Network flows

Introduction to Network Science
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Topic 20

Sources

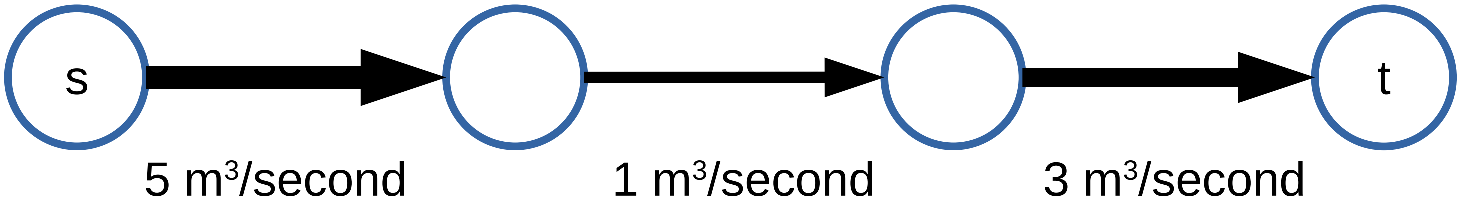
- Barabási 2016 Chapter 9
- Networks, Crowds, and Markets Ch 3
- C. Castillo: Graph partitioning 2017

Splitting into two communities:

Max-flow and Min-cut

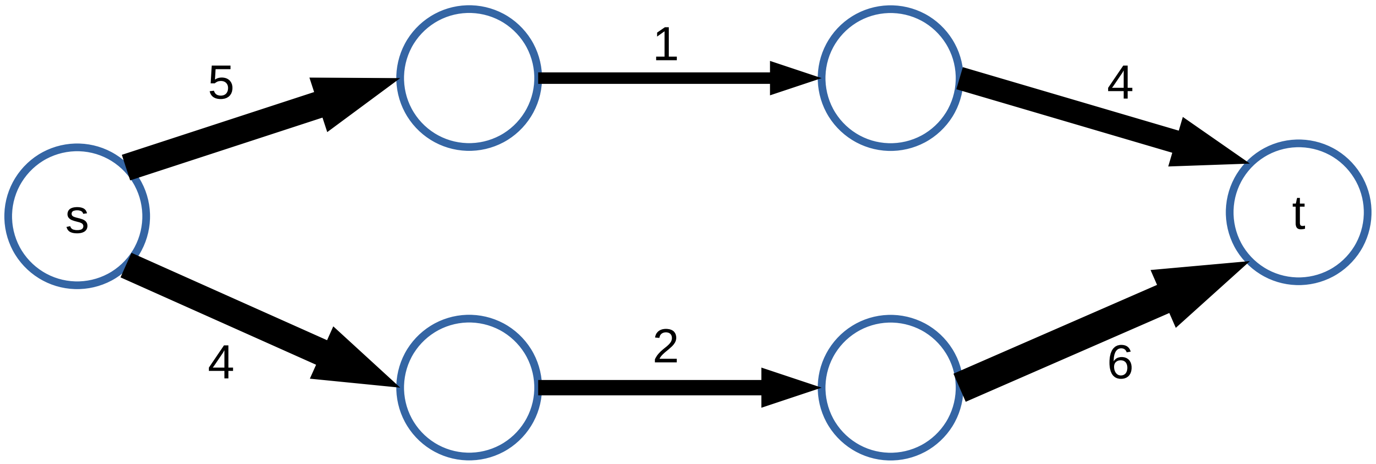
Maximum flow: example 1

- If edge weights were capacities, what is the maximum flow that can be sent from s to t?



Maximum flow: example 2

- If edge weights were capacities, what is the maximum flow that can be sent from s to t?



Maximum flow problem

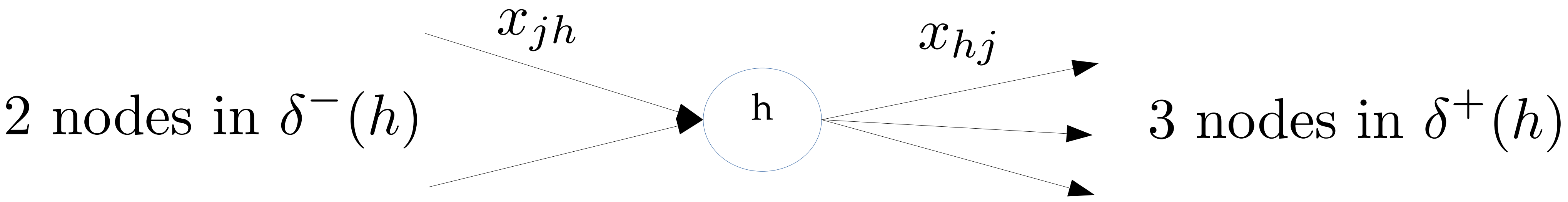
- What is the maximum “flow” that can be carried from s to t?
  - Think of edge weights as capacities (e.g. m^3/s of water)
- What is the flow of an edge?
  - The amount sent through that edge (an assignment)
- What is the net flow of a node?
  - The amount exiting the node minus the amount entering the node

Formulating the max flow problem

- The flow through each edge should be $\leq k_{ij}$
- Net flow at node h:
  flow(h) = out_flow(h) - in_flow(h)
- Node *s* has only *out_flow*, should have positive flow *v*
- Node *t* has only *in_flow*, should have negative flow *-v*
- *What should be the flow of the other nodes?*

Formulating the max flow problem

- Let v be a feasible flow
- Node *s* should have positive flow *v*
- Node *t* should have negative flow *-v*



- *What should be the flow of an arbitrary node h?*

$$\sum_{(h,j)\in\delta^+(h)} x_{hj} - \sum_{(i,h)\in\delta^-(h)} x_{ih} = \ ?$$

Max flow as a linear program
N: set of nodes, A: set of edges

$$\max \quad v \tag{1}$$

$$\sum_{(s,j)\in\delta^+(s)} x_{sj} = v \tag{2}$$

$$-\sum_{(i,t)\in\delta^-(t)} x_{it} = -v \tag{3}$$

$$\sum_{(h,j)\in\delta^+(h)} x_{hj} - \sum_{(i,h)\in\delta^-(h)} x_{ih} = 0, \quad h \in N - \{s,t\} \tag{4}$$

$$x_{ij} \leq k_{ij} \quad (i,j) \in A \tag{5}$$

$$x_{ij} \geq 0 \quad (i,j) \in A \tag{6}$$

http://www.dii.unisi.it/~agnetis/mincutENG.pdf

Primal-Dual in Linear Programming

PRIMAL

$$\min \sum_j c_j x_j \quad \text{subject to}$$

$$\sum_j a_{ij} x_j \geq b_i \quad \forall i \in [m]$$

$$x_j \geq 0 \quad \forall j \in [n]$$

DUAL

$$\max \sum_i y_i b_i \quad \text{subject to}$$

$$\sum_i y_i a_{ij} \leq c_j \quad \forall j \in [n]$$

$$y_i \geq 0 \quad \forall i \in [m]$$

Writing the dual: each constraint will become a variable

$$\max \quad v \tag{1}$$

$$\sum_{(s,j)\in\delta^+(s)} x_{sj} = v \qquad \text{variable } u_s \tag{2}$$

$$-\sum_{(i,t)\in\delta^-(t)} x_{it} = -v \qquad \text{variable } u_t \tag{3}$$

$$\sum_{(h,j)\in\delta^+(h)} x_{hj} - \sum_{(i,h)\in\delta^-(h)} x_{ih} = 0, \quad h \in N - \{s,t\} \qquad \text{variables } u_j \tag{4}$$

$$x_{ij} \le k_{ij} \quad (i,j) \in A \qquad \text{variables } y_{ij} \tag{5}$$

$$x_{ij} \ge 0 \quad (i,j) \in A \tag{6}$$

Writing the dual

- Remember: the infimum of the solutions of the dual is the supremum of the solutions of primal

$$\min \sum_{(i,j)\in A} k_{ij} y_{ij}$$

$$\begin{aligned} u_i - u_j + y_{ij} &\geq 0, (i,j) \in A \\ -u_s + u_t &= 1 \\ y_{ij} &\geq 0 \end{aligned}$$

(Think of y_{ij} as 0 or 1)

- Variables u_i don't enter the objective, only their difference is in the constraints
- We can set them arbitrarily, in particular $u_s = 0$, $u_t = 1$

Dual (after simplification)

$$min \sum_{(i,j)\in A} k_{ij}y_{ij}$$

$$\begin{aligned} u_i - u_j + y_{ij} \quad &\geq \quad 0, (i,j) \in A \\ y_{ij} \quad &\geq \quad 0 \\ u_s = 0, u_t = 1 & \end{aligned}$$

This is a min-cut problem! Every feasible solution represents a cut

k_{ij} are given: capacity of the edges

y_{ij} are unknowns: whether an edge is part of a cut, $y_{ij}=1$, or not, $y_{ij}=0$

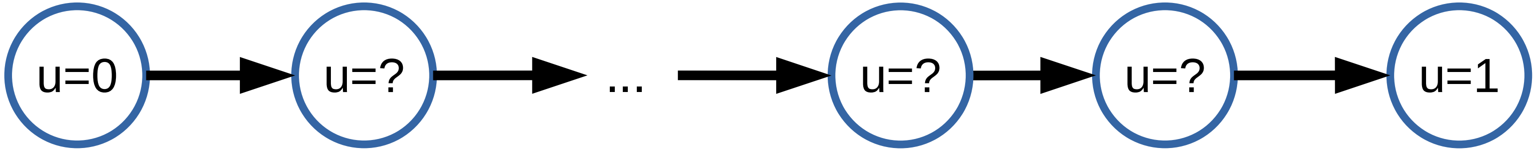
$\sum k_{ij}y_{ij}$ is the cost of the cut

Dual (after simplification)

$$\min \sum_{(i,j)\in A} k_{ij} y_{ij}$$

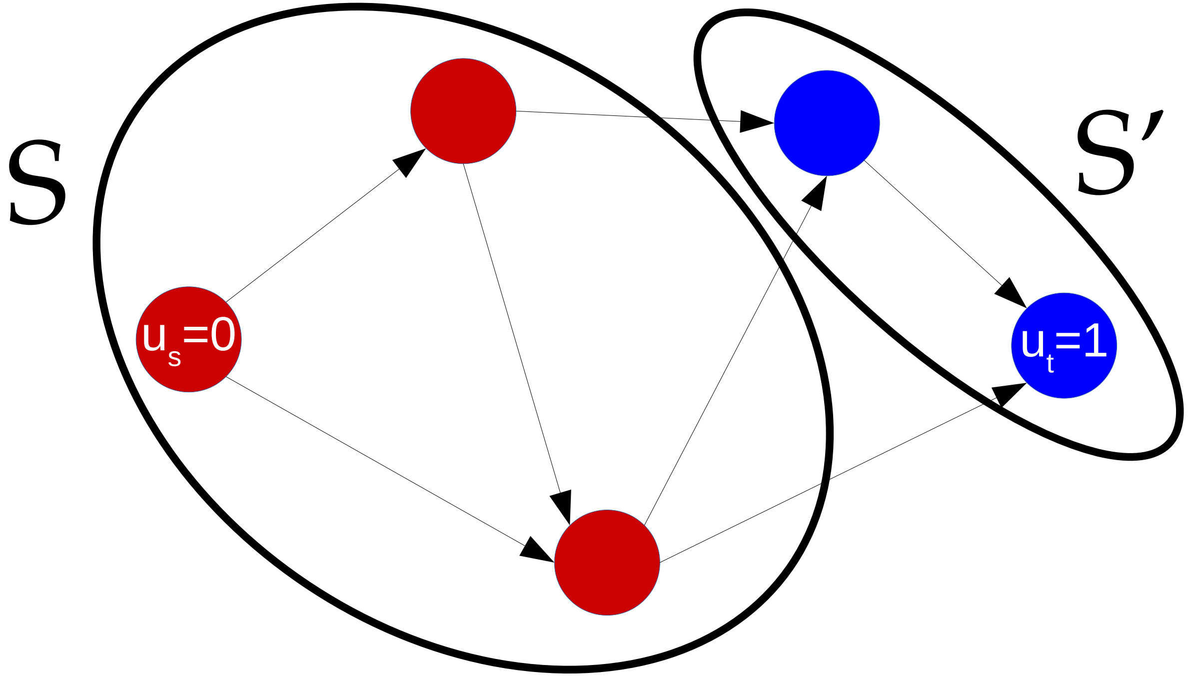
$$\begin{aligned} u_i - u_j + y_{ij} &\geq 0, (i,j) \in A \\ y_{ij} &\geq 0 \\ u_s = 0, u_t &= 1 \end{aligned}$$

- What happens with the values of u in every simple path going from s to t?



Dual solutions are cuts

- Every feasible solution of the dual has the form of a cut (S, S')



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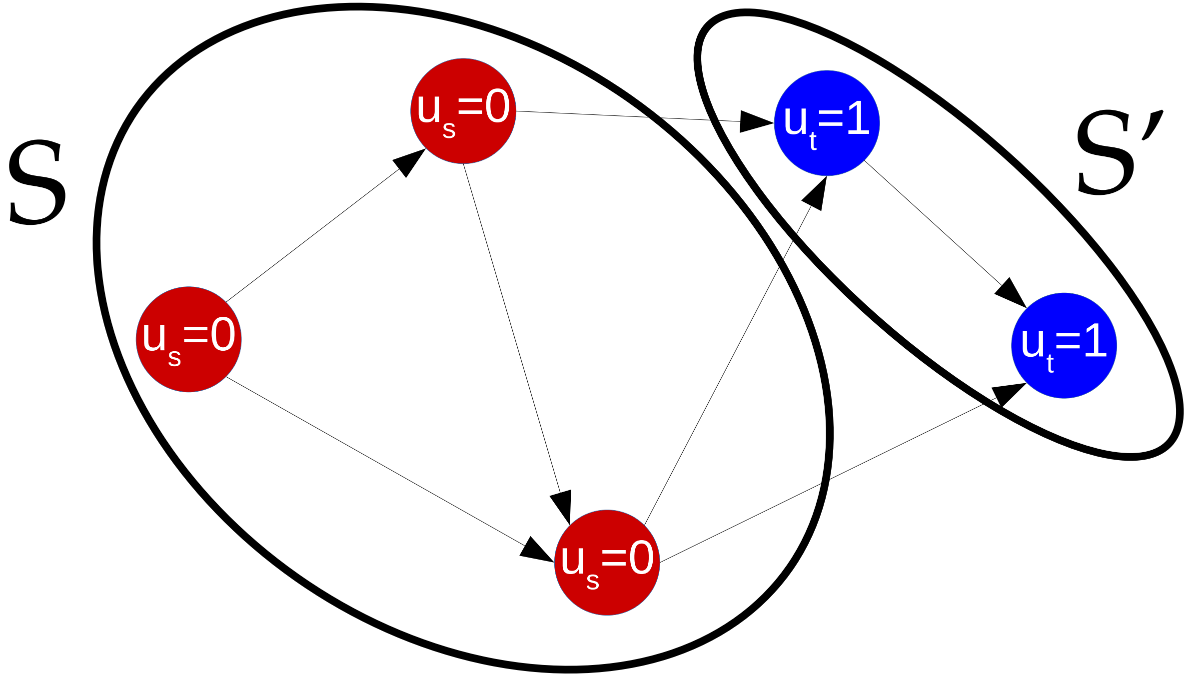
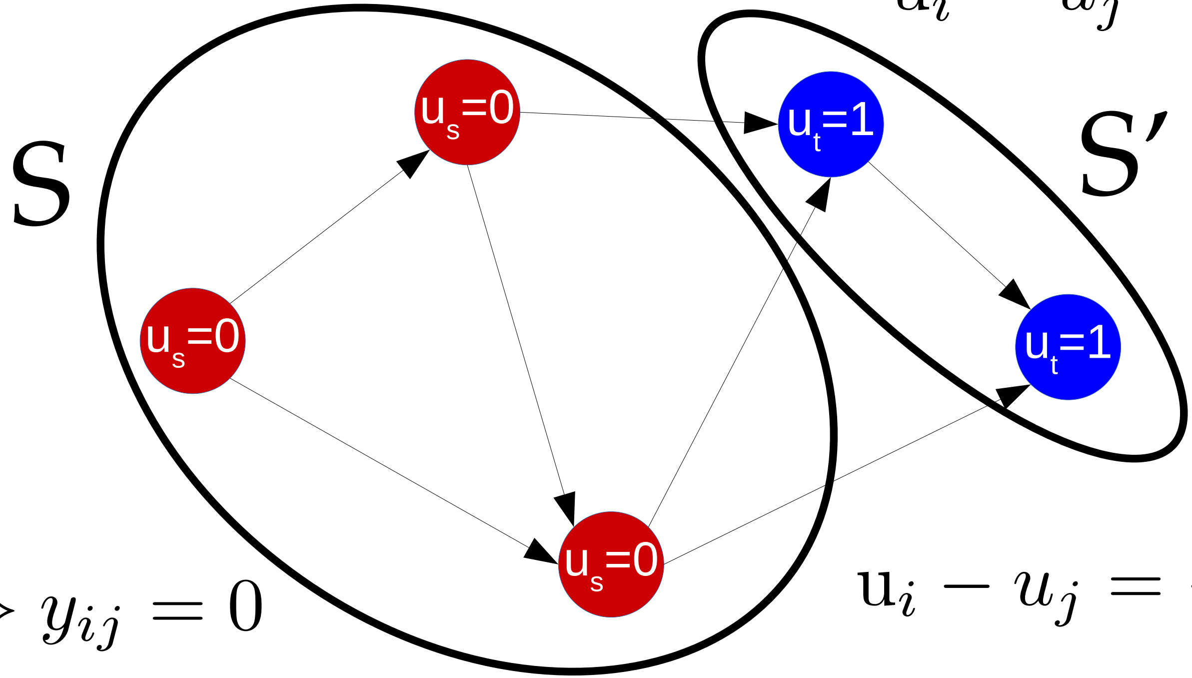


Dual solutions are (s-t)-cuts

$u_i - u_j + y_{ij} \geq 0$ and remember we're trying to minimize $\sum k_{ij} y_{ij}$

$u_i = u_j \Rightarrow y_{ij} = 0$



$u_i = u_j \Rightarrow y_{ij} = 0$

$u_i - u_j = -1 \Rightarrow y_{ij} = 1$

Primal (max flow)

$$
\begin{aligned}
\max \quad & v \\
\sum_{(s,j)\in\delta^+(s)} x_{sj} &= v \\
-\sum_{(i,t)\in\delta^-(t)} x_{it} &= -v \\
\sum_{(h,j)\in\delta^+(h)} x_{hj} - \sum_{(i,h)\in\delta^-(h)} x_{ih} &= 0, \quad h \in N - \{s,t\} \\
x_{ij} &\le k_{ij} \quad (i,j) \in A \\
x_{ij} &\ge 0 \quad (i,j) \in A
\end{aligned}
$$

Dual (min cut)

$$
\begin{aligned}
\min \sum_{(i,j)\in A} k_{ij} y_{ij} & \\
u_i - u_j + y_{ij} &\ge 0, (i,j) \in A \\
y_{ij} &\ge 0 \\
u_s = 0, u_t = 1 &
\end{aligned}
$$

Both problems have the same optimal solution, because one is the dual of the other ⇒ **the maximum flow is equal to the minimum cut**

About the optimal solution
(remember: both problems have the same optimal solution)

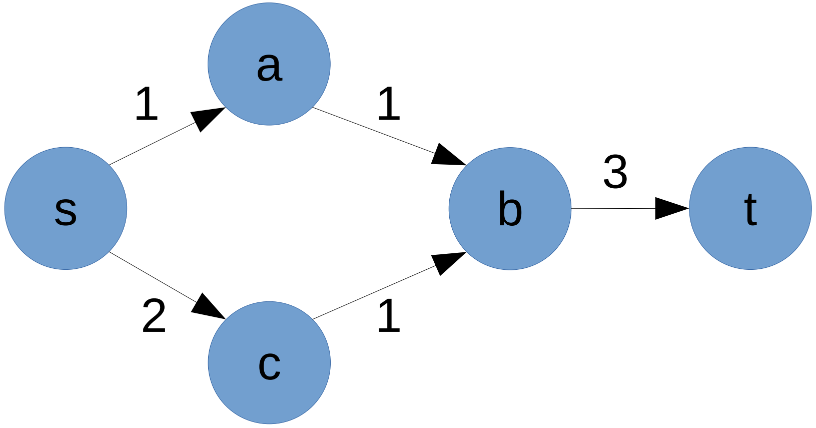
$$
\begin{aligned}
\max \quad & v \\
\sum_{(s,j)\in\delta^+(s)} x_{sj} &= v \\
-\sum_{(i,t)\in\delta^-(t)} x_{it} &= -v \\
\sum_{(h,j)\in\delta^+(h)} x_{hj} - \sum_{(i,h)\in\delta^-(h)} x_{ih} &= 0, \quad h \in N - \{s,t\} \\
\boxed{x_{ij} \le k_{ij} \quad (i,j) \in A} & \\
x_{ij} \ge 0 \quad (i,j) \in A &
\end{aligned}
$$

$$
\begin{aligned}
min \sum_{(i,j)\in A} k_{ij} \boxed{y_{ij}} & \\
u_i - u_j + y_{ij} &\ge 0, (i,j) \in A \\
y_{ij} &\ge 0 \\
u_s = 1, u_t = 0 &
\end{aligned}
$$

y_{ij} is a dual variable corresponding to primal constraint $x_{ij} \le k_{ij}$
If y_{ij} is non-zero, then the corresponding constraint is **tight** (holds with equality)
What does it mean for the edges in the cut?

Exercise

Upload to Nearpod Collaborate
https://nearpod.com/student/
Code to be given during class



Write the primal equations for this graph

- Unknowns: x_{sa}, x_{sc}, x_{ab}, x_{cb}, x_{bt}, v

Write the dual equations for this graph

- Unknowns: u_a, u_b, u_c, y_{sa}, y_{sc}, y_{ab}, y_{cb}, y_{bt}

Guess both solutions, check that you satisfy all constraints

$$
\begin{array}{rcl}
\max & & v \\
\sum_{(s,j)\in\delta^+(s)} x_{sj} & = & v \\
-\sum_{(i,t)\in\delta^-(t)} x_{it} & = & -v \\
\sum_{(h,j)\in\delta^+(h)} x_{hj} - \sum_{(i,h)\in\delta^-(h)} x_{ih} & = & 0, \quad h \in N - \{s,t\} \\
x_{ij} & \leq & k_{ij} \quad (i,j) \in A \\
x_{ij} & \geq & 0 \quad (i,j) \in A
\end{array}
$$

$$
\begin{array}{rcl}
\min \sum_{(i,j)\in A} k_{ij} y_{ij} & & \\
u_i - u_j + y_{ij} & \geq & 0, (i,j) \in A \\
y_{ij} & \geq & 0 \\
u_s = 0, u_t = 1 & &
\end{array}
$$

This is an efficient method

- Min-cut and Max-flow are equivalent problems
  - Their solutions are also equal: the value of the maximum flow is equivalent to the minimum cut
- Think of a chain that breaks at the weakest link
- Both can be solved exactly in polynomial time

Randomized algorithm for (s-t)-cuts (Karger’s Algorithm)

These contents were not covered in 2021

Randomized algorithm for (s-t)-cuts

- Pick an edge at random (u,v)
- Merge u and v in new vertex uv
- Edges between u and v are removed
- Edges pointing to u or v are added as multi-edges to vertex uv
- When only s and t remain, the multi-edges are a cut, <u>probably</u> the minimum one

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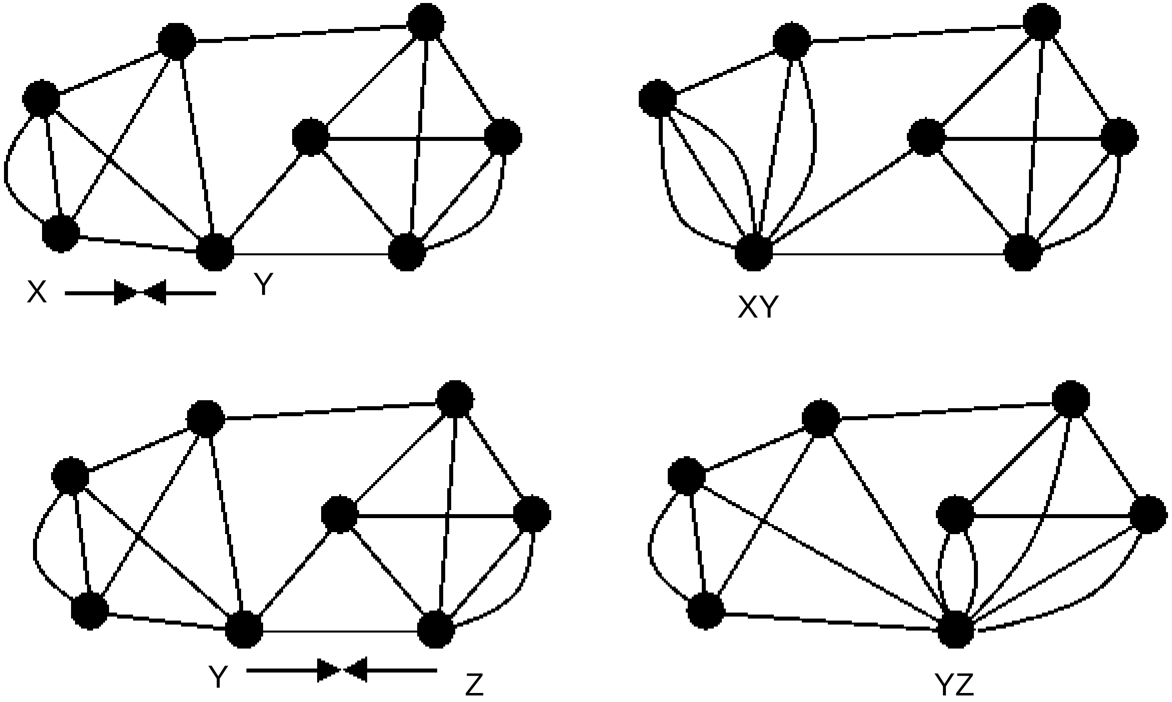
Example merges (“contractions”)



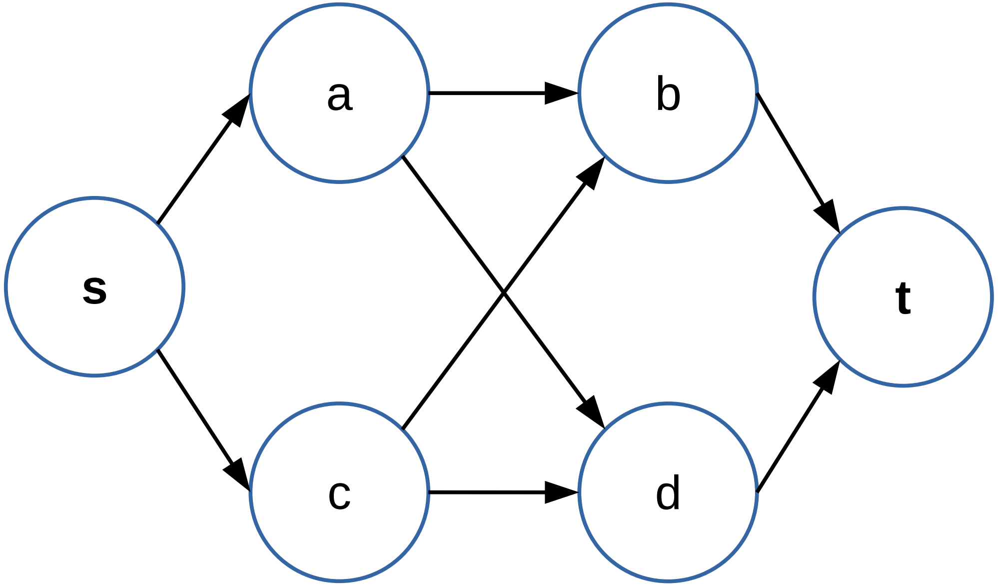
These contents were not covered in 2021

http://www.cs.berkeley.edu/~jfc/cs174lecs/lec18/lec1

Exercise

Run the randomized algorithm on this graph

- Pick an edge at random (u,v)
- Merge u and v in new vertex uv
- Edges between u and v are removed
- Edges pointing to u or v are added as multi-edges to vertex uv
- When only s and t remain, the multi-edges are a cut, probably the minimum one



Upload to N
https://nearp
Code to be

These contents were not covered in 2021

The randomized algorithm might miss the min cut

These contents were not covered in 2021

- Multiple runs are required
- The probability that this finds the min cut in one run is about 1/log(n), so O(log n) iterations are required to find min cut
- Each iteration costs $O(n^2 \log n)$
- $O(n^2 \log^2 n)$ operations needed to find min cut
- Exact algorithm: $O(n^3 + n^2 \log n)$; the n^3 is because of $|V||E|$ operations required

http://www.cs.berkeley.edu/~jfc/cs174lecs/lec18/lec18.html

Summary

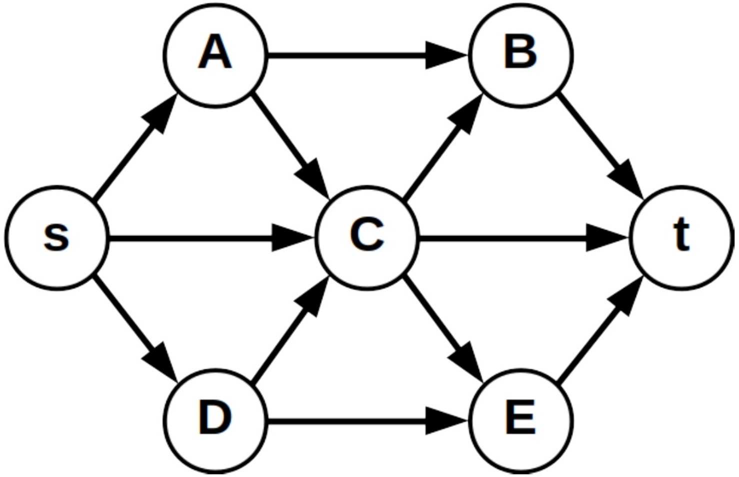
Things to remember

- Minimum s-t cut in a graph = set of edges
- The sum of the capacities of those edges is the maximum s-t flow the graph can carry
- How to write the primal and dual equations for max flow and min cut
- How to run the approximate randomized algorithm

Practice on your own

Consider (s, t)–cuts on the graph on the right, where s is the source node and t is the terminal node. Assume every edge has cost equal to 1.

1. By visual inspection, what is the minimum cost of an (s, t)–cut in this graph, and what is an example of a cut having that cost?
2. Run the algorithm for randomized (s, t)-cuts we saw in class, drawing all intermediate graphs, and indicate the cost of the resulting cut.



Practice on your own (cont.)

- Create a graph
- Write the min-flow, max-cut equations
- Find an optimal solution
- Run the randomized s-t cut algorithm