Using Mathematical Modeling to Improve the Emergency Department Nurse-Scheduling Process

Gerardo Gutierrez

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Introduction

In emergency departments it is Crucial for staff to be appropriately scheduled at all times. Nurse scheduling is complicated since patient needs can be so volatile.

In this article, we examine the Linear Programming model that was produced by Anna Camile Svirsko et al. in "Using Mathematical Modeling to Improve The Emergency Department Nurse-Scheduling Process" to optimize the nurse scheduling process at UPMC's Children Hospital of Pittsburgh. The goal of the model is to minimize the number of shifts required to reach the target nurse-staffing level at each hour.

Furthermore, we will examine a scenario where nurses also have the option to work 4- and 6- hour shifts. We then compare the results of three models: the original LP by Anna and Co, our own LP, and the extended LP with varying shift lengths.

Background

The number of nurses staffed was determined based on the sections of the emergency department that were open, the nurse-to-room ratio for each section, and the additional positions that were staffed at each time. The staffing targets are assigned in 4-hour increments beginning at 7 AM. (see Table 1)

Table 1: Nurse staffing targets by time of day

Time	**11p–3a**	**3a–7a**	**7a–11a**	**11a–3p**	**3p–7p**	**7p–11p**	**11p–12a**
Nurse staffing target	15	8	8	15	19	19	15

When constructing the linear programming model. The following features and assumptions were included:

- Every nurse signs up for his or her own shifts, so they set their own schedule.
- Every nurse works a combination of 8- and 12- hour shifts a week based on his or her own contract.
- The day begins and ends at 7 AM, so no shift spans 7 AM.

Methods

Within the model, meal breaks and the types of shifts were considered. We define the following variables

- x_j: The number of 12-hour shifts beginning at block j.
- y_j: The number of 8-hour shifts beginning at block j.

We then assumed that all nurses take their half hour meal break within the second 4-hr block of their shift. With the exception of nurses who start their shift at 11PM - these nurses take their meal break during the first 4-hr block of their shift.

Linear Programming Model

$$\textbf{Minimize} \quad \sum_{j=1}^{4} x_j + \sum_{j=1}^{5} y_j$$

Subject to:

$$x_1 + y_1 \geq 8 \tag{1}$$

$$\frac{7}{8}x_1 + \frac{7}{8}y_1 + x_2 + y_2 \geq 15 \tag{2}$$

$$x_1 + \frac{7}{8}x_2 + y_2 + \frac{7}{8}x_3 + y_3 \geq 19 \tag{3}$$

$$x_2 + \frac{7}{8}x_3 + y_3 + \frac{7}{8}x_4 + y_4 \geq 19 \tag{4}$$

$$x_3 + \frac{7}{8}x_4 + y_4 + \frac{7}{8}y_5 \geq 15 \tag{5}$$

$$x_4 + y_5 \geq 8 \tag{6}$$

$$\sum_{j=1}^{4} x_j \geq 0.8 \left(\sum_{j=1}^{4} x_j + \sum_{j=1}^{5} y_j \right) \tag{7}$$

$$y_1 \geq 2 \tag{8}$$

$$y_2 \geq 1 \tag{9}$$

$$y_4 \geq 1 \tag{10}$$

$$y_5 \geq 1 \tag{11}$$

Results

Optimal Solution

$$x_1 = 6, \quad x_2 = 7, \quad x_3 = 7, \quad x_4 = 7, \quad y_1 = 2, \quad y_2 = 1, \quad y_3 = 0, \quad y_4 = 1, \quad y_5 = 1, \quad Z = 32$$

Table 2: Comparison of nurse staffing targets and staffing levels from the optimal solution

Time	**7a–11a**	**11a–3p**	**3p–7p**	**7p–11p**	**11p–3a**	**3a–7a**
Nurse staffing target	8	15	19	19	15	8
Nurse staffing levels	8	15	20.125	21	15	8

After solving the linear program, the optimal solution that best lined up with the nurses at CHP can be found in table 2. Which uses a total of 32 nurses in a work day. Where the majority of hours are allocated in the 3pm - 11pm time blocks, as expected.

Exploration for 4-and 6-hour model Extension

To allow for more flexibility, we can add the option for the nurses to take on 4- and 6- hour shifts. These shifts do not require breaks so they can seamlessly be added to a work day. Furthermore, allowing shifts two begin every 2 hours instead of every 4 hours should provide even more flexibility. So, we have a total of 12 2-hour time blocks, starting at 7am. (see Table 3)
And we are assuming that the staffing needs in the 2- hour halves of the original 4- hour time blocks are equal. (see Table 4)

Table 3: Time Blocks for Extended Model

Time	**7a–9a**	**9a–11a**	**11a–1p**	**1p–3p**	**3p–5p**	**5p–7p**	**7p–9p**	**9p–11p**	**11p–1a**	**1a–3a**	**3a–5a**	**5a–7a**
Block	1	2	3	4	5	6	7	8	9	10	11	12

Table 4: Staffing Targets With New Time Blocks

Time	**7a–9a**	**9a–11a**	**11a–1p**	**1p–3p**	**3p–5p**	**5p–7p**	**7p–9p**	**9p–11p**	**11p–1a**	**1a–3a**	**3a–5a**	**5a–7a**
Nurse Staffing Target	8	8	15	15	19	19	19	19	15	15	8	8

This portion is intentionally left blank.

Extended LP

From our updated time blocks, each nurse can pick from a total of eleven 4-hour shifts, ten 6-hour shifts, nine 8-hour shifts, and seven 12-hour shifts in a day.
To construct the extended model, we use a very similar approach as the original model. We seek to minimize the total number of nurses needed to across the entire workday.

Decision Variables

x_j: The number of 12-hour shifts beginning at block j.
y_j: The number of 8-hour shifts beginning at block j.
α_j: The number of 4-hour shifts beginning at block j.
β_j: The number of 6-hour shifts beginning at block j.

$$\text{Minimize } Z = \sum_{j=1}^{11} \alpha_j + \sum_{j=1}^{10} \beta_j + \sum_{j=1}^{7} x_j + \sum_{j=1}^{9} y_j$$

Subject to:

$$\alpha_1 + \beta_1 + x_1 + y_1 \geq 8 \tag{1}$$
$$\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + x_1 + x_2 + y_1 + y_2 \geq 8 \tag{2}$$
$$\alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3 + x_1 + 0.75x_2 + x_3 + y_1 + 0.75y_2 + y_3 \geq 15 \tag{3}$$
$$\alpha_3 + \alpha_4 + \beta_2 + \beta_3 + \beta_4 + x_1 + 0.75x_2 + x_3 + x_4 + y_1 + 0.75y_2 + y_3 + y_4 \geq 15 \tag{4}$$
$$\alpha_4 + \alpha_5 + \beta_3 + \beta_4 + \beta_5 + x_1 + x_2 + 0.75x_3 + x_4 + x_5 + y_2 + 0.75y_3 + y_4 + y_5 \geq 19 \tag{5}$$
$$\alpha_5 + \alpha_6 + \beta_4 + \beta_5 + \beta_6 + x_1 + x_2 + x_3 + 0.75x_4 + x_5 + x_6 + y_3 + 0.75y_4 + y_5 + y_6 \geq 19 \tag{6}$$
$$\alpha_6 + \alpha_7 + \beta_5 + \beta_6 + \beta_7 + x_2 + x_3 + x_4 + 0.75x_5 + x_6 + x_7 + y_4 + 0.75y_5 + y_6 + y_7 \geq 19 \tag{7}$$
$$\alpha_7 + \alpha_8 + \beta_6 + \beta_7 + \beta_8 + x_3 + x_4 + x_5 + x_6 + 0.75x_7 + x_8 + y_5 + y_6 + 0.75y_7 + y_8 \geq 19 \tag{8}$$
$$\alpha_8 + \alpha_9 + \beta_7 + \beta_8 + \beta_9 + x_4 + x_5 + x_6 + x_7 + y_6 + y_7 + 0.75y_8 + y_9 \geq 15 \tag{9}$$
$$\alpha_9 + \alpha_{10} + \beta_9 + \beta_{10} + x_5 + x_6 + x_7 + x_8 + 0.75x_9 + x_{10} + y_7 + y_8 + y_9 + y_{10} \geq 15 \tag{10}$$
$$\alpha_{10} + \alpha_{11} + \beta_9 + \beta_{10} + x_6 + x_7 + y_8 + 0.75y_9 \geq 8 \tag{11}$$
$$\alpha_{11} + \beta_{10} + x_7 + y_9 \geq 8 \tag{12}$$
$$\sum_{j=1}^{11}(-0.8\alpha_j) + \sum_{j=1}^{10}(-0.8\beta_j) + \sum_{j=1}^{7}(0.2x_j) + \sum_{j=1}^{9}(-0.8y_j) \geq 0 \tag{13}$$

$$y_1 + y_2 \geq 2 \tag{14}$$
$$y_3 + y_4 \geq 1 \tag{15}$$
$$y_7 + y_8 \geq 1 \tag{16}$$
$$y_9 + y_{10} \geq 1 \tag{17}$$

Result of Exploration

Unexpectedly, the addition of the options to have 4 and 6 hour shifts did not yield a significantly more optimal result. The model resulted in using only one of the 6 hour shifts with $\alpha_1, ..., \alpha_{11} = 0$, $\beta_2, ..., \beta_{10} = 0$ and $\beta_1 = 1$. And Z = 32, just as in the original LP.

Discussion

Perhaps using a more precise method that uses integer programming would result in a more significant difference between the two models. Furthermore, adding a minimum constraint for the sum of the total 4- and 6-hour shifts could also improve the model.

Appendix: MATLAB Code

Figure 1: MATLAB Code for Original LP

```
%Objective function
%Total of 9 decision variables
f=[1,1,1,1,1,1,1,1,1];




% Left Side Of Constraints; 11 constraints
% Multiply By -1 to convert to <=
A = -1*[1,0,0,0,1,0,0,0,0;
       7/8,1,0,0,7/8,1,0,0,0;
       1,7/8,1,0,0,7/8,1,0,0;
       0,1,7/8,1,0,0,7/8,1,0;
       0,0,1,7/8,0,0,0,7/8,7/8;
       0,0,0,1,0,0,0,0,1;
       .2,.2,.2,.2,-.8,-.8,-.8,-.8,-.8;
       0,0,0,0,1,0,0,0,0;
       0,0,0,0,0,1,0,0,0;
       0,0,0,0,0,0,0,1,0;
       0,0,0,0,0,0,0,0,1;];

% Right Side of Constraints
% Multiply by -1 to convert to <=
b =-1*[8;
      15;
      19;
      19;
      15;
      8;
      0;
      2;
      1;
      1;
      1;];

%non-negativity constraints for 9 decision variables
lb=zeros(9,1);

[x,fval]=linprog(f,A,b,[],[],lb,[])
```

Figure 2: MATLAB Results for Original LP

```
>> nurse_lp

Optimal solution found.


x =

    6.0000
    5.5750
    7.1250
    7.0000
    2.0000
    2.4250
         0
    1.0000
    1.0000


fval =

   32.1250
```

Figure 3: Part 1 of the MATLAB Code for 4- and 6-Hour Shift Extension

```
%LP Extension: Allowing 4 and 6 hour shifts

% Objective function
f = ones(1,37); % Total of 37 decision variables

% Initialize the A Matrix
A = zeros(17,37); % 17 constraints, 37 variables

% Assign values to coefficients of decision variables in A
% for first 12 constraints
for i = 1:12
    % Alpha: Columns 1 to 11, where (0 <= i - n <= 1)
    for n = 1:11
        if (0 <= i - n) && (i - n <= 1)
            A(i,n) = 1; % Alpha_n
        end
    end

    % Beta: Columns 12 to 21, where (0 <= i - n <= 2)
    for n = 1:10
        if (0 <= i - n) && (i - n <= 2)
            A(i,11 + n) = 1; % Beta_n
        end
    end

    % x: Columns 22 to 28
    for n = 1:7
        if (0 <= i - n) && (i - n <= 5)
            A(i,21 + n) = 1; % x_n = 1 for (0 <= i - n <= 5)
        end
        if (i - n == 2) % x_n = 3/4 if (i - n = 2)
            A(i,21 + n) = 3/4;
        end
    end

    % y: Columns 29 to 37
    for n = 1:9
        if (0 <= i - n) && (i - n <= 3)
            A(i,28 + n) = 1; % y_n = 1 for (0 <= i - n <= 3)
        end
        if (i - n == 2) % y_n = 3/4 if (i - n = 2)
            A(i,28 + n) = 3/4;
        end
    end
end
```

Figure 4: Part 2 of the MATLAB Code for 4- and 6-Hour Shift Extension

```
% Constraint 8 adjustments - Since nurses starting their shift past
% 11pm will take their breaks in the first 4 hours of their shift instead
% of hours 4-8.
A(8,28) = 3/4;
A(8,35) = 3/4;
A(9,28) = 1;
A(9,35) = 1;
A(9,36) = 3/4;
A(10,36) = 1;
A(10,37) = 3/4;
A(11,37) = 1;

% Update Constraint 13 to reflect proportion of shifts
A(13,1:11) = -0.8;   % Alpha (columns 1 to 11)
A(13,12:21) = -0.8;  % Beta (columns 12 to 21)
A(13,22:28) = 0.2;  % x (columns 22 to 28)
A(13,29:37) = -0.8;  % y (columns 29 to 37)

% Constraint 14: y1 + y2
A(14,29) = 1; % Coefficient for y1
A(14,30) = 1; % Coefficient for y2

% Constraint 15: y3 + y4
A(15,31) = 1; % Coefficient for y3
A(15,32) = 1; % Coefficient for y4

% Constraint 16: y7 + y8
A(16,35) = 1; % Coefficient for y7
A(16,36) = 1; % Coefficient for y8

% Constraint 17: y9 + y10
A(17,37) = 1; % Coefficient for y9
A(17,37) = 1; % Coefficient for y10



% Multiply By -1 to convert to <=
A = -1*A;

% Right Side of Constraints
% Multiply by -1 to convert to <=
b =-1*[8;
      8;
      15;
      15;
      19;
      19;
      19;
      19;
      15;
      15;
      8;
      8;
      0;
      2;
      1;
      1;
      1];

%non-negativity constraints for 37 decision variables
lb=zeros(37,1);

[x,fval]=linprog(f,A,b,[],[],lb,[])
```

Figure 5: Results from the MATLAB Code for Extended Model

```
>> nurse_lp_4_6_hours_shifts

Optimal solution found.


x =

         0   Alpha_1
         0   Alpha_2
         0   Alpha_3
         0   Alpha_4
         0   Alpha_5
         0   Alpha_6
         0   Alpha_7
         0   Alpha_8
         0   Alpha_9
         0   Alpha_10
         0   Alpha_11
    1.1146   Beta_1
         0   Beta_2
         0   Beta_3
         0   Beta_4
         0   Beta_5
         0   Beta_6
         0   Beta_7
         0   Beta_8
         0   Beta_9
         0   Beta_10
    5.9443   x_1
         0   x_2
    6.6625   x_3
         0   x_4
    6.2500   x_5
         0   x_6
    7.0000   x_7
    0.9411   y_1
    1.0589   y_2
    1.0000   y_3
         0   y_4
         0   y_5
         0   y_6
    0.6500   y_7
    0.3500   y_8
    1.0000   y_9


fval =

   31.9714
```