Graph Theory: Centrality

Introduction to Network Science

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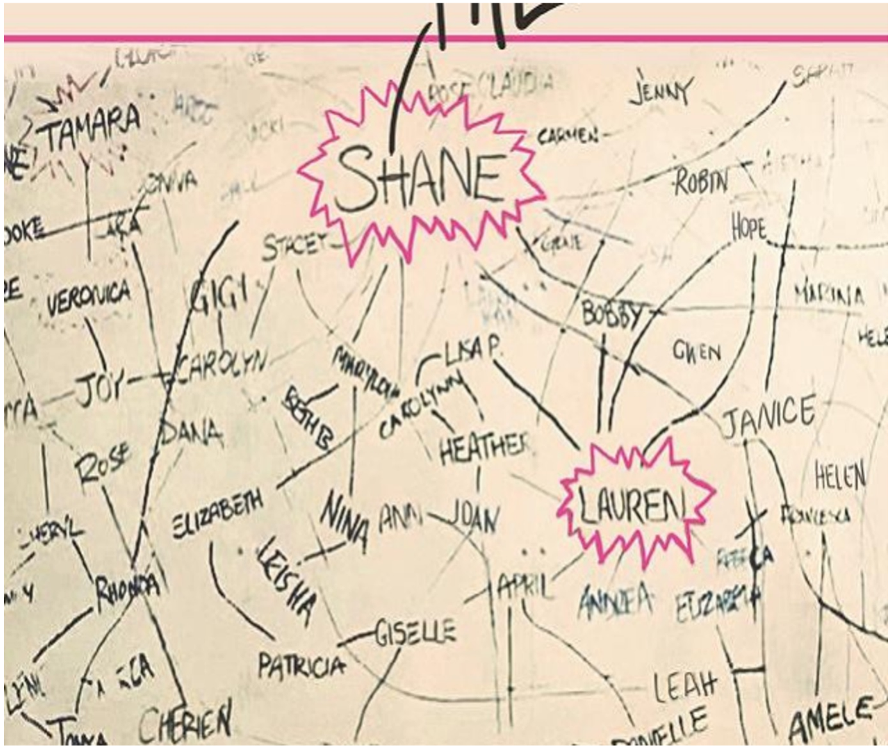
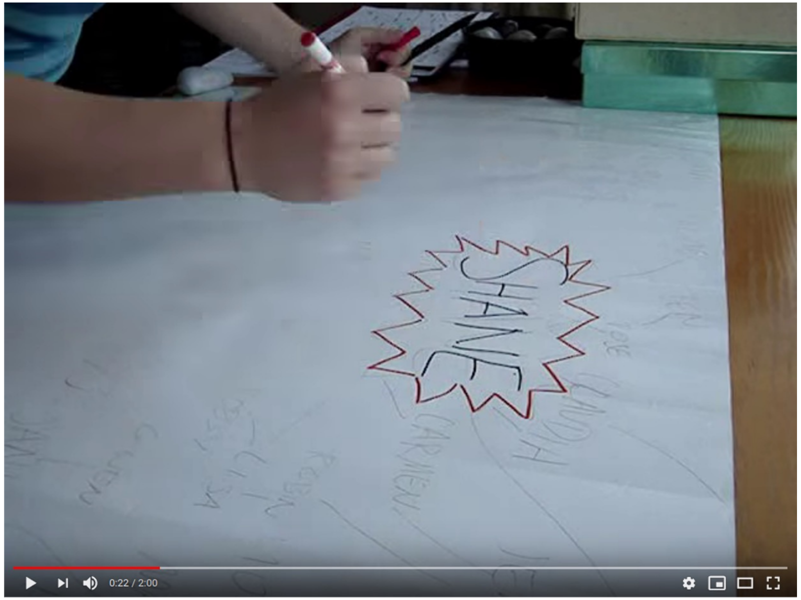
A *central* question in networks is determining who is more ... *central*





https://youtu.be/wQ3TX65MnjM?t=22

"We are all connected through love, loneliness, or one tiny lamentable lapse of judgment"

Types of centrality measure

- **Non-spectral**
  - Degree
  - Closeness and harmonic closeness
  - Betweenness
- Spectral
  - HITS
  - PageRank

Is *u* a well-connected person?

- Degree: *u* has many connections
- **Closeness:** *u* is close to many people
  - Average distance from *u* is small
- **Betweenness**: many connections pass through *u*
  - Large number of shortest paths pass through *u*
- **PageRank**: *u is connected to the well-connected*

Closeness

Closeness

- Distance between two nodes is *d(u,v)*

- **Closeness** is the reciprocal of the sum of distances

$$\text{closeness}(u) = \frac{1}{\sum_{v \in V, v \neq u} d(u, v)}$$

- Some graphs are not connected, in that case *d(u,v)* can be ∞; assuming *1/∞ = 0* one can define the **harmonic closeness**:

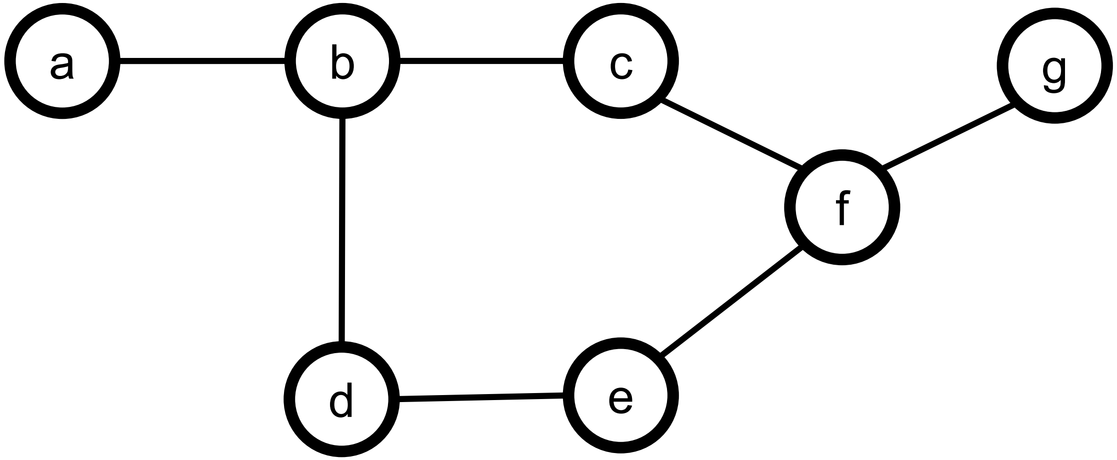
$$\text{hcloseness}(u) = \sum_{v \neq u} \frac{1}{d(u, v)}$$

Exercise

Compute closeness and harmonic closeness for all the nodes; d(u,v) = 1 if v is a neighbor of u

$$\text{closeness}(u) = \frac{1}{\sum_{v \in V, v \neq u} d(u,v)}$$

$$\text{hcloseness}(u) = \sum_{v \in V, v \neq u} \frac{1}{d(u,v)}$$



Spreadsheet links: https://upfbarcelona.padlet.org/chato/shyq9m6f2g2dh1bw

Betweenness

Definitions

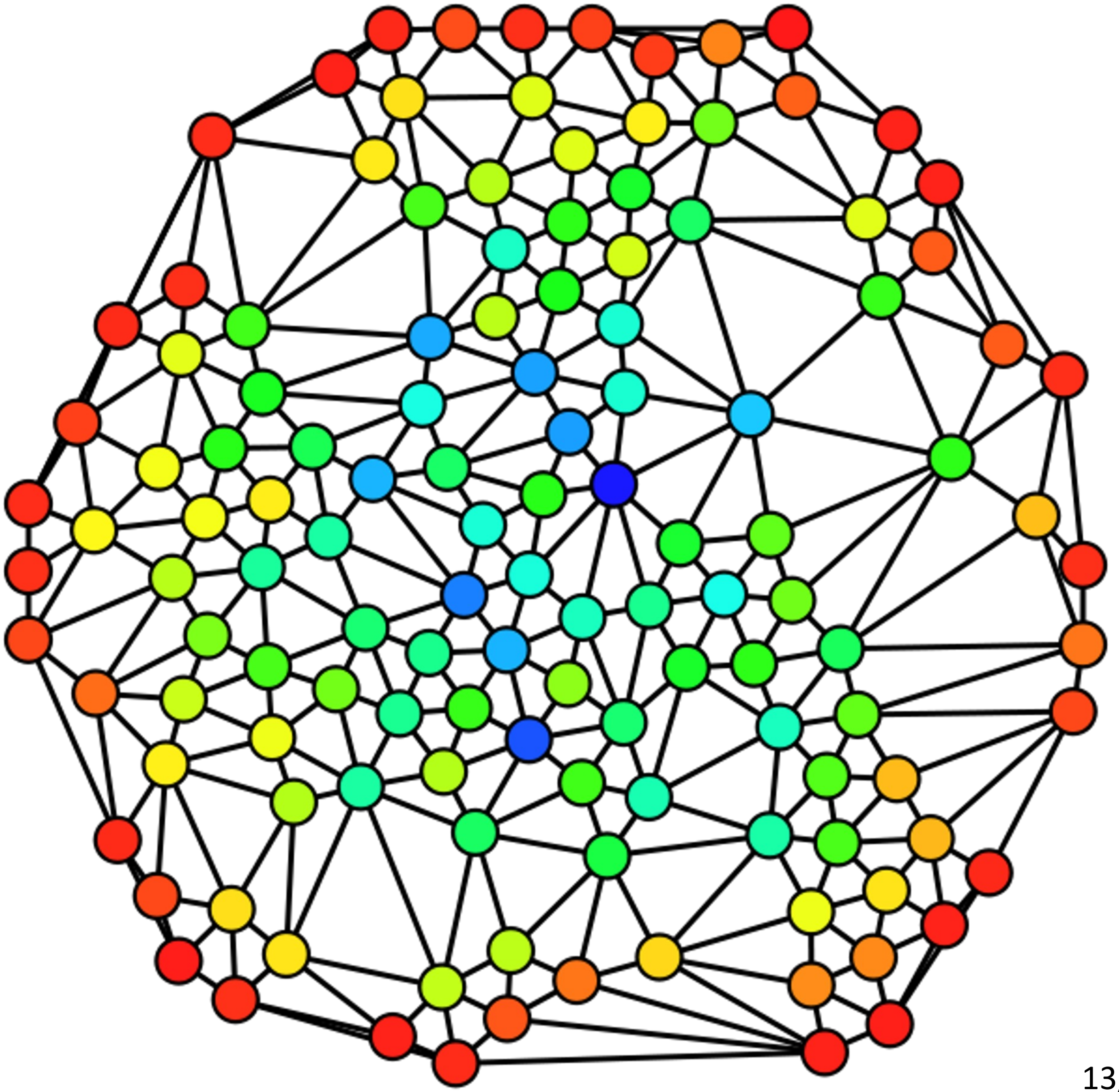
The **betweenness of a node** is the number of shortest paths that cross that node

The **betweenness of an edge** is the number of shortest paths that cross that edge

Node Betweenness

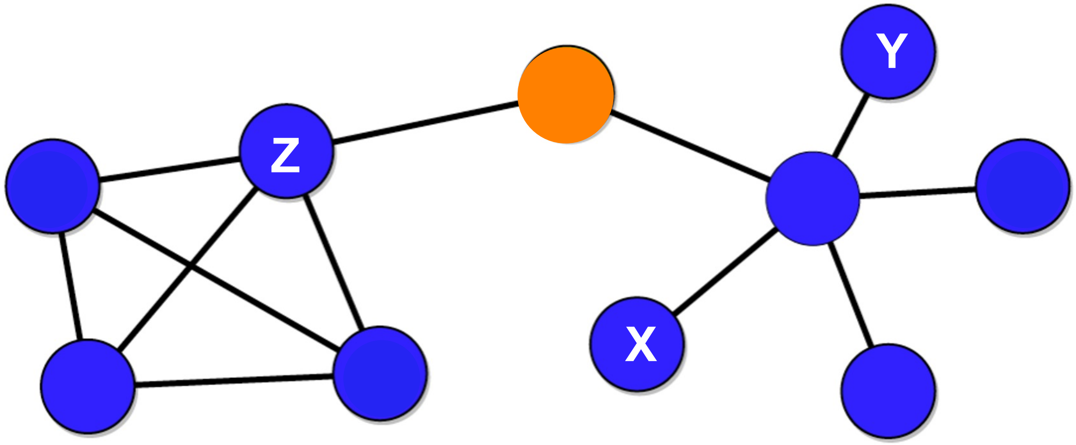
Graph with nodes colored according to node betweenness

red=low, blue=high



[[Wikipedia: Betweenness centrality](https://en.wikipedia.org/wiki/Betweenness_centrality)]

Example 1



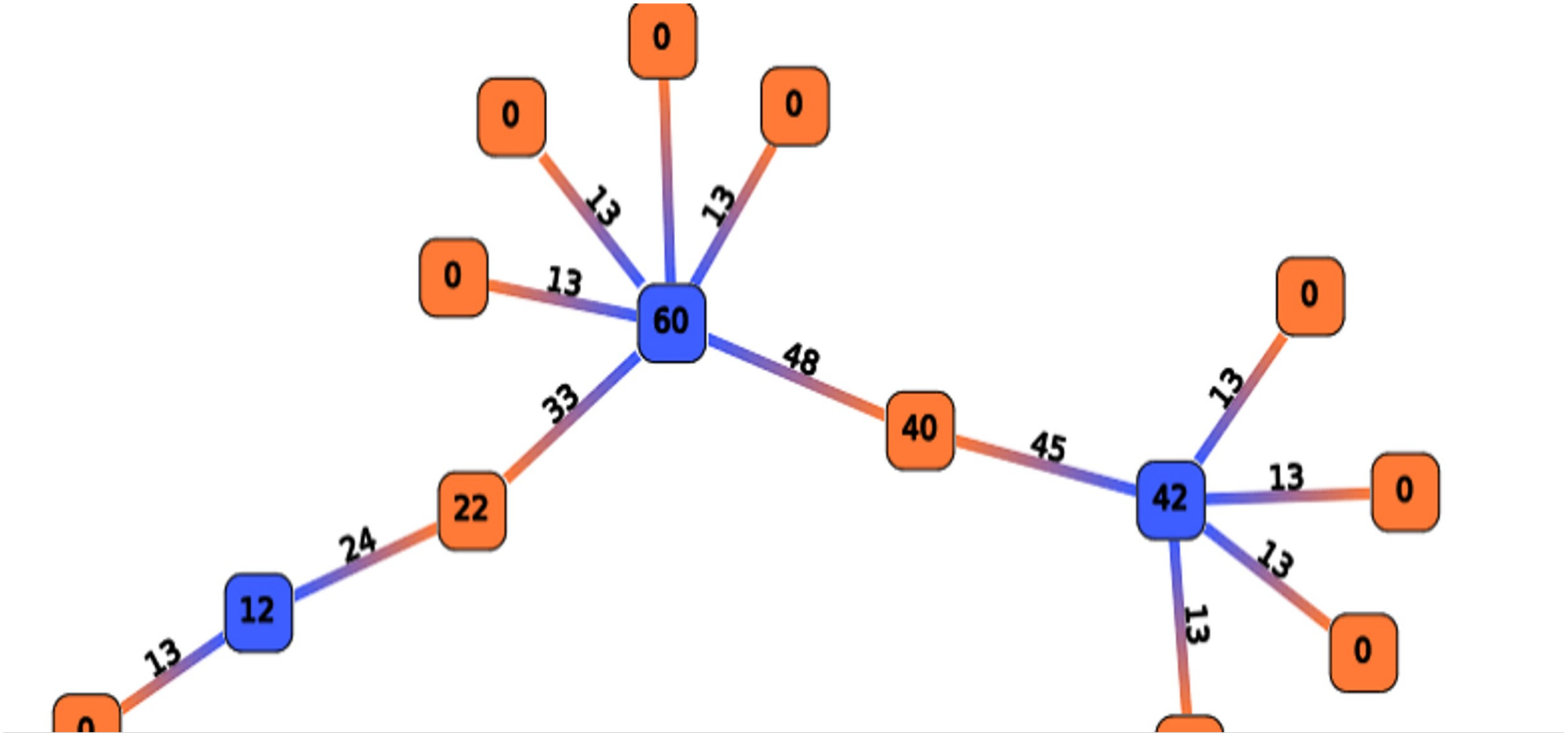
There are 20 shortest paths that cross through the orange node. Why?

The shortest path between nodes X and Y does not cross the orange node, but the shortest path between nodes X and Z does cross the orange node.

Source: Menczer, Fortunato, Davis: A First Course on Networks Science. Cambridge, 2020.

Example 2

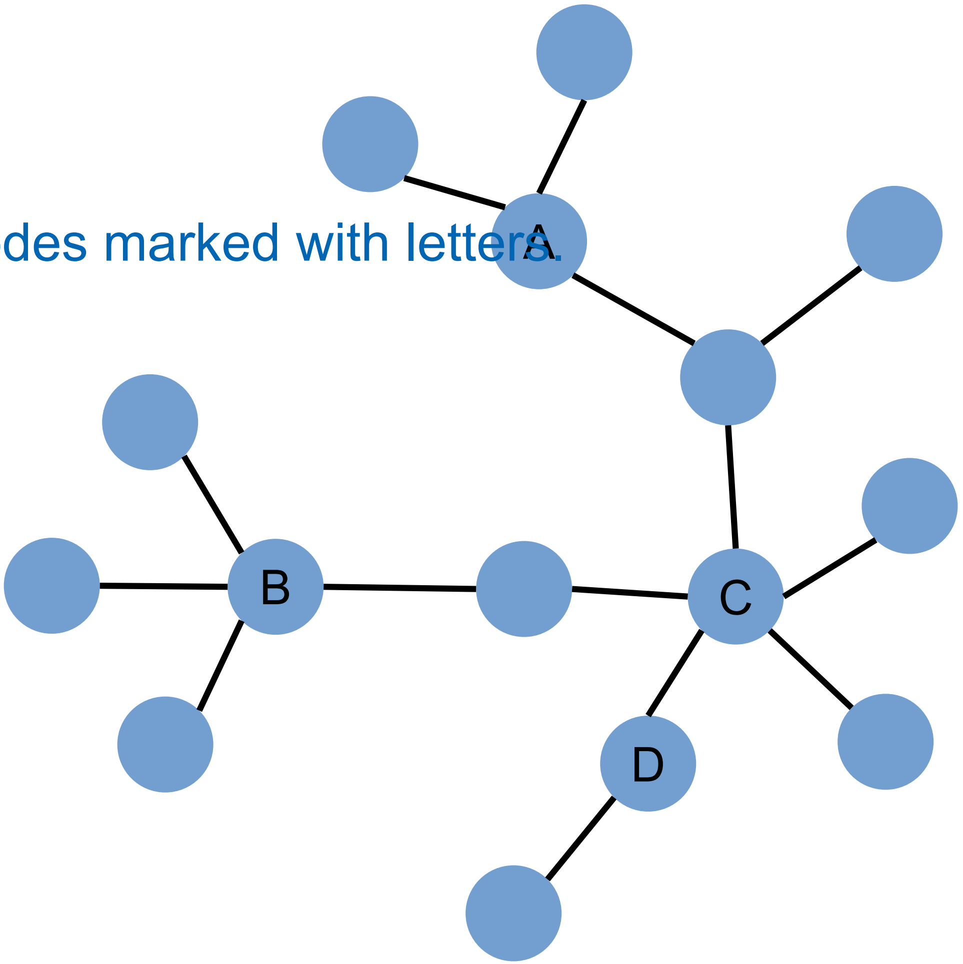
Here, nodes and edges are labeled with their betweenness.



http://rnav.labri.fr/rNAV2.0_public_version/online_doc/_images/ComputeBetweennessCentrality.png

Exercise

Compute the node betweenness of the nodes marked with letters.

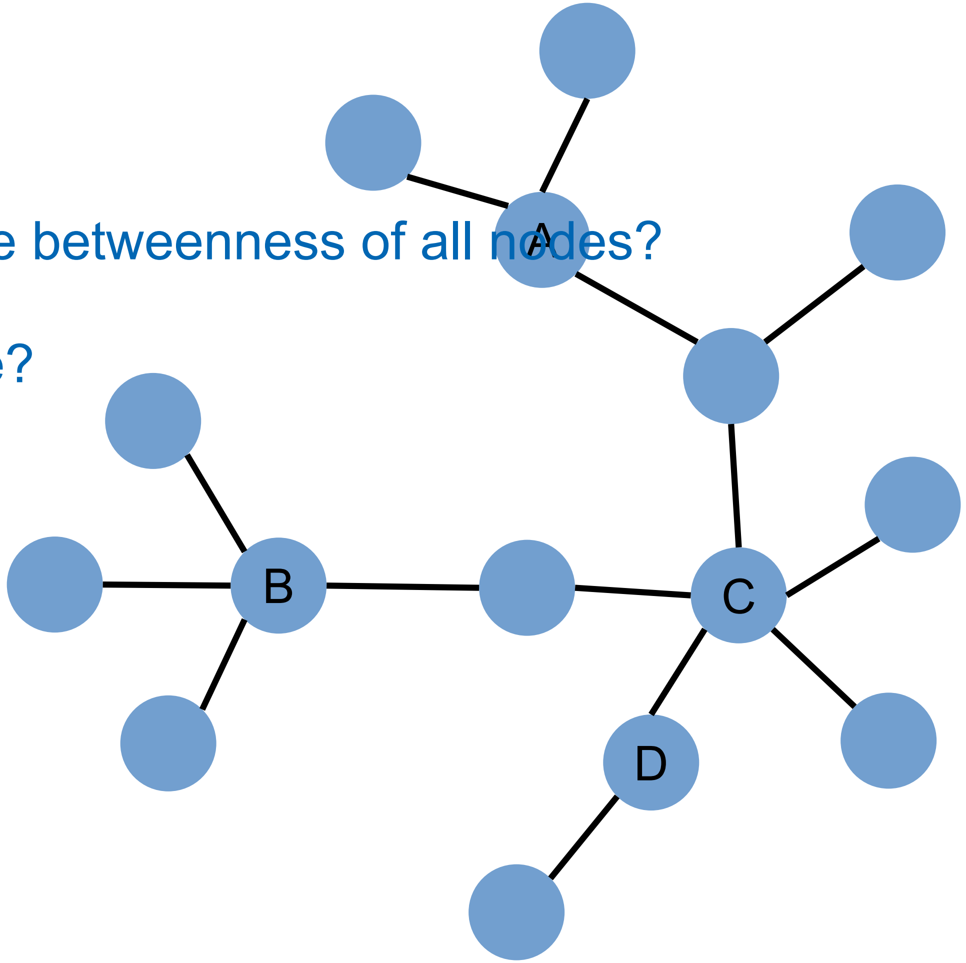


Pin board: https://upfbarcelona.padlet.org/chato/asfs154waxnnkhgo

Exercise (cont.)

What is a good algorithm to compute node betweenness of all nodes?

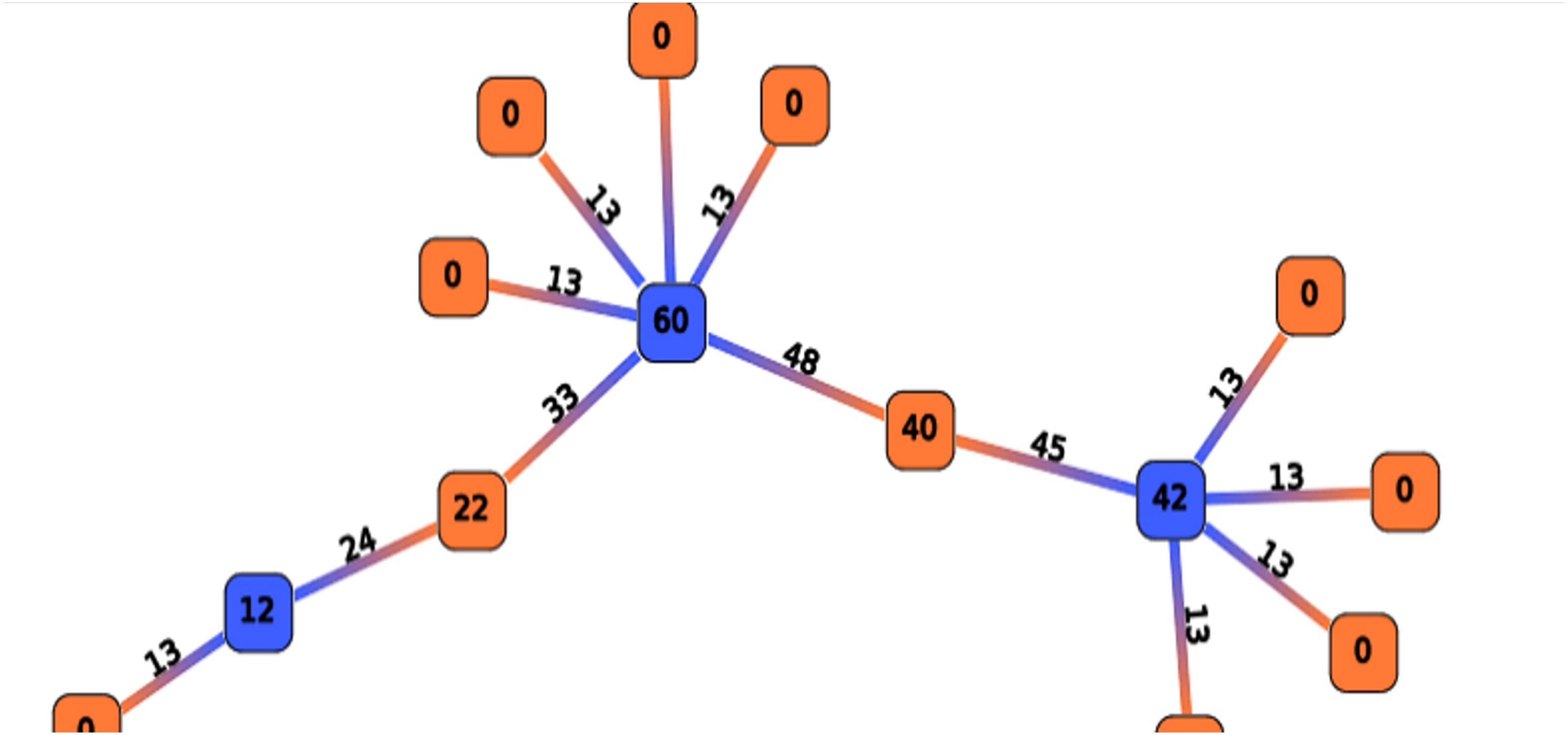
What limitations does your algorithm have?



Edge Betweenness

Edge Betweenness

An **edge** has high betweenness if it is part of many shortest-paths.



http://rnav.labri.fr/rNAV2.0_public_version/online_doc/_images/ComputeBetweennessCentrality.png

Approximate method [sampling]

- Label all edges e with b(e) = 0
- Repeat K times:
  - Pick a random pair of nodes (u,v)
  - Compute shortest path between u and v
  - b(e) ← b(e) + 1 for all edges e along the path
- b(e) is a lower bound for betweenness (e)
- Useful if we only care about finding the edge with the highest betweenness, or finding the top-k edges with the highest betweenness → an early stopping criterion is possible

Riondato, M., & Kornaropoulos, E. M. (2014, February). Fast approximation of betweenness centrality through sampling. In Proceedings of the 7th ACM internati

Exact algorithm [Brandes, Newman]

- For every node *u* in *V*
  - Layer the graph performing a BFS from *u*
  - For every node *v* in *V*, *v*≠*u*, sorted by layer
- Assign to *v* a number *s(v)* indicating how many shortest paths from u arrive to v
  - For every node *v* in *V*, *v*≠*u*, sorted by reverse layer
- Score to distribute = 1 + score from children
- Add score to parent edges in proportion to *s(v)*
- In the end divide all edge scores by two

Ulrich Brandes. A faster algorithm for betweenness centrality. Journal of Mathematical Sociology, 25:163–177, 2001.
Mark E. J. Newman. Scientific collaboration networks: II. Shortest paths, weighted networks, and centrality. Physical Review E, 64:016132, 2001.

Example

For every node u in V

- Layer the graph performing a BFS from u
- For every node v in V, v≠u, sorted by layer
- Assign to v a number s(v) indicating how many shortest p
- For every node v in V, v≠u, sorted by reverse layer
- Score to distribute = 1 + score from children
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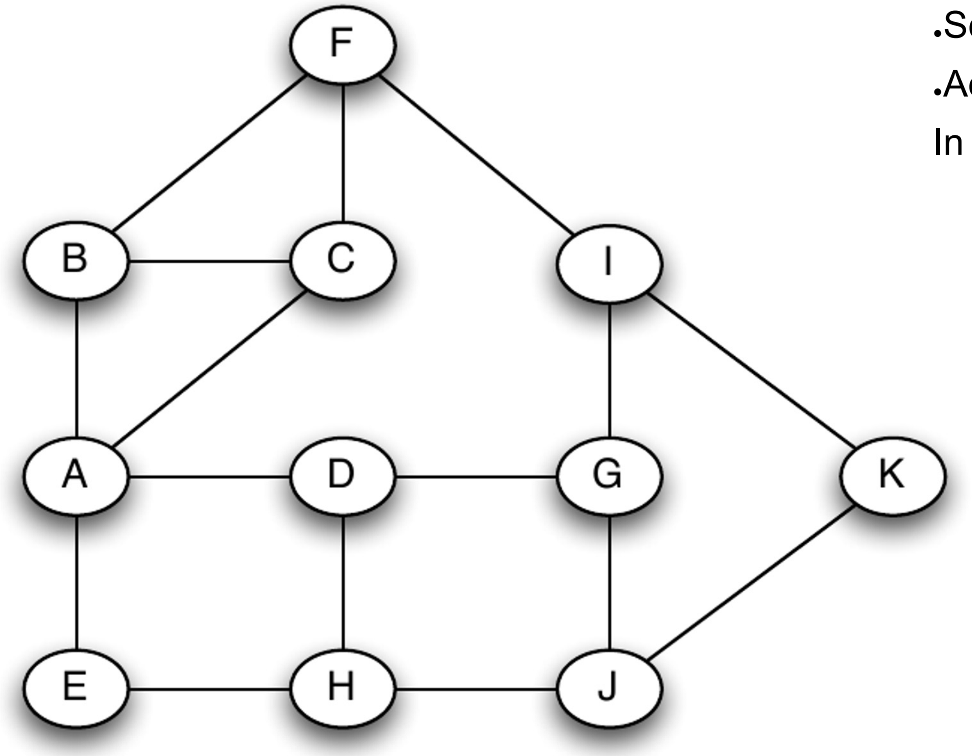
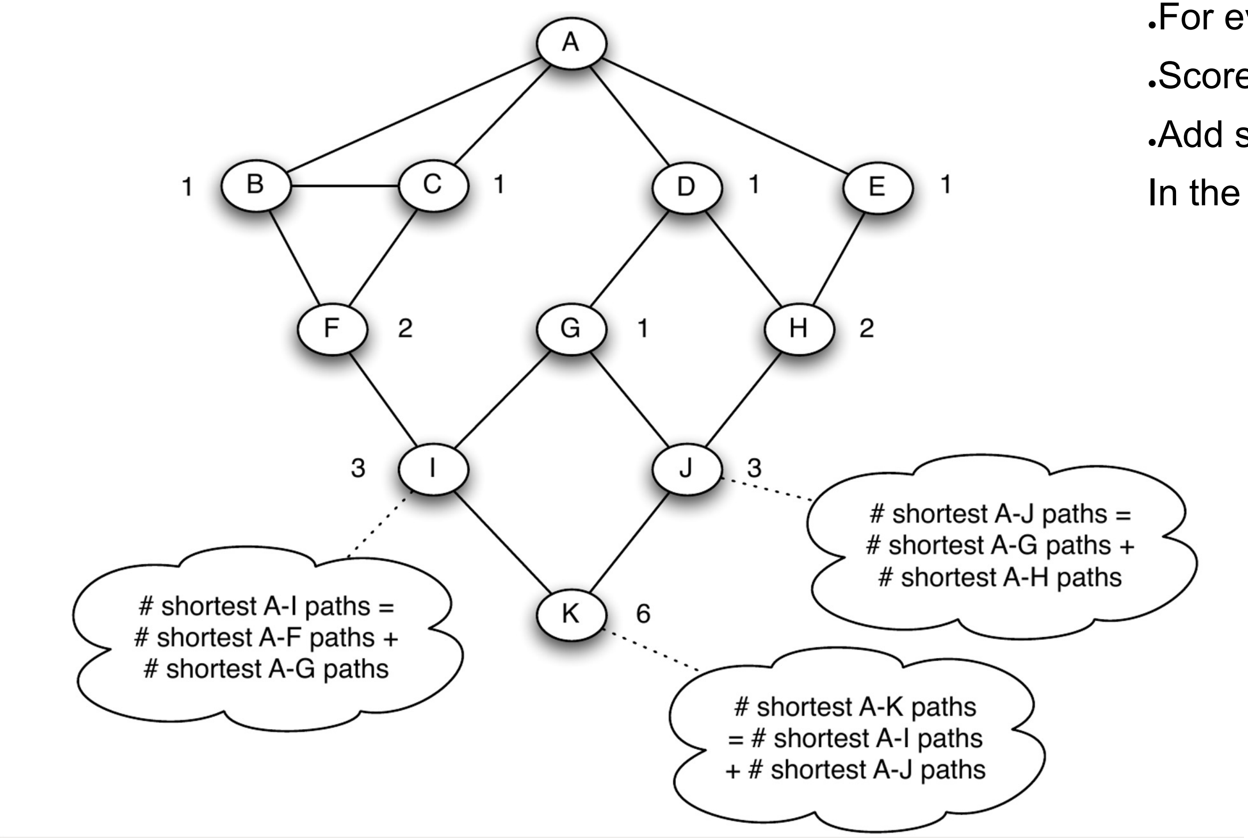
In the end divide all edge scores by two



Example



For every node u in V

- Layer the graph performing a BFS from u
- **For every node v in V, v≠u, sorted by layer**
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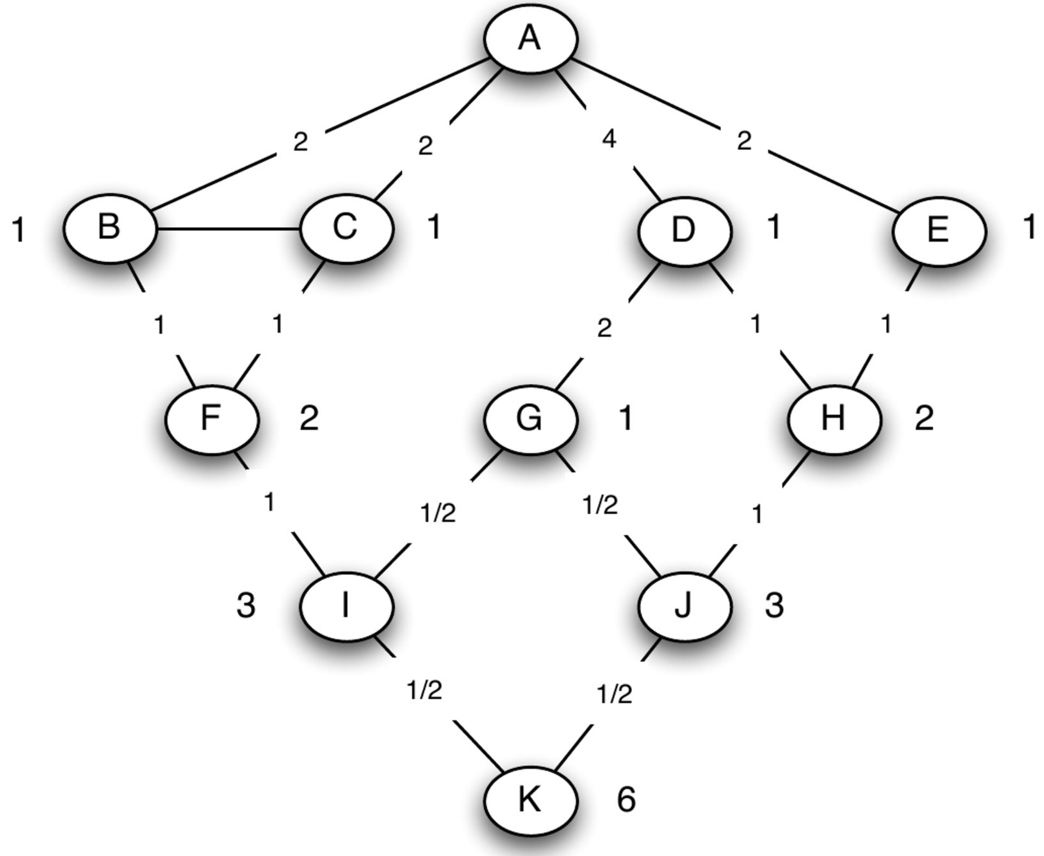
In the end divide all edge scores by two

All nodes in layer 1 get s(v)=1

Remaining nodes: simply add s(.) o

Example



For every node u in V

- Layer the graph performing a BFS from u
- For every node v in V, v≠u, sorted by layer
- Assign to v a number s(v) indicating how many shortest p
- **For every node v in V, v≠u, sorted by rev. layer**
- **Score to distribute = 1 + score from children**
- **Add score to distribute to parent edges in proportion**

In the end divide all edge scores by two

Nodes without children distribute a s

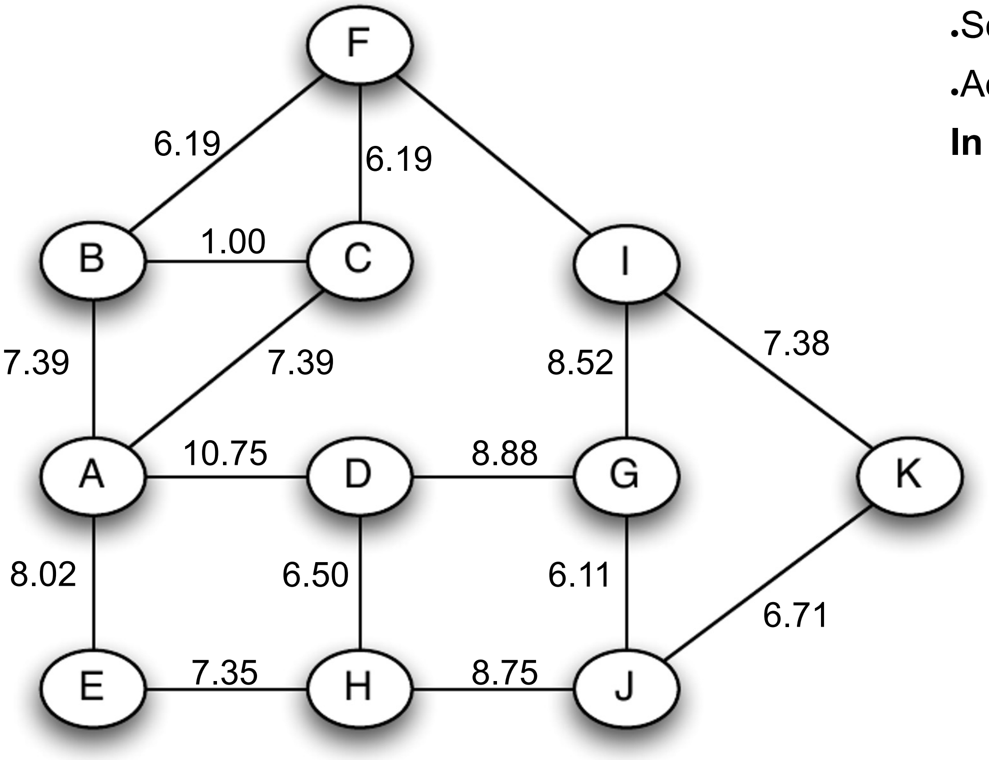
Other nodes distribute 1 + whatever

Result

For every node u in V

- Layer the graph performing a BFS from u
- For every node v in V, v≠u, sorted by layer
- Assign to v a number s(v) indicating how many shortest p
- For every node v in V, v≠u, sorted by reverse layer
- Score to distribute = 1 + score from children
- Add score to distribute to parent edges in proportion to s(

In the end divide all edge scores by two

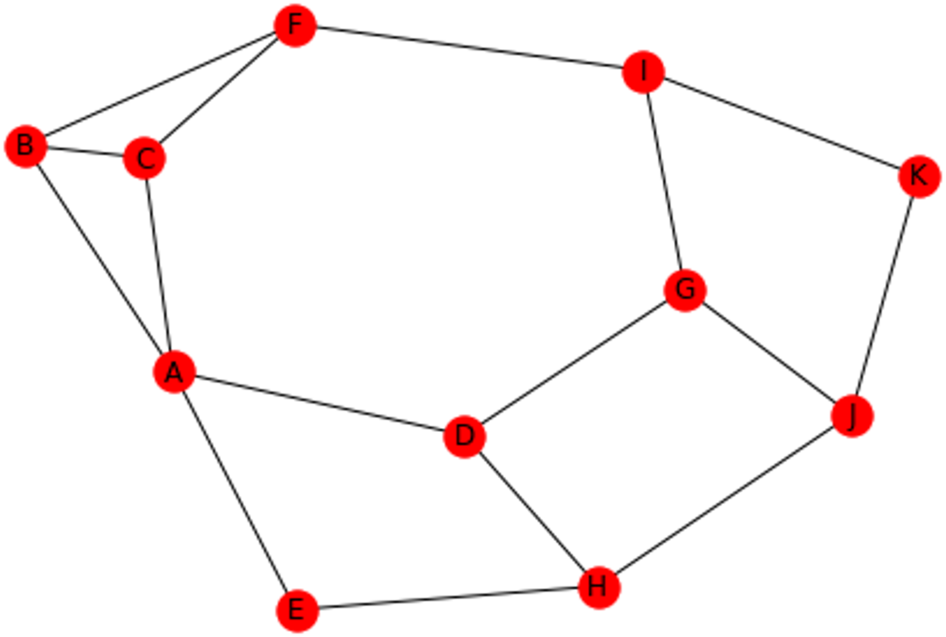


Computed using NetworkX
(edge betweenness)

NetworkX code

```
import networkx as nx
g = nx.Graph()
g.add_edge("A", "B")
g.add_edge("A", "C")
g.add_edge("A", "D")
g.add_edge("A", "E")
g.add_edge("B", "C")
g.add_edge("B", "F")
…
nx.edge_betweenness(g, normalized=False)
```

```
nx.draw_spring(g, with_labels=True)
```



Exercise

*Try to compute **edge betweenness** by inspection first*

Then use the Brandes-Newman algorithm;
you should get the same results

For every node u in V

- Layer the graph performing a BFS from u
- For every node v in V, v≠u, sorted by layer
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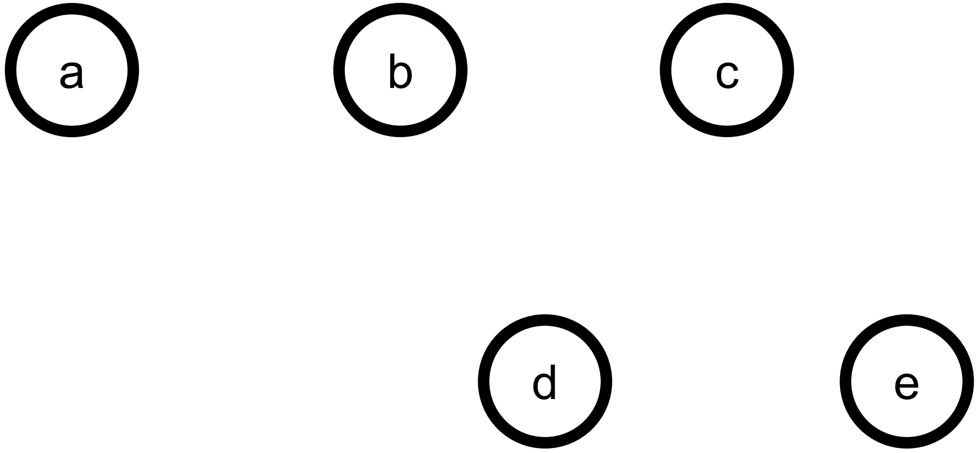
In the end divide all edge scores by two



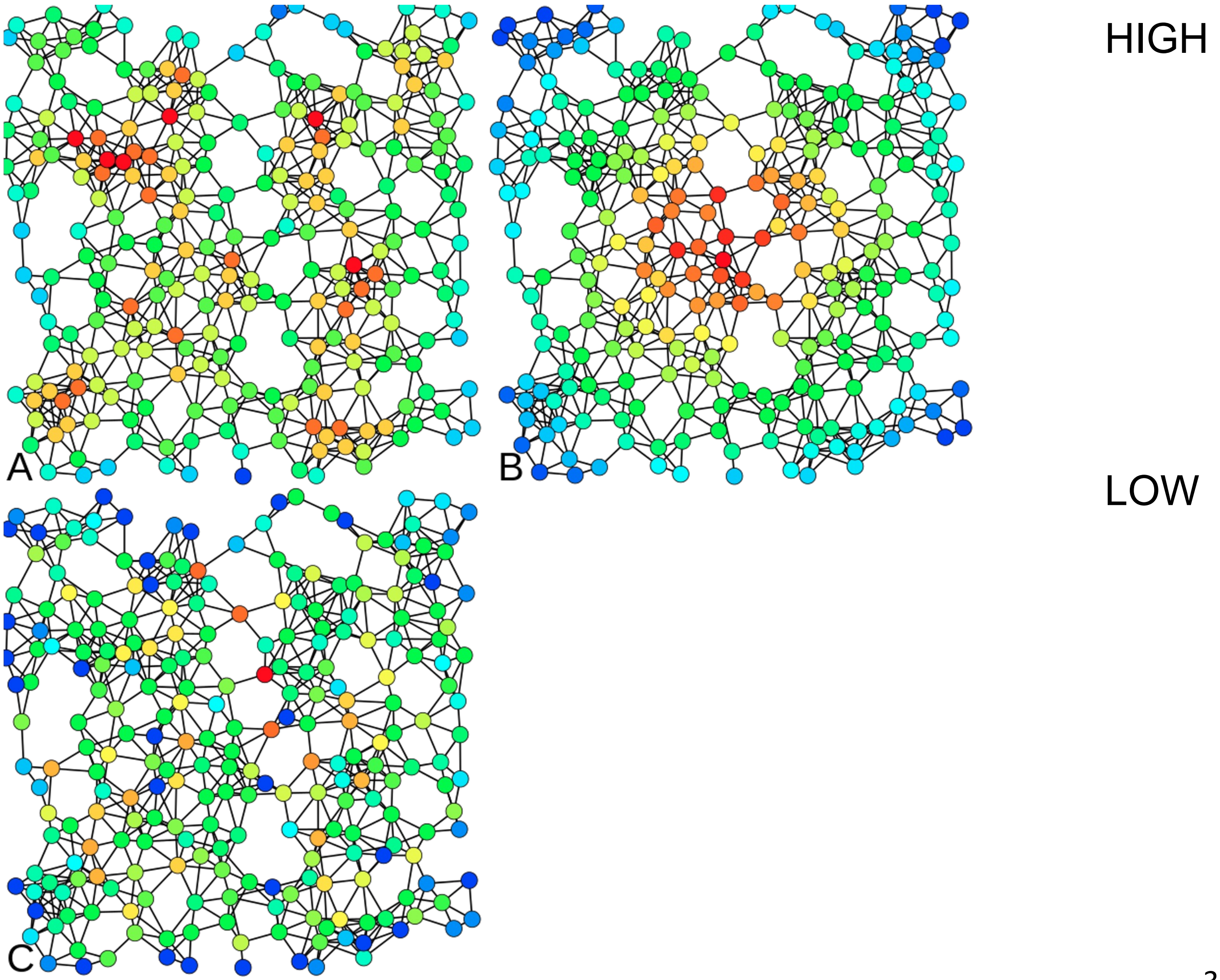
Fractional values?

- In a graph with cycles, you may get **fractional values** of the edge betweenness for an edge
- Conceptually, this is because in a graph with cycles there might be s>1 shortest paths between two nodes, each of them counts 1/s

A: Degree

B: Closeness

C: Betweenness



Summary

Things to remember

- Closeness and harmonic closeness
- Node and edge betweenness
- Practice running the Brandes-Newman algorithm on small graphs
- Write code to execute the Brandes-Newman algorithm

Constructive problems

- Practice drawing examples of graphs in which a chosen node has high degree but low closeness, or viceversa

- Can you find a graph in which there is a node that has the maximum degree and the minimum closeness? If not, why?

Constructive problems

1. Sketch a graph of N nodes in which a node, which you should mark with an asterisk (*), should have betweenness approximately equal to N and closeness approximately 1/N for large N . Explain briefly.

2. Sketch a graph of N nodes in which a node, which you should mark with an asterisk (*), should have betweenness approximately equal to N and closeness approximately $2/N^2$ for large N . Explain briefly.

Do not use a concrete N . Use a general N , for instance by using the ellipsis (. . .) to denote multiple nodes.

Sources

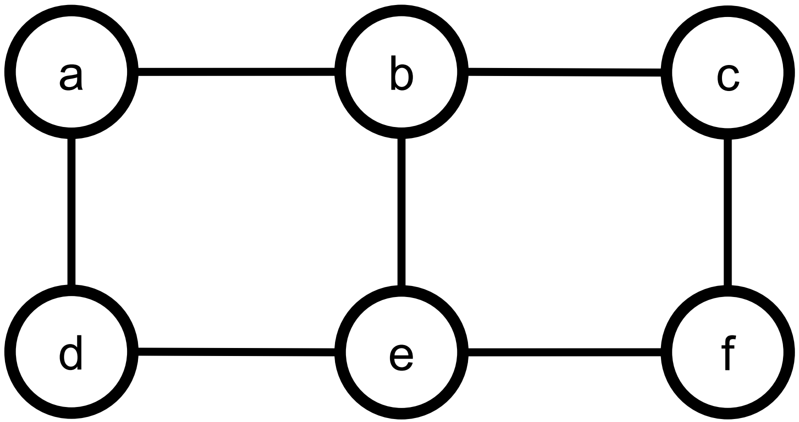
- D. Easly and J. Kleinberg (2010). Networks, Crowds, and Markets – Section 3.6B
- A. L. Barabási (2016). Network Science – Section 9.3
- P. Boldi and S. Vigna (2014). Axioms for Centrality in *Internet Mathematics*
- Esposito and Pesce: Survey of Centrality 2015.
- URLs cited in the footer of slides

Sources

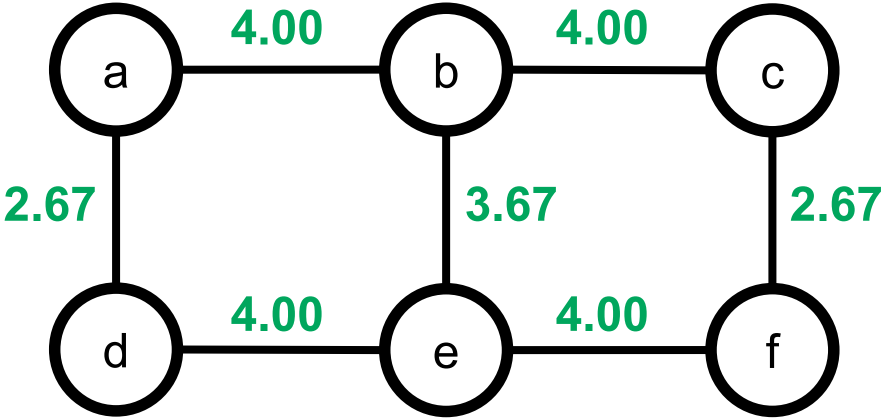
- D. Easley and J. Kleinberg (2010). Networks, Crowds, and Markets – Section 3.6B
- P. Boldi and S. Vigna (2014). Axioms for Centrality in *Internet Mathematics*.
- Esposito and Pesce (2015): Survey of Centrality.
- F. Menczer, S. Fortunato, C. A. Davis (2020). A First Course in Network Science – Chapter 02

Practice on your own

- Compute edge betweenness on this graph



Practice on your own (cont.)



If you don’t get this result, check:
https://www.youtube.com/watch?v=uYjWbp8VC7c