# Random Networks (ER Model)

**Social Networks Analysis and Graph Algorithms**

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# Contents

# Sources

- A. L. Barabási (2016). Network Science – Chapter 03
- Data-Driven Social Analytics course by Vicenç Gómez and Andreas Kaltenbrunner
- URLs cited in the footer of specific slides

# Network Models

# Network models

- Networks of many different types have similar properties:
  - Short paths
  - Many triangles
  - Skewed degree distributions
- Where do such properties come from?
- How do nodes connect to each other? How are triangles formed?
- We will study **network models**, i.e., sets of instructions to create networks

Source: Menczer, Fortunato, Davis: A First Course on Networks Science. Cambridge, 2020.

# Why studying network models?

- Our models will be **stochastic,** i.e., randomized
- Running **stochastic network models** can let us check if they generate networks that **look like real ones**
- Almost invariably, the generated networks will be similar to actual networks in some ways, but different in other ways

# The “Random Network” Erdös-Rényi (ER) Model

*Sounds like “ERDOSH and REGN”*

Paul Erdös
(1913-1996)

Alfred Rényi
(1921-1970)

# Meeting people at a party

- You pick a random person
- Talk to that person for a while, if there are good vibes, you are connected
- Then pick another person
  - And repeat
- The result is what we call a **random network**

# Formalization (Erdös-Rényi or ER)

*Sounds like "ERDOSH and REGN"*

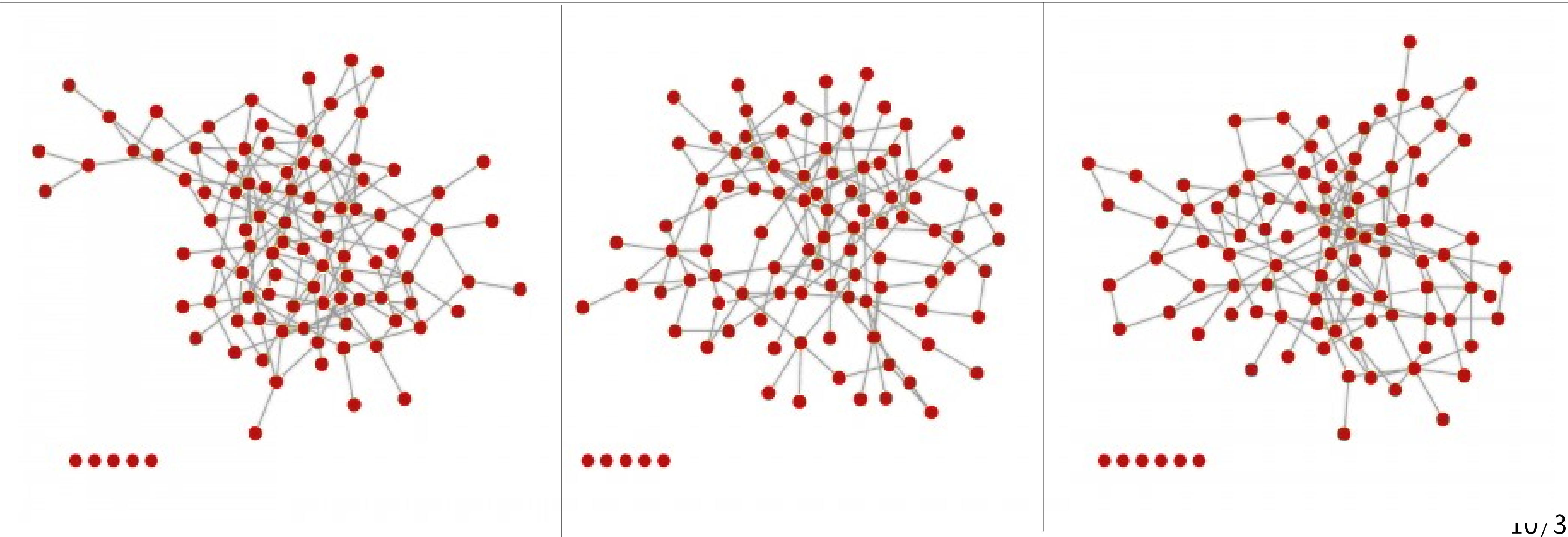
- For each pair of nodes in the graph
  - Perform a Bernoulli trial with probability $p$
    - "Toss a biased coin with probability $p$ of landing heads"
  - If the trial succeeds, connect those nodes
    - "If the coin lands heads, connect those nodes"
- Repeat for all pairs $\dfrac{N(N-1)}{2}$

# Example

# (3 networks, same parameters)

$$N = 100, p = 0.03$$

Nodes at the bottom ended up isolated

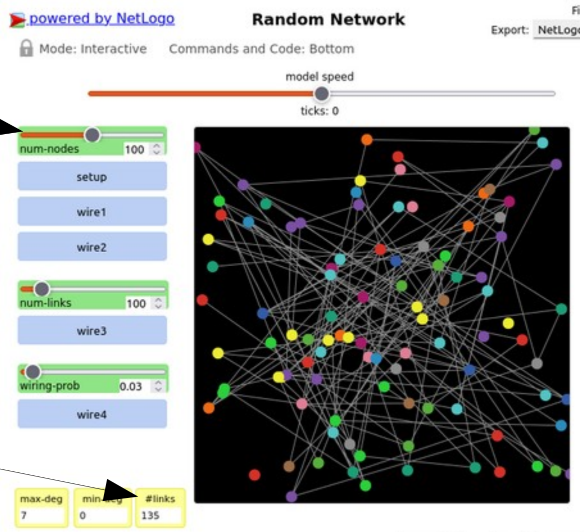


# Exercise

## Guess formula for E[L]

Go to netlogoweb.org/launch and select:
"IABM Textbook / chapter 5 / Random Network"

- Execute the "Random Network" program in Netlogo Web
  - Select num-nodes $N$ (e.g., 100)
  - Click "setup"
  - Select wiring-prob $p$ (e.g., 0.03)
  - Click "wire4"
  - Write down "#links" $L$ somewhere
  - Repeat various times
- **Guess a formula for E[L] as a function of N and p**




Pin board: https://upfbarcelona.padlet.org/chato/84a1nj59pkqpxvh3

# Degree distribution

# A key characteristic of a network: its degree distribution

- One of the most evident characteristics of a network is its **degree distribution**
  - Is this distribution very skewed? Or every node is close to some average? Is there a "typical" degree?
  - Does it look like the degree distribution predicted by a network formation model?
- We will spend a fair amount of time studying the degree distribution under various models

# The binomial distribution

- The distribution of the probability of obtaining x successes in n independent trials, in which each trial has probability of succeeding p

$$p_x = \binom{n}{x} p^x (1-p)^{n-x}$$

$$\langle x \rangle = \sum_{x=0}^{n} x p_x = np$$

# Degree distribution in ER model

- Simply a Binomial distribution
- Note that the maximum number of "successes" (links) of a node is N-1, hence:

$$p_k = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

$$\langle k \rangle = p(N-1)$$

# Expected number of links

- Expected number of links

$$\langle L \rangle = p \cdot L_{\max} = p \frac{N(N-1)}{2}$$

- Average degree

$$\langle k \rangle = \frac{2\,\langle L \rangle}{N} = p(N-1)$$

# Exercise [B. 2016, Ex. 3.11.1]

## Expected number of links and average degree

- Consider an ER graph with N=3,000 $p=10^{-3}$

  1) What is the expected number of links <L>?

  2) What is the average degree <k>?

$$\langle L \rangle = p \cdot L_{\max} = p\frac{N(N-1)}{2}$$

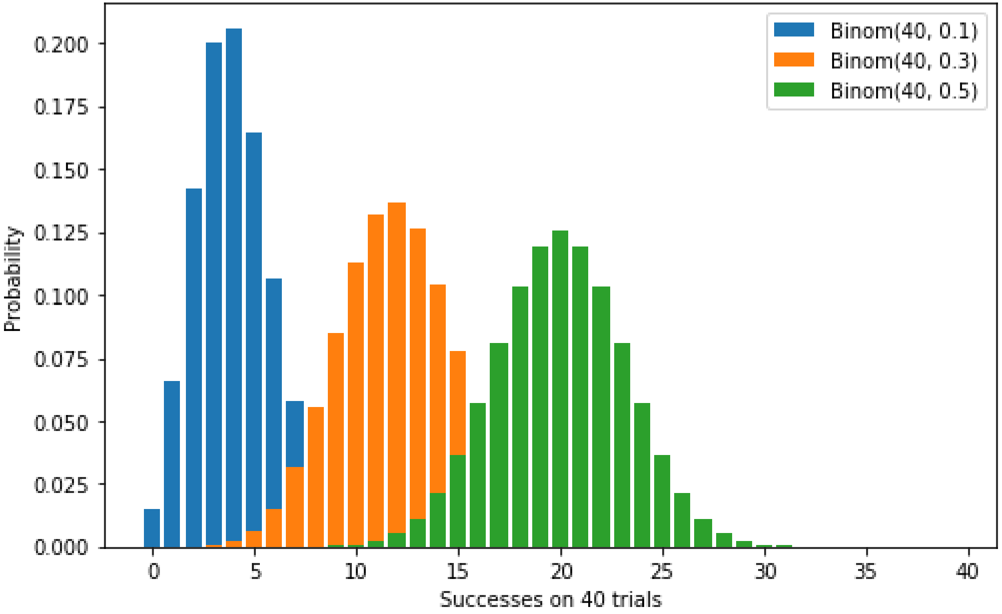
$$\langle k \rangle = \frac{2\langle L \rangle}{N} = p(N-1)$$

# Degree distribution examples

- The peak is always at $\langle k \rangle = p(N-1)$

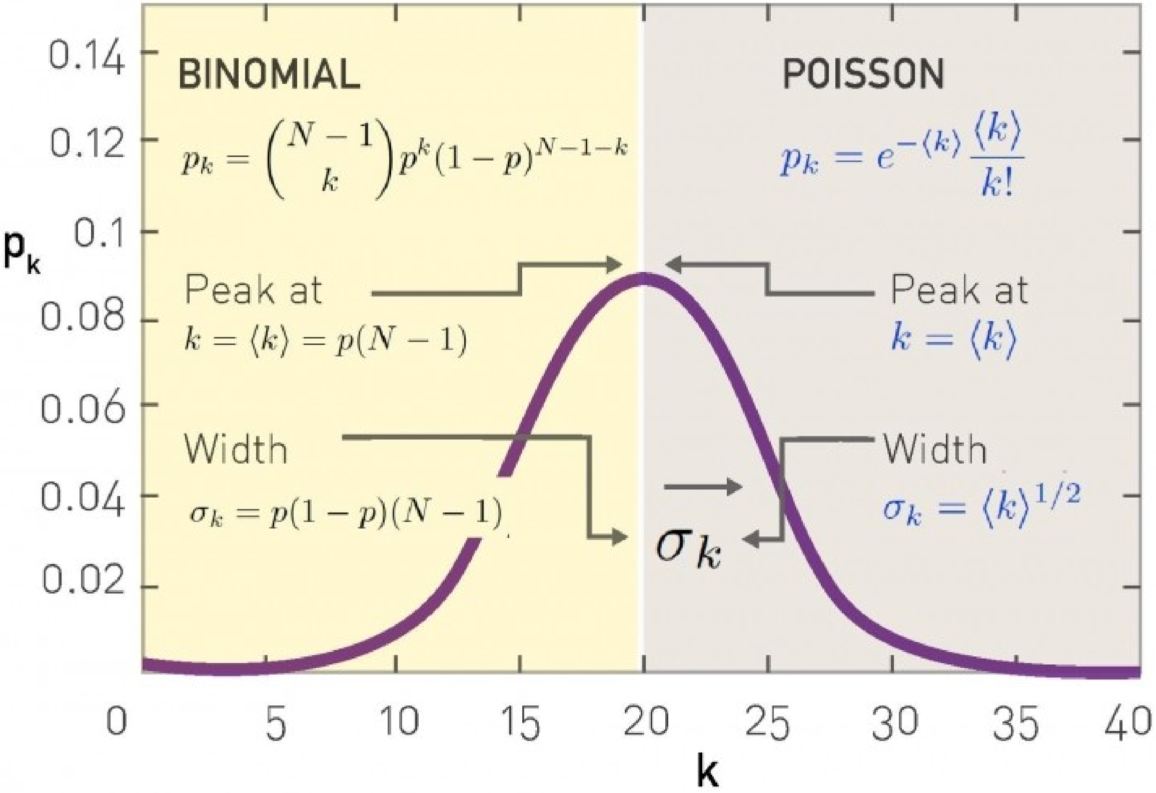
```
import numpy as np
from scipy.stats import binom
from matplotlib import pyplot as plt
```

```
x = np.arange(0, 40)
plt.figure(figsize=(8,5))
plt.bar(x, (binom(40, 0.1)).pmf(x), label='Binom(40, 0.1)')
plt.bar(x, (binom(40, 0.3)).pmf(x), label='Binom(40, 0.3)')
plt.bar(x, (binom(40, 0.5)).pmf(x), label='Binom(40, 0.5)')
plt.gca().legend()
plt.xlabel("Successes on 40 trials")
plt.ylabel("Probability")
plt.show()
```



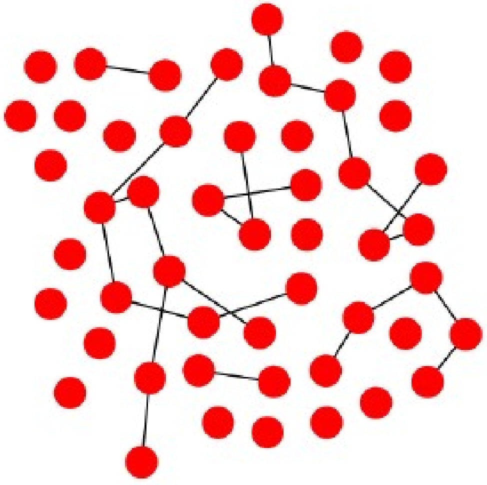

python™

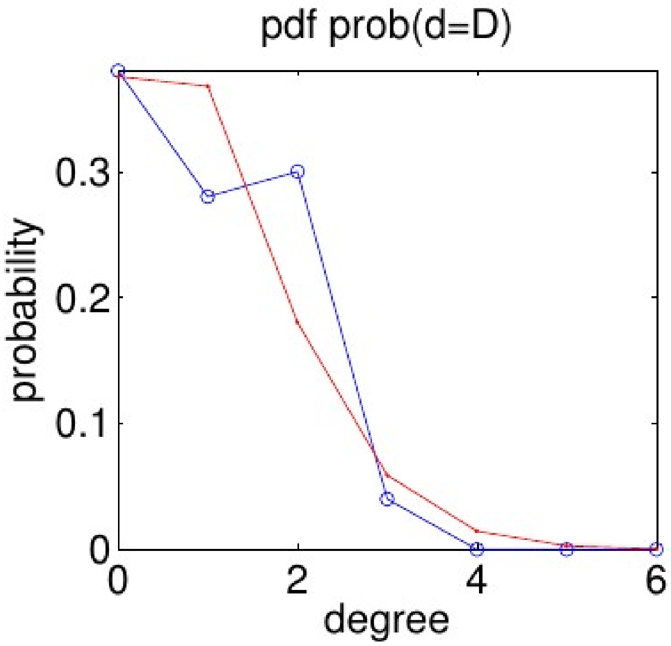
# Approximation with a Poisson distribution for $\langle k \rangle \ll N$

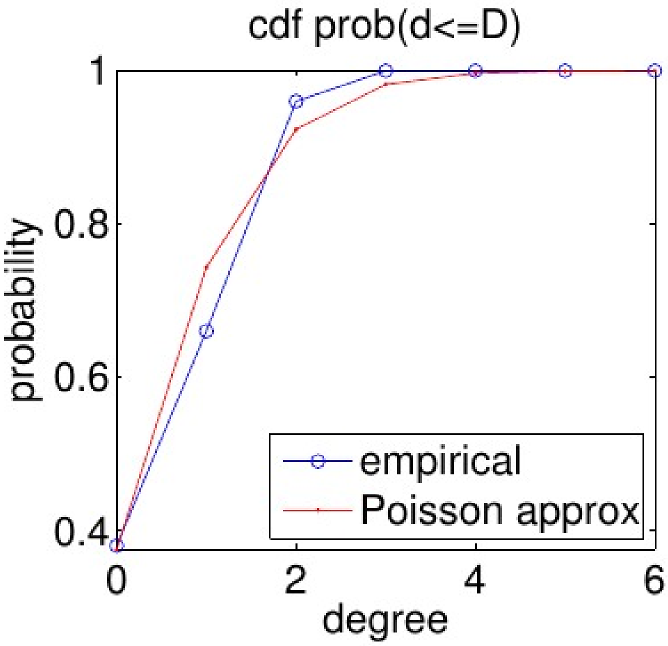


# More examples (1/6)

$$N = 50, p = 0.02, \langle k \rangle \approx 1$$



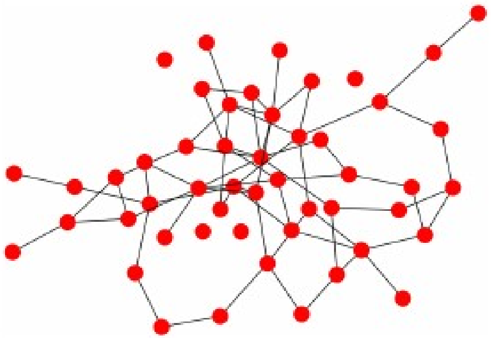


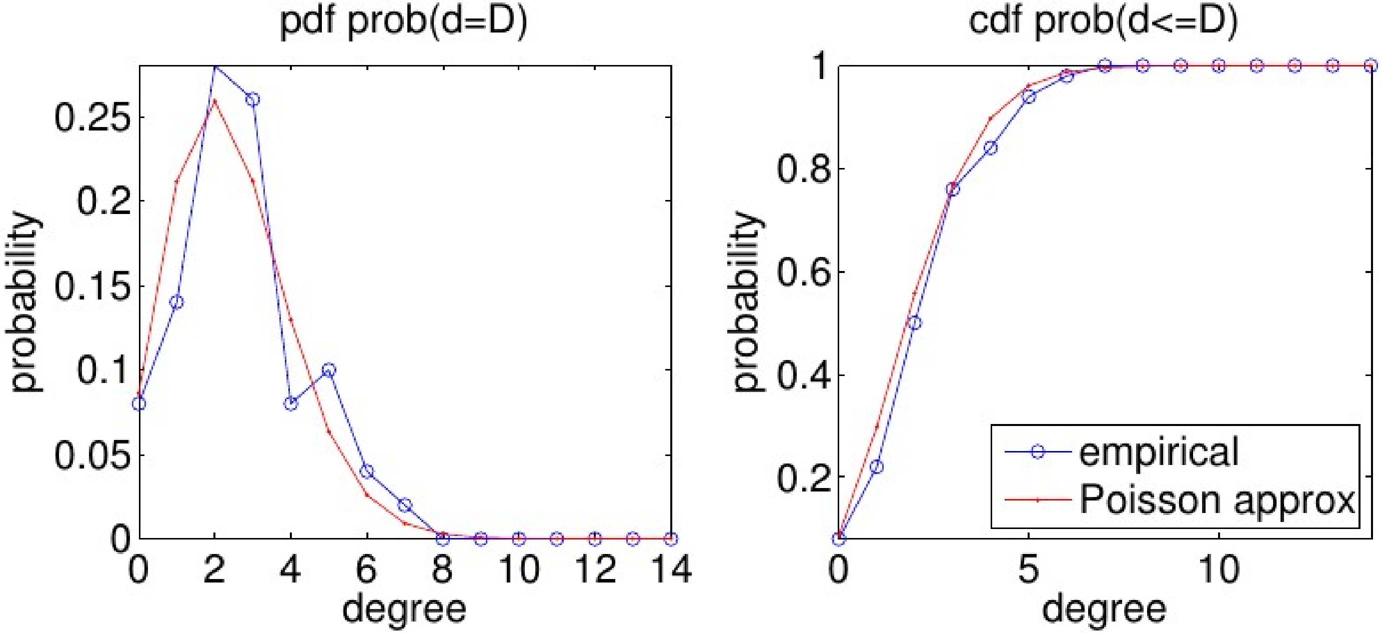




https://www.upf.edu/web/iis/DataDrivenSocialAnalytics

# More examples (2/6)

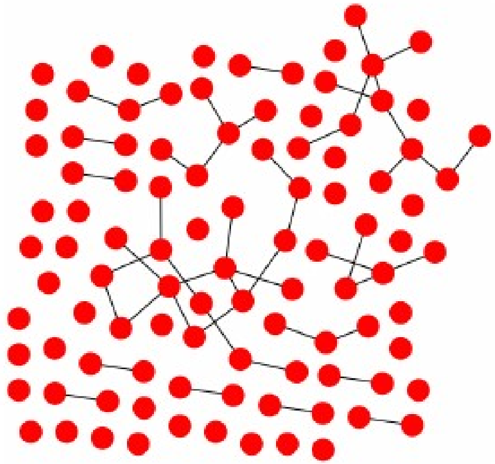
$$N = 50, p = 0.05, \langle k \rangle \approx 2.5$$

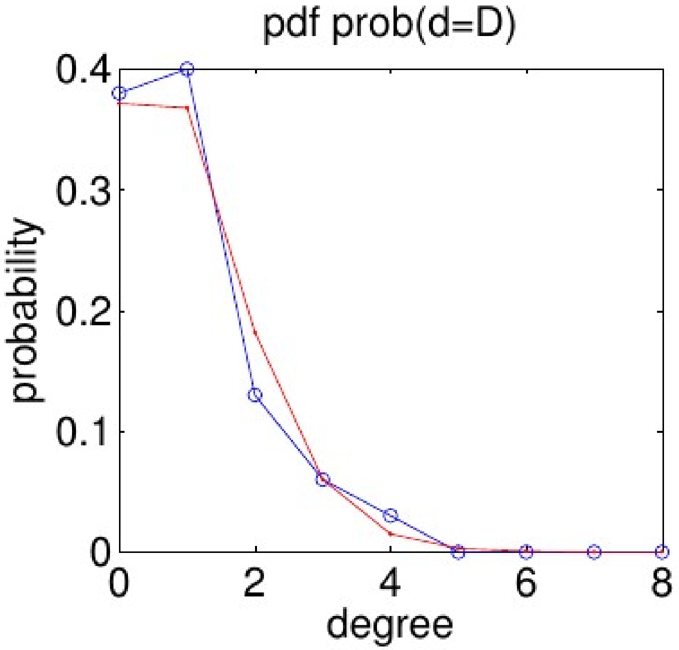


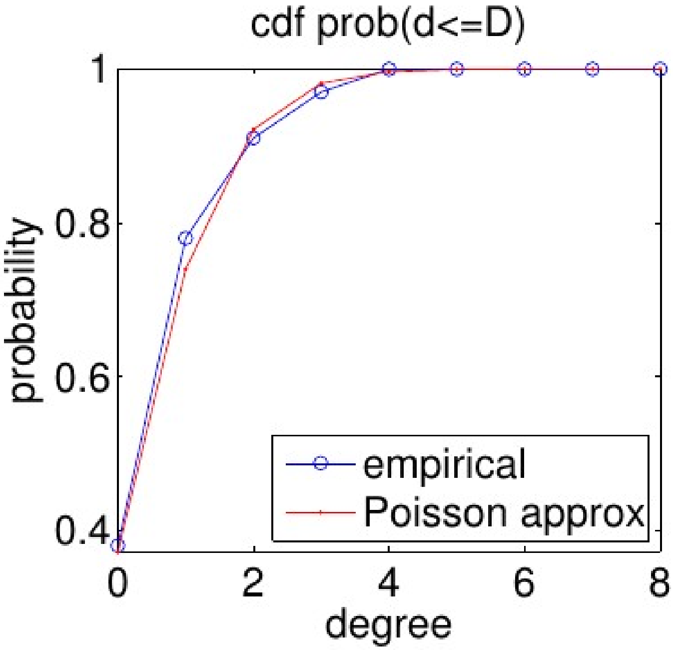


# More examples (3/6)

$$N = 100, p = 0.01, \langle k \rangle \approx 1$$



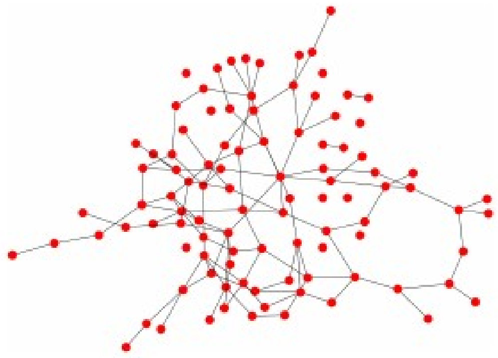


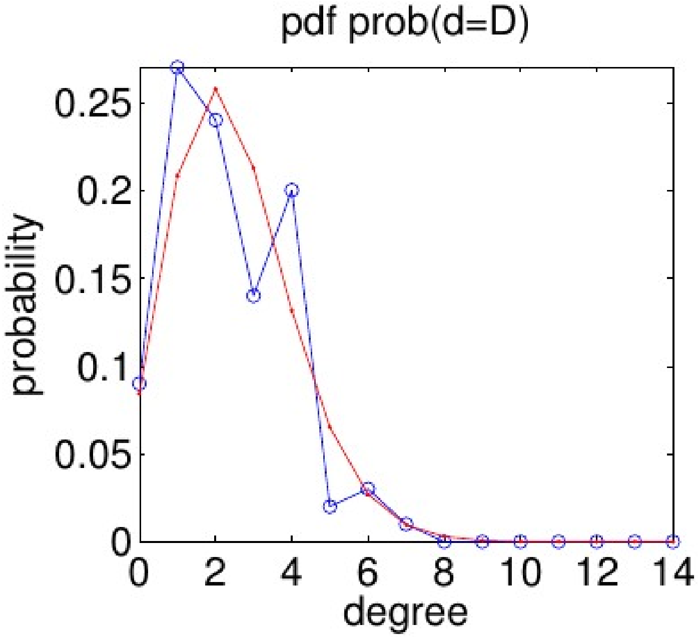



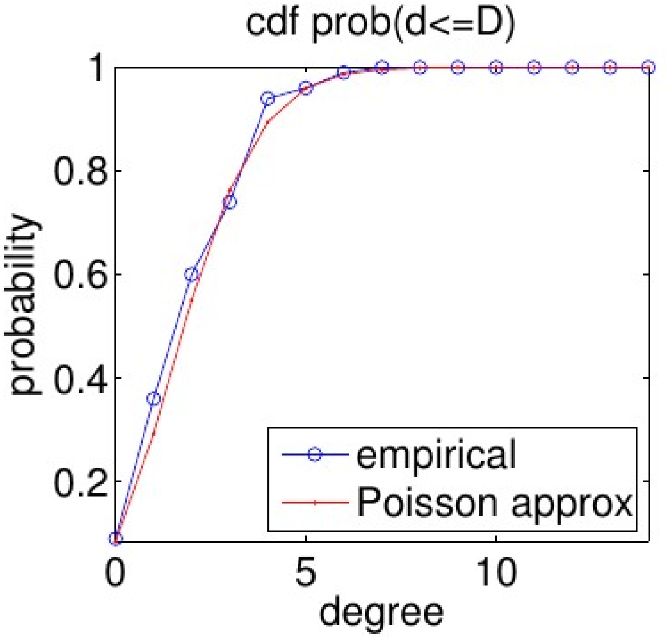

https://www.upf.edu/web/iis/DataDrivenSocialAnalytics

# More examples (4/6)

$$N = 100, p = 0.025, \langle k \rangle \approx 2.5$$



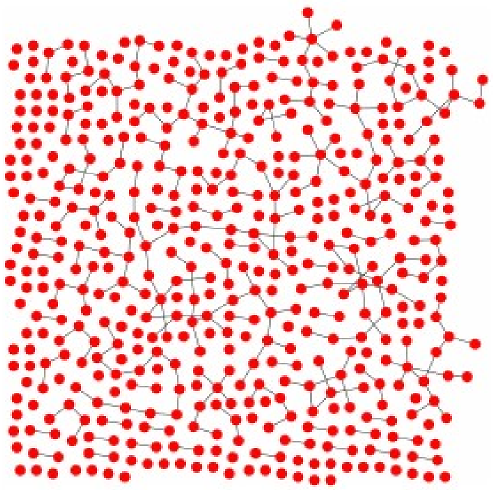


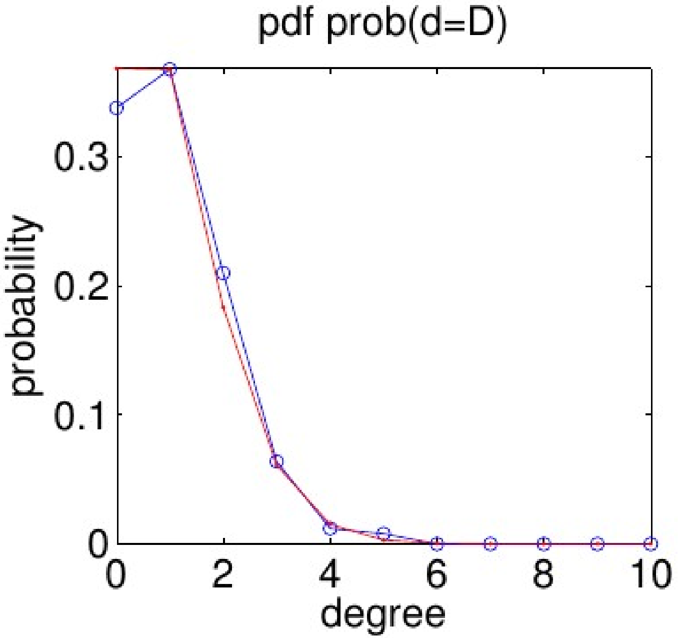



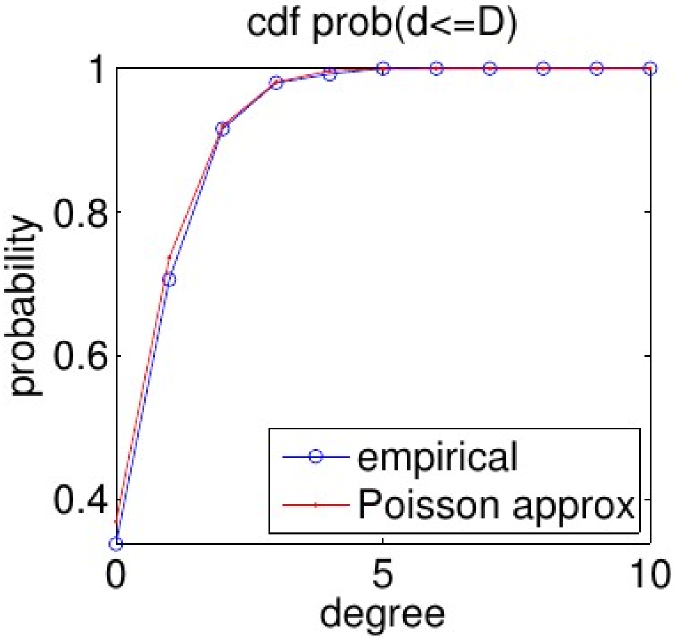

https://www.upf.edu/web/iis/DataDrivenSocialAnalytics

# More examples (5/6)

$$N = 500, p = 0.002, \langle k \rangle \approx 1$$

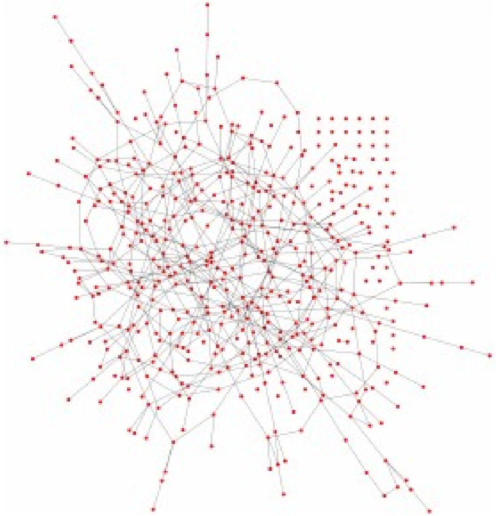


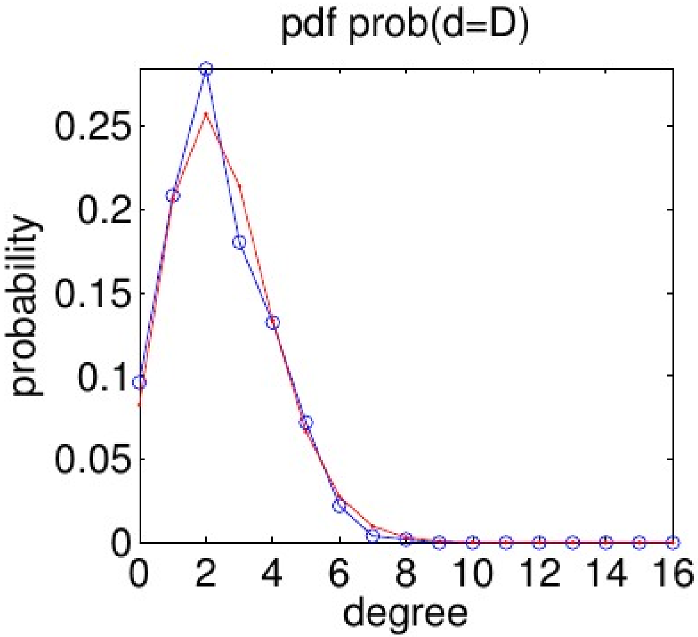


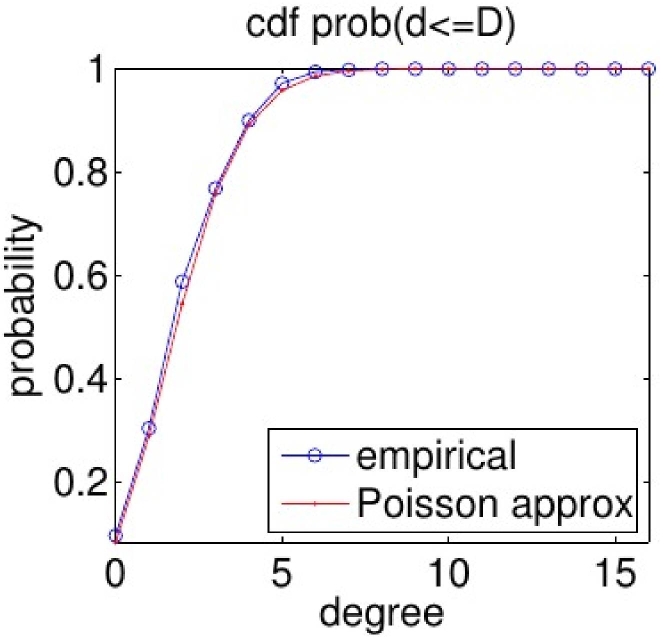



# More examples (6/6)

$$N = 500, p = 0.005, \langle k \rangle \approx 2.5$$



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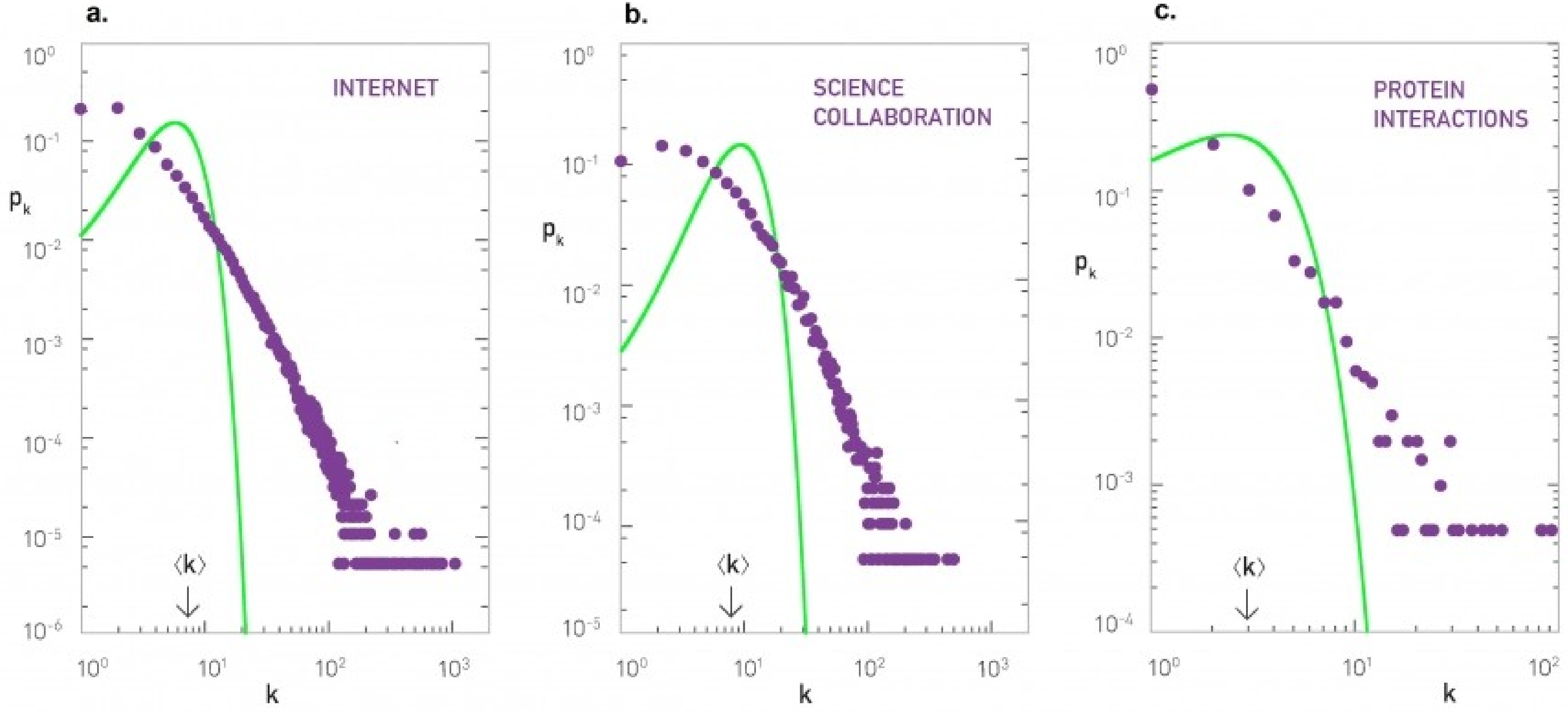
# "Back of the envelope" calculations

- Suppose $N = 7 \times 10^9$
- Suppose $<k> = 1{,}000$
  - A person knows the name of approx. 1,000 others
- Then on expectation $k_{max} = 1{,}185$
- $<k> \pm \sigma$ is the range from 968 to 1,032
- Is this realistic?

# Survey: how many WhatsApp contacts do you have?

https://forms.gle/9xEYhzv2U5NrPQdH8

# Real networks (green = $e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!}$ )



# Summary

# Things to remember

- The ER model
- Degree distribution in the ER model

# Practice on your own

- Indicate the expected number of edges of a network with N=256, p=0.25; then compare your solution with the one on this video:


https://www.youtube.com/watch?v=2DckiyysQy4