

# Epidemics

Introduction to Network Science

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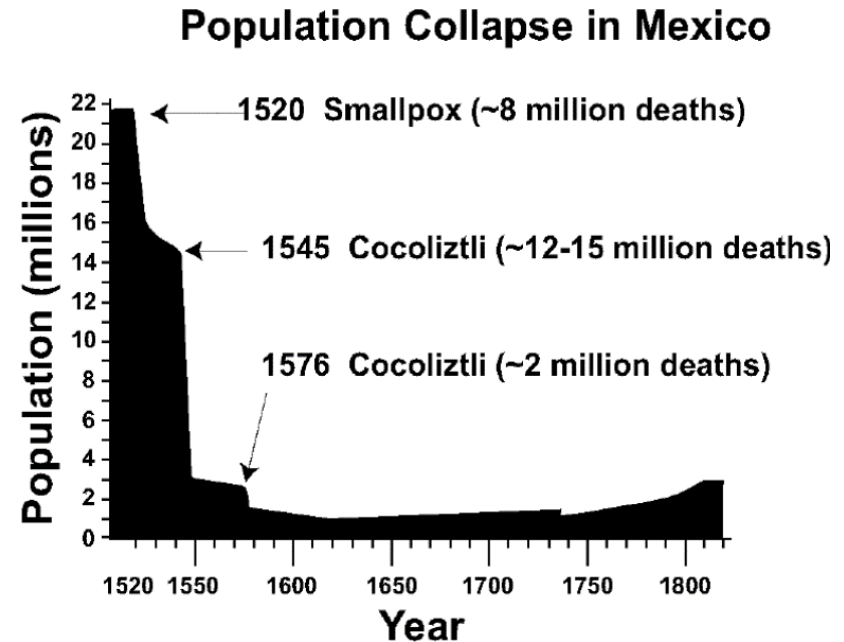
Topic 26

# Sources

- Barabási (2016): [Network Science Ch. 10](#)
- Easley and Kleinberg (2010): Networks, Crowds, and Markets [Ch 21](#).

# Examples: human epidemics

- Influenza, measles, STDs
- The “Black Death”  
[next slide]
- Smallpox and other diseases brought by Europeans to America since early 1500s



# The “Black Death” (Bubonic plague)

1300s

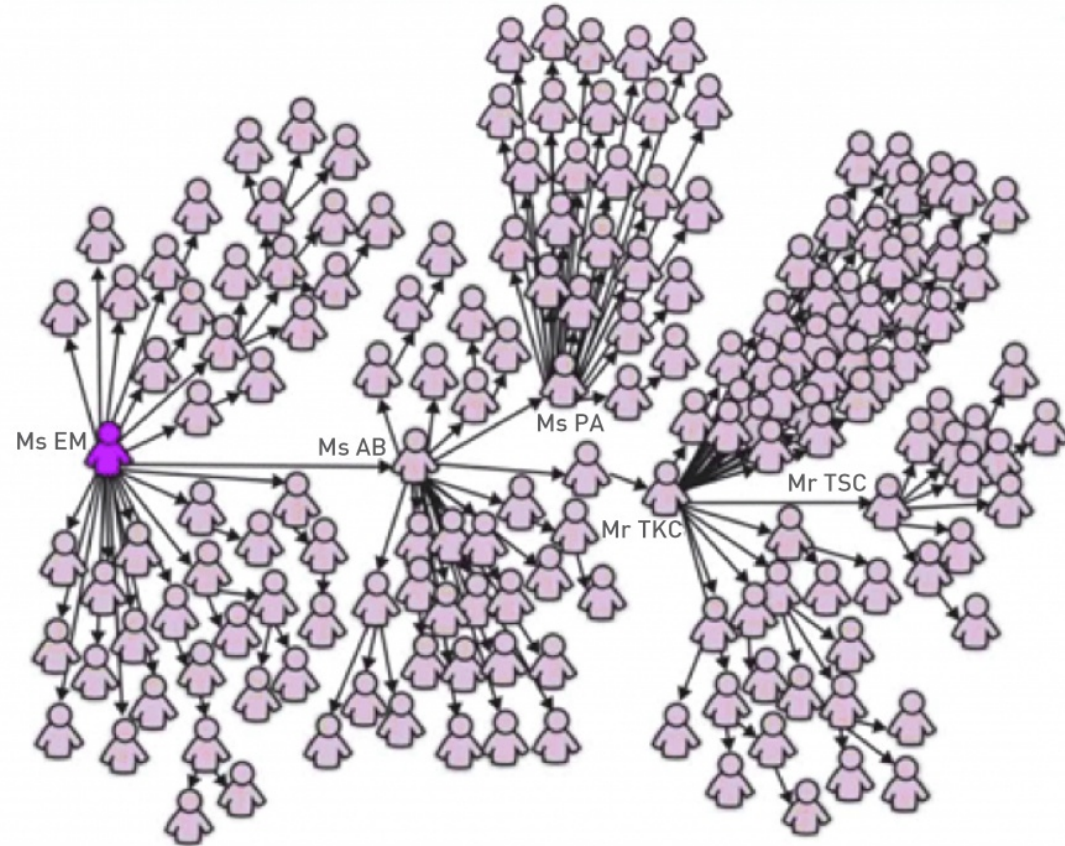
Killed 30%-60% of  
the total population  
of Europe



1346 1347 1348 1349 1350 1351 1352 1353

# SARS Outbreak (2003)

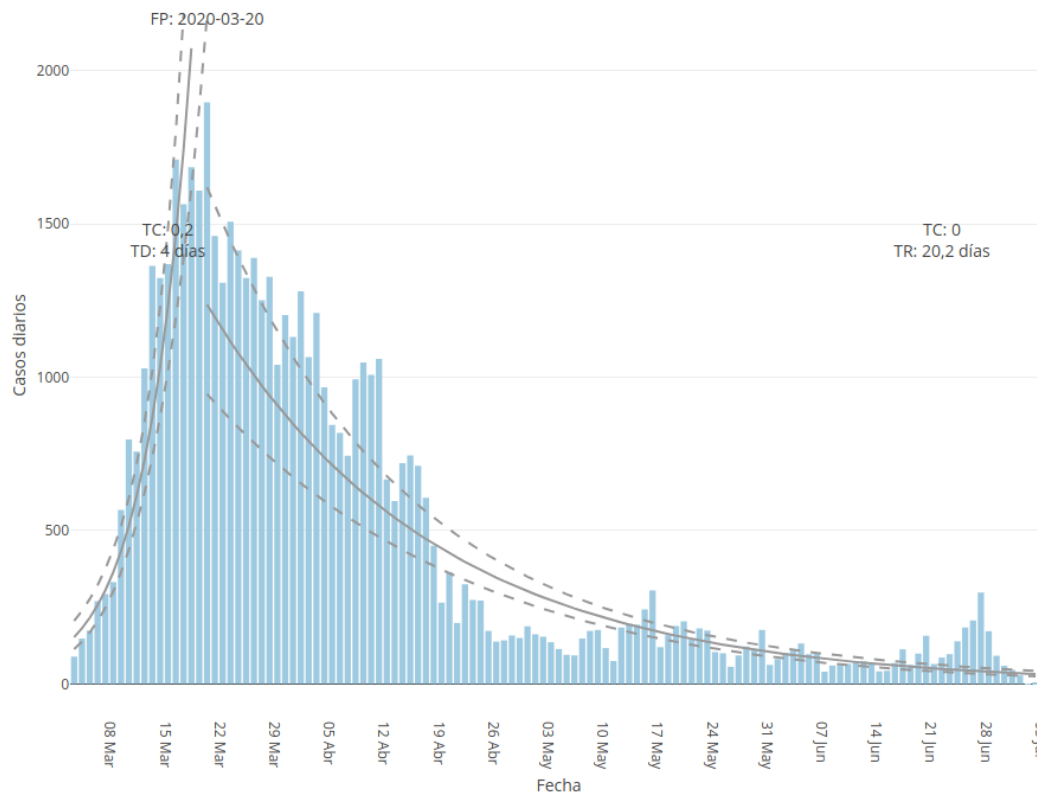
- February 21st: Chinese doctor who have been several treating “atypical pneumonia” cases check-ins into hotel in Hong Kong
  - Hospitalized on Feb 22<sup>nd</sup>
  - Died on March 4<sup>th</sup>
- March 1st: “Ms. E. M.” returns to Singapore after visiting Hong Kong
  - Graph depicts 144 out of the first 206 SARS patients in Singapore
  - Ms. E. M. lived, various of her family members died



# COVID-19

Why this curve?

How can one make  
this kind of  
forecast?





# Diffusion of ideas vs diseases

- Adopting a new idea, behavior, fashion, product, taste, may also spread from person to person: “social contagion”
- There is a certain agency of the receiver
- In diffusion of diseases, we assume **there is no agency: each contagion is random**

Simple model: branching process



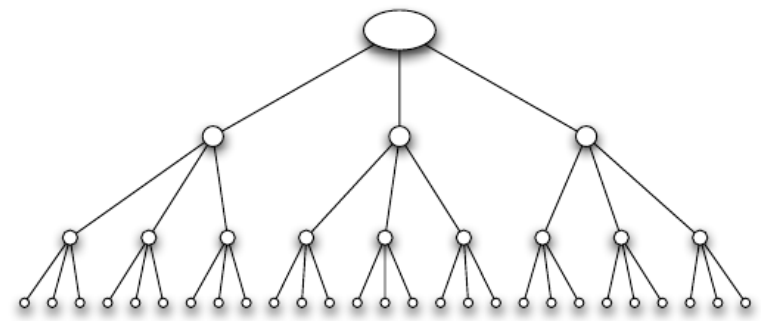
# Modeling epidemics

- There are many factors:
  - Contagiousness
  - Length of infectious period,
  - Severity
  - ...
- Structure of contacts in a population

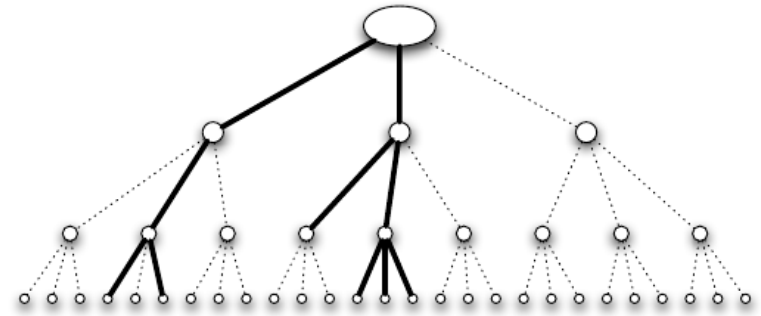
# Simple model: branching process

- Each person interacts with other  $k$  people
- Each interaction ends in infection with probability  $\beta$

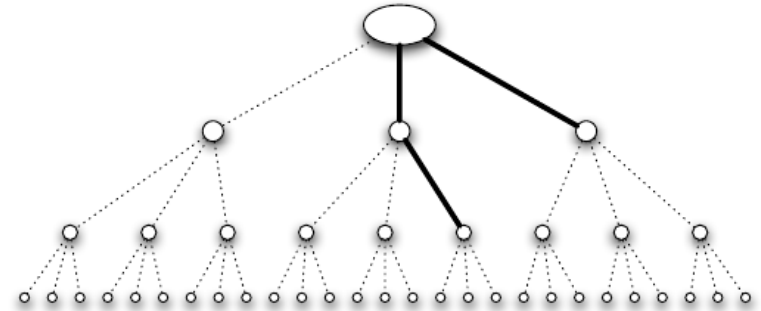
Example:  $k=3$



(a) The contact network for a branching process



(b) With high contagion probability, the infection spreads widely



(c) With low contagion probability, the infection is likely to die out quickly

# Transmission rate or “Basic reproductive number” $R_0$

- Each person interacts with other  $k$  people
- Each interaction ends in infection with probability  $\beta$

- What is the expected number of cases caused by a single individual,  $R_0$ ?
- What do you think happens if  $R_0 < 1$ ?
- What do you think happens if  $R_0 > 1$ ?

# Transmission rate or “Basic reproductive number” $R_0$

- Each person interacts with other  $k$  people
- Each interaction ends in infection with probability  $\beta$

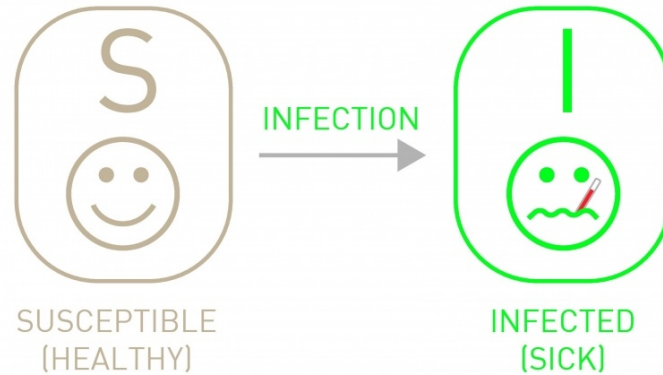
- What is the expected number of cases caused by a single individual?  $R_0 = \beta k$
- What do you think happens if  $R_0 < 1$ ?
  - **The infection dies out**
- What do you think happens if  $R_0 > 1$ ?
  - **The infection persists**

Disease	Transmission	$R_0$
Measles	Airborne	12-18
Pertussis	Airborne droplet	12-17
Diphtheria	Saliva	6-7
Smallpox	Social contact	5-7
Polio	Fecal-oral route	5-7
Rubella	Airborne droplet	5-7
Mumps	Airborne droplet	4-7
HIV/AIDS	Sexual contact	2-5
SARS	Airborne droplet	2-5
Influenza (1918 strain)	Airborne droplet	2-3

# Changing $R_0 = \beta k$

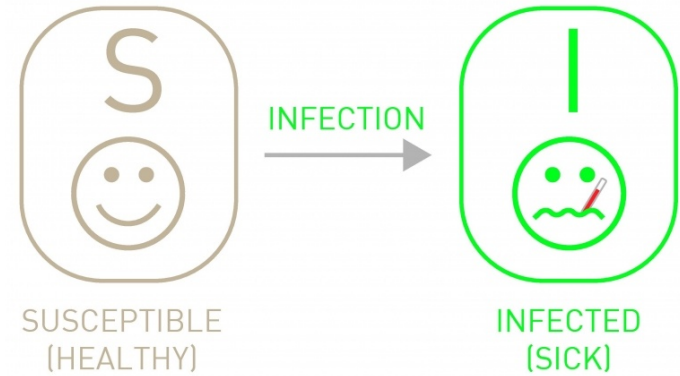
- Sanitary practices  
(reduce  $\beta$ )
- Quarantine  
(reduces  $k$ )

# The SI model



# The SI model

- Susceptible:
  - The node can catch the disease
- Infected:
  - The node has the disease and can spread it
  - It will stay sick forever





# Notation

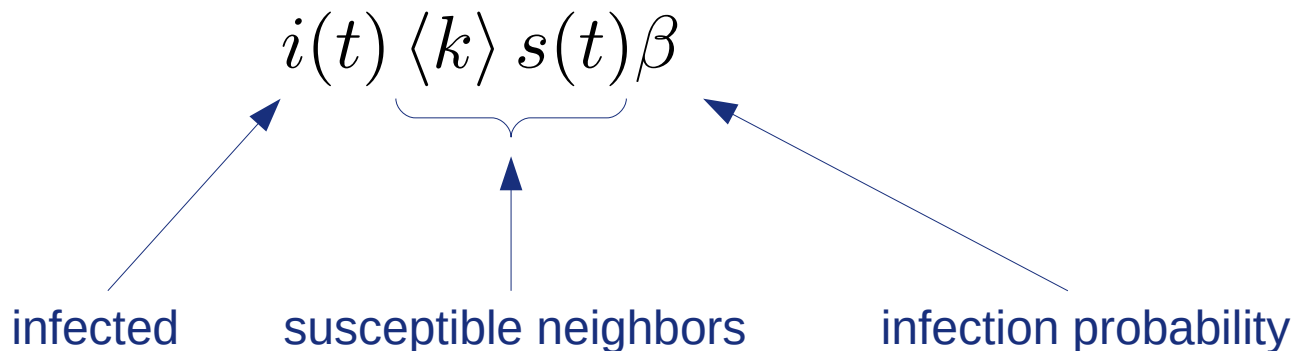
- Number of susceptible  $S(t)$ 
  - Fraction of susceptible  $s(t) = S(t) / N$
- Number of infected  $I(t)$ 
  - Fraction of infected  $i(t) = I(t) / N$
- $s(t) + i(t) = 1$

# How many susceptible neighbors a node has?

$$\langle k \rangle \frac{S(t)}{N} = \langle k \rangle s(t)$$

## How many new infections are produced?

(for every infected, iterate through its susceptible neighbors, infect with probability  $\beta$ )



Prove that  $i(t) = \frac{i_0 e^{\beta \langle k \rangle t}}{1 - i_0 + i_0 e^{\beta \langle k \rangle t}}$

Begin from:  $\frac{di(t)}{dt} = i(t) \langle k \rangle (1 - i(t)) \beta$

First, place all terms with  $i(t)$  on the left side

Second, use  $\frac{1}{x \cdot (1 - x)} = \frac{1}{x} + \frac{1}{1 - x}$

Upload to Nearpod Collaborate  
<https://nearpod.com/student/>  
Code to be given during class

Third, integrate from  $t = 0$  to  $t$  and denote by  $i_0 = i(t = 0)$

$$\int \frac{1}{x} dx = \log x + C$$

$$\int \frac{1}{1 - x} dx = -\log(1 - x) + C$$

# Infected per period of time

$$\begin{aligned}\frac{di(t)}{dt} &= i(t) \langle k \rangle \beta (1 - i(t)) \\ \left( \frac{1}{i(t)(1 - i(t))} \right) \frac{di(t)}{dt} &= \langle k \rangle \beta \\ \left( \frac{1}{i(t)} + \frac{1}{1 - i(t)} \right) \frac{di(t)}{dt} &= \langle k \rangle \beta\end{aligned}$$

Now, integrate on both sides from  $t=0$  to  $t=t \dots$

# Infected per period of time

$$\frac{di(t)}{dt} = i(t) \langle k \rangle \beta (1 - i(t))$$

$$\left( \frac{1}{i(t)(1 - i(t))} \right) \frac{di(t)}{dt} = \langle k \rangle \beta$$

$$\left( \frac{1}{i(t)} + \frac{1}{1 - i(t)} \right) \frac{di(t)}{dt} = \langle k \rangle \beta$$

$$\int_{t=0}^t \frac{1}{i(t)} \frac{di(t)}{dt} dt + \int_{t=0}^t \frac{1}{1 - i(t)} \frac{di(t)}{dt} dt = \int_{t=0}^t \langle k \rangle \beta dt$$

# Infected per period of time ( $i(0) = i_0$ )

$$\int_{t=0}^t \frac{1}{i(t)} \frac{di(t)}{dt} dt + \int_{t=0}^t \frac{1}{1-i(t)} \frac{di(t)}{dt} dt = \int_{t=0}^t \langle k \rangle \beta dt$$
$$\log i(t) - \log i_0 - \log(1-i(t)) + \log(1-i_0) = \langle k \rangle \beta t$$

Now, exponentiate (apply  $e^x$ ) in both sides

# Infected per period of time ( $i(0) = i_0$ )

$$\int_{t=0}^t \frac{1}{i(t)} \frac{di(t)}{dt} dt + \int_{t=0}^t \frac{1}{1-i(t)} \frac{di(t)}{dt} dt = \int_{t=0}^t \langle k \rangle \beta dt$$

$$\log i(t) - \log i_0 - \log(1-i(t)) + \log(1-i_0) = \langle k \rangle \beta t$$

$$\frac{e^{\log i(t)} e^{\log(1-i_0)}}{e^{\log i_0} e^{\log(1-i(t))}} = \frac{i(t)(1-i_0)}{(1-i(t))i_0} = e^{\langle k \rangle \beta t}$$

$$\frac{i(t)}{1-i(t)} = \frac{i_0 e^{\langle k \rangle \beta t}}{1-i_0}$$

$$i(t) = \frac{i_0 e^{\langle k \rangle \beta t}}{1-i_0 + i_0 e^{\langle k \rangle \beta t}}$$



# Behavior in the limit $t \rightarrow \infty$

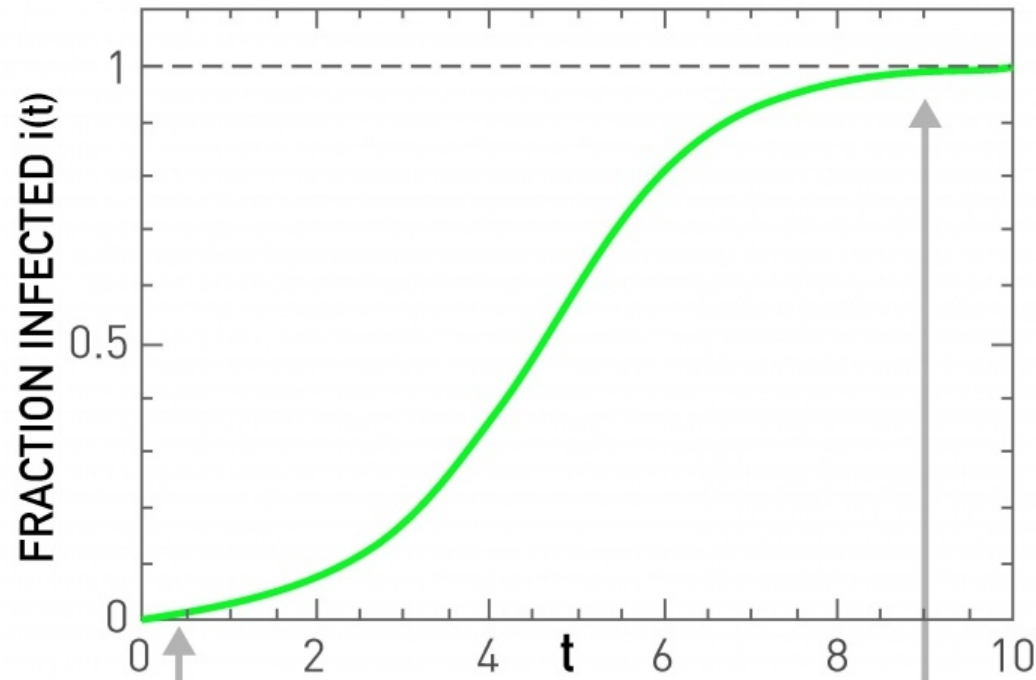
- What is the limit of  $i(t) = \frac{i_0 e^{\beta \langle k \rangle t}}{1 - i_0 + i_0 e^{\beta \langle k \rangle t}}$  when  $t \rightarrow \infty$  ?
- Hint: similar to  $f(t) = \frac{e^t}{1 + e^t}$

# Infected as a function of time (SI)

$$i(t) = \frac{i_0 e^{\beta \langle k \rangle t}}{1 - i_0 + i_0 e^{\beta \langle k \rangle t}}$$

Characteristic time  
(to infect  $1/e \approx 36\%$  of people):

$$\tau = \frac{1}{\beta \langle k \rangle}$$



exponential  
regime

If  $i$  is small,

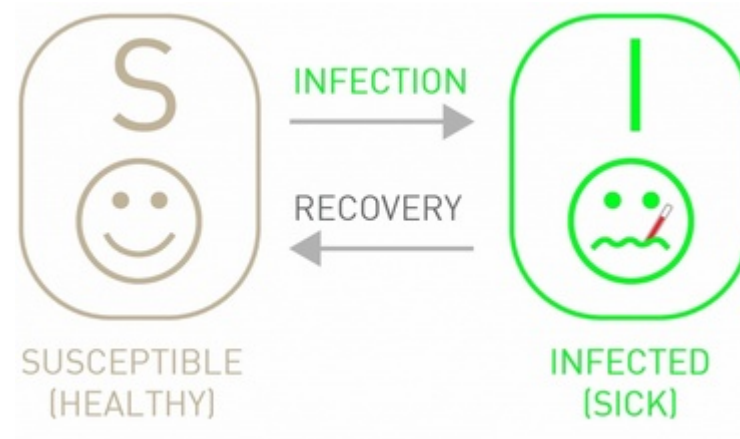
$$i \approx i_0 e^{\beta \langle k \rangle t}$$

saturation  
regime

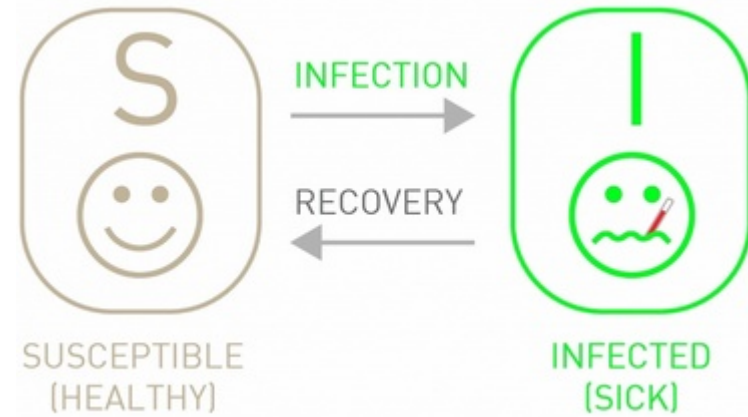
If  $i \rightarrow 1$ ,

$$\frac{di}{dt} \rightarrow 0$$

# The SIS model



# The SIS model



- Susceptible:
  - The node can catch the disease
- Infected:
  - The node has the disease and can spread it
  - After some time, it recovers ... but it becomes susceptible again

# Infection dynamics

$$\frac{di(t)}{dt} = \beta \langle k \rangle i(t)(1 - i(t)) - \mu i(t)$$

- $\mu$  is the recovery rate, i.e., the probability of becoming susceptible again in an unit of time

$$i(t) = \left(1 - \frac{\mu}{\beta \langle k \rangle}\right) \frac{C e^{(\beta \langle k \rangle - \mu)t}}{1 + C e^{(\beta \langle k \rangle - \mu)t}}$$

- $C$  is a constant that depends on  $i_0$

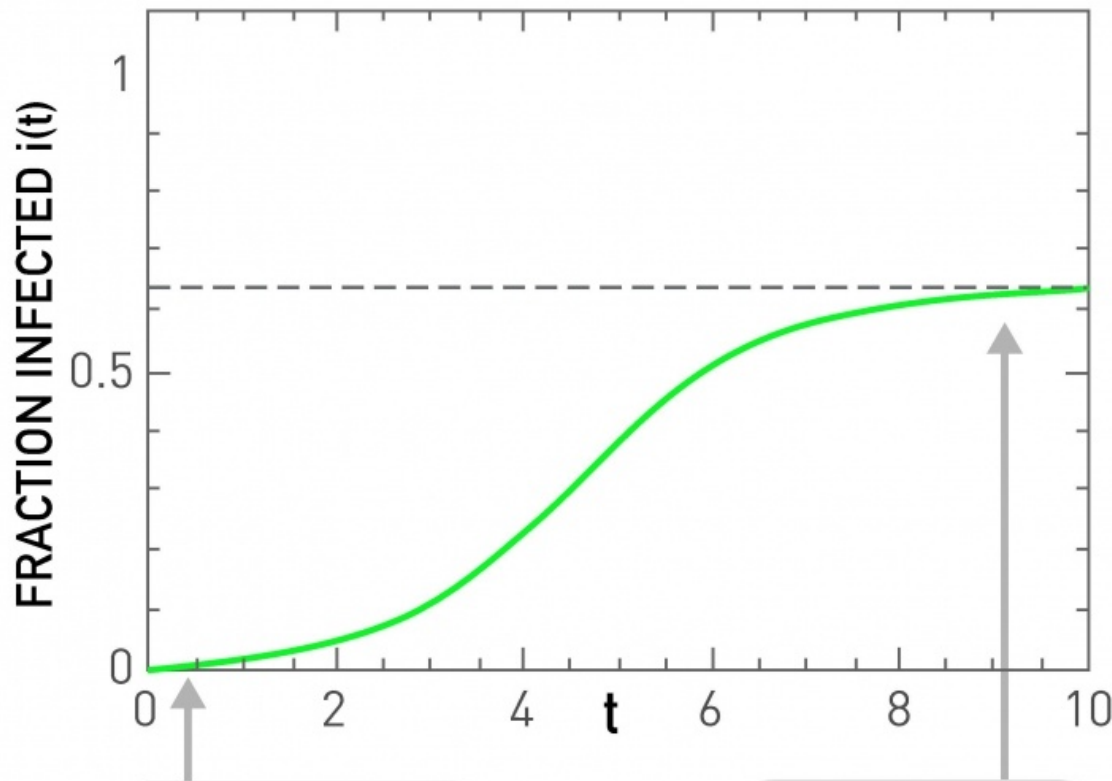
# Behavior in the limit $t \rightarrow \infty$

- What is the limit of  $i(t) = \left(1 - \frac{\mu}{\beta \langle k \rangle}\right) \frac{C e^{(\beta \langle k \rangle - \mu)t}}{1 + C e^{(\beta \langle k \rangle - \mu)t}}$  when  $t \rightarrow \infty$ ?
- Hint: similar to  $f(t) = \alpha \frac{e^t}{1 + e^t}$

# Infected as a function of time (SIS)

$$i(t) = \left(1 - \frac{\mu}{\beta \langle k \rangle}\right) \frac{C e^{(\beta \langle k \rangle - \mu)t}}{1 + C e^{(\beta \langle k \rangle - \mu)t}}$$

This is in the case  $\mu < \beta \langle k \rangle$



**exponential  
outbreak**

If  $i$  is small,  
 $i \approx i_0 e^{(\beta \langle k \rangle - \mu)t}$

**endemic  
state**

$i(\infty) = 1 - \frac{\mu}{\beta \langle k \rangle}$



# What happens if $\mu > \beta \langle k \rangle$ ?

- Remember:  $\frac{di(t)}{dt} = \beta \langle k \rangle i(t)(1 - i(t)) - \mu i(t)$

# What happens if $\mu > \beta \langle k \rangle$ ?

- Remember: 
$$\begin{aligned}\frac{di(t)}{dt} &= \beta \langle k \rangle i(t)(1 - i(t)) - \mu i(t) \\ &= \underbrace{(\beta \langle k \rangle - \mu)}_{\text{negative}} i(t) - \underbrace{\beta \langle k \rangle}_{\text{negative}} (i(t))^2\end{aligned}$$

- The infection dies out**

# The SIR model



# The SIR model



- **Susceptible:**
  - The node can catch the disease
- **Infected:**
  - The node has the disease and can spread it
- **Removed:**
  - The node no longer has the disease, and cannot catch it or propagate it again (could be dead, could be immune)

# Infection dynamics in SIR

$$\frac{di(t)}{dt} = \beta \langle k \rangle i(t)(1 - r(t) - i(t)) - \mu i(t)$$

$$\frac{dr(t)}{dt} = \mu i(t)$$

$$\frac{ds(t)}{dt} = -\frac{di(t)}{dt} - \frac{dr(t)}{dt} = -\beta \langle k \rangle i(t)(1 - r(t) - i(t))$$

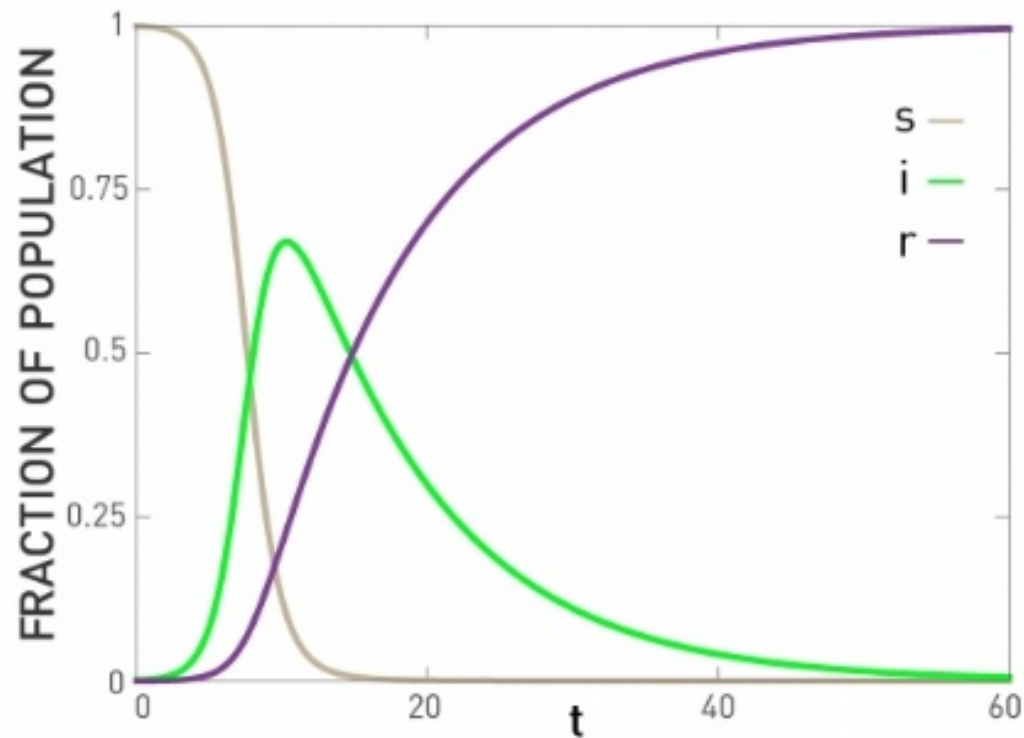
- No closed form solution

# Infection dynamics (SIR)

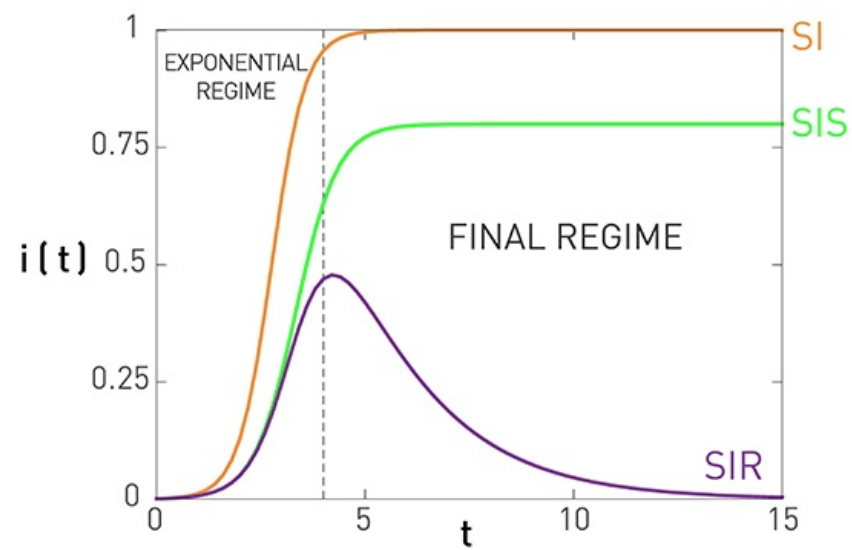
$$\frac{di(t)}{dt} = \beta \langle k \rangle i(t)(1 - r(t) - i(t)) - \mu i(t)$$

$$\frac{dr(t)}{dt} = \mu i(t)$$

$$\frac{ds(t)}{dt} = -\beta \langle k \rangle i(t)(1 - r(t) - i(t))$$



# Comparison of $i(t)$



	SI	SIS	SIR
<b>Exponential Regime:</b> Number of infected individuals grows exponentially	$i = \frac{i_0 e^{\beta\langle k \rangle t}}{1 - i_0 + i_0 e^{\beta\langle k \rangle t}}$	$i = \left(1 - \frac{\mu}{\beta\langle k \rangle}\right) \frac{C e^{(\beta\langle k \rangle - \mu)t}}{1 + C e^{(\beta\langle k \rangle - \mu)t}}$	No closed solution
<b>Final Regime:</b> Saturation at $t \rightarrow \infty$	$i(\infty) = 1$	$i(\infty) = 1 - \frac{\mu}{\beta\langle k \rangle}$	$i(\infty) = 0$
<b>Epidemic Threshold:</b> Disease does not always spread	No threshold	$R_0 = 1$	$R_0 = 1$



# Things to remember

- SI, SIS, SIR models
- Which are the states in each process and which are the possible transitions
- Equations for number of nodes in each state
- Regimes under different parameters
- Practice executing by hand and write code if it helps you remember better each process

# Practice on your own

Under the **SIS** model,  $i(t) = \left(1 - \frac{\mu}{\beta \langle k \rangle}\right) \frac{C e^{(\beta \langle k \rangle - \mu)t}}{1 + C e^{(\beta \langle k \rangle - \mu)t}}$

1. When  $\mu < \beta \langle k \rangle$  what is the limit of  $i(t)$  ?
2. How is this state called?
3. What happens when  $\mu > \beta \langle k \rangle$  ?
4. What conditions lead to large values of  $i(t)$ ?

# Practice on your own (cont.)

- In the **SIRS** epidemic model, there are three possible states for a node: susceptible, infected, and recovered. Susceptible nodes can become infected, infected nodes can become recovered, and *recovered nodes can become susceptible again*.
- During one unit of time, with probability  $\beta$  an infected node can infect one of its contacts, with probability  $\mu$ , an infected node can recover, and with probability  $\sigma$ , a recovered node can become susceptible again.
- Let  $s(t)$  be the fraction of susceptible nodes,  $i(t)$  be the fraction of infected nodes,  $r(t)$  the fraction of recovered nodes, and  $\langle k \rangle$  the average degree of the graph. Write the equations, simplifying them appropriately, for:

$$1. \frac{di(t)}{dt} \quad 2. \frac{dr(t)}{dt} \quad 3. \frac{ds(t)}{dt}$$

4. Is  $\sigma > \mu$  sufficient to say that the recovered will tend to zero in the long run?