AMS 221 Homework 2

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Problems in PRS

1. Let u be a utility function for (incremental) amounts of money x and suppose that u is chosen such that $u(100) = 0$ and $u(300) = 1$. Show that

   i. if $100 \sim \{(0.5 : -25), (0.5 : 300)\}$, then $u(-25) = -1$. Proof:

$$\begin{aligned} u(100) &= (0.5) \cdot u(-25) + (0.5) \cdot u(300) \\ &= (0.5) \cdot u(-25) + (0.5) \\ &= 0 \text{ iff } u(-25) = -1 \end{aligned}$$

   ii. if $300 \sim \{(0.5 : 600), (0.5 : 100)\}$ then $u(600) = 2$. Proof:

$$\begin{aligned} u(300) &= (0.5) \cdot u(600) + (0.5) \cdot u(100) \\ &= (0.5) \cdot u(600) \\ &= 1 \text{ iff } u(600) = 2 \end{aligned}$$

   iii. if $100 \sim \{(0.5 : -100), (0.5 : 600)\}$, then $u(-100) = -2$. Proof:

$$\begin{aligned} u(100) &= (0.5) \cdot u(-100) + (0.5) \cdot u(600) \\ &= (0.5) \cdot u(-100) + (0.5) \cdot 2 \\ &= (0.5) \cdot u(-100) + 1 \\ &= 0 \text{ iff } u(-100) = -2 \end{aligned}$$

   iv. if $-100 \sim \{(0.5 : -200), (0.5 : 300)\}$, then $u(-200) = -5$. Proof:

$$\begin{aligned} u(-100) &= (0.5) \cdot u(-200) + (0.5) \cdot u(300) \\ &= (0.5) \cdot u(-200) + (0.5) \\ &= -2 \text{ iff } u(-200) = -5 \end{aligned}$$

   Plot these six points $\{(-200, -5), (-100, -2), (-25, -1), (100, 0), (300, 1), (600, 2)\}$ on a graph where the horizontal axis is labelled x and the vertical axis u. Fair a smooth curve through these points, and assume for the remainder of this excercise that this utility function is your own.

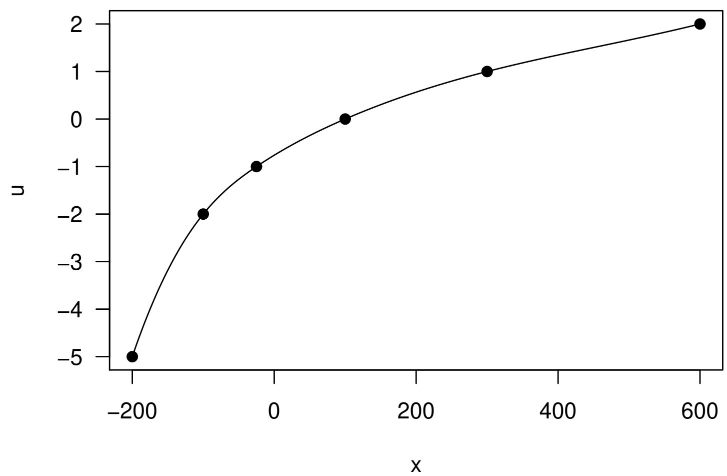

Figure 1: graph of utility function.

Note Figure 1 seems to show concavity of the utility function meaning the decision maker is averse to risk if he considers no lottery more desirable than its EMV, that is CE$\leq$EMV. We will denote $z^* = CE$ and $\bar{z} = EMV$. From the Figure 1, we will be able to answer the following questions:

(a) What is the CE of a lottery that gives a 0.5 chance at \$300 and a 0.5 chance at \$600?
Here the $\bar{z} = (0.5) \cdot (\$300) + (0.5) \cdot (\$600) = \$450$. We need to find z^* such that $u(z^*) = (0.5) \cdot u(\$300) + (0.5) \cdot u(\$600) = (0.5) + (0.5) \cdot 2 = 1.5$. Using Figure 1, we see that $z^* \approx \$445$. Notice $z^* < \bar{z}$ because the utility function is concave.

(b) What is the CE for a lottery that gives a 0.75 chance at \$400 and a 0.25 chance at -\$200?
Here $\bar{z} = (0.75) \cdot (\$400) + (0.25) \cdot (\$200) = \$350$. We need to find z^* such that

$$\begin{aligned}
u(z^*) &= (0.75) \cdot u(\$400) + (0.25) \cdot u(-\$200) \\
&= (0.75) \cdot u(\$400) + (0.25) \cdot -5 \\
&\approx (0.75) \cdot 1.35 - 1.25 \text{ where } u(\$400) \approx 1.35 \\
&= -0.2375
\end{aligned}$$

Using Figure 1, we see that $z^* \approx \$67$. Notice that $z^* < \bar{z}$ and that the difference between the two is larger than in part (a) because the decision maker is more risk averse in this range.

(c) You are offered a compound lottery with a canonical chance at the lottery l of part (b) as one prize and a complementary chance at no net gain. What would your chance of winning lottery l have to be before you would accept the offer?

If I read the question correctly, we want to find π such that

$$\begin{aligned}\pi \cdot u(\$67) &= (1-\pi)\cdot u(\$0)\\ (-0.2375)\cdot \pi &= -0.7 + 0.7\cdot \pi\\ \pi &= \frac{0.7}{0.4625} \approx 0.75\end{aligned}$$

Thus my chance would have to be 0.75 before I would accept the offer.

(d) What is the insurance premium of a lottery that gives a 0.5 chance at \$0 and a 0.5 chance at -\$200?
Here $\bar{z} = (0.5)\cdot \$0 + (0.5)\cdot(-\$200) = -\$100$. We need to find z^* such that

$$\begin{aligned}u(z^*) &= (0.5)\cdot u(\$0) + (0.5)\cdot u(-\$200)\\ &\approx (0.5)\cdot -0.7 + (0.5)\cdot -5 \text{ where } u(\$0) \approx -0.7\\ &= -2.85\end{aligned}$$

Using figure 1, we see that $z^* \approx -\$138$. Hence the insurance premium is

$$-(\bar{z} - z^*) = -(-\$100 - (-\$138)) = \$100 - \$138 = -\$38$$

(e) Given that the CE of a lottery is \$325, and the lottery gives a π chance at \$500 and a $(1-\pi)$ at \$300, find π.
Here we know $z^* = \$325$. Thus $u(\$325) = (\pi)\cdot u(\$500) + (1-\pi)\cdot u(\$300)$. Given $u(\$325) \approx 1.1$, $u(\$500) \approx 1.65$, and $u(\$300) = 1$ then

$$\begin{aligned}1.1 &= \pi \cdot 1.65 + 1 - \pi\\ \implies 0.1 &= 0.65\pi\\ \implies \pi &\approx 0.1538\end{aligned}$$

(f) What is the CE of a lottery that offers a 0.375 chance at \$500, a 0.125 chance at \$600, and a 0.5 chance at \$0?
We want to find z^* such that $u(z^*) = (0.375)\cdot u(\$500) + (0.125)\cdot u(\$600) + (0.5)\cdot u(\$0)$. Given $u(\$500) = 1.65$, $u(\$600) = 2$, and $u(\$0) = -0.7$ then $u(z^*) = (0.375)\cdot 1.65 + (0.125)\cdot 2 + (0.5)\cdot -0.7 = 0.51875$. Using Figure 1, $z^* \approx \$191$.

(g) You are offered the lottery of part (f) for \$200; would you buy it? If you were an EMV'er would your choice change?
If I were offered the lottery of part (f) for \$200 I would not buy it since I would only be willing to spend \$191, the CE in part (f). If I were a EMV'er then I would buy it since the EMV is:

$$\bar{z} = (0.375)\cdot \$500 + (0.125)\cdot \$600 + (0.5)\cdot \$0 = \$262.5$$

which is more than \$200, the cost of the lottery offered. Thus as an EMV'er I would buy the lottery.

(h) Consider the lottery: $l = \{(0.2 : \$0), (0.5 : \$150), (0.3 : \$600)\}$. For how much would you be willing to sell this lottery if you owned it?
We want to find z^* such that $u(z^*) = (0.2) \cdot u(\$0) + (0.5) \cdot u(\$150) + (0.3) \cdot u(\$600)$. Given that $u(\$0) = -0.7$, $u(\$150) = 0.3$, and $u(\$600) = 2$ then $u(z^*) = (0.2) \cdot -0.7 + (0.5) \cdot 0.3 + (0.3) \cdot 2 = 0.61$. Using Figure 1, $z^* \approx \$210$. Recall that z^*denotes CE (cash equivalent) of a lottery. CE of a lottery is the trading or selling price of the lottery assuming you already own the rights and obligations of this lottery. Thus, I would be willing to sell this lottery l for \$210.

(i) For how much would you just be willing to buy the lottery of part (h) if you did not own it?
The amount I am willing to buy the lottery if I did not own it is the CE from part (h), which is \$210.

2. Use the method of section 4.3.1 to obtain your utility function for incremental amounts of money from $-\$10,000$ to $+\$10,000$. Start off by letting $x_0 = \$0$ and $x_1 = \$10,000$. Use graph paper.
Following the method in section 4.3.1, I get the following utility function which shows that I am risk averse since the shape of the curve is roughly concave down.

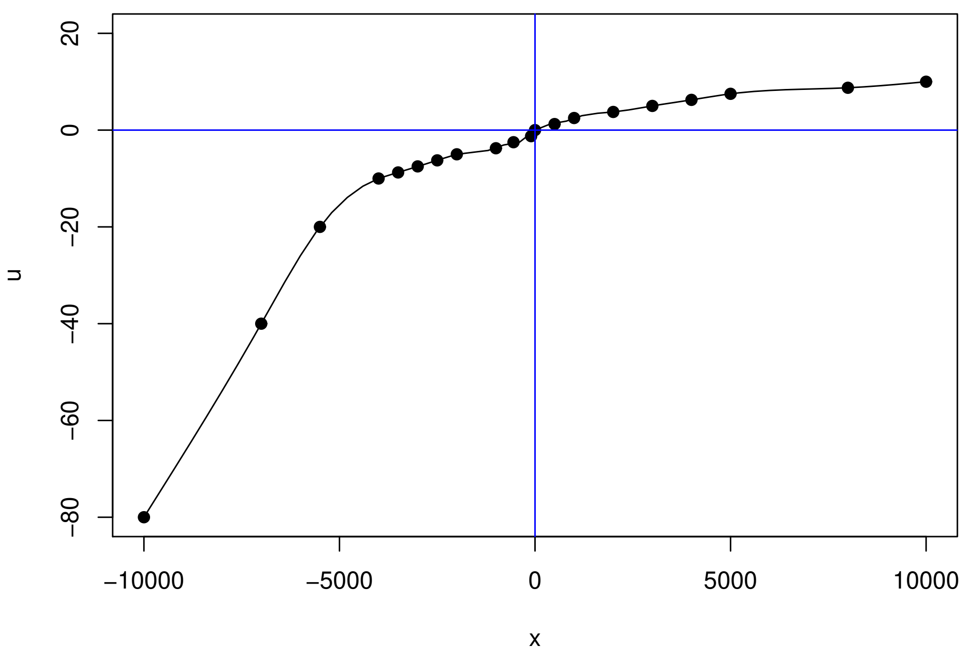

Figure 2: My Utility Function

3. (Problem 4 in PRS) Which act should the wildcatter choose?

- $u(\text{Do not drill}) = u(0) = (0.14) \cdot u(\$1,000,000) + (0.86) \cdot u(-\$50,000)$

- $u(\text{Keep all}) = (0.6) \cdot u(-50) + (0.2) \cdot u(100) + (0.1) \cdot u(200) + (0.07) \cdot u(500) + (0.03) \cdot u(1000)$ where

$$\begin{aligned}
u(-50) &= 0 \cdot u(\$1,000,000) + 1 \cdot u(-\$50,000) \\
u(100) &= 0.35 \cdot u(\$1,000,000) + 0.65 \cdot u(-\$50,000) \\
u(200) &= 0.52 \cdot u(\$1,000,000) + 0.35 \cdot u(-\$50,000) \\
u(500) &= 0.835 \cdot u(\$1,000,000) + 0.165 \cdot u(-\$50,000) \\
u(1000) &= 1 \cdot u(\$1,000,000) + 0 \cdot u(-\$50,000)
\end{aligned}$$

  then $u(\text{Keep all}) = 0.21045 \cdot u(\$1,000,000) + 0.78955 \cdot u(-\$50,000)$

Using the same procedure,

- $u(\text{Sell } 1/4) = (0.6) \cdot u(-37.5) + (0.2) \cdot u(75) + (0.1) \cdot u(150) + (0.07) \cdot u(375) + (0.03) \cdot u(750) = (0.2088) \cdot u(\$1,000,000) + (0.7912) \cdot u(-\$50,000)$
- $u(\text{Drill and sell } 1/2) = (0.6) \cdot u(-25) + (0.2) \cdot u(50) + (0.1) \cdot u(100) + (0.07) \cdot u(250) + (0.03) \cdot u(500) = (0.19805) \cdot u(\$1,000,000) + (0.80195) \cdot u(\$50,000)$
- $u(\text{Sell } 3/4) = (0.6) \cdot u(-12.5) + (0.2) \cdot u(25) + (0.1) \cdot u(50) + (0.07) \cdot u(125) + (0.03) \cdot u(250) = (0.17755) \cdot u(\$1,000,000) + (0.82245) \cdot u(\$50,000)$

Based on the results above, the wildcatter should choose "Keep all" because this option gives the highest expected utility.

4. (Problem 6 in PRS) State an amount z so that you would be indifferent between the following two options:

   Option A: In addition to your regular income you will recieve a tax-free gift of z dollars per year for the rest of your life.

   Option B: A single toss of a fair coin will determine whether you get nothing or the fabulous privilege of an unlimited ability to write checks in any amount you wish for the rest of your natural life.

   In both options you cannot decide to set up your very own Foreign Aid program. The rules of the game - too bad it is only a game - specify that you must spend the money for consumption by you and your family. Incidentally, one point of this exercise is to convince you that u should be bounded from above.

   My Answer: Currently, my monthly income is \$2,000. So in a year I would make about \$24,000. Say as a single adult, my standard of living somewhere in CA would substatially improve had I made \$30,000 per year. I would favor Option A if $z = \$3,000$ since it insures my income will increase to \$27,000 a value close to \$30,000 (in my opinion). If $z = \$2,000$ I would be indifferent to the two options since recieving \$2,000 gives me the same satisfaction as playing the game in Option B.

Problems in PI

1. (Problem 4.4 in PI) Was Bernoulli risk averse? Decreasingly risk averse?
   Let $z \in \mathbb{R}^+$ then Bernoulli's utility function (pg 36. PI) is

$$u(z) = c\log(z) - \log(z_0)$$

   where c is some positive constant and z_0 is the constant of integration and can be interpreted as the wealth necessary to get utility of zero. Then

$$u'(z) = \frac{c}{z},\, u''(z) = -\frac{c}{z^2} < 0,\, \forall\, z > 0$$

   Thus Bernoulli is risk averse. A utility function u is decreasingly risk averse if and only if the function

$$\lambda(z) = -\frac{u''(z)}{u'(z)}$$

   is non-increasing in z. Then

$$\begin{aligned}\lambda(z) &= -\frac{u''(z)}{u'(z)}\\ &= -\left(\frac{-c/z^2}{c/z}\right)\\ &= \frac{1}{z}\end{aligned}$$

$$\lambda'(z) = -\frac{1}{z^2} < 0,\, \forall\, z > 0$$

   and so $\lambda(z)$ is non-increasing in z therefore Bernoulli is decreasingly risk-averse.

2. (Problem 4.6 PI) You are eliciting someone's utility for money. You know this person has constant risk aversion in the range \$0 to \$1000. You propose gambles of the form \$0 with probability p and \$1000 with probability $1-p$ for the following four values of p: 1/10, 1/2, 2/3, and 9/10. You get the following certainty equivalents: 0.25, 0.60, 0.85, and 0.93. Verify that these are not consistent with constant risk aversion. Assuming that the discrepancies are due to difficulty in the exact elicitation on certainty equivalents, rounding, etc., find a utility function with constant risk aversion that closely approximates the elicited certainty equivalents. Justify briefly the method you use for choosing the approximation.

   (Corollary 1 , pp.62) A utility function u is a constantly risk averse if and only if $\lambda(z)$ is constant, in which case there exist constants $a > 0$ and b such that

$$u(z) = \begin{cases} az + b & \text{if } \lambda(z) = 0 \\ -ae^{-\lambda z} + b & \text{if } \lambda(z) = \lambda > 0 \end{cases}$$

The certainty equivalent z^* associated with gambling 0 and 1000 is

$$u(z^*) = u(0)\,p + u(1000)(1-p)$$

Let $u(0) = 0$ and $u(1000) = 1$. Then

$$u(z^*) = (1-p)$$

If $u(z)$ is a constant risk aversion, then $u(z) = -ae^{-\lambda z} + b$ for some constants a, b. Thus

$$0 = u(0) = -ae^{0} + b$$

$$1 = u(1000) = -ae^{-\lambda 1000} + b$$

$$1 - p = u(z^*) = -ae^{-\lambda z^*} + b$$

implies

$$a = b$$

and

$$\begin{aligned} 1 &= -ae^{-\lambda 1000} + b \\ &= a - ae^{-\lambda 1000} \\ \frac{1}{a} &= 1 - e^{-\lambda 1000} \\ e^{-\lambda 1000} &= 1 - \frac{1}{a} \\ &= \frac{a-1}{a} \end{aligned}$$

so $e^{-\lambda 1000} = \frac{a-1}{a}$. Then

$$\begin{aligned} 1 - p &= -ae^{-\lambda z^*} + a \\ \frac{1-p}{a} &= 1 - e^{-\lambda z^*} \\ e^{-\lambda z^*} &= 1 - \frac{1-p}{a} \\ &= \frac{a+p-1}{a} \end{aligned}$$

and $e^{-\lambda z^*} = \frac{a+p-1}{a}$. Then solving for λ in both equations yields

$$\lambda = -\frac{\ln\left(\frac{a-1}{a}\right)}{1000} = -\frac{\ln\left(\frac{a+p-1}{a}\right)}{z^*}$$

which implies λ cannot be a constant function because none of our given z^*, p values will yield the same a value. Then by Corollary 1, $u(z) = -ae^{-\lambda z} + b$ is not constant risk averse.

If we consider $u(z) = az + b$, then setting $u(0) = 0$ and $u(1000) = 1$ as before yields $0 = a \cdot 0 + b$ and $1 = a \cdot 1000 + b$, so $a = \frac{1}{1000}$ and $b = 0$. Not all of our given z^*, p values will satisfy $u(z) = \frac{z}{1000}$, so $\lambda \neq 0$. Again, by Corollary 1, $u(z) = az + b$ is not constant risk averse.

Next "find a utility function with constant risk aversion that closely approximates the elicited certainty equivalents". We know $u(z)$ must be of the form as shown in Corollary 1. Let $u(z) = -ae^{-\lambda z} + b$ for some constants $a > 0$ and b. We have shown previously that $a = b$. If we choose $\lambda = -\frac{\ln\left(\frac{a-1}{a}\right)}{1000}$, then it will not be a great approximation (large margin of error) because it's independent of our given z^*, p values, but it's easier to implement because we can choose the a value that closely approximates those values. Thus our constant risk aversion utility function is $u(z) = a - a\left(\frac{a-1}{a}\right)^{z}$.

If we choose $\lambda = -\frac{\ln\left(\frac{a+p-1}{a}\right)}{z^*}$, we will get a better approximation provided we choose the proper pair of z^*, p values that best represents them, but we would still have to compare with $-\frac{\ln\left(\frac{a-1}{a}\right)}{1000}$ to solve for a. Thus our constant risk aversion utility function is $u(z) = a - a\left(\frac{a+p-1}{a}\right)^{\frac{z}{z^*}}$. Note that calculating a can be computationally expensive, especially for $z^* = 0.93$ and $p = 9/10$.