Spectral Graph Embedding

Social Networks Analysis and Graph Algorithms

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Sources

- **J. Leskovec (2016).** Defining the graph laplacian [video]

- E. Terzi (2013). Graph cuts — The part on spectral graph partitioning

- D. A. Spielman (2009): The Laplacian

- URLs cited in the footer of slides

Graphs are nice, but ...

- They describe only local relationships

- We would like to understand a global structure

- We will try to transform a graph into a more familiar object: a cloud of points in R^k
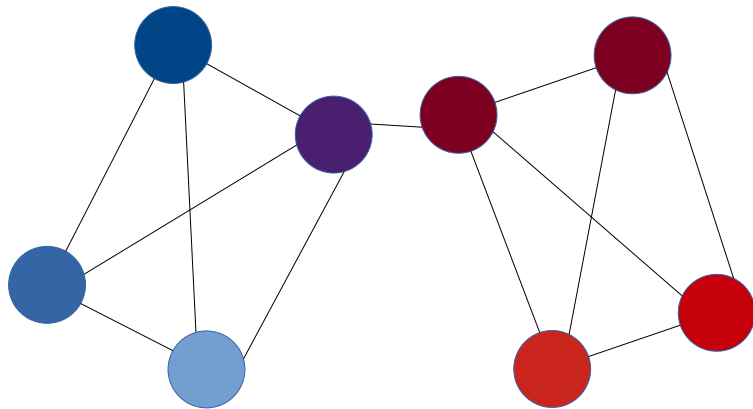
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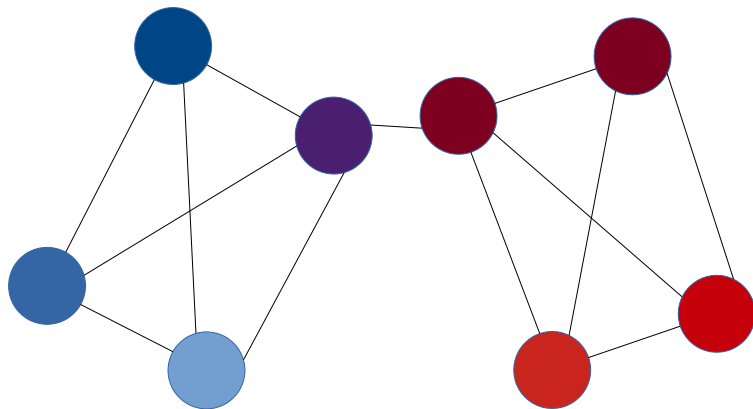
Distances should be somehow preserved

What is a graph embedding?

- A graph **embedding** (or graph **projection**) is a mapping from a graph to a vector space

- If the vector space is $\mathbb{R}^2$ you can think of an embedding as a way of *drawing* a graph on paper

Exercise: draw this graph

V = {v1, v2, ..., v8}

E = { (v1, v2), (v2, v3), (v3, v4), (v4, v1), (v5, v6), (v6, v7), (v7, v8),

(v8, v5), (v1,v5), (v2, v6), (v3, v7), (v4, v8) }

Draw this graph on paper, upload a photo

What constitutes a good drawing?

In a good graph embedding ...

- Pairs of nodes that are **connected** to each other should be **close**

- Pairs of nodes that are not connected should be far

- Compromises will need to be made

Random projections

Random graph projection (2D)

- Start a BFS from a random node, that has x=1, and nodes visited have ascending x

- Start a BFS from another random node, which has y=1, and nodes visited have ascending y

- Project node i to position (x_i, y_i)

Exercise: random projection

- Given this graph

- Pick a random node *u*
  - Distances from *u* are the x positions

- Pick a random node v
  - Distances from *v* are the y positions

- Draw the graph in an $\mathbb{R}^2$ plane
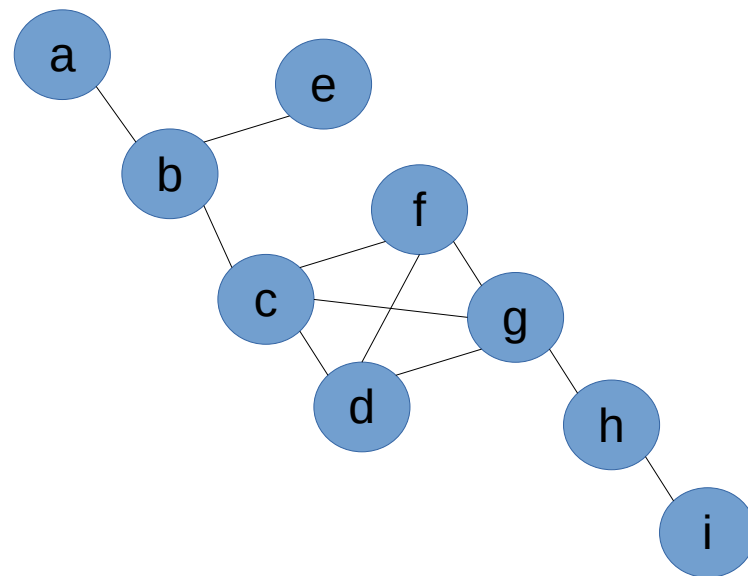
Refresher about eigenvectors/eigenvalues

Eigenvectors and eigenvalues

- In general $Av = \lambda v$ means A has an eigenvector v of eigenvalue λ

- In **symmetric** matrices, **eigenvectors are orthogonal**
  Suppose $\lambda_1 \neq \lambda_2$

$$\lambda_1 v_1^T v_2 = v_1^T A v_2 = v_1^T \lambda_2 v_2 = \lambda_2 v_1^T v_2$$

- This implies $v_1^T v_2 = 0$

In symmetric matrices

- The **multiplicity** of an eigenvalue λ is the dimension of the space of eigenvectors of eigenvalue λ

- Every n x n symmetric matrix has n eigenvalues counted with multiplicity

- Hence, it has an orthonormal basis of eigenvectors

Rayleigh quotient

In symmetric matrices M, the second eigenvalue is

$$\lambda_2 = \min_x \frac{x^T M x}{x^T x}$$

https://en.wikipedia.org/wiki/Rayleigh_quotient

Eigenvectors of the adjacency matrix

Properties of adjacency matrix

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- How many non-zeros are in every **row** of A?

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

https://www.youtube.com/watch?v=AR7iFxM-NkA

Adjacency matrix of G=(V,E)

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Can you write $\mathbf{y_i}$ using E ?

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

https://www.youtube.com/watch?v=AR7iFxM-NkA

Adjacency matrix of G=(V,E)

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- **What is Ax?** Think of x as a set of labels/values:

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$y_i = \sum_{j:(j,i) \in E} x_j$$

Ax is a vector whose i[th] coordinate contains the sum of the x_j who are in-neighbors of i

Spectral graph theory ...

- Studies the eigenvalues and eigenvectors of a graph matrix

  – Adjacency matrix $Ax = \lambda x$

  – Laplacian matrix (next)

- Suppose graph is d-regular: $k_i = d \ \forall i$

- What is the value of

- What does that imply?

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = ?$$

An eigenvector of a d-regular graph

- Suppose graph is d-regular, i.e. all nodes have degree d:

$$
\begin{bmatrix}
a_{11} & \dots & a_{1n} \\
a_{21} & \dots & a_{2n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \dots & a_{nn}
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
=
\begin{bmatrix}
d \\
d \\
\vdots \\
d
\end{bmatrix}
= d
\begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
$$

- Hence, $[1, 1, \dots, 1]^\top$ is an eigenvector of eigenvalue d

Disconnected graphs

- Suppose the graph is regular **and disconnected**
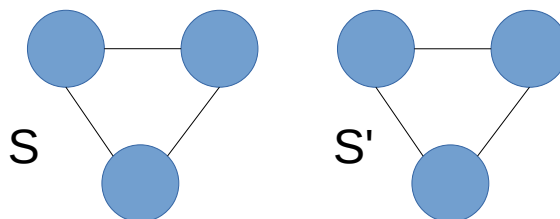
- Then its adjacency matrix has block structure:

$$A = \begin{bmatrix} S & 0 \\ 0 & S' \end{bmatrix}$$

Disconnected graphs
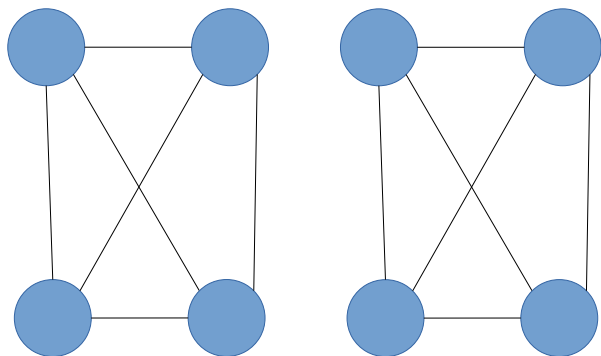
- Suppose the graph is regular **and disconnected**

S   S'

$$\text{Let } x_i^S = \left\{ \begin{array}{ll} 1 & \text{if } i \in S \\ 0 & \text{if } i \in S' \end{array} \right.$$
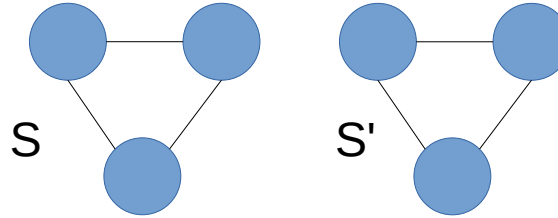
$$\begin{bmatrix} S & 0 \\ 0 & S' \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} = ?$$

Disconnected graphs

- Suppose the graph is regular **and disconnected**
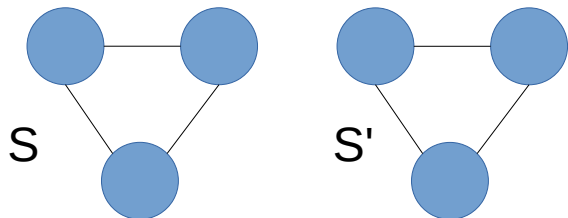
$$Ax^S = dx^S$$

$$Ax^{S'} = dx^{S'}$$

- What is the multiplicity of eigenvalue d?

- What happens if there are more than 2 connected components?

In general

Disconnected graph

$$\lambda_1 = \lambda_2$$

Almost disconnected graph
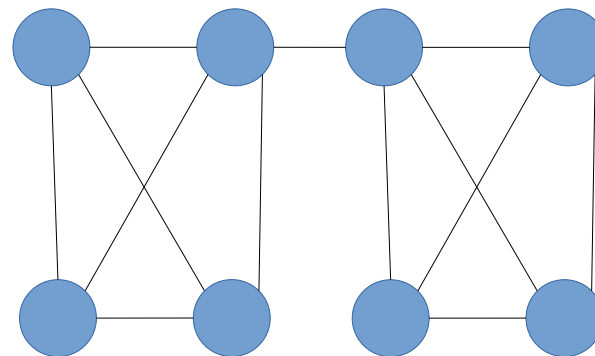
$$\lambda_1 \approx \lambda_2$$

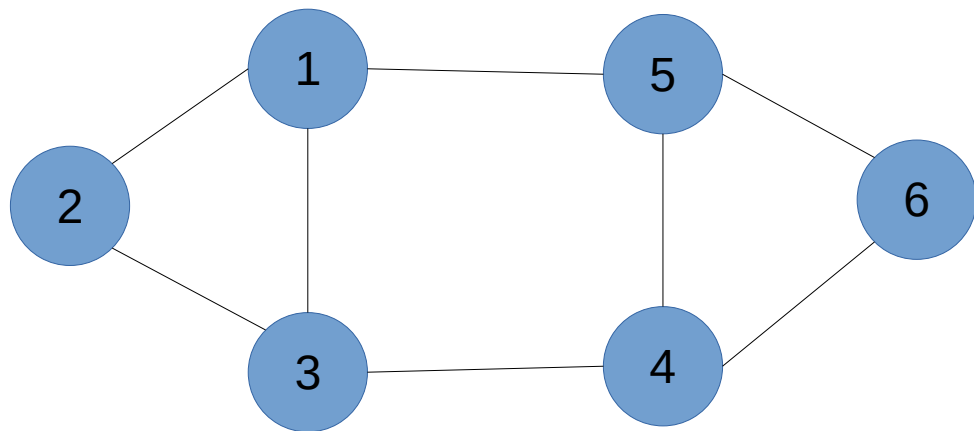Small "**eigengap**"

Graph Laplacian

Adjacency matrix

$$A_{ij} = \left\{ \begin{array}{ll} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{array} \right.$$
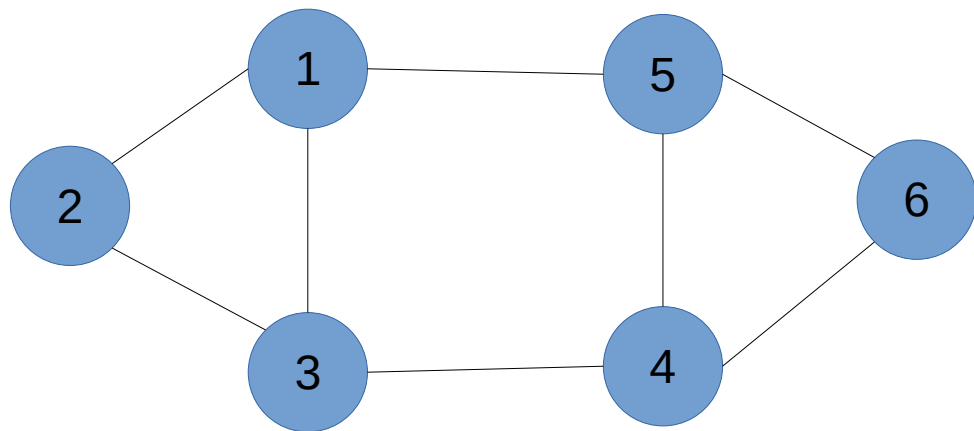
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Degree matrix

$$D_{ij} = \begin{cases} k_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Laplacian matrix

$$L = D - A$$
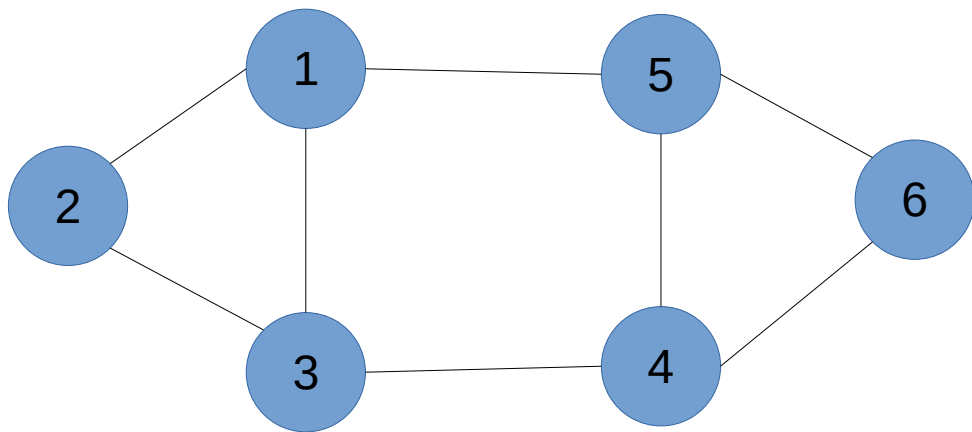
Given A is symmetric, **L is symmetric.**
They only differ in the diagonal.

$$L = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Laplacian matrix L = D - A

$$L\vec{1} = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = ?$$

The constant vector is an eigenvector of L

The constant vector $x=[1,1,...,1]^T$ is an eigenvector of the Laplacian, and has eigenvalue 0

$$Lx = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Does it need to be this specific graph? Why?

Does it need to be the vector [1, 1, …, 1]? Why?

If the graph is disconnected

- If the graph is disconnected into two components, the same argument as for the adjacency matrix applies, and $\lambda_1 = \lambda_2 = 0$

- The multiplicity of eigenvalue 0 is equal to the number of connected components

$$x^T L x$$

Prove this!

- Prove that $x^T L x = \sum_{(i,j) \in E} (x_i - x_j)^2$

$$L_{ij} = D_{ij} - A_{ij}$$

$$D_{ij} = \begin{cases} k_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Think of this quantity as the "stress" produced by the assignment of node labels x

As shown before, the constant vector is one of the eigenvectors of **L**, with eigenvalue 0

- If x is such that $x_i = x_j$ for all i,j:

$$x^T L x = \sum_{(i,j) \in E} (x_i - x_j)^2 = 0 \Rightarrow Lx = 0$$

- This means the constant vector is an eigenvector of L with eigenvalue 0

The eigenvector x of $\lambda=0$ is the constant vector if the graph is connected

- If x is the eigenvector of eigenvalue 0, $Lx = 0$

- Then $x^T L x = \displaystyle\sum_{(i,j)\in E} (x_i - x_j)^2 = 0$

From this, we deduct that $x_i = x_j$ for any pair i, j even if i and j are not directly connected by an edge. Why?

The eigenvector x of $\lambda=0$ is the constant vector if the graph is connected

- If x is the eigenvector of eigenvalue 0, $Lx = 0$

- Then $x^T L x = \displaystyle\sum_{(i,j) \in E} (x_i - x_j)^2 = 0$

- Hence, for any pair of nodes (i,j) connected by an edge, $x_i = x_j$

- Given the graph is connected, there is a path between any two nodes $\Rightarrow$

  for **any** pair of nodes (i,j), **even the ones not connected by an edge**, $x_i = x_j$

- Hence x is a constant vector

All the eigenvalues of the Laplacian are non-negative

- If v is an eigenvector of L of eigenvalue λ:

$$\lambda v^T v = v^T L v = \sum_{(i,j) \in E} (v_i - v_j)^2 \geq 0$$

- This means all eigenvalues λ are non-negative

In summary, the Laplacian matrix L = D - A

- Is symmetric, eigenvectors are orthogonal

- Has N eigenvalues that are non-negative

- 0 is one eigenvalue $0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$

- The multiplicity of eigenvalue 0 equals the number of connected components of the graph

The second eigenvector of the Laplacian in a connected graph

xᵀLx and graph cuts

- Suppose (S, S') is a cut of graph G

- Set $x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \in S' \end{cases}$
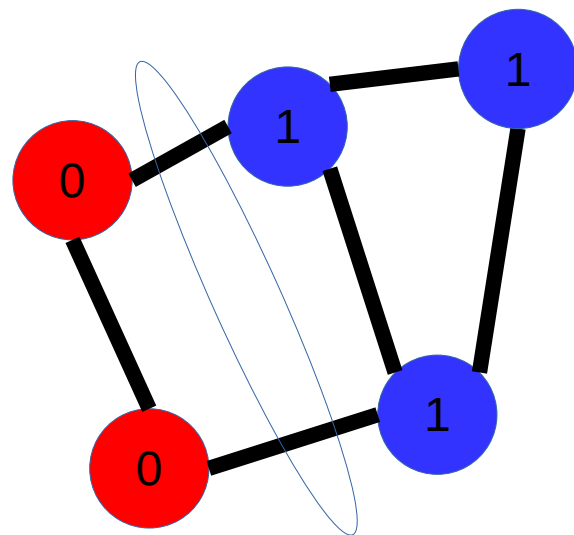
$$|c(S, S')| = 2$$

$$x^T L x = \sum_{(i,j) \in E} (x_i - x_j)^2 = \sum_{(i,j) \in c(S,S')} 1^2 = |c(S, S')|$$

Remember

- For symmetric matrices

$$\lambda_2 = \min_x \frac{x^T M x}{x^T x}$$

- If x is an eigenvector, $\frac{x^T M x}{x^T x}$ is its eigenvalue

https://en.wikipedia.org/wiki/Rayleigh_quotient

Second eigenvector

- Orthogonal to the first one: $x \cdot \vec{1} = 0 \Rightarrow \sum_i x_i = 0$

- Normal: $\sum_i x_i^2 = 1$

$$\lambda_2 = \min_x \frac{x^T L x}{x^T x} = \min_{x: \sum x_i = 0} \frac{x^T L x}{\sum x_i^2} = \min_{x: \sum x_i = 0 \wedge \sum x_i^2 = 1} \sum_{(i,j) \in E} (x_i - x_j)^2$$

The second eigenvalue tells us how well the graph can be partitioned into two

If the graph is divided into two connected components of sizes N_1 and N_2, you can use this assignment
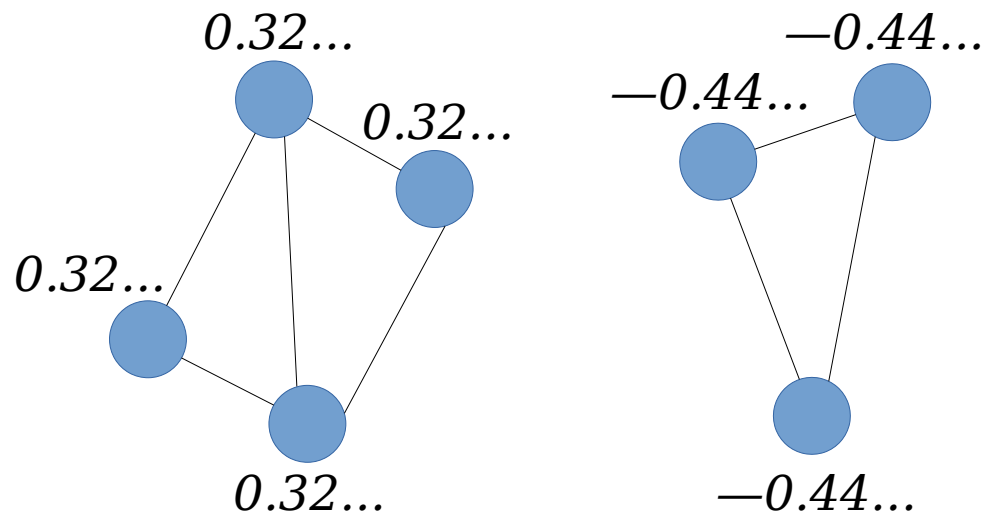
$$\lambda_2 = \min_{x:\sum x_i = 0 \wedge \sum x_i^2 = 1} \sum_{(i,j) \in E} (x_i - x_j)^2$$

The second eigenvalue tells us how well the graph can be partitioned into two

$$\lambda_2 = \min_{x:\sum x_i=0 \wedge \sum x_i^2=1} \sum_{(i,j)\in E} (x_i - x_j)^2$$

If the graph is connected but almost partitioned into two component, the optimal x should have values similar to each other in each partition
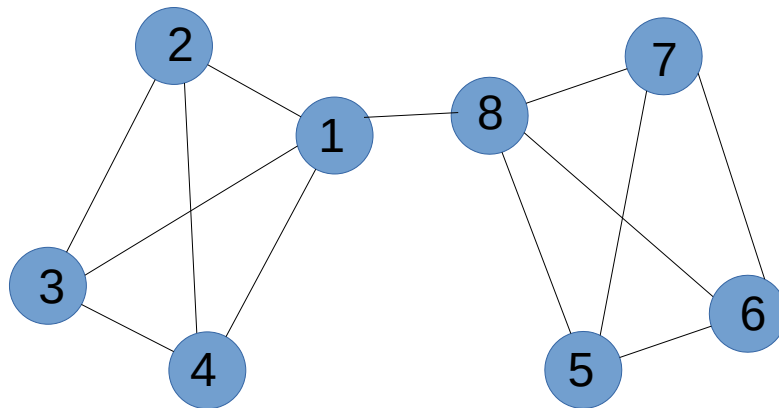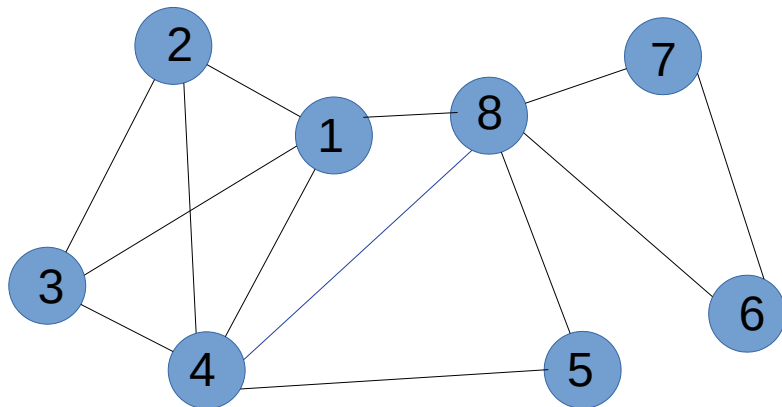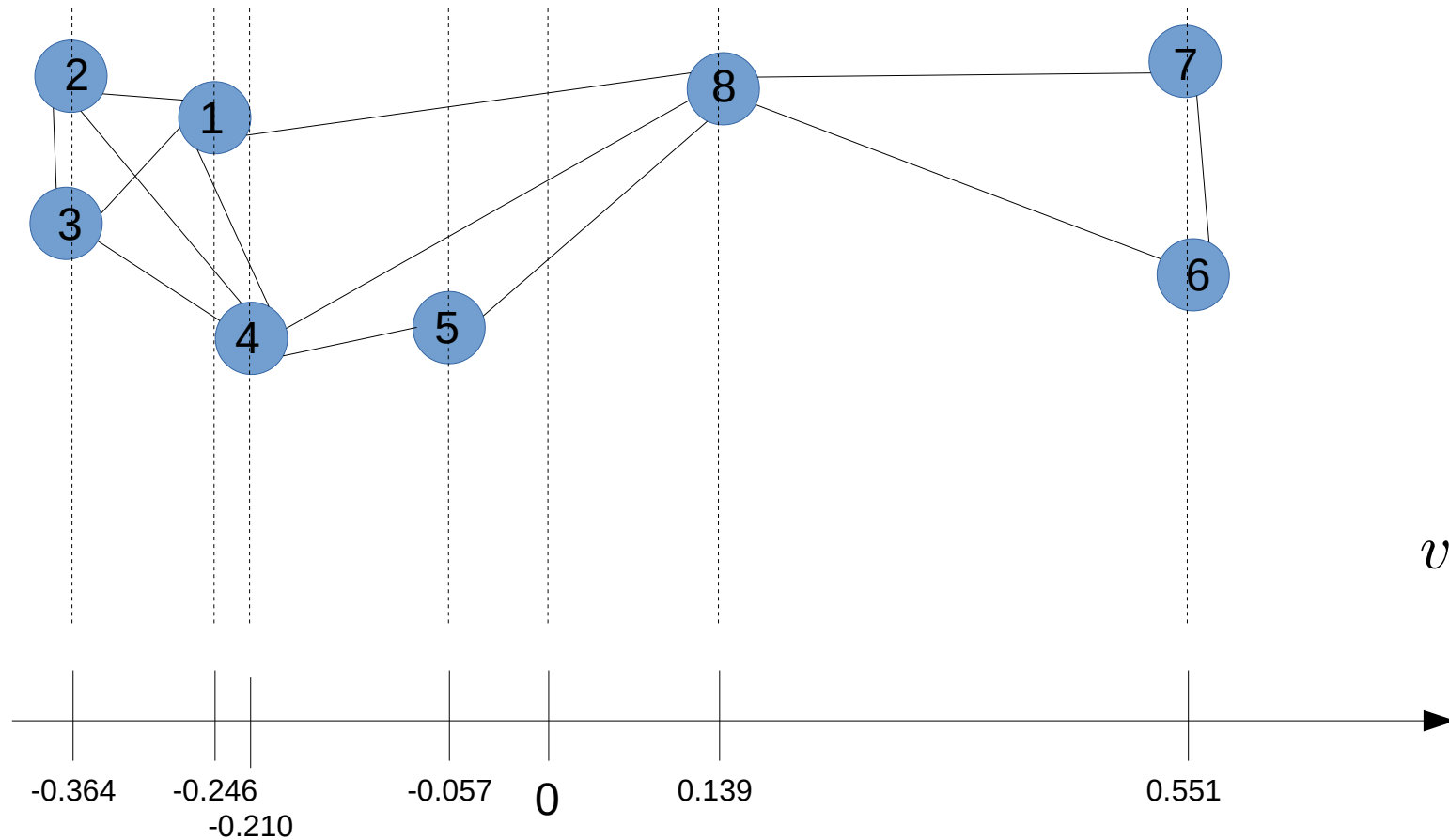

x_i x_j

0

Example Graph 1

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 & -1 & 4 \end{bmatrix}$$

Example Graph 1 (second eigenvalue of L)
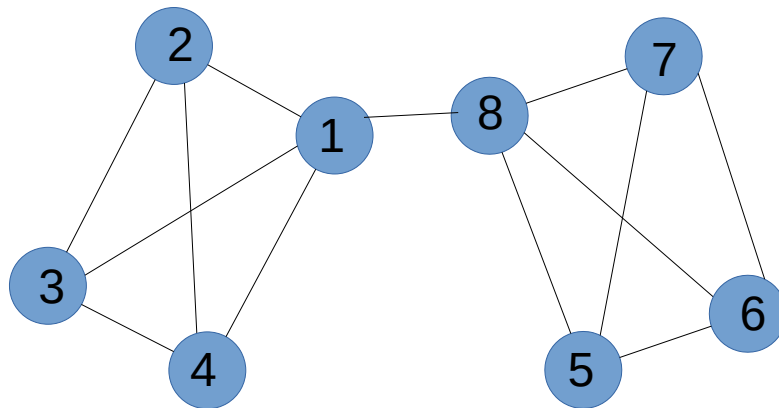
$$\lambda_1 = 0$$
$$\lambda_2 = 0.354$$

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 & -1 & 4 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0.247 \\ 0.383 \\ 0.383 \\ 0.383 \\ -0.383 \\ -0.383 \\ -0.383 \\ -0.247 \end{bmatrix}$$
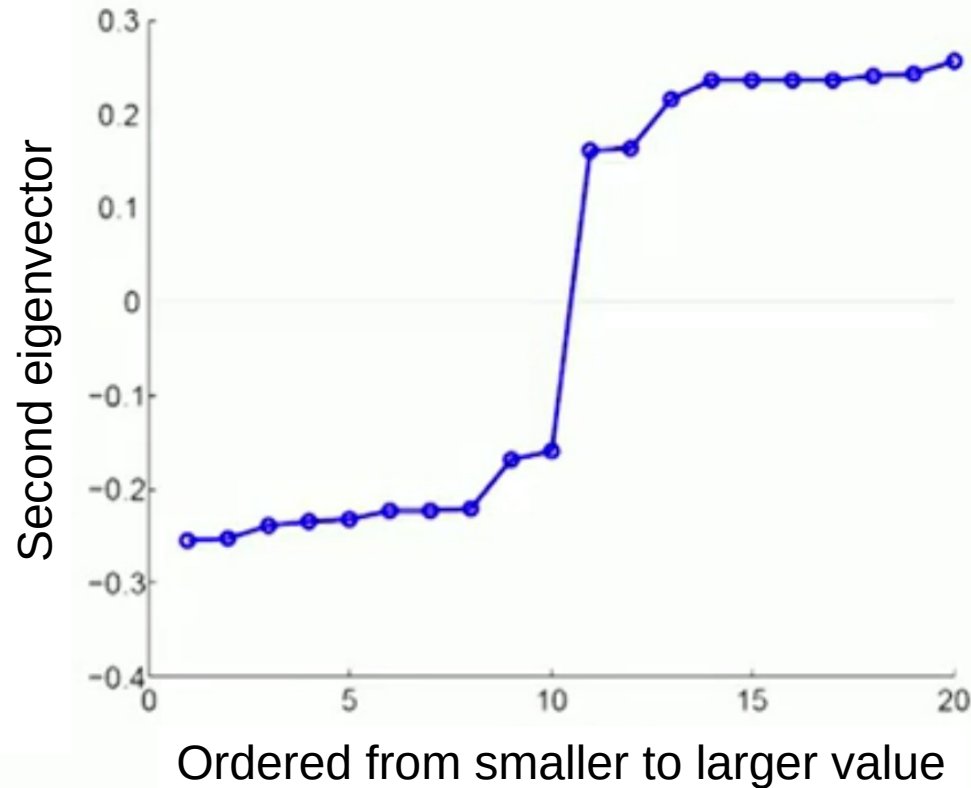
Example Graph 1, projected in R^1

$$\lambda_1 = 0$$
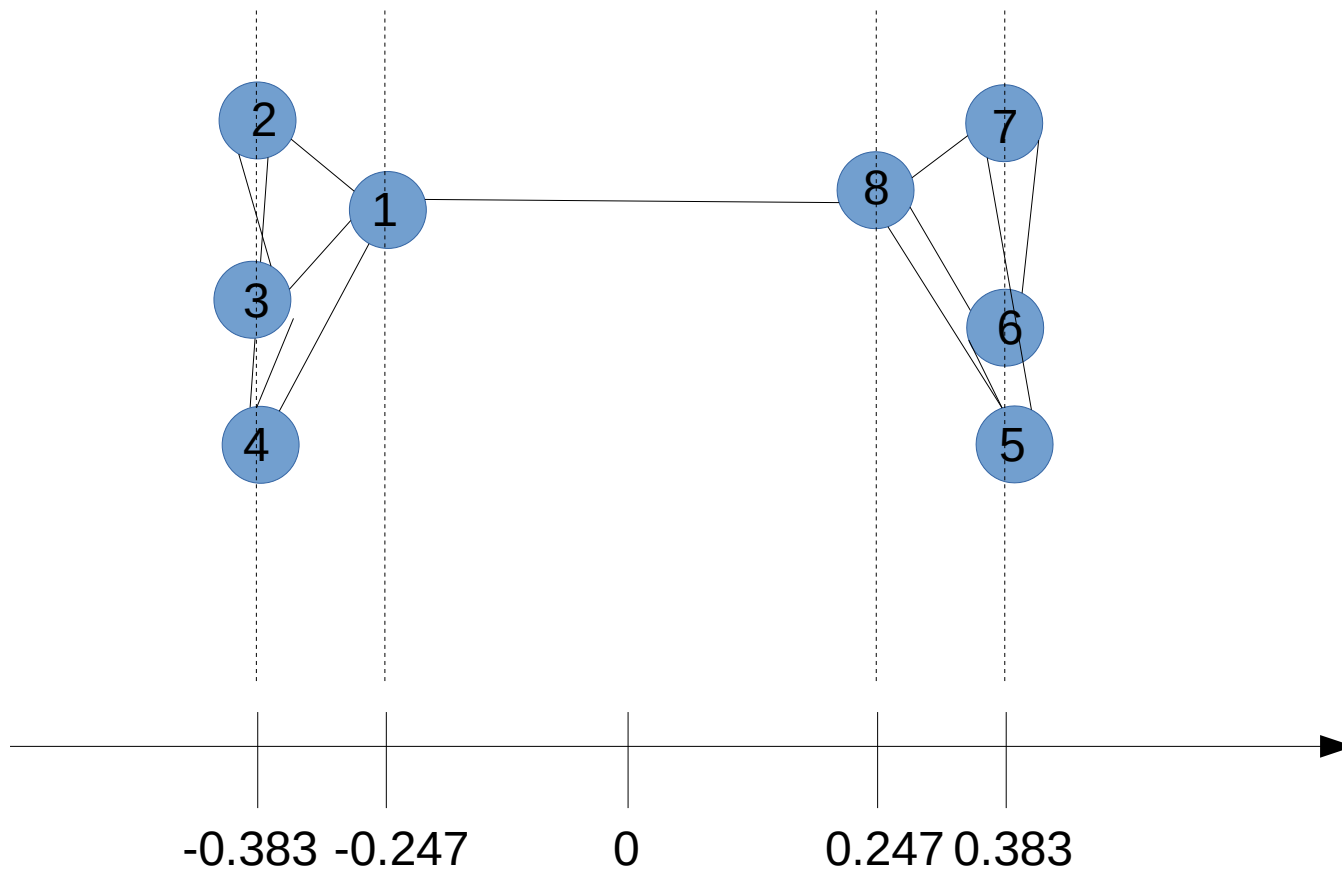
$$\lambda_2 = 0.354$$

$$v_2 = \begin{bmatrix} 0.247 \\ 0.383 \\ 0.383 \\ 0.383 \\ -0.383 \\ -0.383 \\ -0.383 \\ -0.247 \end{bmatrix}$$

-0.383 -0.247   0   0.247 0.383

Example Graph 1, communities

$$\lambda_1 = 0$$
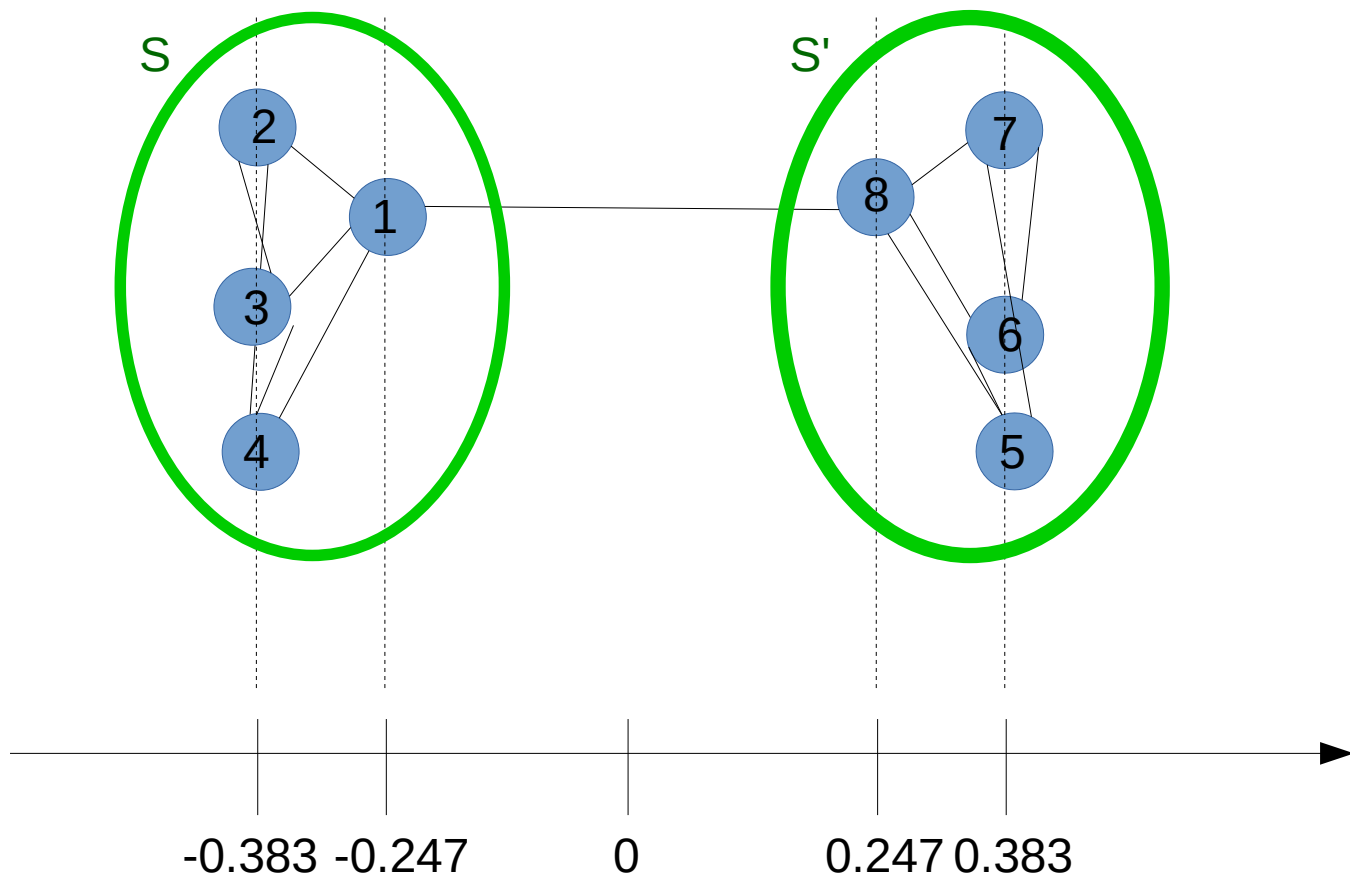
$$\lambda_2 = 0.354$$

$$v_2 = \begin{bmatrix} 0.247 \\ 0.383 \\ 0.383 \\ 0.383 \\ -0.383 \\ -0.383 \\ -0.383 \\ -0.247 \end{bmatrix}$$

Example Graph 2
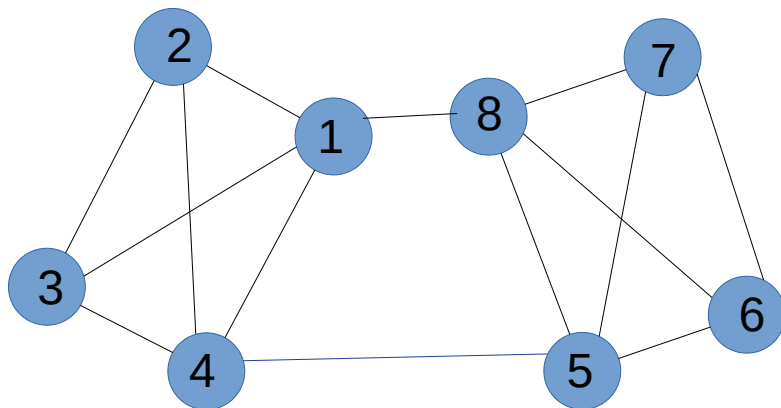
$$\lambda_1 = 0$$

$$\lambda_2 = 0.764$$

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 & -1 & 4 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0.263 \\ 0.425 \\ 0.425 \\ 0.263 \\ -0.263 \\ -0.425 \\ -0.425 \\ -0.263 \end{bmatrix}$$

Example Graph 3

$$\lambda_1 = 0$$

$$\lambda_2 = 0.748$$

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 5 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & -1 & -1 & -1 & 5 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -0.246 \\ -0.364 \\ -0.364 \\ -0.210 \\ -0.057 \\ 0.551 \\ 0.551 \\ 0.139 \end{bmatrix}$$

Example Graph 3, projected (where to cut?)

$$\lambda_1 = 0$$
$$\lambda_2 = 0.748$$

$$v_2 = \begin{bmatrix} -0.246 \\ -0.364 \\ -0.364 \\ -0.210 \\ -0.057 \\ 0.551 \\ 0.551 \\ 0.139 \end{bmatrix}$$

-0.364   -0.246   -0.057   0   0.139   0.551
          -0.210

Example Graph 3, projected (where to cut?)

$\lambda_1 = 0$

$\lambda_2 = 0.748$

$$v_2 = \begin{bmatrix} -0.246 \\ -0.364 \\ -0.364 \\ -0.210 \\ -0.057 \\ 0.551 \\ 0.551 \\ 0.139 \end{bmatrix}$$

-0.364   -0.246   -0.057   0   0.139   0.551
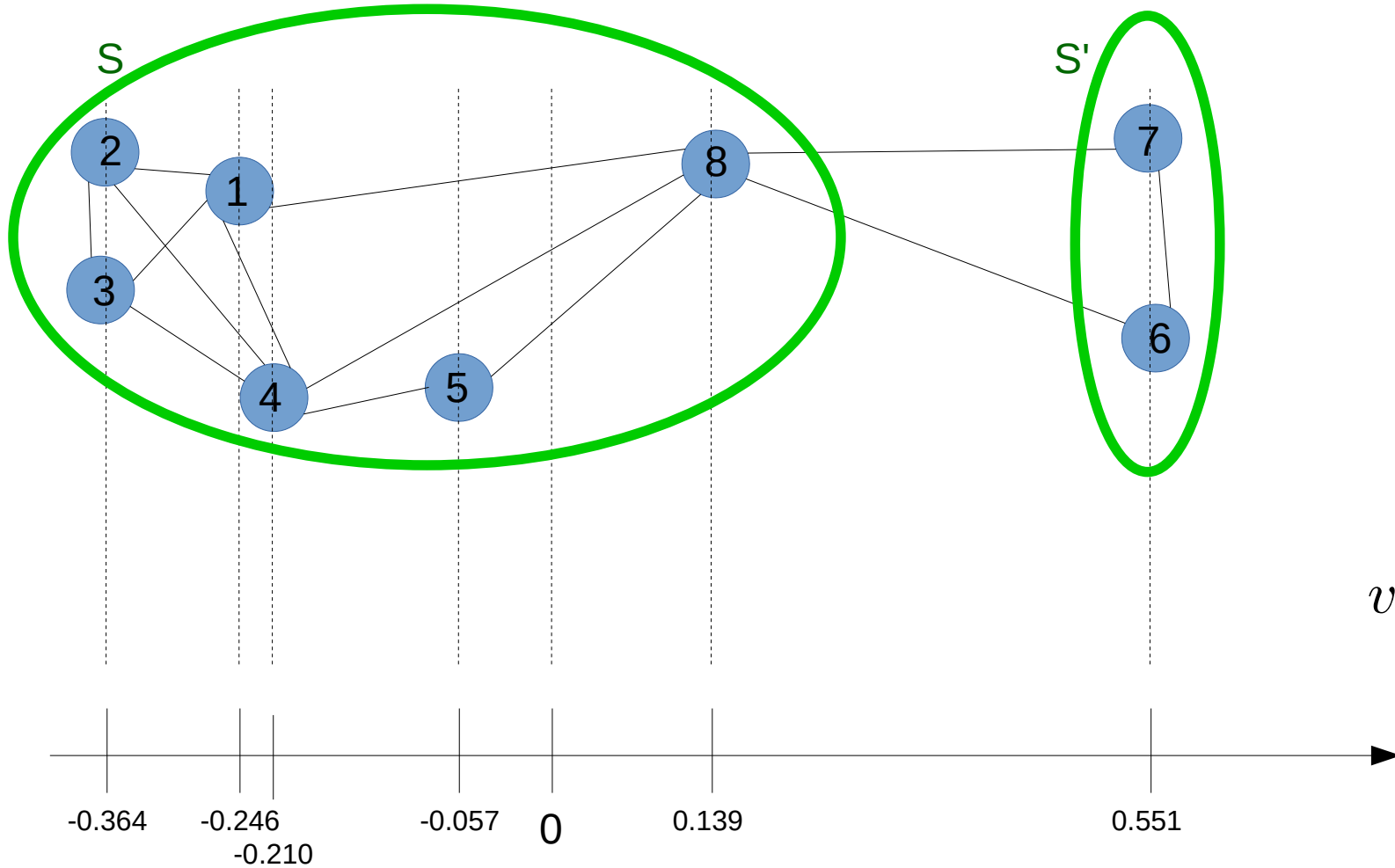
-0.210

A graph with two communities in $\mathbb{R}^1$

Second eigenvector

Ordered from smaller to larger value
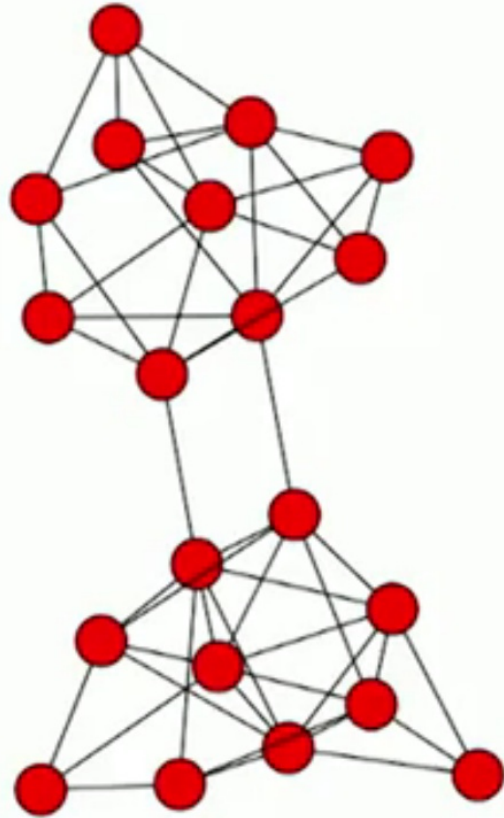
A graph with four communities in $\mathbb{R}^1$
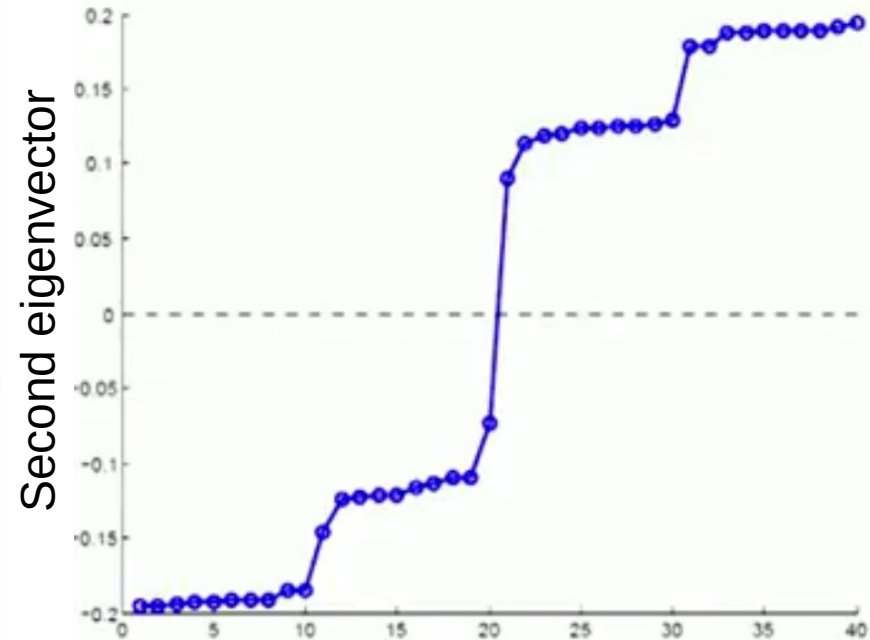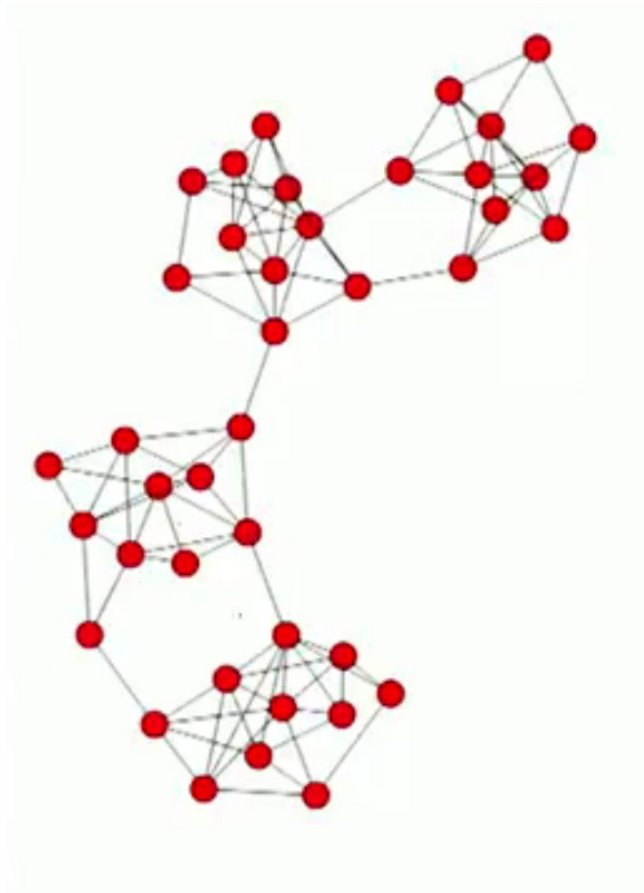
Ordered from smaller to larger value

Application: graph drawing

A graph with four communities in $\mathbb{R}^2$

Second eigenvector

Third eigenvector
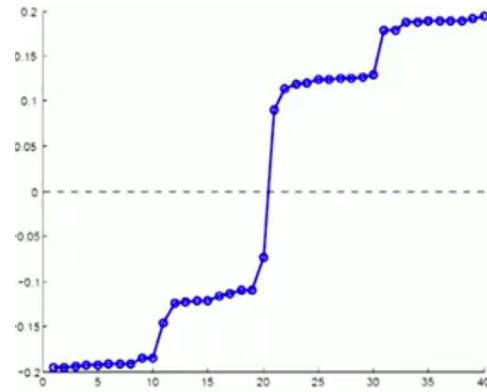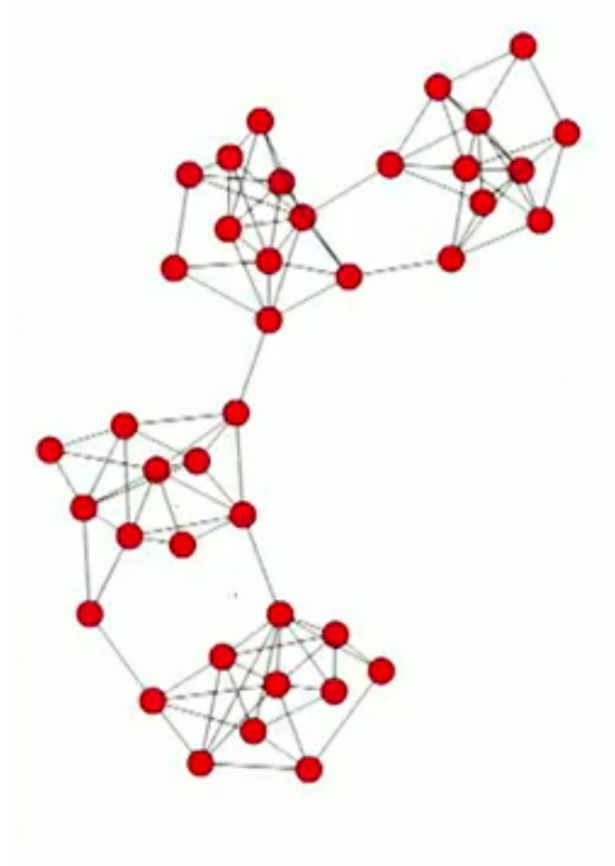
Value in third eigenvector
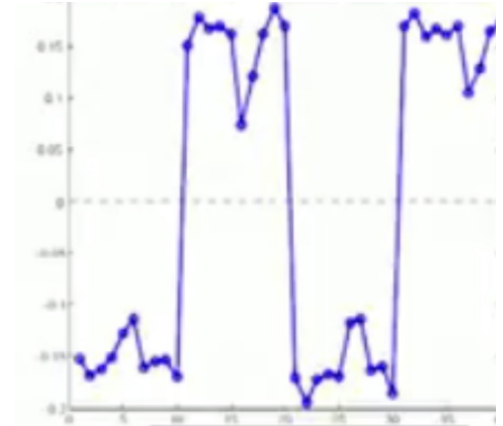
Value in second eigenvector
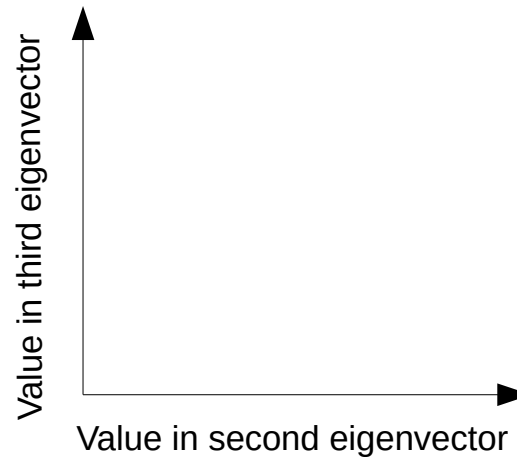
A graph with four communities in R² (cont)

Second eigenvector

Third eigenvector
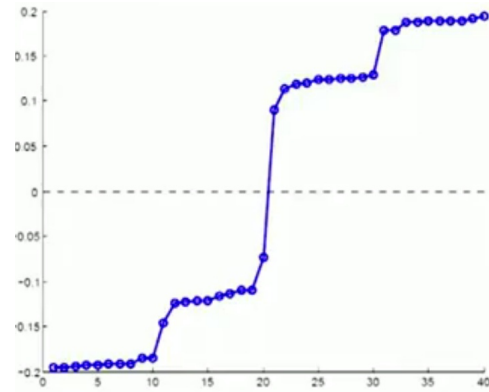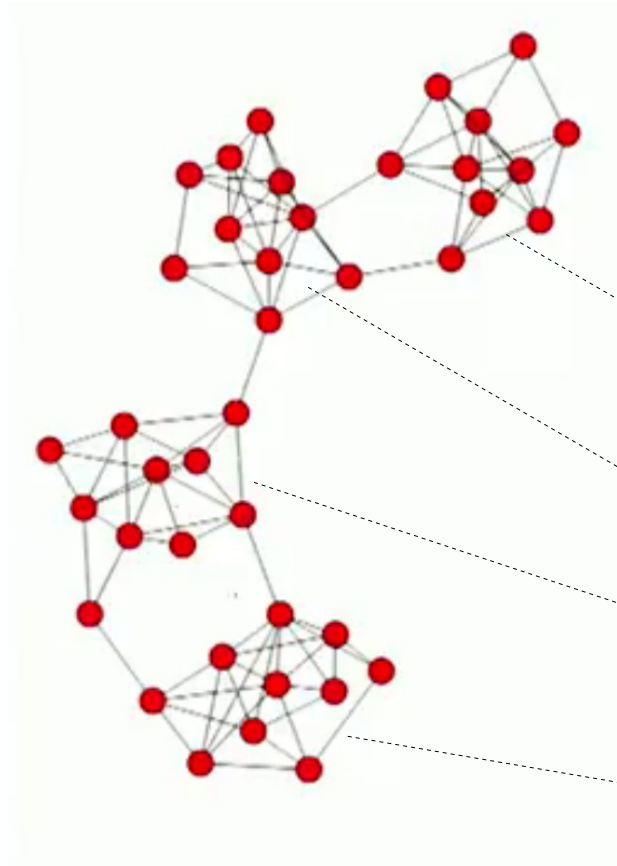
Value in third eigenvector
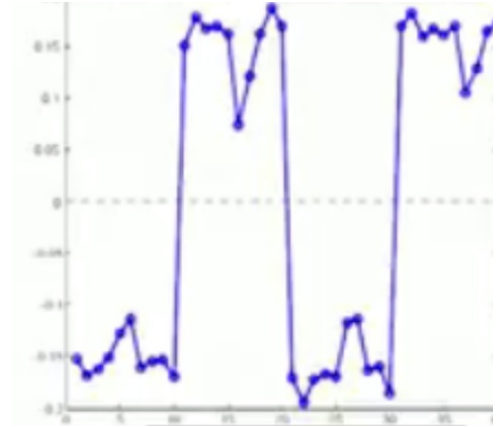
Value in second eigenvector
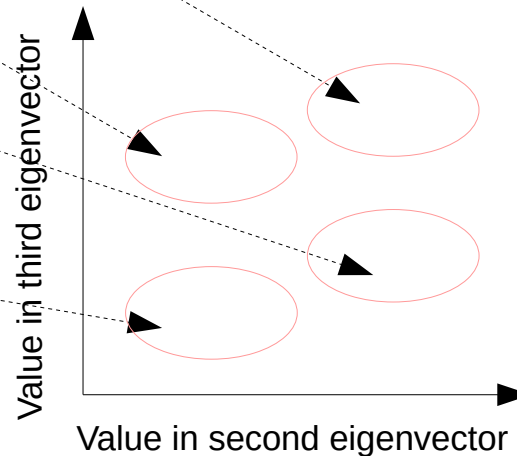
This can be
used to draw
the graph in R²

The graph from the initial exercise
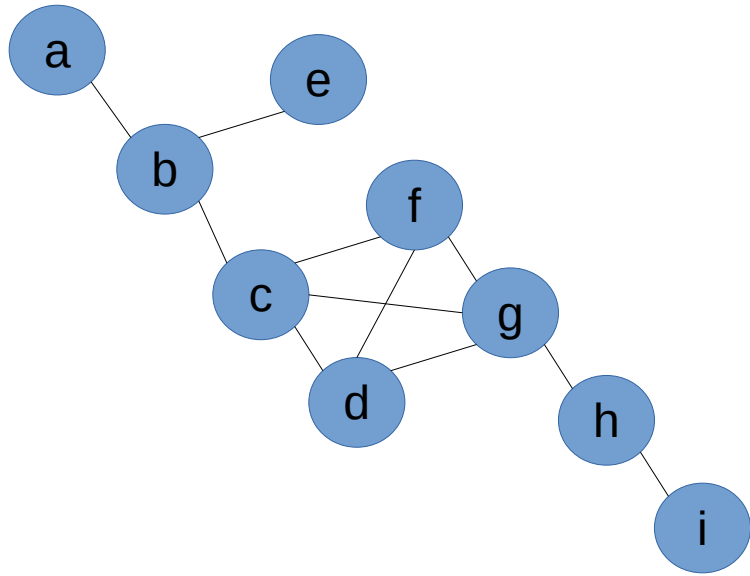
Input nodes and edges
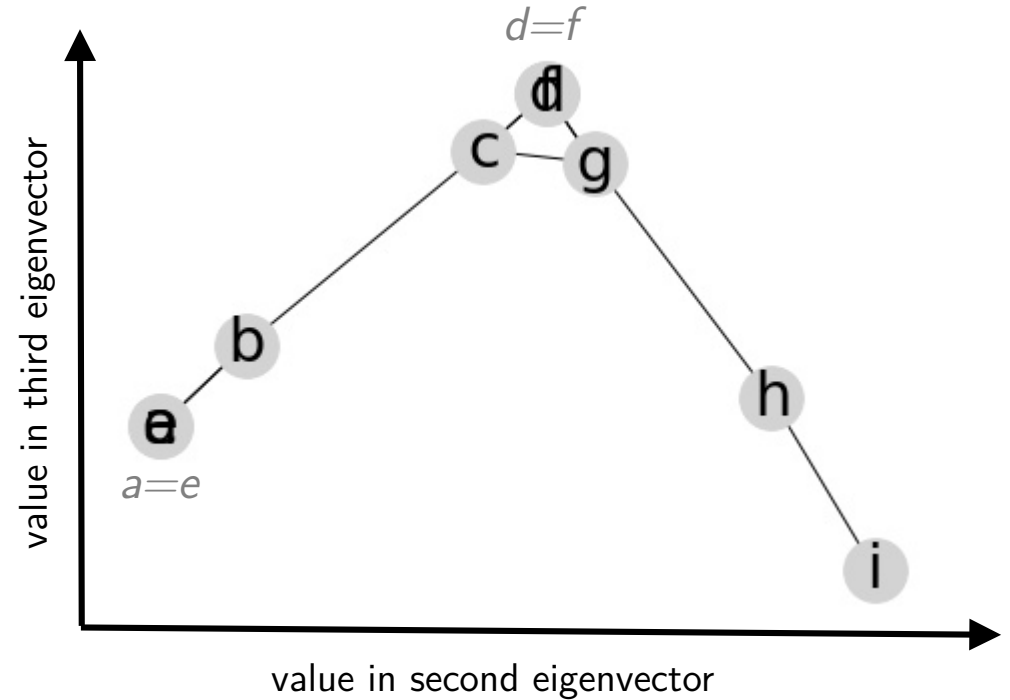
Spectral embedding

Exercise: spectral projection

- Write the Laplacian

- Get the second and third eigenvector

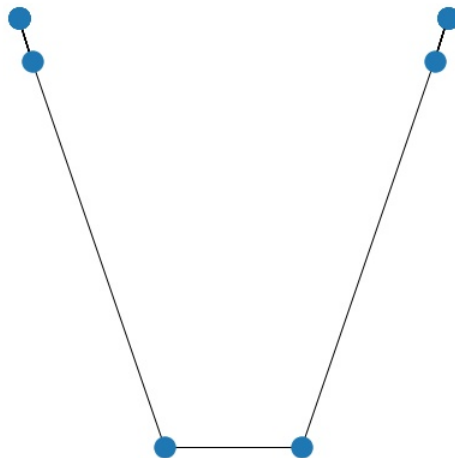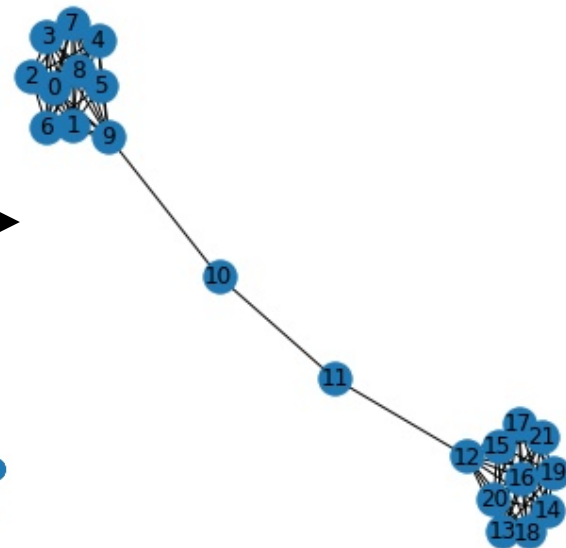 (e.g., "online eigenvector calculator")

- Obtain projection
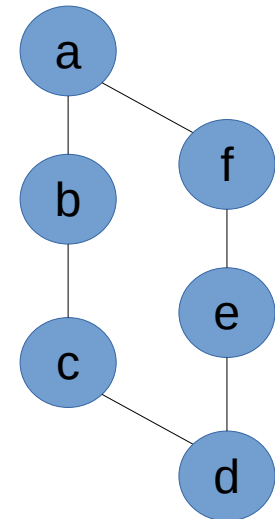
A barbell graph in R^2 (code)

```
B = nx.barbell_graph(10,2)

plt.figure(figsize=(6,6))
nx.draw_networkx(B)
_ = plt.show()
```
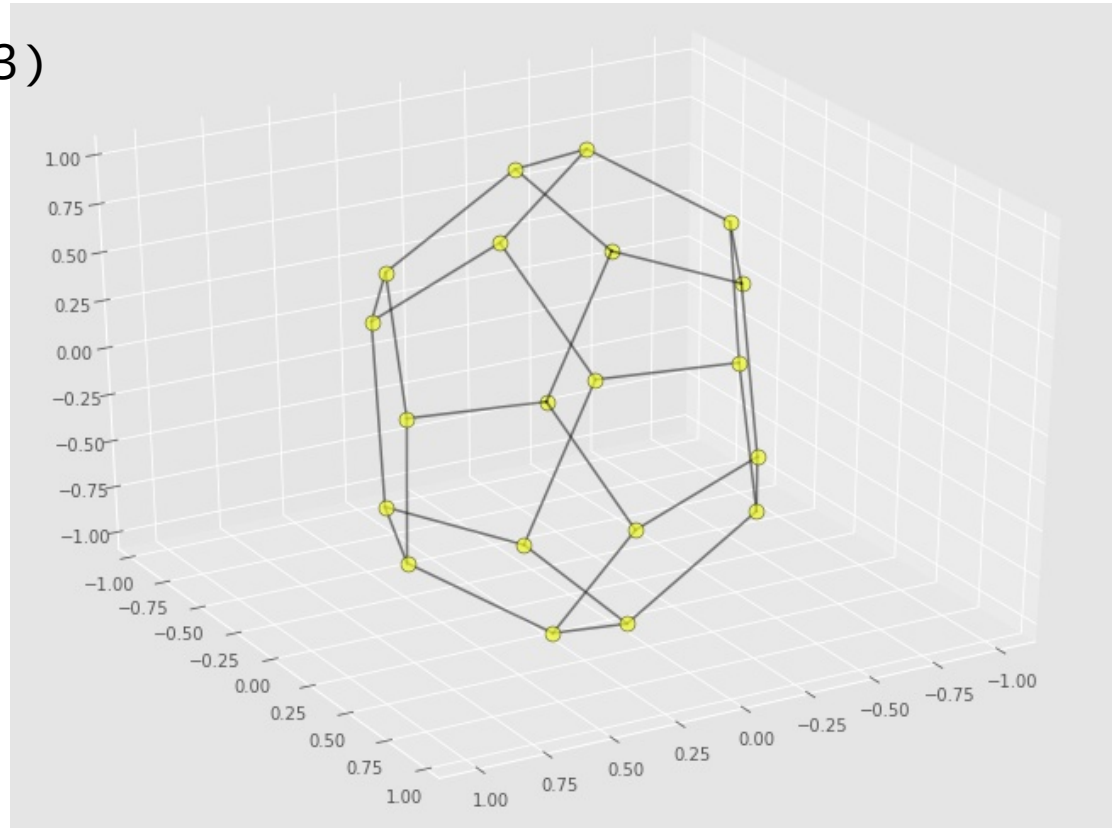
```
plt.figure(figsize=(6,6))
nx.draw_spectral(B)
_ = plt.show()
```

Graph Laplacian

Dodecahedral graph in 3D

```python
g = nx.dodecahedral_graph()
pos = nx.spectral_layout(g, dim=3)
network_plot_3D_alt(g, 60, pos)
```


https://www.idtools.com.au/3d-network-graphs-python-mplot3d-toolkit/

Application: spectral clustering

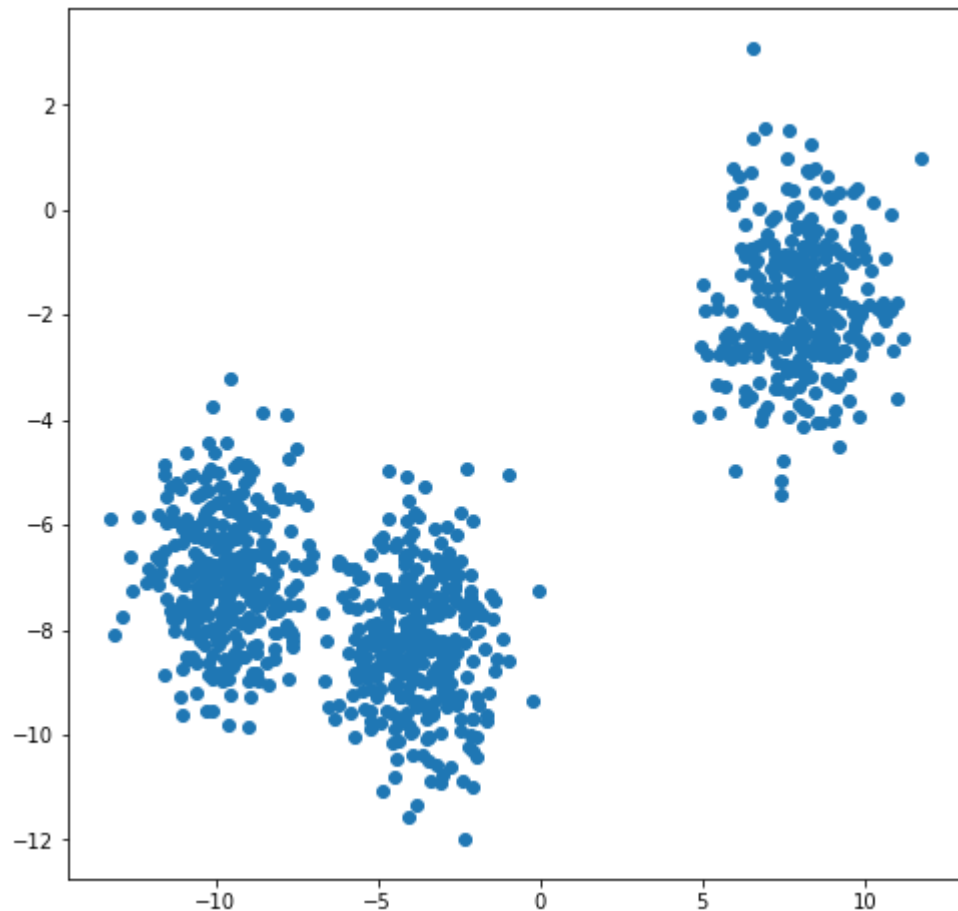# Generating data

```python
from sklearn.datasets import
    make_blobs

N = 1000

x, _ = make_blobs(
    n_samples=N,
    centers=3,
    cluster_std=1.2)


plt.figure(figsize=(8,8))

plt.scatter(x[:,0], x[:,1])

plt.show()
```

# Connect nodes to k=5 nearest neighbors

```python
from sklearn.neighbors
    import NearestNeighbors

nbrs = NearestNeighbors(
    n_neighbors=6,          # includes self
    algorithm='ball_tree')
    .fit(x)

distances, neighbors =
    nbrs.kneighbors(x)

G = nx.Graph()

for neighbor_list in neighbors:

    source_node = neighbor_list[0]

    for target_index in range(1,
        len(neighbor_list)):

        target_node = neighbor_list[target_ind

        G.add_edge(source_node, target_node)
```
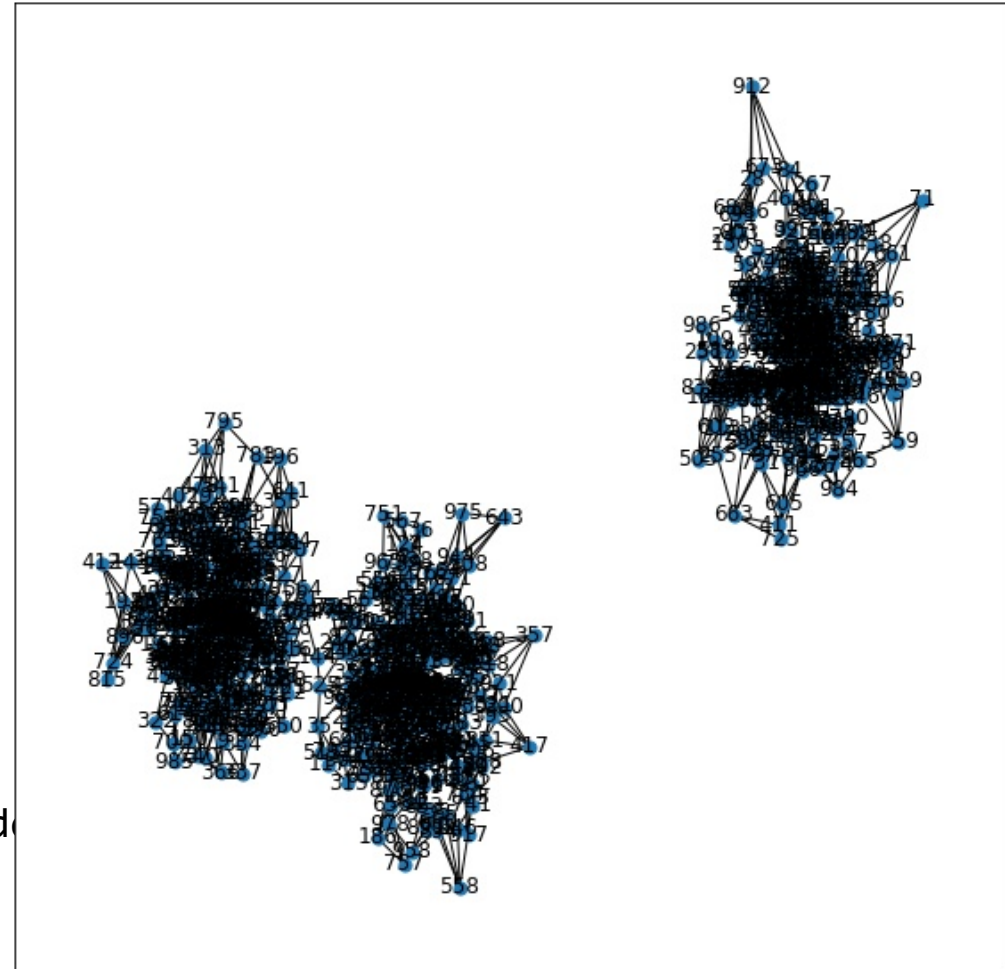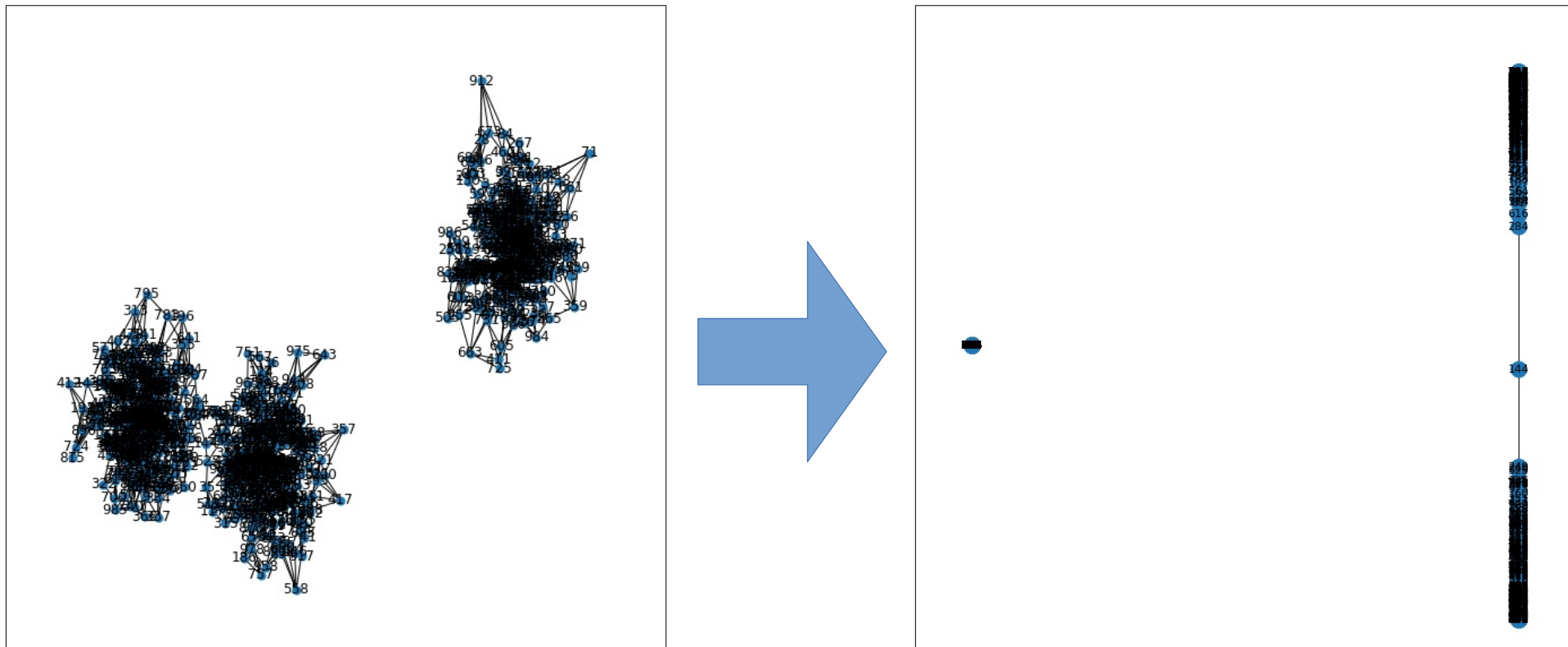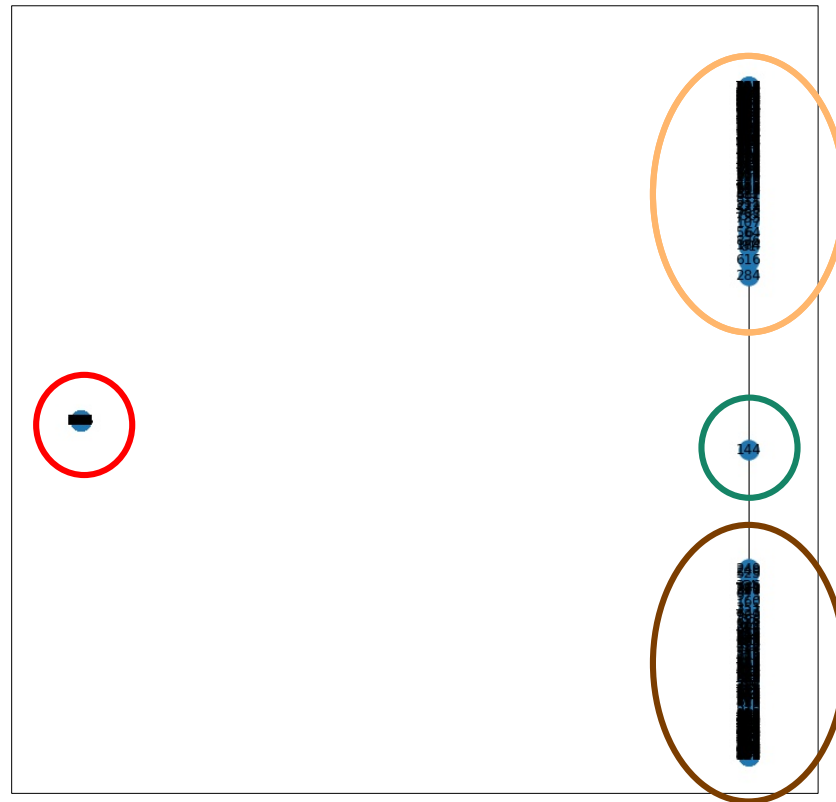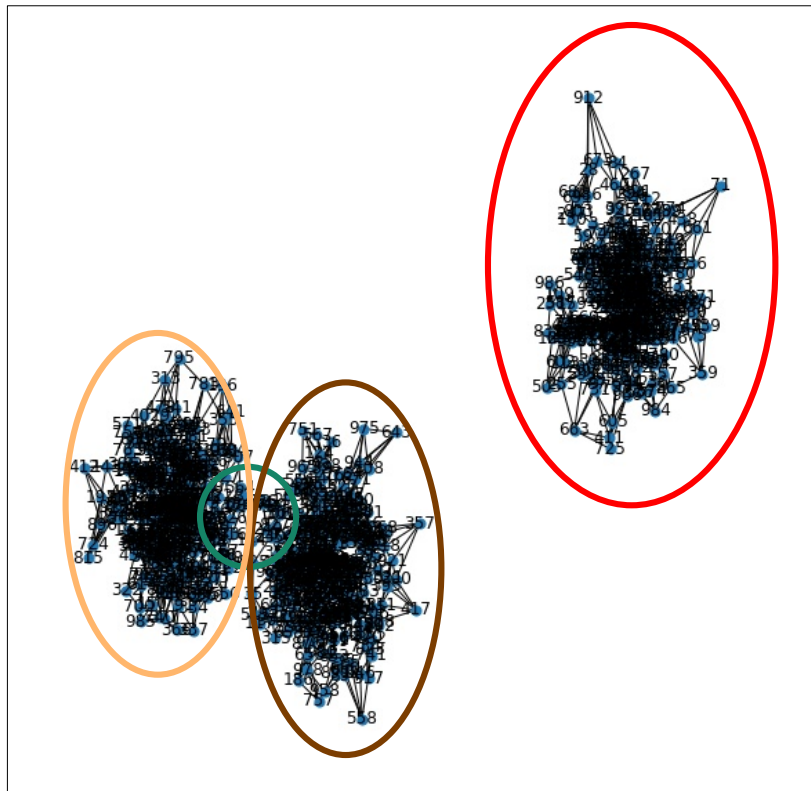
Perform spectral embedding

nx.draw_spectral(G, with_labels=True)

Perform spectral embedding

nx.draw_spectral(G, with_labels=True)

Summary

Things to remember

- Graph Laplacian

- Laplacian and graph components

- Spectral graph embedding

Exercises for this topic

- Mining of Massive Datasets (2014) by Leskovec et al.

    - Exercises 10.4.6