# The Friendship Paradox

**Social Networks Analysis and Graph Algorithms**

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# Contents

- Sampling nodes and edges
- Average degree of friends

# Sources

- A. L. Barabási (2016). Network Science – Chapter 04
- F. Menczer, S. Fortunato, C. A. Davis (2020). A First Course in Network Science – Chapter 03
- URLs cited in the footer of specific slides

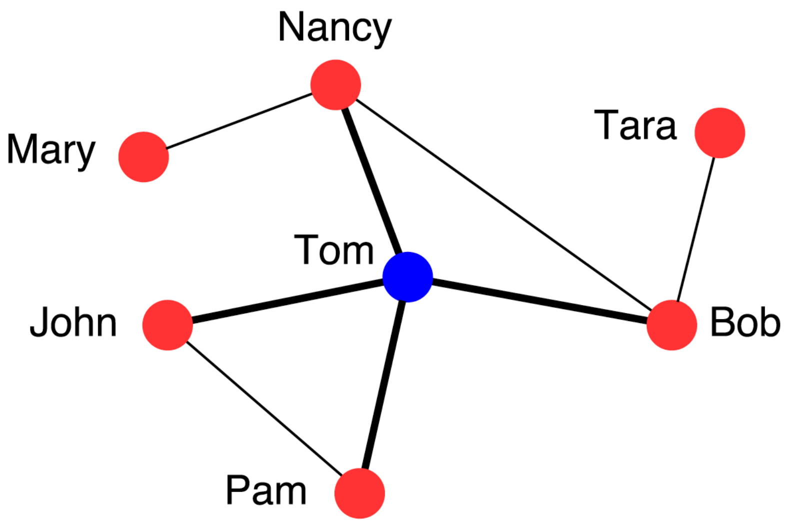
**Sampling a random node**

**vs**

**sampling at random one of the two nodes attached to a random edge**

# Exercise Numerical calculation of friendship paradox

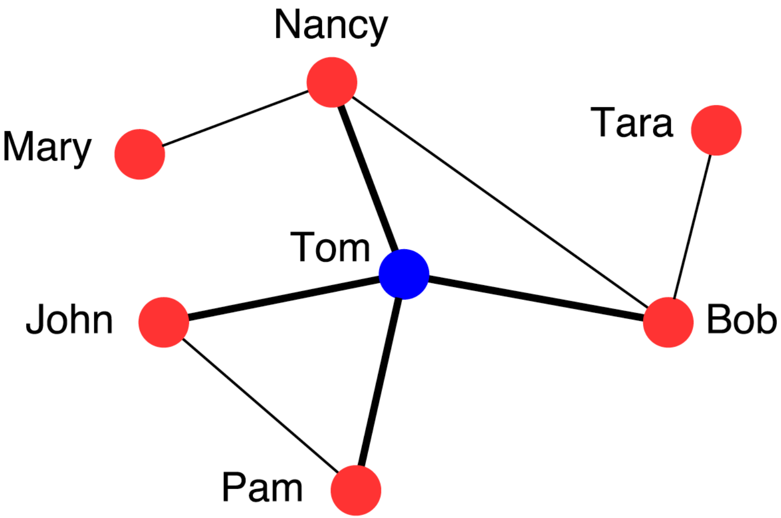
- What is the probability of selecting Tom **if we select a random node**?
- What is the probability of selecting Tom **if we select a random edge and then randomly one of the two nodes attached to it**?




Pin board: https://upfbarcelona.padlet.org/chato/ocpl5n14i8hrkr4i

# Sampling a random node
# vs
# sampling a random friend
# of a random node

# Average degree of friends




- Average degree

  $(1+3+3+1+4+2+2)/7 = 16/7 \simeq 2.29$

- Average degree of friends of ...

  ... Mary: 3

  ... Nancy: $(1{+}4{+}3)/3 = 8/3$

  ... Tara: 3

  ... Bob: $(1{+}3{+}4)/3 = 8/3$

  ... Tom: $(3{+}3{+}2{+}2)/4 = 10/4$

  ... John: $(4{+}2)/2 = 3$

  ... Pam: $(4{+}2)/2 = 3$

  Average of degree of friends $\simeq$ 2.83 $\quad (> 2.29)$

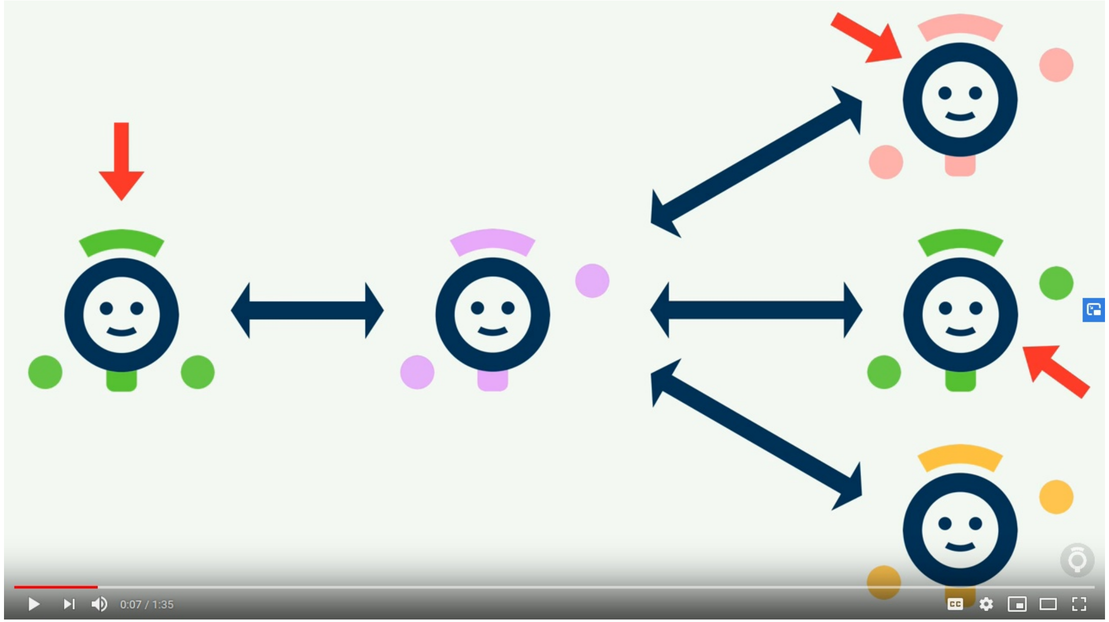
# The friendship paradox

- Take a random person x; what is the expected degree of this person? $<k>$
- Take a random person x, now pick one of x's neighbors, let's say y; what is the expected degree of y?

  **It is not $<k>$**

- This "paradox" is a useful:
  - As a marketing strategy: if *u* invites a friend *v* to buy/use a product, it is likely that *v* has many friends, and hence it is relevant for marketing that v buys/use the product
  - As a vaccination strategy: instead of offering a vaccine to random people, ask them to name a friend, offer the vaccine to those people, who will have larger degree

# Sampling bias and the friendship paradox (1'35'')




https://www.youtube.com/watch?v=httLvVufAYs

# Imagine you're at a random airport on earth

- Is it more likely to be …

  a large airport or a small airport?

- If you take a random flight out of it …

  will it go to a large airport or a small airport?

# An example of friendship paradox

- Pick a random airport on Earth
  - Most likely it will be a small airport
- However, no matter how small it is, it **will** have flights to big airports
- On average those airports will have much larger degree

# Exercise [B. 2016, Ex. 4.10.2]: "Friendship Paradox"

- If random variable $K$ represents the degree of a randomly chosen node, we denoted as $p_k$ the probability that a randomly chosen node has degree $k$
  - $p_k = Pr(K=k)$ Note that for simplicity we always denote by ⟨k⟩ what we should have named ⟨K⟩
- Random variable $K_F$ will represent the degree of a randomly chosen neighbor ("friend") of a randomly chosen node; we will denote by $q_k$ the probability that a randomly chosen neighbor of a randomly chosen node has degree $k$

  $q_k = Pr(K_F=k)$
- The formula is: $q_k = C\ k\ p_k$ where $C$ is a normalization factor

  (a) Find $C$ (hint: sum of $q_k$ must be 1)

# Exercise [B. 2016, Ex. 4.10.2]: "Friendship Paradox"

Random variable $K_F$ is the degree of a randomly chosen neighbor of a randomly chosen node; we denote by $q_k$ the probability that a randomly chosen neighbor of a randomly chosen node has degree $k$

$$q_k = Pr(K_F{=}k) = C\ k\ p_k$$

(b) Find the expectation $\langle K_F \rangle$

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Hints: $E[X] = \sum_x x \cdot P(X = x) \quad E[X^2] = \sum_x x^2 \cdot P(X = x)$

# Exercise [B. 2016, Ex. 4.10.2]: "Friendship Paradox"

For the scale-free network described below:

(c) Compute $\langle K_F \rangle$: the expected number of friends of a randomly chosen neighbor of a randomly chosen node

(d) Compare with $\langle k \rangle$: the expected number of friends of a randomly chosen node

$$N = 10000$$
$$\gamma = 2.3$$
$$k_{\min} = 1$$
$$k_{\max} = 1000$$

You can use this formula for the **moments** ($\langle k \rangle$, $\langle k^2 \rangle$, $\langle k^3 \rangle$, ...) of the degree distribution in a scale-free network:

$$\langle k^n \rangle = (\gamma - 1)\, k_{\min}^{\gamma - 1} \frac{\left(k_{\max}^{n-\gamma+1} - k_{\min}^{n-\gamma+1}\right)}{n - \gamma + 1}$$

# Code

```python
def degree_moment(kmin, kmax, moment, gamma):
    C = (gamma-1.0)*(kmin**(gamma-1.0))
    numerator = (kmax**(moment-gamma+1.0) - kmin**(moment-gamma+1.0))
    denominator = (moment-gamma+1.0)
    return C * numerator / denominator
```

```python
kavg = degree_moment(kmin=1, kmax=1000, moment=1, gamma=2.3)
print(kavg)
```

```
3.787798988222529
```

```python
ksqavg = degree_moment(kmin=1, kmax=1000, moment=2, gamma=2.3)
print(ksqavg)
```

```
231.94329076177414
```

```python
print(ksqavg / kavg)
```

```
61.23431879119234
```

# Summary

# Summary

- Your friends have more friends than you

$$\langle K_F \rangle > \langle k \rangle$$

- This can be quite strong in scale-free networks

# Practice on your own

- Draw a small graph, and sample from that graph until you're convinced $\langle K_F \rangle > \langle k \rangle$