

# PageRank

## Social Networks Analysis and Graph Algorithms

Prof. Carlos “ChaTo” Castillo — <https://chato.cl/teach>



Universitat  
Pompeu Fabra  
*Barcelona*

# Sources

- D. Easley and J. Kleinberg (2010): Networks, Crowds, and Markets – Chapter 14
- Fei Li's lecture on PageRank (2011)
- Evimaria Terzi's lecture on link analysis (2013)
- URLs in the footer of specific slides





Back to the 1990s ...



# The early days of the web

- March 1989: proposal by Tim Berners-Lee at CERN
- Early 1993: NCSA Mosaic graphical browser
- Jan 1994: Yahoo! Web directory (manual)
- 1994: WebCrawler, Lycos (automated, crawlers)
- End of 1994: the web has about 10,000 sites
- 1995-1996: Altavista, Inktomi, and many others ...

# Backrub!

Part of a research project that started in 1995 ...



Search The Web (type only necessary words):

10 results

clustering on

Search

Current Repository Size: ~25 million pages (searchable index slightly smaller)

## [Research Papers about Google and the WebBase](#)

## Credits

Current Development: [Sergey Brin](#) and [Larry Page](#)

Design and Implementation Assistance: [Scott Hassan](#) and [Alan Sterenberg](#)

Faculty Guidance: [Hector Garcia-Molina](#), [Rajeev Motwani](#), [Jeffrey D. Ullman](#), and [Terry Winograd](#)

Equipment Donations: [IBM](#), [Intel](#), and [Sun](#)

Software: [GNU](#), [Linux](#), and [Python](#)

Collaborating Groups in the [Computer Science Department](#) at [Stanford University](#): [The Digital Libraries Project](#), [The Project on People Computers and Design](#), [The Database Group](#), [The MIDAS Data Mining Group](#), and [The Theory Division](#)

Outside Collaborators: [Interval Research Corporation](#) and the [IBM Almaden Research Center](#)

Technical Assistance: [The Computer Science Department's Computer Facilities Group](#), [Stanford's Distributed Computing](#) and [Intra-Networking Systems Group](#)

Note: Google is research in progress and there are only a few of us so expect some downtimes and malfunctions. This system used to be called Backrub.

New! Wonder what your search runs on? Here are some [pictures and stats](#) for the Google Hardware.

1. This new index contains only a very limited number of international pages because we do not want to congest busy international links.
2. When no documents match your query, the system will return 20000 random web pages.
3. For improved speed, try to avoid common words unless they are necessary, and use as few search terms as possible.

Before emailing a question please read the [FAQ](#). Thanks! We can be reached at [google@google.stanford.edu](mailto:google@google.stanford.edu) and we appreciate your comments.

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# PageRank

The **PageRank** citation ranking: Bringing order to the web.

[\[link\]](#)

L Page, S Brin, R **Motwani**, T Winograd - 1999 - [ilpubs.stanford.edu](#)

... We compare **PageRank** to an idealized random Web surfer. We show how to efficiently compute **PageRank** for large numbers of pages. And, we show how to apply **PageRank** to search ...

★ Save  Cite Cited by 16682 Related articles All 16 versions 

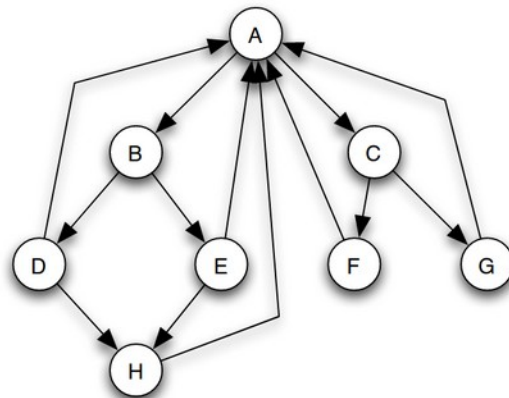
- Today, PageRank and its variants are probably part of most ranking systems in linked collections of data
- $\text{Relevance} = \text{links} + \text{content} + \text{interactions} + \dots$

# Simplified PageRank



# (Simplified) PageRank

- All nodes start with score  $1/N$
- Repeat  $t$  times:
  - Divide equally and “send” its score to out-links
  - Add received scores



# Exercise

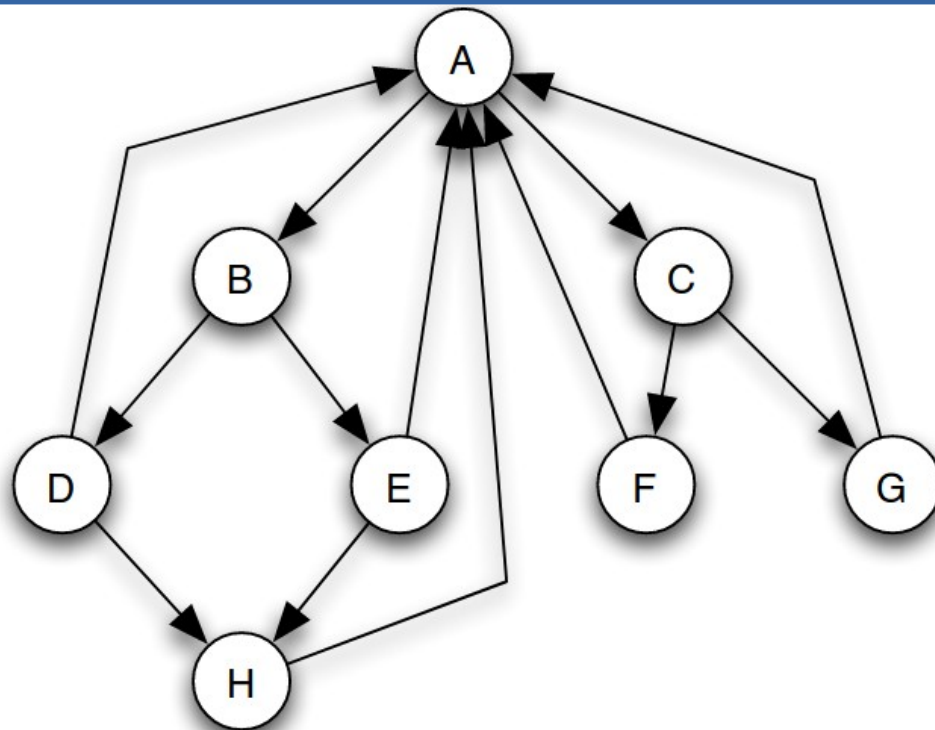
## Execute simplified PageRank

All nodes start with score  $1/N$

- Repeat  $t$  times:
  - Divide equally and “send” the score of each node to out-links
  - Add received scores

Keep intermediate values in a table

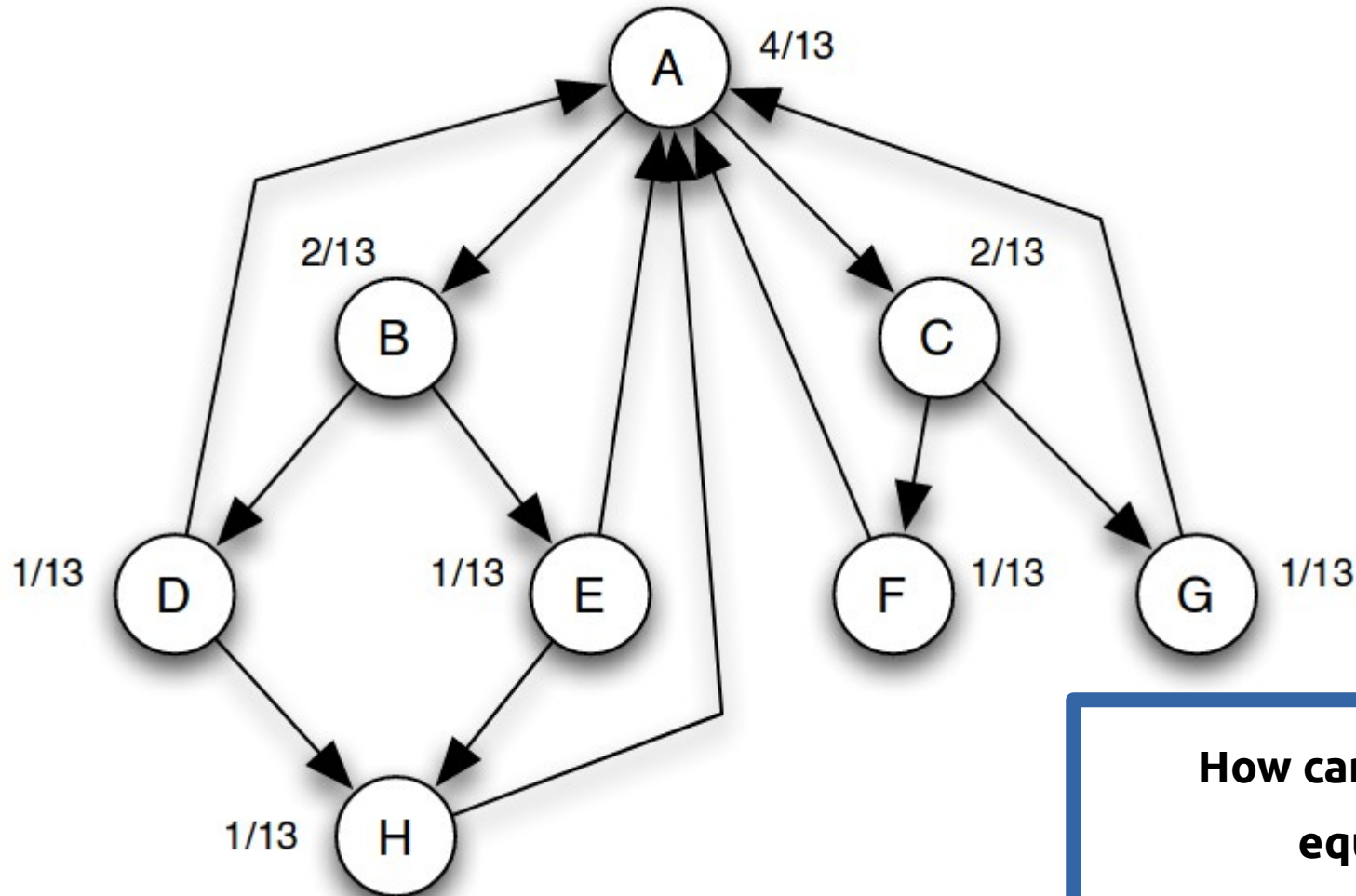
Try to arrive to equilibrium values



Spreadsheet links: <https://upfbarcelona.padlet.org/chato/shyq9m6f2g2dh1bw>



# Equilibrium values



**How can you prove these are  
equilibrium values?**

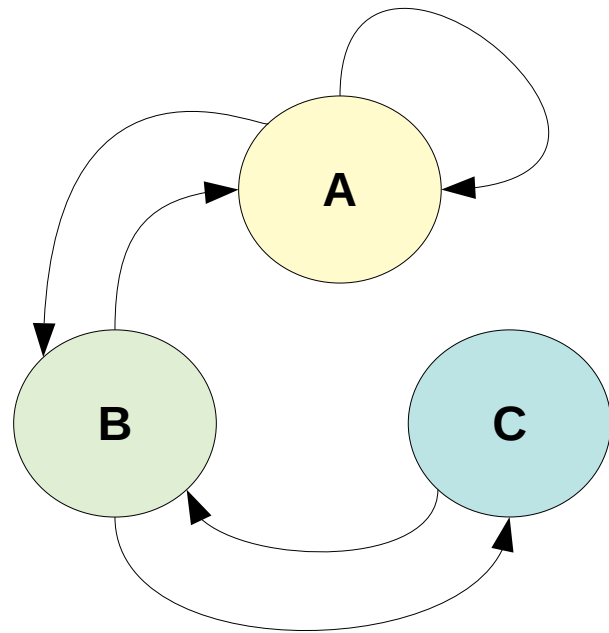
# (Simplified) PageRank

$$P_i = c \sum_{j \rightarrow i} \frac{P_j}{k_j^{\text{out}}}$$

- $k_j^{\text{out}}$  is the number of out-links of page  $j$
- $c$  is a normalization factor to ensure

$$||P||_1 = |P_1| + |P_2| + \cdots + |P_N| = 1$$

# Running simplified PageRank on a graph

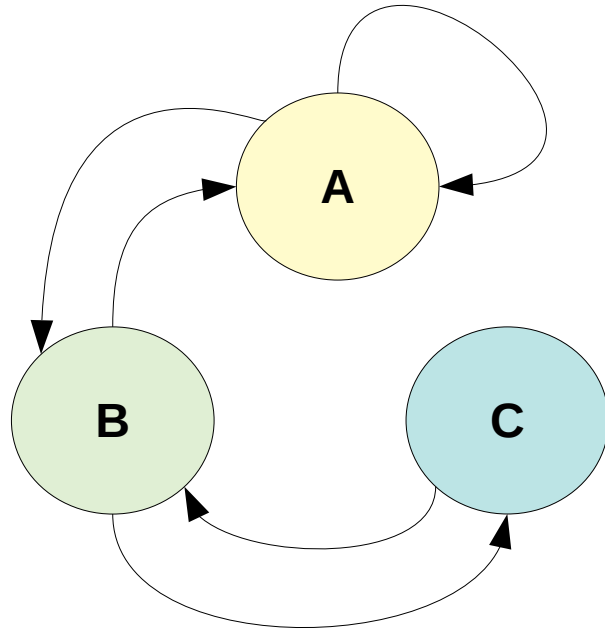


$$M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{Adjacency matrix}$$

$$\hat{M} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{Row-stochastic adjacency matrix}$$



# Another example of Simplified PageRank

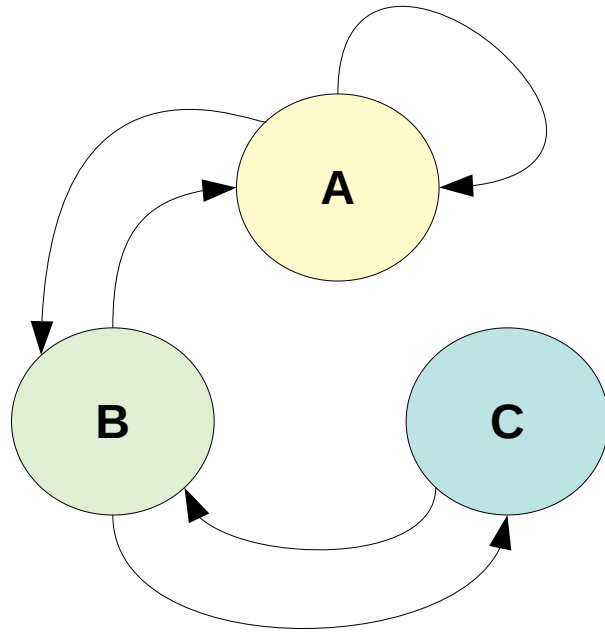


$$\hat{M}^T = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix}$$

First iteration of calculation:

$$\begin{bmatrix} 1/3 \\ 1/2 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

# Another example of Simplified PageRank

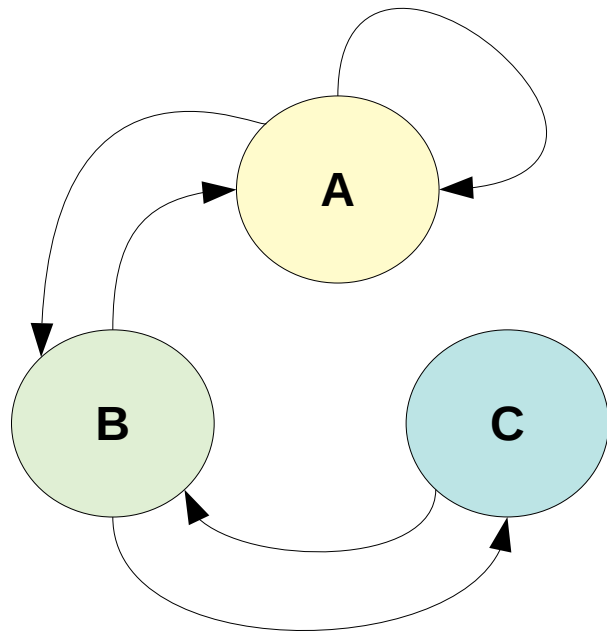


$$\hat{M}^T = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix}$$

Second iteration:

$$\begin{bmatrix} 5/12 \\ 1/3 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/2 \\ 1/6 \end{bmatrix}$$

# Another example of Simplified PageRank

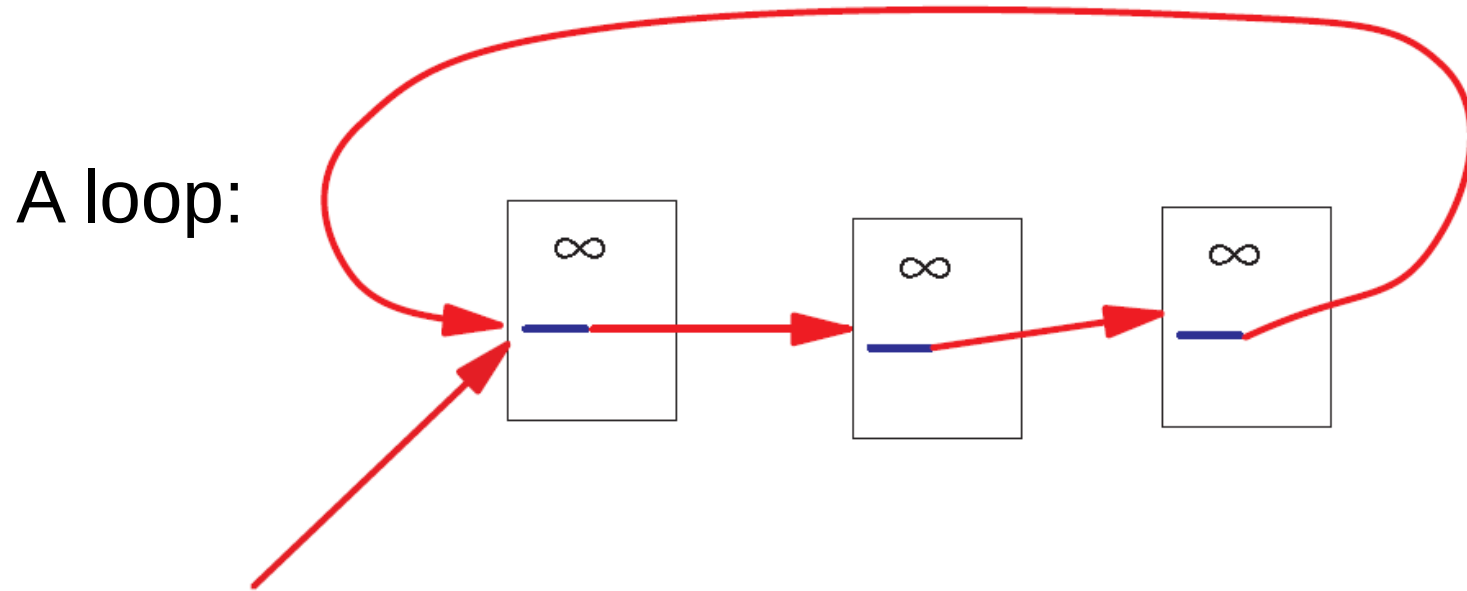


$$\hat{M}^T = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix}$$

Following iterations:

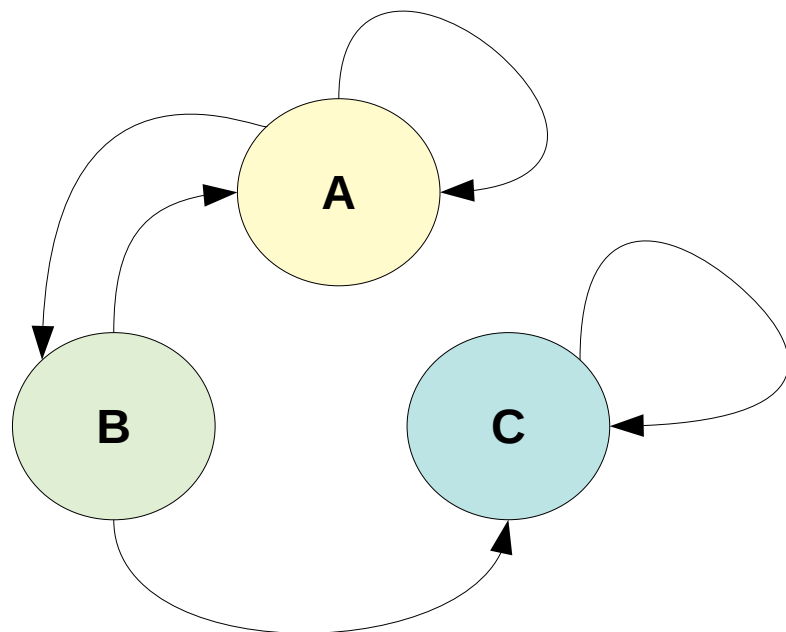
$\begin{bmatrix} 3/8 \\ 11/24 \\ 1/6 \end{bmatrix}$	$\begin{bmatrix} 5/12 \\ 17/48 \\ 11/48 \end{bmatrix}$	...	$\begin{bmatrix} 2/5 \\ 2/5 \\ 1/5 \end{bmatrix}$	Final score
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# A Problem with Simplified PageRank



During each iteration, the loop accumulates score but never distributes score to other pages!

# Example of the problem ...



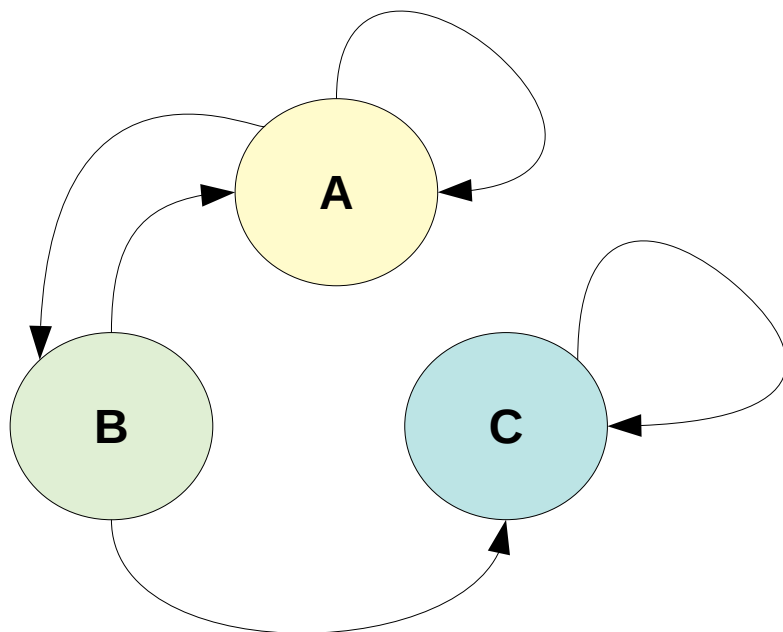
$$\hat{M}^T = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}^*$$

First iteration of calculation:

$$\begin{bmatrix} 1/3 \\ 1/6 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}^*$$



# Example of the problem ...

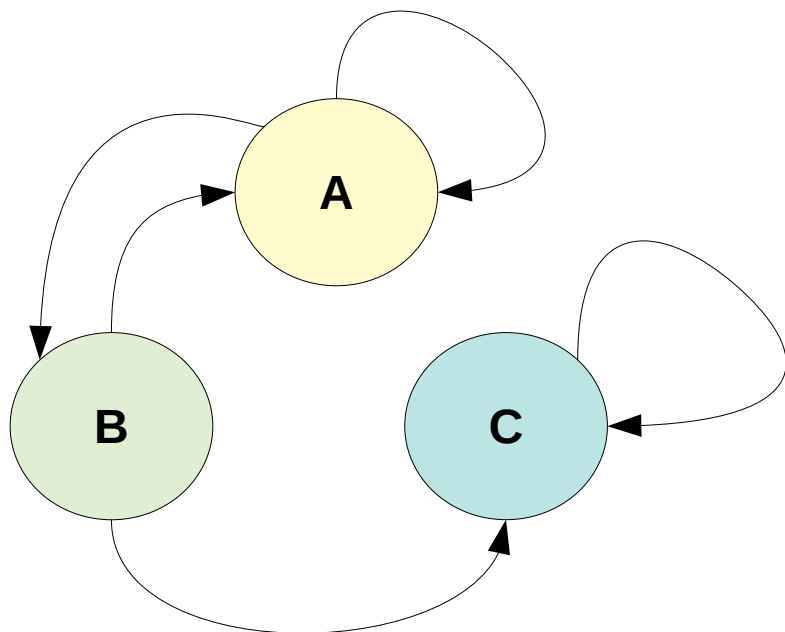


$$\hat{M}^T = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}^*$$

Second iteration:

$$\begin{bmatrix} 1/4 \\ 1/6 \\ 7/12 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/6 \\ 1/2 \end{bmatrix}^*$$

# Example of the problem ...



$$\hat{M}^T = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}^*$$

Following iterations:

$$\begin{bmatrix} 5/24 \\ 1/8 \\ 2/3 \end{bmatrix} \begin{bmatrix} 1/6 \\ 5/48 \\ 35/48 \end{bmatrix} \dots \boxed{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

The winner takes all!

Why is PageRank also referred to as  
“Eigen” centrality

# What are we computing?

$$p^t = Ap^{t-1}$$

after convergence :  $p = Ap$

A is the transposed row-stochastic adjacency matrix

What is p?

How do you call this method to compute p?

# What are we computing?

$$p^t = Ap^{t-1}$$

$$\text{after convergence : } p = Ap$$

- This will converge if A is:
  - **Left-stochastic** (each column adds up to one)
  - **Irreducible** (represents a strongly connected graph)
  - **Aperiodic** (does not represent a bipartite graph)



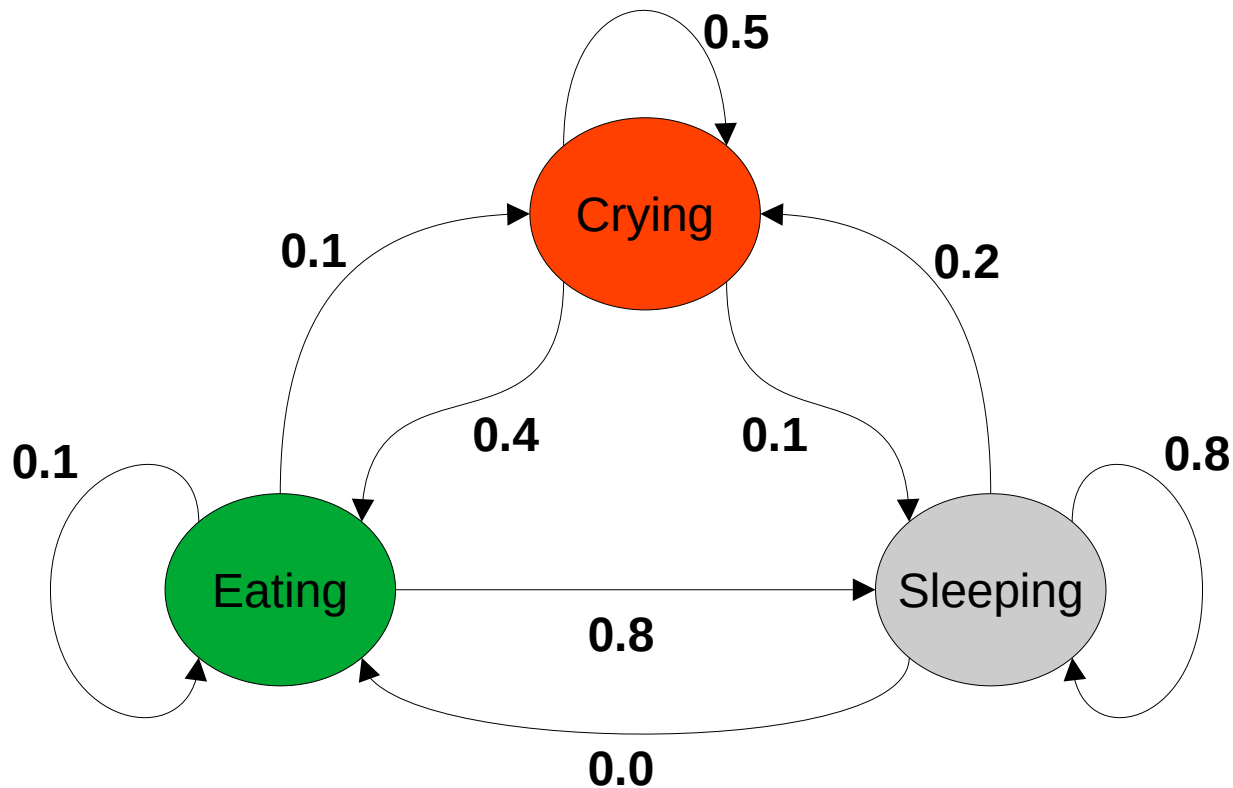
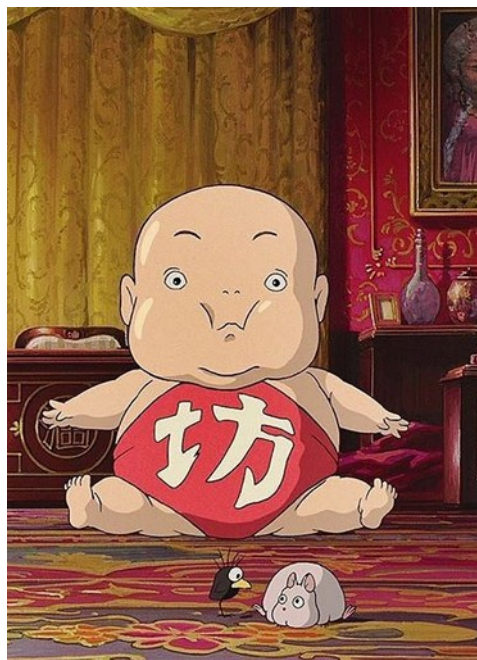
# **“Random walk” interpretation**

# Markov Chain

- Discrete process over a set of **states**
- Next state computed from **current state** only (**no memory** of older states)
  - Higher-order Markov chains can be defined
- **Stationary distribution** of Markov chain is a probability distribution such that  $p = Ap$
- Intuitively,  $p$  represents “**the average time spent**” at each node if the process continues forever

# Example Markov Chain: a baby

(think of 1-hour time steps)



# Random Walks in Graphs

- **Random Surfer Model** → Simplified PageRank
  - The simplified model: the standing probability distribution of a random walk on the graph of the web. simply keeps clicking successive links at random
- **Modified Random Surfer** → PageRank
  - The modified model: the “random surfer” simply keeps clicking successive links at random, but periodically “gets bored” and jumps to a random page based on a distribution  $R$  (e.g., uniform)
  - This guarantees **irreducibility**
  - Pages without out-links (dangling nodes) are a row of zeros, can be replaced by  $R$ , or by a row of  $1/N$

# PageRank

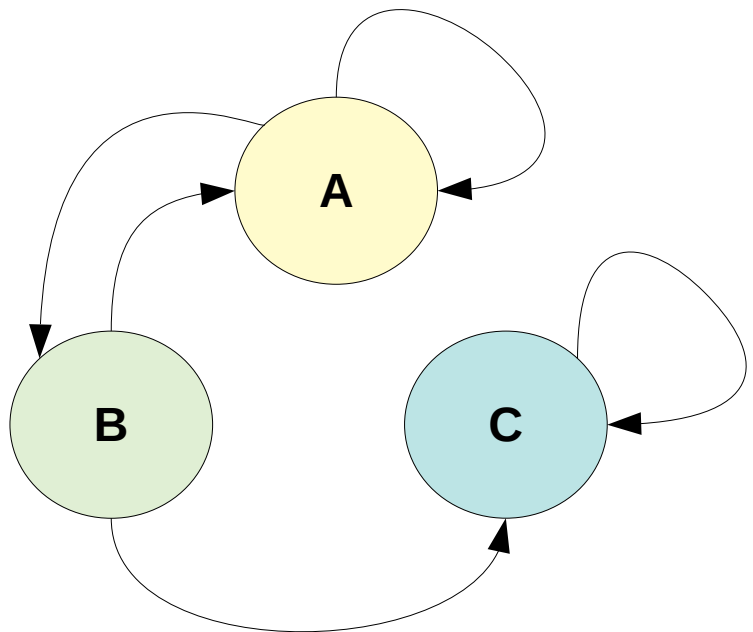
$$P_i = \alpha \sum_{j \rightarrow i} \frac{P_j}{k_j^{\text{out}}} + (1 - \alpha)R(i)$$

$R(i)$ : web pages that “users” jump to when they “get bored”;  
Uniform preferences  $\Rightarrow R(i) = 1/N$



# An example of PageRank

$$\alpha = 0.8$$



$$\hat{M}^T = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}$$

$$\alpha \hat{M}^T + (1 - \alpha)R = 0.8 \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} + 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\begin{bmatrix} 0.333 \\ 0.333 \\ 0.333 \end{bmatrix}$$

$$\begin{bmatrix} 0.333 \\ 0.200 \\ 0.467 \end{bmatrix}$$

$$\begin{bmatrix} 0.280 \\ 0.200 \\ 0.520 \end{bmatrix}$$

$$\begin{bmatrix} 0.259 \\ 0.179 \\ 0.563 \end{bmatrix}$$

...

$$\begin{bmatrix} 7/33 \\ 5/33 \\ 21/33 \end{bmatrix}$$

Was:  $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

# Summary

# Things to remember

- Simplified PageRank
- PageRank

# Practice on your own

- Consider a directed graph  $G = (V, E)$  in which  $V = \{1, 2, \dots, N\}$  and  $(i, j) \in E \iff i \in V \wedge j \in V \wedge (j = i + 1 \vee j = i = N)$ 
  - 1. Indicate the value of Simplified PageRank  $S(i)$  for each node  $i$  in the graph, justifying your answer.
  - 2. Indicate the value of PageRank  $P(i)$  for each node  $i$  in the graph as a function of  $i$  and the parameter  $\alpha$ .
- Tip: write  $P(1)$ , then write  $P(2)$ , then write  $P(3)$ , then write  $P(i)$ .