

# Properties of Random Networks

## Social Networks Analysis and Graph Algorithms

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# Contents

- Connectedness under the ER model
- Distances under the ER model
- Clustering coefficient under the ER model

# Sources

- Albert-László Barabási: Network Science. Cambridge University Press, 2016.
  - Follows almost section-by-section [chapter 03](#)
- [Data-Driven Social Analytics](#) course by Vicenç Gómez and Andreas Kaltenbrunner
- URLs cited in the footer of specific slides

# The “Magtension” game

- Take turns placing one magnet inside an enclosed space
- You lose if, after your play, any two magnets stick to each other

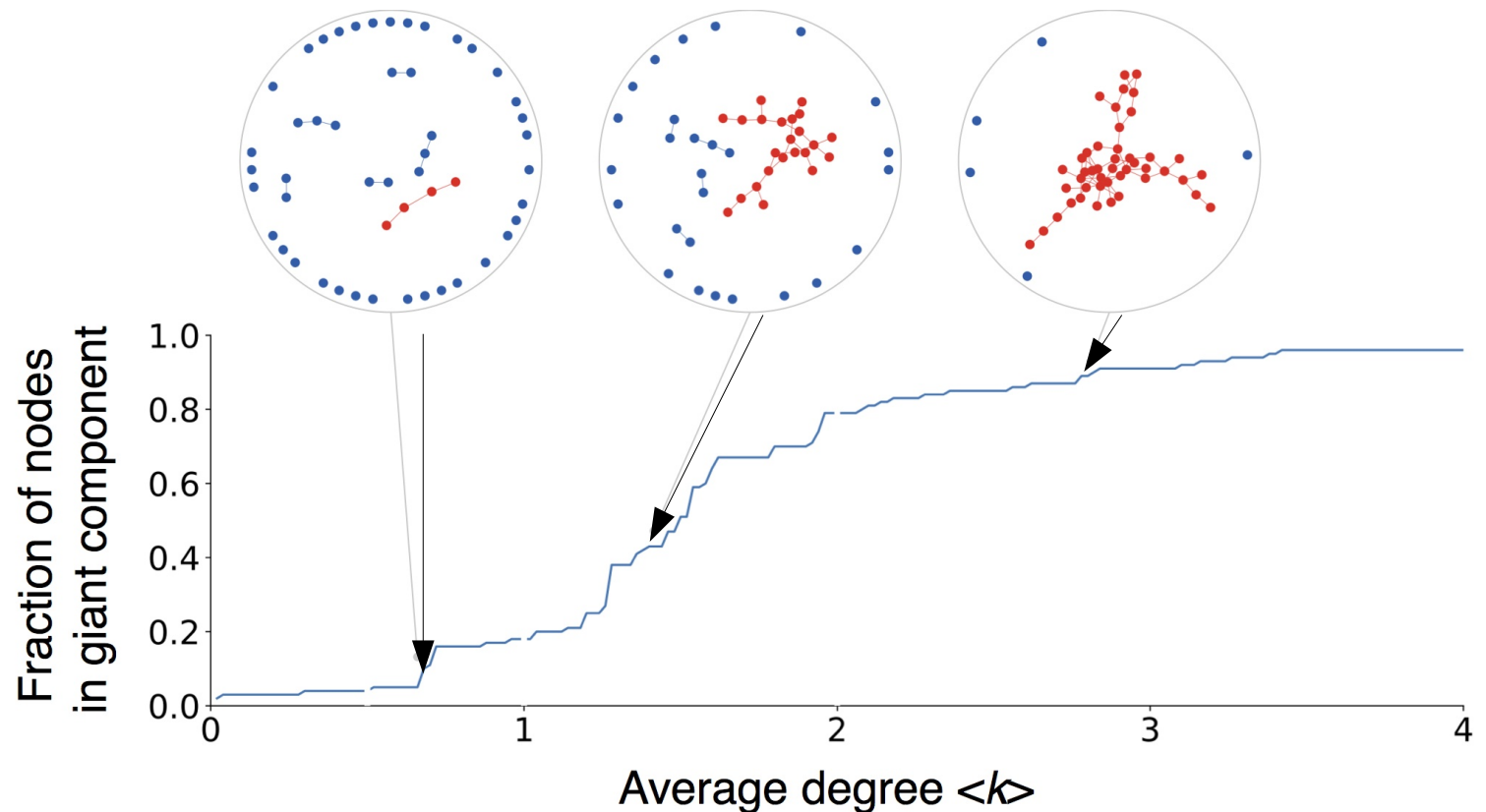


<https://www.youtube.com/watch?v=PDyadRTCSE>

# Connectivity in ER networks

# An interesting property of ER networks

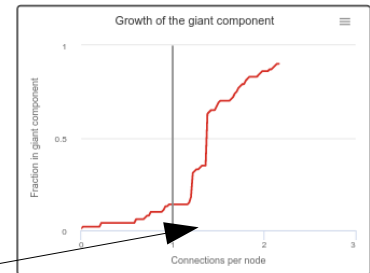
Red = nodes in  
largest connected  
component



# Exercise

Answer in Nearpod Poll  
<https://nearpod.com/student/>  
Code to be given during class

- Execute the “Giant Component” program in Netlogo Web
  - Select num-nodes  $N$  (e.g., 100)
  - Click “setup”
  - Click “go”
  - Write down the point at which there is an “elbow” in the distribution of links (mouseover the graphics to be more accurate)
  - Repeat various times
- Indicate approximately where, on average, you find the “elbow”



Go to [netlogoweb.org/launch](http://netlogoweb.org/launch) – run “Sample Models / Networks / Giant component”

# ER network as $\langle k \rangle$ increases

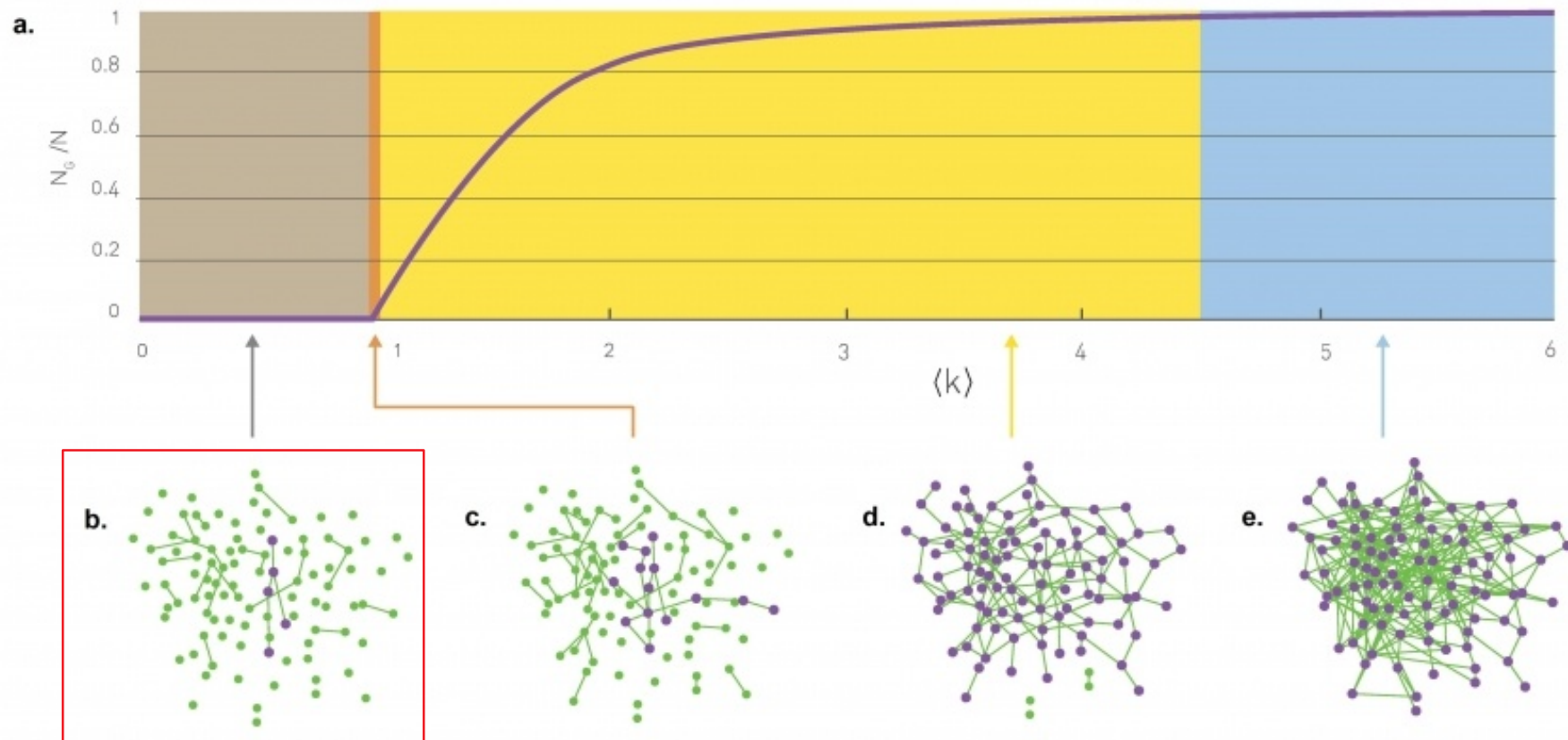
- When  $\langle k \rangle = 0$ : only singletons
- When  $\langle k \rangle < 1$ : disconnected
- When  $\langle k \rangle > 1$ : giant connected component
- When  $\langle k \rangle = N - 1$  complete graph

It's obvious that to have a giant connected it is **necessary** that  $\langle k \rangle = 1$   
Erdős and Rényi proved it is **sufficient** in 1959

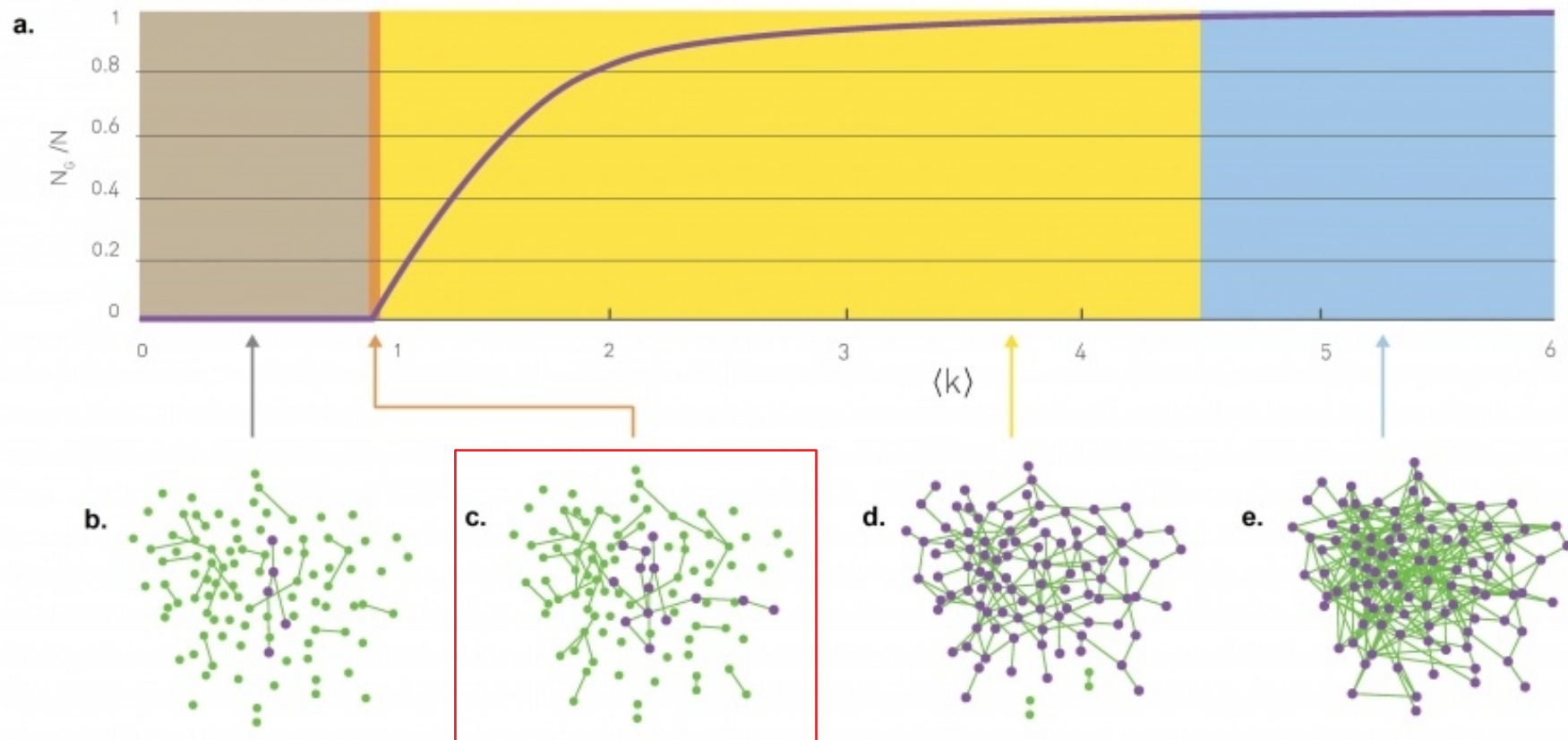
This result holds **on average**, not on every execution of the model



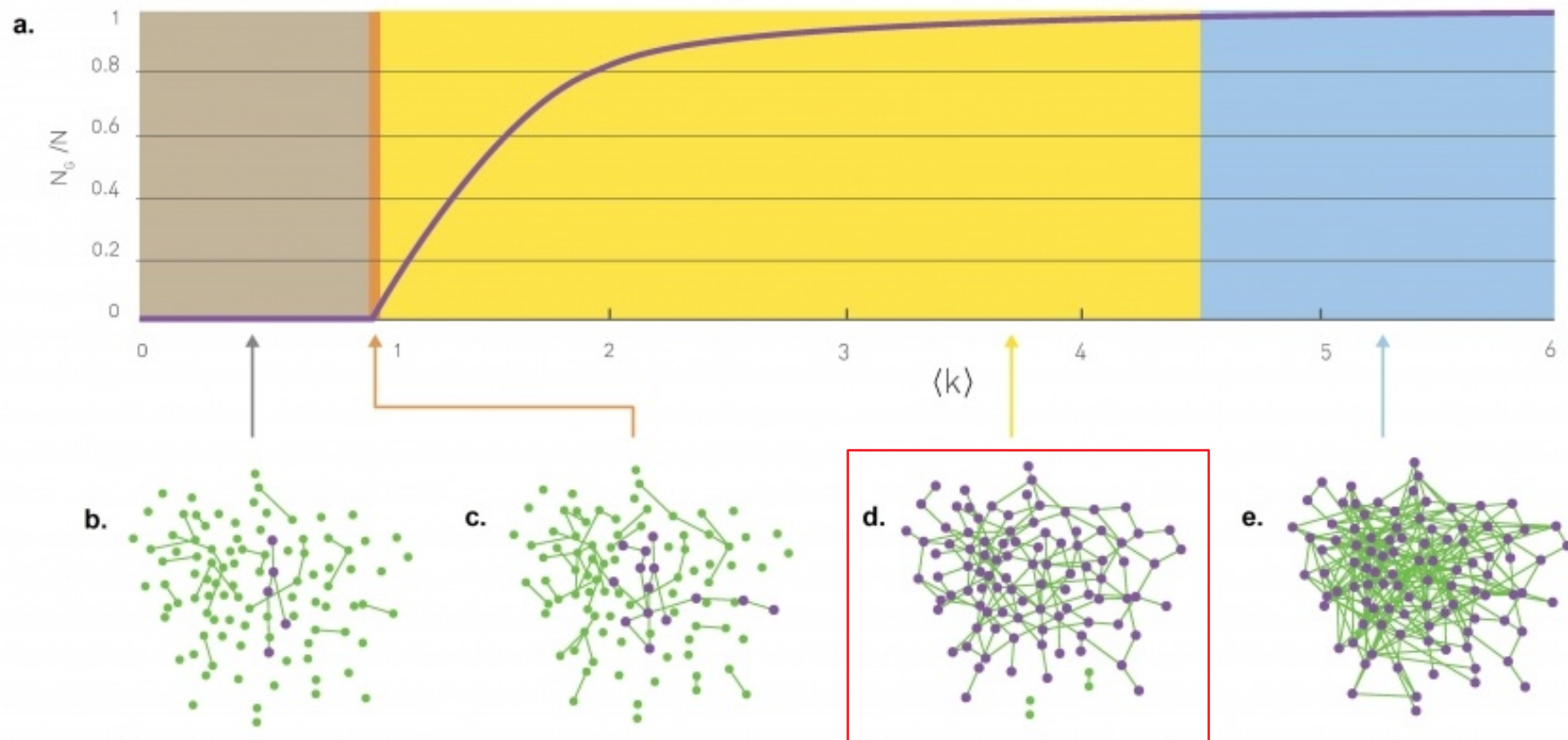
# Sub-critical regime: $\langle k \rangle < 1$



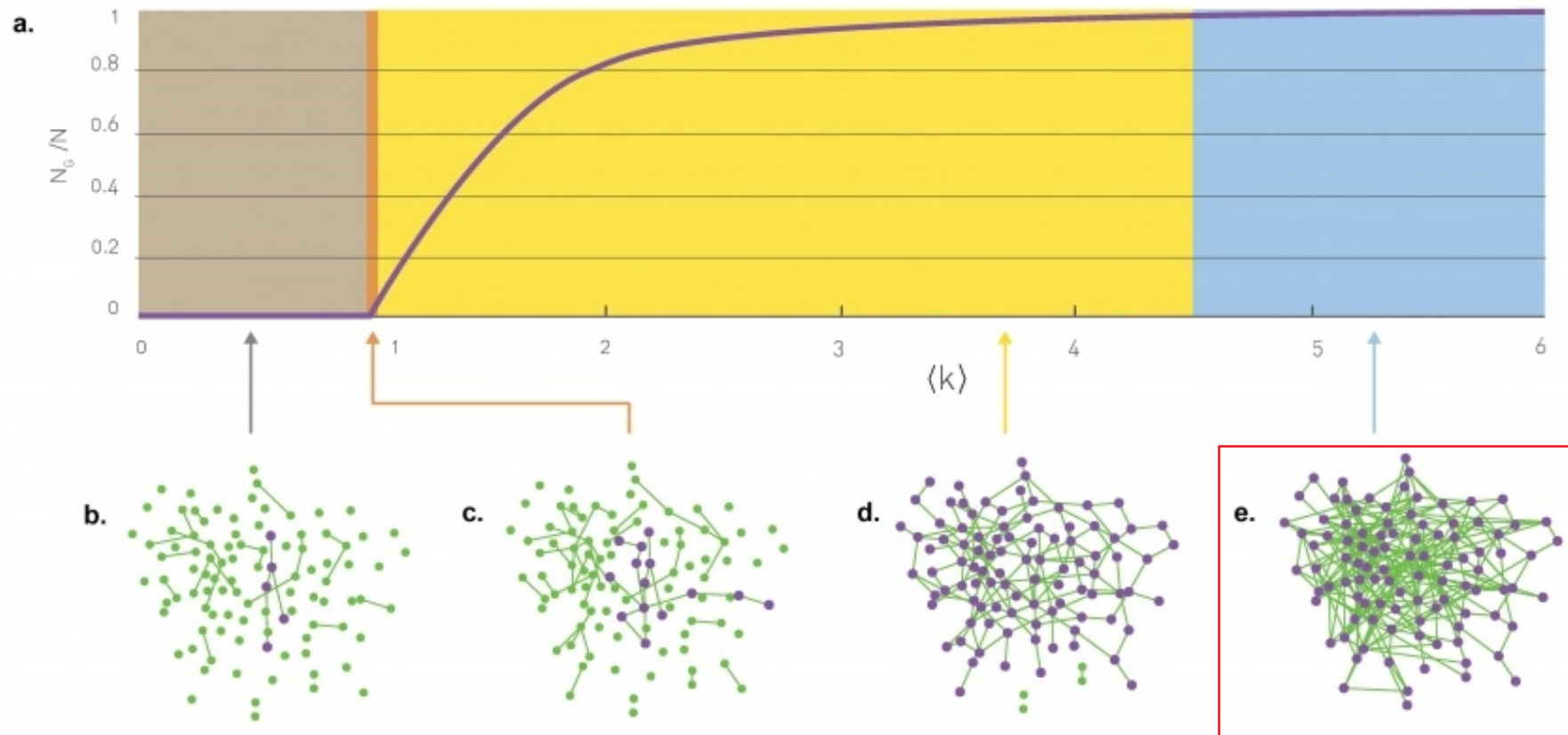
# Critical point: $\langle k \rangle = 1$



# Supercritical regime: $\langle k \rangle > 1$



# Connected regime: $\langle k \rangle > \log N$



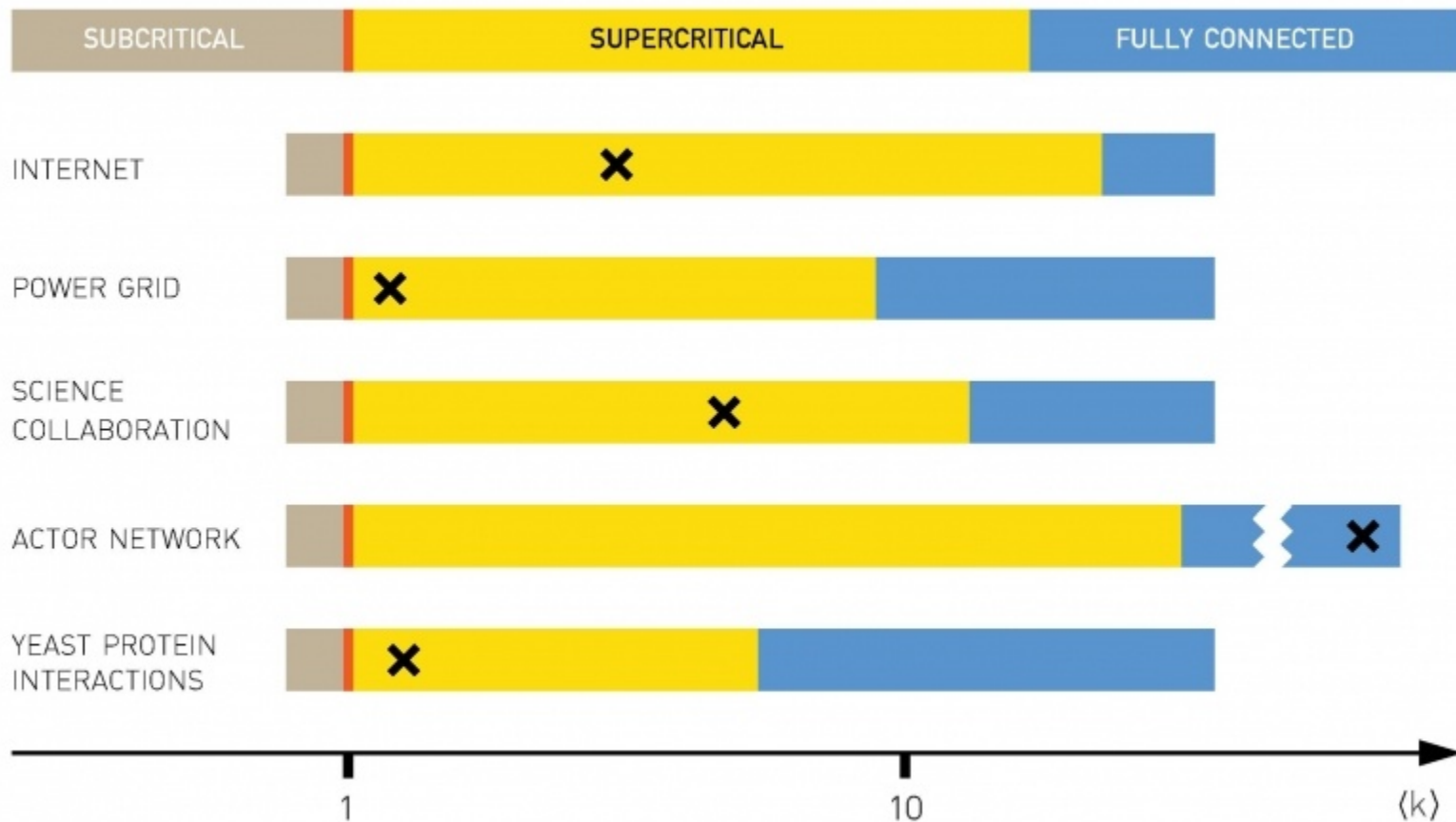
# Most real networks are supercritical:

$$\langle k \rangle > 1$$

Network	N	L	$\langle k \rangle$	$\ln N$
Internet	192,244	609,066	6.34	12.17
Power Grid	4,941	6,594	2.67	8.51
Science Collaboration	23,133	94,437	8.08	10.05
Actor Network	702,388	29,397,908	83.71	13.46
Protein Interactions	2,018	2,930	2.90	7.61

# Most real networks are supercritical:

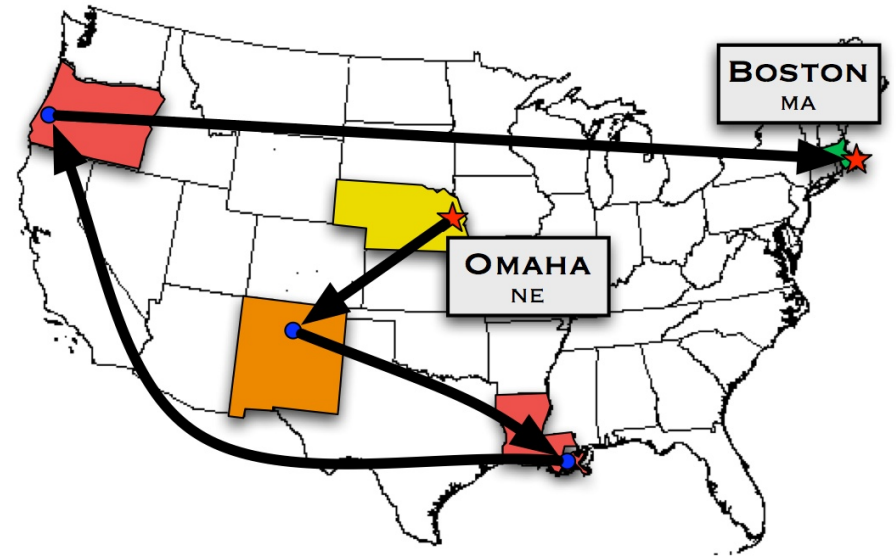
$$\langle k \rangle > 1$$



**Small-world phenomenon**  
**a.k.a. “six degrees of separation”**

# Milgram's experiment in 1967

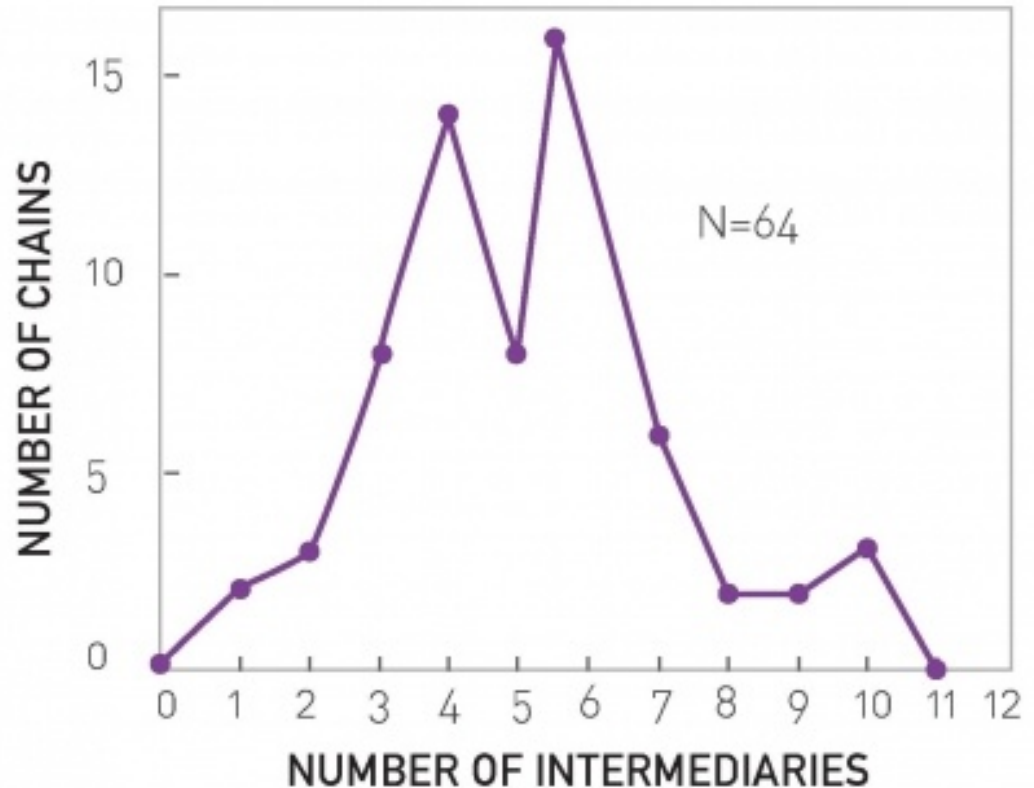
- Instructions: send to personal acquaintance most likely to know the target
  - Sources: 160 people in Wichita and Omaha
  - Targets: (1) a stock broker in Boston, MA and (2) a student in Sharon, MA
- Materials: short summary of study purpose, target photograph, name, address and information

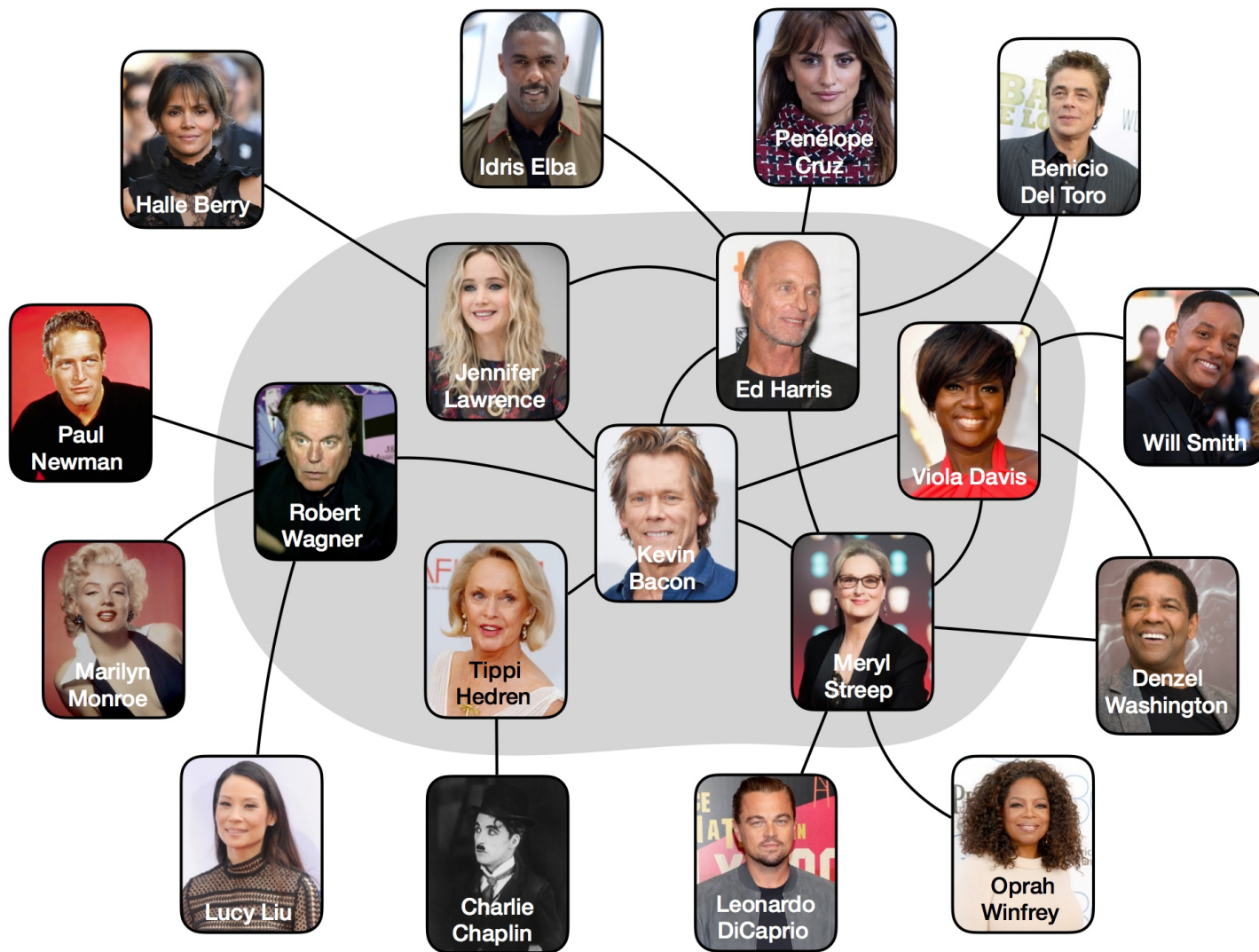




# Milgram's experiment in 1967 (results)

- 64 of 296 (22%) of the letters reached their destination
- Average 6.5 steps, much lower than expected





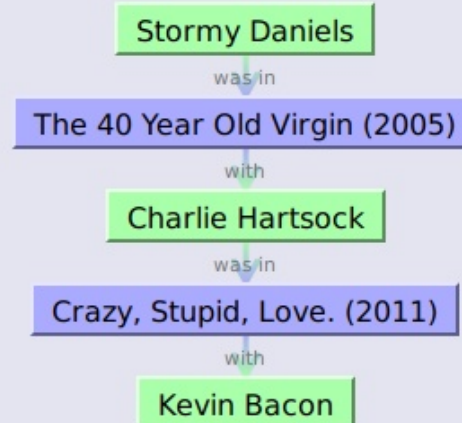
<https://oracleofbacon.org/>

# THE ORACLE OF BACON



Stormy Daniels has a Bacon number of 2.

[Find a different link](#)



Kevin Bacon

to

Stormy Daniels

[Find link](#)

[More options >>](#)

# “Small-world phenomenon”

- If you choose any two individuals on Earth, they are connected by a relatively **short path** of acquaintances
- Formally
  - The expected distance between two randomly chosen nodes in a network grows much slower than its number of nodes

# How many nodes at distance $\leq d$ ?

In an ER graph:

$\langle k \rangle$  nodes at distance 1

$\langle k \rangle^2$  nodes at distance 2

...

$\langle k \rangle^d$  nodes at distance  $d$

$$N(d) = 1 + \langle k \rangle + \langle k \rangle^2 + \dots + \langle k \rangle^d = \frac{\langle k \rangle^{d+1} - 1}{\langle k \rangle - 1}$$

# What is the maximum distance?

- Assuming  $\langle k \rangle \gg 1$   $N(d_{\max}) = \frac{\langle k \rangle^{d_{\max}+1} - 1}{\langle k \rangle - 1} \approx N$

$$\langle k \rangle^{d_{\max}} \approx N$$

$$d_{\max} \approx \log_{\langle k \rangle} N$$

$$d_{\max} \approx \frac{\log N}{\log \langle k \rangle}$$

# Empirical average and maximum distances

Network	N	L	$\langle k \rangle$	$\langle d \rangle$	$d_{\max}$	$\ln N / \ln \langle k \rangle$
Internet	192,244	609,066	6.34	6.98	26	6.58
WWW	325,729	1,497,134	4.60	11.27	93	8.31
Power Grid	4,941	6,594	2.67	18.99	46	8.66
Mobile-Phone Calls	36,595	91,826	2.51	11.72	39	11.42
Email	57,194	103,731	1.81	5.88	18	18.4
Science Collaboration	23,133	93,437	8.08	5.35	15	4.81
Actor Network	702,388	29,397,908	83.71	3.91	14	3.04
Citation Network	449,673	4,707,958	10.43	11.21	42	5.55
E. Coli Metabolism	1,039	5,802	5.58	2.98	8	4.04
Protein Interactions	2,018	2,930	2.90	5.61	14	7.14

# Approximation

- Given that  $d_{\max}$  is dominated by a few long paths, while  $\langle d \rangle$  is averaged over all paths, in general we observe that in an ER graph:

$$\langle d \rangle \approx \frac{\log N}{\log \langle k \rangle}$$



# Exercise

Go to <https://oracleofbacon.org/> and find a famous actress or actor that has a distance from Kevin Bacon larger than

$$\langle d \rangle \approx \frac{\log N}{\log \langle k \rangle} = \frac{\log 702388}{\log 83.71} \approx 3$$

Write the name of the actress/actor and its distance

Write in Nearpod Collaborate  
<https://nearpod.com/student/>  
Code to be given during class

# Clustering coefficient

or

”a friend of a friend is my friend”

# Clustering coefficient $C_i$ of node $i$

- Remember
  - $C_i = 0 \Rightarrow$  neighbors of  $i$  are disconnected
  - $C_i = 1 \Rightarrow$  neighbors of  $i$  are fully connected

# Links between neighbors in ER graphs

- The number of nodes that are neighbors of node  $i$  is  $k_i$
- The number of distinct pairs of nodes that are neighbors of  $i$  is  $k_i(k_i-1)/2$
- The probability that any of those pairs is connected is  $p$
- Then, the expected links  $L_i$  between neighbors of  $i$  are:

$$\langle L_i \rangle = p \frac{k_i(k_i - 1)}{2}$$

# Clustering coefficient in ER graphs

- Expected links  $L_i$  between

neighbors of  $i$ :  $\langle L_i \rangle = p \frac{k_i(k_i - 1)}{2}$

- Clustering coefficient

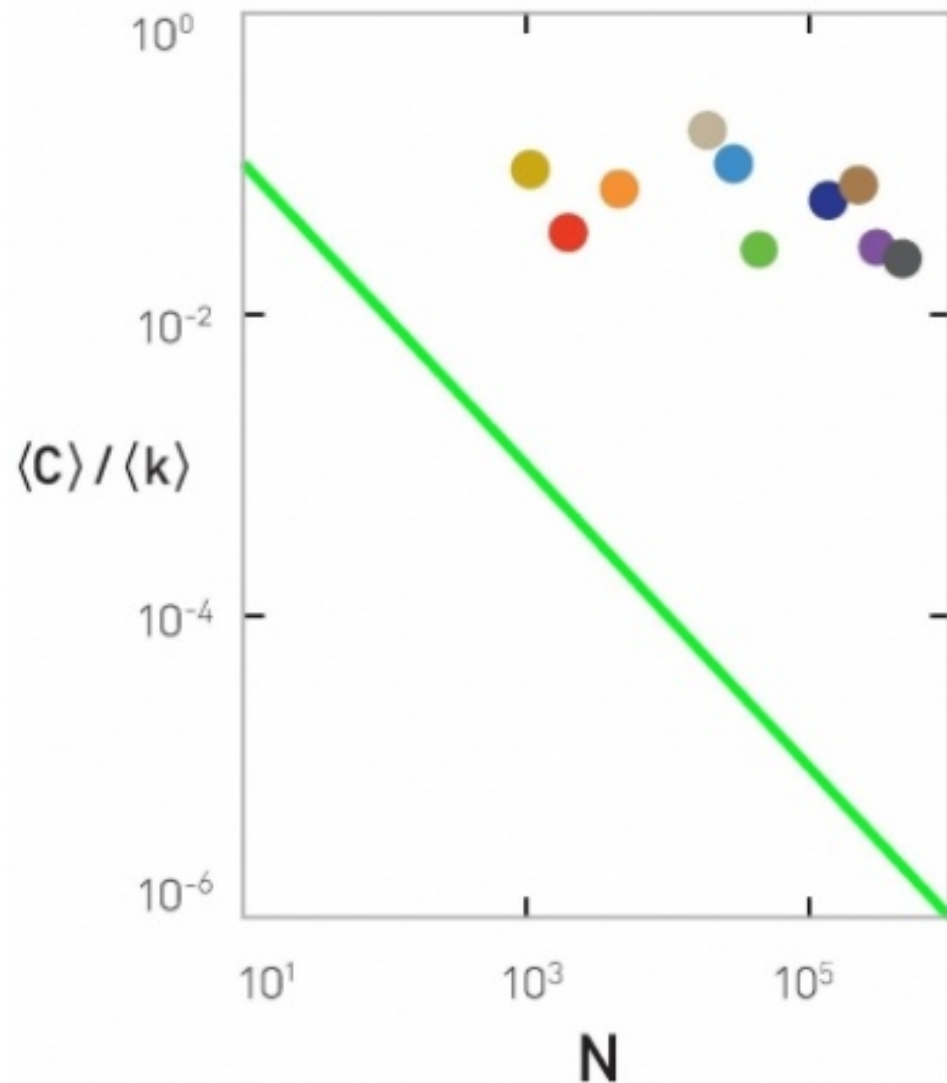
$$\begin{aligned} C_i &= \frac{2 \langle L_i \rangle}{k_i(k_i - 1)} \\ &= \frac{2p \frac{k_i(k_i - 1)}{2}}{k_i(k_i - 1)} = \frac{\langle k \rangle}{N} \end{aligned}$$

# In an ER graph

$$C_i = \langle k \rangle / N$$

If  $\langle k \rangle$  is fixed, large networks should have smaller clustering coefficient

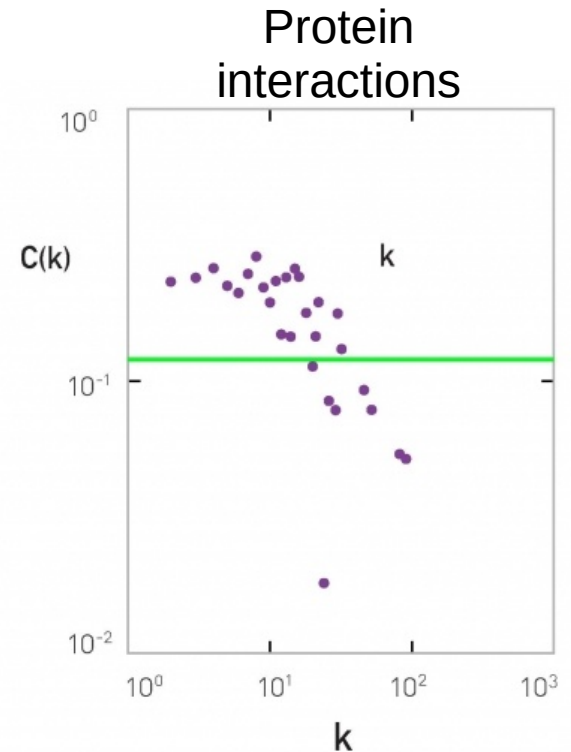
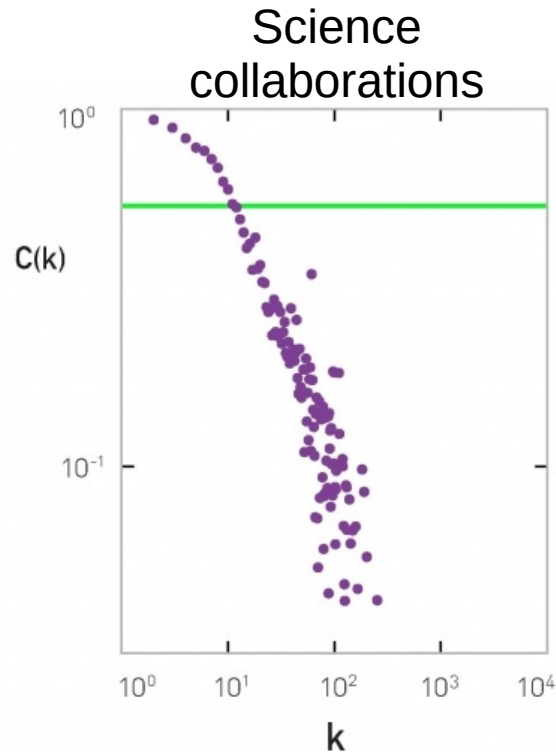
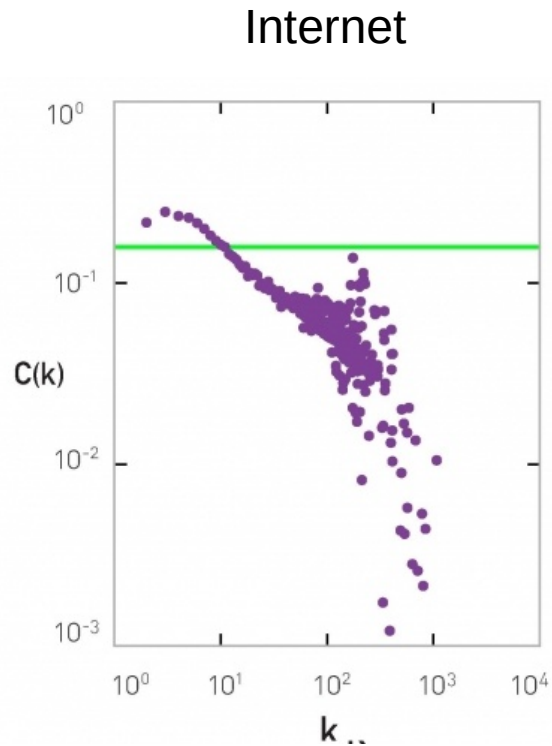
We should have that  $\langle C \rangle / \langle k \rangle$  follows  $1/N$



# If in an ER graph

$$C_i = \langle k \rangle / N$$

Then the clustering coefficient of a node should be independent of the degree



**To re-cap ...**



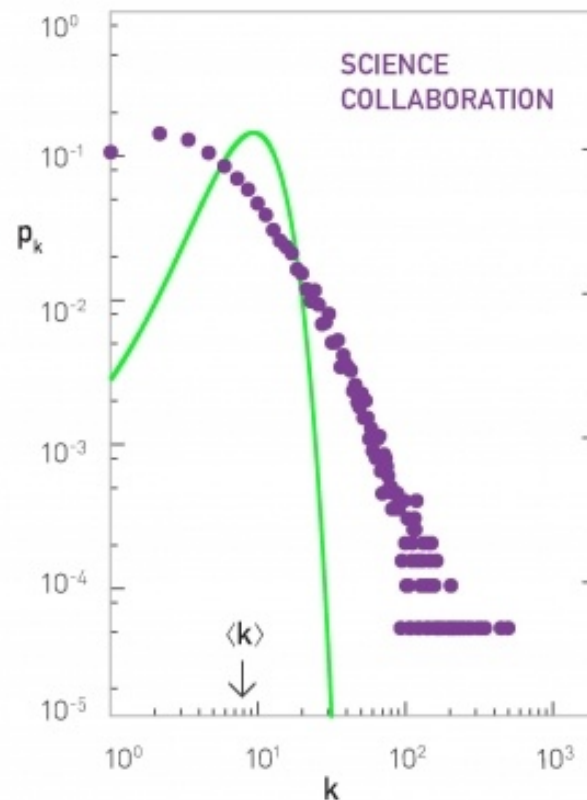
# The ER model is a **bad model** of degree distribution

- Predicted

$$p_k = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$$

- Observed

*Many nodes with larger degree than predicted*



# The ER model is a **good model** of path length

- Predicted

$$d_{\max} \approx \frac{\log N}{\log \langle k \rangle}$$

- $\langle d \rangle \approx \frac{\log N}{\log \langle k \rangle}$

$\langle d \rangle$	$d_{\max}$	$\ln N / \ln \langle k \rangle$
6.98	26	6.58
11.27	93	8.31
18.99	46	8.66
11.72	39	11.42
5.88	18	18.4
5.35	15	4.81
3.91	14	3.04
11.21	42	5.55
2.98	8	4.04
5.61	14	7.14

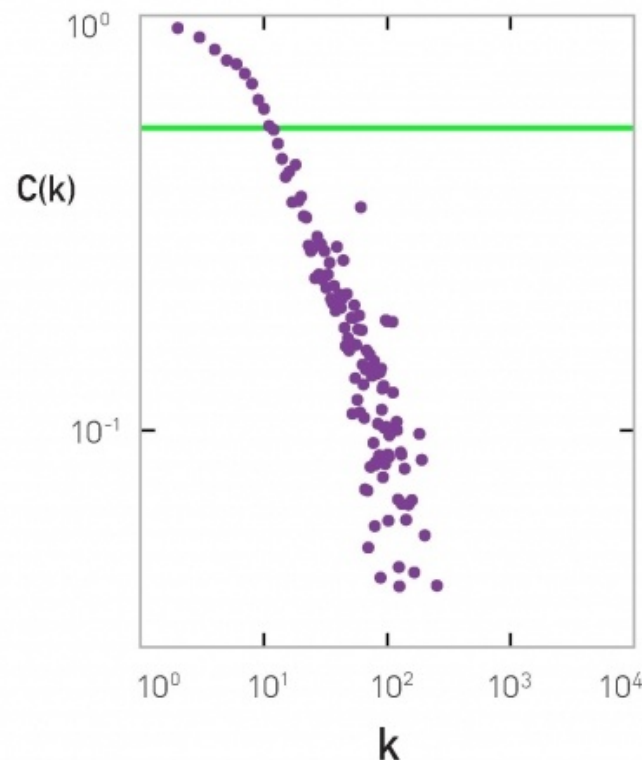
# The ER model is a **bad model** of clustering coefficient

- Predicted

$$C_i = \langle k \rangle / N$$

- Observed

*Clustering coefficient decreases  
if degree increases*



# Why do we study the ER model?

- Starting point
- Simple
- Instructional
- Historically important, and gained prominence only when large datasets started to become available  $\Rightarrow$  relevant to Data Science!

# Exercise [B. 2016, Ex. 3.11.1]

Write in Nearpod Collaborate  
<https://nearpod.com/student/>  
Code to be given during class

Consider an ER graph with  $N=3,000$   $p=10^{-3}$

1)  $\langle k \rangle \simeq ?$

$$\langle k \rangle < 1, \langle k \rangle = 1, \langle k \rangle > 1, \langle k \rangle > \log N$$

2) In which regime is the network?

3) Suppose we want to increase  $N$  until there is only one connected component

3.1) What is  $\langle k \rangle$  as a function of  $p$  and  $N$ ?

$$\langle k \rangle \approx \log N$$

3.2) What should  $N$  be, then? Let's call that value  $N^{\text{cr}}$

Write the equation and solve by trial and error

4) What is  $\langle k \rangle$  if the network has  $N^{\text{cr}}$  nodes?

$$\langle d \rangle \approx \frac{\log N}{\log \langle k \rangle}$$

5) What is the expected distance  $\langle d \rangle$  with  $N^{\text{cr}}$  nodes?

# Summary

# Things to remember

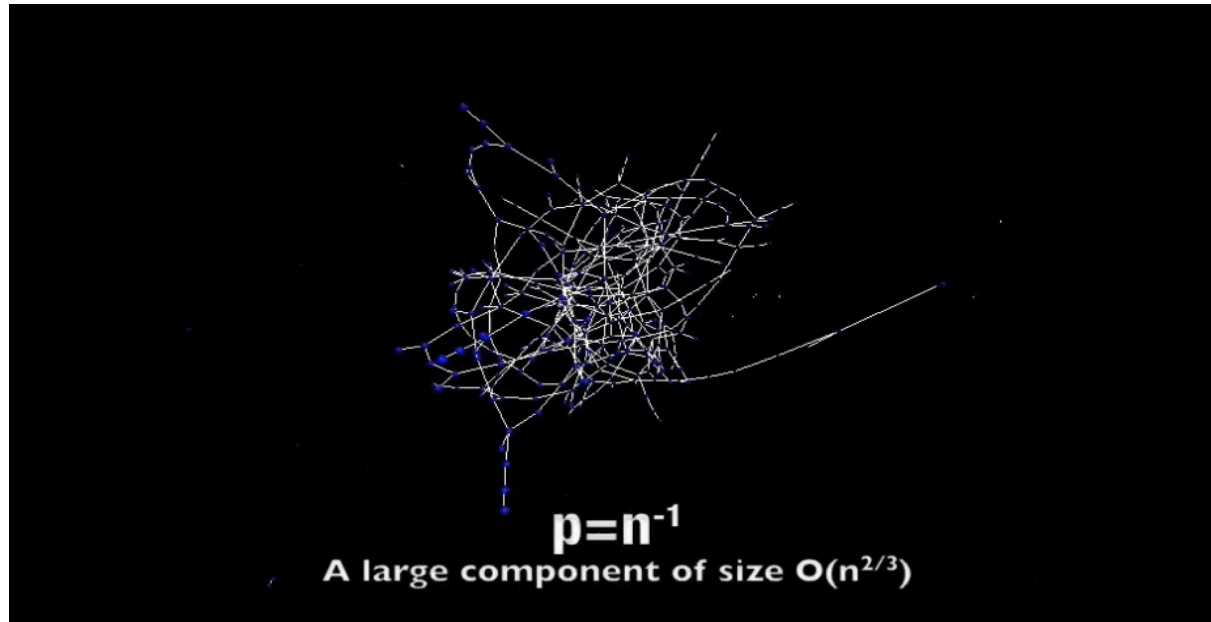
- The ER model
- Degree distribution in the ER model
- Distance distribution in the ER model
- Connectivity regimes in the ER model

# Practice on your own

- Take an existing network
  - (e.g., from the slide “Empirical average and maximum distances”)
  - Assume it is an ER network
  - Indicate in which regime is the network
  - Estimate expected distance
  - Compare to actual distances, if available
- Write code to create ER networks



# Another visualization of the emergence of a giant connected component



<http://networksciencebook.com/images/ch-03/video-3-2.m4v>