The Friendship Paradox

Social Networks Analysis and Graph Algorithms

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Contents

- Sampling nodes and edges
- Average degree of friends

Sources

- A. L. Barabási (2016). Network Science – Chapter 04
- F. Menczer, S. Fortunato, C. A. Davis (2020). A First Course in Network Science – Chapter 03
- URLs cited in the footer of specific slides

Sampling a random node

vs

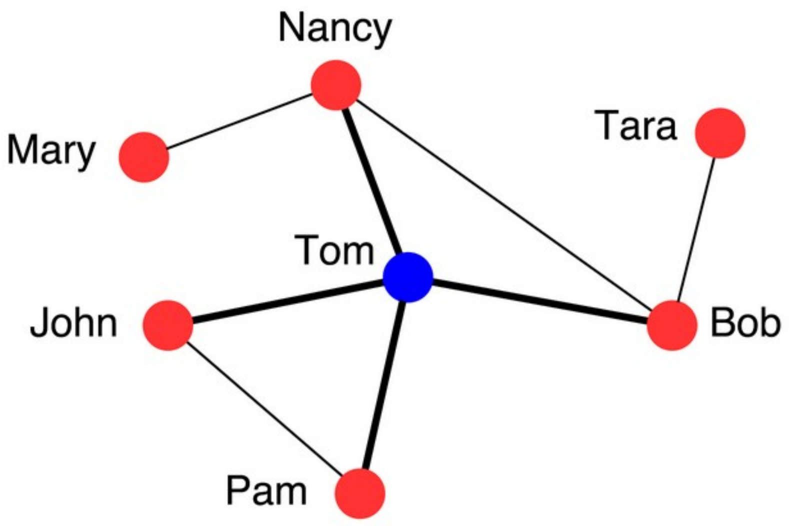
sampling at random one of the two nodes

attached to a random edge

Exercise

Numerical calculation of friendship paradox

- What is the probability of selecting Tom **if we select a random node**?
- What is the probability of selecting Tom **if we select a random edge and then randomly one of the two nodes attached to it**?



Pin board: https://upfbarcelona.padlet.org/chato/ocpl5n14i8hrkr4i

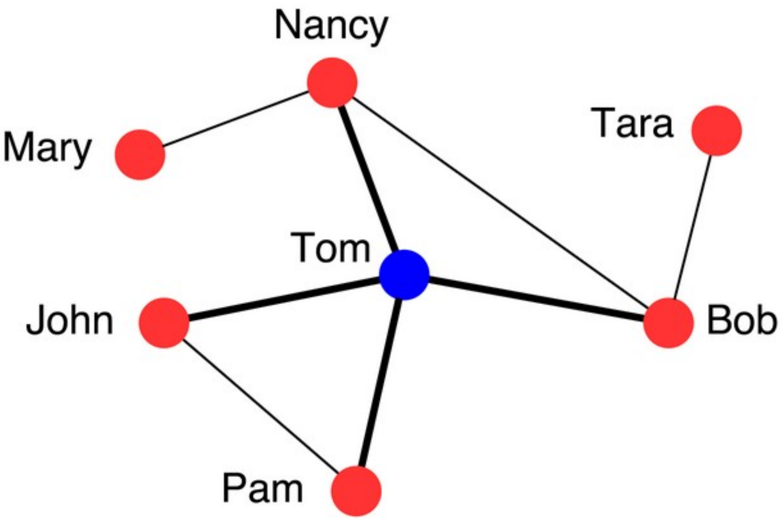
Sampling a random node

vs

sampling a random friend

of a random node

Average degree of friends



- Average degree

  $$(1+3+3+1+4+2+2)/7 = 16/7 \simeq \mathbf{2.29}$$

- Average degree of friends of:
  - Mary: 3
  - Nancy: $(1+4+3)/3 = 8/3$
  - Tara: 3
  - Bob: $(1+3+4)/3 = 8/3$
  - Tom: $(3+3+2+2)/4 = 10/4$
  - John: $(4+2)/2 = 3$
  - Pam: $(4+2)/2 = 3$
  - Average degree of friends $\simeq$ **2.83** (> 2.29)

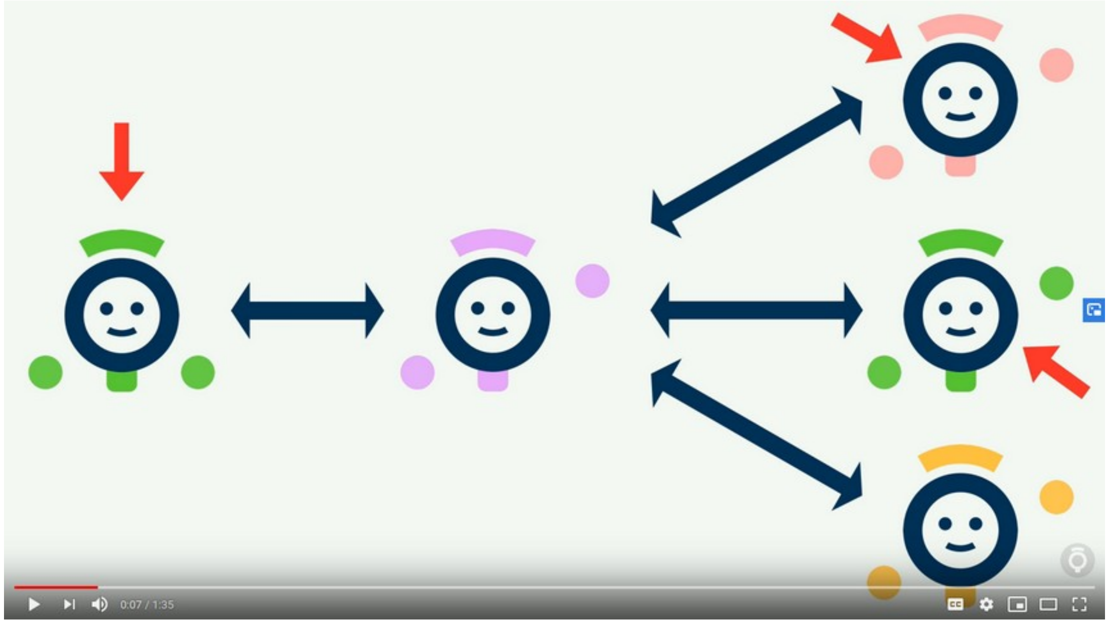
The friendship paradox

- Take a random person x; what is the expected degree of this person? $<k>$
- Take a random person x, now pick one of x's neighbors, let's say y; what is the expected degree of y?

  It is not $<k>$

- This "paradox" is a useful vaccination strategy

Sampling bias and the friendship paradox (1'35'')



https://www.youtube.com/watch?v=httLvVufAYs

Imagine you're at a random airport on earth

- Is it more likely to be …

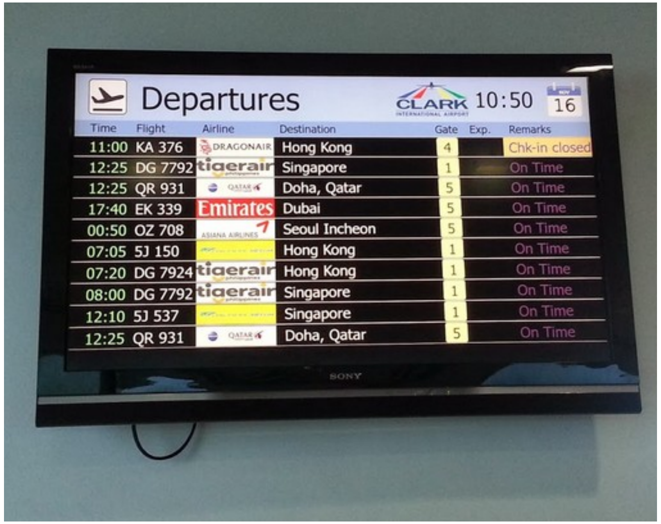
  a large airport or a small airport?

- If you take a random flight out of it …

  will it go to a large airport or a small airport?

An example of friendship paradox

- Pick a random airport on Earth
  - Most likely it will be a small airport
- However, no matter how small it is, it **will** have flights to big airports
- On average those airports will have much larger degree



Exercise [B. 2016, Ex. 4.10.2]
"Friendship Paradox"

- Remember p_k is the probability that a node has k "friends"
- If we randomly select a link, the probability that a node at any end of the link has k friends is $q_k = C\, k\, p_k$ where C is a normalization factor
  - (a) Find C (the sum of q_k must be 1)

Exercise [B. 2016, Ex. 4.10.2] "Friendship Paradox"

- If we randomly select a link, the probability that a node at any end of the link has k friends is $q_k = C\, k\, p_k$ where C is a normalization factor
- q_k is also the prob. that a randomly chosen node has a neighbor of degree k

(b) Find its expectation $E[q_k]$ which we will call $\langle k_F \rangle$

Remember $$E[X] = \sum_{X_{\min}}^{X_{\max}} x \cdot P(X = x)$$

Exercise [B. 2016, Ex. 4.10.2]
"Friendship Paradox"

(c) Compute the expected number of friends of a neighbor of a randomly chosen node in the case below

(d) compare with the expected number of friends of a randomly chosen node

$$N = 10000$$
$$\gamma = 2.3$$
$$k_{\min} = 1$$
$$k_{\max} = 1000$$

$$\langle k^n \rangle = C \frac{k_{\max}^{n-\gamma+1} - k_{\min}^{n-\gamma+1}}{n-\gamma+1}$$

$$C = (\gamma - 1) k_{\min}^{\gamma-1}$$

Code

```python
def degree_moment(kmin, kmax, moment, gamma):
    C = (gamma-1.0)*(kmin**(gamma-1.0))
    numerator = (kmax**(moment-gamma+1.0) - kmin**(moment-gamma+1.0))
    denominator = (moment-gamma+1.0)
    return C * numerator / denominator
```

```python
kavg = degree_moment(kmin=1, kmax=1000, moment=1, gamma=2.3)
print(kavg)
```

```
3.787798988222529
```

```python
ksqavg = degree_moment(kmin=1, kmax=1000, moment=2, gamma=2.3)
print(ksqavg)
```

```
231.94329076177414
```

```python
print(ksqavg / kavg)
```

```
61.23431879119234
```

Summary

Practice on your own

- Draw a small graph, and sample from that graph until you're convinced $\langle k_F \rangle > \langle k \rangle$