Spectral Graph Embedding

Social Networks Analysis and Graph Algorithms

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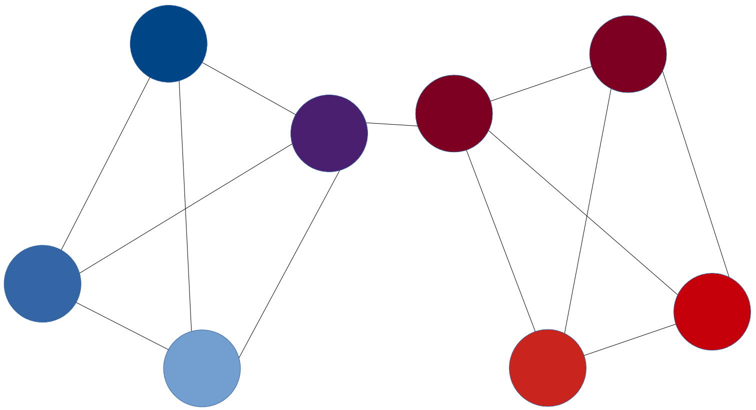
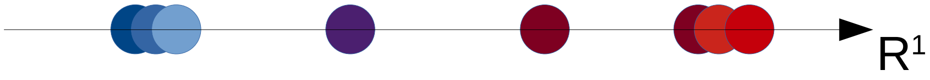
Sources

- **J. Leskovec (2016). Defining the graph laplacian** [video]
- E. Terzi (2013). Graph cuts — The part on spectral graph partitioning
- D. A. Spielman (2009): The Laplacian
- URLs cited in the footer of slides

Graphs are nice, but ...

- They describe only local relationships
- We would like to understand a global structure
- We will try to transform a graph into a more familiar object: a cloud of points in R^k





Graphs are nice, but ...

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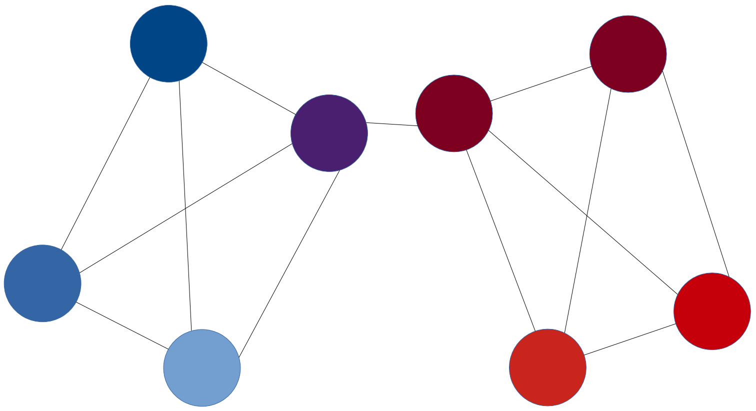
- We will try to transform a graph into a more familiar object: a cloud of points in R^k



Distances should be somehow preserved

What is a graph embedding?

- A graph **embedding** (or graph **projection**) is a mapping from a graph to a vector space
- If the vector space is $\mathbb{R}^2$ you can think of an embedding as a way of ***drawing*** a graph on paper

Exercise: draw this graph

V = {v1, v2, ..., v8}

E = { (v1, v2), (v2, v3), (v3, v4), (v4, v1), (v5, v6), (v6, v7), (v7, v8),
(v8, v5), (v1,v5), (v2, v6), (v3, v7), (v4, v8) }

Draw this graph on paper, upload a photo

What constitutes a good drawing?

Padlet: https://upfbarcelona.padlet.org/chato/3hwnctvqb7p1z6xx

In a good graph embedding ...

- Pairs of nodes that are **connected** to each other should be **close**
- Pairs of nodes that are not connected should be far
- Compromises will need to be made

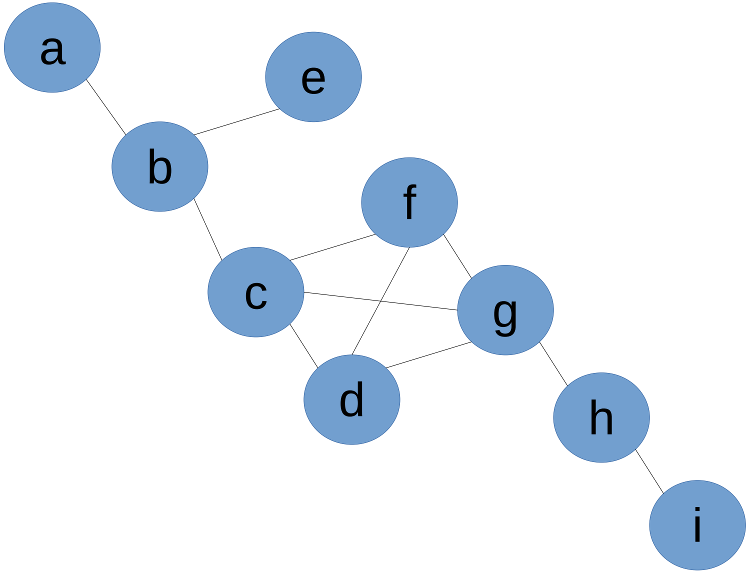
Random projections

Random graph projection (2D)

- Start a BFS from a random node, that has x=1, and nodes visited have ascending x
- Start a BFS from another random node, which has y=1, and nodes visited have ascending y
- Project node i to position (x_i, y_i)

Exercise: random projection

- Given this graph
- Pick a random node u
  - Distances from u are the x positions
- Pick a random node v
  - Distances from v are the y positions
- Draw the graph in an $\mathbb{R}^2$ plane



Padlet: https://upfbarcelona.padlet.org/chato/9pd56scbpko5svdj

Refresher about eigenvectors/eigenvalues

Eigenvectors and eigenvalues

- In general $Av = \lambda v$ means A has an eigenvector v of eigenvalue λ
- In **symmetric** matrices, **eigenvectors are orthogonal**

  Suppose $\lambda_1 \neq \lambda_2$

$$\lambda_1 v_1^T v_2 = v_1^T A v_2 = v_1^T \lambda_2 v_2 = \lambda_2 v_1^T v_2$$

- This implies $v_1^T v_2 = 0$

In symmetric matrices

- The **multiplicity** of an eigenvalue λ is the dimension of the space of eigenvectors of eigenvalue λ
- Every n x n symmetric matrix has n eigenvalues counted with multiplicity
- Hence, it has an orthonormal basis of eigenvectors

Rayleigh quotient

In symmetric matrices M, the second eigenvalue is

$$\lambda_2 = \min_x \frac{x^T M x}{x^T x}$$

https://en.wikipedia.org/wiki/Rayleigh_quotient

Eigenvectors of the adjacency matrix

Properties of adjacency matrix

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- How many non-zeros are in every **row** of A?

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

https://www.youtube.com/watch?v=AR7iFxM-NkA

Adjacency matrix of G=(V,E)

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Can you write $\boldsymbol{y_i}$ using $\boldsymbol{E}$?

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

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Adjacency matrix of G=(V,E)

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- **What is $\boldsymbol{Ax}$?** Think of x as a set of labels/values:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$y_i = \sum_{j:(j,i)\in E} x_j$$

Ax is a vector whose i^{th} coordinate contains the sum of the x_j who are in-neighbors of i

https://www.youtube.com/watch?v=AR7iFxM-NkA

Spectral graph theory ...

- Studies the eigenvalues and eigenvectors of a graph matrix
  - Adjacency matrix $Ax = \lambda x$
  - Laplacian matrix (next)

- Suppose graph is d-regular: $k_i = d \; \forall i$
- What is the value of
- What does that imply?

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = ?$$

An eigenvector of a d-regular graph

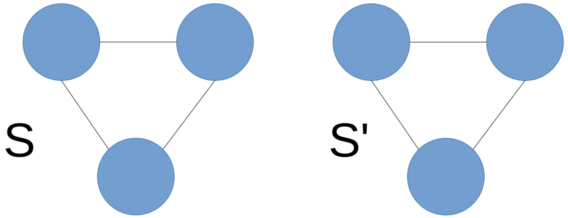
- Suppose graph is d-regular, i.e. all nodes have degree d:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} d \\ d \\ \vdots \\ d \end{bmatrix} = d \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

- **Hence, $[1, 1, \ldots, 1]^T$ is an eigenvector of eigenvalue d**

Disconnected graphs

- Suppose the graph is regular **and disconnected**

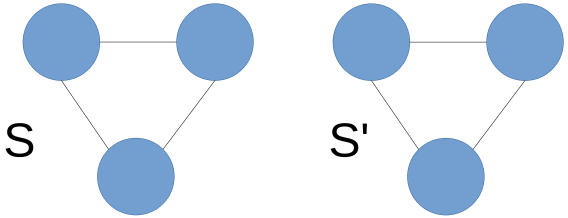


- Then its adjacency matrix has block structure:

$$A = \begin{bmatrix} S & 0 \\ 0 & S' \end{bmatrix}$$

Disconnected graphs

- Suppose the graph is regular **and disconnected**

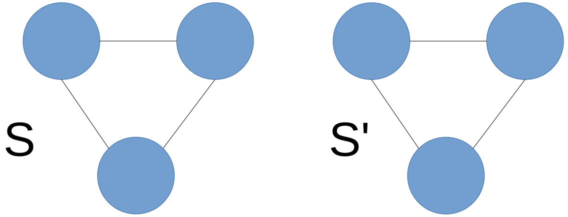


$$\text{Let } x_i^S = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \in S' \end{cases}$$

$$\begin{bmatrix} S & 0 \\ 0 & S' \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = ?$$

Disconnected graphs

- Suppose the graph is regular **and disconnected**



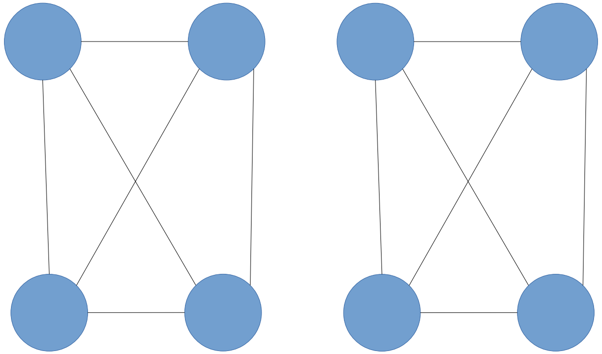
$$Ax^S = dx^S$$

$$Ax^{S'} = dx^{S'}$$

- What is the multiplicity of eigenvalue d?
- What happens if there are more than 2 connected components?

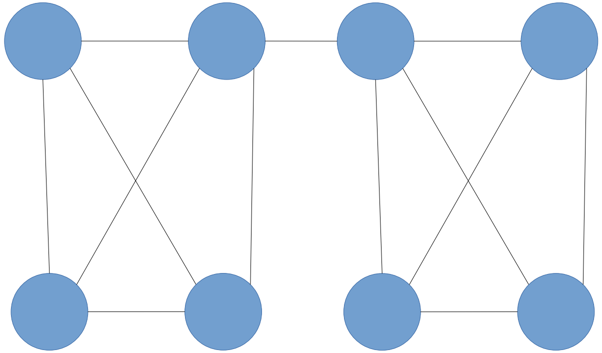
In general

Disconnected graph



$$\lambda_1 = \lambda_2$$

Almost disconnected graph



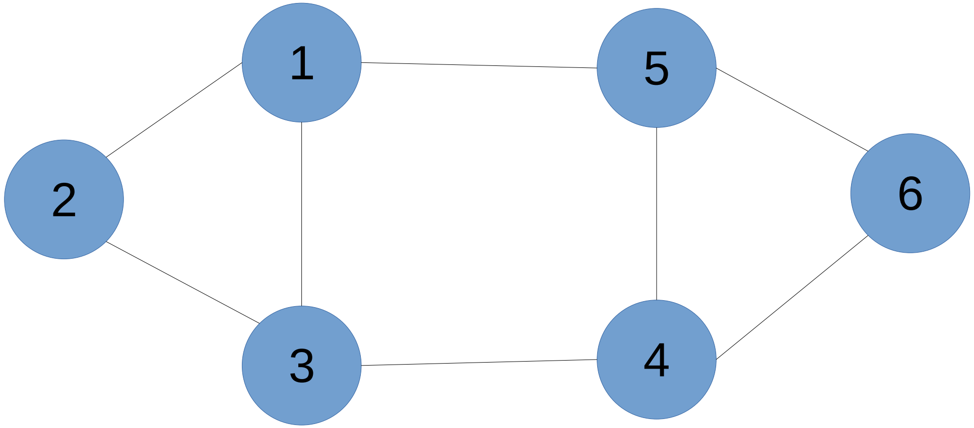
$$\lambda_1 \approx \lambda_2$$

Small "**eigengap**"

Graph Laplacian

Adjacency matrix

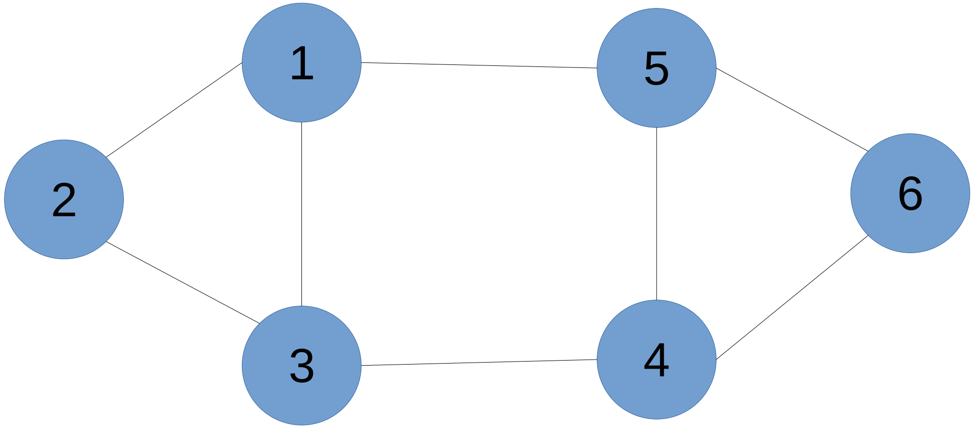
$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Degree matrix

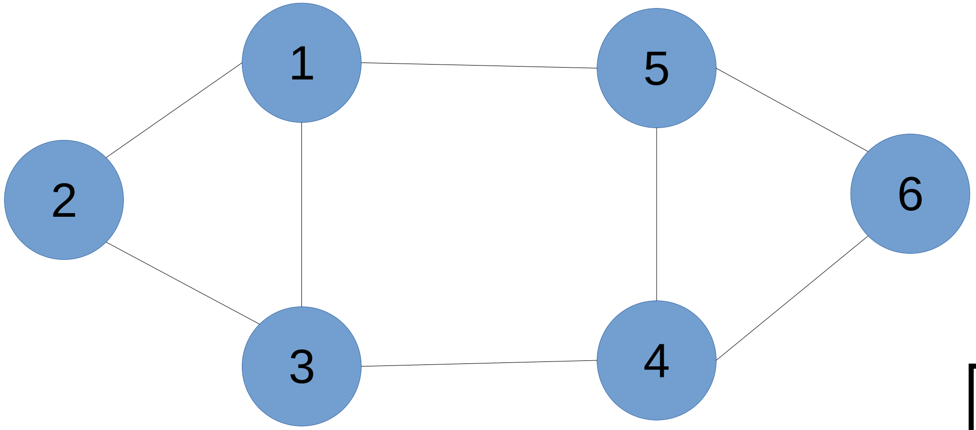
$$D_{ij} = \begin{cases} k_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$



$$D = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Laplacian matrix

$$L = D - A$$



$$L = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Given A is symmetric, **L is symmetric.**

They only differ in the diagonal.

Laplacian matrix $\mathbf{L} = \mathbf{D}$ - $\mathbf{A}$

$$L\vec{1} = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = ?$$

The constant vector is an eigenvector of L

The constant vector $x = [1, 1, \ldots, 1]^T$ is an eigenvector of the Laplacian, and has eigenvalue 0

$$Lx = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Does it need to be this specific graph? Why?

Does it need to be the vector [1, 1, …, 1]? Why?

If the graph is disconnected

- If the graph is disconnected into two components, the same argument as for the adjacency matrix applies, and $\lambda_1 = \lambda_2 = 0$
- The multiplicity of eigenvalue 0 is equal to the number of connected components

$$x^T L x$$

Prove this!

- Prove that $\mathrm{x}^T L x = \sum_{(i,j)\in E}(x_i - x_j)^2$

$$L = D - A$$

$$D_{ij} = \begin{cases} k_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Think of this quantity as the “stress” produced by the assignment of node labels x

One of the eigenvalues of L is 0

- If x is such that $x_i = x_j$ for all i,j:

$$x^T L x = \sum_{(i,j)\in E} (x_i - x_j)^2 = 0 \Rightarrow Lx = 0$$

- This means 0 is an eigenvalue of L

In the Laplacian of a connected graph, the eigenvector x of $\lambda=0$ must be the constant vector

- If x is the eigenvector of eigenvalue 0, $Lx = 0$
- Then $x^T Lx = \sum_{(i,j)\in E} (x_i - x_j)^2 = 0$

From this, we deduct that $x_i = x_j$ for any pair i, j
even if i and j are not directly connected by an edge. Why?

In the Laplacian of a connected graph, the eigenvector x of λ=0 must be the constant vector

- If x is the eigenvector of eigenvalue 0, $Lx = 0$
- Then $x^T L x = \sum_{(i,j)\in E} (x_i - x_j)^2 = 0$

- Hence, for any pair of nodes *(i,j)* connected by an edge, $x_i = x_j$
- Given the graph is connected, there is a path between any two nodes ⇒

  for **any** pair of nodes *(i,j)*, **even the ones not connected by an edge**, $x_i = x_j$
- Hence x is a constant vector

Eigenvalues of the Laplacian are non-negative

- If v is an eigenvector of L of eigenvalue λ:

$$\lambda v^T v = v^T L v = \sum_{(i,j)\in E} (v_i - v_j)^2 \geq 0$$

- This means all eigenvalues are non-negative

In summary, the Laplacian matrix L = D - A

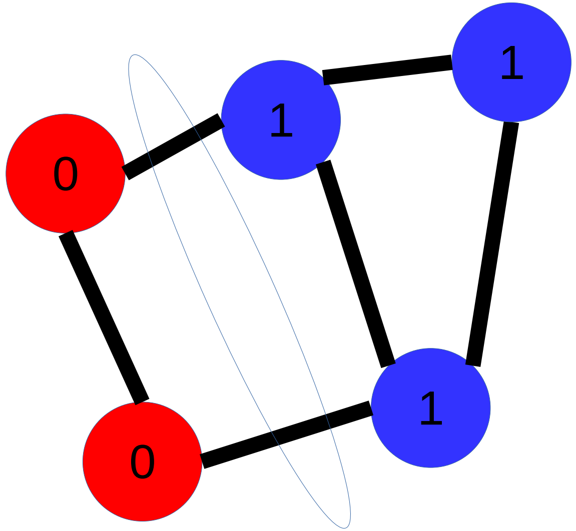
- Is symmetric, eigenvectors are orthogonal
- It has N eigenvalues that are non-negative
- 0 is one eigenvalue, with multiplicity equal to the number of connected components of the graph

$$0 = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$$

The second eigenvector in a connected graph

$x^T Lx$ and graph cuts

- Suppose (S, S') is a cut of graph G
- Set $x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \in S' \end{cases}$



$$|c(S, S')| = 2$$

$$x^T Lx = \sum_{(i,j)\in E} (x_i - x_j)^2 = \sum_{(i,j)\in c(S,S')} 1^2 = |c(S, S')|$$

Remember

- For symmetric matrices

$$\lambda_2 = \min_x \frac{x^T M x}{x^T x}$$

https://en.wikipedia.org/wiki/Rayleigh_quotient

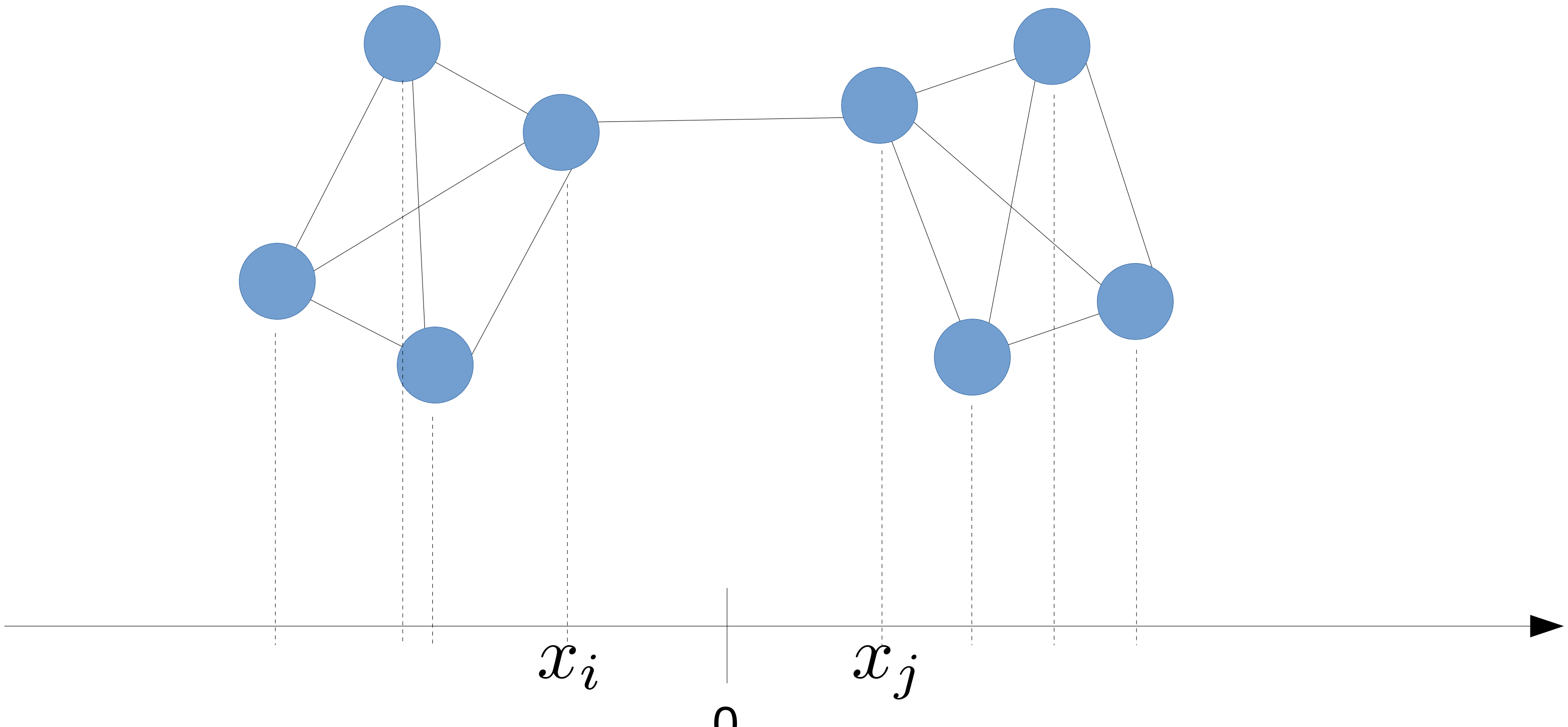
Second eigenvector

- Orthogonal to the first one: $x \cdot \vec{1} = 0 \Rightarrow \sum_i x_i = 0$
- Normal: $\sum_i x_i^2 = 1$

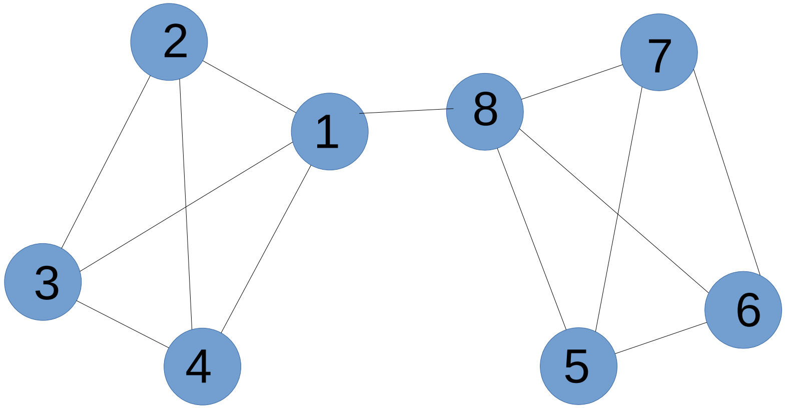
$$\lambda_2 = \min_x \frac{x^T L x}{x^T x} = \min_{x: \sum x_i = 0} \frac{x^T L x}{\sum x_i^2} = \min_{x: \sum x_i = 0} \sum_{(i,j) \in E} (x_i - x_j)^2$$

What does this mean?

$$\lambda_2 = \min_{x:\sum x_i = 0} \sum_{(i,j)\in E} (x_i - x_j)^2$$

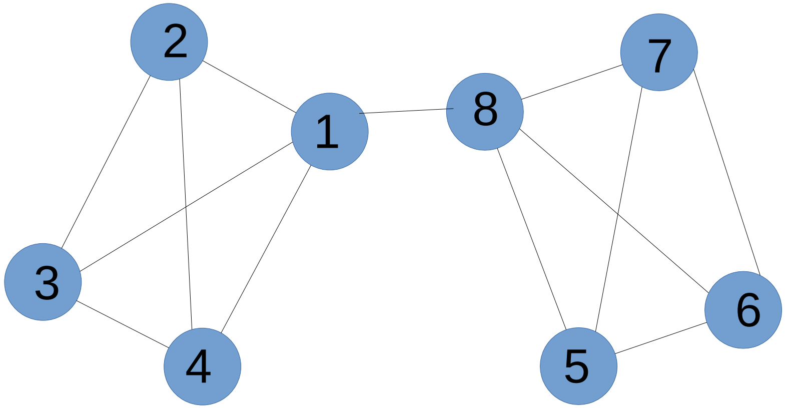


Example Graph 1



$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 & -1 & 4 \end{bmatrix}$$

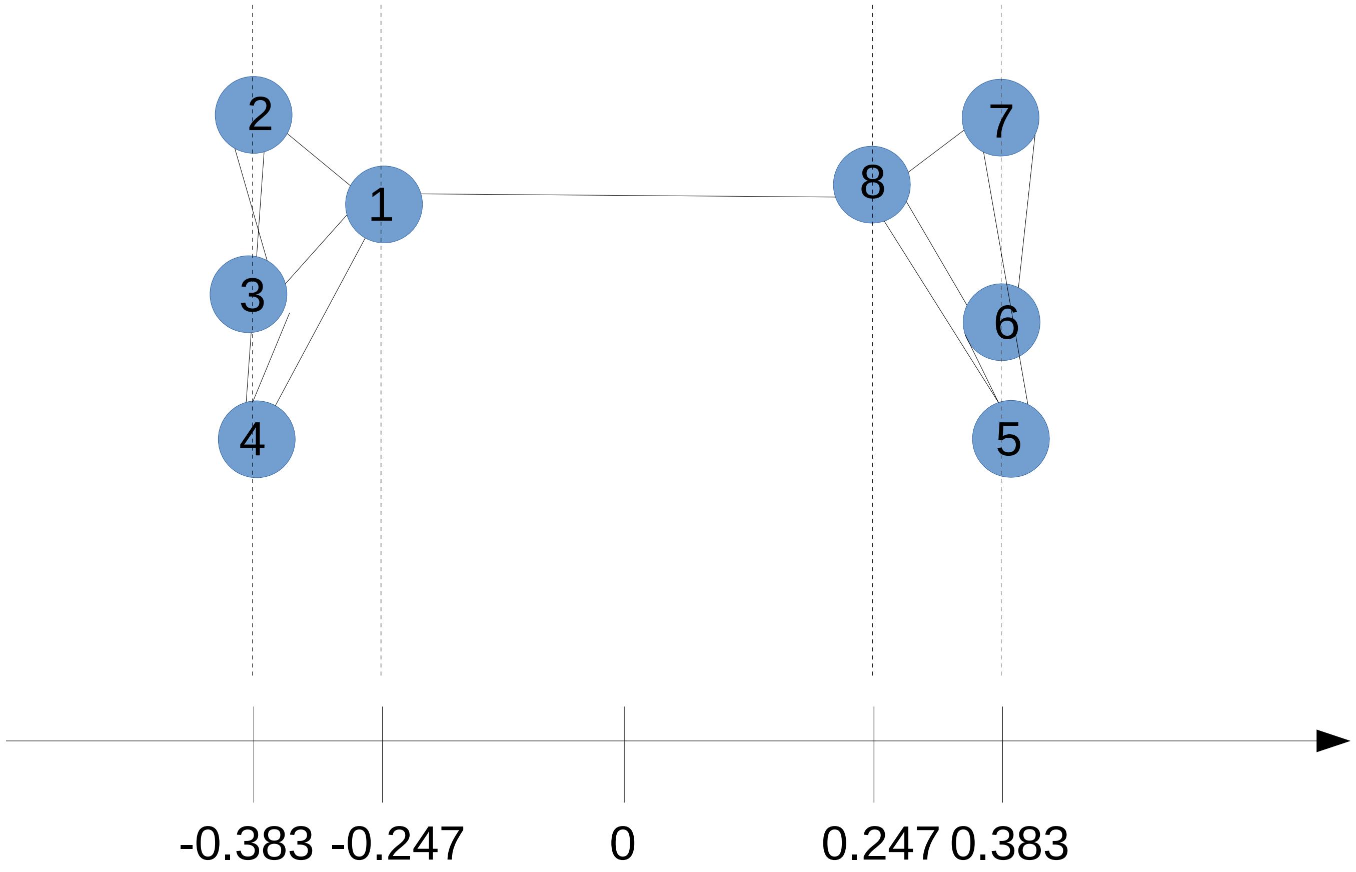
Example Graph 1 (second eigenvalue)



$$\lambda_1 = 0$$
$$\lambda_2 = 0.354$$

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 & -1 & 4 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0.247 \\ 0.383 \\ 0.383 \\ 0.383 \\ -0.383 \\ -0.383 \\ -0.383 \\ -0.247 \end{bmatrix}$$

Example Graph 1, projected in $\mathbb{R}^1$

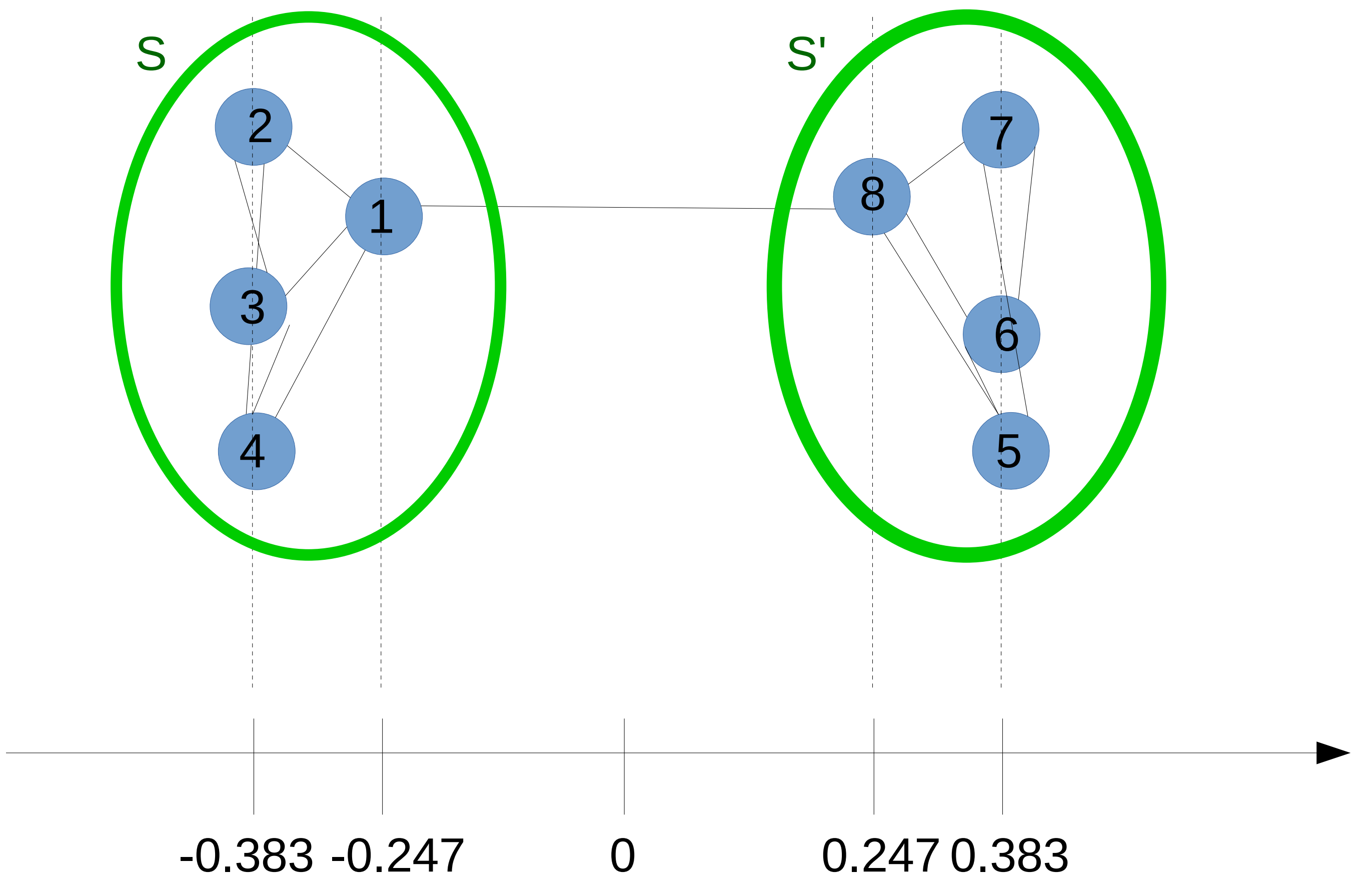


$$\lambda_1 = 0$$

$$\lambda_2 = 0.354$$

$$v_2 = \begin{bmatrix} 0.247 \\ 0.383 \\ 0.383 \\ 0.383 \\ -0.383 \\ -0.383 \\ -0.383 \\ -0.247 \end{bmatrix}$$

Example Graph 1, communities

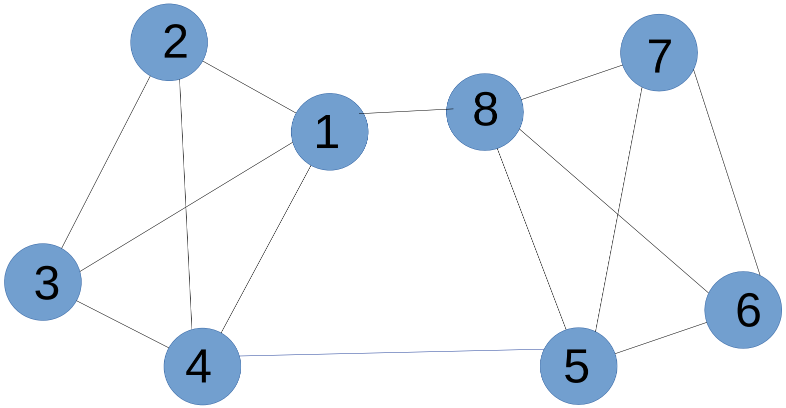


$$\lambda_1 = 0$$

$$\lambda_2 = 0.354$$

$$v_2 = \begin{bmatrix} 0.247 \\ 0.383 \\ 0.383 \\ 0.383 \\ -0.383 \\ -0.383 \\ -0.383 \\ -0.247 \end{bmatrix}$$

Example Graph 2



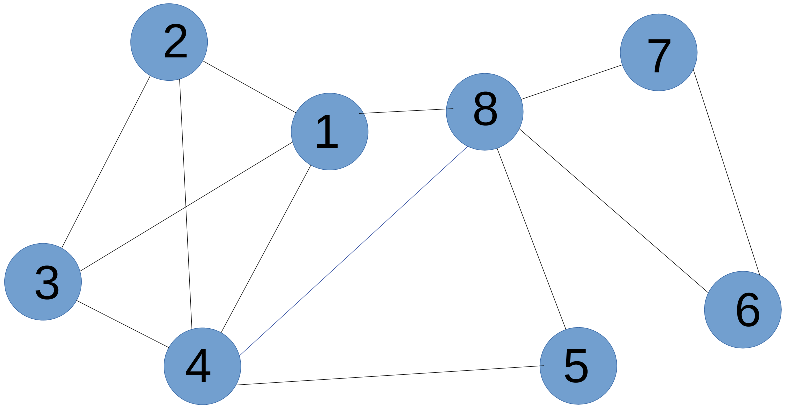
$$\lambda_1 = 0$$

$$\lambda_2 = 0.764$$

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 & -1 & 4 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0.263 \\ 0.425 \\ 0.425 \\ 0.263 \\ -0.263 \\ -0.425 \\ -0.425 \\ -0.263 \end{bmatrix}$$

Example Graph 3

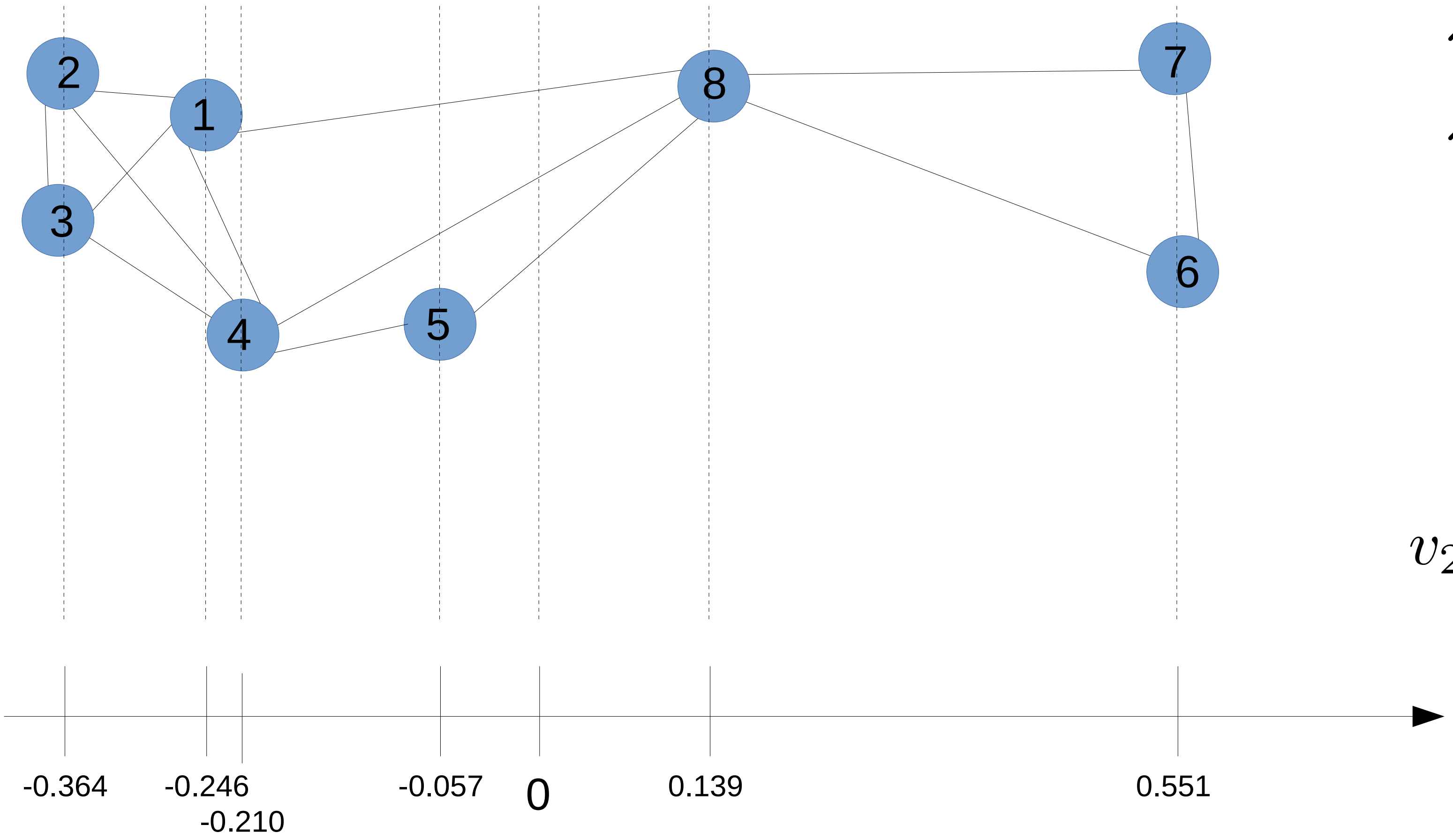


$$\lambda_1 = 0$$
$$\lambda_2 = 0.748$$

$$L = \begin{bmatrix} 4 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 5 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & -1 & -1 & -1 & 5 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -0.246 \\ -0.364 \\ -0.364 \\ -0.210 \\ -0.057 \\ 0.551 \\ 0.551 \\ 0.139 \end{bmatrix}$$

Example Graph 3, projected (where to cut?)



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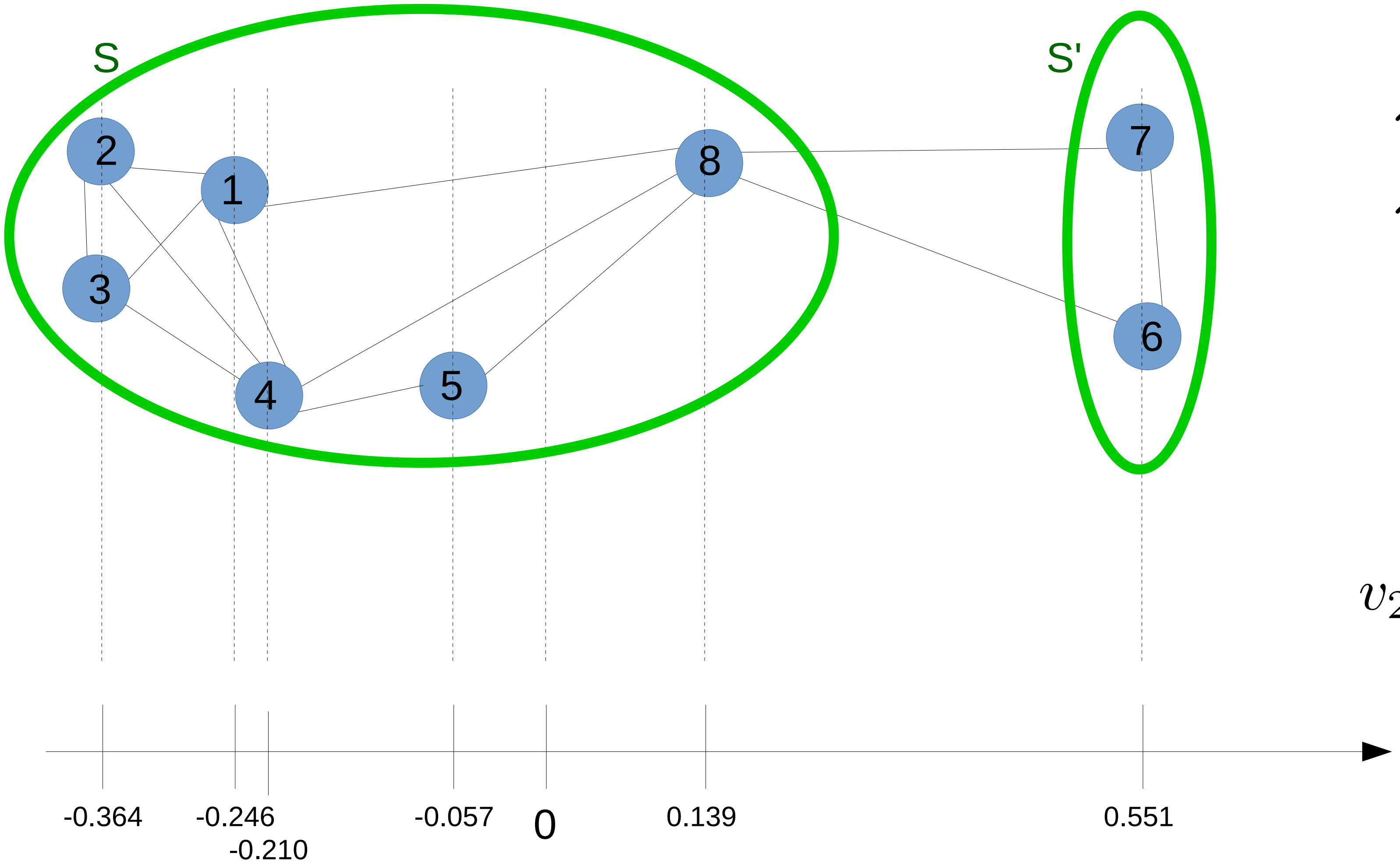

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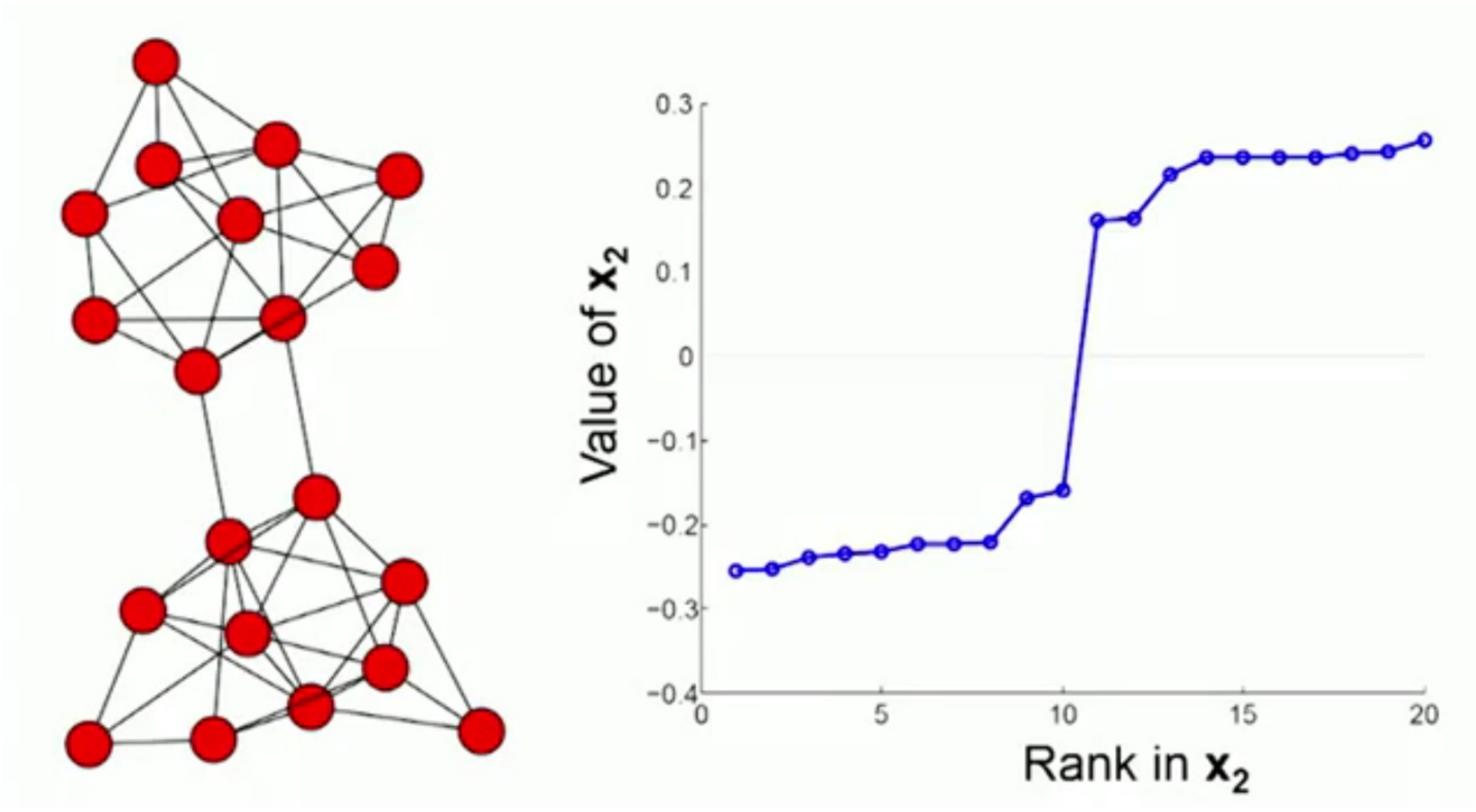
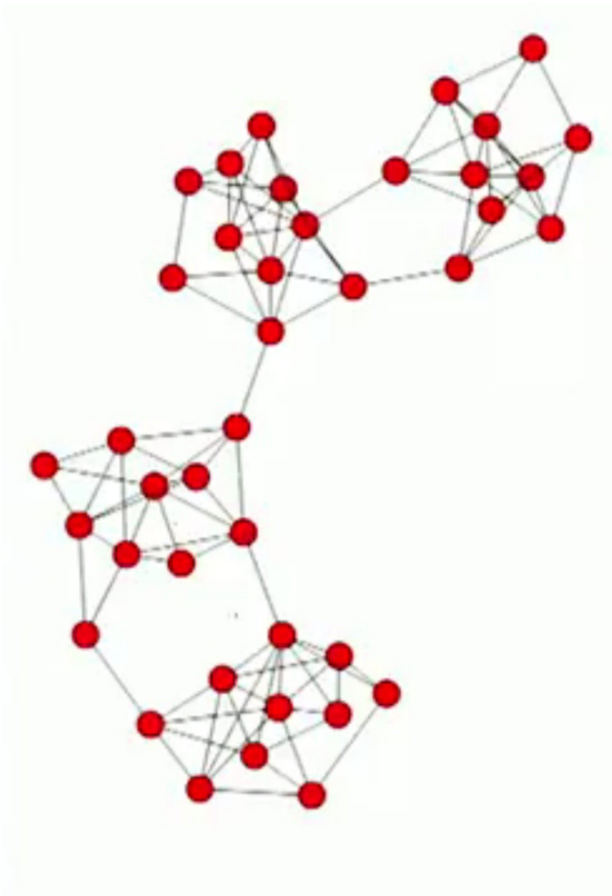
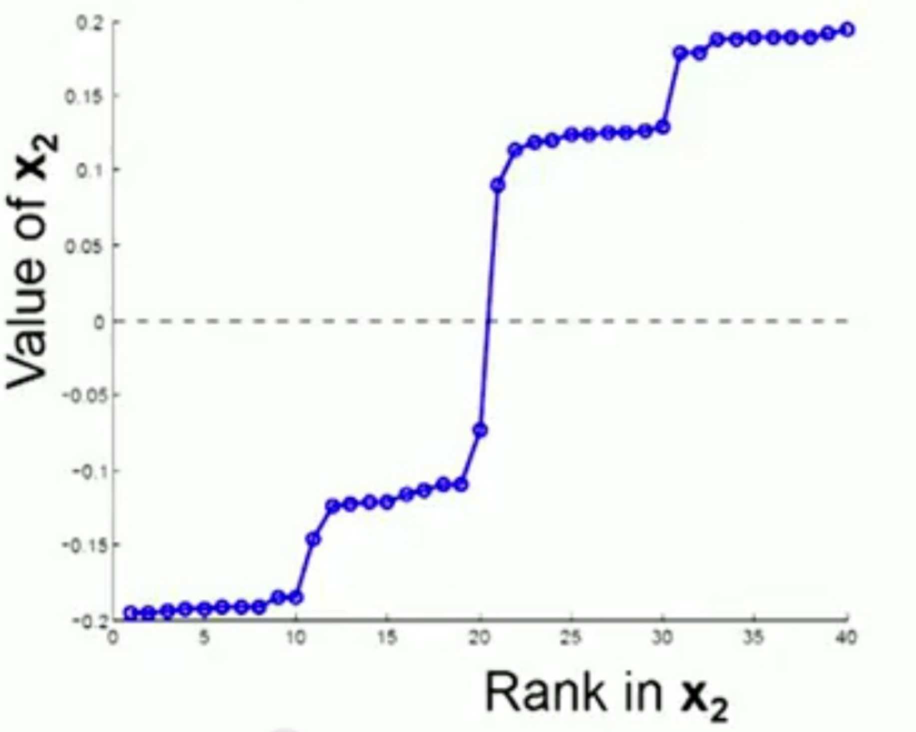
A graph with two communities in R^1



https://www.youtube.com/watch?v=jpTjj5PmcMM

A graph with four communities in R^1

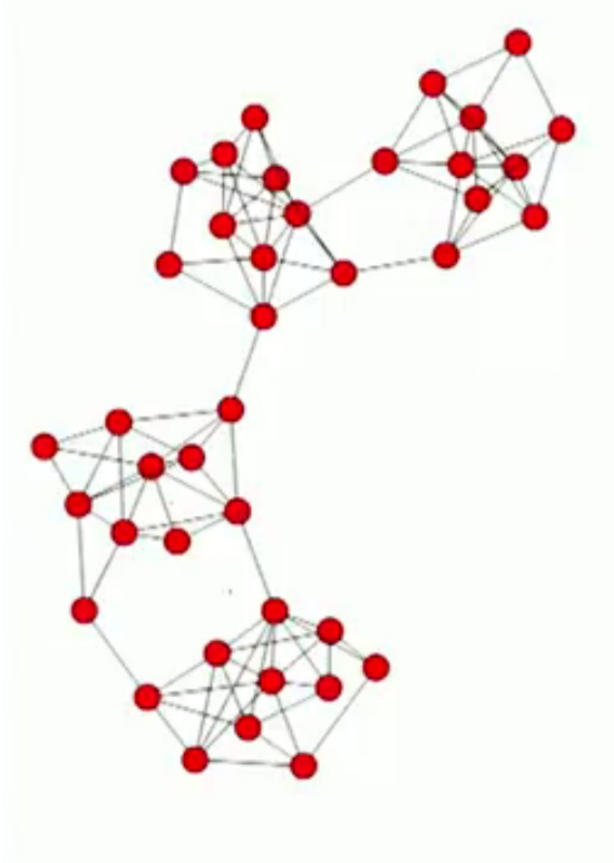


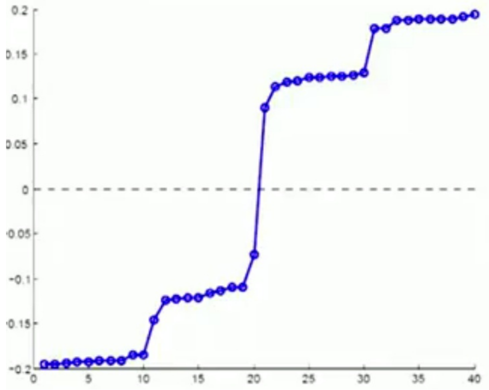


https://www.youtube.com/watch?v=jpTjj5PmcMM

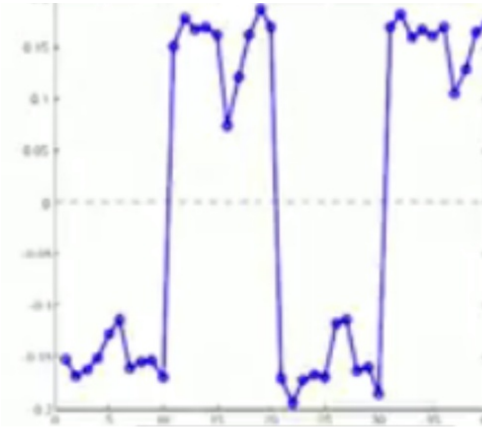
Application: graph drawing

A graph with four communities in R^2



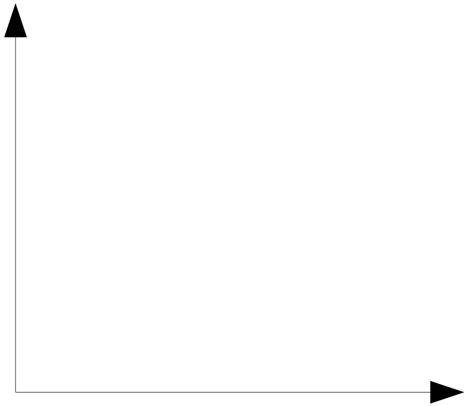


Second eigenvector



Third eigenvector

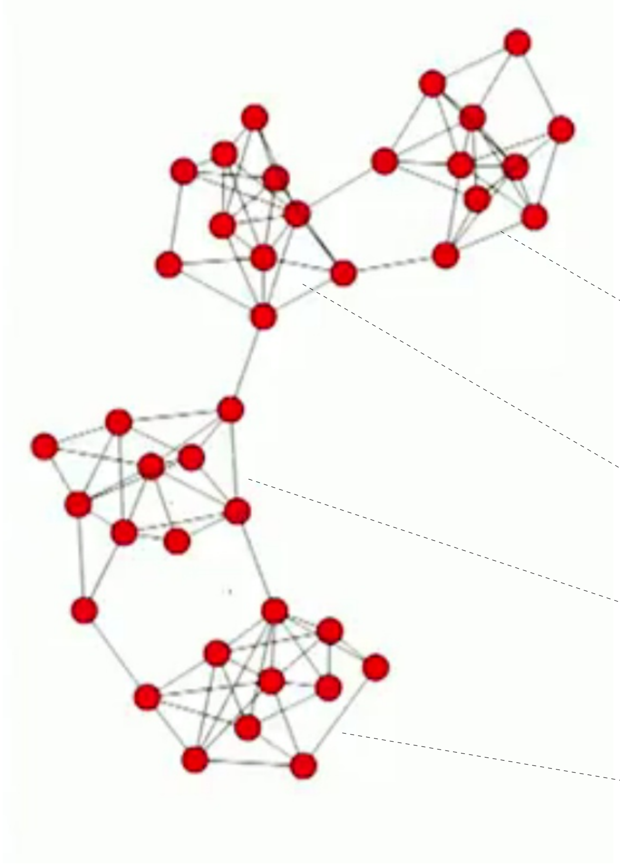
Value in third eigenvector

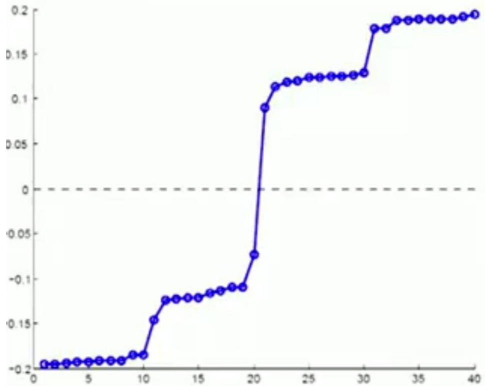


Value in second eigenvector

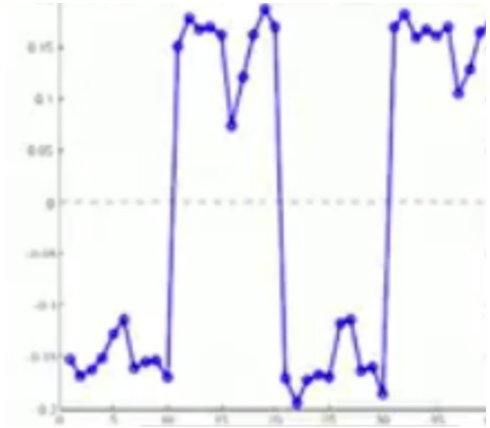
https://www.youtube.com/watch?v=jpTjj5PmcMM

A graph with four communities in R^2 (cont)

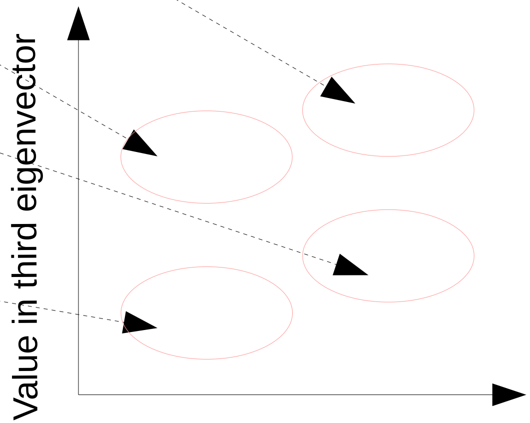




Second eigenvector



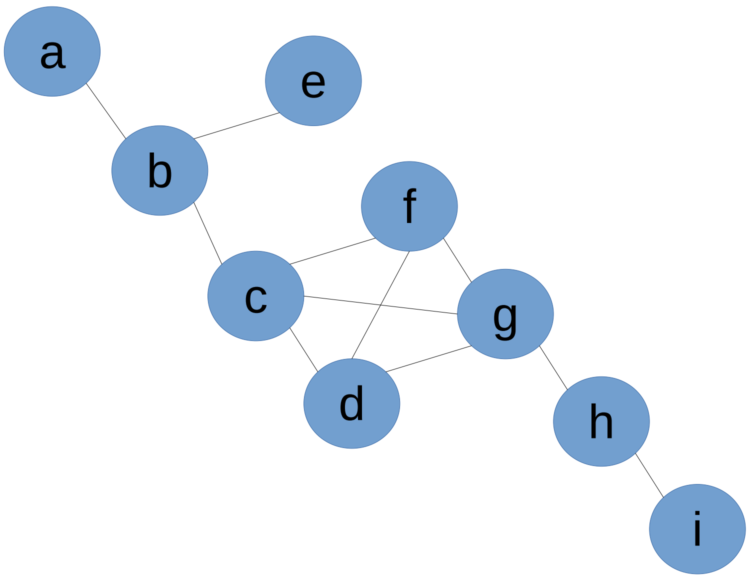
Third eigenvector



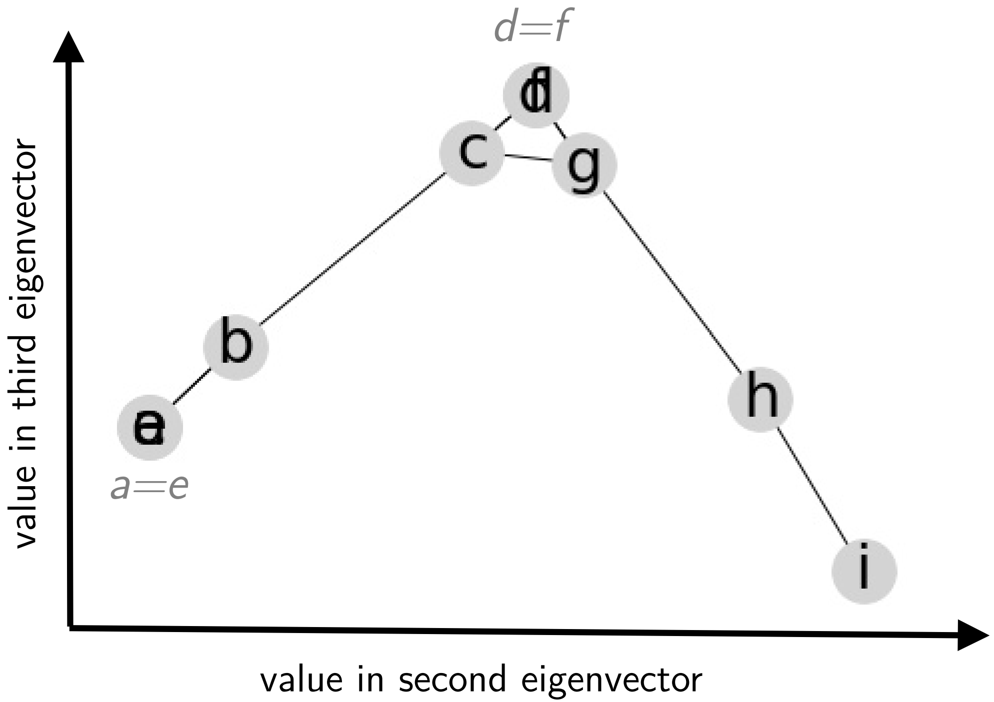
This can be used to draw the graph in R^2

https://www.youtube.com/watch?v=jpTjj5PmcMM

The graph from the initial exercise



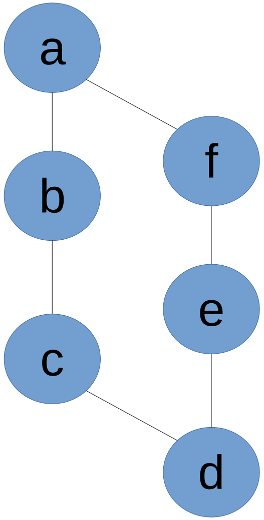
Input nodes and edges



Spectral embedding

Exercise: spectral projection

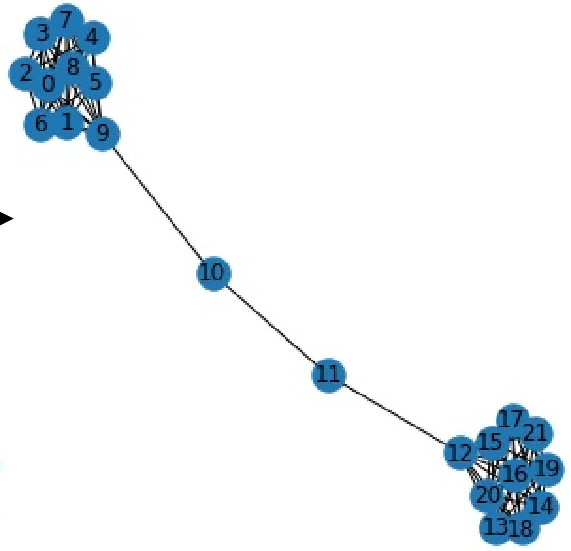
- Write the Laplacian
- Get the second and third eigenvector
- Obtain projection



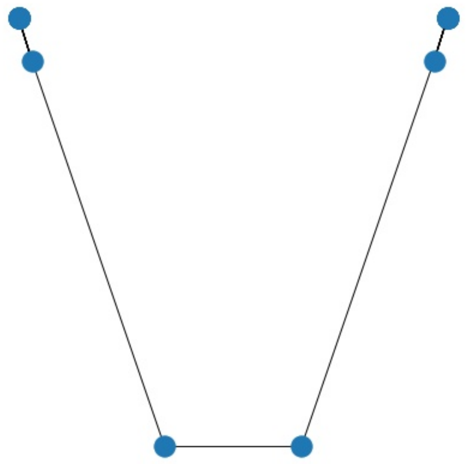
Link to spreadsheet: https://upfbarcelona.padlet.org/chato/shyq9m6f2g2dh1bw

A barbell graph in R^2 (code)

```
B = nx.barbell_graph(10,2)

plt.figure(figsize=(6,6))
nx.draw_networkx(B)
_ = plt.show()
```

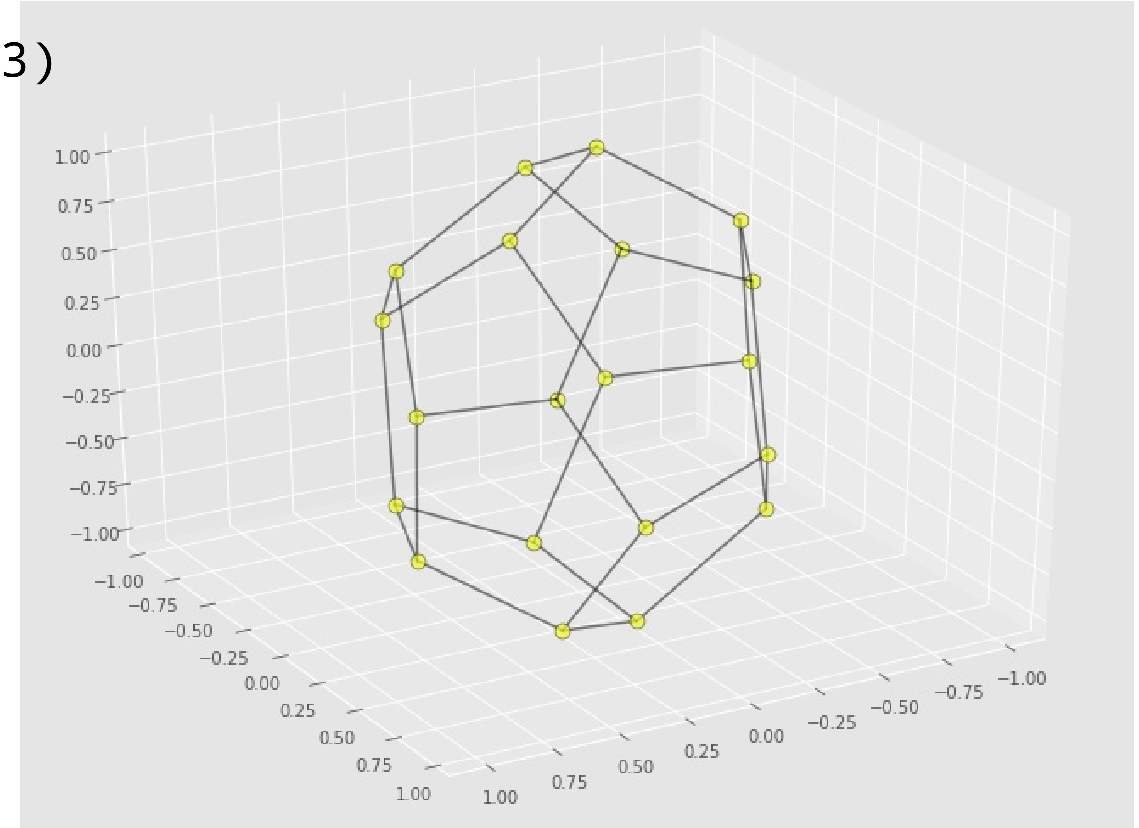



```
plt.figure(figsize=(6,6))
nx.draw_spectral(B)
_ = plt.show()
```



Dodecahedral graph in 3D

```
g = nx.dodecahedral_graph()
pos = nx.spectral_layout(g, dim=3)
network_plot_3D_alt(g, 60, pos)
```

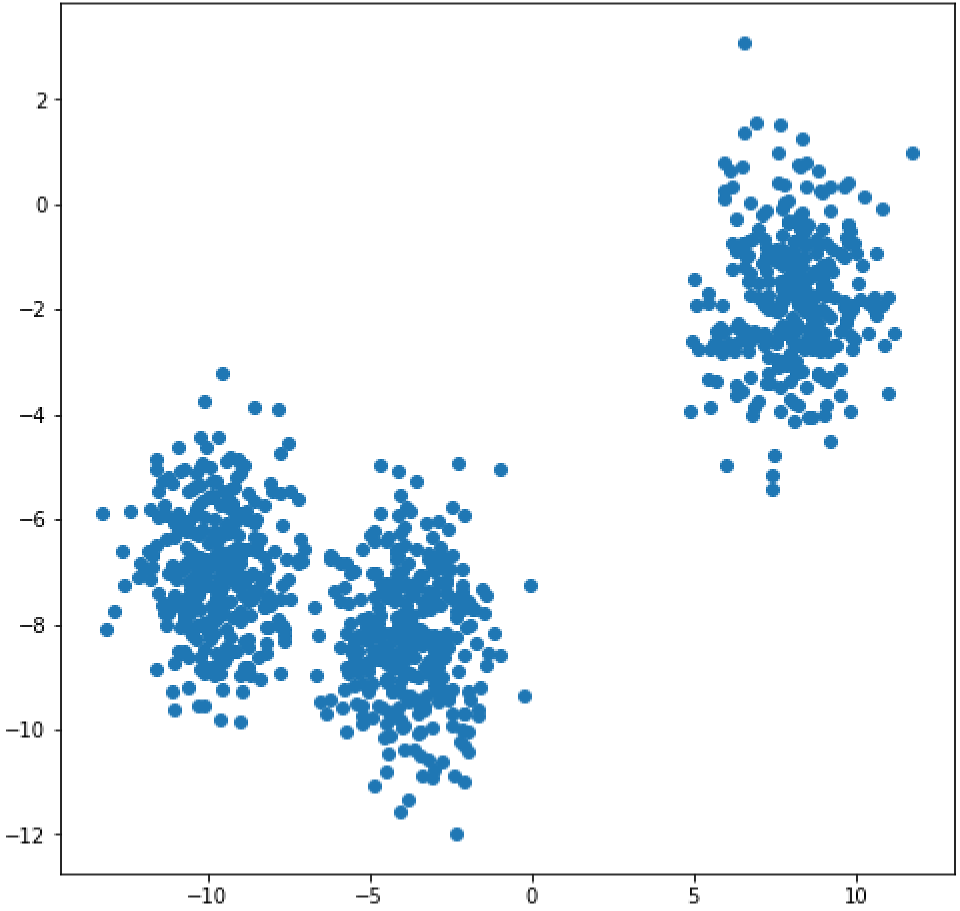


https://www.idtools.com.au/3d-network-graphs-python-mplot3d-toolkit/

Application: spectral clustering

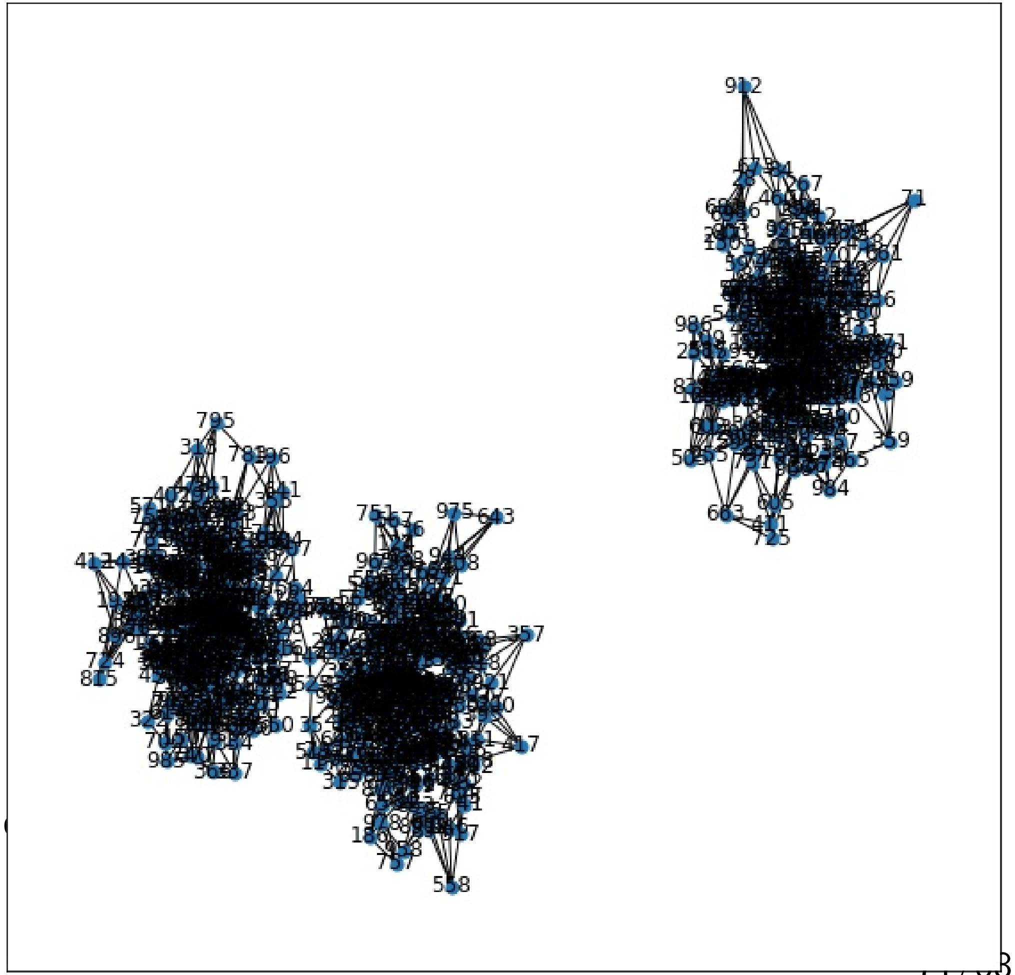
Generating data

```
from sklearn.datasets import
   make_blobs

N = 1000

x, _ = make_blobs(
   n_samples=N,
   centers=3,
   cluster_std=1.2)


plt.figure(figsize=(8,8))

plt.scatter(x[:,0], x[:,1])

plt.show()
```



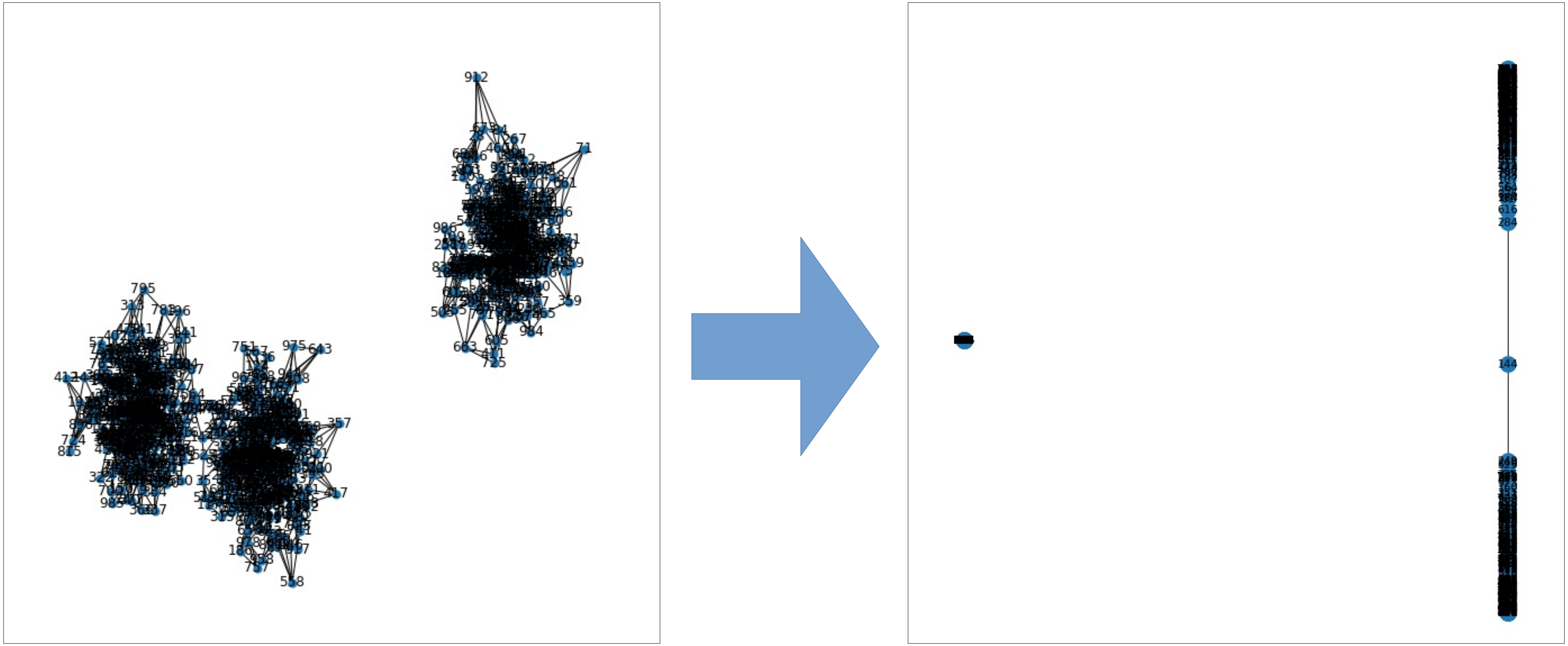
Connect nodes to k=5 nearest neighbors

```
from sklearn.neighbors
   import NearestNeighbors

nbrs = NearestNeighbors(
    n_neighbors=6,          # includes self
    algorithm='ball_tree')
    .fit(x)

distances, neighbors =
    nbrs.kneighbors(x)

G = nx.Graph()

for neighbor_list in neighbors:

    source_node = neighbor_list[0]

    for target_index in range(1,
        len(neighbor_list)):

        target_node = neighbor_list[target_ind

        G.add_edge(source_node, target_node)
```



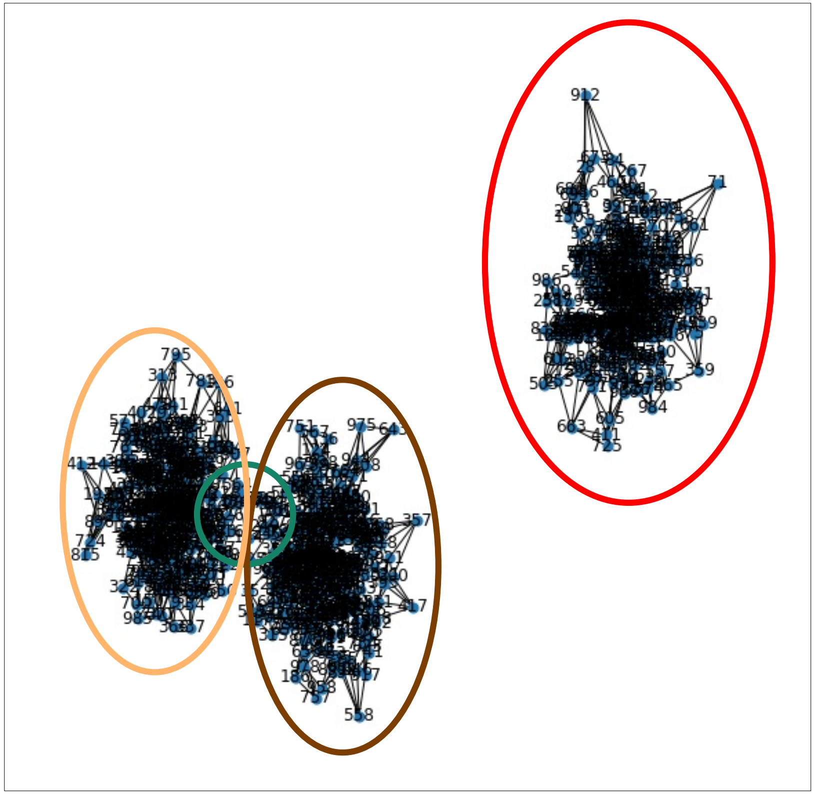
Perform spectral embedding

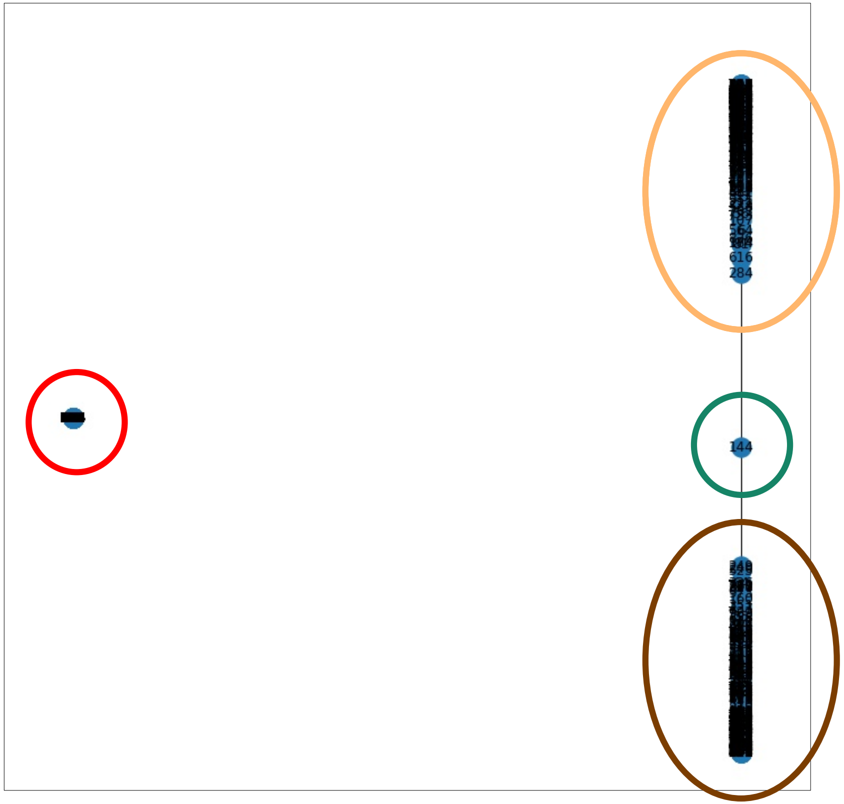
```
nx.draw_spectral(G, with_labels=True)
```



Perform spectral embedding

```
nx.draw_spectral(G, with_labels=True)
```





Summary

Things to remember

- Graph Laplacian
- Laplacian and graph components
- Spectral graph embedding

Exercises for this topic

- Mining of Massive Datasets (2014) by Leskovec et al.
  - Exercises 10.4.6