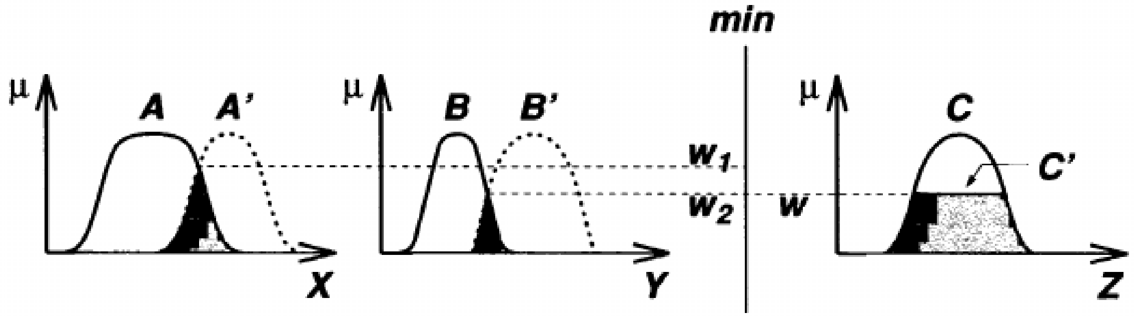


Fuzzy reasoning

Multiple antecedents:

premise 1 (fact):	x is A' and y is B',
premise 2 (rule):	if x is A and y is B then z is C,
consequence (conclusion):	z is C'.

$$
\begin{aligned}
\mu_{C'}(z) &= \vee_{x,y}[\mu_{A'}(x) \wedge \mu_{B'}(y)] \wedge [\mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z)] \\
&= \vee_{x,y}\{[\mu_{A'}(x) \wedge \mu_{B'}(y) \wedge \mu_A(x) \wedge \mu_B(y)]\} \wedge \mu_C(z) \\
&= \underbrace{\{\vee_x[\mu_{A'}(x) \wedge \mu_A(x)]\}}_{w_1} \wedge \underbrace{\{\vee_y[\mu_{B'}(y) \wedge \mu_B(y)]\}}_{w_2} \wedge \mu_C(z) \\
&= (w_1 \wedge w_2) \wedge \mu_C(z),
\end{aligned}
$$

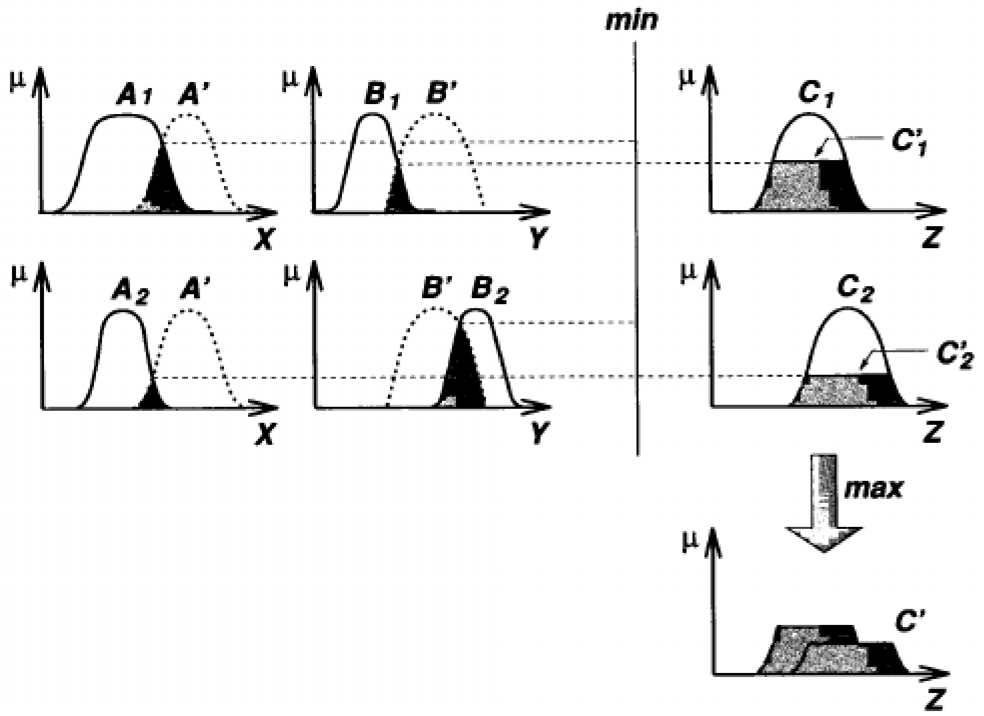


Fuzzy reasoning

Multiple antecedents and multiple rules:

premise 1 (fact):	x is A' and y is B',
premise 2 (rule 1):	if x is A_1 and y is B_1 then z is C_1,
premise 3 (rule 2):	if x is A_2 and y is B_2 then z is C_2,
consequence (conclusion):	z is C',

$$
\begin{aligned}
C' &= (A' \times B') \circ (R_1 \cup R_2) \\
&= [(A' \times B') \circ R_1] \cup [(A' \times B') \circ R_2] \\
&= C_1' \cup C_2',
\end{aligned}
$$



HUGE

Done.