

Network flows

Introduction to Network Science

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Topic 20

Sources

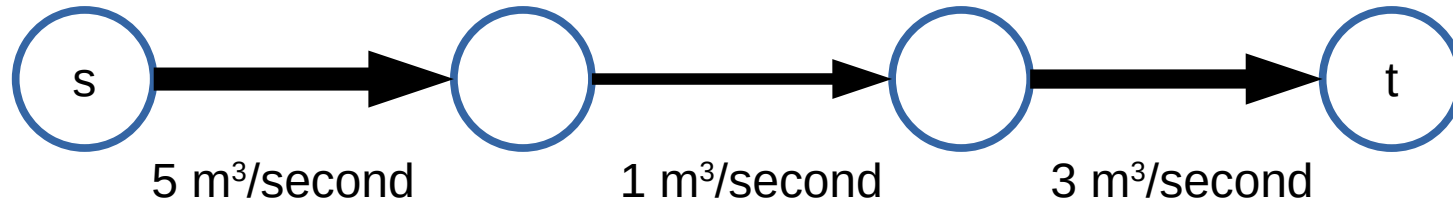
- Barabási 2016 Chapter 9
- [Networks, Crowds, and Markets](#) Ch 3
- C. Castillo: [Graph partitioning](#) 2017

Splitting into two communities:

Max-flow and Min-cut

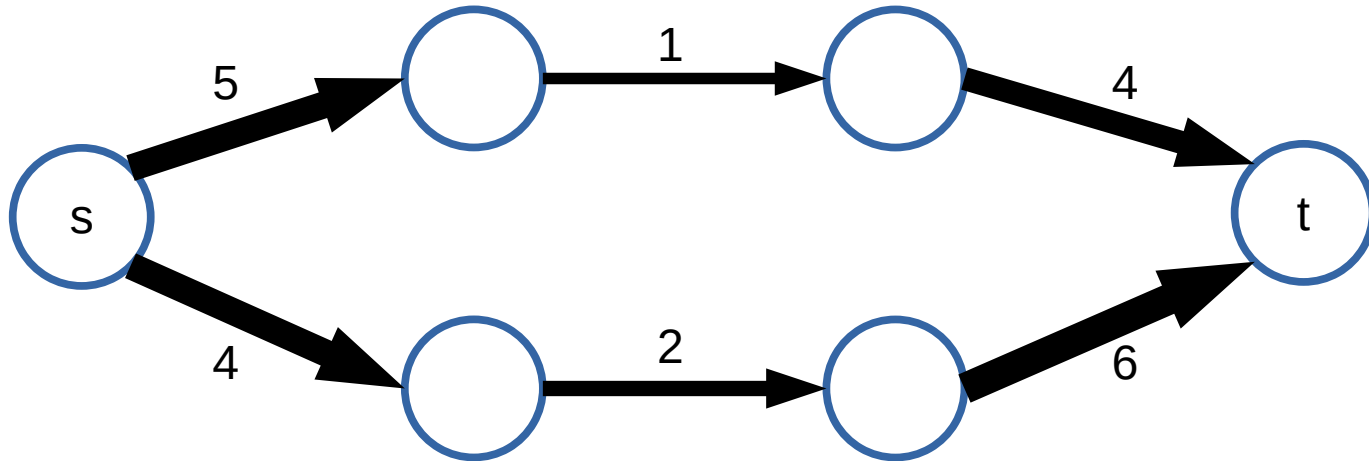
Maximum flow: example 1

- If edge weights were capacities, what is the **maximum flow** that can be sent from s to t ?



Maximum flow: example 2

- If edge weights were capacities, what is the **maximum flow** that can be sent from s to t?



Maximum flow problem

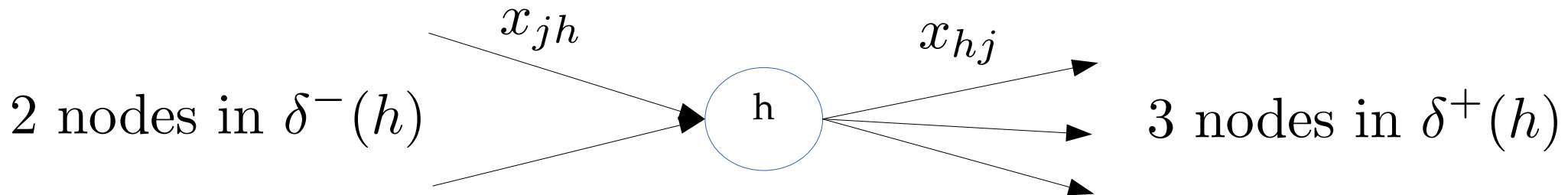
- What is the maximum “flow” that can be carried from s to t ?
 - Think of edge weights as capacities (e.g. m^3/s of water)
- What is the flow of an edge?
 - The amount sent through that edge (an assignment)
- What is the net flow of a node?
 - The amount exiting the node minus the amount entering the node

Formulating the max flow problem

- The flow through each edge should be $\leq k_{ij}$
- Net flow at node h :
$$flow(h) = out_flow(h) - in_flow(h)$$
- Node s has only *out_flow*, should have positive flow v
- Node t has only *in_flow*, should have negative flow $-v$
- *What should be the flow of the other nodes?*

Formulating the max flow problem

- Let v be a feasible flow
- Node s should have positive flow v
- Node t should have negative flow $-v$



- *What should be the flow of an arbitrary node h ?*

$$\sum_{(h,j) \in \delta^+(h)} x_{hj} - \sum_{(i,h) \in \delta^-(h)} x_{ih} = ?$$

Max flow as a linear program

N: set of nodes, A: set of edges

$$\max \quad v \quad (1)$$

$$\sum_{(s,j) \in \delta^+(s)} x_{sj} = v \quad (2)$$

$$- \sum_{(i,t) \in \delta^-(t)} x_{it} = -v \quad (3)$$

$$\sum_{(h,j) \in \delta^+(h)} x_{hj} - \sum_{(i,h) \in \delta^-(h)} x_{ih} = 0, \quad h \in N - \{s, t\} \quad (4)$$

$$x_{ij} \leq k_{ij} \quad (i, j) \in A \quad (5)$$

$$x_{ij} \geq 0 \quad (i, j) \in A \quad (6)$$

Primal-Dual in Linear Programming

PRIMAL

$$\begin{aligned} & \min \sum_j c_j x_j \quad \text{subject to} \\ & \sum_j a_{ij} x_j \geq b_i \quad \forall i \in [m] \\ & x_j \geq 0 \quad \forall j \in [n] \end{aligned}$$

DUAL

$$\begin{aligned} & \max \sum_i y_i b_i \quad \text{subject to} \\ & \sum_i y_i a_{ij} \leq c_j \quad \forall j \in [n] \\ & y_i \geq 0 \quad \forall i \in [m] \end{aligned}$$

Writing the dual: each constraint will become a variable

$$\max \quad v \tag{1}$$

$$\sum_{(s,j) \in \delta^+(s)} x_{sj} = v \quad \text{variable } u_s \tag{2}$$

$$- \sum_{(i,t) \in \delta^-(t)} x_{it} = -v \quad \text{variable } u_t \tag{3}$$

$$\sum_{(h,j) \in \delta^+(h)} x_{hj} - \sum_{(i,h) \in \delta^-(h)} x_{ih} = 0, \quad h \in N - \{s, t\} \quad \text{variables } u_j \tag{4}$$

$$x_{ij} \leq k_{ij} \quad (i, j) \in A \quad \text{variables } y_{ij} \tag{5}$$

$$x_{ij} \geq 0 \quad (i, j) \in A \tag{6}$$

Writing the dual

- Remember: the infimum of the solutions of the dual is the supremum of the solutions of primal

$$\min \sum_{(i,j) \in A} k_{ij} y_{ij}$$

$$u_i - u_j + y_{ij} \geq 0, (i, j) \in A$$

$$-u_s + u_t = 1$$

$$y_{ij} \geq 0$$

(Think of y_{ij} as
0 or 1)

- Variables u_i don't enter the objective, only their difference is in the constraints
- We can set them arbitrarily, in particular $u_s = 0, u_t = 1$

Dual (after simplification)

$$\min \sum_{(i,j) \in A} k_{ij} y_{ij}$$

$$u_i - u_j + y_{ij} \geq 0, (i, j) \in A$$

$$y_{ij} \geq 0$$

$$u_s = 0, u_t = 1$$

This is a min-cut problem! Every feasible solution represents a cut

k_{ij} are given: capacity of the edges

y_{ij} are unknowns: whether an edge is part of a cut, $y_{ij}=1$, or not, $y_{ij}=0$

$\sum k_{ij} y_{ij}$ is the cost of the cut

Dual (after simplification)

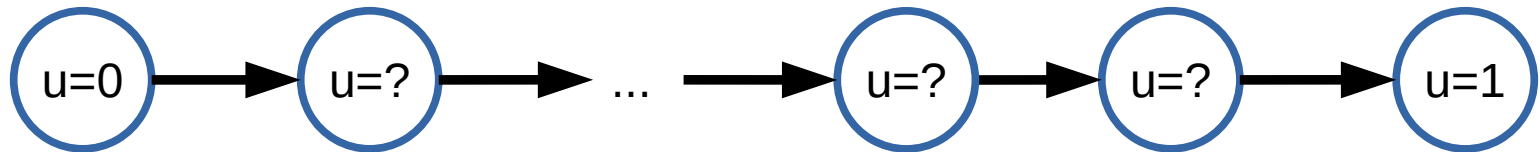
$$\min \sum_{(i,j) \in A} k_{ij} y_{ij}$$

$$u_i - u_j + y_{ij} \geq 0, (i, j) \in A$$

$$y_{ij} \geq 0$$

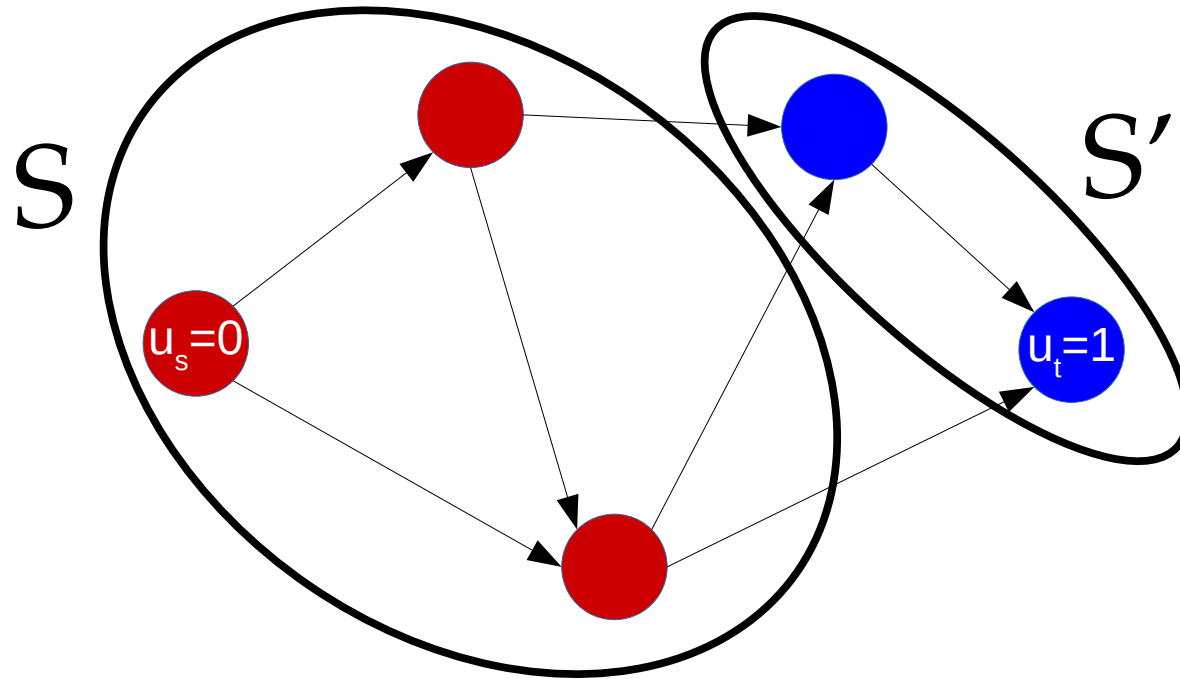
$$u_s = 0, u_t = 1$$

- What happens with the values of u in every simple path going from s to t ?



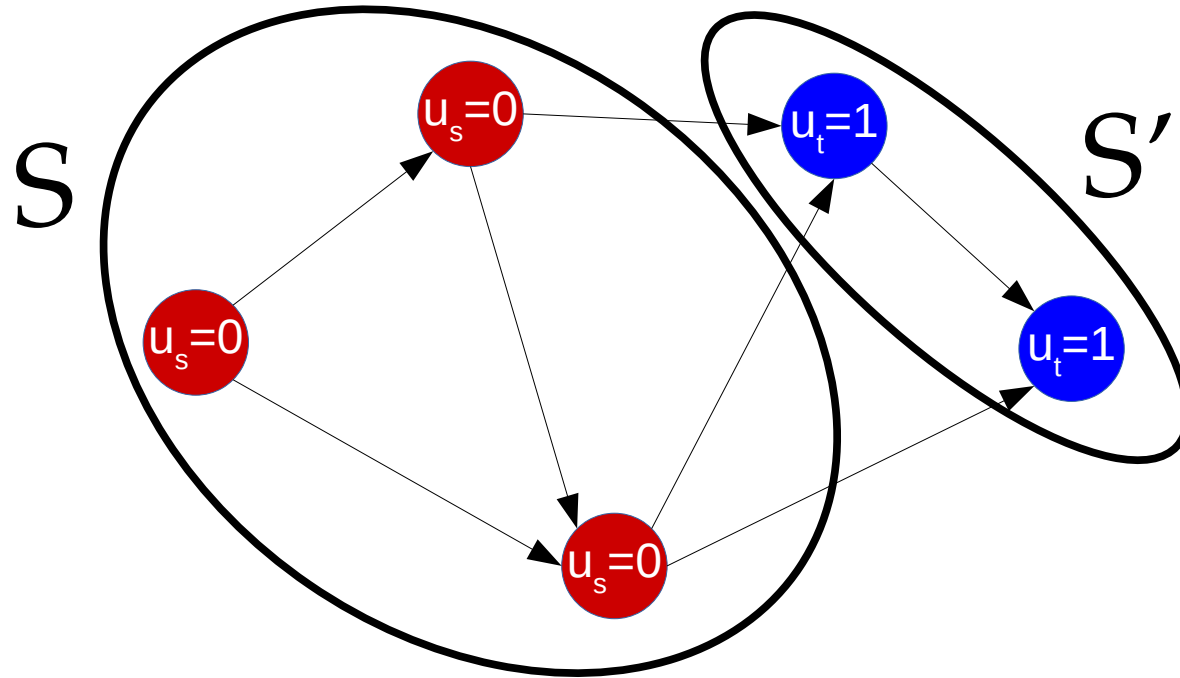
Dual solutions are cuts

- Every feasible solution of the dual has the form of a cut (S, S')



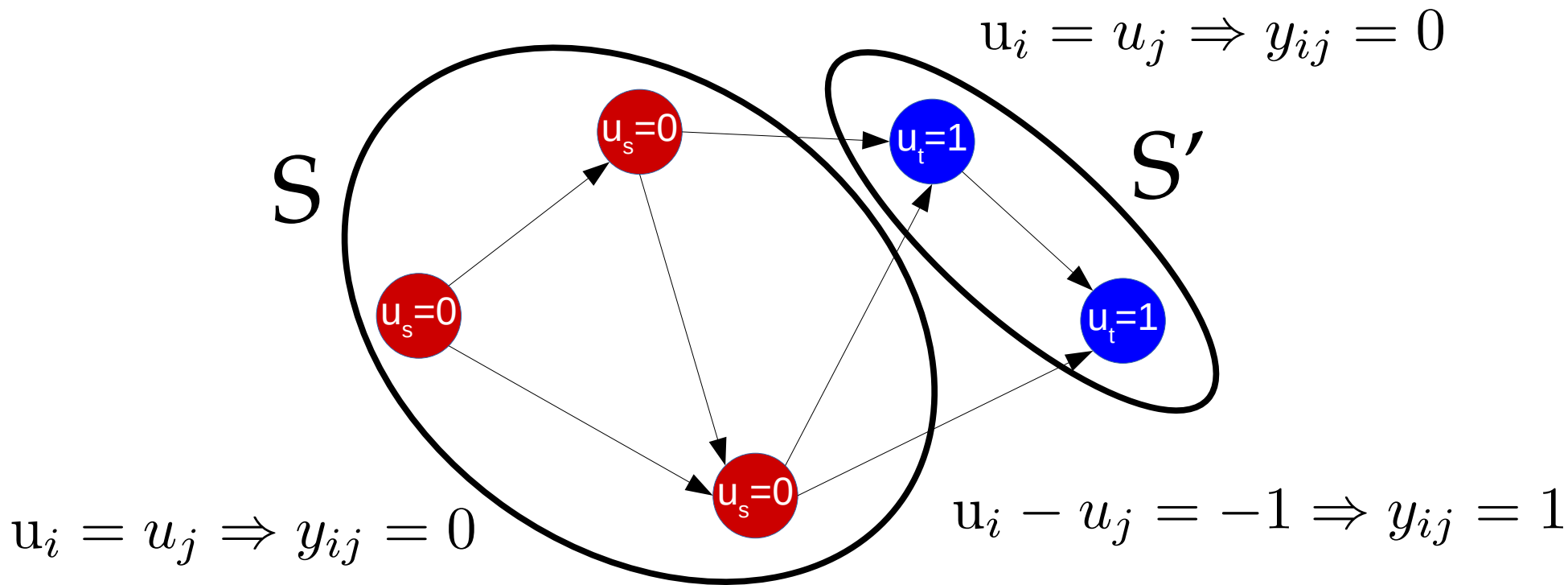
Dual solutions are cuts

- Every feasible solution of the dual has the form of a cut (S, S')



Dual solutions are (s-t)-cuts

$u_i - u_j + y_{ij} \geq 0$ and remember we're trying to minimize $\sum k_{ij} y_{ij}$



Primal (max flow)

$$\begin{aligned} \max \quad & v \\ \sum_{(s,j) \in \delta^+(s)} x_{sj} &= v \\ - \sum_{(i,t) \in \delta^-(t)} x_{it} &= -v \\ \sum_{(h,j) \in \delta^+(h)} x_{hj} - \sum_{(i,h) \in \delta^-(h)} x_{ih} &= 0, \quad h \in N - \{s, t\} \\ x_{ij} &\leq k_{ij} \quad (i, j) \in A \\ x_{ij} &\geq 0 \quad (i, j) \in A \end{aligned}$$

Dual (min cut)

$$\begin{aligned} \min \quad & \sum_{(i,j) \in A} k_{ij} y_{ij} \\ u_i - u_j + y_{ij} &\geq 0, \quad (i, j) \in A \\ y_{ij} &\geq 0 \\ u_s = 0, u_t = 1 \end{aligned}$$

Both problems have the same optimal solution, because one is the dual of the other \Rightarrow **the maximum flow is equal to the minimum cut**

About the optimal solution

(remember: both problems have the same optimal solution)

$$\begin{aligned} \max \quad & v \\ \sum_{(s,j) \in \delta^+(s)} x_{sj} &= v \\ - \sum_{(i,t) \in \delta^-(t)} x_{it} &= -v \\ \sum_{(h,j) \in \delta^+(h)} x_{hj} - \sum_{(i,h) \in \delta^-(h)} x_{ih} &= 0, \quad h \in N - \{s, t\} \\ x_{ij} &\leq k_{ij} \quad (i, j) \in A \\ x_{ij} &\geq 0 \quad (i, j) \in A \end{aligned}$$

$$\begin{aligned} \min \quad & \sum_{(i,j) \in A} k_{ij} y_{ij} \\ u_i - u_j + y_{ij} &\geq 0, \quad (i, j) \in A \\ y_{ij} &\geq 0 \\ u_s = 1, u_t &= 0 \end{aligned}$$

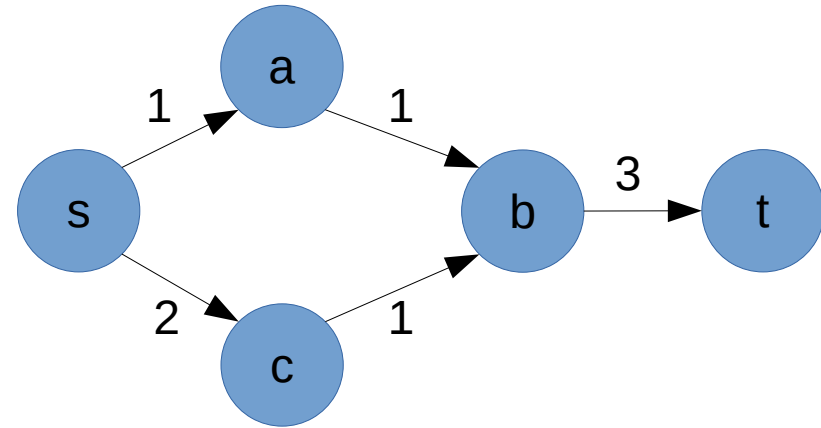
y_{ij} is a dual variable corresponding to primal constraint $x_{ij} \leq k_{ij}$

If y_{ij} is non-zero, then the corresponding constraint is **tight** (holds with equality)

What does it mean for the edges in the cut?

Upload to Nearpod Collaborate
<https://nearpod.com/student/>
 Code to be given during class

Exercise



Write the primal equations for this graph

- Unknowns: $x_{sa}, x_{sc}, x_{ab}, x_{cb}, x_{bt}, v$

Write the dual equations for this graph

- Unknowns: $u_a, u_b, u_c, y_{sa}, y_{sc}, y_{ab}, y_{cb}, y_{bt}$

Guess both solutions, check that you satisfy all constraints

$$\begin{aligned}
 \max \quad & v \\
 \sum_{(s,j) \in \delta^+(s)} x_{sj} &= v \\
 - \sum_{(i,t) \in \delta^-(t)} x_{it} &= -v \\
 \sum_{(h,j) \in \delta^+(h)} x_{hj} - \sum_{(i,h) \in \delta^-(h)} x_{ih} &= 0, \quad h \in N - \{s, t\} \\
 x_{ij} &\leq k_{ij} \quad (i, j) \in A \\
 x_{ij} &\geq 0 \quad (i, j) \in A
 \end{aligned}$$

$$\begin{aligned}
 \min \quad & \sum_{(i,j) \in A} k_{ij} y_{ij} \\
 u_i - u_j + y_{ij} &\geq 0, \quad (i, j) \in A \\
 y_{ij} &\geq 0 \\
 u_s = 0, u_t &= 1
 \end{aligned}$$

This is an efficient method

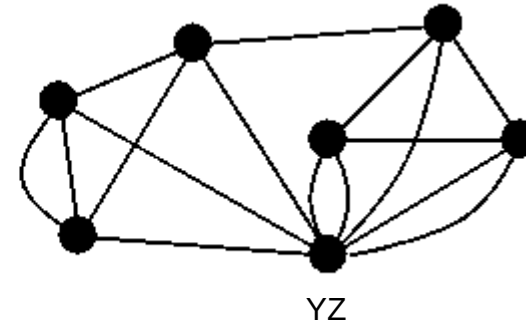
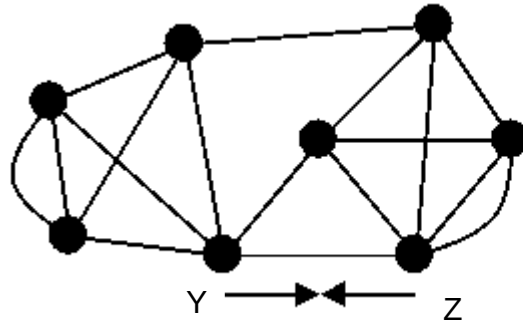
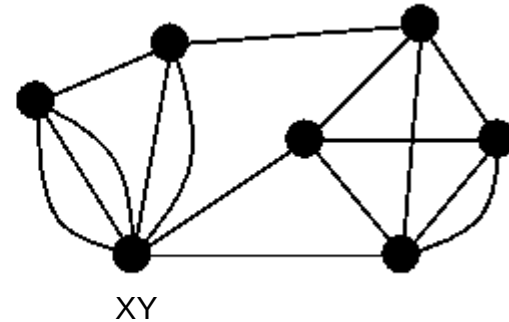
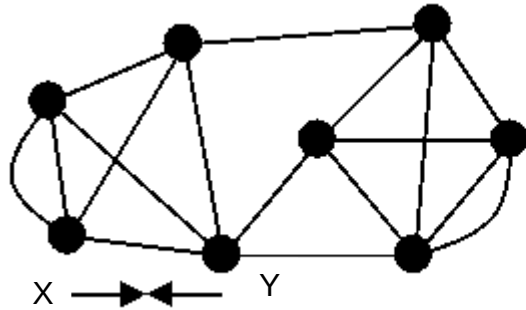
- Min-cut and Max-flow are equivalent problems
 - Their solutions are also equal: the value of the maximum flow is equivalent to the minimum cut
- Think of a chain that breaks at the weakest link
- Both can be solved exactly in polynomial time

Randomized algorithm for (s-t)-cuts (Karger's Algorithm)

Randomized algorithm for (s-t)-cuts

- Pick an edge at random (u, v)
- Merge u and v in new vertex uv
- Edges between u and v are removed
- Edges pointing to u or v are added as multi-edges to vertex uv
- When only s and t remain, the multi-edges are a cut, probably the minimum one

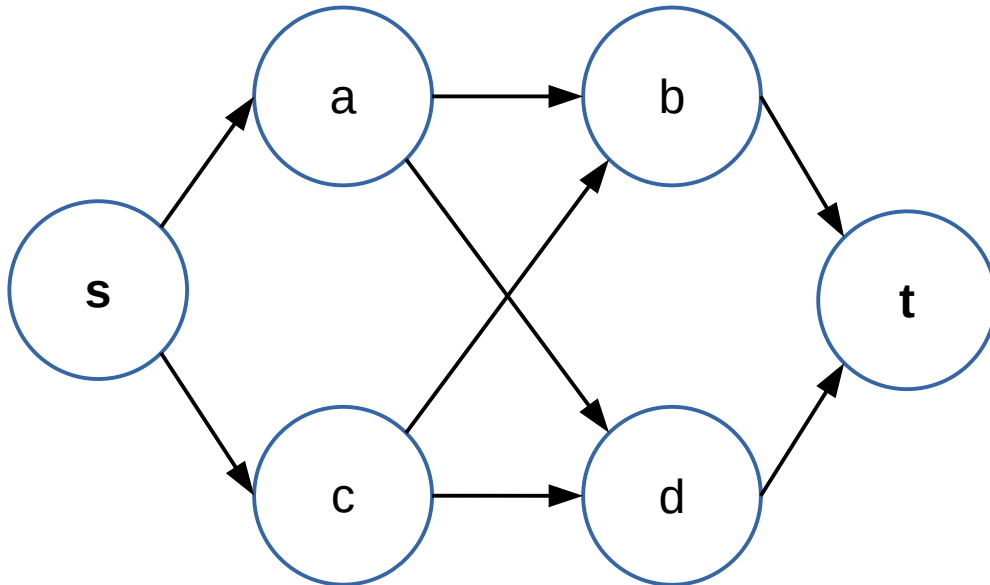
Example merges (“contractions”)



Exercise

Run the randomized algorithm on this graph

- Pick an edge at random (u, v)
- Merge u and v in new vertex uv
- Edges between u and v are removed
- Edges pointing to u or v are added as multi-edges to vertex uv
- When only s and t remain, the multi-edges are a cut, probably the minimum one



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The randomized algorithm might miss the min cut

- Multiple runs are required
- The probability that this finds the min cut in one run is about $1/\log(n)$, so $O(\log n)$ iterations are required to find min cut
- Each iteration costs $O(n^2 \log n)$
- $O(n^2 \log^2 n)$ operations needed to find min cut
- Exact algorithm: $O(n^3 + n^2 \log n)$; the n^3 is because of $|V||E|$ operations required

Summary

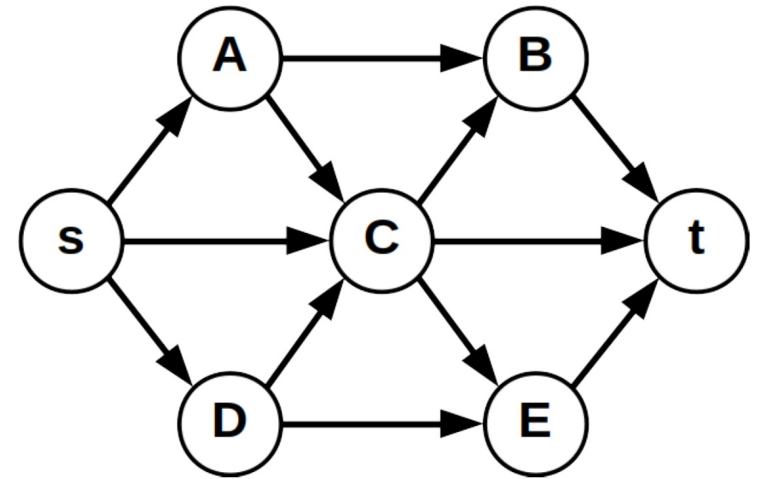
Things to remember

- Minimum s-t cut in a graph = set of edges
- The sum of the capacities of those edges is the maximum s-t flow the graph can carry
- How to write the primal and dual equations for max flow and min cut
- How to run the approximate randomized algorithm

Practice on your own

Consider (s, t) -cuts on the graph on the right, where s is the source node and t is the terminal node. Assume every edge has cost equal to 1.

1. By visual inspection, what is the minimum cost of an (s, t) -cut in this graph, and what is an example of a cut having that cost?
2. Run the algorithm for randomized (s, t) -cuts we saw in class, drawing all intermediate graphs, and indicate the cost of the resulting cut.



Practice on your own (cont.)

- Create a graph
- Write the min-flow, max-cut equations
- Find an optimal solution
- Run the randomized s-t cut algorithm