

# Scale-Free Networks

## Social Networks Analysis and Graph Algorithms

Prof. Carlos Castillo — <https://chato.cl/teach>



Universitat  
Pompeu Fabra  
*Barcelona*

# Contents

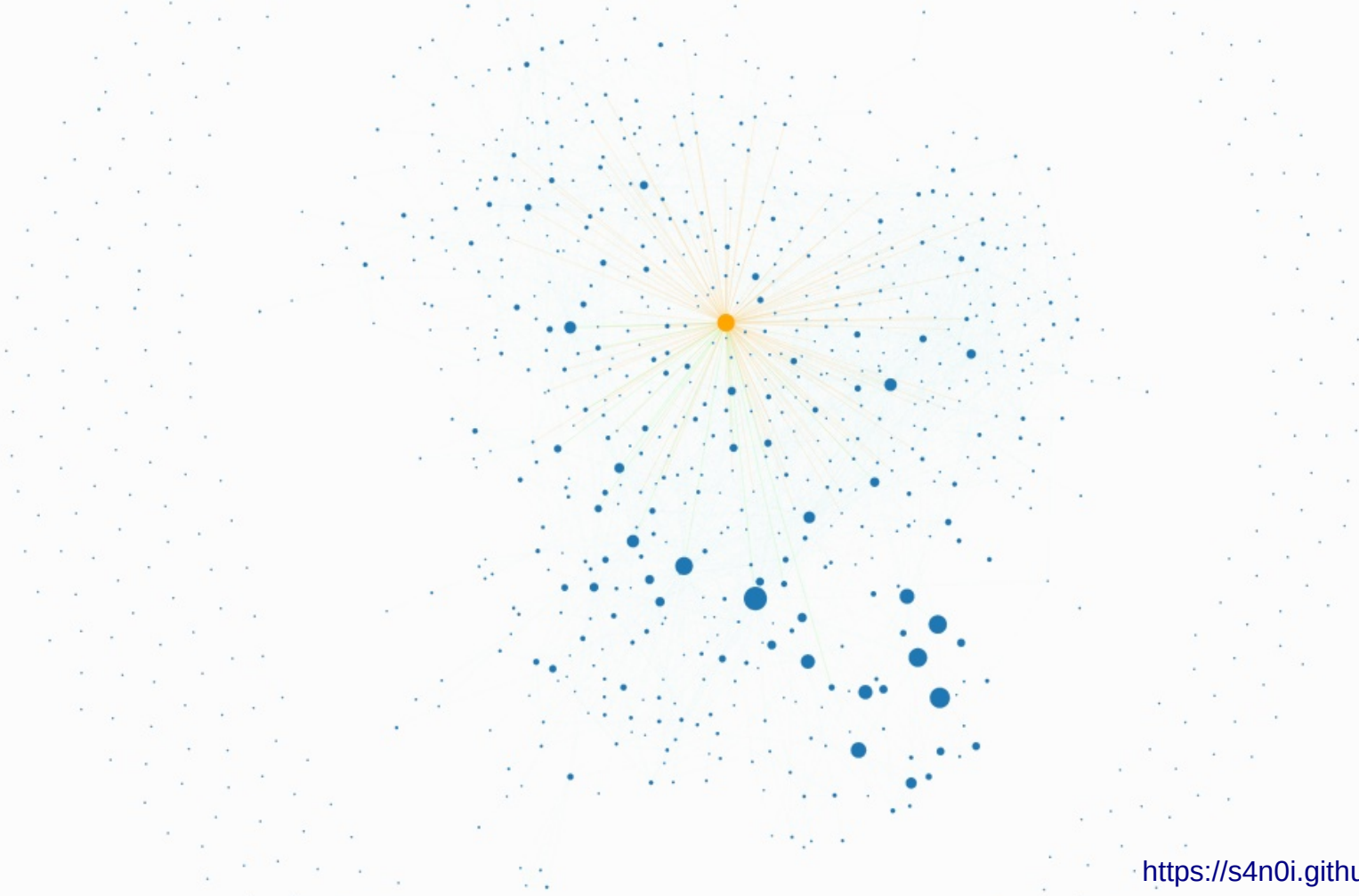
- Characteristics of scale-free networks
- Degree distribution of scale-free networks
- Distance distribution of scale-free networks

# Sources

- A. L. Barabási (2016). Network Science – Chapter 04
- URLs cited in the footer of specific slides

# Observed degree distributions

# Immanuel Kant in Wikipedia

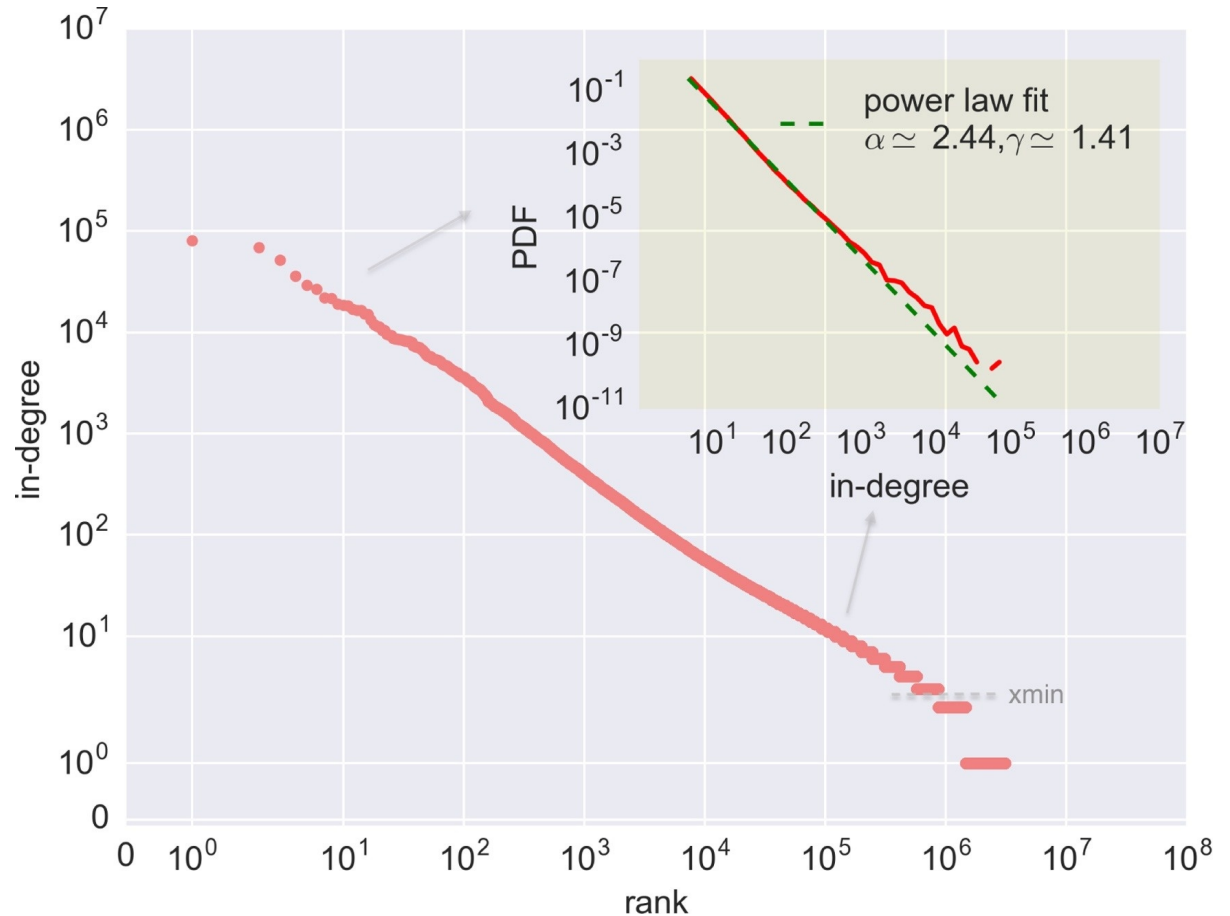


# Wikipedia

2017; first link only

Article	In-degree
United States	~80 K
village	~70 K
moth	~52 K
Communes of France	~37 K
species	~29 K
...	...

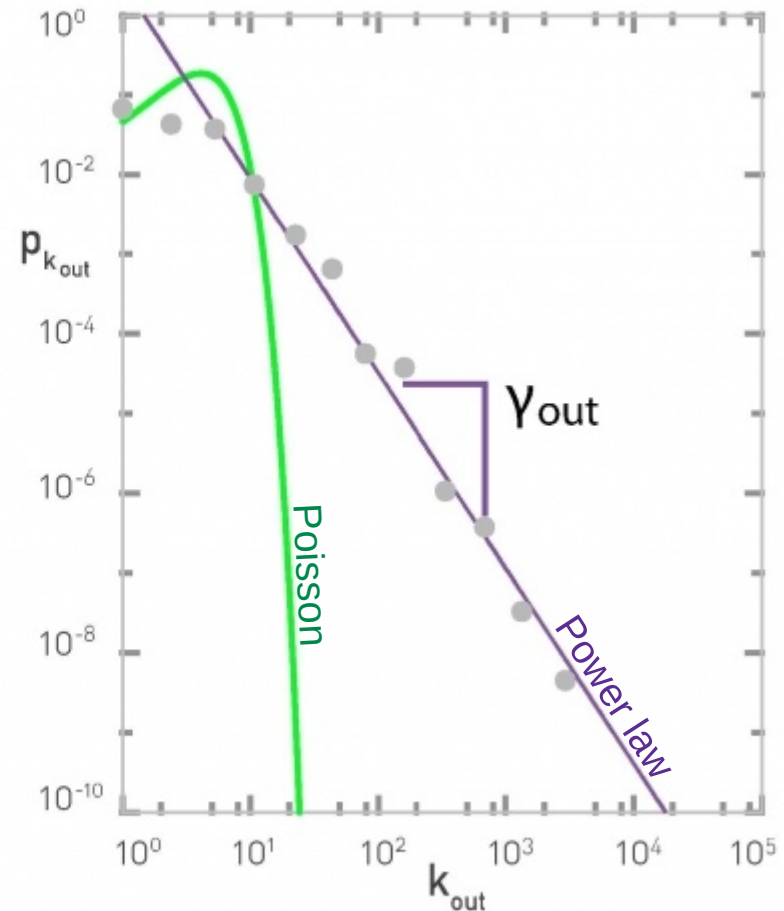
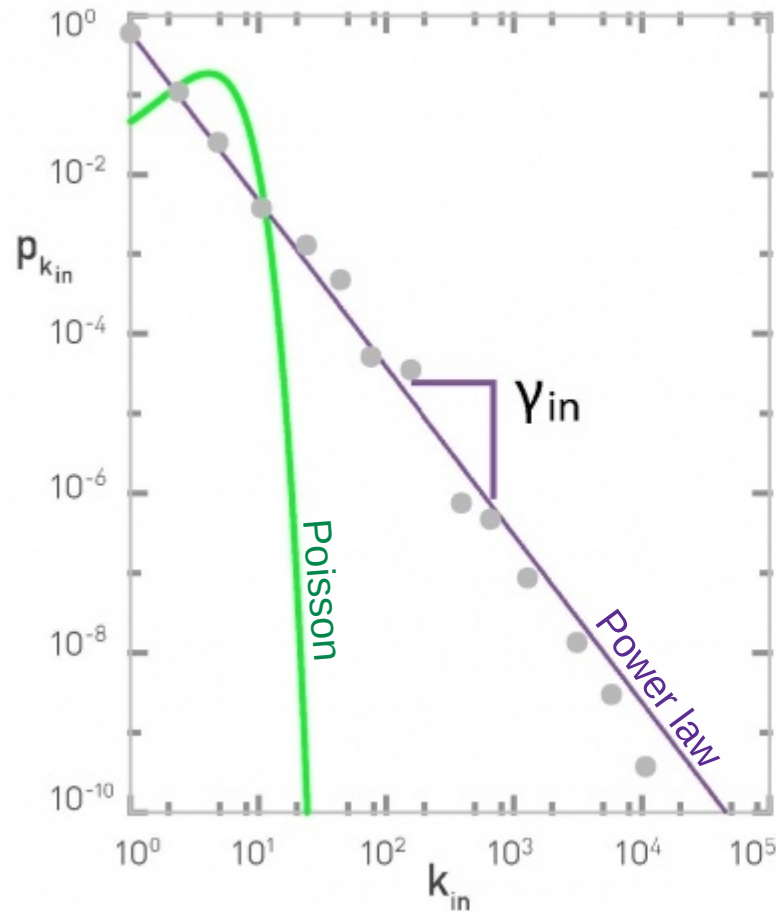
In-degree of k-th page with most (first) in-links



# Degree distributions in Web graphs

- Web graphs have large “hubs”:
  - This is, nodes with very large degree
- This does not happen in a random (ER) graph
- We have already seen the Poisson distribution is a bad approximation of the observed degree distribution

# Degree distributions in a web graph [nd.edu]





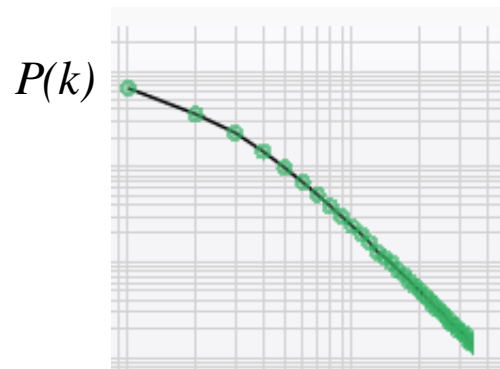
# Modeling degree distributions

# Degree distribution in real-world networks

- Straight descending line in log-log plot

$$\log p_k \sim -\gamma \log k$$

$$p_k \sim k^{-\gamma}$$

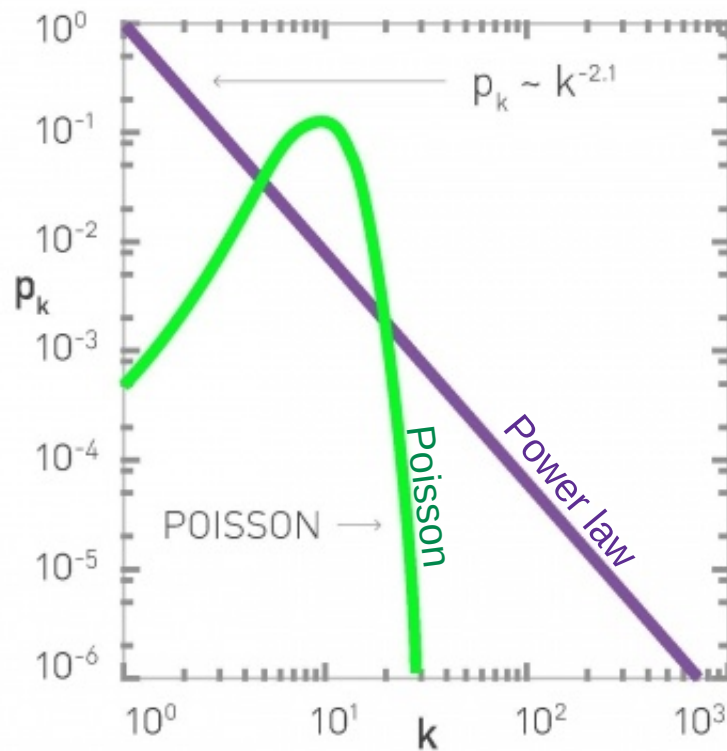
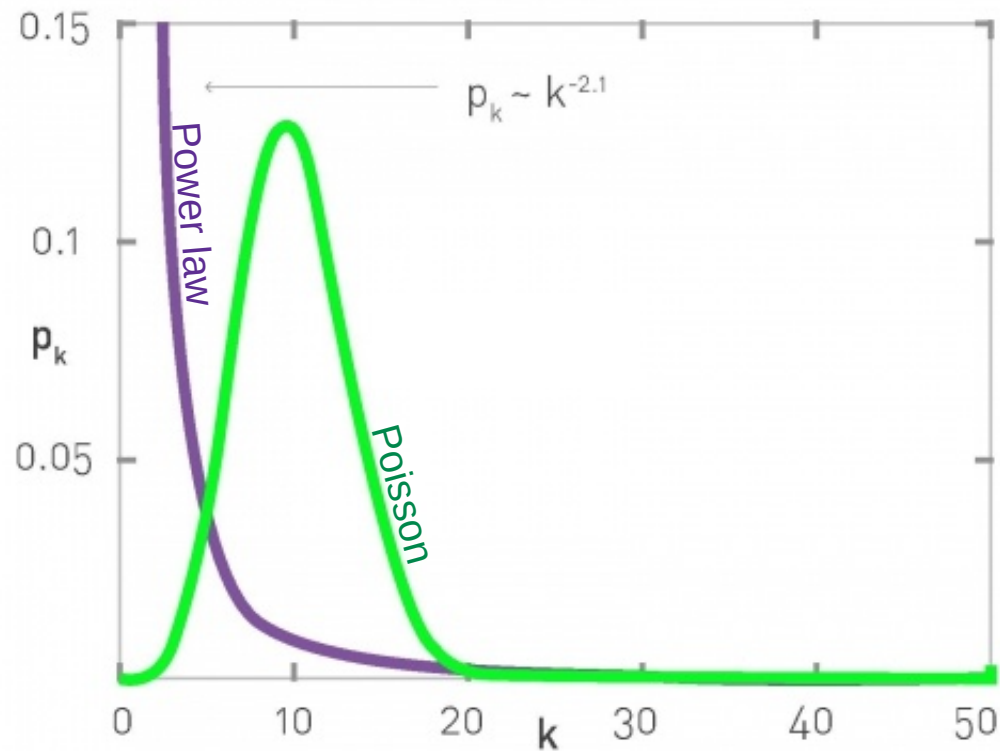


Degree  $k$

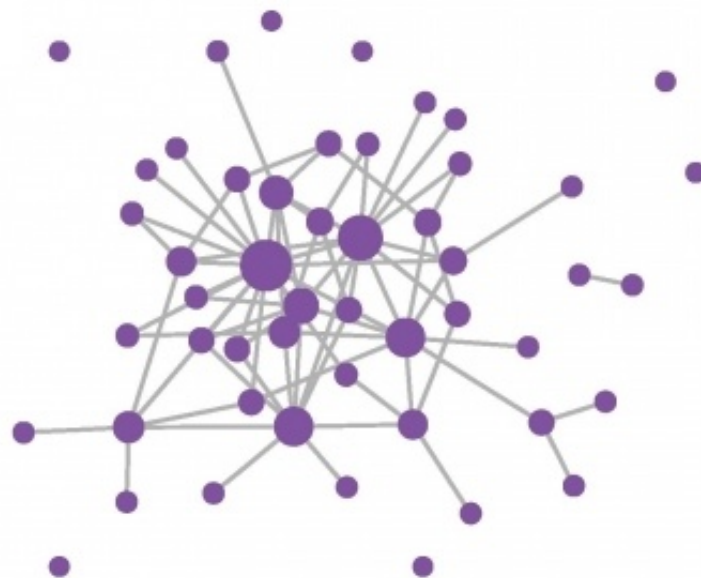
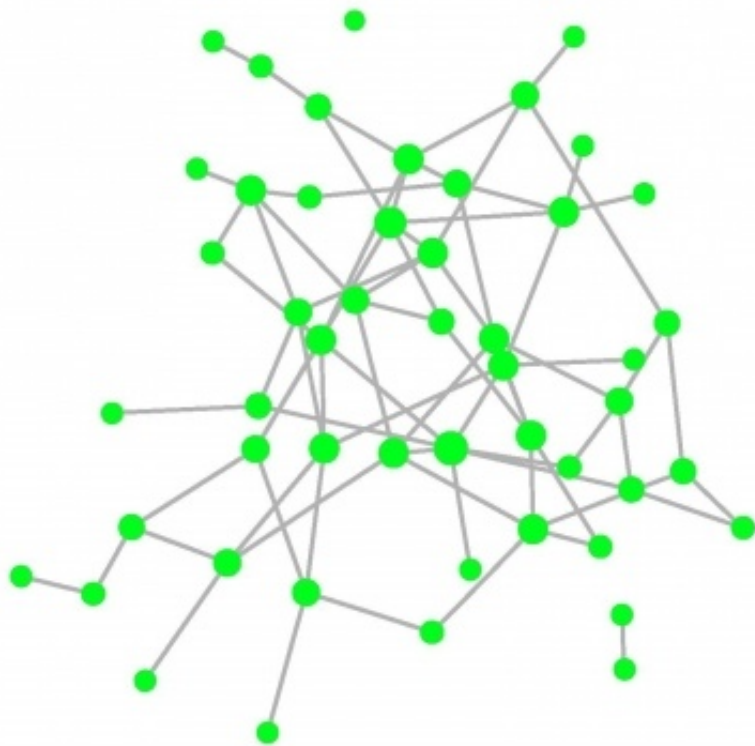
- Parameter  $\gamma$  is the exponent of the power law

**A scale-free network is a network whose degree distribution follows a power law**

# Comparing Poisson to power law



# Comparing Poisson to power law



# Random vs scale-free networks



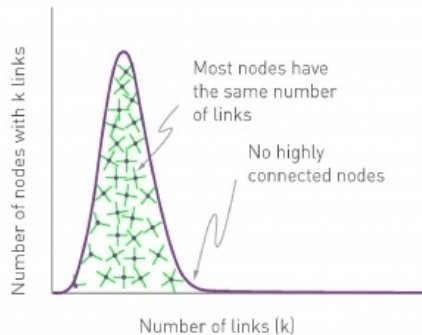
High-speed trains



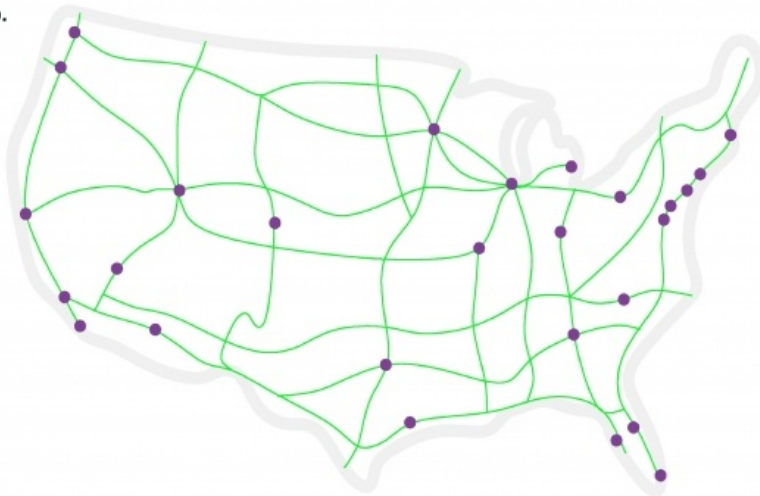
Air transportation

# Random vs scale-free networks (cont.)

a. POISSON

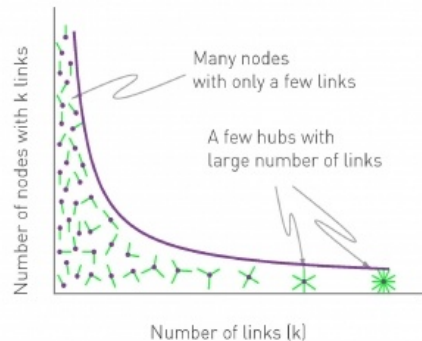


b.



Ground transportation

c. POWER LAW



d.



Air transportation

# Formally (discrete)

$$p_k = Ck^{-\gamma}$$

$$\sum_{k=1}^{\infty} p_k = 1 \longrightarrow C = \frac{1}{\sum_{k=1}^{\infty} k^{-\gamma}} = \frac{1}{\zeta(\gamma)}$$

$$p_k = \frac{k^{-\gamma}}{\zeta(\gamma)}$$

Riemann's  
zeta

This formalism assumes there are no nodes with degree zero

# Formally (continuous approx.)

$$p_k = Ck^{-\gamma}$$

$$\int_{k=k_{\min}}^{\infty} p_k = 1 \longrightarrow C = \frac{1}{\int_{k=k_{\min}}^{\infty} k^{-\gamma}} = (\gamma - 1)k_{\min}^{\gamma-1}$$

$$p_k = (\gamma - 1)k_{\min}^{\gamma-1} k^{-\gamma}$$

$k_{\min}$  is the smaller degree found in the network



# Exercise

In the actor network,  $N=702,388$ ,  $\gamma=2.12$

1. How many actors do we expect to have ...

1 other co-star?

<https://www.wolframalpha.com/> recognizes  $x*y$ ,  $x/y$ ,  $\text{Zeta}(x)$ ,  $x^{(-y)}$ , etc.

10 other co-stars?

100 other co-stars?

$$p_k = \frac{k^{-\gamma}}{\zeta(\gamma)} \quad \zeta(2.12) \approx 1.5452$$

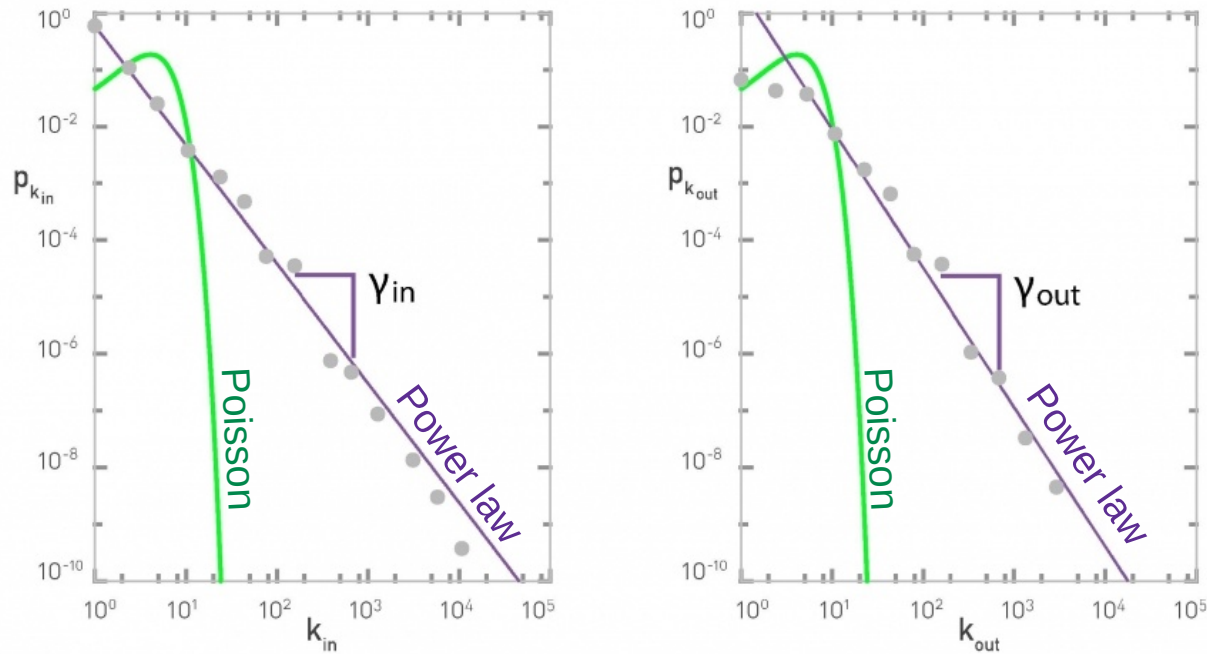
2. For how many co-stars do we still expect to have one actor that has that many co-stars?

# Directed graphs

# In-Degree vs Out-Degree

In TikTok or Instagram (or any online social networking site), which distribution is likely to have a larger range, that of in-degree or out-degree?

# In directed graphs the exponent might be different for in-degree and out-degree



$$p_k \sim k^{-\gamma}$$

What kind of values of gamma make the power law “tail” look more like the Poisson “tail”?

# In directed networks ...

- Each node has two degrees:  $k_{\text{in}}$  and  $k_{\text{out}}$
- In general they may **differ**, hence

$$p_{k_{\text{in}}} \sim k^{-\gamma_{\text{in}}}$$

$$p_{k_{\text{out}}} \sim k^{-\gamma_{\text{out}}}$$

- In nd1998,  $\gamma_{\text{in}} \approx 2.1$ ,  $\gamma_{\text{out}} \approx 2.4$

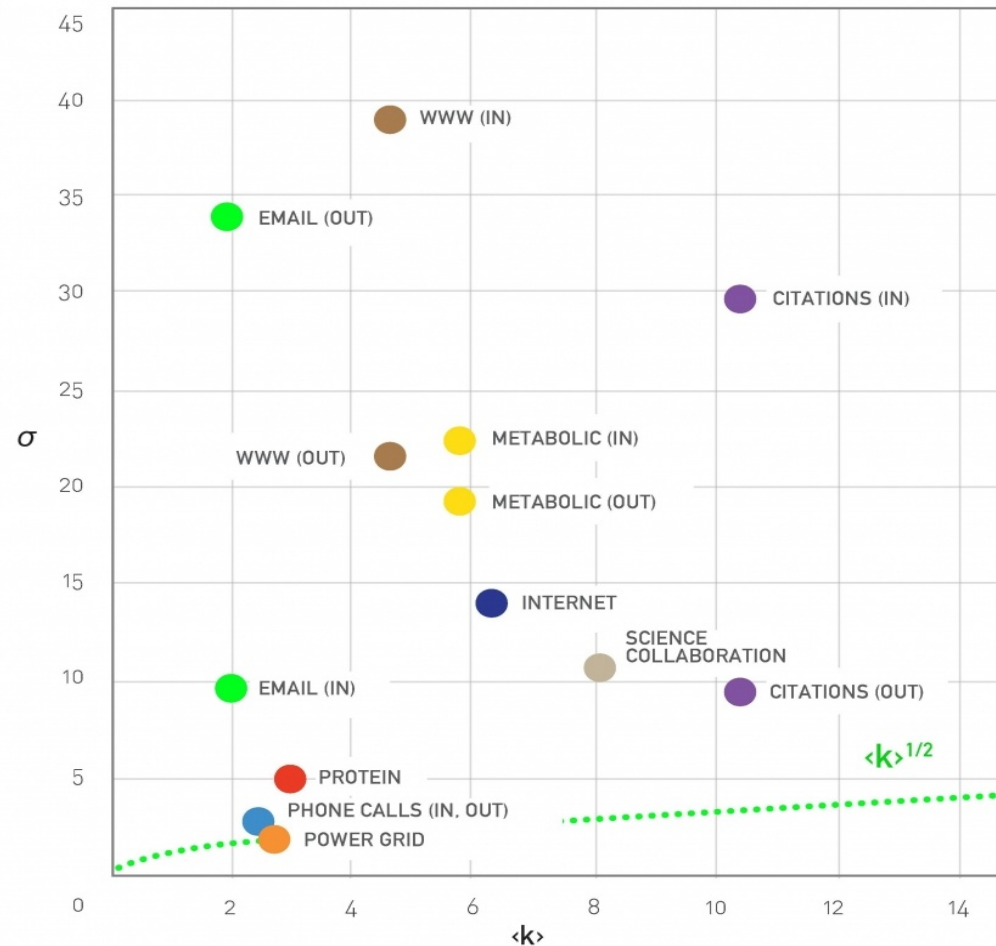
# Dispersion of the degree distribution

# What does it mean “scale-free”?

- A distribution **has a “scale”** if values are close to each other, for instance in a random network  $\sigma_k = \langle k \rangle^{1/2}$
- Hence, most nodes are in the range  $\langle k \rangle \pm \langle k \rangle^{1/2}$
- However in scale-free network with  $\gamma < 3$

$$\sigma_k \rightarrow \infty$$

# Real network examples

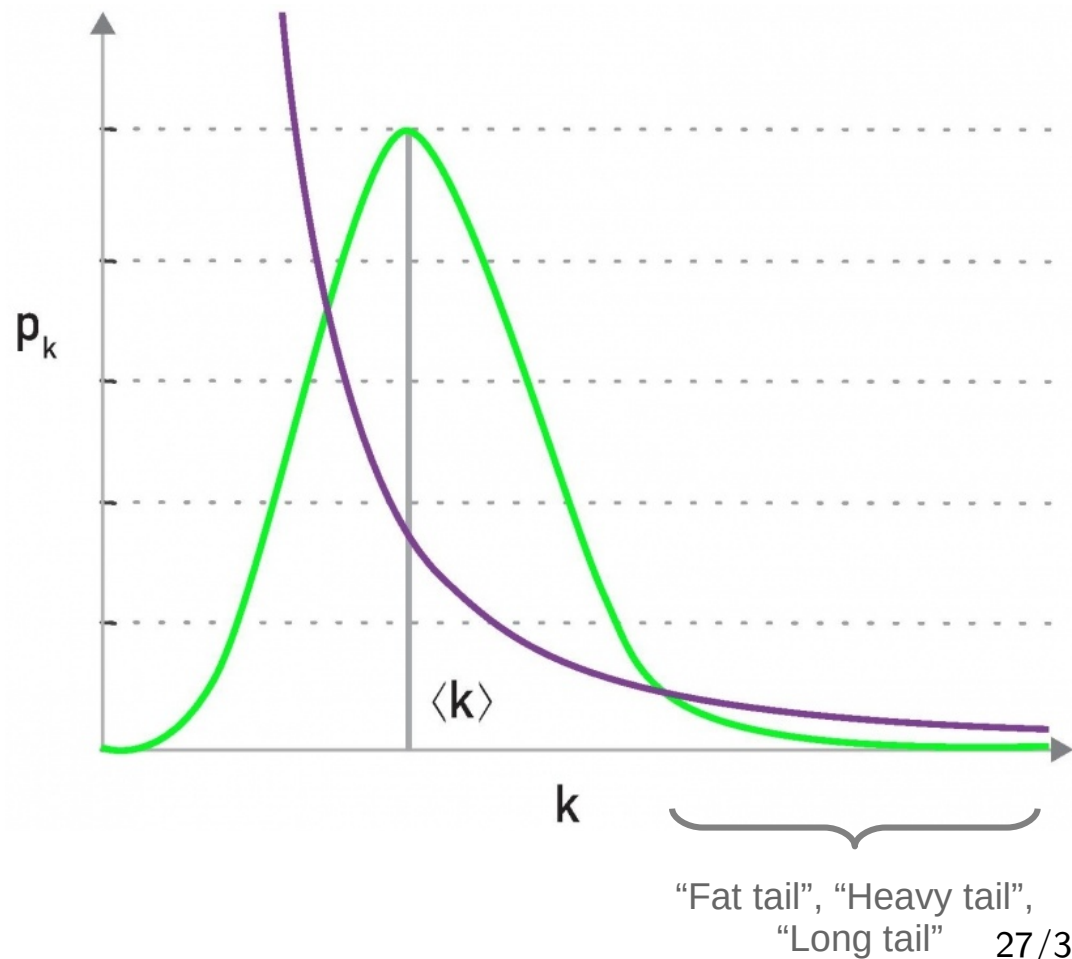




# Example: citations to scientific papers

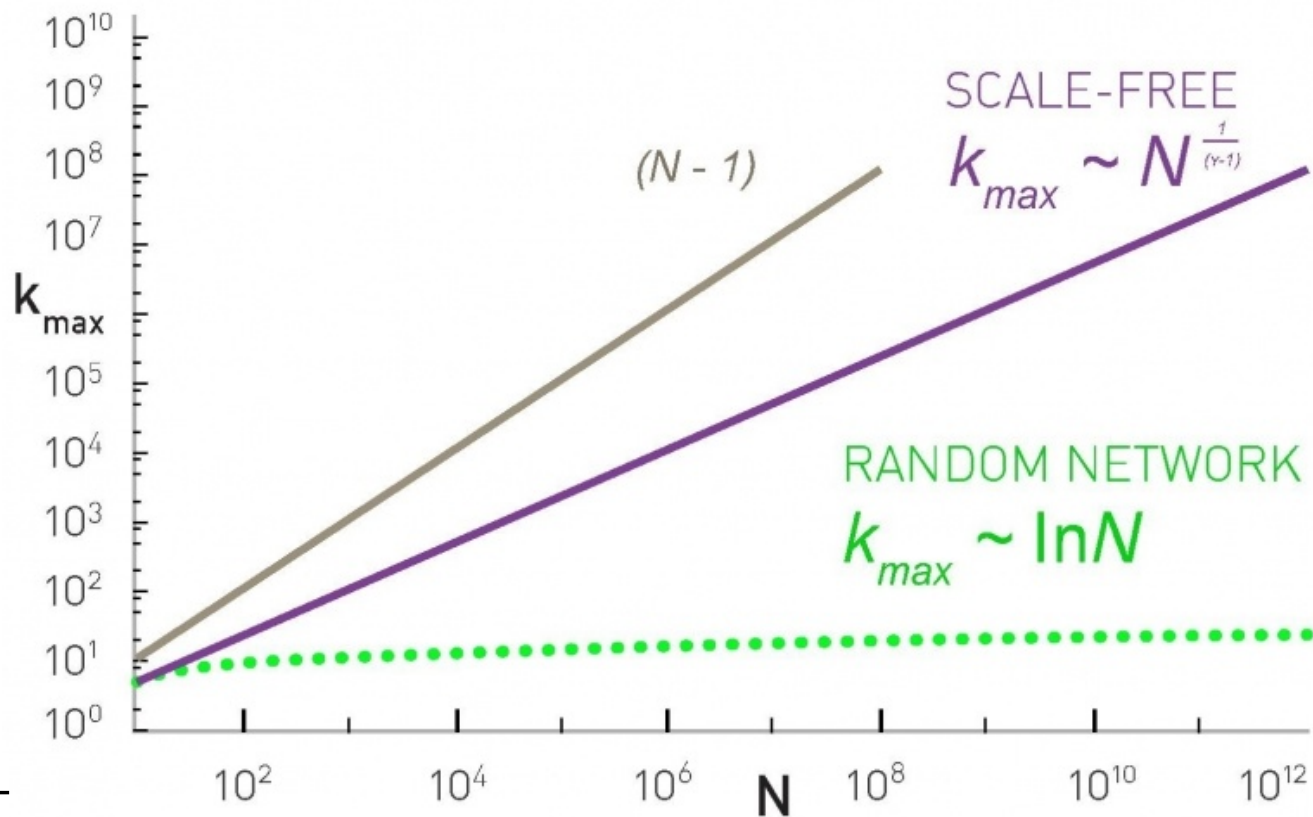
$$k_{\text{in}} \approx 10 \pm 900$$

In general, the average degree is not very informative in scale-free networks



# There is a natural cut-off of the degree

The largest hub cannot have more than  $N-1$  connections



$$k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$$

# Real network examples

Network	$N$	$L$	$\langle k \rangle$	$\langle k_{in}^2 \rangle$	$\langle k_{out}^2 \rangle$	$\langle k^2 \rangle$	$Y_{in}$	$Y_{out}$	$Y$
Internet	192,244	609,066	6.34	-	-	240.1	-	-	3.42*
WWW	325,729	1,497,134	4.60	1546.0	482.4	-	2.00	2.31	-
Power Grid	4,941	6,594	2.67	-	-	10.3	-	-	Exp.
Mobile-Phone Calls	36,595	91,826	2.51	12.0	11.7	-	4.69*	5.01*	-
Email	57,194	103,731	1.81	94.7	1163.9	-	3.43*	2.03*	-
Science Collaboration	23,133	93,437	8.08	-	-	178.2	-	-	3.35*
Actor Network	702,388	29,397,908	83.71	-	-	47,353.7	-	-	2.12*
Citation Network	449,673	4,689,479	10.43	971.5	198.8	-	3.03*	4.00*	-
E. Coli Metabolism	1,039	5,802	5.58	535.7	396.7	-	2.43*	2.90*	-
Protein Interactions	2,018	2,930	2.90	-	-	32.3	-	-	2.89*-

# Summary

# Things to remember

- Power law
- Formulas for degree distribution
  - Discrete formula
  - Continuous formula
- Definition of scale-free

# Practice on your own

- (Somewhat) difficult, try to solve it ON YOUR OWN
- Imagine a connected scale-free graph with 1 million nodes and average degree 5

If we draw 100 nodes from this graph, how many will have degree 1?

*Remember, if the graph is connected,  $k_{min}=1$*

*If you cannot clear the unknown in a formula, plot it*

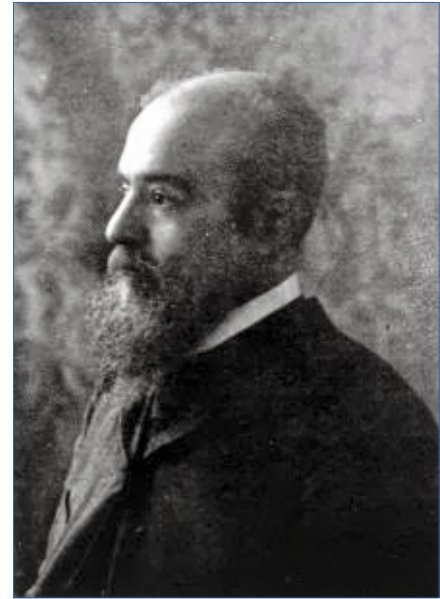
- Solution in next slide (shown only in .odp, not .pdf)

**Additional contents  
(not included in exams)**

**EXTRA**

# Pareto's Law

- Italian economist Vilfredo Pareto in the 19<sup>th</sup> century noted 80% of money was earned by 20% of people
- More recently ...
  - 80 percent of links on the Web point to only 15 percent of pages;
  - 80 percent of citations go to only 38 percent of scientists;
  - 80 percent of links in Hollywood are to 30 percent of actors
- A debate that is still open: the wealth of the 1% and the 0.1%





# Moments of degree distribution

- Moments of degree distribution

$$\langle k^n \rangle = \int_{k_{\min}}^{k_{\max}} k^n p_k dk = C \frac{k_{\max}^{n-\gamma+1} - k_{\min}^{n-\gamma+1}}{n - \gamma + 1}$$

$$C = (\gamma - 1) k_{\min}^{\gamma-1}$$

# Moments of degree distribution (cont.)

$$\sigma_k^2 = \langle k^2 \rangle - \langle k \rangle^2$$

- In a scale-free network

$$\langle k^2 \rangle = \int_{k_{\min}}^{k_{\max}} k^2 p_k dk = C \frac{k_{\max}^{3-\gamma} - k_{\min}^{3-\gamma}}{3 - \gamma}$$

- This diverges as  $k_{\max} \rightarrow \infty$  if  $\gamma < 3$
- Hence there is no “typical” scale

# When you do not observe the scale-free property

- In general, when there is a **limit** to  $k_{\max}$
- Out-degree in some social networks
- Materials/crystals

