STEADY-STATE SINUSOIDAL ANALYSIS

So far, we have considered circuits in which the sources are D.C. We now investigate circuits where the sources deliver sinusoidal (AC) currents and voltages.

- methods of analysis are identical.
- arithmetic changes from real to complex.

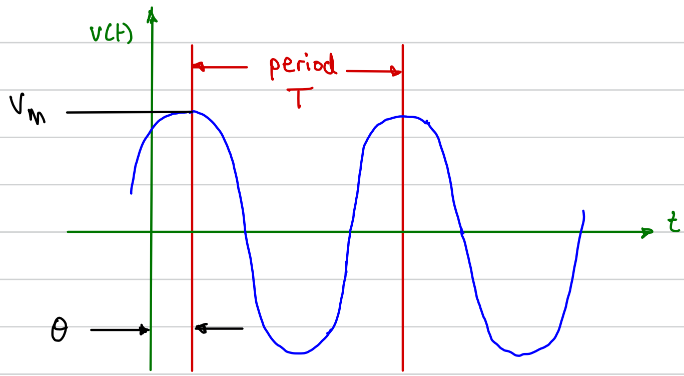
Sinusoidal voltages and currents

Let $v(t) = V_m \cos(\omega t + \theta)$

V_m: peak value

ω: radian frequency (radians/sec)

θ: phase angle (radians)



The sinusoid is periodic with period T. We have one complete period when the angle increases by 2π radians.

$$\left.\omega t\right|_{t=T} = 2\pi, \quad \text{so} \quad \omega T = 2\pi \quad \text{and} \quad T = \frac{2\pi}{\omega}$$

Frequency is defined as the number of complete periods (cycles) per second.

$$f = \frac{1}{T}, \quad f = \text{frequency in Hertz (Hz)}$$

We also have

$$\omega = \frac{2\pi}{T}, \quad \text{so} \quad \omega = 2\pi f.$$

By convention, we use <u>cosine</u> and not <u>sin</u>. They are related by a simple phase difference:

$$\sin(\omega t) = \cos(\omega t - \pi/2)$$
$$= \cos(\omega t - 90°)$$

We say that $\sin(\omega t)$ has the phase angle of $-90°$ (or $-\pi/2$ rads).

<u>Root-mean-square values</u>

We often express voltages and currents in terms of their <u>peak</u> values (V_m, I_m), but also in terms of their <u>root-mean-square</u> (rms) values.

Consider power in a resistor over one period of the waveform.
Instantaneous power is simply

$$p(t) = v(t)\,i(t) = \frac{v(t)}{R} \cdot v(t) = \frac{v^2(t)}{R}$$

The energy over one period is $E_T = \int_0^T p(t)\,dt$, where $p(t) = v^2(t)/R$.

An important measure is <u>average power</u> over one period.

$$P_{avg} = \frac{E_T}{T} = \frac{1}{T}\int_0^T \frac{v^2(t)}{R}\,dt$$

which can be expressed as

$$P_{avg} = \frac{\left[\sqrt{\frac{1}{T}\int_0^T \frac{v^2(t)}{R}\,dt}\,\right]^2}{R} = \frac{V_{rms}^2}{R}$$

Thus, we define rms voltage as

$$V_{rms} = \underbrace{\sqrt{}}_{\text{root}}\!\!\!\!\underbrace{\frac{1}{T}\int_0^T \overbrace{\frac{v^2(t)}{R}}^{\text{square}}}_{\text{mean}}$$

Rms values are sometimes called <u>effective values</u>. In the real world,

- AC voltages are specified as <u>rms</u>, not peak (e.g., household voltages are in the range 110-120 V_{rms}).
- Power is <u>average power</u>, not instantaneous power (e.g., a 100 W light bulb uses 100 W of average power).

For sinusoidal voltages and currents,

$$V_{rms} = \sqrt{\int_0^T V_m^2 \cos^2(\omega t + \theta)\, dt}$$

This can be used to show that

$$\boxed{V_{rms} = \frac{V_m}{\sqrt{2}}}$$

SINUSOIDAL RMS VOLTAGE

If the voltage in your home is $V_{rms} = 120\ V_{rms}$, then

$$v(t) = 120\sqrt{2}\cos(\omega t + \theta)\ V, \quad \text{where } \omega = 2\pi f = 2\pi \times 60$$

$$v(t) = 169.7\cos(120\pi t + \theta)\ V.$$

<u>Example</u>: Let $v(t) = 10\sin(1000\pi t + 30°)$

Express this as a cosine, give angular frequency, frequency in Hz, and average power in a $10\,\Omega$ resistor.

We have
$$v(t) = 10\sin(1000\pi t + 30°)$$
$$= 10\cos(\underbrace{1000\pi t}_{\text{angular frequency}} + 30° - 90°)$$

$\omega = 1000\pi$ radians/sec

$f = \omega/2\pi = 500$ Hz

and $V_{rms} = \dfrac{V_m}{\sqrt{2}} = 7.071$ V.

and average power $P_{avg} = \dfrac{V_{rms}^2}{R} = \dfrac{(7.071)^2}{10} = 5W$

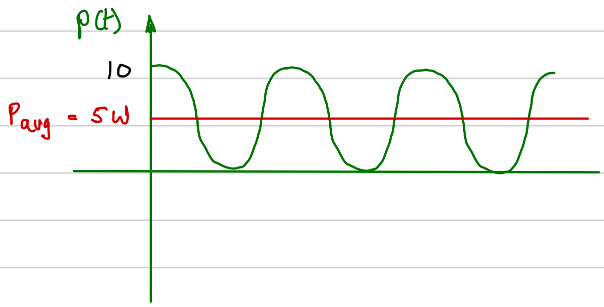
Let's also sketch the instantaneous power p(t)

$$p(t) = \frac{V^2(t)}{R} = \frac{100}{10} \cos^2(1000\pi t - 60°)$$

Using the identity $\cos^2(x) = \frac{1}{2}[1 + \cos(2x)]$

$$p(t) = 5 + 5\cos(2000\pi t - 120°)$$

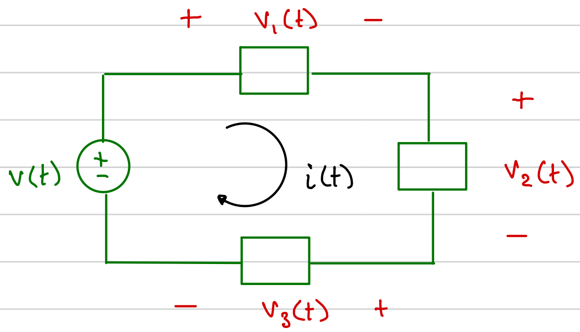
makes the average 5!



Phasors

For sinusoidal voltages and currents, we need a convenient way to add them to satisfy KVL, KCL.

Consider a simple loop:



and assume that:

$$v_1(t) = 10\cos(\omega t)$$
$$v_2(t) = 5\cos(\omega t - 30^\circ)$$
$$v_3(t) = 5\cos(\omega t + 90^\circ)$$

We wish to find $v(t)$ in the form $v(t) = V_m \cos(\omega t + \theta)$

KVL must be satisfied by this circuit over all time.

$$-v(t) + v_1(t) + v_2(t) + v_3(t) = 0$$

So $v(t) = 10\cos(\omega t) + 5\cos(\omega t - 30^\circ) + 5\cos(\omega t + 90^\circ)$

How do we manipulate this to get the desired form $V_m \cos(\omega t + \theta)$?
What will V_m and θ be?

For this, we instead express voltages and currents in terms of phasors.

Let $v_1(t) = V_1 \cos(\omega t + \theta_1)$

ω is usually fixed in value throughout the circuit and is ignored in phasor notation.

We have a pair of parameters that describe this voltage:

V_1 – magnitude (i.e., peak value, or amplitude)
θ_1 – phase angle

Basic idea — Represent as a vector on a plane; then add as vectors.

Examples of phasor notation

Voltage	phasor notation
$v_a(t) = V_a \cos(\omega t + \theta_a)$	$\bar{V}_a = V_a \angle \theta_a$
	magnitude, phase angle
$V_b(t) = V_b \sin(\omega t + \theta_b)$ $= V_b \cos(\omega t + \theta_b - 90^\circ)$	$\bar{V}_b = V_b \angle \theta_b - 90^\circ$

Similarly, for currents:

$$i_c(t) = I_c \cos(\omega t + \theta_c) \longleftrightarrow \bar{I}_c = I_c \angle \theta_c$$

To manipulate phasors, we will need to use complex numbers.

Complex numbers ~ review

We express and manipulate phasors as complex numbers. Complex numbers involve "imaginary" numbers:

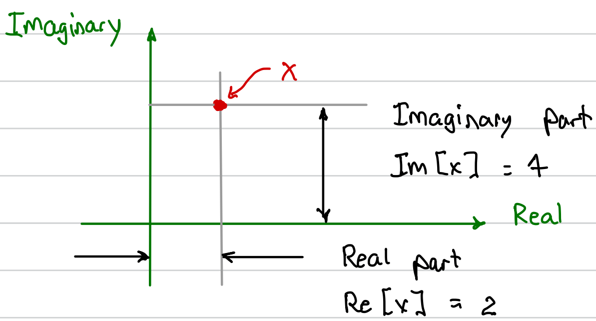
Mathematicians : $i = \sqrt{-1}$

Engineers : i is current, so $j = \sqrt{-1}$

or $j^2 = -1$

E.g., complex number: $x = 2 + j4$

real part, imaginary part



x is a point in the complex plane

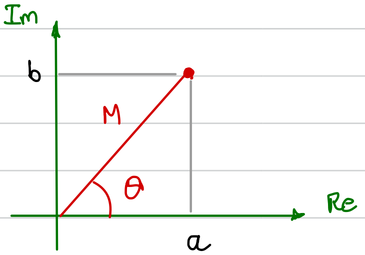
The complex conjugate of x is : $x^* = 2 - j4$

denotes complex conjugate

flip the sign of Im [x]

Rectangular and polar forms of complex numbers:

$$x = a + jb$$
$$x = M\angle\theta$$



Conversion between forms:

Polar M

$$M = \sqrt{a^2 + b^2}$$
$$\theta = \tan^{-1}(b/a)$$
$$= \arctan(b/a)$$

Rectangular $a + jb$

$$a = M \cos\theta$$
$$b = M \sin\theta$$

Complex arithmetic

Let $x = 2 + j4$ and $y = 4 + j5$

- Addition and subtraction must be done in rectangular form:

$$x + y = (2 + j4) + (4 + j5) = 6 + j9$$
$$x - y = (2 + j4) - (4 + j5) = -2 - j$$

- Multiplication and division can be done in either form

Rectangular:

$$x \cdot y = (2 + j4)(4 + j5)$$
$$= 8 + j10 + j16 + j^2 20 \qquad j^2 = -1$$
$$= -12 + j26$$

$$x/y = \frac{2 + j4}{4 + j5} = \left(\frac{2 + j4}{4 + j5}\right)\left(\frac{4 - j5}{4 - j5}\right)$$

$$= \frac{8 - j10 + j16 - j^2 20}{16 - j20 + j20 - j^2 25} = \frac{28 + j6}{41}$$

$$= \frac{28}{41} + j\,\frac{6}{41}$$

- <u>Polar</u>: First convert x and y to polar

$$x = \sqrt{2^2 + 4^2}\ \angle \tan^{-1}(4/2) = \sqrt{20}\ \angle 63.44^\circ$$

$$y = \sqrt{4^2 + 5^2}\ \angle \tan^{-1}(5/4) = \sqrt{41}\ \angle 51.34^\circ$$

Then: $$x \cdot y = \left(\sqrt{20}\right)\left(\sqrt{41}\right) \angle 63.44^\circ + 51.34^\circ$$ MULTIPLY MAGNITUDES, ADD ANGLES

$$x / y = \left(\sqrt{20}\right) / \left(\sqrt{41}\right) \angle 63.44^\circ - 51.34^\circ$$ DIVIDE MAGNITUDES, SUBTRACT ANGLES

Phasors and Euler's identity

Key to the functioning of phasors is Euler's identity

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Multiply both sides by M

$$\underbrace{Me^{j\theta}}_{\text{complex exponential, another way of expressing } M\angle\theta} = \underbrace{M\cos\theta + jM\sin\theta}_{\text{rectangular form}}$$

e.g., $x = 10e^{j30^\circ} = 10\angle 30^\circ$

Using Euler's identity,

$$x = 10e^{j30^\circ} = 10\cos(30^\circ) + j\,10\sin(30^\circ)$$

and where

$$\mathrm{Re}[x] = \mathrm{Re}[10e^{j30^\circ}] = 10\cos(30^\circ) = 8.66$$
$$\mathrm{Im}[x] = \mathrm{Im}[10e^{j30^\circ}] = 10\sin(30^\circ) = 5$$

so $10e^{j30^\circ} = 8.66 + j5$

Writing KVL/KCL equations using phasors

The key step: Express cosines as complex exponentials using Euler's identity.

$$\boxed{\cos(x) = \mathrm{Re}[e^{jx}]}$$

Finally, back to our original KVL example

$$v(t) = 10\cos(\omega t) + 5\cos(\omega t - 30^\circ) + 5\cos(\omega t + 90^\circ)$$

Express each term as a complex exponential.

$$v_1(t) = 10\cos(\omega t) = \text{Re}\left[10e^{j\omega t}\right]$$
$$v_2(t) = 5\cos(\omega t - 30^\circ) = \text{Re}\left[5e^{j(\omega t - 30^\circ)}\right]$$
$$v_3(t) = 5\cos(\omega t + 90^\circ) = \text{Re}\left[5e^{j(\omega t + 90^\circ)}\right]$$

so $$v(t) = \text{Re}\left[10e^{j\omega t}\right] + \text{Re}\left[5e^{j(\omega t - 30^\circ)}\right] + \text{Re}\left[5e^{j(\omega t + 90^\circ)}\right]$$

The sum of the real parts is equal to the real part of the sum.

$$v(t) = \text{Re}\left[10e^{j\omega t} + 5e^{j\omega t}e^{-j30^\circ} + 5e^{j\omega t}e^{j90^\circ}\right]$$

$$= \text{Re}\left[\underbrace{(10 + 5e^{-j30^\circ} + 5e^{j90^\circ})}_{\text{addition of three complex constants (i.e., phasors)!}} e^{j\omega t}\right] \quad (1)$$

factor out

Complex addition:

$$10 = 10 + j0$$
$$5e^{-j30^\circ} = 5\cos(-30^\circ) + j5\sin(-30^\circ) = 4.33 - j2.5$$
$$5e^{j90^\circ} = 5\cos(90^\circ) + j5\sin(90^\circ) = 0 + j5$$

so $$(10 + j0) + (4.33 - j2.5) + (0 + j5) = 14.33 + j2.5$$

In polar notation:
$$M = \sqrt{14.33^2 + 2.5^2} = 14.54$$
$$\theta = \tan^{-1}(2.5/14.33) = 9.90^\circ$$

Finally, equation (1) simplifies to

$$v(t) = \text{Re}\left[(14.54\,\underline{/9.90^\circ})\, e^{j\omega t}\right] = \text{Re}\left[14.54\, e^{j9.90^\circ} e^{j\omega t}\right]$$

$$= \text{Re}\left[14.54\, e^{j(\omega t + 9.90^\circ)}\right]$$

The final step using Euler's identity: $\text{Re}\left[e^{jx}\right] = \cos(x)$

$$v(t) = 14.54\cos(\omega t + 9.90^\circ)$$ "TIME-DOMAIN EXPRESSION"

<u>Summary of phasor-domain calculation method</u>

1. Express cosine functions as phasors

$$\bar{V}_1 = 10\cos(\omega t) = 10\angle 0^\circ = 10 + j0$$
$$\bar{V}_2 = 5\cos(\omega t - 30^\circ) = 5\angle -30^\circ = 4.33 - j2.5$$
$$\bar{V}_3 = 5\cos(\omega t + 90^\circ) = 5\angle 90^\circ = 0 + j5$$

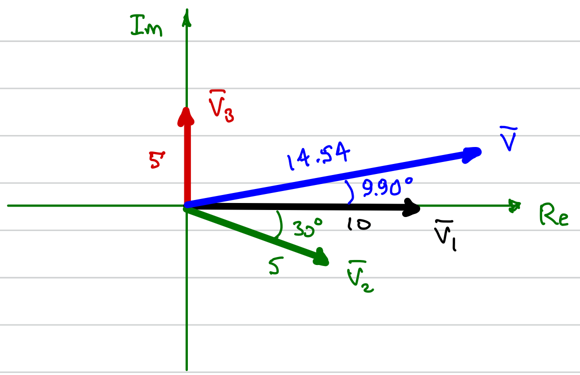
2. Use these phasors in the KVL equation

$$\bar{V} = \bar{V}_1 + \bar{V}_2 + \bar{V}_3 = 14.33 + j2.5$$
$$= 14.54\angle 9.90^\circ$$

3. If required, convert your combined phasor back into the time domain as a cosine function.

$$v(t) = 14.54\cos(\omega t + 9.90^\circ)$$

A <u>phasor diagram</u> is often used to represent the sinusoidal components.



PHASOR DIAGRAM