Degree Distributions in Preferential Attachment

Introduction to Network Science

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Topic 12

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Contents

- Degree distribution under the BA model
- Distance distribution under the BA model
- Clustering coefficient under the BA model

BA model means Barabási-Albert model (preferential attachment)

Sources

- Albert László Barabási (2016) Network Science
  - Preferential attachment follows chapter 05
- Ravi Srinivasan 2013 Complex Networks Ch 12
- Networks, Crowds, and Markets Ch 18
- Data-Driven Social Analytics course by Vicenç Gómez and Andreas Kaltenbrunner

Remember the BA model

- Network starts with m_0 nodes connected arbitrarily as long as their degree is ≥ 1
- At every time step we add 1 node
- This node will have m outlinks $(m \leq m_0)$
- The probability of an existing node of degree k_i to gain one such link is $$\Pi(k_i) = \frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

Degree $k_i(t)$ as a function of time

$$\frac{d}{dt}k_i = m\Pi(k_i) = m\frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

$$\sum_{j=1}^{N-1} k_j = L(0) + 2m(t-1) \approx 2m(t-1)$$

(For large t)

$$\frac{d}{dt}k_i = \frac{mk_i}{2m(t-1)} = \frac{k_i}{2t-2} \approx \frac{k_i}{2t}$$

Degree $k_i(t)$ … continued

$$\frac{d}{dt}k_i(t) = \frac{k_i(t)}{2t}$$

$$\frac{1}{k_i(t)}\frac{d}{dt}k_i(t) = \frac{1}{2t}$$

$$\int_{t=t_i}^{t} \frac{1}{k_i(t)}\frac{d}{dt}k_i(t)dt = \int_{t=t_i}^{t} \frac{1}{2t}dt$$

$$\log k_i(t) - \log k_i(t_i) = \frac{1}{2}\log t - \frac{1}{2}\log t_i$$

$$\log k_i(t) = \frac{1}{2}\log t - \frac{1}{2}\log t_i + \log m$$

Note: in exams for this course, you will **not** be asked to solve differential equations on your own

(t_i is the creation time of node i)

Degree $k_i(t)$ ... continued

$$\log k_i(t) = \frac{1}{2}\log t - \frac{1}{2}\log t_i + \log m$$

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{1}{2}}$$

Is the degree growth linear, super-linear, or sub-linear? Intuitively, why?

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{1}{2}} = m\left(\frac{t}{t_i}\right)^{\beta}$$

$\beta = 1/2$ is called the dynamical exponent

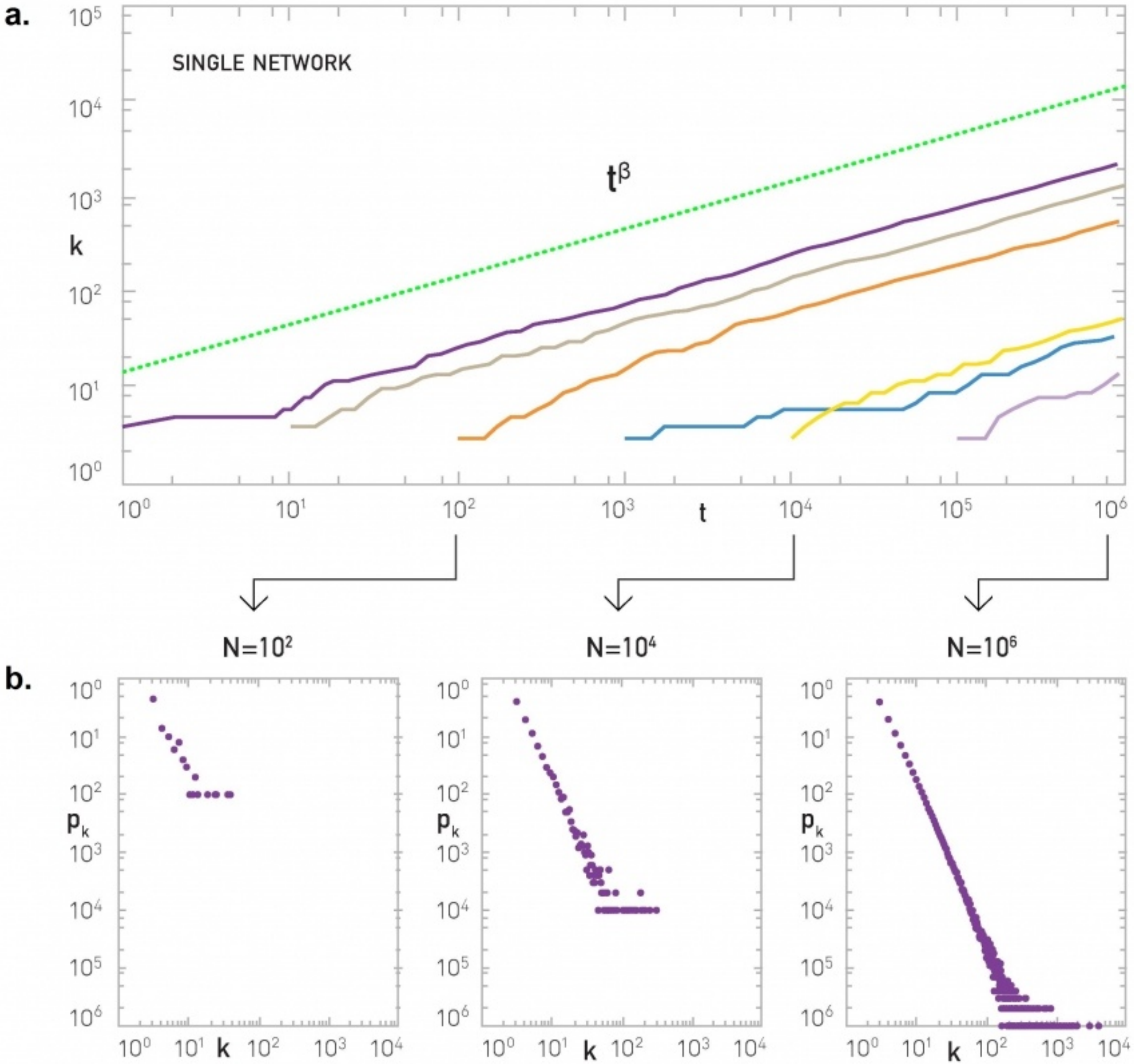
Degree $k_i(t)$ ... consequences

$$\log k_i(t) = \frac{1}{2}\log t - \frac{1}{2}\log t_i + \log m$$

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\frac{1}{2}}$$

$$\frac{dk_i(t)}{dt} = \frac{k_i(t)}{2t} = \frac{m\left(\frac{t}{t_i}\right)^{\frac{1}{2}}}{2t} = \frac{m}{2\left(t \cdot t_i\right)^{\frac{1}{2}}}$$

If $t_i < t_j$ (node i is older than node j), what do we expect of k_i and k_j?



Simulation results

Model

Nodes with t_i = 1, 10, 100, 1000, 10000, ...

Cumulative Distribution Function

Let’s calculate the CDF of the degree distribution

By definition of CDF, this is equal to:

$$Pr(k_i(t) \leq k) = 1 - Pr(k_i(t) > k)$$

CDF (cont.)

Let's calculate $Pr(k_i(t) > k)$

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\beta}$$

$$k_i(t) > k \Rightarrow m\left(\frac{t}{t_i}\right)^{\beta} > k$$

$$m^{\frac{1}{\beta}}\left(\frac{t}{t_i}\right) > k^{\frac{1}{\beta}}$$

$$\left(\frac{m}{k}\right)^{\frac{1}{\beta}}\left(\frac{t}{t_i}\right) > 1$$

$$\left(\frac{m}{k}\right)^{\frac{1}{\beta}} > \left(\frac{t_i}{t}\right)$$

This means that nodes i with degree larger than k were created at time t_i **before** a certain timestep, which is expected because older nodes have larger degree.

$$t_i < t\left(\frac{m}{k}\right)^{\frac{1}{\beta}}$$

CDF (cont.)

From the previous slide, we have: $$Pr(k_i(t) > k) = Pr\left(\left(\frac{m}{k}\right)^{\frac{1}{\beta}} > \frac{t_i}{t}\right)$$

Remember there is one node created at each timestep, so by time t there are $N(t) = m_o + t$ nodes, and for large t, we have $N(t) \approx t$

Now, what is $Pr(x > t_i/t)$ if you pick a node i at random?

It is x, because $\boldsymbol{t_i/t}$ **is distributed uniformly in** $\boldsymbol{[0,1]}$

Imagine the following "game", in which the larger number wins

- You pick a number x in $[0,1]$
- Your opponent picks a number y uniformly at random in $[0,1]$

The probability that $x > y$ and hence you win is exactly x

Hence:

$$Pr(k_i(t) > k) = \left(\frac{m}{k}\right)^{\frac{1}{\beta}}$$

CDF (cont.)

Hence:

$$\begin{aligned} Pr\left(k_i(t) \leq k\right) &= 1 - Pr\left(k_i(t) > k\right) \\ &= 1 - \left(\frac{m}{k}\right)^{\frac{1}{\beta}} \end{aligned}$$

Probability Density Function (PDF)

Now let's take the derivative of the CDF to obtain the PDF

$$p_k = \frac{d}{dk} Pr(k_i \leq k) = \frac{d}{dk}\left(1 - \left(\frac{m}{k}\right)^{1/\beta}\right)$$

$$= -\frac{d}{dk}\left(\left(\frac{m}{k}\right)^{1/\beta}\right) = -m^{1/\beta}\frac{d}{dk}\left(\frac{1}{k^{1/\beta}}\right)$$

$$= \frac{1}{\beta}\frac{m^{1/\beta}}{k^{1/\beta+1}} \qquad (\beta = 1/2)$$

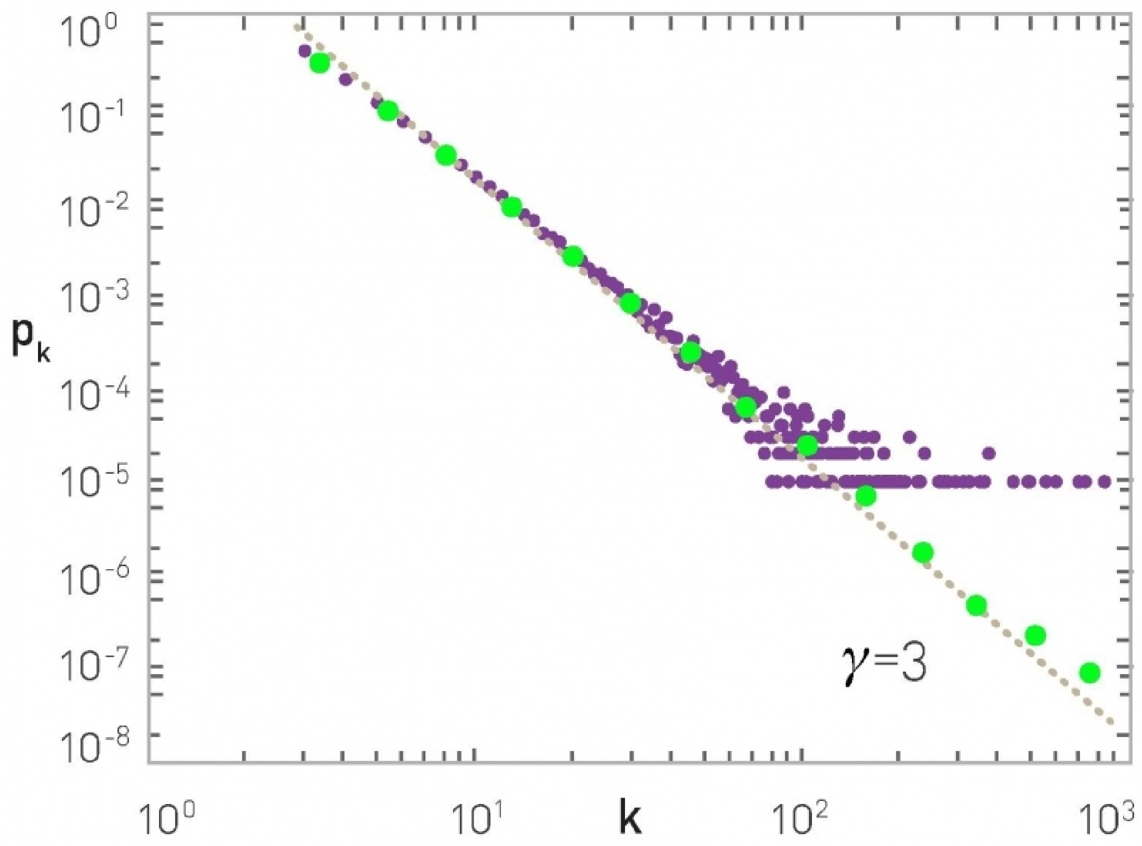
$$= 2\frac{m^2}{k^3} \longrightarrow p(k) \propto k^{-3}$$

Degree distribution

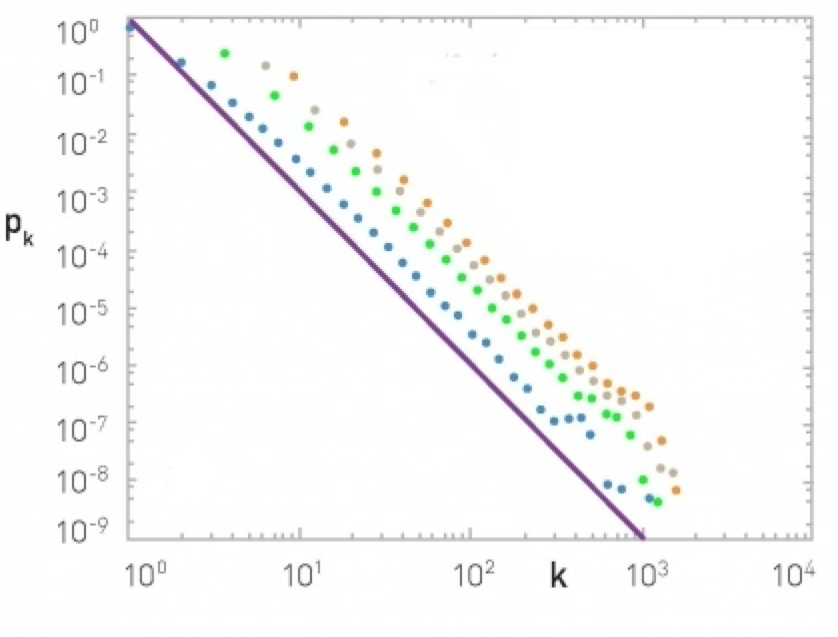
- $\beta = 1/2$ is called the dynamical exponent
- $\gamma = \frac{1}{\beta} + 1 = 3$ is the power-law exponent

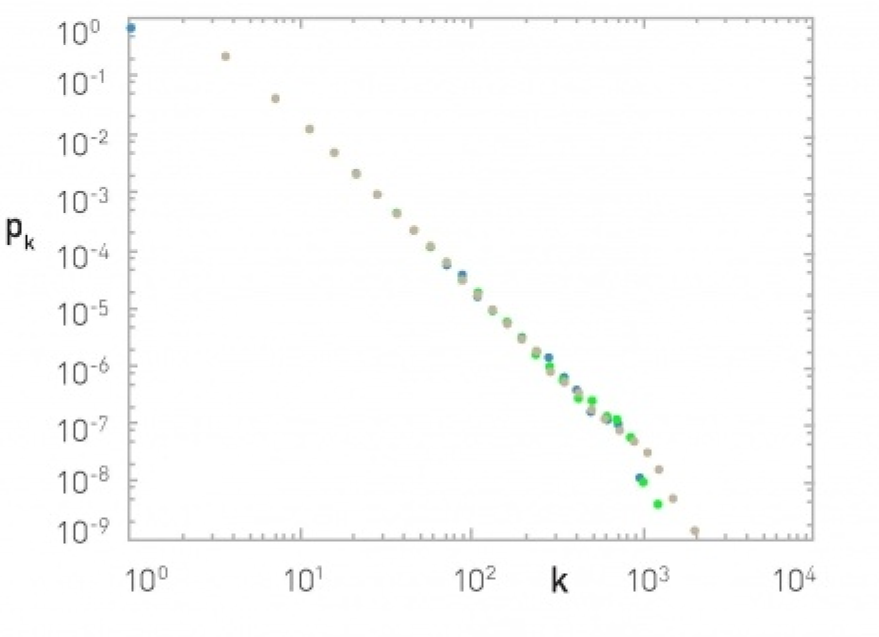
- Note that $p(k) \approx 2m^2/k^3$
  does not depend on t
  hence, it describes a stationary network

Degree distribution, simulation results
N=100,000   m=3



More simulations





N = 100,000; m_0 = m =
1 (blue), 3 (green), 5 (gray), 7 (orange)

Observe γ is independent of m (and m_0)

The slope of the purple line is -3

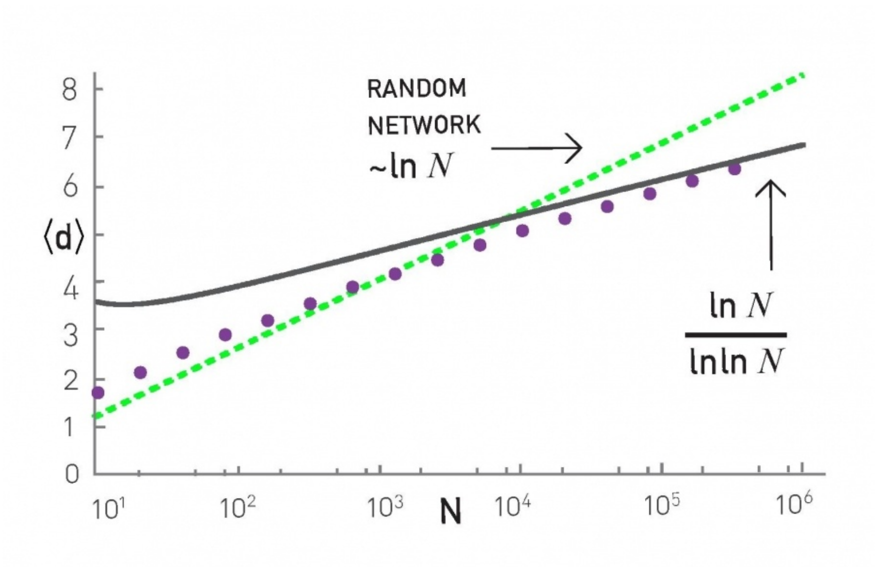
m_0 = m = 3; N =
50K (blue), 100K (green), 200K (gray)

Observe p_k is independent of N

Average distance

- Distances grow slower than log N

$$\langle d \rangle \approx \frac{\log N}{\log \log N}$$

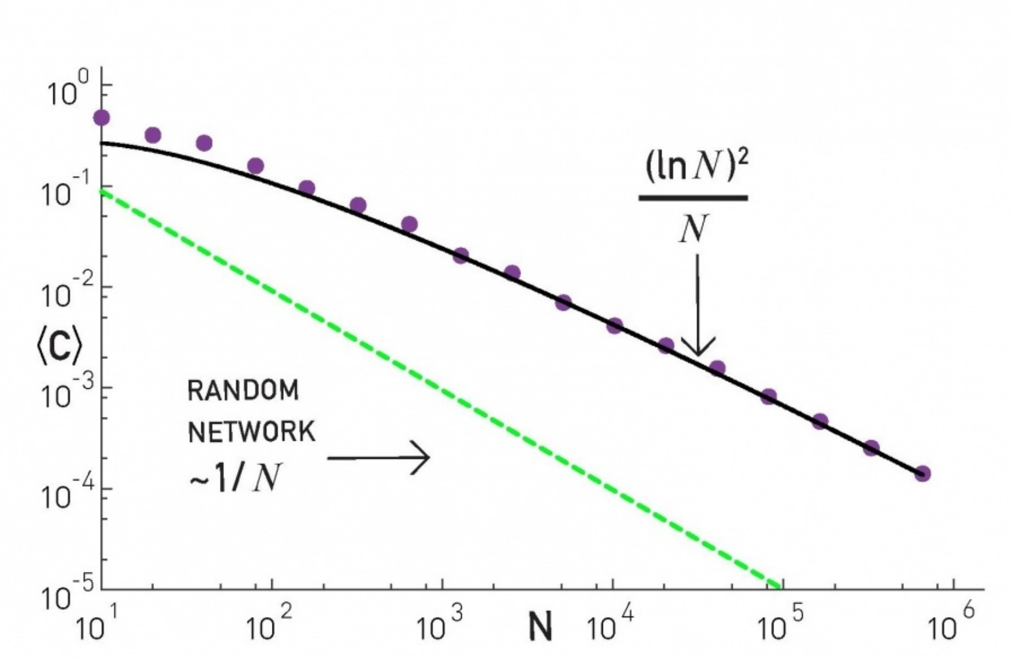


(Why: scale free network with $\gamma = 3$)

Clustering coefficient

- BA networks are locally more clustered than ER networks

$$\langle C \rangle \approx \frac{(\log N)^2}{N}$$



Limitations of the BA model

- Predicts a fixed exponent of -3
- Assumes an undirected network, while many real complex networks are directed
- Does not consider node deletions or edge deletions which are common in practice
- Considers that all nodes are equal except for their arrival times

Other processes that generate scale-free networks

- Link-selection model — step:
  - Add one new node v to the network
  - Select an existing link (u,w) at random and connect v to either u or w
- Copy model — step:
  - Add one new node v to the network
  - Pick a random existing node u
  - With probability p link to u
  - With probability 1-p link to a neighbor of u

Exercise: the copy model

In the copy model, start at $t=1$ with one node, and at every step t :

- Add one new node v to the network
- Pick a random existing node u
- If u has no out-links, link to u
- If u has out-links choose one of the following:
  - With probability p link to u
  - With probability $1-p$ link to one of the out-neighbors of u chosen at random

- Simulate it on paper (directed graph) for 7 nodes with $p=0.5$
  - Make sure you understand the model fully!
- What is $N(t)$ and $L(t)$? What is k_i^{out} ?

Answer in Nearpod Draw-it
https://nearpod.com/student/
Access to be provided during class

Degree distribution in the copy model

Proven in the paper by Kumar et al. (FOCS 2000) “Stochastic models for the web graph” and developed in the advanced materials.

$$\gamma = \frac{2-p}{1-p} \in [2,3] \quad \text{if } p \in [0, 1/2]$$

The copy model can generate any exponent between 2 and 3!

Summary

Things to remember

- Degree distribution in the BA model
- Distances and clustering coefficient in BA
- The copy model

Practice on your own

- Try to reconstruct the derivations we have done in class
  - Try to understand every step
- Insert a small change in the model and try to recalculate what we have done

Advanced materials: Copy model degree (not included in the exam)

In the copy model, at every step t:
1) Add one new node v to the network
2) Pick a random existing node u
3) With probability p link to u
4) With probability 1-p link to a neighbor of u

Answer in Nearpod Draw-it
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- We will compute k_i^{in} but first ...
- How many links on average gets node *i* at time *t*? In other words, what is:

$$\frac{d}{dt}k_i^{\text{in}}(t)$$

- Hint: it has a term with p and a term with *1-p*

- Integrate between t_i and t to obtain an expression for $k_i(t_i)$
(***we drop the "in" superscript just for simplicity during this exercise***)
- Note that now $k_i(t_i) = 0$

- Once you have a expression for $k_i(t_i)$
- Compute $Pr(k_i(t_i) > k)$
- Now write the cumulative distribution function of $k_i(t_i)$
- And compute its derivative to obtain

$$p_k = Pr(k_i(t) = k) = \frac{d}{dk} Pr(k_i(t) \leq k)$$

- It should show exponent $\gamma = \dfrac{2-p}{1-p}$