Scale-Free Networks

Social Networks Analysis and Graph Algorithms

Prof. Carlos “ChaTo” Castillo — https://chato.cl/teach

Universitat Pompeu Fabra Barcelona

Contents

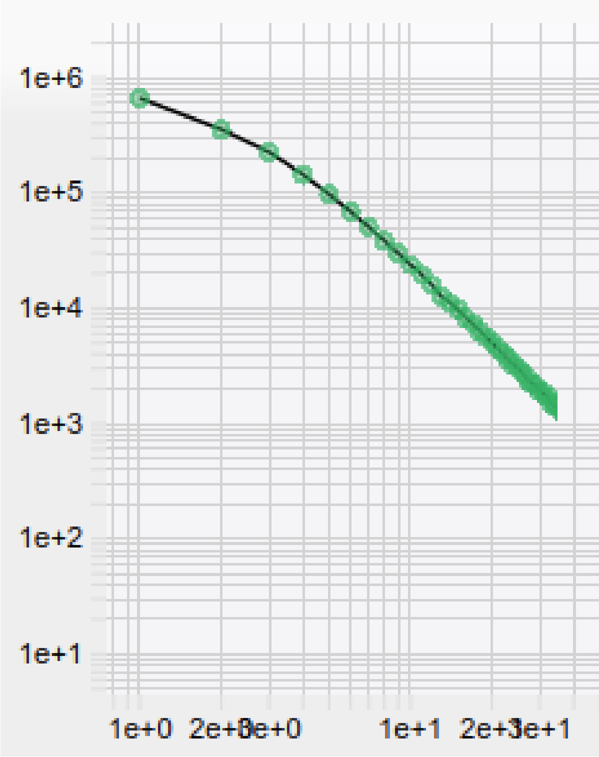
- Characteristics of scale-free networks
- Degree distribution of scale-free networks
- Distance distribution of scale-free networks

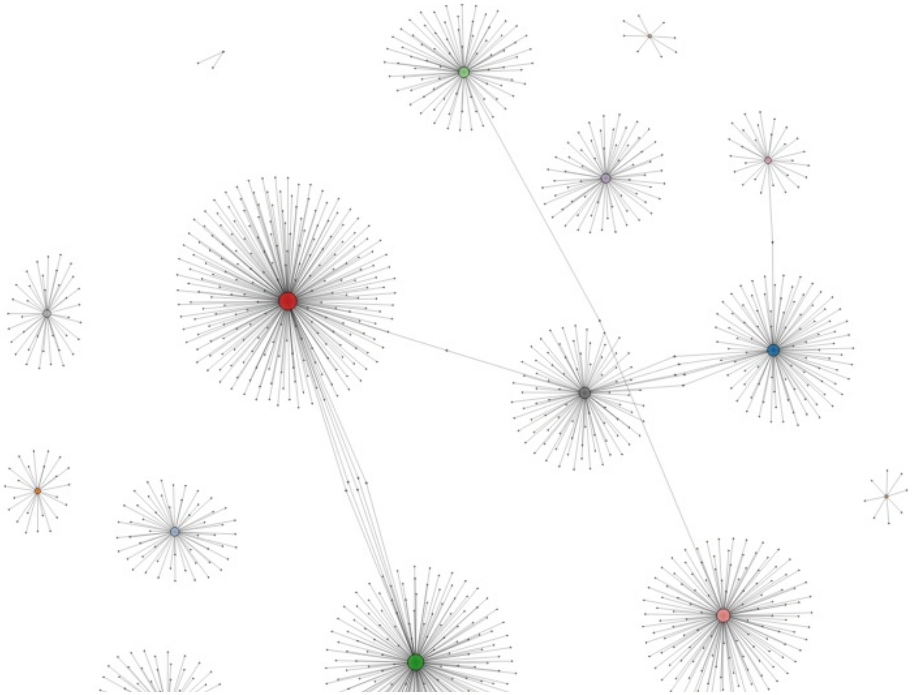
Sources

- A. L. Barabási (2016). Network Science – Chapter 04
- URLs cited in the footer of specific slides

Observed degree distributions

Wikipedia 2009 (N=1.9M, L=4.5M)



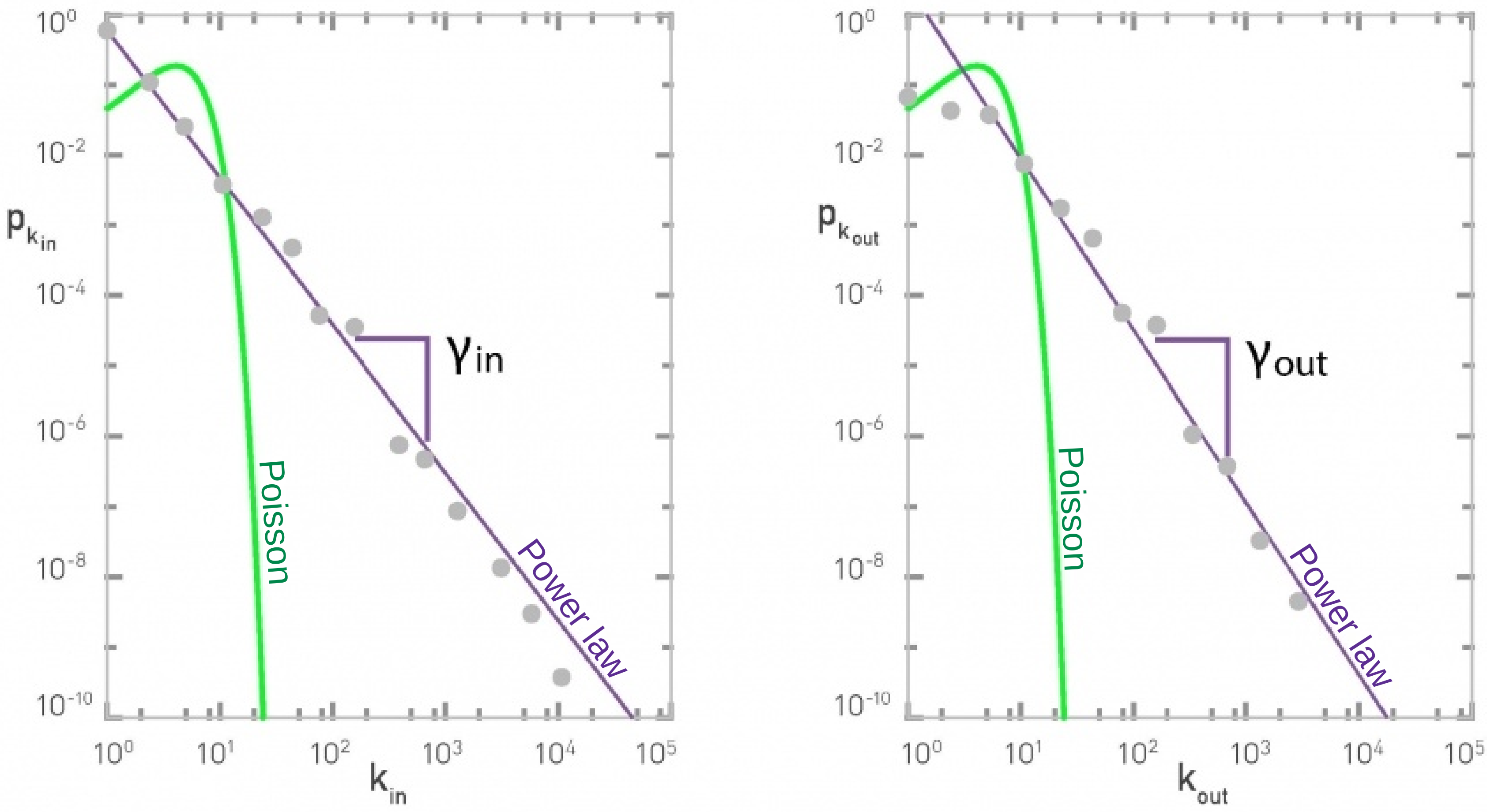


https://networkrepository.com/web-wikipedia2009.php

Degree distributions in Web graphs

- Web graphs have large “hubs”:
  - This is, nodes with very large degree
- This does not happen in a random (ER) graph
- We have already seen the Poisson distribution is a bad approximation of the observed degree distribution

Degree distributions in a web graph [nd.edu]

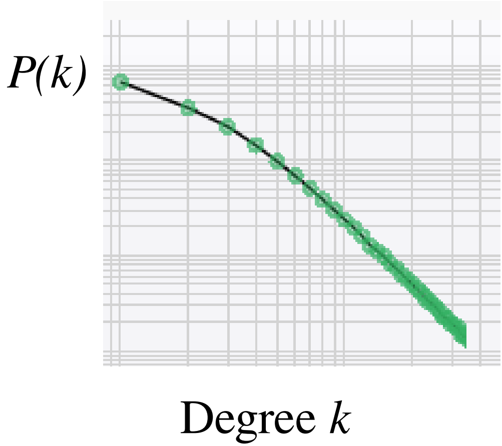


Modeling degree distributions

Degree distribution in real-world networks

- Straight descending line in log-log plot

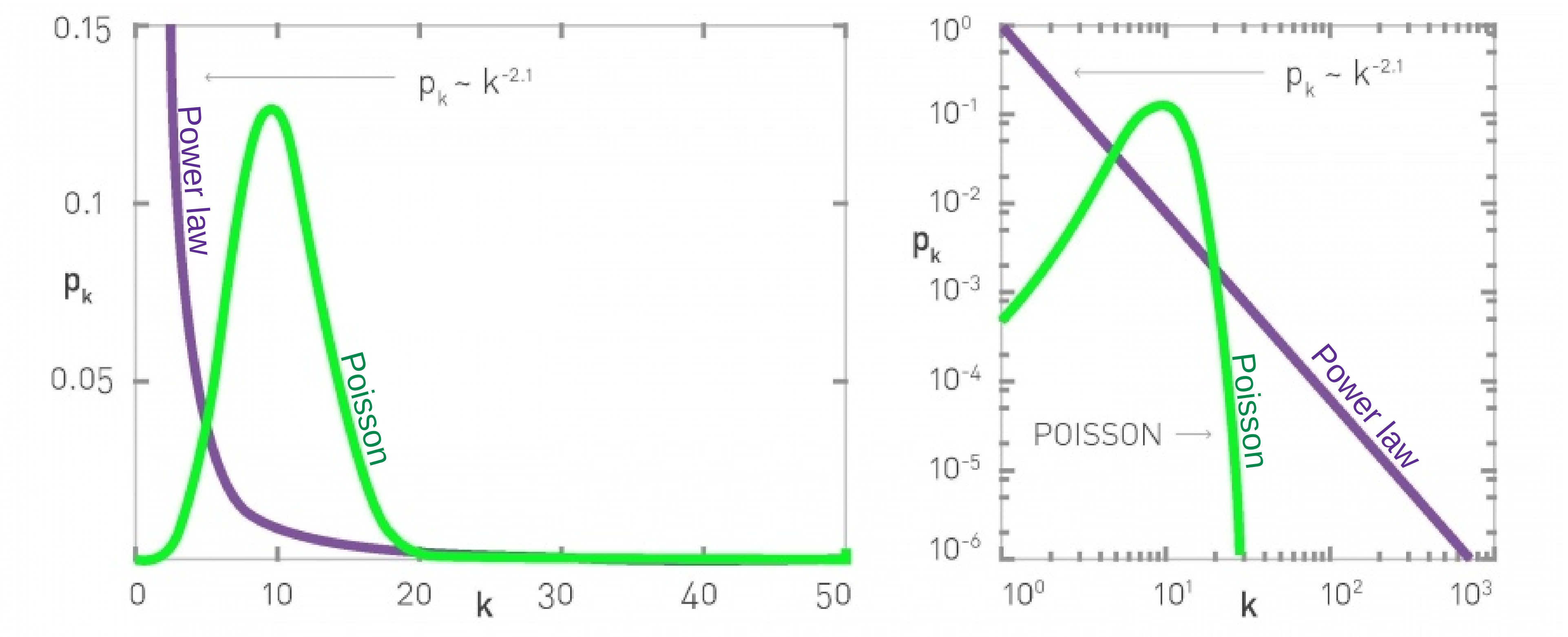
$$\log p_k \sim -\gamma \log k$$
$$p_k \sim k^{-\gamma}$$



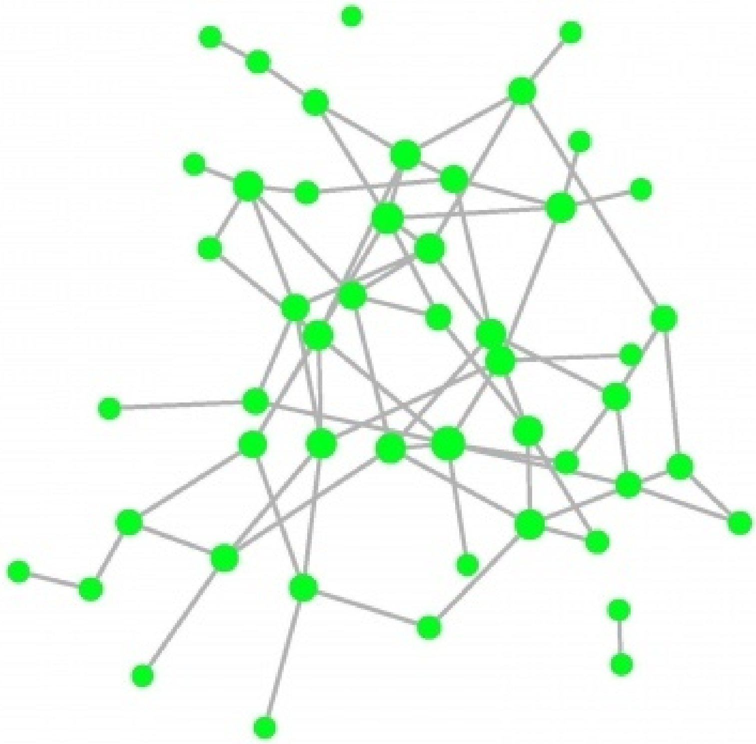
- Parameter ɣ is the exponent of the power law

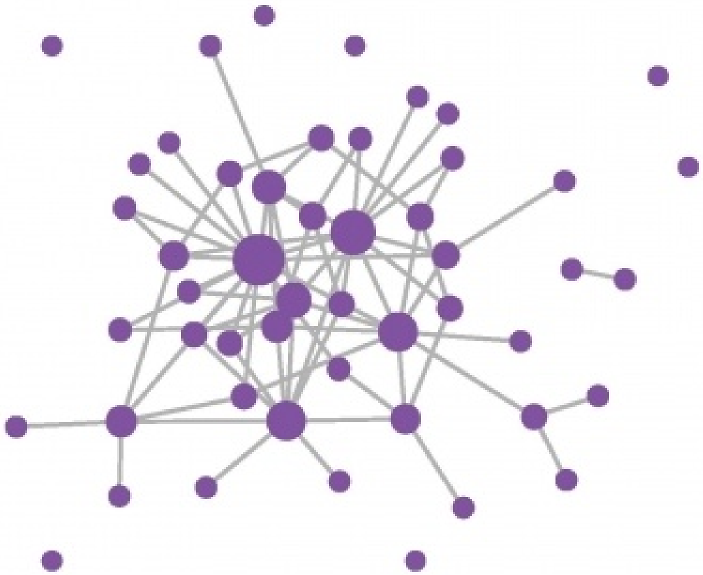
A scale-free network is a network whose degree distribution follows a power law

Comparing Poisson to power law

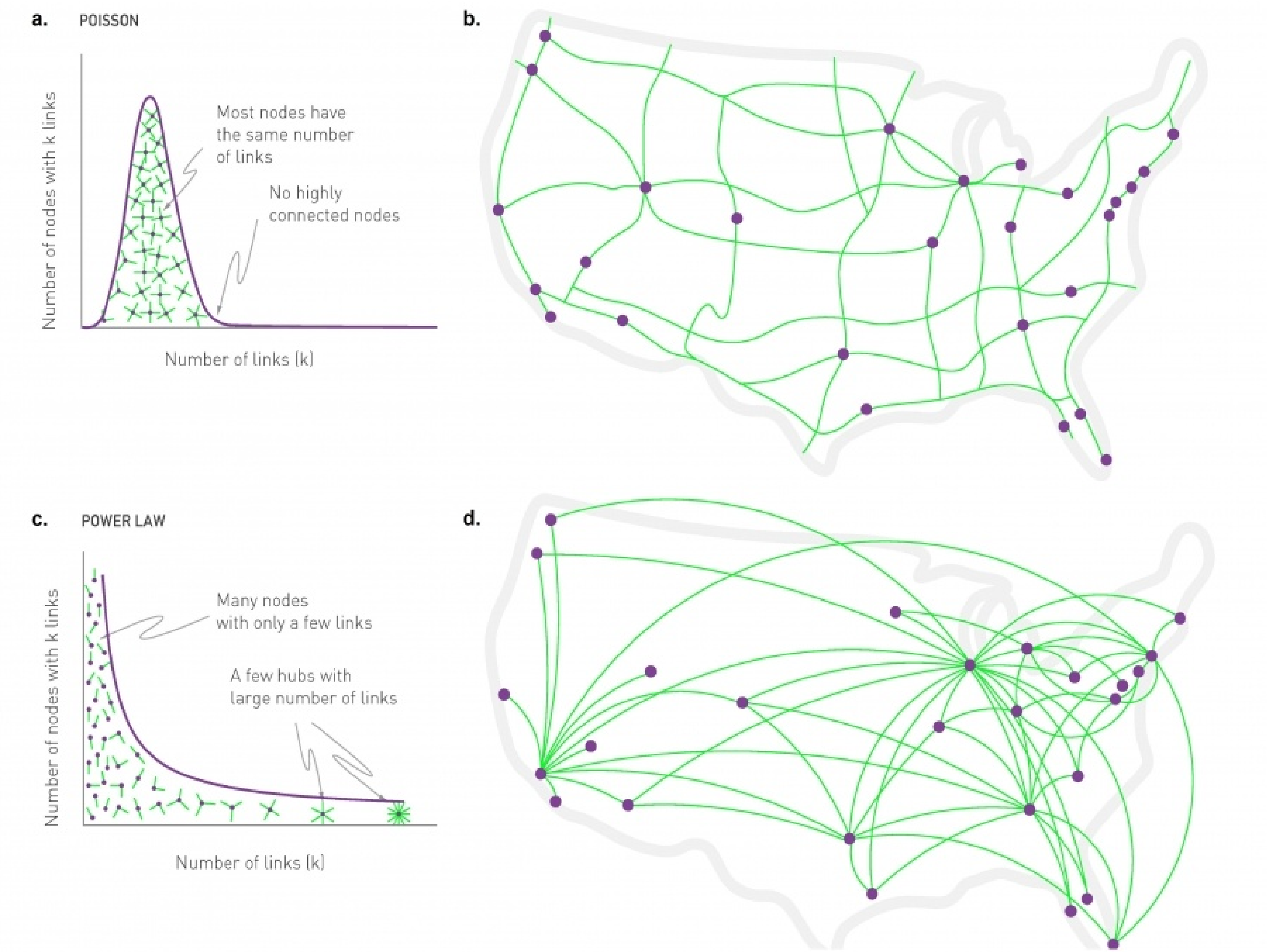


Comparing Poisson to power law





Random vs scale-free networks



Ground transportation

Air transportation

Random vs scale-free networks (cont.)

High-speed trains

Air transportation

Formally (discrete)

$$p_k = Ck^{-\gamma}$$

$$\sum_{k=1}^{\infty} p_k = 1 \longrightarrow C = \frac{1}{\sum_{k=1}^{\infty} k^{-\gamma}} = \frac{1}{\zeta(\gamma)}$$

Riemann's zeta

$$p_k = \frac{k^{-\gamma}}{\zeta(\gamma)}$$

This formalism assumes there are no nodes with degree zero

Formally (continuous approx.)

$$p_k = Ck^{-\gamma}$$

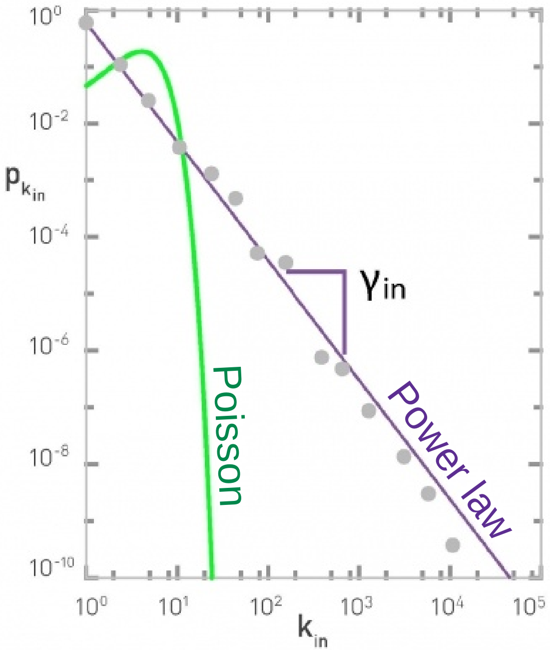
$$\int_{k=k_{\min}}^{\infty} p_k = 1 \longrightarrow C = \frac{1}{\int_{k=k_{\min}}^{\infty} k^{-\gamma}} = (\gamma - 1)k_{\min}^{\gamma-1}$$

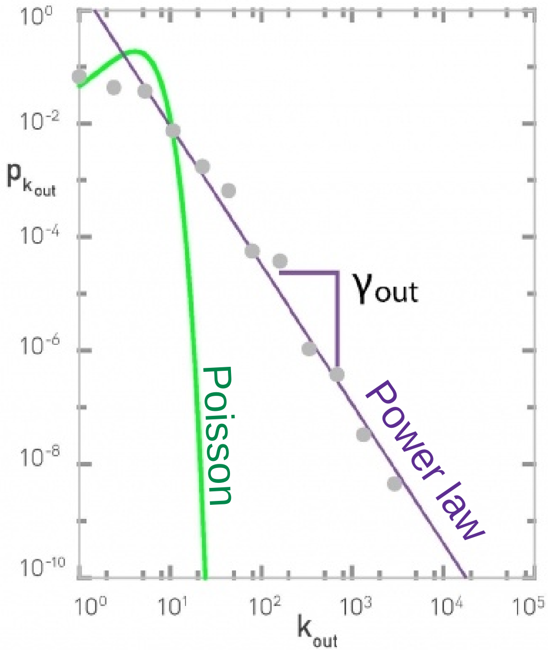
$$p_k = (\gamma - 1)k_{\min}^{\gamma-1}k^{-\gamma}$$

k_{min} is the smaller degree found in the network

Directed graphs

In directed graphs the exponent might be different for in-degree and out-degree





$$\mathrm{p}_k \sim k^{-\gamma}$$

What kind of values of gamma make the power law “tail” look more like the Poisson “tail”?

In directed networks ...

- Each node has two degrees: k_{in} and k_{out}
- In general they may differ, hence

$$p_{k_{in}} \sim k^{-\gamma_{in}}$$

$$p_{k_{out}} \sim k^{-\gamma_{out}}$$

- In nd1998, $\gamma_{in} \simeq 2.1$, $\gamma_{out} \simeq 2.4$

Dispersion of the degree distribution

What does it mean “scale-free”?

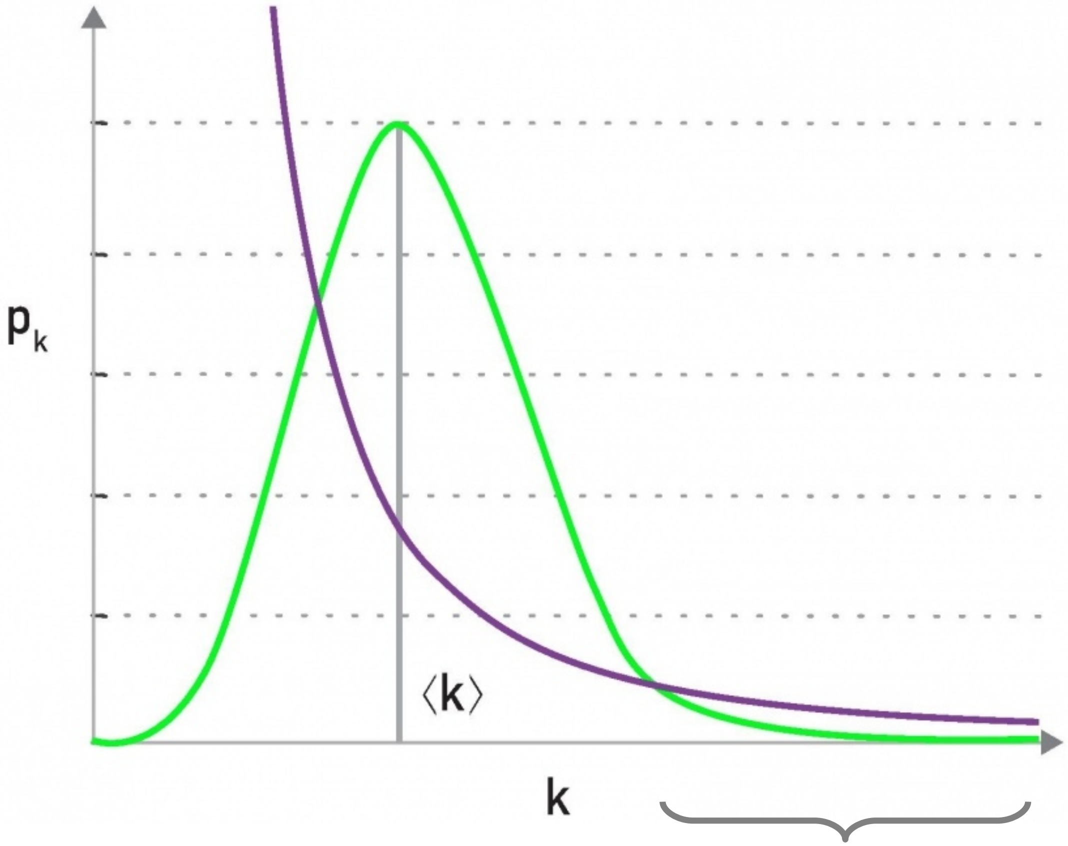
- A distribution **has a “scale”** if values are close to each other, for instance in a random network $\sigma_k = \langle k \rangle^{1/2}$
- Hence, most nodes are in the range $\langle k \rangle \pm \langle k \rangle^{1/2}$
- However in scale-free network with $\gamma < 3$

$$\sigma_k \to \infty$$

Example: citations to scientific papers

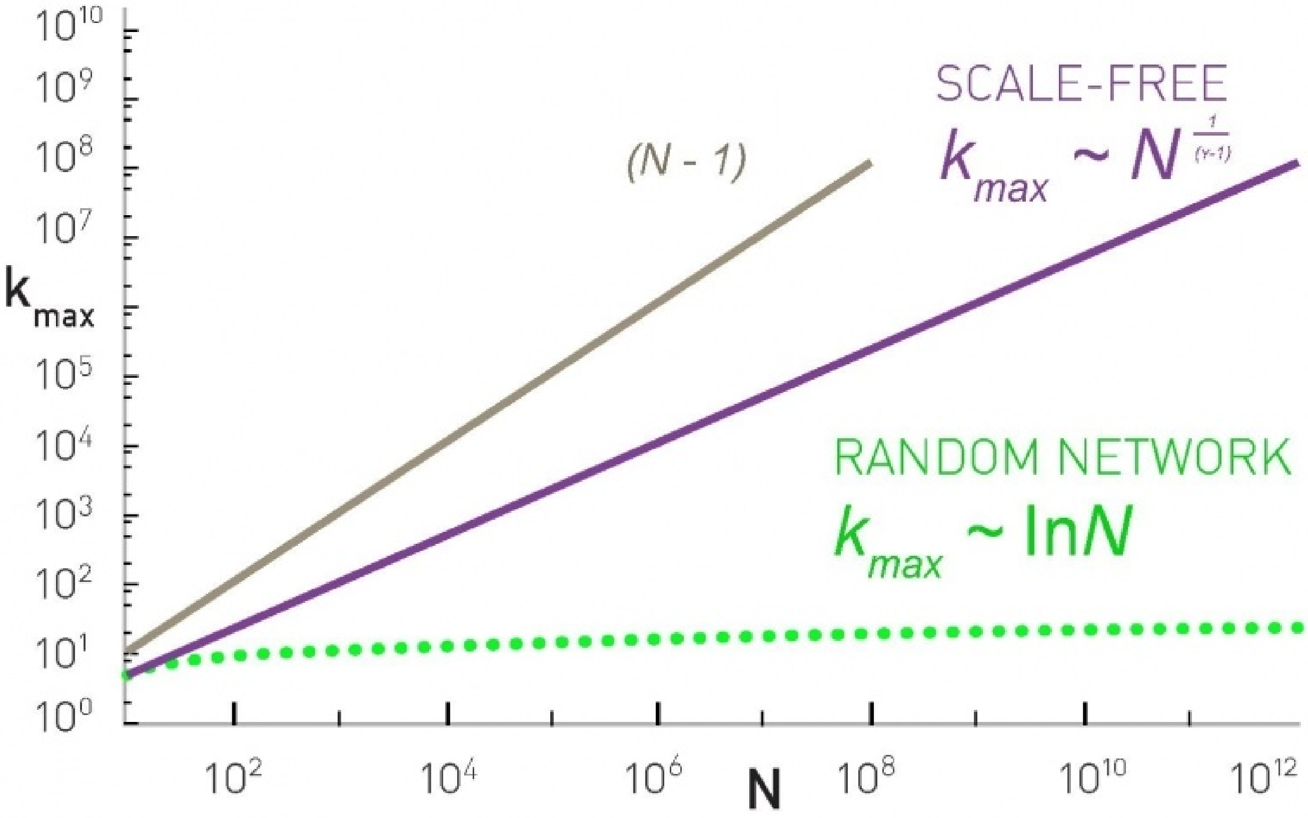
$$k_{\text{in}} \approx 10 \pm 900$$

In general, the average degree is not very informative in scale-free networks



There is a natural cut-off of the degree

The largest hub cannot have more than N-1 connections



$$k_{\max} = k_{\min} N^{\frac{1}{\gamma-1}}$$

Real network examples

Network	N	L	$\langle k \rangle$	$\langle k_{in}^2 \rangle$	$\langle k_{out}^2 \rangle$	$\langle k^2 \rangle$	γ_{in}	γ_{out}	γ
Internet	192,244	609,066	6.34	-	-	240.1	-	-	3.42*
WWW	325,729	1,497,134	4.60	1546.0	482.4	-	2.00	2.31	-
Power Grid	4,941	6,594	2.67	-	-	10.3	-	-	Exp.
Mobile-Phone Calls	36,595	91,826	2.51	12.0	11.7	-	4.69*	5.01*	-
Email	57,194	103,731	1.81	94.7	1163.9	-	3.43*	2.03*	-
Science Collaboration	23,133	93,437	8.08	-	-	178.2	-	-	3.35*
Actor Network	702,388	29,397,908	83.71	-	-	47,353.7	-	-	2.12*
Citation Network	449,673	4,689,479	10.43	971.5	198.8	-	3.03*	4.00*	-
E. Coli Metabolism	1,039	5,802	5.58	535.7	396.7	-	2.43*	2.90*	-
Protein Interactions	2,018	2,930	2.90	-	-	32.3	-	-	2.89*-

Exercise

In the actor network, N=702,388, $\gamma = 2.12$

1. How many actors do we expect to have ...

   1 other co-star?

   10 other co-stars?

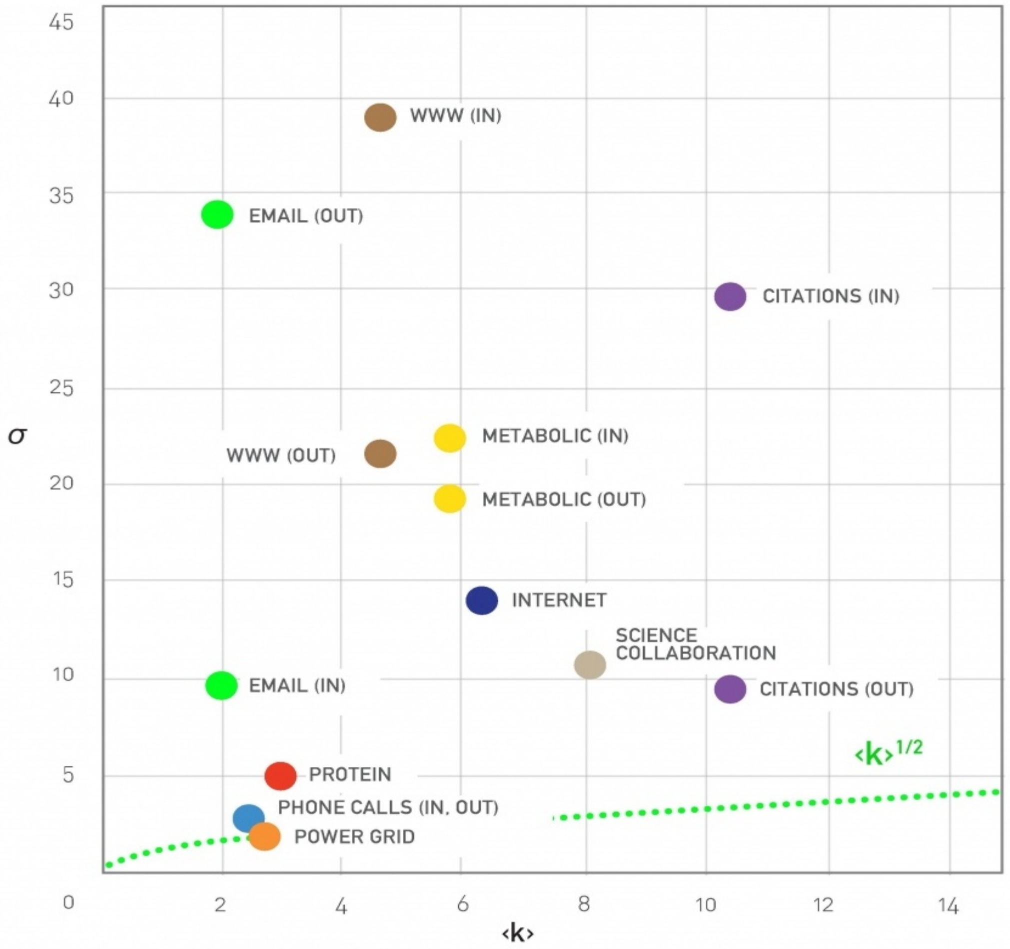
   100 other co-stars?

https://www.wolframalpha.com/ recognizes `x*y, x/y, Zeta(x), x^(-y),` etc.

$$p_k = \frac{k^{-\gamma}}{\zeta(\gamma)} \qquad \zeta(2.12) \approx 1.5452$$

2. For how many co-stars do we still expect to have one actor that has that many co-stars?

Real network examples



Summary

Things to remember

- Definition of scale-free
- Power law
- Formulas for degree distribution
  - Discrete formula
  - Continuous formula
- Formula for k_{max}

Practice on your own

- **(Somewhat) difficult, <u>try to solve it ON YOUR OWN</u>**
- Imagine a connected scale-free graph with 1 million nodes and average degree 5

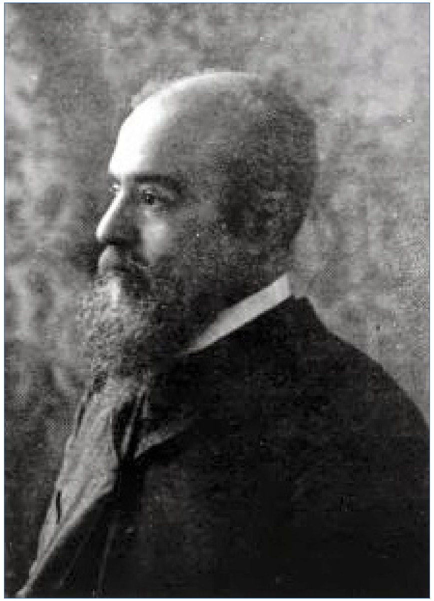
  If we draw 100 nodes from this graph, how many will have degree 1?

  Remember, if the graph is connected, k_{min}=1

  If you cannot clear the unknown in a formula, plot it
- Solution in next slide (shown only in .odp, not .pdf)

Additional contents
(not included in exams)

Pareto's Law



- Italian economist Vilfredo Pareto in the 19th century noted 80% of money was earned by 20% of people

- More recently ...
  - 80 percent of links on the Web point to only 15 percent of pages;
  - 80 percent of citations go to only 38 percent of scientists;
  - 80 percent of links in Hollywood are to 30 percent of actors

- A debate that is still open: the wealth of the 1% and the 0.1%

Moments of degree distribution

- Moments of degree distribution

$$\langle k^n \rangle = \int_{k_{\min}}^{k_{\max}} k^n p_k dk = C \frac{k_{\max}^{n-\gamma+1} - k_{\min}^{n-\gamma+1}}{n-\gamma+1}$$

$$C = (\gamma - 1) k_{\min}^{\gamma-1}$$

Moments of degree distribution (cont.)

$$\sigma_k^2 = \langle k^2 \rangle - \langle k \rangle^2$$

- In a scale-free network

$$\langle k^2 \rangle = \int_{k_{\min}}^{k_{\max}} k^2 p_k dk = C \frac{k_{\max}^{3-\gamma} - k_{\min}^{3-\gamma}}{3-\gamma}$$

- This diverges as $k_{\max} \to \infty$ if $\gamma < 3$
- Hence there is no "typical" scale

When you do not observe the scale-free property

- In general, when there is a limit to k_{max}
- Out-degree in some social networks
- Materials/crystals