Graph theory basics

Introduction to Network Science

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Topic 04

Contents

- Notation for graphs
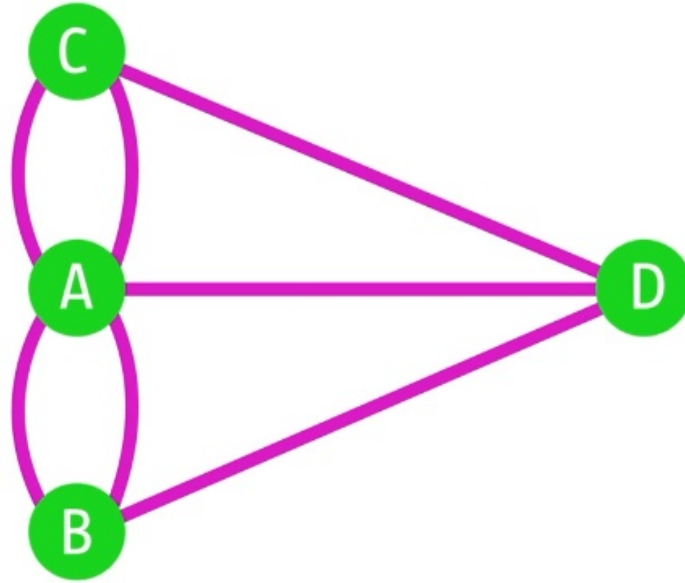- Degree distributions
- Adjacency matrices

Sources

- Albert László Barabási: Network Science. Cambridge University Press, 2016.
  - Follows almost section-by-section chapter 02
- URLs cited in the footer of specific slides

The seven bridges of Königsberg

http://networksciencebook.com/images/ch-02/video-2-1.m4v
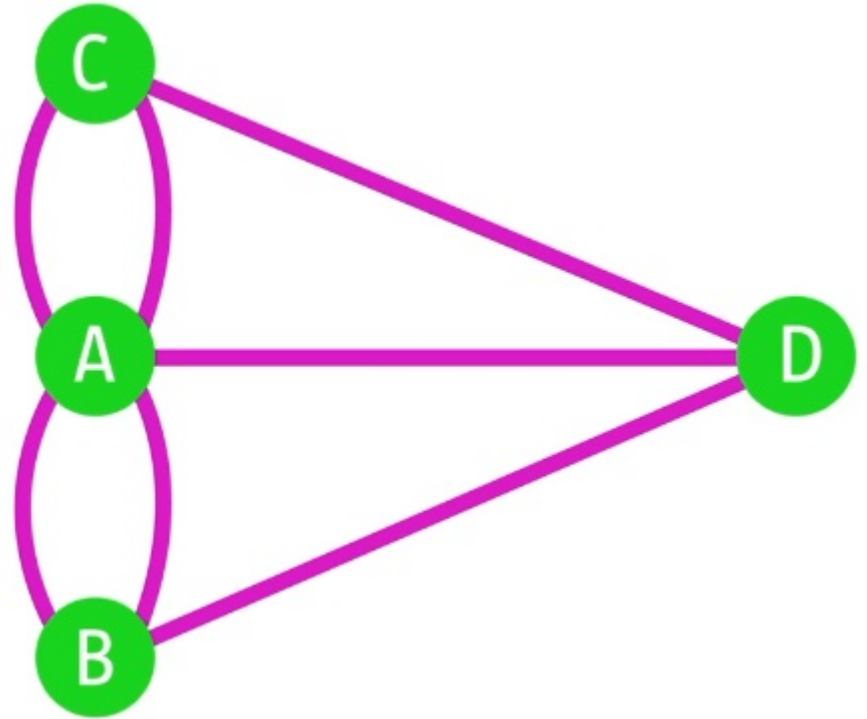
Quick Question

Can one walk across the 7 bridges without crossing the same bridge twice?

Basic concepts

Notation for a graph

- G = (V,E)
  - V: nodes or vertices
  - E: links or edges
- |V| = N size of graph
- |E| = L number of links

Typical notation variations

- You may find that G is denoted by (N, A), this is typical of directed graphs, means "*nodes, arcs*"

- You may find that
  - |V| is denoted by n or N
  - |E| is denoted by m, M, or L

Directed vs undirected graphs

- In an undirected graph
  - E is a symmetric relation
    $$(u, v) \in E \Rightarrow (v, u) \in E$$

- In a directed graph, also known as "digraph"
  - E is not a symmetric relation
    $$(u, v) \in E \nRightarrow (v, u) \in E$$

Example graphs we will use

Network	|V|	|E|
Zachary's Karate Club (karate.gml)	34	78
Les Misérables (lesmiserables.gml)	77	254
E-mail exchanges (email-eu-core.csv)	868	25K
US companies ownership	1351	6721
Marvel comics (hero-network.csv)	6K	167K

https://github.com/chatox/networks-science-course/tree/master/practicum/data

Degree

- Node i has degree k_i
  - This is the number of links incident on this node
  - The total number of links L is given by $L = \dfrac{1}{2}\sum\limits_{i=1}^{N} k_i$

- Average degree $\langle k \rangle = \dfrac{1}{N}\sum\limits_{i=1}^{N} k_i = \dfrac{2L}{N}$

In directed networks

- We distinguish in-degree from out-degree
  - Incoming and outgoing links, respectively
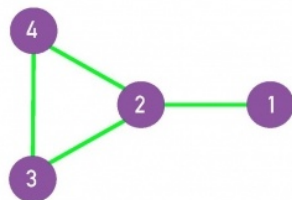- Degree is the sum of both $k_i = k_i^{\text{in}} + k_i^{\text{out}}$
- Counting total number of links:

$$L = \sum_{i=1}^{N} k_i^{\text{in}} = \sum_{i=1}^{N} k_i^{\text{out}}$$

Degree distribution

- If there are N_k nodes with degree k

- The degree distribution is given by $p_k = \dfrac{N_k}{N}$

- The average degree is then $\langle k \rangle = \displaystyle\sum_{k=0}^{\infty} k p_k$
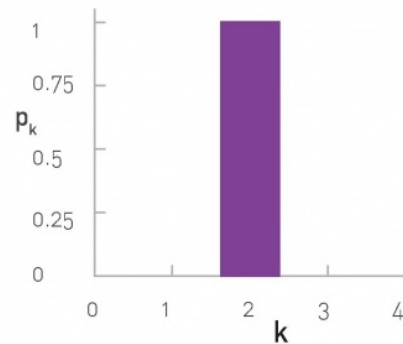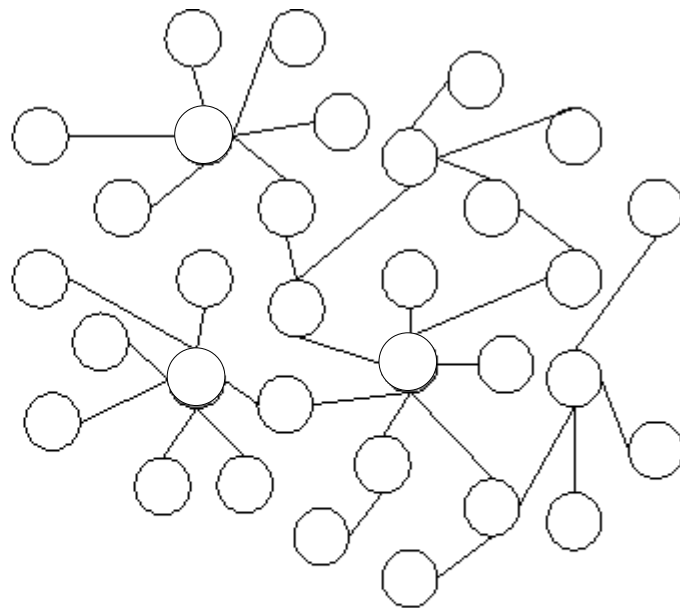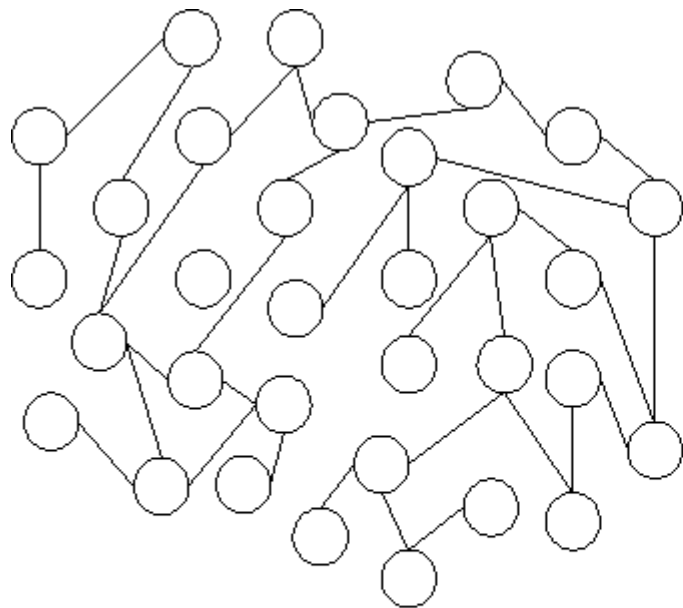
Degree distribution; two toy graphs
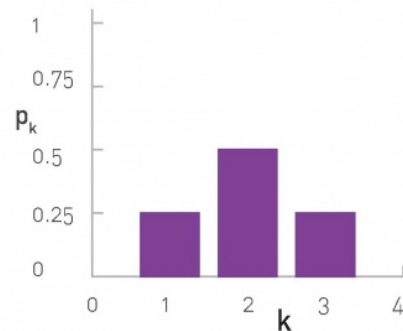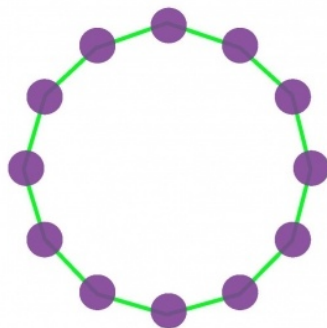
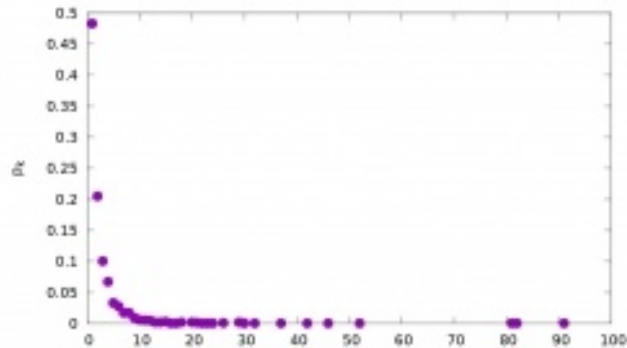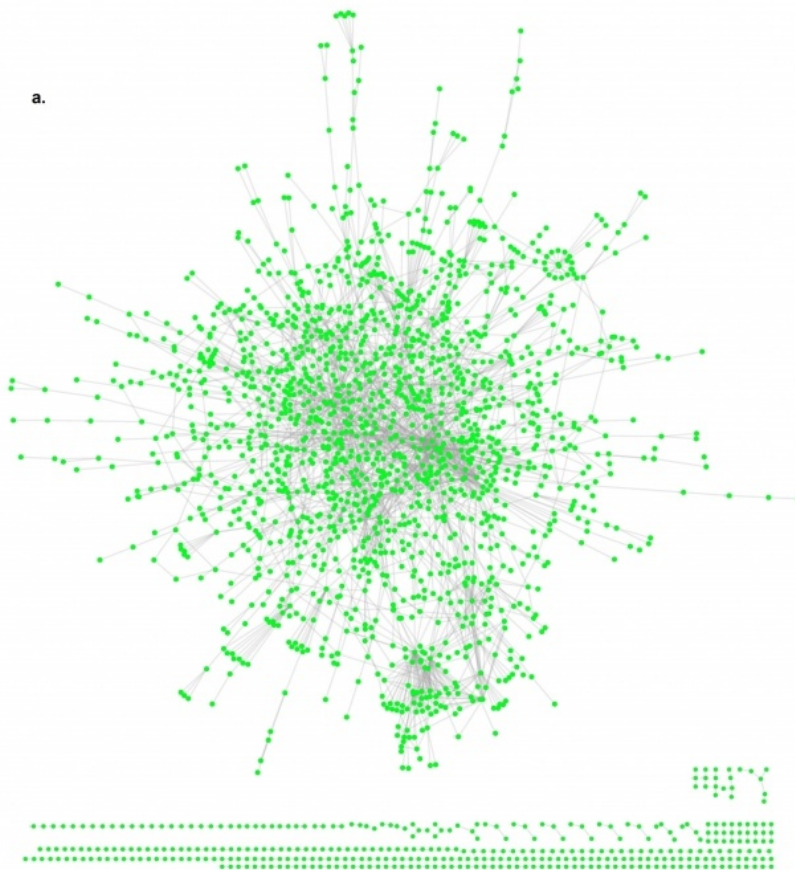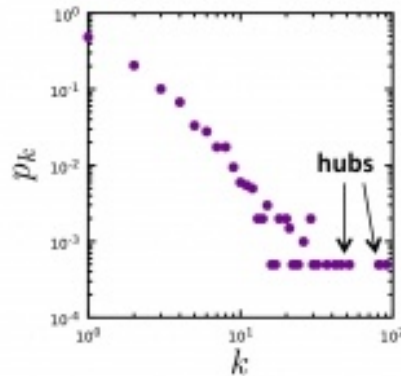# Exercise

Draw the degree distribution of these graphs

Degree distribution; real graph

Linear scale
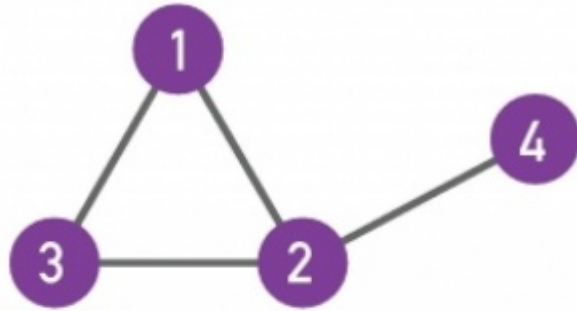
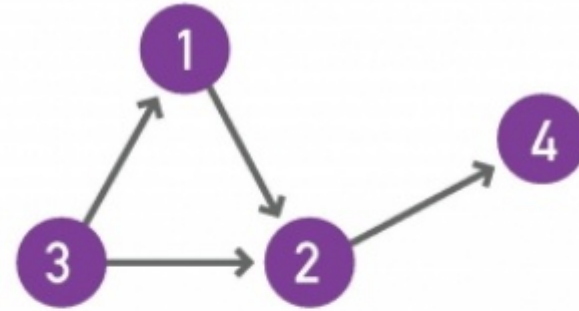Log-log scale

Adjacency matrix

What is an adjacency matrix

- A is the adjacency matrix of G = (V, E) iff:
  - A has |V| rows and |V| columns
  - $A_{ij} = 1$ if $(i,j) \in E$
  - $A_{ij} = 0$ if $(i,j) \notin E$
- **A_{ij} always means row i, column j**
  - Sometimes Barabási's book has this wrong
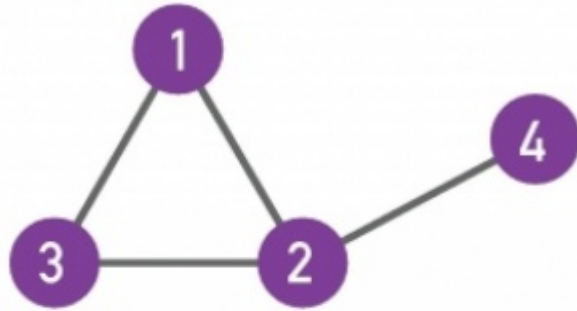
Examples

Undirected graph

$$A_{ij} = \begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$
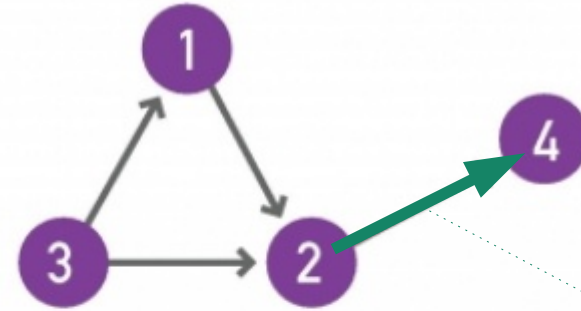
Directed graph

$$A_{ij} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

A$_{ij}$ always means row i, column j

Undirected graph

$$A_{ij} = \begin{matrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$

Directed graph

Row 2
Column 4

$$A_{ij} = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Quick Question

- In terms of A, what is the expression for:

$$k_i^{\mathrm{in}} =$$

$$k_i^{\mathrm{out}} =$$

Some "graphology" …

- G is undirected ⇔ A is symmetric

- G has a self-loop
  ⇔ A has a non-zero element in the diagonal

- G is complete ⇔ $A_{ij} \neq 0$ (except if i=j)

Summary

Things to remember

- Definitions:
  - Degree, in-degree, out-degree
- Writing the adjacency matrix of a graph and drawing a graph given its adjacency matrix

Practice on your own

Draw the indegree, outdegree, degree distribution

Write the adjacency matrix