Simulator justification file

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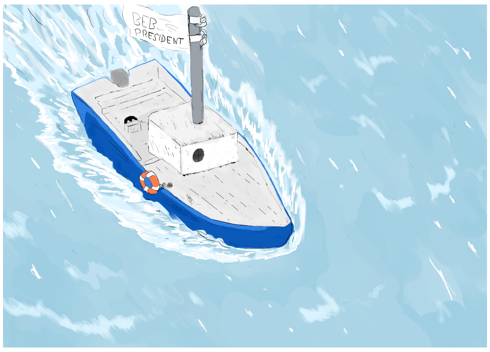

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Chapter 1

Simulator design & introduction

This file gathers the mathematical equations behind the simulator of the ship. The objective is to be able to control the motion of the ship, its position, and its heading. In fact, by controlling the heading, the position can be controlled. Therefore, the choice has been made to dynamically control the heading only.

In order to do this, several classes of algorithms must be designed and implemented on a microprocessor. The latter will communicate with the sensors and actuators to execute actions in the scope of following a pre-defined trajectory. For a description of these algorithms and their connections, see [1].

However, the algorithms designed can only serve an existing system. Due to the fact that direct implementation is not possible due to the lack of availability to the ship, a mathematical model of the vessel has been built to simulate its behavior. As a result, based on the assumption that the simulator is close enough to the real boat, the tuning of the Guidance, Navigation, and Control algorithms has been possible.

As we only want to control the position of the ship, by actively controlling its heading, the outputs of the ship will be the heading, the position along a local horizontal North axis and a local vertical East axis. The motion is supposed to be planar and can be described as follows if no rudder action is applied:

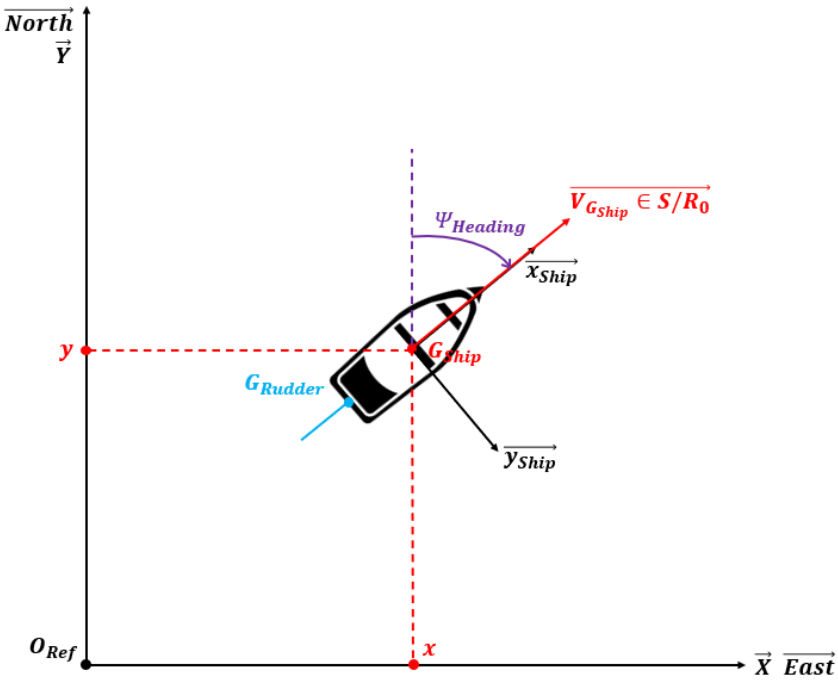

Figure 1.1: Planar motion description without rudder action - Without heading change.

On the other hand, if a rudder angle is applied to allow the ship to turn, the velocity is no longer perfectly aligned with the bow of the ship, and two components of the velocity can be observed on the longitudinal axes of the vessel.

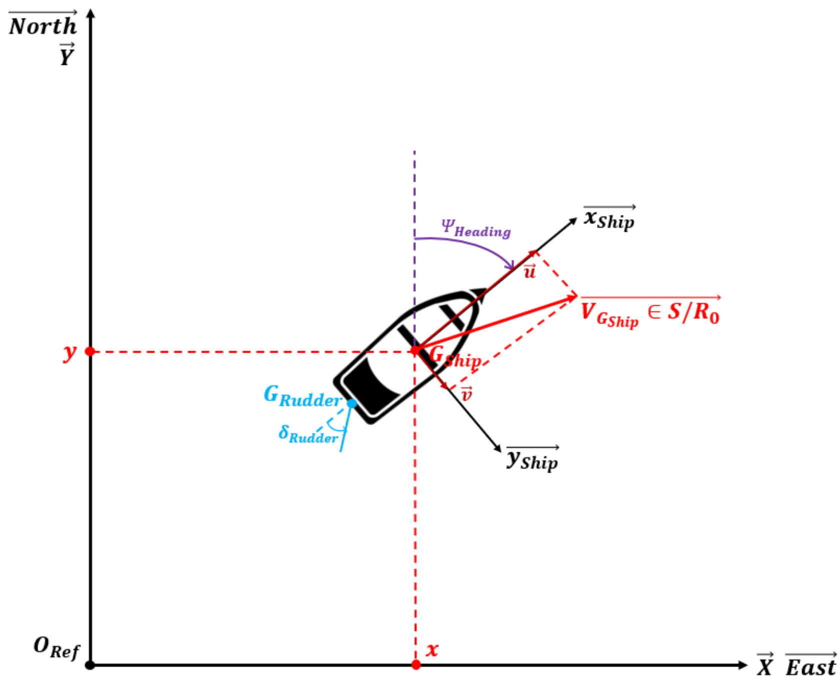

Figure 1.2: Planar motion description with rudder action - Imposing heading change.

In the previous figures, two frames are used to express physical parameters. First, a fixed inertial frame whose origin O_{Ref} coincides with the initial point of the trajectory and whose axes point towards the East, North, and Up directions. That's the ENU frame: $(O_{Ref}; \vec{East}, \vec{North}, \vec{Up}) = (O_{Ref}; \vec{X}, \vec{Y}, \vec{Z})$. On the other hand, the frame attached to the main axes of the ship is moving with respect to the fixed inertial frame. Its origin is fixed to be at the center of gravity of the ship. The frame is $(G_{Ship}; \vec{x_{Ship}}, \vec{y_{Ship}}, \vec{z_{Ship}})$.

However, as will be presented in the next sections, as the rudder turning angular rate is very limited (a few degrees per second), a good approximation is to stay on the Figure 1.1 supposing that the velocity component on the longitudinal ship axis will always be much higher than the one on the transversal ship axis.

The inputs and outputs of the simulator shall be as follows:

Inputs	Rudder angle δ
Outputs	Heading angle ψ North Y-Axis position East X-Axis position

Table 1.1: Inputs and outputs of the boat simulator.

In the very next section, the global 6 Degrees of Freedom (DoF) equations of motion of a surface rigid-body ship will be derived using Newtonian vectorial equations and rigid-body dynamics. We will see that, for the purpose of controlling the heading, they can be further simplified.

Chapter 2

Newtonian Motion of the ship

2.1 6 DoF non linear dynamics

The main equations and rationale in the following sections are based on the technical book on Guidance & Control of Ocean Vehicles [2].

The 6 DoF non-linear dynamic of the ship can be described using the Newton's second law of translational and rotational motions in vectorial form.

Fundamental Principle of Dynamics

$$\sum \mathbf{F}_{\overline{S}/S} = m_S \cdot \mathbf{a}_S \tag{2.1}$$

$$\sum \boldsymbol{M}_{G,\boldsymbol{F}_{\overline{S}/S}} = \bar{\bar{\mathbf{I}}}_{G_{Ship}} \cdot \dot{\boldsymbol{\Omega}}_{R_{\text{Ship}}/R_{\text{Inertial}}} \tag{2.2}$$

Denoting:

- $\boldsymbol{a}_S$ The acceleration vector of the ship.
- m_S The mass of the ship.
- $\boldsymbol{F}_{\overline{S}/S}$ The forces applied to the ship.
- $\boldsymbol{M}_{G,\boldsymbol{F}_{\overline{S}/S}}$ The moments applied to the ship taken at its gravity point.
- $\bar{\bar{\boldsymbol{I}}}_{G_{Ship}}$ The inertia matrix of the ship at its gravity point.
- $\dot{\boldsymbol{\Omega}}_{R_{\text{Ship}}/R_{\text{Inertial}}}$ The angular acceleration of the moving ship with respect to an inertial fixed frame.
- S The abbreviation for Ship.

Note that in the literature, the general vectorial form of the Newton's second law applied to a Surface vessel in motion through the water has a specific notation. In vectorial form, the equations can be presented as follows:

$$M \cdot \dot{\boldsymbol{\nu}} = \boldsymbol{\tau} - C(\boldsymbol{\nu}) \cdot \boldsymbol{\nu} - D(\boldsymbol{\nu}) \cdot \boldsymbol{\nu} - \boldsymbol{g}(\boldsymbol{\eta}) \tag{2.3}$$

With the following elements:

- M The Inertia Matrix.
- $\boldsymbol{\nu}$ The linear and angular ship velocity vector.
- $C(\boldsymbol{\nu})$ The Coriolis and Centripetal terms matrix.
- $D(\boldsymbol{\nu})$ The damping matrix.
- $\boldsymbol{g}(\boldsymbol{\eta})$ The vector of gravitational Forces and Moments.

- $\boldsymbol{\tau}$ Control inputs vector.
- $\boldsymbol{\eta}$ Position and attitude vector with coordinates in Earth's fixed frame.

Note that a precise description of the Coriolis and Centripetal terms matrix and the Inertia matrix will be done in the following development.

The linear and angular velocities of the ship representing its motion can be summarized bellow:

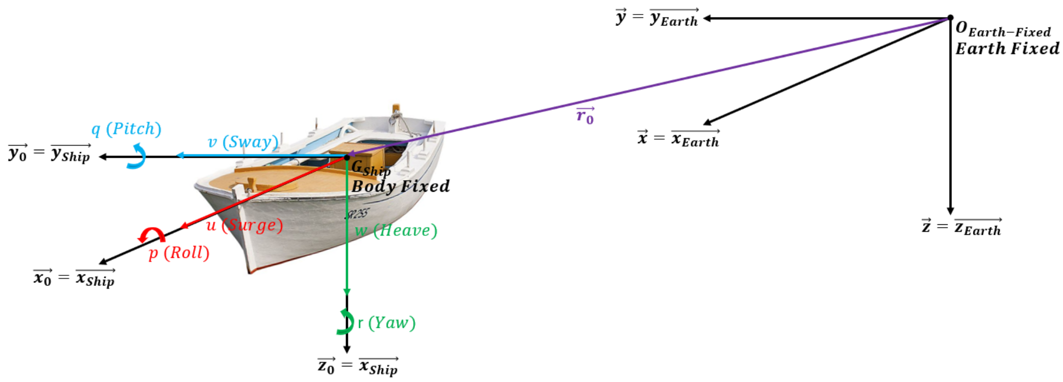

Figure 2.1: 6 DoF motion problem parametrization.

The Body-Fixed frame R_{Ship} is fixed to the vehicle. The origin of this frame is chosen to coincide with the center of gravity of the ship. The motion of the moving Body-Fixed frame is usually described with respect to an inertial reference frame. This inertia frame $R_{Inertia}$ is considered to be Earth-fixed. The following parameters will be used to model the ship motion:

DOF	Motion	Forces and Moments	Linear and Angular Velocities	Positions and Euler Angles
1	Motions in the x-direction (surge)	X	u	x
2	Motions in the y-direction (sway)	Y	v	y
3	Motions in the z-direction (heave)	Z	w	z
4	Rotation about the x-axis (roll)	K	p	ϕ
5	Rotation about the y-axis (pitch)	M	q	θ
6	Rotation about the z-axis (yaw)	N	r	ψ

Table 2.1: DoF and associated quantities in marine dynamics.

Additionally, the general parameter notations are recalled here for the next mathematical development and for convenience:

$$\boldsymbol{\eta_1} = \begin{pmatrix} x & y & z \end{pmatrix}^{\mathrm{T}} \qquad \boldsymbol{\eta_2} = \begin{pmatrix} \phi & \theta & \psi \end{pmatrix}^{\mathrm{T}} \qquad \boldsymbol{\eta} = \begin{pmatrix} \boldsymbol{\eta_1}^{\mathrm{T}} & \boldsymbol{\eta_2}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$$

$$\boldsymbol{\nu_1} = \begin{pmatrix} u & v & w \end{pmatrix}^{\mathrm{T}} \qquad \boldsymbol{\nu_2} = \begin{pmatrix} p & q & r \end{pmatrix}^{\mathrm{T}} \qquad \boldsymbol{\nu} = \begin{pmatrix} \boldsymbol{\nu_1}^{\mathrm{T}} & \boldsymbol{\nu_2}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$$

$$\boldsymbol{\tau_1} = \begin{pmatrix} X & Y & Z \end{pmatrix}^{\mathrm{T}} \qquad \boldsymbol{\tau_2} = \begin{pmatrix} K & M & N \end{pmatrix}^{\mathrm{T}} \qquad \boldsymbol{\tau} = \begin{pmatrix} \boldsymbol{\tau_1}^{\mathrm{T}} & \boldsymbol{\tau_2}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$$

2.2 Newton-Euler formulation

The following formulation of the mechanical aspect of the problem will be useful when using the Rigid-Body Dynamics equations to write the set of 6 DoF non-linear equations. This formulation is based on Newton's second law, as depicted by equation 2.1, which can be written in a simpler form as follows:

$$m_S \cdot \dot{\boldsymbol{v_G}} = \boldsymbol{f_G} \tag{2.4}$$

With:

- $\dot{\boldsymbol{v_G}}$ The acceleration of the ship at its gravity center.
- $\boldsymbol{f_G}$ The force acting on the ship at its gravity center.

Euler's first and second axioms

Newton's second law has been interpreted by Euler on terms of conservation of linear and angular momentum, respectively $\boldsymbol{p_G}$ and $\boldsymbol{h_G}$.

$$\dot{\boldsymbol{p_G}} \triangleq \boldsymbol{f_G} \qquad \boldsymbol{p_G} \triangleq m_S \cdot \boldsymbol{v_G} \tag{2.5}$$

$$\dot{\boldsymbol{h_G}} \triangleq \boldsymbol{M_G} \qquad \boldsymbol{h_G} \triangleq \overline{\overline{\boldsymbol{I}}}_G \cdot \boldsymbol{\Omega} \tag{2.6}$$

For notation ease, we shall next consider the point $G_{Ship} = G$ and $S = Ship$. The term $\mathbf{F}_{\overline{S}/S}$ will represent the resulting force. Applying our notation to the previously introduced general mathematical expression will lead to the following:

$$m_S \cdot \dot{\boldsymbol{V}}_G \in (S/R_{Inertia}) = \mathbf{F}_{\overline{S}/S} \tag{2.7}$$

$$\dot{\boldsymbol{p}}_{\boldsymbol{G}} \in \boldsymbol{(S/R_{Inertia})} \triangleq \mathbf{F}_{\overline{S}/S} \qquad \boldsymbol{p}_G \in (S/R_{Inertia}) \triangleq m_S \cdot \boldsymbol{V}_G \in (S/R_{Inertial}) \tag{2.8}$$

$$\dot{\boldsymbol{L}}_{\boldsymbol{G}} \in \boldsymbol{(S/R_{Inertia})} \triangleq \boldsymbol{M}_{G,\boldsymbol{F}_{\overline{S}/S}} \qquad \boldsymbol{L_G} \in \boldsymbol{(S/R_{Inertia})} \triangleq \overline{\overline{\boldsymbol{I}}}_{G_{Ship}} \cdot \boldsymbol{\Omega}_{R_{\text{Ship}}/R_{\text{Inertial}}} \tag{2.9}$$

Chapter 3

General Rigid-Body Dynamics

The main results of Rigid-Body dynamics will be derived and recalled here. The general expressions obtained will then be directly used to establish the 6 DoF equations representing the motion of a surface ship through water in the next chapter.

Our reasoning will be based upon the following comprehensive scheme illustrating an amorphous rigid body in motion with respect to a fixed inertial frame:

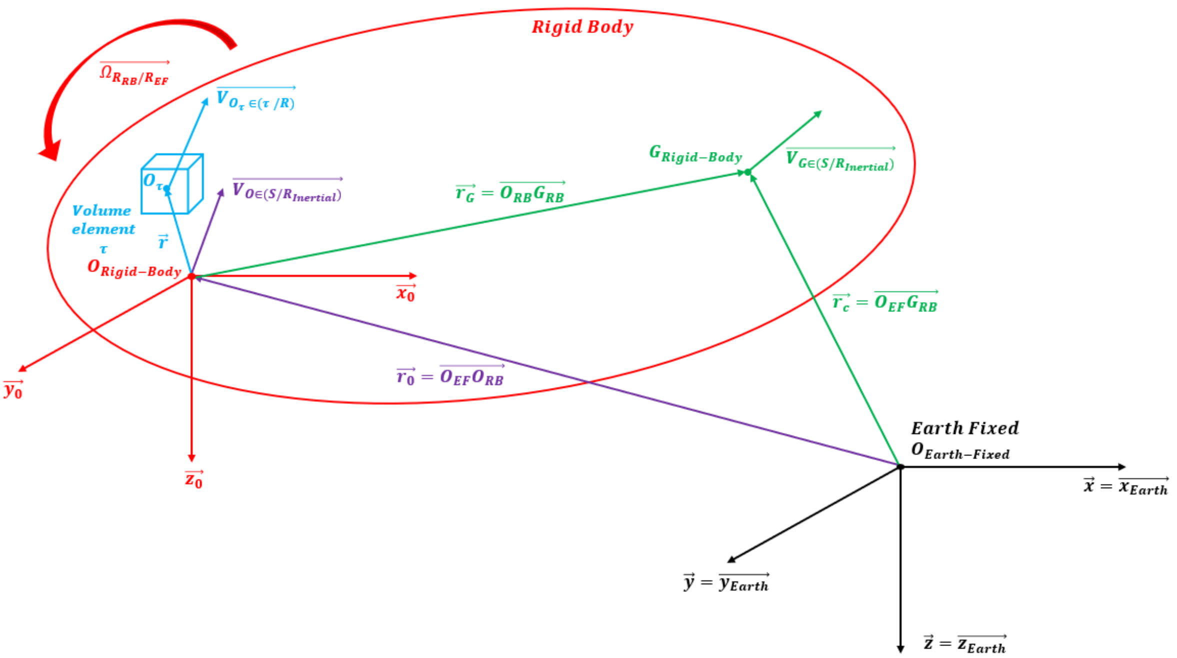

Figure 3.1: Rigid-Body Dynamics problem parametrization.

Note that the origin of the frame attached to the rigid-body is not necessarily coincident to its center of gravity. In the next development, the term RB will be used whenever the red rigid body frame will be used. The term EF or $Inertial$ will refer to the black frame assumed to be inertial.

3.1 Translational motion

Based on the previous figure, it is possible to state that:

$$\boldsymbol{O_{EF}G_{RB}} = \boldsymbol{O_{EF}O_{RB}} + \boldsymbol{O_{RB}G_{RB}} \Leftrightarrow \boldsymbol{r_C} = \boldsymbol{r_O} + \boldsymbol{r_G}$$

Therefore, the velocity vector at G will be:

$$\boldsymbol{V}_G \in (S/R_{Inertial}) = \frac{d\boldsymbol{O_{EF}G_{RB}}}{dt} = \frac{d(\boldsymbol{r_O} + \boldsymbol{r_G})}{dt} = \dot{\boldsymbol{r_O}} + \dot{\boldsymbol{r_G}}$$

Additionally:

$$\frac{d\boldsymbol{r_O}}{dt}^{[R_{Inertial}]} = \boldsymbol{V}_O \in (S/R_{Inertial}) = \dot{\boldsymbol{r_O}} = \boldsymbol{v_O}, \qquad \frac{d\boldsymbol{r_G}}{dt}^{[R_{RB}]} = \boldsymbol{0}$$

This is due to the fact that the body RB is rigid. Therefore, a vector expressed inside the rigid body will move or evolve. Its derivative with respect to time is then null. Furthermore:

$$\dot{\boldsymbol{r_G}} = \dot{\boldsymbol{O_{RB}G_{RB}}} = \frac{d\boldsymbol{r_G}}{dt}^{[R_{Inertial}]} = \frac{d\boldsymbol{r_G}}{dt}^{[R_{RB}]} + \boldsymbol{\Omega}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times \boldsymbol{r_G} \Rightarrow \dot{\boldsymbol{r_G}} = \boldsymbol{\Omega}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times \boldsymbol{r_G} \tag{3.1}$$

Therefore:

$$\boldsymbol{V}_G \in (S/R_{Inertial}) = \boldsymbol{V}_O \in (S/R_{Inertial}) + \boldsymbol{\Omega}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times \boldsymbol{r_G} \tag{3.2}$$

The acceleration at the gravity point will then be:

$$\dot{\boldsymbol{V}}_G \in (S/R_{Inertial}) = \dot{\boldsymbol{V}}_O \in (S/R_{Inertial}) + \dot{\boldsymbol{\Omega}}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times \boldsymbol{r_G} + \boldsymbol{\Omega}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times \dot{\boldsymbol{r_G}} \tag{3.3}$$

The expression of the time derivative of the velocity vector in the previous equation is:

$$\dot{\boldsymbol{V}}_O \in (S/R_{Inertial}) = \dot{\boldsymbol{V}}_O \in (S/R_{RB}) + \boldsymbol{\Omega}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times \boldsymbol{V}_O,$$

Using equation 3.1 and the previous determined velocity vector, equation 3.3 can be written as follows:

$$\dot{\boldsymbol{V}}_G \in (S/R_{Inertial}) = \dot{\boldsymbol{V}}_O \in (S/R_{RB}) + \boldsymbol{\Omega}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times \boldsymbol{V}_O + \dot{\boldsymbol{\Omega}}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times \boldsymbol{r_G} + \boldsymbol{\Omega}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times (\boldsymbol{\Omega}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times \boldsymbol{r_G}) \tag{3.4}$$

Eventually, injecting equation 3.4 in equation 2.7, we can therefore state that:

$$\boxed{\begin{aligned} m_{RB} \cdot (\dot{\boldsymbol{V}}_O \in (S/R_{\mathrm{RB}}) + \boldsymbol{\Omega}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times \boldsymbol{V}_O \\ + \dot{\boldsymbol{\Omega}}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times \boldsymbol{r_G} + \boldsymbol{\Omega}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times (\boldsymbol{\Omega}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times \boldsymbol{r_G})) = \boldsymbol{F}_{\overline{S}/S} \end{aligned}} \tag{3.5}$$

Using this result in the next sections, the first 3 equations related to the 3 DoF translational motion will be derived.

3.2 Inertia tensor and mass

Before deriving the main equation for the rotational motion, the concept of inertia of a rigid body needs to be introduced. For that, we are considering the situation depicted in Figure 3.1 where a rigid body RB is rotating around an inertial fixed frame EF. The rigid body has its frame fixed at its origin $(O_{RB}; \vec{x_0}, \vec{y_0}, \vec{z_0})$ while the inertial frame is $(O_{EF}; \vec{x}, \vec{y}, \vec{z})$. In further development, the vector $\boldsymbol{V}_{O_\tau} \in (\tau/R_{\mathrm{Inertial}})$ in cyan will be noted $\boldsymbol{v}$ to ease the notations. The inertia tensor of a rigid body expressed at its origin point O_{RB}, denoted $\overline{\overline{\boldsymbol{I}}}_{O_{RB}}$, is defined as follows:

$$\overline{\overline{\boldsymbol{I}}}_{O_{RB}} \triangleq \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix}, \qquad \overline{\overline{\boldsymbol{I}}}_{O_{RB}} = \overline{\overline{\boldsymbol{I}}}^{\mathrm{T}}_{O_{RB}} \tag{3.6}$$

Additionally, according to equation 2.9 that we recall hereafter:

$$\boldsymbol{L_G} \triangleq \overline{\overline{\boldsymbol{I}}}_{G_{RB}} \cdot \boldsymbol{\Omega}_{R_{RB}/R_{\mathrm{Inertial}}}$$

And according to the fact that the general expression of the angular momentum is:

$$\boldsymbol{L_G} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times (m_{RB} \cdot \boldsymbol{\Omega}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times \boldsymbol{r}) = \int_V \boldsymbol{r} \times (\boldsymbol{\Omega}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times \boldsymbol{r}) \cdot \rho_{RB} \cdot dV$$

It is therefore possible to state that:

$$\overline{\overline{\boldsymbol{I}}}_{O_{RB}} \cdot \boldsymbol{\Omega}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} = \int_V \boldsymbol{r} \times (\boldsymbol{\Omega}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} \times \boldsymbol{r}) \cdot \rho_{RB} \cdot dV \tag{3.7}$$

With ρ_{RB} the mass density of the rigid body. Furthermore, it is possible to define the mass of the body as follows:

$$m_{RB} = \int_V \rho_{RB} \cdot dV \tag{3.8}$$

It will also be considered that the mass of the rigid body is constant in time, i.e $\dot{m} = 0$. Therefore, for a rigid body under this condition, the distance from its origin to its center of gravity is:

$$\boldsymbol{r_G} = \boldsymbol{O_{RB}G_{RB}} = \frac{1}{m_{RB}} \int_V \boldsymbol{r} \cdot \rho_{RB} \cdot dV \tag{3.9}$$

Hence, in the next section, the main equations for the rotational motion will be derived in order to determine their contribution to the global 6 DoF motion modeling.

3.3 Rotational motion

In a similar manner as for the translational motion, we will base our calculations on the general expression on the absolute angular momentum about O_{RB}. Additionally, to ease the expression, $\boldsymbol{L_O} \in (\boldsymbol{S}/\boldsymbol{R_{Inertia}}) = \boldsymbol{L_O}$.

$$\boldsymbol{L_O} \triangleq \int_V (\boldsymbol{r} \times \boldsymbol{v}) \cdot \rho_{RB} \cdot dV \tag{3.10}$$

Therefore:

$$\dot{\boldsymbol{L_O}} = \int_V (\dot{\boldsymbol{r}} \times \boldsymbol{v}) \cdot \rho_{RB} \cdot dV + \int_V (\boldsymbol{r} \times \dot{\boldsymbol{v}}) \cdot \rho_{RB} \cdot dV \tag{3.11}$$

The second term in the previous equation is homogeneous to a momentum vector. It will be established for further development that:

$$\boldsymbol{M_O} \triangleq \int_V (\boldsymbol{r} \times \dot{\boldsymbol{v}}) \cdot \rho_{RB} \cdot dV \tag{3.12}$$

Furthermore, in the Figure 3.1, it can be seen that:

$$\boldsymbol{v} = \frac{d(\boldsymbol{r_O} + \boldsymbol{r})}{dt} = \boldsymbol{V}_{O_\tau} \in (\tau/R_{\mathrm{Inertial}}) = \frac{d(\boldsymbol{O_{EF}O_{RB}} + \boldsymbol{O_{RB}O_{\tau}})}{dt}^{[R_{Inertial}]} = \dot{\boldsymbol{r_O}} + \dot{\boldsymbol{r}} \tag{3.13}$$

Then:

$$\dot{\boldsymbol{r}} = \boldsymbol{v} - \boldsymbol{v_O} \tag{3.14}$$

In that case, it is possible to establish that:

$$\dot{\boldsymbol{L_O}} = \int_V ((\boldsymbol{v} - \boldsymbol{v_O}) \times \boldsymbol{v}) \cdot \rho_{RB} \cdot dV + \boldsymbol{M_O}$$

$$\dot{\boldsymbol{L_O}} = -\int_V (\boldsymbol{v_O} \times \boldsymbol{v}) \cdot \rho_{RB} \cdot dV + \boldsymbol{M_O}$$

$$\dot{\boldsymbol{L_O}} = -\boldsymbol{v_O} \times \int_V \boldsymbol{v} \cdot \rho_{RB} \cdot dV + \boldsymbol{M_O} \tag{3.15}$$

Equivalently, by noticing according to equation 3.14 that $\boldsymbol{v} = \dot{\boldsymbol{r}} + \dot{\boldsymbol{r_O}} = \dot{\boldsymbol{r}} + \boldsymbol{v_O}$, one can establish that:

$$\dot{\boldsymbol{L_O}} = -\boldsymbol{v_O} \times \int_V (\dot{\boldsymbol{r}} + \boldsymbol{v_O}) \cdot \rho_{RB} \cdot dV + \boldsymbol{M_O}$$

$$\dot{\boldsymbol{L_O}} = -\boldsymbol{v_O} \times \int_V \dot{\boldsymbol{r}} \cdot \rho_{RB} \cdot dV + \boldsymbol{M_O} \tag{3.16}$$

However, using the time derivative of the distance from the origin to the gravity center of the rigid body equation 3.9, one can notice that:

$$m_{RB} \cdot \dot{\boldsymbol{r_G}} = \int_V \dot{\boldsymbol{r}} \cdot \rho_{RB} \cdot dV$$

And according to equation 3.1, one has:

$$\dot{\boldsymbol{r_G}} = \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \times \boldsymbol{r_G}$$

Hence:

$$m_{RB} \cdot \dot{\boldsymbol{r_G}} = m_{RB} \cdot \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \times \boldsymbol{r_G} = \int_V \dot{\boldsymbol{r}} \cdot \rho_{RB} \cdot dV$$

By injecting the previous equation into the equation 3.16, it can be obtained that:

$$\dot{\boldsymbol{L_O}} = \boldsymbol{M_O} - m_{RB} \cdot \boldsymbol{v_O} \times (\boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \times \boldsymbol{r_G}) \tag{3.17}$$

Now, it is possible to start again from the equation 3.10 and to write:

$$\boldsymbol{L_O} = \int_V (\boldsymbol{r} \times \boldsymbol{v}) \cdot \rho_{RB} \cdot dV = \int_V (\boldsymbol{r} \times (\dot{\boldsymbol{r}} + \boldsymbol{v_O})) \cdot \rho_{RB} \cdot dV = \int_V (\boldsymbol{r} \times \boldsymbol{v_O}) \cdot \rho_{RB} \cdot dV + \int_V (\boldsymbol{r} \times \dot{\boldsymbol{r}}) \cdot \rho_{RB} \cdot dV$$

In addition, analogically to what was done in equation 3.1, it is possible to state that:

$$\dot{\boldsymbol{r}} = \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \times \boldsymbol{r} \Rightarrow \boldsymbol{r} \times \dot{\boldsymbol{r}} = \boldsymbol{r} \times (\boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \times \boldsymbol{r})$$

Therefore:

$$\boldsymbol{L_O} = \int_V (\boldsymbol{r} \times \boldsymbol{v_O}) \cdot \rho_{RB} \cdot dV + \int_V \boldsymbol{r} \times (\boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \times \boldsymbol{r}) \cdot \rho_{RB} \cdot dV \tag{3.18}$$

Here, the first integral term in the right-hand side of the previous equation can be simplified using the equation of the origin to the center of gravity of the rigid body 3.9. Indeed:

$$\int_V (\boldsymbol{r} \times \boldsymbol{v_O}) \cdot \rho_{RB} \cdot dV = (\int_V \boldsymbol{r} \cdot \rho_{RB} \cdot dV) \times \boldsymbol{v_O} = m_{RB} \cdot \boldsymbol{r_G} \times \boldsymbol{v_O}$$

The second term of the right-hand side of equation 3.18 is recognized through equation 3.7. Therefore, it is possible to rewrite it in the following form:

$$\boldsymbol{L_O} = \overline{\overline{\boldsymbol{I}}}_{O_{RB}} \cdot \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} + m_{RB} \cdot \boldsymbol{r_G} \times \boldsymbol{v_O} \tag{3.19}$$

It is then possible to differentiate the previous equation with respect to time by considering the inertia term to be constant. This leads to the following equation:

$$\dot{\boldsymbol{L_O}} = \frac{d(\overline{\overline{\boldsymbol{I}}}_{O_{RB}} \cdot \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}})}{dt} + m_{RB} \cdot \dot{\boldsymbol{r_G}} \times \boldsymbol{v_O} + m_{RB} \cdot \boldsymbol{r_G} \times \dot{\boldsymbol{v_O}} \tag{3.20}$$

For the first right-hand term of the previous equation, it is possible to notice in fact that:

$$\frac{d(\overline{\overline{\boldsymbol{I}}}_{O_{RB}} \cdot \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}})}{dt} = \frac{d(\overline{\overline{\boldsymbol{I}}}_{O_{RB}} \cdot \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}})}{dt}^{[R_{Inertial}]}$$

$$\frac{d(\overline{\overline{\boldsymbol{I}}}_{O_{RB}} \cdot \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}})}{dt}^{[R_{Inertial}]} = \frac{d(\overline{\overline{\boldsymbol{I}}}_{O_{RB}} \cdot \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}})}{dt}^{[R_{RB}]} + \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \times \overline{\overline{\boldsymbol{I}}}_{O_{RB}} \cdot \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}}$$

Furthermore, for the other two right-terms of the equation, it is possible to state that:

$$\dot{\boldsymbol{r_G}} \times \boldsymbol{v_O} = ((\boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \times \boldsymbol{r_G}) \times \boldsymbol{v_O})$$

$$\boldsymbol{r_G} \times \dot{\boldsymbol{v_O}} = \boldsymbol{r_G} \times (\dot{\boldsymbol{v_O}}^{[R_{RB}]} + \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \times \boldsymbol{v_O})$$

By injecting all these equations into equation 3.20, one obtains:

$$\begin{aligned}\dot{\boldsymbol{L_O}} = \frac{d(\overline{\overline{\boldsymbol{I}}}_{O_{RB}} \cdot \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}})}{dt}^{[R_{RB}]} + \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \times \overline{\overline{\boldsymbol{I}}}_{O_{RB}} \cdot \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \\ m_{RB} \cdot ((\boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \times \boldsymbol{r_G}) \times \boldsymbol{v_O}) + m_{RB} \cdot \boldsymbol{r_G} \times (\dot{\boldsymbol{v_O}}^{[R_{RB}]} + \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \times \boldsymbol{v_O})\end{aligned} \tag{3.21}$$

However, according to equation 3.17, we know that the first form of the time derivative of the angular momentum found was:

$$\dot{\boldsymbol{L_O}} = \boldsymbol{M_O} - m_{RB} \cdot \boldsymbol{v_O} \times (\boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \times \boldsymbol{r_G}) = \boldsymbol{M_O} + m_{RB} \cdot (\boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \times \boldsymbol{r_G}) \times \boldsymbol{v_O}$$

Eventually, by identifying the terms, one can conclude that the overall momentum of the resulting forces applied on the rigid body at the origin of the rigid body is:

$$\boxed{\begin{aligned}\boldsymbol{M_O} = \overline{\overline{\boldsymbol{I}}}_{O_{RB}} \cdot \dot{\boldsymbol{\Omega}}^{[R_{RB}]}_{R_{\text{RB}}/R_{\text{Inertial}}} + \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \times \overline{\overline{\boldsymbol{I}}}_{O_{RB}} \cdot \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} + \\ m_{RB} \cdot \boldsymbol{r_G} \times (\dot{\boldsymbol{v_O}}^{[R_{RB}]} + \boldsymbol{\Omega}_{R_{\text{RB}}/R_{\text{Inertial}}} \times \boldsymbol{v_O})\end{aligned}} \tag{3.22}$$

Finally, by using this result, the other 3 equations for the last 3 DoF for the rotational motion can be retrieved.

Chapter 4

Application to the ship

4.1 6 DoF equations of a moving ship

We will now apply the relations 3.5 and 3.22 to the following *SNAME* official notations:

- $\boldsymbol{\tau_1} = \mathbf{F}_{\overline{S}/S} = \begin{pmatrix} X & Y & Z \end{pmatrix}^{\mathrm{T}}$, The resulting of external forces applied on the ship.
- $\boldsymbol{\tau_2} = \boldsymbol{M_O} = \begin{pmatrix} K & M & N \end{pmatrix}^{\mathrm{T}}$, The moment of the resulting of the external forces about the O_{RB} point.
- $\boldsymbol{\nu_1} = \boldsymbol{v_O} = \begin{pmatrix} u & v & w \end{pmatrix}^{\mathrm{T}}$, The linear velocity associated with the RB frame.
- $\boldsymbol{\nu_2} = \boldsymbol{\Omega}_{R_{\mathrm{RB}}/R_{\mathrm{Inertial}}} = \begin{pmatrix} p & q & r \end{pmatrix}^{\mathrm{T}}$, Angular velocity associated with the RB frame.
- $\boldsymbol{r_G} = \begin{pmatrix} x_G & y_G & z_G \end{pmatrix}^{\mathrm{T}}$, Center of gravity of the ship.

We will now show the application of the previously determined equation for the first DoF concerning the motion along the horizontal x-axis. Therefore, this concerns the translational motion relation 3.5. By expanding it and applying the introduced notations, one has:

$$m_S \cdot \left(\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} + \begin{pmatrix} p \\ q \\ r \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix} \times \begin{pmatrix} x_G \\ y_G \\ z_G \end{pmatrix} + \begin{pmatrix} p \\ q \\ r \end{pmatrix} \times \left(\begin{pmatrix} p \\ q \\ r \end{pmatrix} \times \begin{pmatrix} x_G \\ y_G \\ z_G \end{pmatrix} \right) \right) = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

Developing for the first DoF, this leads to:

$$m_S \cdot (\dot{u} - v \cdot r + w \cdot q - x_G \cdot (q^2 + r^2) + y_G \cdot (p \cdot q - \dot{r}) + z_G \cdot (p \cdot r + \dot{q})) = X$$

Eventually, the 6 DoF equations that describes the motion of the ship on the surface of water are the following:

$$m_S \cdot (\dot{u} - v \cdot r + w \cdot q - x_G \cdot (q^2 + r^2) + y_G \cdot (p \cdot q - \dot{r}) + z_G \cdot (p \cdot r + \dot{q})) = X \tag{4.1}$$

$$m_S \cdot (\dot{v} - w \cdot p + u \cdot r - y_G \cdot (r^2 + p^2) + z_G \cdot (q \cdot r - \dot{p}) + x_G \cdot (q \cdot p + \dot{r})) = Y \tag{4.2}$$

$$m_S \cdot (\dot{w} - u \cdot q + v \cdot p - z_G \cdot (p^2 + q^2) + x_G \cdot (r \cdot p - \dot{q}) + y_G \cdot (r \cdot q + \dot{p})) = Z \tag{4.3}$$

$$\begin{split} I_x \cdot \dot{p} + (I_z - I_y) \cdot q \cdot r - (\dot{r} + p \cdot q) \cdot I_{xz} + (r^2 - q^2) \cdot I_{yz} + (p \cdot r - \dot{q}) \cdot I_{xy} + \\ m_S \cdot (y_G \cdot (\dot{w} - u \cdot q + v \cdot p) - z_G \cdot (\dot{v} - w \cdot p + u \cdot r)) = K \end{split} \tag{4.4}$$

$$\begin{split} I_y \cdot \dot{q} + (I_x - I_z) \cdot r \cdot p - (\dot{p} + q \cdot r) \cdot I_{xy} + (p^2 - r^2) \cdot I_{zx} + (q \cdot p - \dot{r}) \cdot I_{yz} + \\ m_S \cdot (z_G \cdot (\dot{u} - v \cdot r + w \cdot q) - x_G \cdot (\dot{w} - u \cdot q + v \cdot p)) = M \end{split} \tag{4.5}$$

$$I_z \cdot \dot{r} + (I_y - I_x) \cdot p \cdot q - (\dot{q} + r \cdot p) \cdot I_{yz} + (q^2 - p^2) \cdot I_{xy} + (r \cdot q - \dot{p}) \cdot I_{zx} + \\ m_S \cdot (x_G \cdot (\dot{v} - w \cdot p + u \cdot r) - y_G \cdot (\dot{u} - v \cdot r + w \cdot q)) = N \tag{4.6}$$

One can note that for the purpose of controlling the heading in order to control the position of the ship, there is the need to keep only 3 equations from the ones presented above. In our case, the equations 4.1, 4.2 and 4.6 are of interest. A first simplification can also be made to handle a more simple set of equations.

4.2 A first simplification: diagonal inertia matrix

To simplify several dynamics-related calculations, it is of interest to have a diagonal inertia tensor. For that, a special choice of O_{RB} must be made to ensure that the body axes of the ship shall coincide with the main axis of inertia. If the matrix is diagonal, in addition to facilitating the computations, it implies that there are no coupling between the rotations along different axes. Some unwanted behavior are then avoided.
For that, the Huygens–Steiner theorem will be of interest:

Huygens–Steiner Theorem

The Inertia tensor I_O about an arbitrary origin O is defined as:

$$I_O = I_C - m \cdot (\boldsymbol{r_G} \cdot \boldsymbol{r_G}^{\mathrm{T}} - \boldsymbol{r_G}^{\mathrm{T}} \cdot \boldsymbol{r_G} \cdot I_{3\times3}) \tag{4.7}$$

With:

- $I_{3\times3}$ The identity matrix of size 3 by 3.
- I_C The inertia tensor about the body's center of gravity.
- r_G The vector representing the distance from the origin to the center of gravity of the body.

Using this theorem and expanding the expression of 4.7 leads to:

$$\begin{pmatrix} I_{x_O} & 0 & 0 \\ 0 & I_{y_O} & 0 \\ 0 & 0 & I_{z_O} \end{pmatrix} = \begin{pmatrix} I_{x_C} & -I_{xy_C} & -I_{xz_C} \\ -I_{yx_C} & I_{y_C} & -I_{yz_C} \\ -I_{zx_C} & -I_{zy_C} & I_{z_C} \end{pmatrix} - \\ m_S \cdot (\begin{pmatrix} x_G \\ y_G \\ z_G \end{pmatrix} \cdot \begin{pmatrix} x_G & y_G & z_G \end{pmatrix} - \begin{pmatrix} x_G & y_G & z_G \end{pmatrix} \cdot \begin{pmatrix} x_G \\ y_G \\ z_G \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) \tag{4.8}$$

By simplifying the previous equation, it can be established that the diagonal terms are:

$$I_{x_O} = I_{x_C} + m_S \cdot (y_G^2 + z_G^2) \tag{4.9}$$

$$I_{y_O} = I_{y_C} + m_S \cdot (x_G^2 + z_G^2) \tag{4.10}$$

$$I_{z_O} = I_{z_C} + m_S \cdot (x_G^2 + y_G^2) \tag{4.11}$$

The equality terms of the off-diagonal equation 4.8 lead to the following equations:

$$m_S \cdot I_{yz_C} \cdot x_G^2 = -I_{xy_C} \cdot I_{xz_C} \tag{4.12}$$

$$m_S \cdot I_{xz_C} \cdot y_G^2 = -I_{xy_C} \cdot I_{yz_C} \tag{4.13}$$

$$m_S \cdot I_{xy_C} \cdot z_G^2 = -I_{xz_C} \cdot I_{yz_C} \tag{4.14}$$

By respecting this condition and the ones from the other off-diagonal terms non presented here but retrieved from 4.8, the block equations from 4.1 - 4.6 can be simplified as follows:

$$m_S \cdot (\dot{u} - v \cdot r + w \cdot q - x_G \cdot (q^2 + r^2) + y_G \cdot (p \cdot q - \dot{r}) + z_G \cdot (p \cdot r + \dot{q})) = X \tag{4.15}$$

$$m_S \cdot (\dot{v} - w \cdot p + u \cdot r - y_G \cdot (r^2 + p^2) + z_G \cdot (q \cdot r - \dot{p}) + x_G \cdot (q \cdot p + \dot{r})) = Y \tag{4.16}$$

$$m_S \cdot (\dot{w} - u \cdot q + v \cdot p - z_G \cdot (p^2 + q^2) + x_G \cdot (r \cdot p - \dot{q}) + y_G \cdot (r \cdot q + \dot{p})) = Z \tag{4.17}$$

$$I_x \cdot \dot{p} + (I_z - I_y) \cdot q \cdot r + m_S \cdot (y_G \cdot (\dot{w} - u \cdot q + v \cdot p) - z_G \cdot (\dot{v} - w \cdot p + u \cdot r)) = K \tag{4.18}$$

$$I_y \cdot \dot{q} + (I_x - I_z) \cdot r \cdot p + m_S \cdot (z_G \cdot (\dot{u} - v \cdot r + w \cdot q) - x_G \cdot (\dot{w} - u \cdot q + v \cdot p)) = M \tag{4.19}$$

$$I_z \cdot \dot{r} + (I_y - I_x) \cdot p \cdot q + m_S \cdot (x_G \cdot (\dot{v} - w \cdot p + u \cdot r) - y_G \cdot (\dot{u} - v \cdot r + w \cdot q)) = N \tag{4.20}$$

Of course, note that only the equations related to the rotational motion were simplified. This is because the products of inertia (off-diagonal terms of the inertia tensor) are now null.

4.3 A second simplification: Nomoto's ship model - First degree of freedom

4.3.1 Assumption and simplification of the equations

The equations can be further simplified by notifying several additional assumptions. First, the ship has a xz plane symmetry leading to null inertia terms $I_{xy} = I_{yz} = 0$. According to this and to equation 4.13, it is possible to conclude that the coordinate origin should be set in the center line of the ship $y_G = 0$.

Most importantly, as already notified before, several DoF can be neglected. Referring to table 2.1, the heave, roll and pitch modes can be neglected. Therefore, by taking equations 4.15 to 4.20 and settling $w = p = q = 0 = \dot{w} = \dot{p} = \dot{q}$ and all the previously stated hypotheses, this leads to:

$$m_S \cdot (\dot{u} - v \cdot r - x_G \cdot r^2) = X \tag{4.21}$$

$$m_S \cdot (\dot{v} + u \cdot r + x_G \cdot \dot{r}) = Y \tag{4.22}$$

$$I_z \cdot \dot{r} + m_S \cdot x_G \cdot (\dot{v} + u \cdot r) = N \tag{4.23}$$

4.3.2 Perturbed ship equation of motion

Further assumptions need to be taken into account when it comes to the perturbed motion of the ship. Indeed, the sway velocity v, the yaw rate r and the rudder angle δ are small. As we started explaining it at the beginning of this report, it is indeed the case due to the slowly varying motion of the ship. The longitudinal surge motion (expressed through the u velocity) can therefore be decoupled from the lateral sway motion and the yaw motion as well.

We will suppose that for a constant thrust, the surge velocity is a constant of value u_0. We can neglect the mean velocities in terms of sway and yaw: $v_0 = r_0 = 0$. Therefore, for the velocities, one has:

$$u = u_0 + \Delta u \qquad v = v_0 + \Delta v = \Delta v \qquad r = r_0 + \Delta r = \Delta r$$

The similar approach is taken for the forces and moments:

$$X = X_0 + \Delta X \qquad Y = Y_0 + \Delta Y = \Delta Y \qquad N = N_0 + \Delta N = \Delta N$$

Here, we assume that $\Delta(\cdot)$ is a small perturbation from the nominal $(\cdot)$ value. Then it is possible to express the equations of motion in the following form. The explanation will be given for the first equation, the other final ones will be given directly based on the same technique.

$$m_S \cdot (\dot{u}_{eq} - v_{eq} \cdot r_{eq} - x_G \cdot r_{eq}^2) = X_{eq}$$

$$m_S \cdot ((\dot{u_0} + \dot{\Delta u}) - \Delta v \cdot \Delta r - x_G \cdot \Delta r^2) = X_0 + \Delta X$$

One can notice that $\dot{u_0} = 0$, that $\Delta v << 1$ and $\Delta r << 1$ hence $\Delta v \cdot \Delta r << 1$ and $\Delta r^2 << 1$. Therefore:

$$m_S \cdot \dot{\Delta u} = X_0 + \Delta X \tag{4.24}$$

The other two perturbed equations of motion for the sway and yaw modes are:

$$m_S \cdot (\Delta \dot{v} + u_0 \cdot \Delta r + x_G \cdot \Delta \dot{r}) = \Delta Y \tag{4.25}$$

$$I_z \cdot \Delta \dot{r} + m_S \cdot x_G \cdot (\Delta \dot{v} + u_0 \cdot \Delta r) = \Delta N \tag{4.26}$$

Applying again the perturbation equations to the equations 4.24 to 4.26, one obtains:

$$m_S \cdot \dot{u} = X \tag{4.27}$$

$$m_S \cdot (\dot{v} + u_0 \cdot r + x_G \cdot \dot{r}) = Y \tag{4.28}$$

$$I_z \cdot \dot{r} + m_S \cdot x_G \cdot (\dot{v} + u_0 \cdot r) = N \tag{4.29}$$

Note that equation 4.27 refers to the speed equation. The equations 4.28 and 4.29 are called the steering equations.

4.3.3 The linear ship steering equations

For the next development, we shall consider the equations determined in the previous section, referred to as the steering equations. These equations involve the main parameters that we want to control in this project: v, r, δ_R and ψ. Even if our main focus is to control the heading ψ by using the rudder δ_R on the vessel.

Starting from equations 4.28 and 4.29 and from the Davidson & Schiff theory, the hydrodynamics forces and moments can be linearly modeled as follows:

$$Y = Y_{\dot{v}} \cdot \dot{v} + Y_{\dot{r}} \cdot \dot{r} + Y_v \cdot v + Y_r \cdot r + Y_{\delta_R} \cdot \delta_R \tag{4.30}$$

$$N = N_{\dot{v}} \cdot \dot{v} + N_{\dot{r}} \cdot \dot{r} + N_v \cdot v + N_r \cdot r + N_{\delta_R} \cdot \delta_R \tag{4.31}$$

Where the terms in the previous two equations are the hydrodynamics coefficients that must be carefully calculated as a function of many ship parameters and hydrodynamics features (as will be presented in further section of this note).

Therefore, the equations of motion can be put in the following form:

$$\boldsymbol{M} \cdot \dot{\boldsymbol{\nu}} + \boldsymbol{N}(\boldsymbol{u_0}) \cdot \boldsymbol{\nu} = \boldsymbol{b} \cdot \delta_R \tag{4.32}$$

With:

- $\boldsymbol{\nu} = \begin{pmatrix} v & r \end{pmatrix}^{\mathrm{T}}$ The state vector.
- δ_R The rudder angle input scalar.
- $\boldsymbol{M}$ The added mass and Inertia matrix.
- $\boldsymbol{N}$ The Coriolis and Centripetal terms matrix.

The two matrices of the linear model are composed as follows:

$$\boldsymbol{M} = \begin{pmatrix} m_S - Y_{\dot{v}} & m_S \cdot x_G - Y_{\dot{r}} \\ m_S \cdot x_G - N_{\dot{v}} & I_z - N_{\dot{r}} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \tag{4.33}$$

$$\boldsymbol{N} = \begin{pmatrix} -Y_v & m_S \cdot u_0 - Y_r \\ -N_v & m_S \cdot x_G \cdot u_0 - N_r \end{pmatrix} = \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \tag{4.34}$$

4.3.4 The Nomoto's model: First DoF of the simulator

The Nomoto's model can be presented as a transfer function between the input rudder angle δ_R to the yaw rate r or the heading ψ. Therefore, it constitutes the best model that we can wish for our design of a controller capable of piloting the heading of the ship. The general transfer function of Nomoto can be written as:

$$\frac{r}{\delta_R}(s) = \frac{K_r \cdot (1 + T_3 \cdot s)}{(1 + T_1 \cdot s) \cdot (1 + T_2 \cdot s)} = \frac{K_r \cdot (1 + T_3 \cdot s)}{1 + (T_1 + T_2) \cdot s + T_1 \cdot T_2 \cdot s^2} \tag{4.35}$$

This transfer function can be obtained starting from equations 4.32 to 4.34. An attempt of the proof has been made hereafter; the result is quite close to what was expected and given by the author. However, a difference has been observed in the final sign of the K_r gain. A crosscheck has been performed using *ChatGPT*, resulting in the same result as the one presented here, against the one proposed by the author in [2]. Expanding 4.32 using 4.33 and 4.34 leads to:

$$\begin{cases} m_{11} \cdot \dot{v} + m_{12} \cdot \dot{r} + n_{11} \cdot v + n_{12} \cdot r = b_1 \cdot \delta_R \\ m_{21} \cdot \dot{v} + m_{22} \cdot \dot{r} + n_{21} \cdot v + n_{22} \cdot r = b_2 \cdot \delta_R \end{cases}$$

The goal here is to eliminate the velocity v from the equations in order to keep only the yaw rate r and the input rudder angle δ_R. For that, the Laplace transform will be used:

$$\begin{cases} (m_{11} \cdot s + n_{11}) \cdot v + (m_{12} \cdot s + n_{12}) \cdot r = b_1 \cdot \delta_R \\ (m_{21} \cdot s + n_{21}) \cdot v + (m_{22} \cdot s + n_{22}) \cdot r = b_2 \cdot \delta_R \end{cases}$$

$$\begin{cases} v = \frac{b_1 \cdot \delta_R}{(m_{11} \cdot s + n_{11})} - \frac{(m_{12} \cdot s + n_{12})}{(m_{11} \cdot s + n_{11})} \cdot r \\ (m_{21} \cdot s + n_{21}) \cdot (\frac{b_1 \cdot \delta_R}{(m_{11} \cdot s + n_{11})} - \frac{(m_{12} \cdot s + n_{12})}{(m_{11} \cdot s + n_{11})} \cdot r) + (m_{22} \cdot s + n_{22}) \cdot r = b_2 \cdot \delta_R \end{cases}$$

The second equation will therefore give:

$$\frac{(m_{21} \cdot s + n_{21})}{(m_{11} \cdot s + n_{11})} \cdot b_1 \cdot \delta_R - \frac{(m_{21} \cdot s + n_{21}) \cdot (m_{12} \cdot s + n_{12})}{(m_{11} \cdot s + n_{11})} \cdot r + (m_{22} \cdot s + n_{22}) \cdot r = b_2 \cdot \delta_R$$

$$r \cdot (m_{22} \cdot s + n_{22} - \frac{(m_{21} \cdot s + n_{21}) \cdot (m_{12} \cdot s + n_{12})}{(m_{11} \cdot s + n_{11})}) = (b_2 - b_1 \cdot \frac{(m_{21} \cdot s + n_{21})}{(m_{11} \cdot s + n_{11})}) \cdot \delta_R$$

$$\frac{r}{\delta_R} = \frac{b_2 - b_1 \cdot \frac{(m_{21} \cdot s + n_{21})}{(m_{11} \cdot s + n_{11})}}{m_{22} \cdot s + n_{22} - \frac{(m_{21} \cdot s + n_{21}) \cdot (m_{12} \cdot s + n_{12})}{(m_{11} \cdot s + n_{11})}}$$

After further development, this leads to the following equation:

$$\frac{r}{\delta_R}(s) = \frac{(n_{11} \cdot b_2 - n_{21} \cdot b_1) \cdot (1 + \frac{m_{11} \cdot b_2 - m_{21} \cdot b_1}{n_{11} \cdot b_2 - n_{21} \cdot b_1} \cdot s)}{(m_{22} \cdot m_{11} - m_{12} \cdot m_{21}) \cdot s^2 + (m_{22} \cdot n_{11} + n_{22} \cdot m_{11} - m_{12} \cdot n_{21} - n_{12} \cdot m_{21}) \cdot s + n_{22} \cdot n_{11} - n_{12} \cdot n_{21}} \tag{4.36}$$

By noting $D_{r/\delta}$ the denominator of the previous equation, we can see that:

$$D_{r/\delta} = det(\boldsymbol{M}) \cdot s^2 + (m_{22} \cdot n_{11} + n_{22} \cdot m_{11} - m_{12} \cdot n_{21} - n_{12} \cdot m_{21}) \cdot s + det(\boldsymbol{N})$$

$$D_{r/\delta} = det(\boldsymbol{N}) \cdot (\frac{det(\boldsymbol{M})}{det(\boldsymbol{N})} \cdot s^2 + \frac{m_{22} \cdot n_{11} + n_{22} \cdot m_{11} - m_{12} \cdot n_{21} - n_{12} \cdot m_{21}}{det(\boldsymbol{N})} \cdot s + 1)$$

Injecting this in equation 4.36 leads to the following transfer function:

$$\frac{r}{\delta_R}(s) = \frac{(n_{11} \cdot b_2 - n_{21} \cdot b_1) \cdot (1 + \frac{m_{11} \cdot b_2 - m_{21} \cdot b_1}{n_{11} \cdot b_2 - n_{21} \cdot b_1} \cdot s)}{det(\boldsymbol{N}) \cdot (\frac{det(\boldsymbol{M})}{det(\boldsymbol{N})} \cdot s^2 + \frac{m_{22} \cdot n_{11} + n_{22} \cdot m_{11} - m_{12} \cdot n_{21} - n_{12} \cdot m_{21}}{det(\boldsymbol{N})} \cdot s + 1)} \tag{4.37}$$

$$\frac{r}{\delta_R}(s) = \frac{\frac{n_{11} \cdot b_2 - n_{21} \cdot b_1}{det(\boldsymbol{N})} \cdot (1 + \frac{m_{11} \cdot b_2 - m_{21} \cdot b_1}{n_{11} \cdot b_2 - n_{21} \cdot b_1} \cdot s)}{\frac{det(\boldsymbol{M})}{det(\boldsymbol{N})} \cdot s^2 + \frac{m_{22} \cdot n_{11} + n_{22} \cdot m_{11} - m_{12} \cdot n_{21} - n_{12} \cdot m_{21}}{det(\boldsymbol{N})} \cdot s + 1} \tag{4.38}$$

Eventually, according to 4.38 and 4.35, we can identify each terms:

$$T_1 \cdot T_2 = \frac{det(\boldsymbol{M})}{det(\boldsymbol{N})} \tag{4.39}$$

$$T_1 + T_2 = \frac{m_{22} \cdot n_{11} + n_{22} \cdot m_{11} - m_{12} \cdot n_{21} - n_{12} \cdot m_{21}}{det(\boldsymbol{N})} \tag{4.40}$$

$$K_r = \frac{n_{11} \cdot b_2 - n_{21} \cdot b_1}{det(\boldsymbol{N})} \tag{4.41}$$

$$K_r \cdot T_3 = \frac{m_{11} \cdot b_2 - m_{21} \cdot b_1}{det(\boldsymbol{N})} \tag{4.42}$$

The author of [2] obtains the same first two equations. However, we have a different result by the -1 sign for 4.41. As will be explained in the technical report GNC justification algorithms [1], tuning the control algorithms to ensure the stability of the plant highly depends on the choice of the physical and numerical parameters of the system. That is why concept such as robust control will be explored in order to ensure robustness in the establishment of the controllers. For the release of the first version of the simulators and algorithms although, the equations given in [2] were kept, even if they seem to be inexact at one sign.

4.3.5 Nomoto's first order model

One last simplification of the model can be explored as it reduces the complexity of the model of the ship. Indeed, instead of a transfer function such as the one presented in equation 4.35, the transfer function is now of the form of a first order system:

$$\frac{r}{\delta}(s) = \frac{K}{1 + T \cdot s} \tag{4.43}$$

What must be understood is that this simplification is only possible considering a sufficiently low frequency response system, which is the case for a small pleasure ship. The high frequency terms were neglected, for the benefit of a simplification of the model. This is justified since the associated frequency response of the ship can be considered to be low.

Indeed, if one expresses the complex transfer function from equation 4.35, one has:

$$\frac{r}{\delta_R}(j\omega) = K_r \cdot \frac{1 + T_3 \cdot j\omega}{1 + (T_1 + T_2) \cdot j\omega + T_1 \cdot T_2 \cdot (j\omega)^2}$$

As previously stated, by neglecting the high order frequencies, it can be written that:

$$\frac{r}{\delta_R}(j\omega) \approx K_r \cdot \frac{1 + T_3 \cdot j\omega}{1 + (T_1 + T_2) \cdot j\omega}$$

We are then focusing our analysis in the domain of relatively low frequencies. As such, we can consider that $\omega \to 0$. As a result, one can use the Taylor series expansion of the following mathematical function [3]:

$$\frac{1}{1+x} \underset{x \to 0}{=} 1 - x + x^2 + ... + x^n + o(x^n) \tag{4.44}$$

By replacing x to ω and stopping the development for $n = 1$, one obtains that:

$$\begin{aligned}
\frac{r}{\delta_R}(j\omega) &\underset{\omega\to 0}{=} K_r \cdot (1 + j\omega T_3) \cdot (1 - j\omega(T_1 + T_2) + o(\omega)) \\
\frac{r}{\delta_R}(j\omega) &\approx K_r \cdot (1 + j\omega T3 - (1 + j\omega T3) \cdot (j\omega(T_1 + T_2))) \\
\frac{r}{\delta_R}(j\omega) &\approx K_r \cdot (1 + j\omega T3 - j\omega(T_1 + T_2) + j^2\omega^2 \cdot T_3 \cdot (T_1 + T_2)) \\
\frac{r}{\delta_R}(j\omega) &\approx K_r \cdot (1 + j\omega T3 - j\omega(T_1 + T_2)) \\
\frac{r}{\delta_R}(j\omega) &\approx K_r \cdot (1 + j\omega \cdot (T3 - T_1 - T_2))
\end{aligned} \tag{4.45}$$

The same reasonning can be applyed to the equation 4.43, leading to the following expression:

$$\frac{r}{\delta}(j\omega)_{aimed} \approx K \cdot (1 + j\omega \cdot T) \tag{4.46}$$

Before being able to totally identify the two previously introduced equations, there is a need to establish an equality between the parameters δ, K and δ_R, K_R. That is to say that the rudder angle δ_R is defined positively in the compass circle, the one used for the heading measurements. As our positive reference is therefore the compass unit circle, from 0° to 360° in the inverse trigonometric rotation, a positive turning angle $\delta_R > 0$ implies a negative yaw rate angle $r < 0$. As such, it is often convenient to define $\delta \triangleq -\delta_R$. This means that a positive turning angle $\delta_R > 0$ is, in fact, a negative turning angle $\delta < 0$ implying a negative turning rate $r < 0$. Therefore, a positive turning angle $\delta > 0$ implies a positive turning rate $r > 0$. For convenience, it is also useful to consider $K \triangleq -K_r$. Therefore:

$$\frac{r}{-\delta}(j\omega) \approx -K \cdot (1 + j\omega \cdot (T3 - T_1 - T_2)) \Rightarrow \frac{r}{\delta}(j\omega) \approx K \cdot (1 + j\omega \cdot (T3 - T_1 - T_2)) \tag{4.47}$$

The identification is now possible, resulting in the ability to use the simplified Nomoto model 4.43 under the condition of having the time constant:

$$T = T_1 + T_2 - T_3 \tag{4.48}$$

NOTA - The simplification of the second order transfer function to the benefit of a transfer function of the first order comes with the condition on the equivalent time constant of the model. As one can clearly understand, the physical meaning of this equivalent time is limited. Indeed, if T_3 is higher than $T_1 + T_2$, the equivalent time constant T can be negative. Special attention on the interpretation of the simplified first order transfer function must then be done.

4.3.6 Proposition of calculation of Nomoto's model coefficient

According to the previous subsection, we now have at our disposal the transfer function of the ship relating the input rudder angle and the output heading of the ship, such that:

Heading in function of rudder angle - Nomoto's first order model

$$\frac{\psi}{\delta}(s) = \frac{K}{s \cdot (1 + T \cdot s)} \tag{4.49}$$

This constitute the first DoF that can be controllable. The challenge now lies in the calculation of the coefficients K and T. As one can easily understand, the tuning of the controller of the ship will be based on this model of the ship's dynamics. Therefore, if the model is inexact, the tuned controllers might not work on the real system after the implementation. Worse, it could cause the ship response to diverge and could lead to the failure of the electromechanical actuators. To avoid that, robust control using H_∞ and

$\mu-$synthesis will be studied in the next version of the repositories, to add some uncertainties on these two parameters and allow for a robust tuning of the control algorithms.

For this first issue of the algorithms and simulator, an analytical calculation of the hydrodynamics coefficients behind these two parameters was done based on the Strip Theory explained in [2]. As can be understood, the expressions of T and K depend on the coefficients in equations 4.39 to 4.42. These coefficients depends themselves on the terms of the matrices 4.33 and 4.34. Eventually, the terms inside these matrices are the hydrodynamics coefficients present in equations 4.30 and 4.31.

As such, we will first give here the relations allowing to calculate these hydrodynamics coefficients. Note that the script of the project allowing to calculate the hydrodynamics coefficient of the ship in MATLAB can be accesses through the following link.

The following relations for the hydrodynamics coefficients have been established according to [2]:

$$Y_v^{'} = \frac{Y_v}{\frac{1}{2} \cdot \rho \cdot L \cdot T \cdot U} = -(\frac{\pi \cdot T}{L} - C_{D0}) \tag{4.50}$$

$$Y_r^{'} = \frac{Y_r}{\frac{1}{2} \cdot \rho \cdot L^2 \cdot T \cdot U} = X_{\dot{u}}^{'} + \frac{x_P}{L} \cdot Y_v^{'} \tag{4.51}$$

$$N_v^{'} = \frac{N_v}{\frac{1}{2} \cdot \rho \cdot L^2 \cdot T \cdot U} = -(X_{\dot{u}}^{'} - Y_{\dot{v}}^{'}) + \frac{x_P}{L} \cdot Y_v^{'} \tag{4.52}$$

$$N_r^{'} = \frac{N_r}{\frac{1}{2} \cdot \rho \cdot L^3 \cdot T \cdot U} = \frac{1}{4} \cdot Y_v^{'} \tag{4.53}$$

$$Y_\delta^{'} = \frac{Y_\delta}{\frac{1}{2} \cdot \rho \cdot L \cdot T \cdot U^2} = \frac{\pi}{4} \cdot \frac{A_\delta}{L \cdot T} \tag{4.54}$$

$$N_\delta^{'} = \frac{N_\delta}{\frac{1}{2} \cdot \rho \cdot L^2 \cdot T \cdot U^2} = -\frac{1}{2} \cdot Y_\delta^{'} \tag{4.55}$$

The following added mass derivatives have been established:

$$X_{\dot{u}} \in [-0.10 \cdot m_S; -0.05 \cdot m_S] \tag{4.56}$$

$$Y_{\dot{v}} \in [-0.70 \cdot m_S; 1.00 \cdot m_S] \tag{4.57}$$

$$Y_{\dot{r}} = 0 \tag{4.58}$$

$$N_{\dot{v}} = 0 \tag{4.59}$$

$$N_{\dot{r}} \in [-0.1 \cdot I_z; -0.01 \cdot I_z] \tag{4.60}$$

As expected in equations 4.30 and 4.31, 10 hydrodynamics coefficients and added mass derivatives must be calculated: $Y_{\dot{v}}$, $Y_{\dot{r}}$, Y_v, Y_r, $Y_{\delta_R} = Y_\delta$ and $N_{\dot{v}}$, $N_{\dot{r}}$, N_v, N_r, $N_{\delta_R} = N_\delta$. However, the proposed relations to calculate Y_r and N_v (that is, equations 4.51 and 4.52) imply the calculation of two more added mass derivatives $X_{\dot{u}}^{'}$ and $Y_{\dot{v}}^{'}$. Their expressions were not given and were therefore established using the $PRIME$ system for non-dimensional parameters determination.

As one can see in the previous equations, the $'$ exponant was used. This comes from the fact that the hydrodynamics coefficients and added mass derivative terms are more easily calculated under some assumptions that simplifie the problem. That is, the use of normalized parameters and nominal values of length L (hull length), sea water density ρ, draft depth T and velocity of the ship U. Other parameters are used for the calculations in these equations, such as the drag coefficient of the ship at zero angle of attack C_{D0}, the rudder area A_δ, the moment of inertia I_z and the distance between the center of gravity and the center of pressure x_P. The various equations allowing to calculate these parameters are the following:

$$I_z = m_S \cdot x_G^2 + I_r \tag{4.61}$$

With r the radius of gyration defined as:

$$I_r = m_S \cdot r^2 \qquad r \in [0.15 \cdot L; 0.3 \cdot L] \tag{4.62}$$

The distance between the center of gravity and center of pressure is usually given as:

$$x_P = x_G \pm 0.1 \cdot L \tag{4.63}$$

The $PRIME$ System nomalizes the parameters as settled in the table.

Unit	$PRIME$ **System** I	$PRIME$ **System** II
Length	L	L
Mass	$\frac{\rho}{2} \cdot L^3$	$\frac{\rho}{2} \cdot L^2 \cdot T$
Inertia Moment	$\frac{\rho}{2} \cdot L^5$	$\frac{\rho}{2} \cdot L^4 \cdot T$
Time	$\frac{L}{U}$	$\frac{L}{U}$
Reference Area	L^2	$L \cdot T$
Position	L	L
Angle	1	1
Linear Velocity	U	U
Angular Velocity	$\frac{U}{L}$	$\frac{U}{L}$
Linear Acceleration	$\frac{U^2}{L}$	$\frac{U^2}{L}$
Angular Acceleration	$\frac{U^2}{L^2}$	$\frac{U^2}{L^2}$
Force	$\frac{\rho}{2} \cdot U^2 \cdot L^2$	$\frac{\rho}{2} \cdot U^2 \cdot L \cdot T$
Moment	$\frac{\rho}{2} \cdot U^2 \cdot L^3$	$\frac{\rho}{2} \cdot U^2 \cdot L^2 \cdot T$

Table 4.1: $PRIME$ System notation and normalization.

To illustrate the normalization over a simple example, that is to say that the $PRIME\ I$ normalization of the yaw rate r must be done according to the line of angular velocity of the previous frame. As such, r' being the normalized yaw rate, one can note that:

$$r = \frac{U}{L} \cdot r'$$

This is how the calculations of the added mass derivatives $X_{\dot{u}}^{'}$ and $Y_{\dot{v}}^{'}$ has been done indeed. The calculation will be derived for $X_{\dot{u}}^{'}$ hereafter, since the same reasoning can be applied to $Y_{\dot{v}}^{'}$. In terms of unit analysis, according to equation 4.27, we recall here that:

$$m_S \cdot \dot{u} = X$$

However, according to [2], it can be noted that the first term of the series expansion of X is:

$$X \approx X_{\dot{u}} \cdot \dot{u} \Rightarrow m_S \cdot \dot{u} \approx X_{\dot{u}} \cdot \dot{u}$$

Hence, the following approximation (only for unit analyses purpose) holds using the $PRIME\ II$ System:

$$[X_{\dot{u}}] = [m_S] = \frac{\rho}{2} \cdot L^2 \cdot T$$

Therefore, it was possible to establish that:

$$X_{\dot{u}} = (\frac{\rho}{2} \cdot L^2 \cdot T) \cdot X_{\dot{u}}^{'} \tag{4.64}$$

Hence, the following added mass derivatives relations were established:

$$X_{\dot{u}} = \frac{X'_{\dot{u}}}{\frac{\rho}{2} \cdot L^2 \cdot T} \tag{4.65}$$

$$Y'_{\dot{v}} = \frac{Y_{\dot{v}}}{\frac{\rho}{2} \cdot L^2 \cdot T} \tag{4.66}$$

4.3.7 Release V0.0.1: Numerical values and first ship model

In this subsection, the chosen numerical values for the model will be discussed. Please note that those values were determined at first glance and serve as a mean to be able to realize fast algorithms prototyping. One can easily see that the parameters are subject to high uncertainty and that their precise measurement is complicated to obtain. Therefore, even if it was not the case for the first $V0.0.1$ release of algorithms and simulator, Robust Control method will be explored further in the next release before starting any implementation on the real system.

- $\boldsymbol{L}$ **-** The length of the hull of the ship. This parameter has been approximated to be the length of the ship. The first reasonable considered value is **5.3 meters**.
- $\boldsymbol{\rho}$ **-** The density of salted water. The approximated value of $\mathbf{1000}\ \boldsymbol{kg \cdot m^3}$ has been selected.
- $\boldsymbol{T}$ **-** The depth of the draft. It has been estimated to be around **0.35 meters** of length.
- $\boldsymbol{U}$ **-** The mean velocity of the ship. It has been agreed that the velocity won't exceed **5 to 10 km/h** during the mission.
- $\boldsymbol{C_{D0}}$ **-** The drag coefficient at zero angle of attack of the ship. This parameter has been chosen to be in between values from **0.05 to 0.1**. A first choice of 0.05 has been tested.
- $\boldsymbol{A_\delta}$ **-** The longeron area. No rudder on the small ship, only a longeron. It has been estimated that its area is about $\mathbf{0.2}\ \boldsymbol{m^2}$
- $\boldsymbol{m_S}$ **-** The mass of the ship. A first value of around **550 kg** has been selected.
- $\boldsymbol{x_G}$ **-** The position of the center of gravity on the central symmetry line of the ship. A value of $\mathbf{0.5} \cdot \boldsymbol{L}$ has been taken as a start, supposing that the center of gravity will be in the middle of the ship.

The other physical parameters given by equations 4.61 to 4.63 have been determined in concequence, thanks to the previously presented values.

Therefore, there will be a need to create some dispersion for each parameters previously presented in order to intend to produce a robust controller. The values of the previous parameters directly impact the values of the hydrodynamics coefficients and added mass derivatives terms. Which at their turn, directly impact the values of the static gain and time constant of the Nomoto's first order transfer function between the heading of the ship and the rudder input angle.

For this first release, the following Nomoto first order transfer function has been used to tune some first Guidance, Navigation & Control algorithms:

$$\frac{\psi}{\delta}(s) = \frac{K}{s \cdot (1 + T \cdot s)} = \frac{0.604}{s \cdot (1 + (-5.12) \cdot s)} \tag{4.67}$$

A rapid physical analysis can be performed by considering the yaw rate and rudder angle transfer function, corresponding to the integration of the previous equation:

$$\frac{r}{\delta}(s) = \frac{K}{1 + T \cdot s} = \frac{0.604}{1 + (-5.12) \cdot s} \tag{4.68}$$

This leads to the differential equation:

$$T \cdot \dot{r} + r = K \cdot \delta \tag{4.69}$$

If we consider no angular acceleration, that is $\dot{r} = 0$. Then:

$$r = K \cdot \delta$$

This means that for a turning angle of 15° or 0.26 rad done by the rudder, using the obtained value of K, this leads to a yaw rate of around 9 °/s. At first glance and physically speaking, this seems to be realistic in terms of reaction. However, this reasoning is only valuable by considering ourselves in a regime that differs from the transitional one. A better analysis of the choice of this first model will be done in the justification file related to the GNC algorithms. The fact is that the physical interpretation can be easier by evaluating the response of the system to a specific simple input, a step order by instance.

4.4 Degrees of freedom in positions

The other two degrees of freedom are not controlled and are used only to propagate the state of the ship. There are its position on a local planar x-axis and y-axis. This is exactly the situation depicted in Figure 1.2. Indeed, one can clearly see that:

$$\dot{\boldsymbol{V}}_G \in (S/R_{Inertia}) = u \cdot \vec{x}_{Ship} + v \cdot \vec{y}_{Ship} = u \cdot (cos(\psi) \cdot \boldsymbol{y} + sin(\psi) \cdot \boldsymbol{x}) + v \cdot (-sin(\psi) \cdot \boldsymbol{y} + cos(\psi) \cdot \boldsymbol{x})$$

$$\dot{\boldsymbol{V}}_G \in (S/R_{Inertia}) = \dot{x} \cdot \boldsymbol{x} + \dot{y} \cdot \boldsymbol{y}$$

Therefore, by a simple identification, one can establish the other two degrees of freedom equations. This leads to the following equations:

$$\dot{x} = u \cdot sin(\psi) + v \cdot cos(\psi) \tag{4.70}$$

$$\dot{y} = u \cdot cos(\psi) - v \cdot sin(\psi) \tag{4.71}$$

As mentioned earlier in this technical report, it has been considered that $v << u$ and that if we denote V as the mean norm velocity of the ship, then $u \approx V$. This leads to the final used equations in the simulator V0.0.1 to propagate the position of the ship, considering a constant velocity V:

$$\dot{x} = V \cdot sin(\psi) \tag{4.72}$$

$$\dot{y} = V \cdot cos(\psi) \tag{4.73}$$

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[3] J.-L. Rouget, “Développements limités usuels,” Université Côte d’Azur, Sheet lesson, 2007. [Online]. Available: https://math.univ-cotedazur.fr/~frapetti/analyse/FormulaireDL.pdf.