Exploration: Distance Between Graphs

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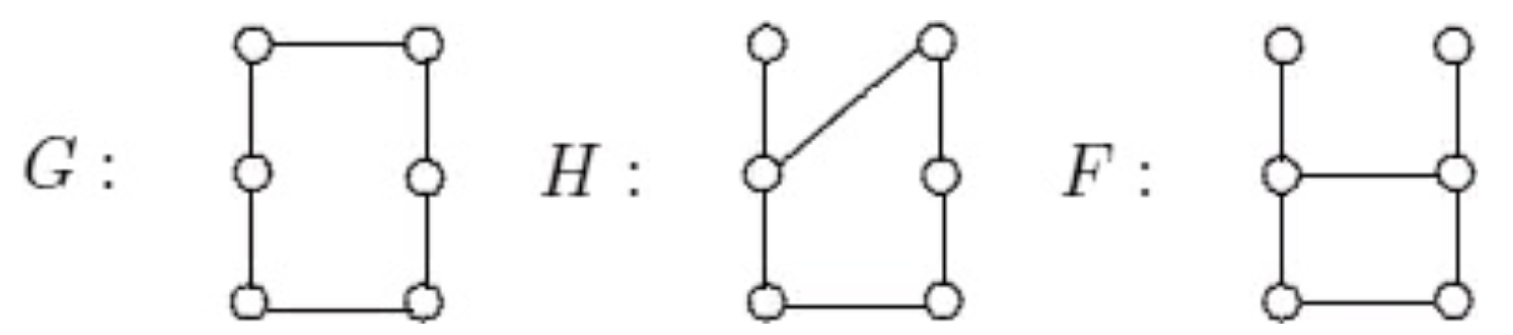
Introduction

How close to being isomorphic are two non-isomorphic graphs?
This suggest the problem of comparing two graphs, at least two graphs of the same order and same size.

Edge Rotations

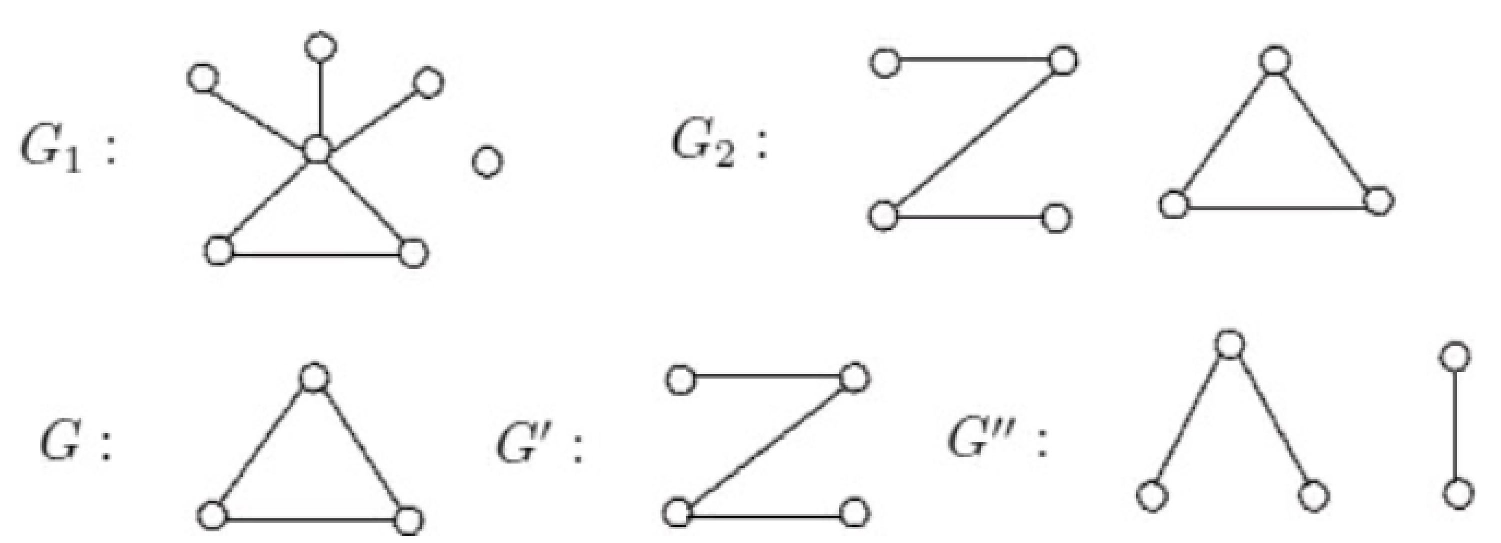
Edge Rotations: Suppose that $G \ncong H$. We say that G can be **transformed into** H by an **edge rotation** (or G can be *rotated into* H) if G contains distinct vertices u, v, and w such that $uw \in E(G)$, $uw \notin E(G)$ and $H = G - uv + uw$.



If we let G and H be graph of order 6 and size 6. The distance between them can be define as d(*H*,*G*). For example the distance between two isomorphic graph would be d(*H*,*G*)=0. Assume G and H are not isomorphic then we **transform** G into H by **rotating the edges** (or vise versa H into G). Suppose G contains two vertices *u*,*v* and *w* such that $uv \in E(G)$ but $uw \notin E(G)$. Then $H \cong G - uv + uw$. In the figure above the graph G can be rotated into H but G cannot be rotated into F. Since the d(G,H)=1 but d(G,F)=2.

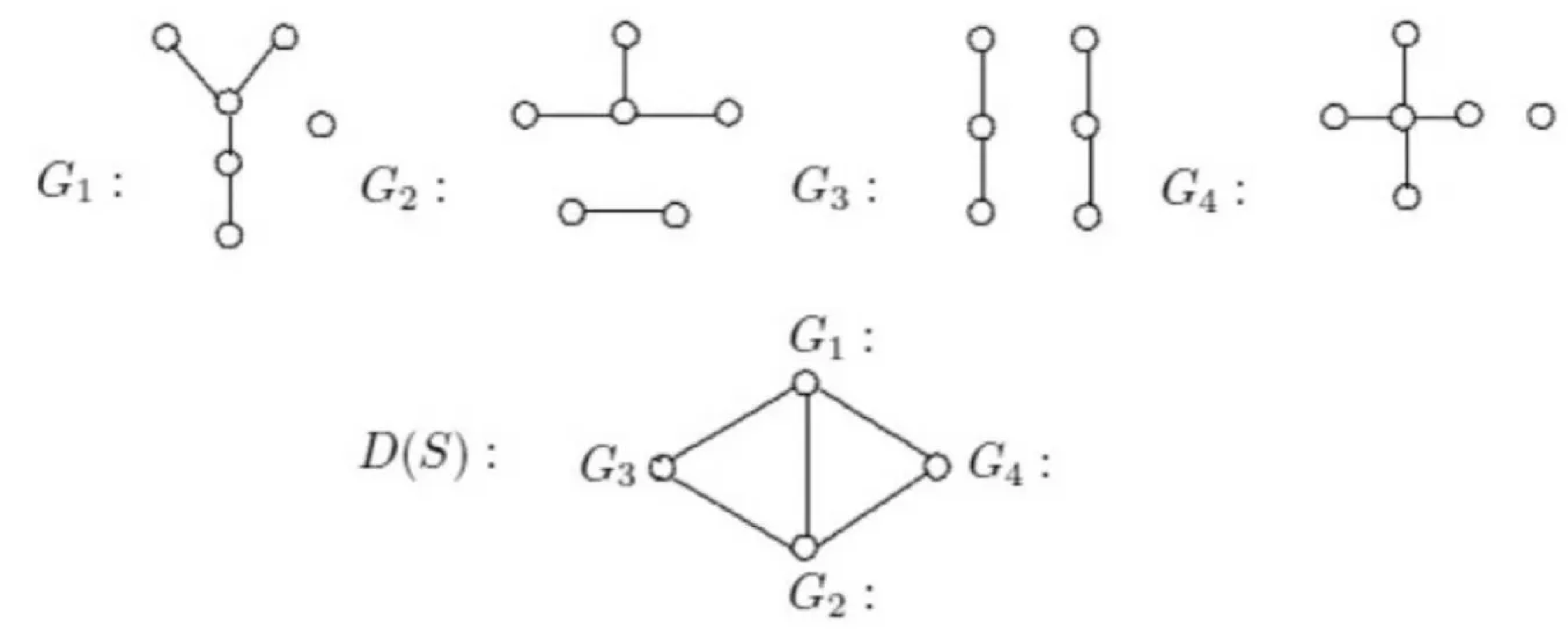
Greatest Common Subgraphs

For two nonempty graphs G_1 and G_2 (not necessarily having the same order or same size), a graph G is called a greatest common subgraph of G_1 and G_2. If G is a graph of maximum size that is isomorphic to both an edge-induced subgraph of G_1 and an edge-induced subgraph of G_2. The graphs G_1 and G_2 of have three distinct greatest common subgraphs, namely G, G' and G''



Rotation Distance

Rotation Distance: For two graphs G and H of the same order and same size, the **rotation distance** $d(G, H)$ between G and H is defined as the *smallest nonnegative integer* k for which there exists a sequence $G_0, G_1, \ldots, G_k$ of graphs such that $G_0 \cong G$, $G_k \cong H$ and G_i can be rotated into G_{i+1} for $i = 0, 1, \ldots, k-1$.



Theorem 12.21

Theorem 12.21 Let G and H be graphs of order n and size m for positive integers n and m and let F be a greatest common subgraph of G and H, where F has size s. Then $d(G, H) \leq 2(m - s)$

Proof. If $s = m$, then $G = H$ and $d(G, H) = 0$. Hence we may assume that $1 \leq s < m$. Let G^* and H^* be edge-induced subgraphs of G and H, respectively, such that $G^* \cong H^* \cong F$. Furthermore, assume that $V(G) = V(H) = v_1, v_2, ..., v_n$ and that the subgraphs G^* and H^* are identically labeled. Since $G \ncong H$, the graph G contains an edge v_iv_j that is not in H and H contains an edge v_pv_q that is not in G. Suppose that $v_i, v_j \cap v_p, v_q \neq \emptyset$, say $v_j = v_p$. Then G can be rotated into $G_1 = G - v_iv_j + v_jv_q$. and $d(G, G_1) = 1$. Next, suppose that $v_i, v_j \cap v_p, v_q = \emptyset$.

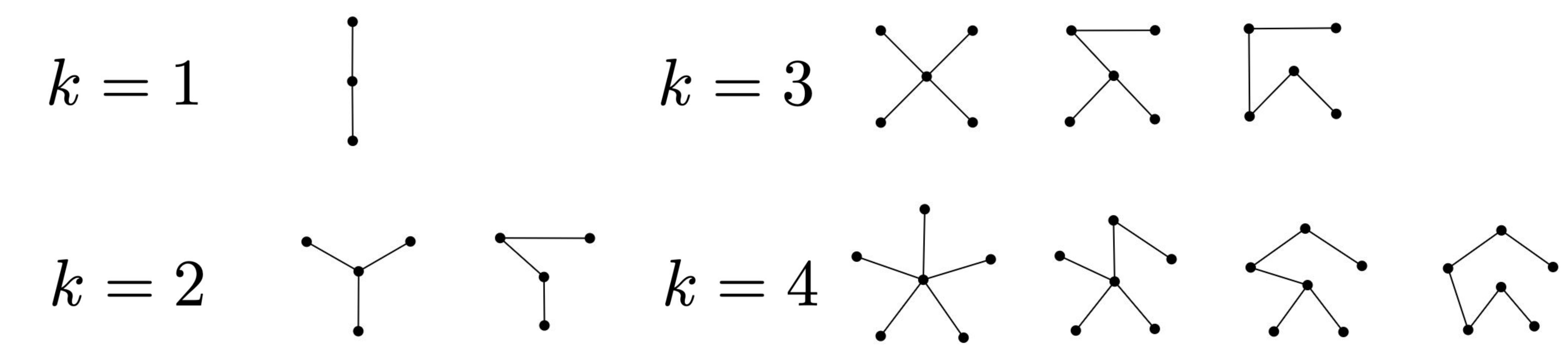
Suppose that at least one of v_i and v_j is not adjacent in G to at least one of v_p and v_q, say $v_iv_p \notin E(G)$. Then G can be rotated into $G' = G - v_iv_j + v_iv_p$ and G' can be rotated into $G'' = G' - v_iv_p + v_pv_q$ and so $d(G, G'') \leq 2$.

If, on the other hand, each of v_i and v_j is adjacent to both v_p and v_q, then G can be rotated into $G_1 = G - v_iv_p + v_pv_q$ and G_1 can be rotated into $G_2 = G_1 - v_iv_j + v_iv_p$ and so $d(G, G_2) \leq 2$.

In any case, G can be transformed into $H' = G - v_iv_j + v_pv_q$ by at most two rotations and so $d(G, H') \leq 2$. The graphs H' and H have $s + 1$ edges in common. Continuing in this manner, we have $d(G, H) \leq 2(m - s)$.

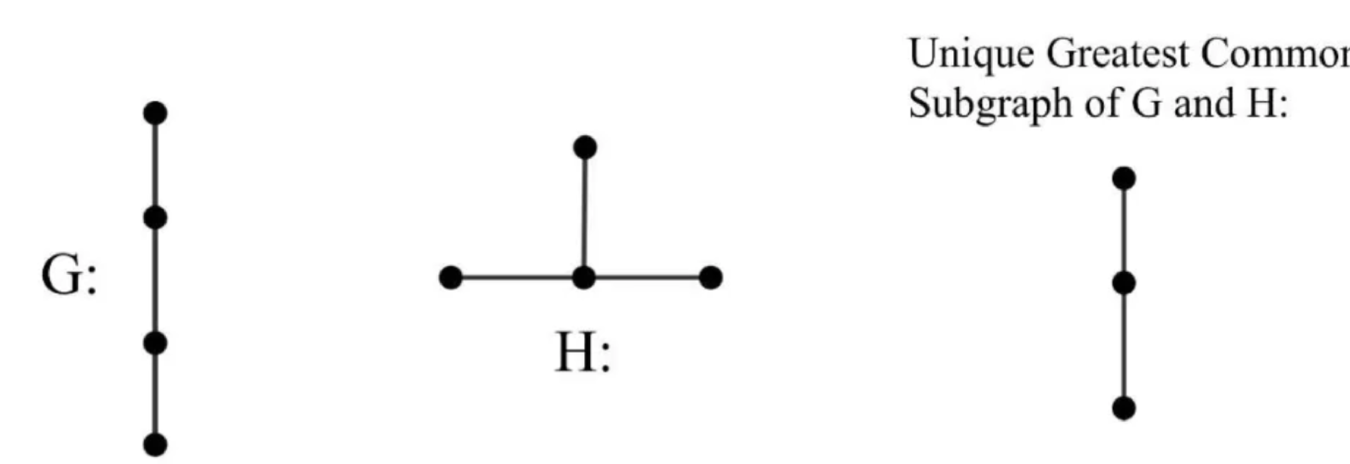
Exercise 12.57

For each positive integer k, show that there exists two graphs G and H such that $d(G, H) = k$.
Proof. Let $G = S_n$ and $H = P_n$ where $n \geq 3$. So, $d(G, H) = k = n - 3$



Exercise 12.58

Give an example of two graphs G and H that have a unique greatest common subgraph.



Exercise 12.59

For each positive integer k, give an example of two graphs G and H that have exactly k greatest common subgraphs.
Proof. Consider any integer $k \geq 2$.

Let $G = P_2C_{2k-1}$ and $H = P_2C_{2k}$.