The Friendship Paradox

Social Networks Analysis and Graph Algorithms

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Sources

- A. L. Barabási (2016). Network Science – Chapter 04

- F. Menczer, S. Fortunato, C. A. Davis (2020). A First Course in Network Science – Chapter 03

- URLs cited in the footer of specific slides

Sampling a random node

vs

sampling at random one of the two nodes attached to a random edge

Exercise
Numerical calculation of friendship paradox

- What is the probability of selecting Tom **if we select a random node**?

- What is the probability of selecting Tom **if we select a random edge and then randomly one of the two nodes attached to it**?
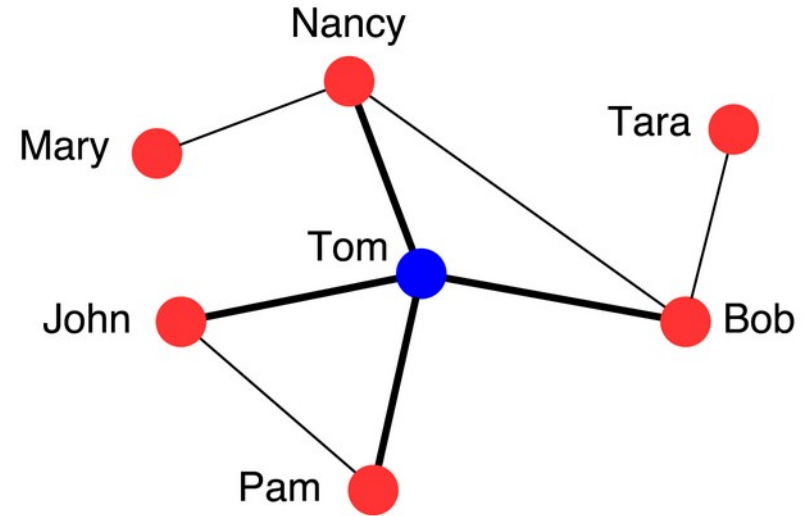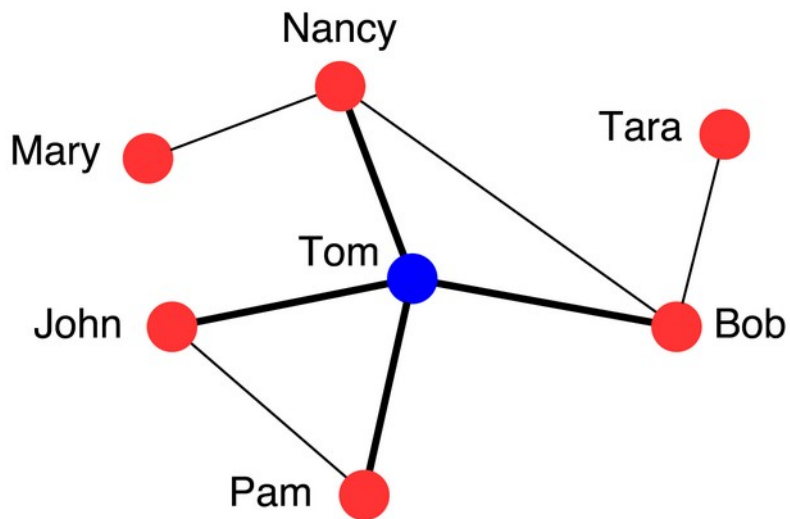
Sampling a random node

vs

sampling a random friend
of a random node

Average degree of friends

- Average degree

$$(1 + 3 + 3 + 1 + 4 + 2 + 2)/7 = 16/7 \simeq 2.29$$

- Average degree of friends of:

  - Mary: 3

  - Nancy: $(1+4+3)/3 = 8/3$

  - Tara: 3

  - Bob: $(1+3+4)/3 = 8/3$

  - Tom: $(3+3+2+2)/4 = 10/4$

  - John: $(4+2)/2 = 3$

  - Pam: $(4+2)/2 = 3$

  - Average degree of friends $\simeq$ **2.83** (> 2.29)
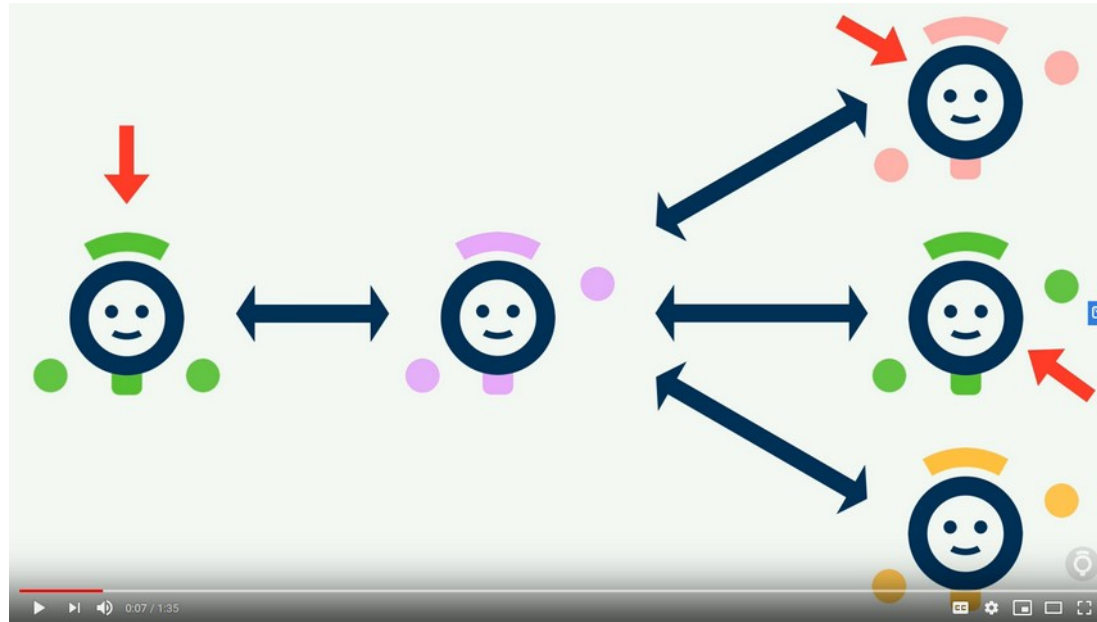
The friendship paradox

- Take a random person x; what is the expected degree of this person? $<k>$

- Take a random person x, now pick one of x's neighbors, let's say y; what is the expected degree of y?

  It is not $<k>$

- This "paradox" is a useful vaccination strategy

Sampling bias and the friendship paradox (1'35'')

https://www.youtube.com/watch?v=httLvVufAYs

Imagine you're at
a random airport on earth

- Is it more likely to be ...

  a large airport or a small airport?

- If you take a random flight out of it ...

  will it go to a large airport or a small airport?

An example of friendship paradox

- Pick a random airport on Earth
  - Most likely it will be a small airport

- However, no matter how small it is, it **will** have flights to big airports

- On average those airports will have much larger degree

Exercise [B. 2016, Ex. 4.10.2]
"Friendship Paradox"

- Remember p_k is the probability that a node has k "friends"

- If we randomly select a link, the probability that a node at any end of the link has k friends is
  $q_k = C \; k \; p_k$ where C is a normalization factor

  - (a) Find C (the sum of q_k must be 1)

Exercise [B. 2016, Ex. 4.10.2]

"Friendship Paradox"

- If we randomly select a link, the probability that a node at any end of the link has k friends is

  $q_k = C \, k \, p_k$ where C is a normalization factor

- q_k is also the prob. that a randomly chosen node has a neighbor of degree k

  (b) Find its expectation $E[a_v]$ which we will call $<k_F>$

  Remember $\quad E[X] = \sum_{X_{\min}}^{X_{\max}} x \cdot P(X = x)$

Exercise [B. 2016, Ex. 4.10.2]
"Friendship Paradox"

(c) Compute the expected number of friends of a neighbor of a randomly chosen node in the case below

(d) compare with the expected number of friends of a randomly chosen node

$$N = 10000$$

$$\gamma = 2.3$$

$$k_{\min} = 1$$

$$k_{\max} = 1000$$

$$\langle k^n \rangle = C \frac{k_{\max}^{n-\gamma+1} - k_{\min}^{n-\gamma+1}}{n - \gamma + 1}$$

$$C = (\gamma - 1)k_{\min}^{\gamma-1}$$

Code

```python
def degree_moment(kmin, kmax, moment, gamma):
    C = (gamma-1.0)*(kmin**(gamma-1.0))
    numerator = (kmax**(moment-gamma+1.0) - kmin**(moment-gamma+1.0))
    denominator = (moment-gamma+1.0)
    return C * numerator / denominator
```

```python
kavg = degree_moment(kmin=1, kmax=1000, moment=1, gamma=2.3)
print(kavg)
```

3.787798988222529

```python
ksqavg = degree_moment(kmin=1, kmax=1000, moment=2, gamma=2.3)
print(ksqavg)
```

231.94329076177414

```python
print(ksqavg / kavg)
```

61.23431879119234

Summary

Practice on your own

- Draw a small graph, and sample from that graph until you're convinced $\langle k_F \rangle > \langle k \rangle$