Other Graph Evolution Models

Social Networks Analysis and Graph Algorithms

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Actual network growth is complex

A snapshot of the Autodesk organizational hierarchy was taken each day between May 2007 and June 2011, a span of 1498 days.

Each day the entire hierarchy of the company is constructed as a tree with each employee represented by a circle, and a line connecting each employee with his or her manager.

Larger circles represent managers with more employees working under them. The tree is then laid out using a force-directed layout algorithm.

From day to day, there are three types of changes that are possible:

- Employees join the company
- Employees leave the company
- Employees change managers

https://www.youtube.com/watch?v=mkJ-Uy5dt5g

Other graph evolution models

- Uniform random attachment
- Sub-linear and super-linear preferential attachment
- Good-get-richer
- Aging effects
- Link selection
- Copy model
- No preference and no growth

Sources

- A. L. Barabási (2016). Network Science – Chapter 05 and Chapter 06

Uniform Random Attachment

Growth in an ER network

- Two assumptions in ER networks:
  - There are N nodes that **pre-exist**
  - Nodes connect **at random**
- Let's challenge the first assumption

Uniform Attachment

- Network starts with m fully-connected nodes
- Time starts at $t_0=m$
- At every time step we add 1 node
- This node will have m outlinks

Expected degree over time

- Probability of obtaining one link: m/t
  - Decreases over time
- Expected degree of node born at $m < i < t$

$$m + \frac{m}{i} + \frac{m}{i+1} + \frac{m}{i+2} + \cdots + \frac{m}{t} \approx m\left(1 + \log\left(\frac{t}{i}\right)\right)$$

Tail of degree distribution

- How many nodes of degree larger than K are there at time t? (Computation in "Advanced materials" at the end of these slides)

$$e^{-\frac{K-m}{m}}$$

- Decreases exponentially with *K*: it's vanishingly rare to find high-degree nodes

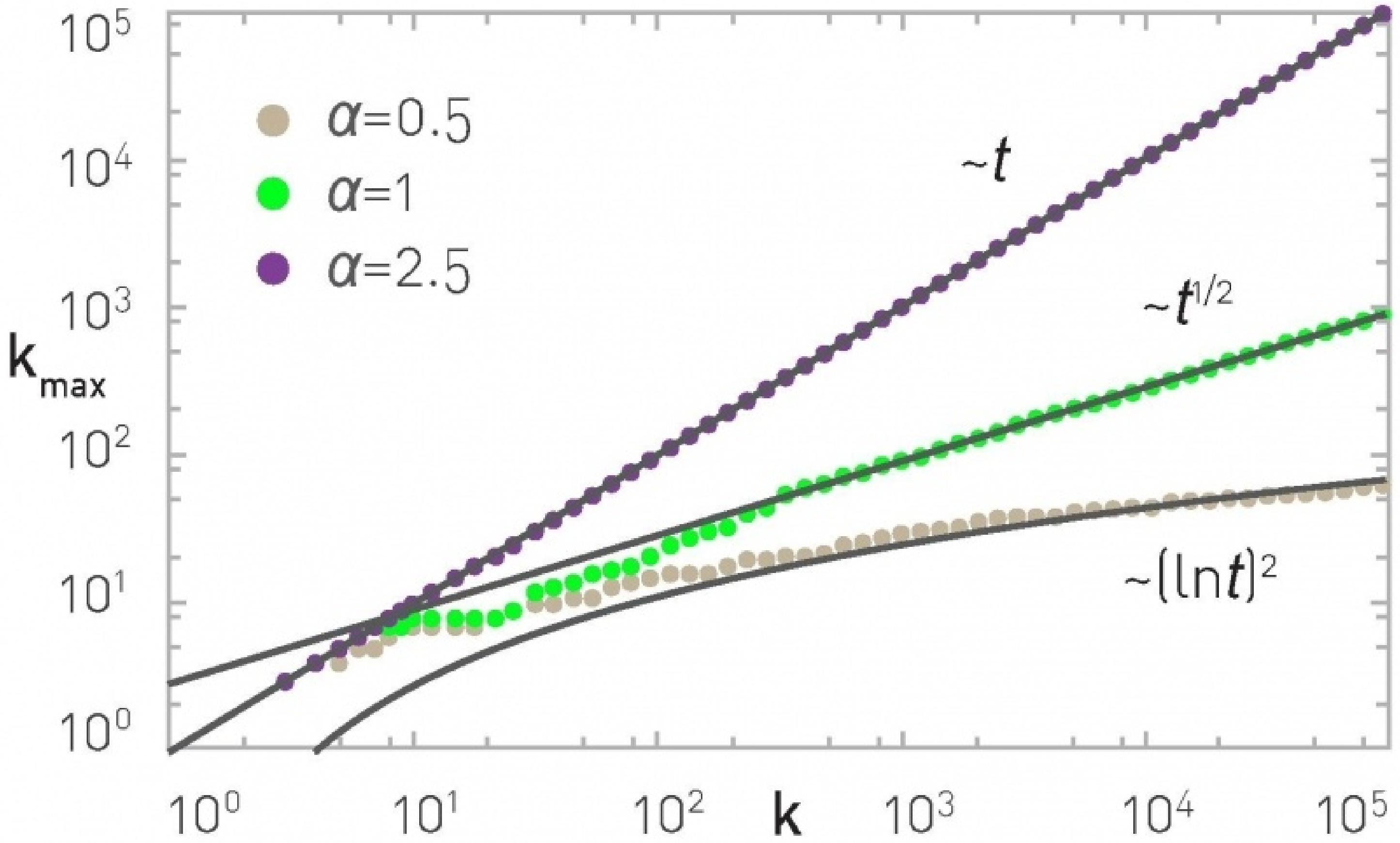
Sub-linear and super-linear preferential attachment

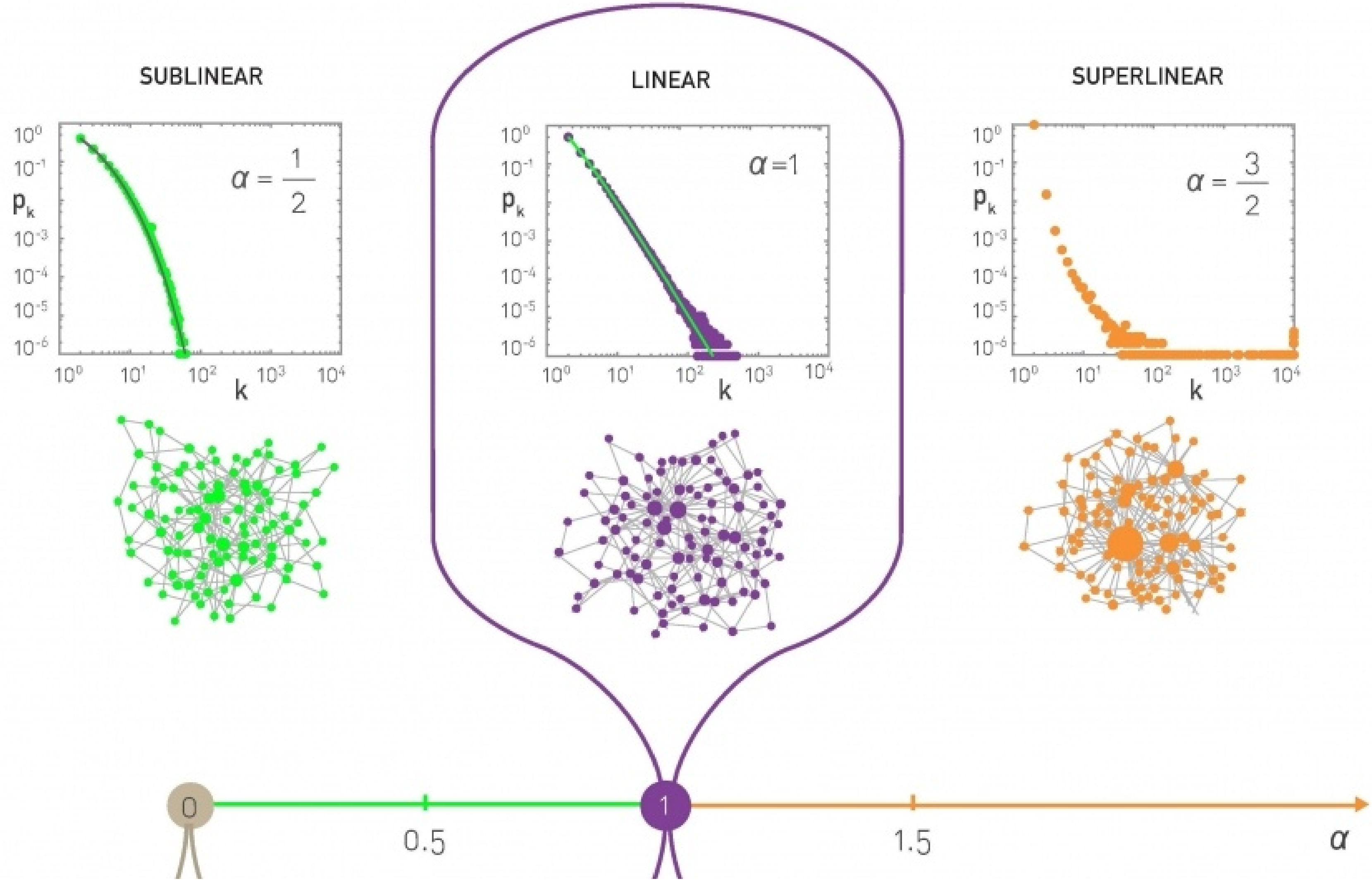
Sub-linear and super-linear preferential attachment

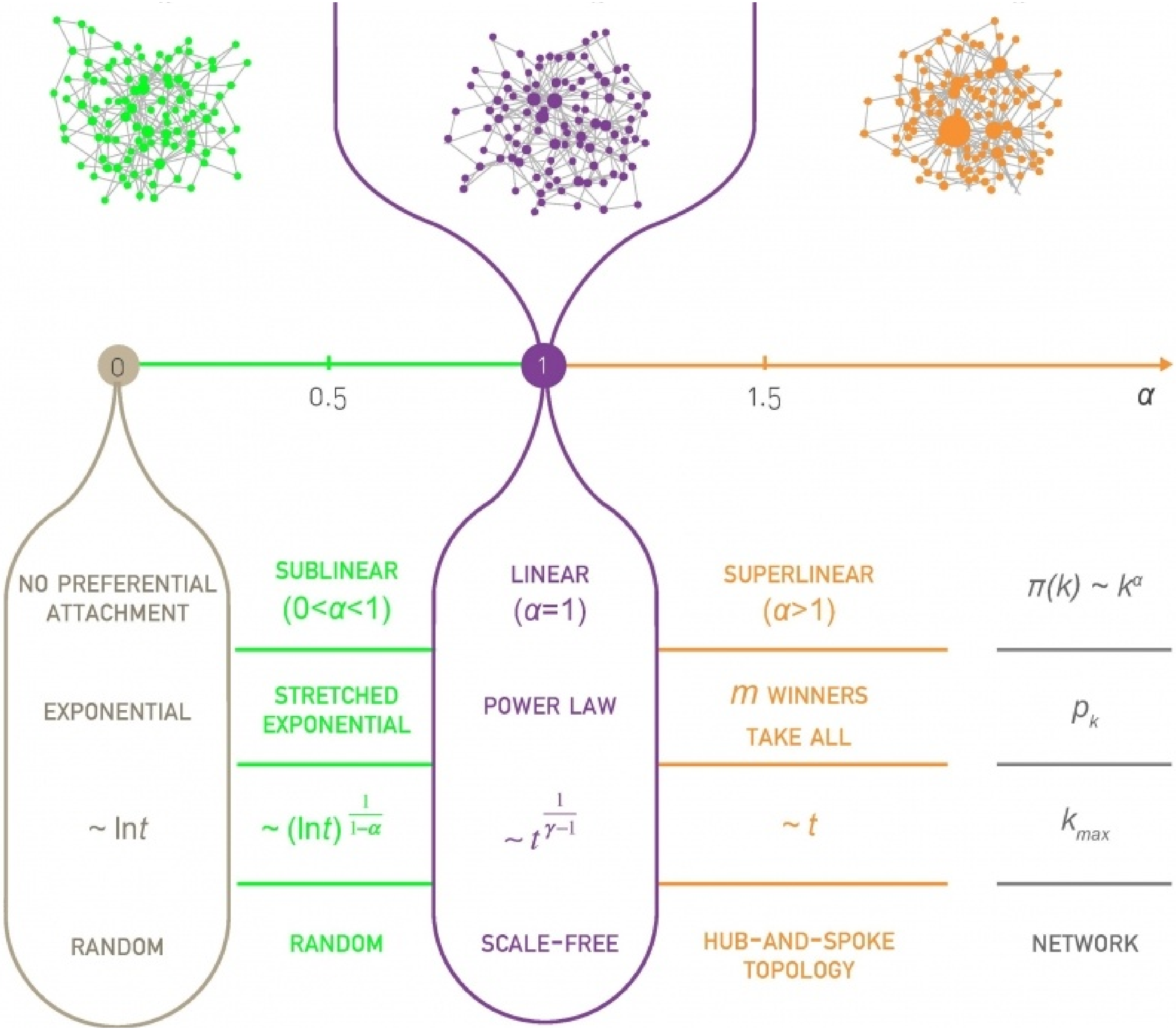
- The model we have studied so far has **linear preferential attachment** because $$\frac{d}{dt}k_i \propto k_i$$
- We could imagine cases where $$\frac{d}{dt}k_i \propto k_i^{\alpha}$$ for $\alpha > 1$ or $\alpha < 1$

$\boxed{\text{What do you think should happen in each case?}}$

The degree of the largest hub k_{max}




SUBLINEAR
LINEAR
SUPERLINEAR
$\alpha = \frac{1}{2}$
$\alpha = 1$
$\alpha = \frac{3}{2}$
p_k
k
0
0.5
1
1.5
α



0
0.5
1
1.5
α
NO PREFERENTIAL ATTACHMENT
SUBLINEAR
$(0<\alpha<1)$
LINEAR
$(\alpha=1)$
SUPERLINEAR
$(\alpha>1)$
$\pi(k) \sim k^{\alpha}$
EXPONENTIAL
STRETCHED EXPONENTIAL
POWER LAW
m WINNERS TAKE ALL
p_k
$\sim \ln t$
$\sim (\ln t)^{\frac{1}{1-\alpha}}$
$\sim t^{\frac{1}{\gamma-1}}$
$\sim t$
k_{max}
RANDOM
RANDOM
SCALE-FREE
HUB-AND-SPOKE TOPOLOGY
NETWORK

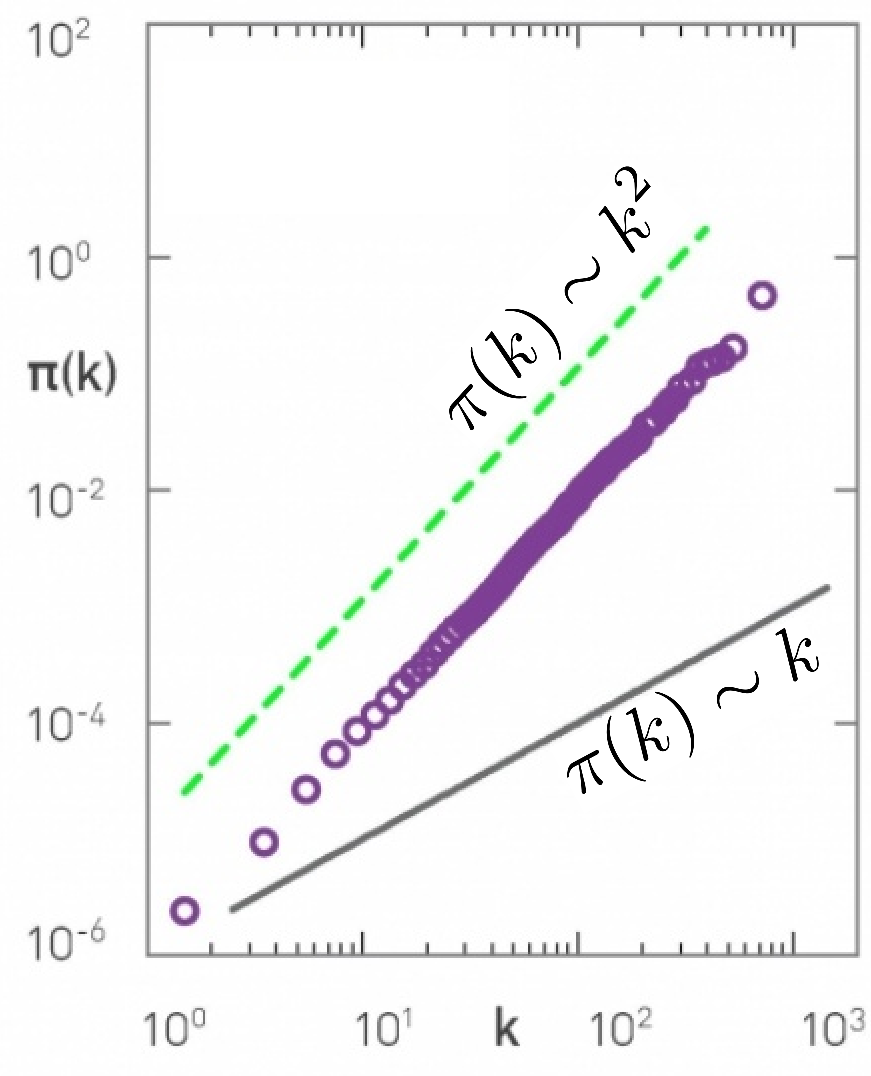
Measuring preferential attachment

Measuring preferential attachment

- We should try to measure $$\Pi(k_i) \approx \frac{\Delta k_i}{\Delta t}$$
- This can be too noisy
  - Why?
- Instead we will measure $$\pi(k) = \sum_{k_i=0}^{k} \Pi(k_i)$$
- If $\Pi(k_i)$ is constant $\pi(k) \propto k$
- If $\Pi(k_i) \propto k$ then $\pi(k) \propto k^2$

Preferential attachment in a citation network

- We observe it follows preferential attachment (with $\alpha = 1$) in this case



Aging effects

Sick Boy's unified theory of life from *Trainspotting* (1996)

In English: https://www.youtube.com/watch?v=pQD-dXfHrvk
In Spanish: https://www.youtube.com/watch?v=cN_WbiuqyQU
English (bad audio) subs in Spanish: https://www.youtube.com/watch?v=4xTWD9GNRFA

Aging effects

- Models without fitness but with a negative effect of age

$$\Pi(k_i, t - t_i) \approx k_i (t - t_i)^{-v}$$

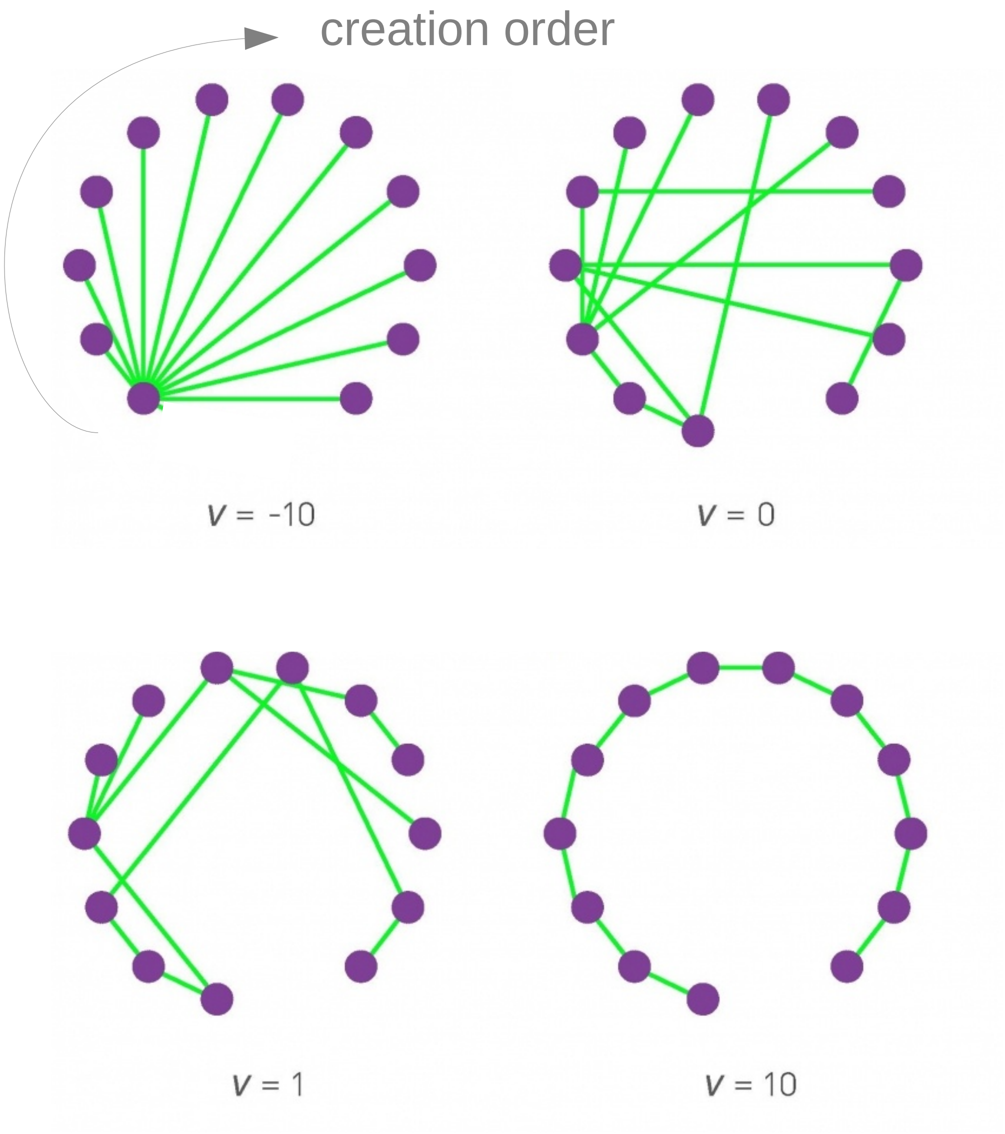
- Older nodes accumulate links more slowly
- Parameter v is the decay factor

Qualitatively, what would you expect if:

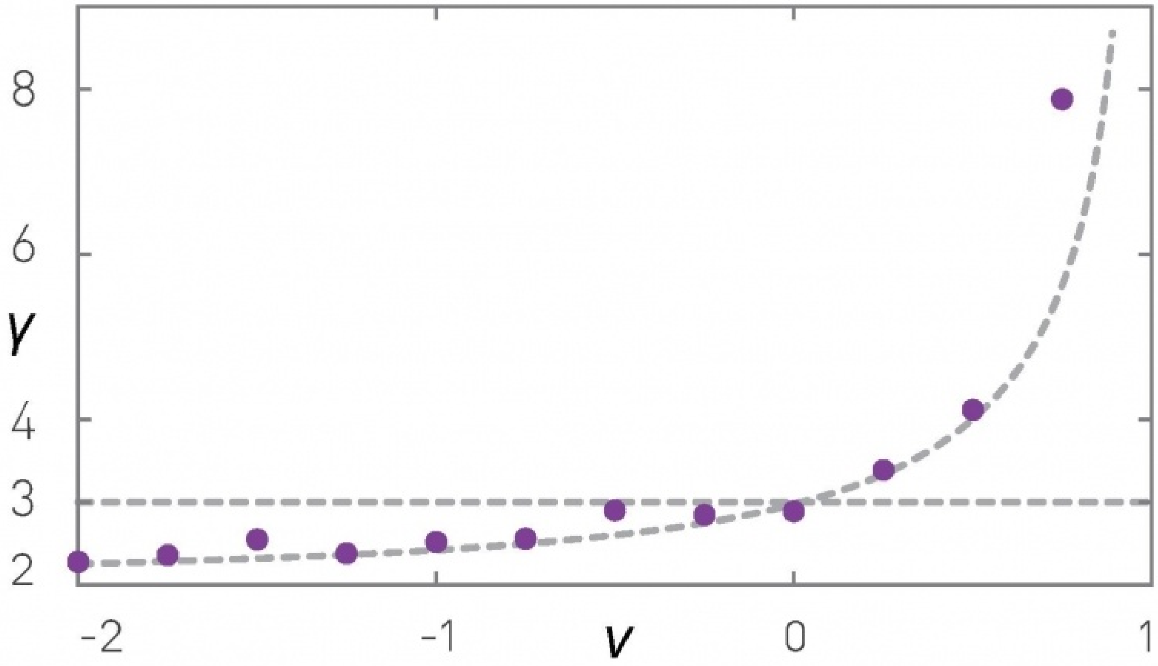
$$v < 0 \qquad v = 0 \qquad v \approx 1 \qquad v \gg 1$$

Aging effects

- $v < 0$ favors older nodes
- $v = 0$ is simply preferential attachment
- $v \gg 1$ means only youngest are linked



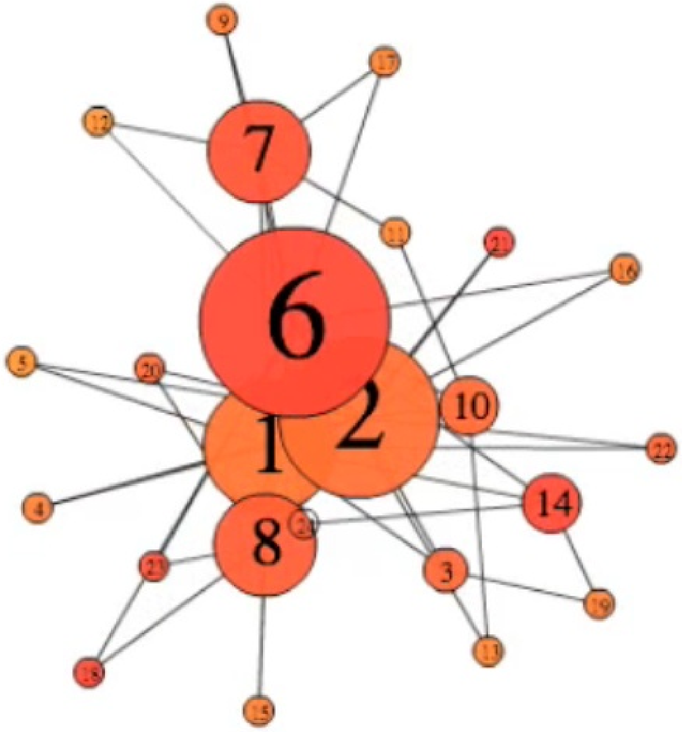
Power-law exponent in models with aging (N=10K, m=1)



More heterogeneous ⟷ More homogeneous

“Good get richer” (incl. Bianconi-Barabási model)

“Good get richer” simulation (number is attractiveness)



http://networksciencebook.com/images/ch-06/video-6-1.mov

"Good get richer"

- A "good get richer" model is one where
  - Each node has an "attractiveness" (called "fitness")
  - Preferential attachment is guided by this fitness

$$\eta_i$$

- The probability of $\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}$ o node i is:

Degree dynamics

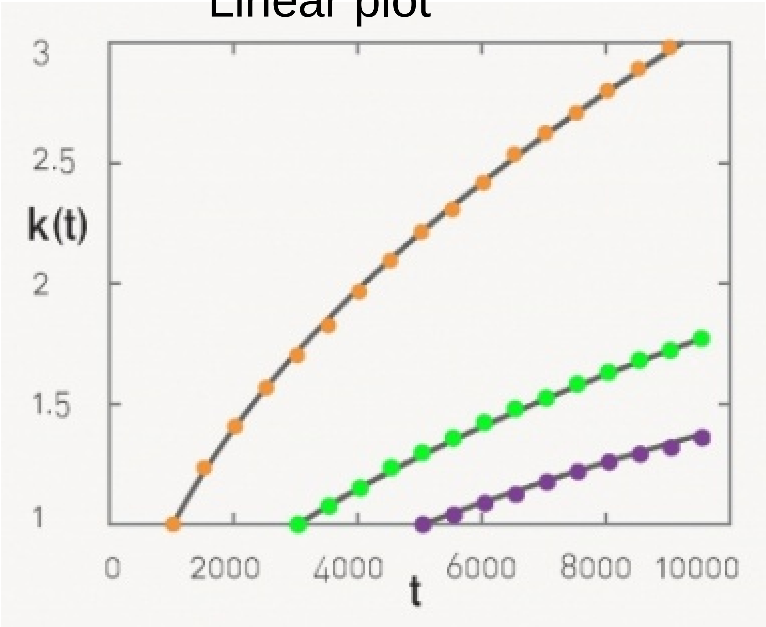
$$\frac{d}{dt}k_i = m\frac{\eta_i k_i}{\sum_j \eta_j k_j}$$

$$k_i(t; t_i, \eta_i) = m\left(\frac{t}{t_i}\right)^{\beta(\eta_i)}$$

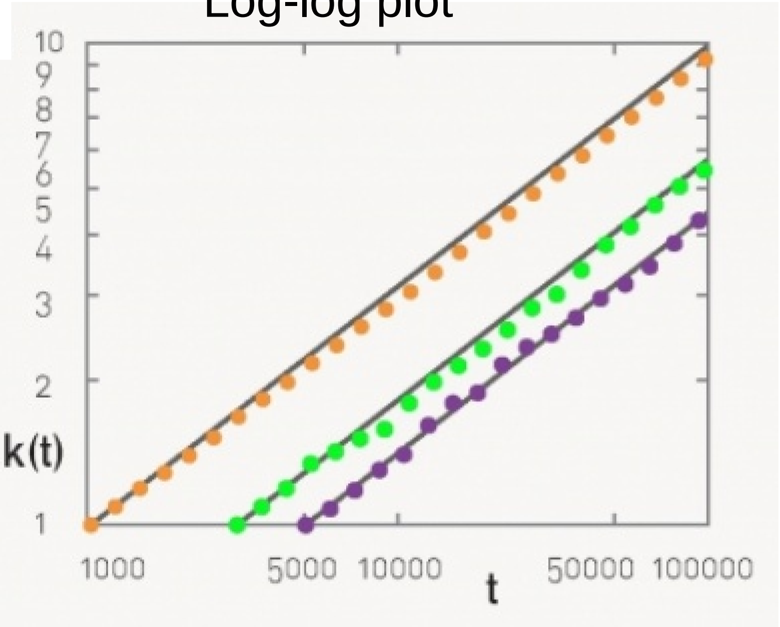
- With the dynamic exponent $\beta(\eta_i) \propto \eta_i$
- Remember that in linear preferential attachment $\beta = 1/2$ (for all nodes)

In preferential attachment (BA) a "younger" node cannot overtake an "older" node

Linear plot

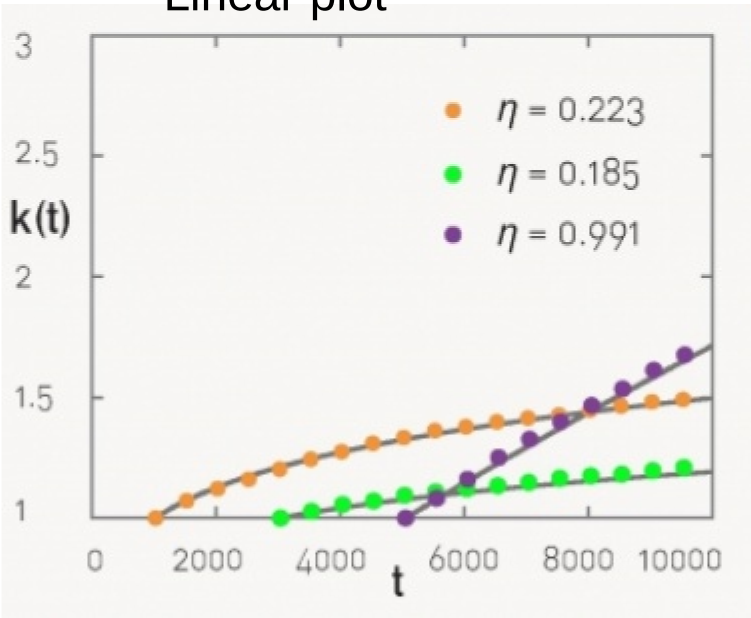


Log-log plot

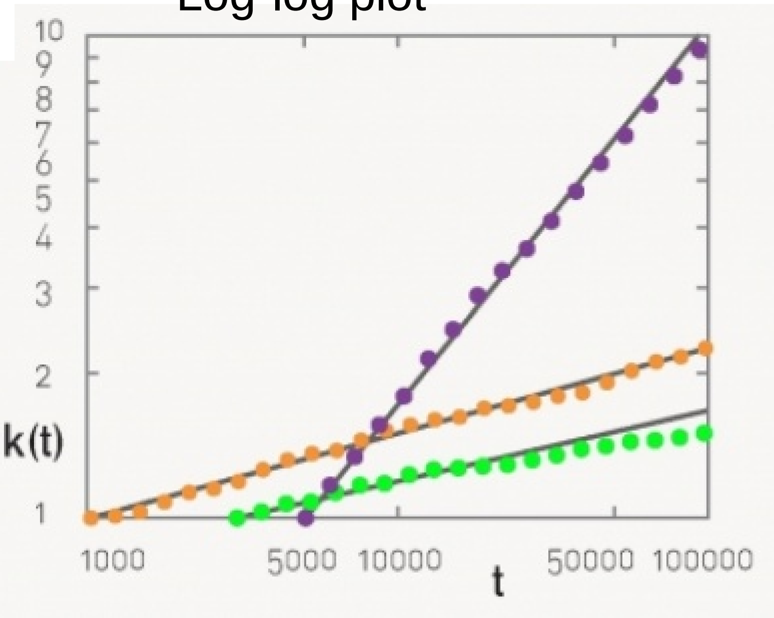


In good-get-richer (Bianconi-Barabási) this depends on node fitness

Linear plot



Log-log plot



Degree distribution

$$p_k \propto \int \frac{\rho(\eta)}{\eta} \left(\frac{m}{k}\right)^{\frac{c}{\eta}+1} d\eta \qquad\qquad \eta \sim \rho(\eta)$$

- When η is constant this reduces to BA
- When η is uniformly distributed in [0, 1] this also yields a power law but instead of $\gamma = 3$ we get $\gamma \approx 2.3$

Which distribution is more heterogeneous?

Link selection model / copy model

Other processes that generate scale-free networks

- Link-selection model — step:
  - Add one new node v to the network
  - Select an existing link (u,w) at random and connect v to either u or w
- Copy model — step:
  - Add one new node v to the network
  - Pick a random existing node u
  - With probability p link to u
  - With probability $1\text{-}p$ link to a neighbor of u

Exercise: the copy model

In the copy model, start at $t=1$ with one node, and at every step t :

- Add one new node v to the network
- Pick a random existing node u
- If u has no out-links, link to u
- If u has out-links choose one of the following:
  - With probability p link to u
  - With probability $1\text{-}p$ link to one of the out-neighbors of u chosen at random

- Simulate it on paper (directed graph) for 7 nodes with $p=0.5$
  - Make sure you understand the model fully!
- What is $N(t)$ and $L(t)$? What is k_i^{out}?

Degree distribution in the copy model

Proven in the paper by Kumar et al. (FOCS 2000) “Stochastic models for the web graph” and developed in the advanced materials.

$$\gamma = \frac{2-p}{1-p} \in [2,3] \quad \text{if } p \in [0, 1/2]$$

The copy model can generate any exponent between 2 and 3!

In the copy model, at every step t:

1) Add one new node v to the network
2) Pick a random existing node u
3) With probability p link to u
4) With probability 1-p link to a neighbor of u

- We will compute k_i^{in} but first ...
- How many links on average gets node i at time t? In other words, what is:

$$\frac{d}{dt}k_i^{\text{in}}(t)$$

- Hint: it has a term with p and a term with *1-p*

- Integrate between t_i and t to obtain an expression for $k_i(t_i)$

  (***we drop the "in" superscript just for simplicity during this exercise***)
- Note that now $k_i(t_i) = 0$

- Once you have a expression for $k_i(t_i)$
- Compute $Pr(k_i(t_i) > k)$
- Now write the cumulative distribution function of $k_i(t_i)$
- And compute its derivative to obtain

$$p_k = Pr(k_i(t) = k) = \frac{d}{dk} Pr(k_i(t) \leq k)$$

- It should show exponent $\gamma = \dfrac{2-p}{1-p}$

Summary

Things to remember

- Uniform attachment
- Sub-linear and super-linear preferential attachment
- Measuring preferential attachment
- “Good-get-richer” and aging effects
- The copy model

Practice on your own

- Practice creating graphs using the different models
  - By hand
  - Or write your own code (it's not a lot of code)

Advanced materials: expected degree under uniform random attachment

Expected degree in uniform random attachment using a differential equation

$$\frac{d}{dt}k_i(t) = \frac{m}{t}$$

Obtain $\boldsymbol{k_i}$

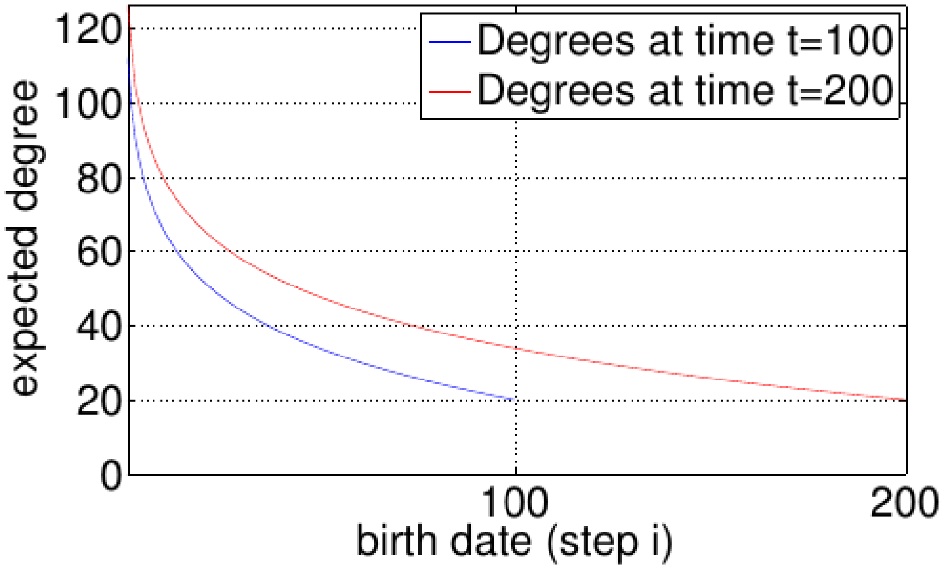
(1) Integrate between time i and time t

(2) Use initial condition $k_i(i) = m$

$$\int \frac{1}{t} = \log t + C$$

Degree distribution over time is not static

Degree of node born at time $m < i < t = m\left(1 + \log\left(\frac{t}{i}\right)\right)$



Tail of degree distribution

How many nodes of degree larger than K are there
at time t?

$$m\left(1+\log\left(\frac{t}{i}\right)\right) > K$$

$$1+\log\left(\frac{t}{i}\right) > \frac{K}{m}$$

$$\log\left(\frac{t}{i}\right) > \frac{K-m}{m}$$

$$\frac{t}{i} > e^{\frac{K-m}{m}}$$

$$i < te^{-\frac{K-m}{m}}$$

The fraction is $$\frac{te^{-\frac{K-m}{m}}}{t} = e^{-\frac{K-m}{m}}$$

Decreases exponentially with K: it's vanishingly rare to find high-degree nodes

Advanced materials:

(1) No preference (2) No growth

Remember preferential attachment

- Start with m_0 nodes
- At every time step
  - Add one new node u
  - Repeat m times
    - Pick a node v with probability
    - Connect u to v

$$\Pi(k_v) = \frac{k_v}{\sum_j k_j}$$

Two simple variants

- No preference
  - Nodes receiving inlinks are picked uniformly at random
- No growth
  - The network starts with N nodes
  - No new nodes are created

No preference model

- Write the process on paper
- Write $\Pi(k_i)$
- Noting that $\frac{d}{dt}k_i = m\Pi(k_i)$ obtain $k_i(t)$

$$\int \frac{a}{b+x} = a\log(b+x) + C$$

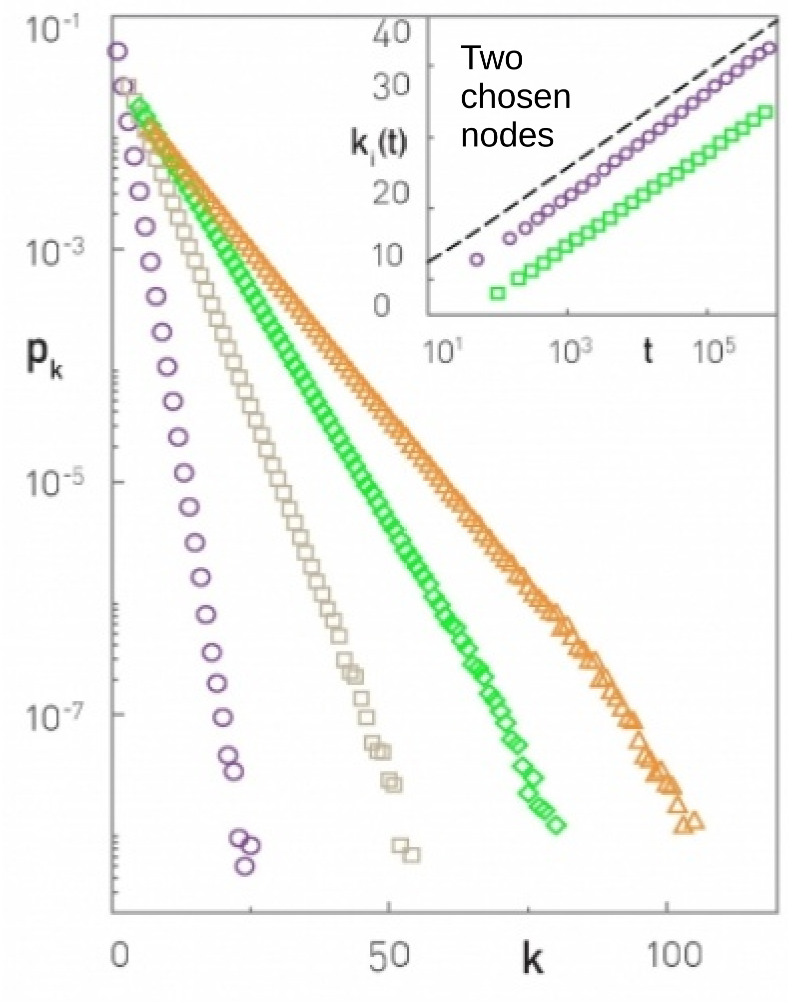
No preference model (cont.)

- Compute $Pr(k_i(t) > k)$ assuming large t, t_i
- Use it to compute

$$Pr(k_i(t) \leq k) = 1 - Pr(k_i(t) > k)$$

$$p_k = Pr(k_i(t) = k)$$

- Derive to obtain



m=1, m=3, m=5, m=7

Consequences of the "no preference" model

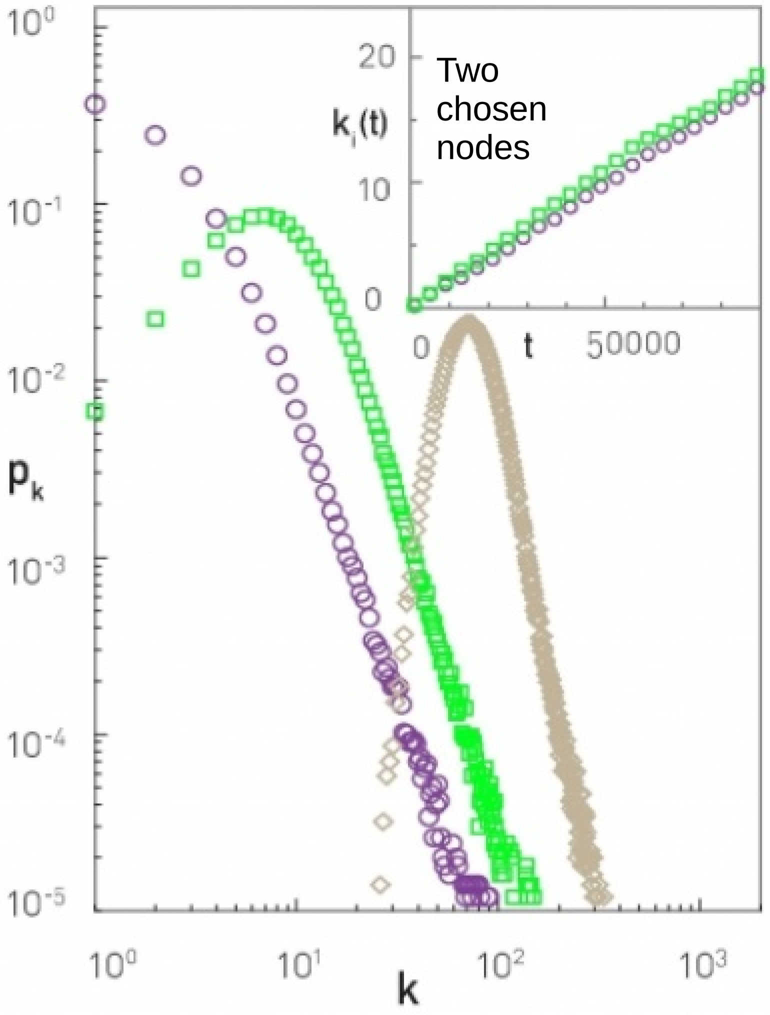
- Degree decays exponentially

$$p_k \propto e^{-k/m}$$

- No power-law
- No large hubs

No growth model

- Write the process on paper
- You will need to impose $k_i(t_i) \neq 0$ why?
- Write $\Pi(k_i)$
- Noting that $\frac{d}{dt}k_i = \Pi(k_i)$ obtain $k_i(t)$



N=100K t=N, t=5N, t=40N

Consequences of the "no growth" model

- Degree grows linearly $k_i(t) \propto t$
- Degree distribution is not stationary