Epidemic Analysis on Graphs

Introduction to Network Science
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Topic 27

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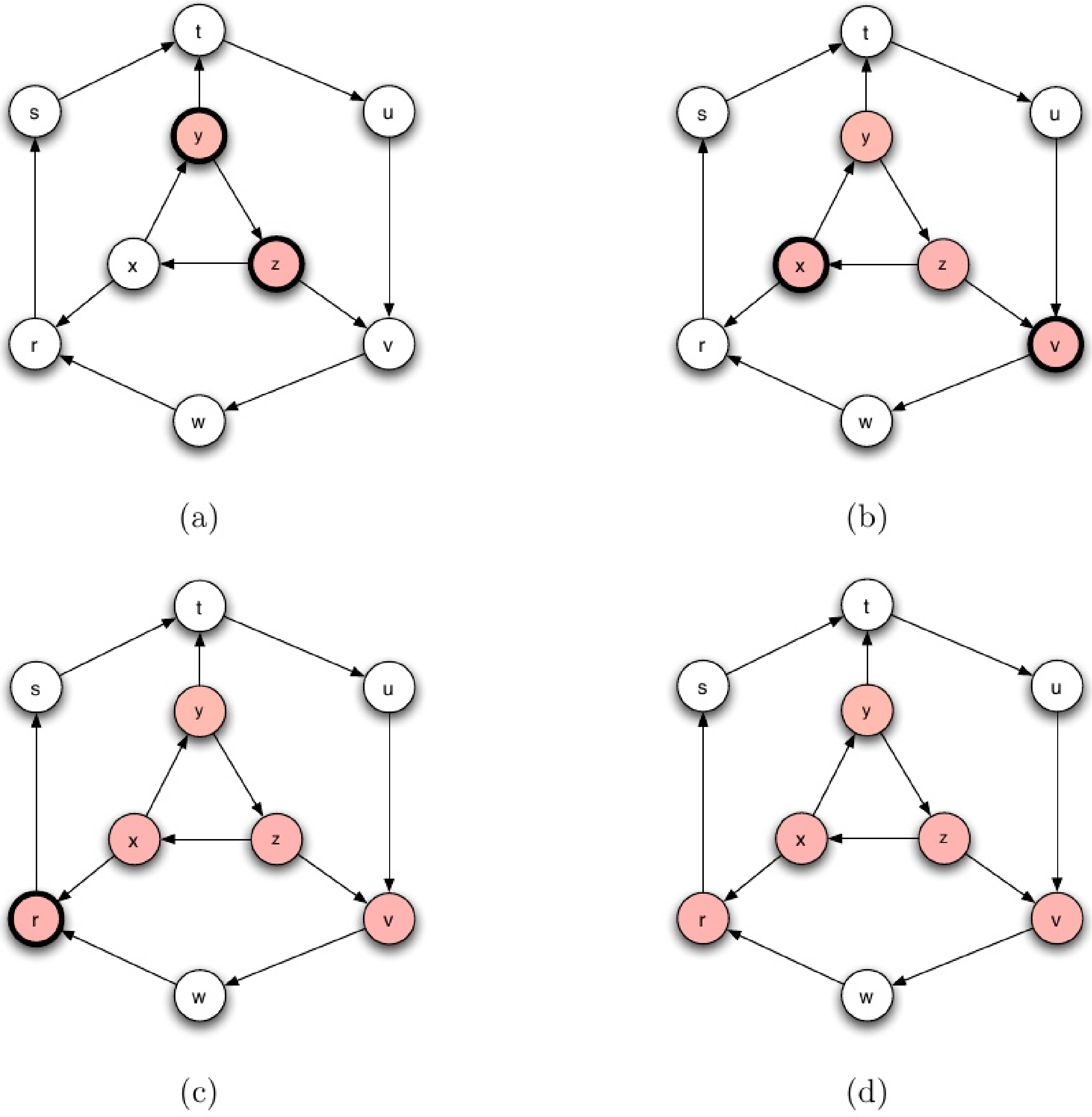
Sources

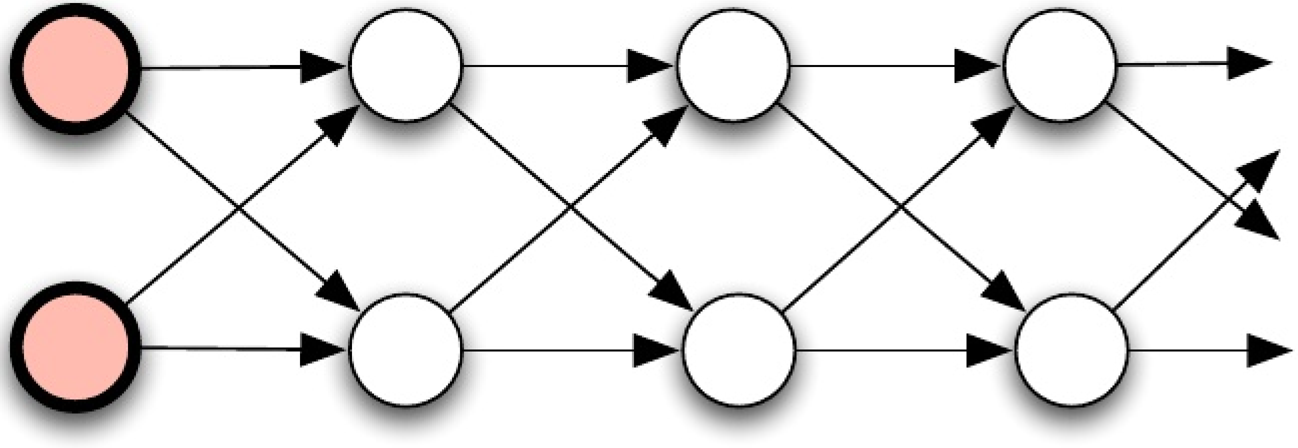
- Barabási (2016): Network Science Ch. 10
- Easley and Kleinberg (2010): Networks, Crowds, and Markets Ch 21.

SIR on a graph

- In this simulation we assume recovery takes one timestep
- Infected nodes have thick borders
- Recovered nodes have thin borders



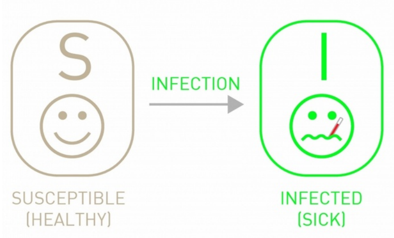
Social distancing



In this network, the epidemic is forced to pass through a narrow "channel" of nodes. In such a structure, even a highly contagious disease will tend to die out relatively quickly (fig 21.3 Easley+Kleinberg)

Computing infection dynamics in known graph models

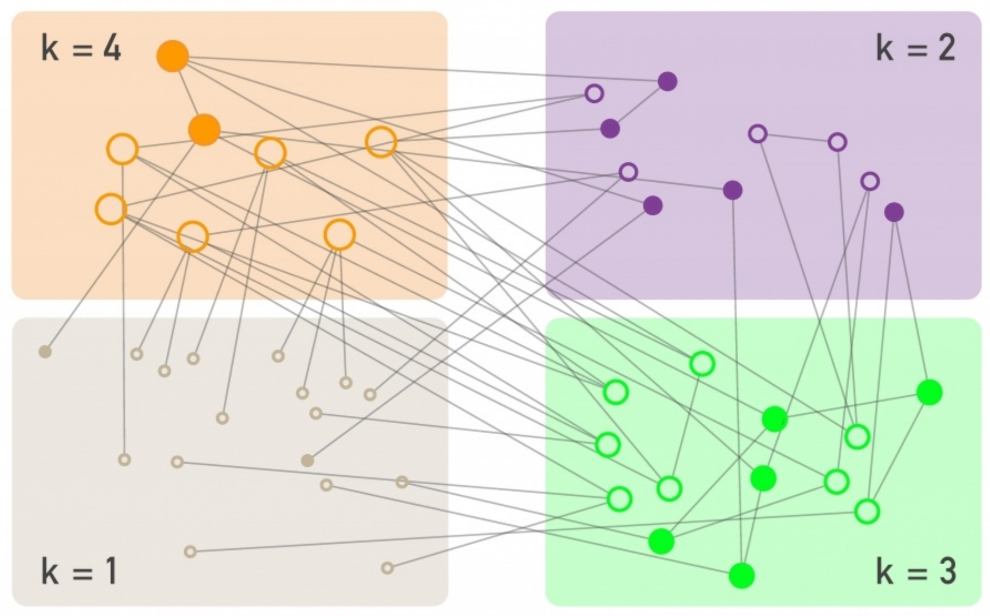
SI dynamics on a graph



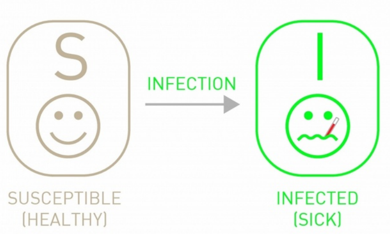
- Degree block approximation: all nodes with the same degree are jointly analyzed

$$i_k(t) = \frac{I_k(t)}{N_k}$$

$$i(t) = \sum_k i_k(t) p_k$$



SI dynamics on a graph



- Degree block approximation: all nodes with the same degree are jointly analyzed

$$\frac{di_k(t)}{dt} = \Theta_k k(1 - i_k(t))\beta$$

Similar to simple SI model, except:

Θ_k is the fraction of infected neighbors of a susceptible node of degree k

Compare with simple SI model:

$$\frac{di(t)}{dt} = i(t) \langle k \rangle (1 - i(t))\beta$$

SI model on a graph: infected as a function of time

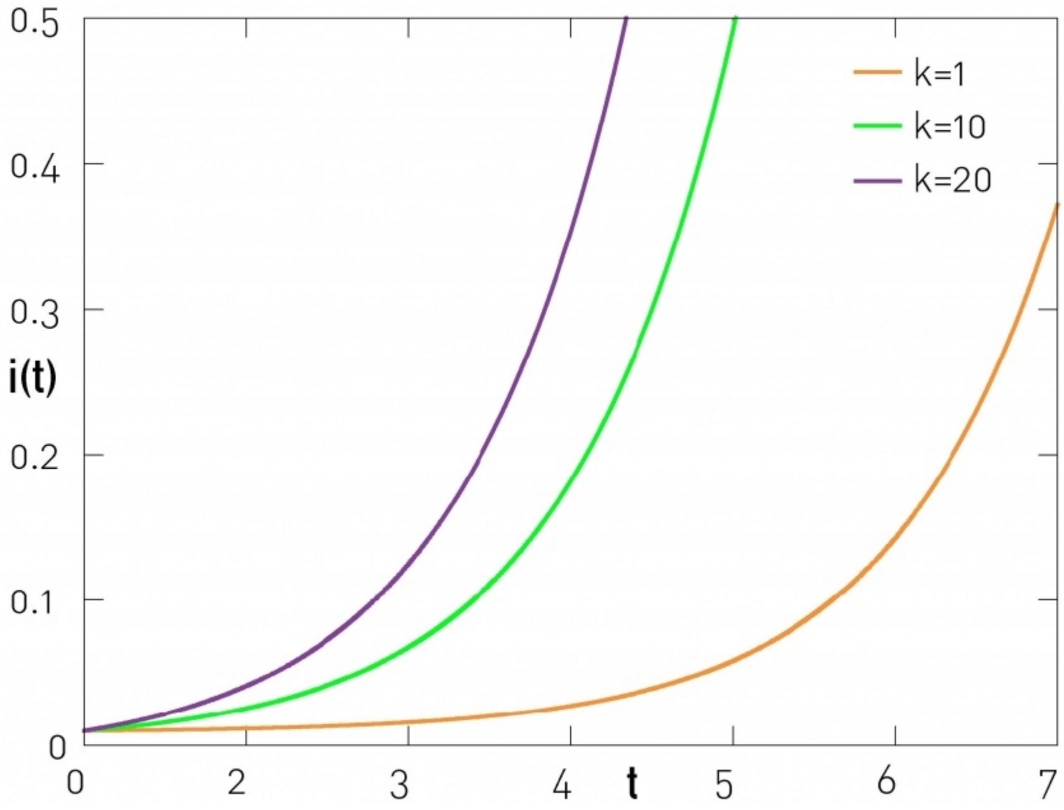
$$i_k(t) \approx i_0 \left(1 + k \frac{\langle k \rangle - 1}{\langle k^2 \rangle - \langle k \rangle} \left(e^{t/\tau^{SI}} - 1\right)\right)$$

What can you say about $i_k(t)$? (as a function of k,t)

$$\tau^{SI} = \frac{\langle k \rangle}{\beta \left(\langle k^2 \rangle - \langle k \rangle\right)}$$

Characteristic time, i.e., the time to infect $1/e \simeq 36\%$ of nodes

Higher degree nodes are more likely to become infected



$$i_k(t) = i_0 \left(1 + \frac{k\left(\langle k \rangle - 1\right)}{\langle k^2 \rangle - \langle k \rangle}\left(e^{t/\tau^{SI}} - 1\right)\right)$$

Characteristic time

(time to infect *1/e ≃ 36%* of nodes)

$$\tau^{SI} = \frac{\langle k \rangle}{\beta \left(\langle k^2 \rangle - \langle k \rangle \right)}$$

- Random network

$$\left\langle k^2 \right\rangle = \langle k \rangle \left(\langle k \rangle + 1 \right) \Rightarrow$$

(Do it on paper)

- Scale-free network with *γ ≥ 3*

$$\langle k \rangle , \left\langle k^2 \right\rangle \text{ are finite} \Rightarrow$$

- Scale-free network with *γ < 3*

$$\left\langle k^2 \right\rangle \xrightarrow[N \to \infty]{} \infty \Rightarrow$$

Characteristic time

(time to infect $1/e \simeq 36\%$ of nodes)

$$\tau^{SI} = \frac{\langle k \rangle}{\beta \left(\langle k^2 \rangle - \langle k \rangle \right)}$$

- Random network

$$\langle k^2 \rangle = \langle k \rangle \left(\langle k \rangle + 1 \right) \Rightarrow \tau_{ER}^{SI} = \frac{1}{\beta \langle k \rangle}$$

- Scale-free network with $\gamma \geq 3$

$$\langle k \rangle , \langle k^2 \rangle \ \text{are finite} \Rightarrow \tau_{BA}^{SI} \ \text{is finite}$$

- Scale-free network with $\gamma < 3$

$$\langle k^2 \rangle \xrightarrow[N \to \infty]{} \infty \Rightarrow$$

What happens?

Characteristic time

$$\tau^{SI} = \frac{\langle k \rangle}{\beta \left(\langle k^2 \rangle - \langle k \rangle \right)}$$

(time to infect *1/e ≃ 36%* of nodes)

- Random network

$$\left\langle k^2 \right\rangle = \langle k \rangle \left(\langle k \rangle + 1 \right) \Rightarrow \tau_{ER}^{SI} = \frac{1}{\beta \langle k \rangle}$$

- Scale-free network with $\gamma \geq 3$

$$\langle k \rangle , \left\langle k^2 \right\rangle \text{ are finite} \Rightarrow \tau_{BA}^{SI} \text{ is finite}$$

- Scale-free network with $\gamma < 3$

$$\left\langle k^2 \right\rangle \xrightarrow[N \to \infty]{} \infty \Rightarrow \lim_{N \to \infty} \tau_{BA}^{SI} = 0$$

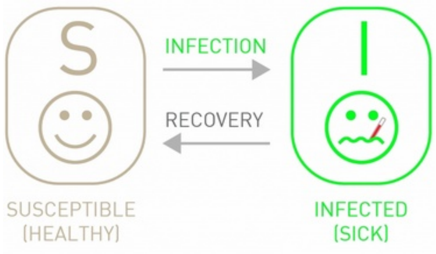
Vanishing characteristic time

$$\tau^{SI} = \frac{\langle k \rangle}{\beta \left(\langle k^2 \rangle - \langle k \rangle \right)}$$

- If $\lim_{N \to \infty} \frac{\langle k \rangle}{\langle k^2 \rangle} = 0$ the characteristic time goes to 0

- Networks with skewed degree distributions allow infections with the same β to spread faster

SIS dynamics on a graph



Similar to SI dynamics but allowing recovery

$$\frac{di_k(t)}{dt} = k(1 - i_k(t))\Theta_k\beta - \mu i_k(t) \qquad \tau^{SIS} = \frac{\langle k \rangle}{\beta \langle k^2 \rangle - \mu \langle k \rangle}$$

If people recover quickly, $\tau < 0$ and the infection dies out ... unless $\langle k^2 \rangle \to \infty$

Epidemic threshold

- A key quantity is the spreading rate $\lambda = \frac{\beta}{\mu}$
- The critical spreading rate λ_c called the epidemic threshold, is such that $\tau > 0$
- Spreading rate larger than epidemic threshold:
  - infection becomes endemic
- Spreading rate smaller than epidemic threshold:
  - Infection dies out

Epidemic threshold

- A key quantity is the spreading rate $\lambda = \dfrac{\beta}{\mu}$
- The critical spreading rate λ_c called the epidemic threshold, is such that $\tau > 0$

Compute the epidemic threshold β/μ for an ER graph where $\langle k^2 \rangle = \langle k \rangle (\langle k \rangle + 1)$

Upload to Nearpod Collaborate
https://nearpod.com/student/
Code to be given during class

Begin from: $\tau^{SIS} = \dfrac{\langle k \rangle}{\beta \langle k^2 \rangle - \mu \langle k \rangle}$ and use that $\tau^{SIS} > 0$

Epidemic threshold in a scale-free network

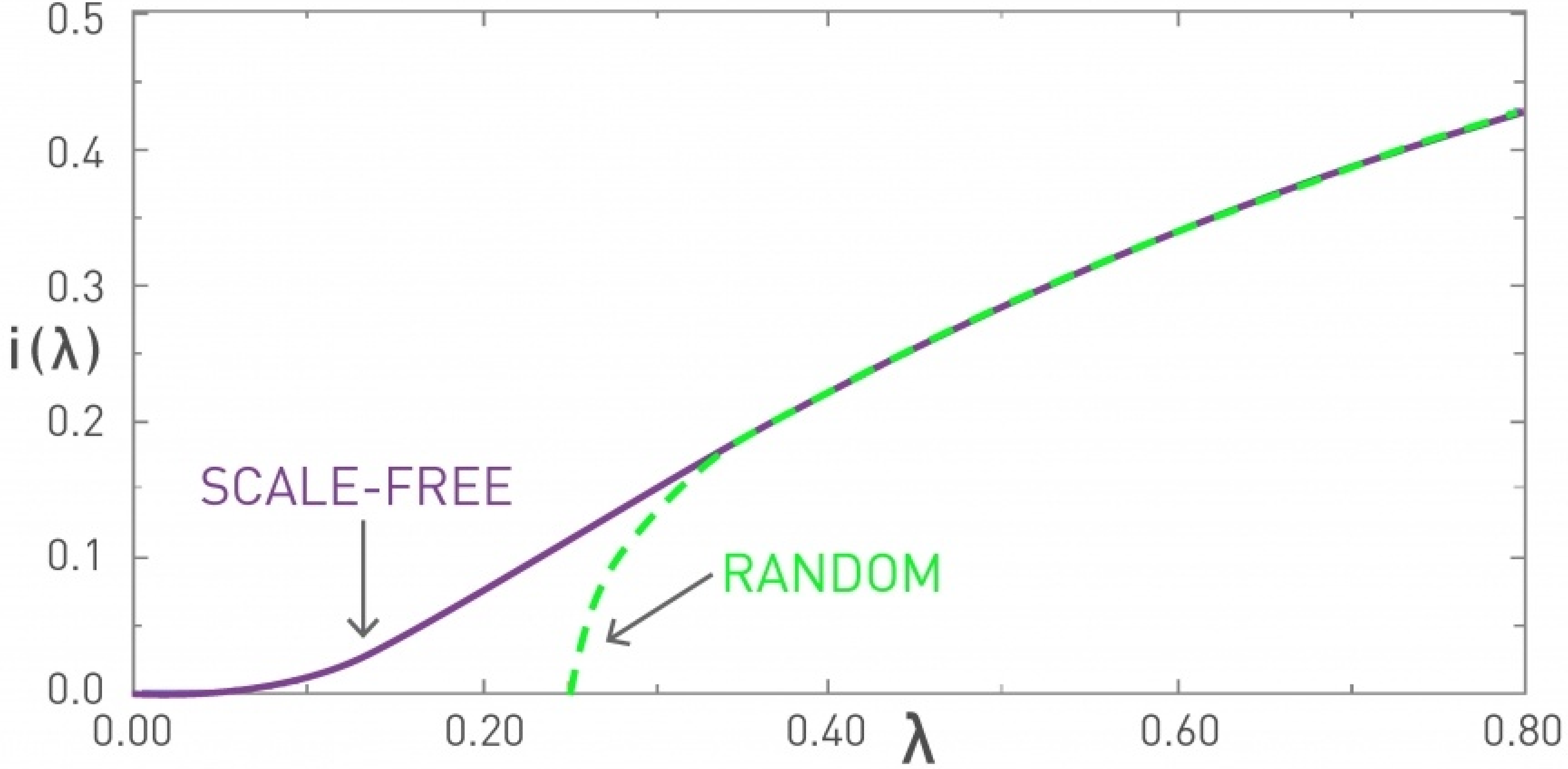
$$\tau^{SIS} = \frac{\langle k \rangle}{\beta \langle k^2 \rangle - \mu \langle k \rangle} > 0 \Rightarrow \frac{\beta}{\mu} > \frac{\langle k \rangle}{\langle k^2 \rangle} = \lambda_c$$

Epidemic threshold in BA graph

- In a scale-free network with $\gamma < 3$

$$\langle k^2 \rangle \xrightarrow[N \to \infty]{} \infty \Rightarrow \lim_{N \to \infty} \tau^{SIS} = 0$$

Infected (in the limit) as a function of the epidemic threshold



Two key results for SI and SIS models on a BA graph

In a large scale-free network with $\gamma < 3$

- An infection may reach everybody in a very short time: $\tau = 0$
- An infection may become endemic even if it is not very contagious and even if people recover fast: $\lambda_c = 0$

Summary

Things to remember

- Degree block approximation
- Epidemic thresholds in scale-free networks