# Graph theory basics

Introduction to Network Science

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Topic 04

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# Contents

- Notation for graphs
- Degree distributions
- Adjacency matrices

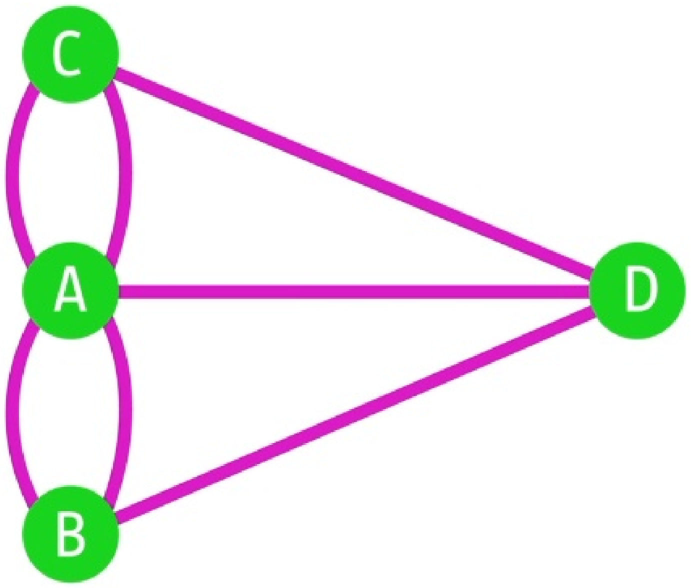
# Sources

- Albert László Barabási: Network Science. Cambridge University Press, 2016.
  - Follows almost section-by-section chapter 02
- URLs cited in the footer of specific slides

# The seven bridges of Königsberg

http://networksciencebook.com/images/ch-02/video-2-1.m4v

# Quick Question

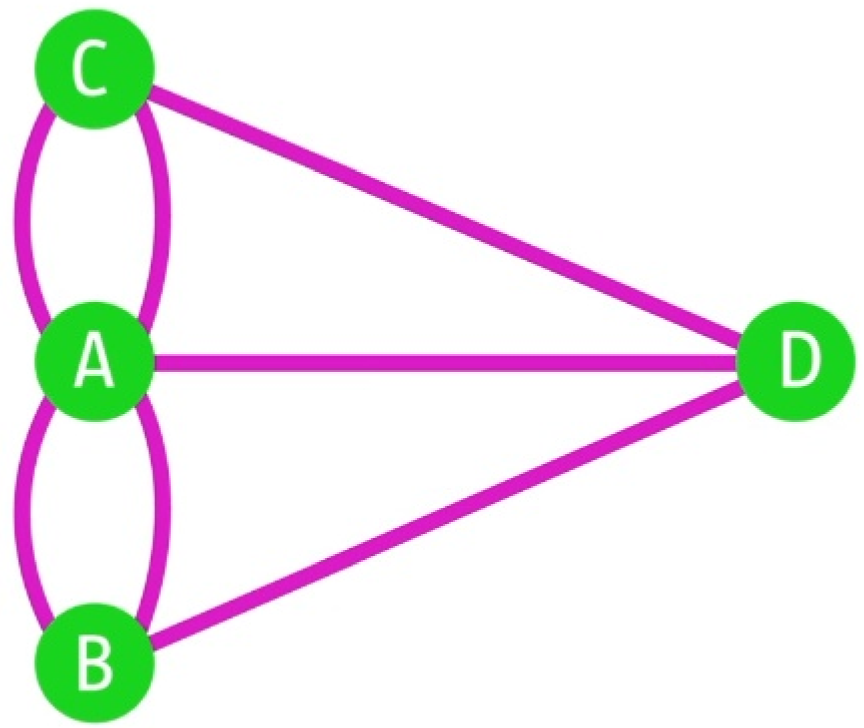



Can one walk across the 7 bridges without crossing the same bridge twice?

# Basic concepts

# Notation for a graph

- G = (V,E)
  - V: nodes or vertices
  - E: links or edges
- |V| = N size of graph
- |E| = L number of links



# Typical notation variations

- You may find that G is denoted by (N, A), this is typical of directed graphs, means “*nodes, arcs*”
- You may find that
  - |V| is denoted by n or N
  - |E| is denoted by m, M, or L

# Directed vs undirected graphs

- In an undirected graph
  - E is a symmetric relation

$$(u, v) \in E \Rightarrow (v, u) \in E$$

- In a directed graph, also known as “digraph”
  - E is not a symmetric relation

$$(u, v) \in E \not\Rightarrow (v, u) \in E$$

# Example graphs we will use

| Network | \|V\| | \|E\| |
|---|---|---|
| Zachary’s Karate Club (karate.gml) | 34 | 78 |
| Game of Thrones (got-relationships.csv) | 84 | 216 |
| US companies ownership | 1351 | 6721 |
| Marvel comics (hero-network.csv) | 6K | 167K |

https://github.com/chatox/networks-science-course/tree/master/practicum/data

# Degree

- Node i has degree $k_i$
  - This is the number of links incident on this node
  - The total number of links L is given by $L = \frac{1}{2}\sum_{i=1}^{N} k_i$

- Average degree $\langle k \rangle = \frac{1}{N}\sum_{i=1}^{N} k_i = \frac{2L}{N}$

# In directed networks

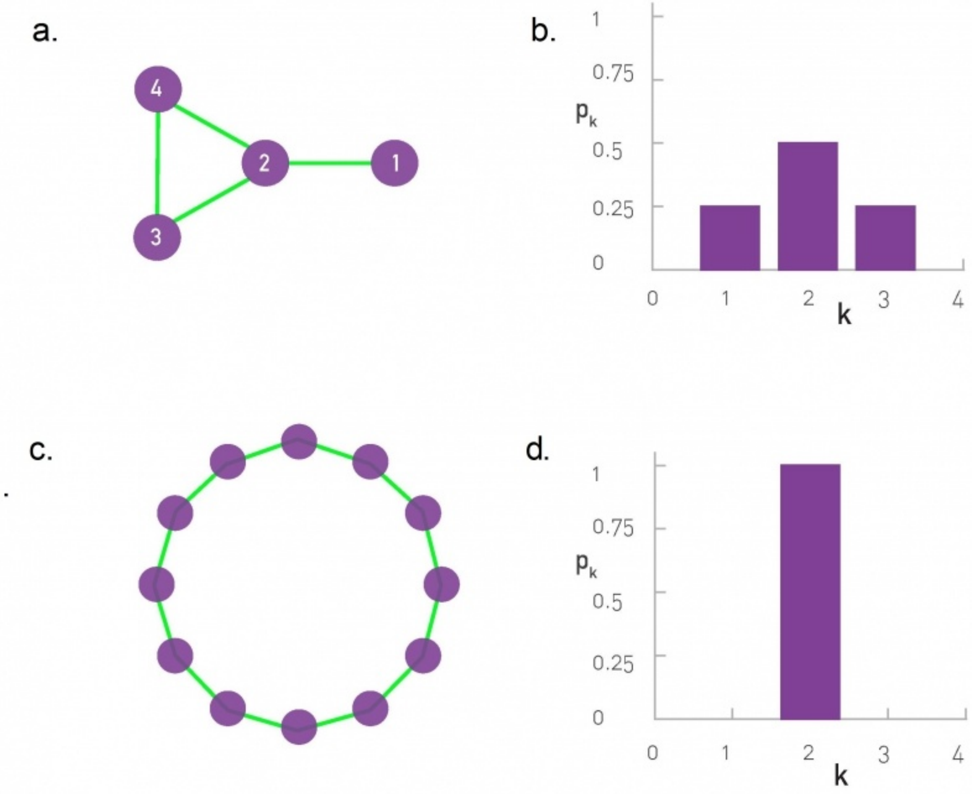
- We distinguish in-degree from out-degree
  - Incoming and outgoing links, respectively
- Degree is the sum of both $k_i = k_i^{\mathrm{in}} + k_i^{\mathrm{out}}$
- Counting total number of links:

$$L = \sum_{i=1}^{N} k_i^{\mathrm{in}} = \sum_{i=1}^{N} k_i^{\mathrm{out}}$$

# Degree distribution

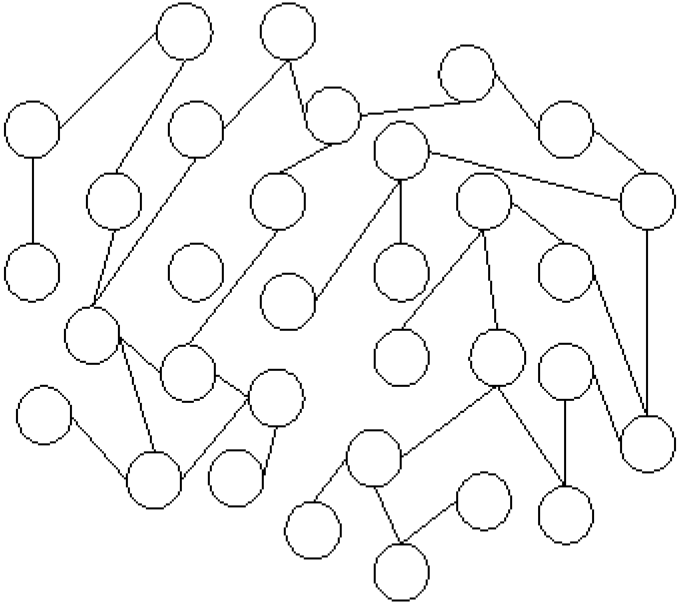
- If there are $N_k$ nodes with degree k
- The degree distribution is given by $p_k = \frac{N_k}{N}$
- The average degree is then $\langle k \rangle = \sum_{k=0}^{\infty} k p_k$

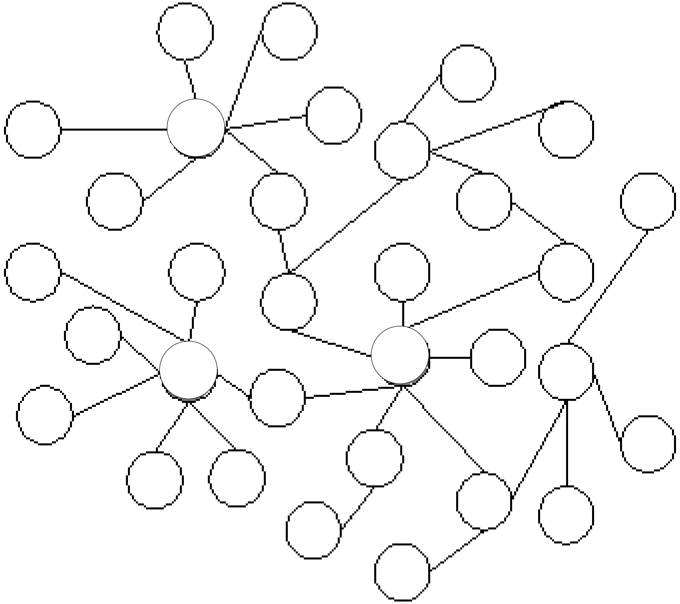
# Degree distribution; two toy graphs



# Exercise

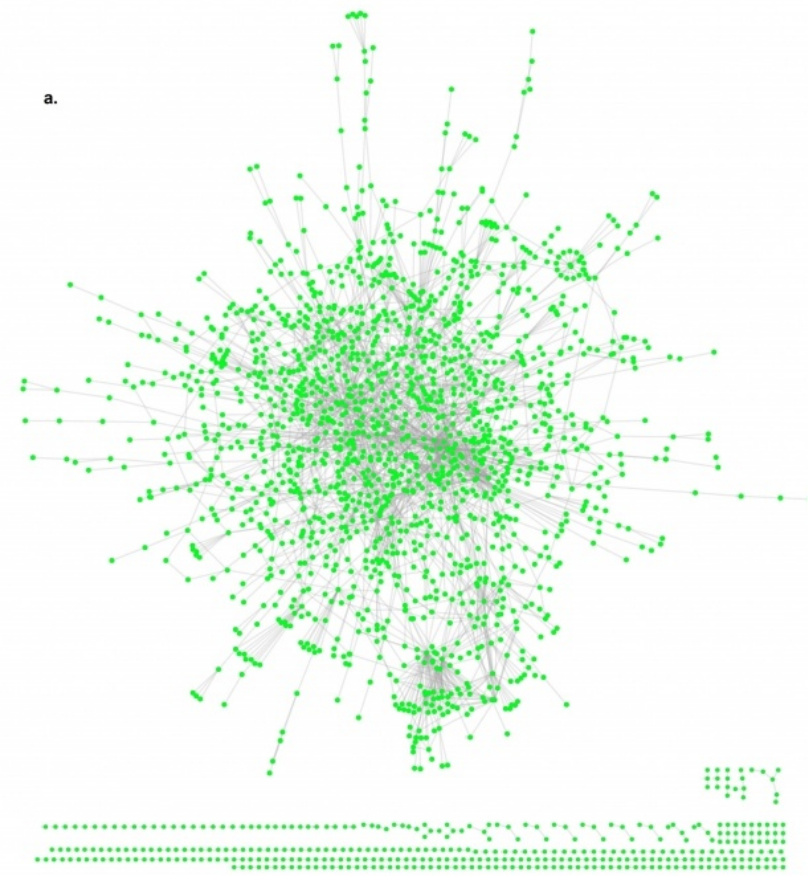
Answer in
Google Spreadsheet
(Link to be provided during class)

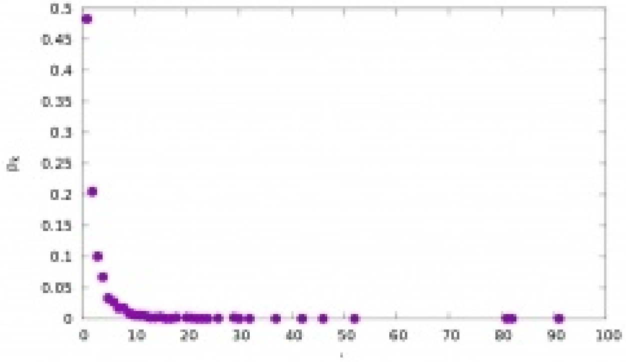




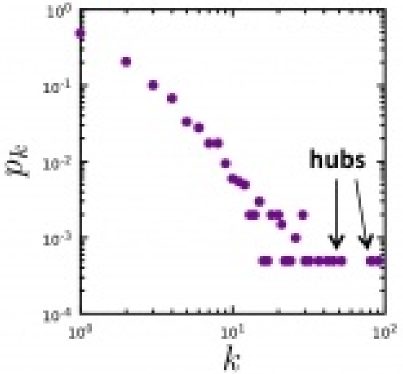
Draw the degree distribution of these graphs

# Degree distribution; real graph




Linear scale



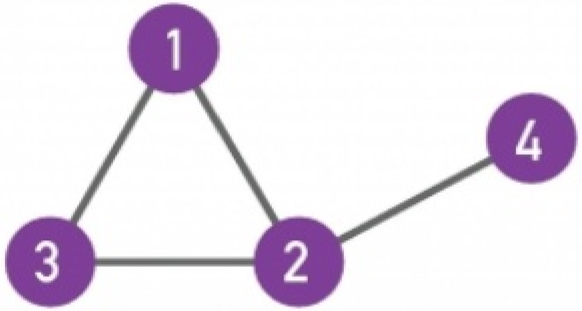

Log-log scale

# Adjacency matrix

# What is an adjacency matrix

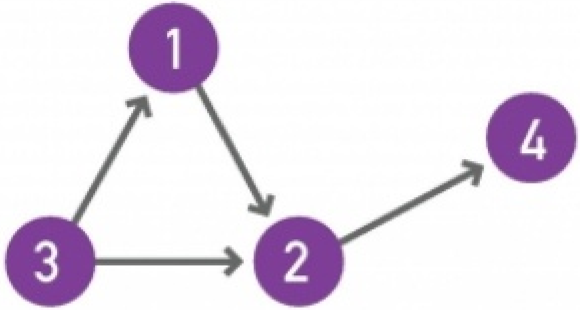
- A is the adjacency matrix of G = (V, E) iff:
  - A has |V| rows and |V| columns
  - $A_{ij} = 1$ if $(i,j) \in E$
  - $A_{ij} = 0$ if $(i,j) \notin E$
- **$A_{ij}$ always means row i, column j**
  - Sometimes Barabási's book has this wrong

# Examples




**Undirected graph**

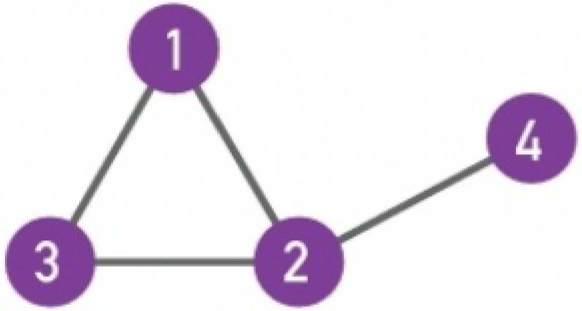
$$A_{ij} = \begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$




**Directed graph**

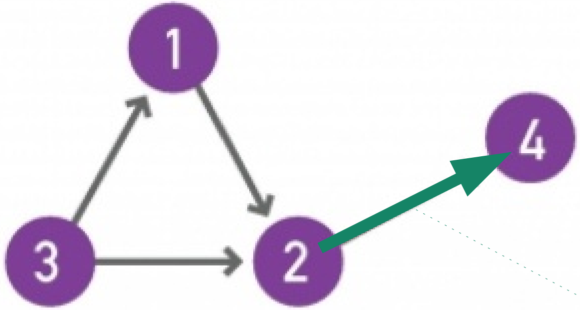
$$A_{ij} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

# $A_{ij}$ always means row i, column j




**Undirected graph**

$$A_{ij} = \begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$




**Directed graph**

Row 2
Column 4

$$A_{ij} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

# Quick Question

- In terms of A, what is the expression for:

$$k_i^{\text{in}} =$$

$$k_i^{\text{out}} =$$

# Some “graphology” …

- G is undirected $\Leftrightarrow$ A is symmetric
- G has a self-loop
  $\Leftrightarrow$ A has a non-zero element in the diagonal
- G is complete $\Leftrightarrow$ $A_{ij} \neq 0$ (except if i=j)

# Summary

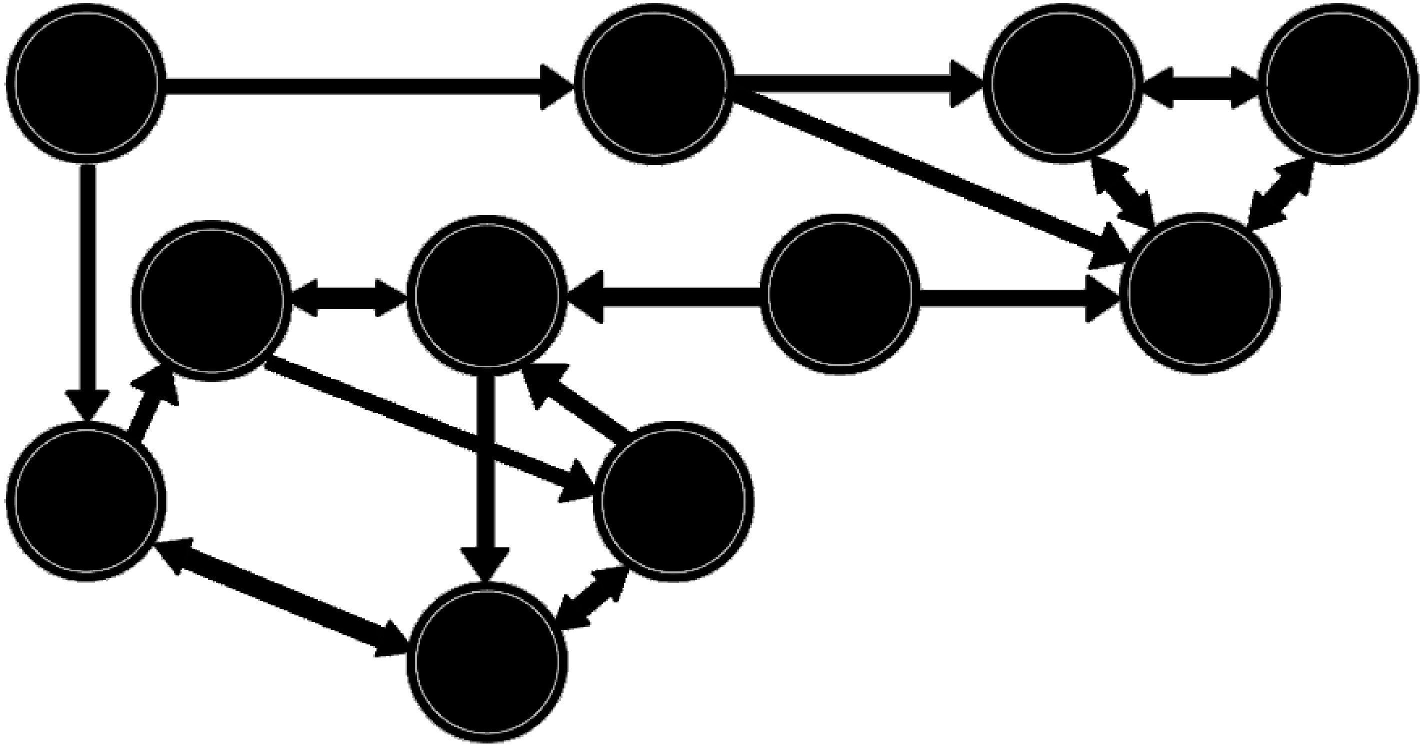
# Things to remember

- Definitions:
  - Degree, in-degree, out-degree
- Writing the adjacency matrix of a graph and drawing a graph given its adjacency matrix

# Practice on your own

Draw the indegree, outdegree, degree distribution

Write the adjacency matrix



https://www.6seconds.org/2017/07/03/20125/