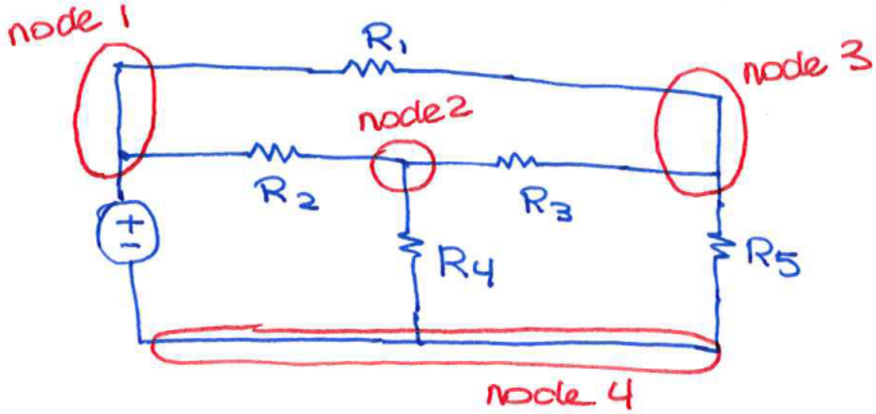
Node Voltage Analysis

- There are limits to what we can do with series/parallel equivalents and voltage/current division:



Nothing in series or parallel!

- the node voltage method allows analysis of <u>any circuit</u>

Steps: 1) Identify nodes & known node voltages (if any)

2) Apply KCL at nodes; Develop a system of equations.

Unknowns: Node Voltages.

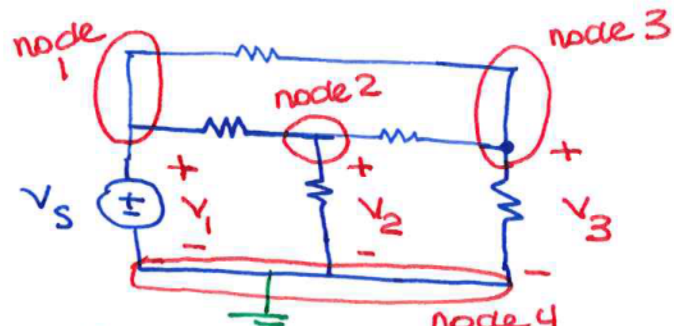
3) Solve system of equations to determine node voltages.

- Let's consider our example circuit:

Step 1: Identify nodes

Node Voltage: Potential difference between a node & the reference node.

e.g. V_1, V_2, V_3



reference node

This is our 0V reference.

We can eliminate one unknown node voltage by selecting one end of a voltage source as the reference.

By selecting node 4 as the reference, V_1 automatically becomes $V_1 = V_S$

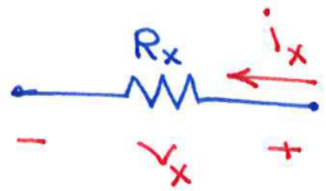
Step 2: Apply KCL at nodes

- New KCL convention: current leaving the node adds,
  current entering the node subtracts.

- For resistors, always assume the current is leaving the node of interest

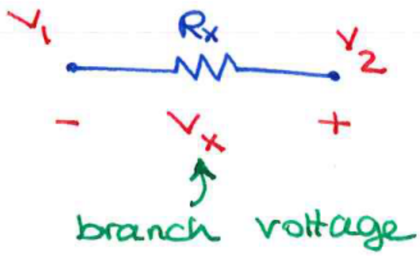
Writing equations for resistor currents in terms of node voltages:

- Node voltage vs. branch voltage:
  - So far, we have labelled branch voltages for circuit elements:



$$v_x = i_x \cdot R$$

  - In node voltage method, node voltages are our only variable so we must express v_x and i_x in terms of node voltages:



v_1 & v_2 are node voltages of nodes 1 & 2 on either side of R_x

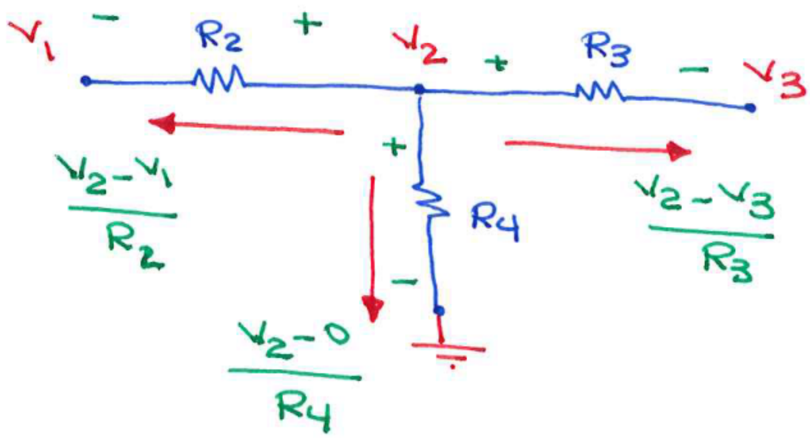
voltage v_2 is labelled as being at a higher potential than v_1 (v_2 is the + terminal of v_x, v_1 is the − terminal) therefore, the branch voltage v_x is the difference between higher (v_2) and lower (v_1) nodes:

$$v_x = v_2 - v_1$$

$$\boxed{\text{branch voltage} = \text{node labelled as } (+) - \text{node labelled as } (-)}$$

Therefore, for resistor above : $i_x = \dfrac{V_2 - V_1}{R}$ current in terms of node voltages

Now, consider node 2 in the original circuit:

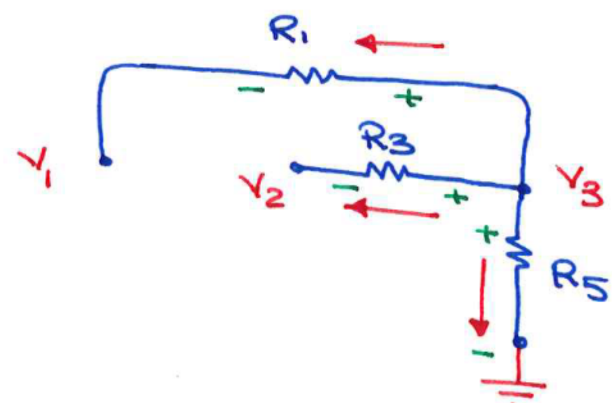


Sum the currents at node 2 by KCL:

$$\frac{V_2 - V_1}{R_2} + \frac{V_2}{R_4} + \frac{V_2 - V_3}{R_3} = 0$$

- In summary, at each node n, calculate the outgoing current to node k through R_m is $\boxed{\dfrac{V_n - V_k}{R_m}}$

- Now, <u>forget about node 2</u> and repeat for node 3:

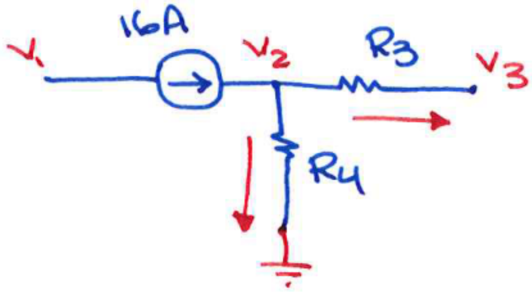


$$\frac{V_3 - V_1}{R_1} + \frac{V_3 - V_2}{R_3} + \frac{V_3}{R_5} = 0$$

Node eq for node 3

Things other than resistors connected to nodes

1)



Node eq at node 2:

node of interest comes first

subtract connecting node

$$\frac{v_2 - v_3}{R_3} + \frac{v_2 - 0}{R_4} - 16 = 0$$

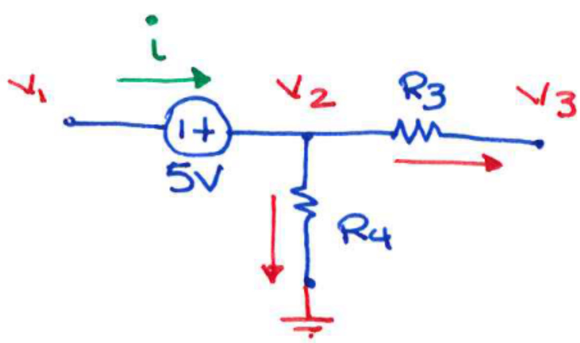
connecting R

a constant

(−) since entering the node.

i.e. current sources are ok!

2)



Node eq at 2:

$$\frac{v_2 - v_3}{R_3} + \frac{v_2}{R_4} - i = 0$$

another unknown if we must write an eq at node 2. (Help is coming!)

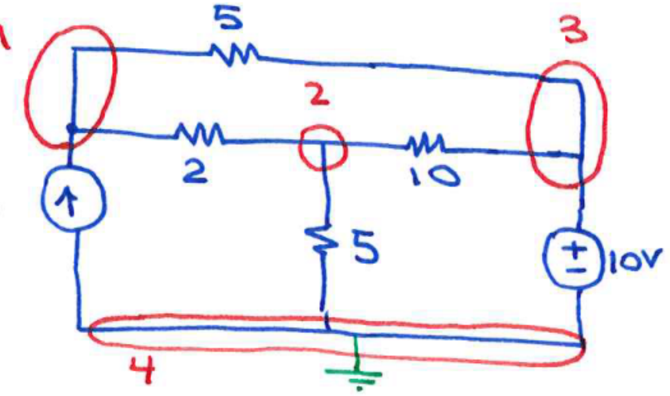
. we can't express i in terms of node voltages.

Step 3: Solve sys of equations

- we now have 2 equations for our example circuit and 2 unknown node voltages v_2 & v_3.

  $v_1 = v_s$ is already known.

Ex: Find node voltages



Looks like 3 nodes + 1 ref node

We can get V_3 immediately:

$$V_3 = 10\ V$$

if we select node 4 as the ref node.
(if we select node 3 as the ref node, then $V_4 = -10\ V$)

KCL at node 1: $$\frac{V_1 - \overset{10V}{V_3}}{5} + \frac{V_1 - V_2}{2} - 1 = 0 \qquad \text{(i)}$$

1A into the node ∴ subtract

KCL at node 2: $$\frac{V_2 - V_1}{2} + \frac{V_2 - 0}{5} + \frac{V_2 - \overset{10V}{V_3}}{10} = 0 \qquad \text{(ii)}$$

- Don't write KCL at nodes 3 or 4 when solving for node voltages. because we don't know the current through 10V source.

- can simplify (i) and (ii):

(i) x10 : $$2V_1 - 20 + 5V_1 - 5V_2 - 10 = 0$$
$$7V_1 - 5V_2 = 30 \qquad \text{(iii)}$$

(ii) x 10 : $$5V_2 - 5V_1 + 2V_2 + V_2 - 10 = 0$$
$$-5V_1 + 8V_2 = 10 \qquad \text{(iv)}$$

Solve (iii) and (iv) with the method of your choosing to find V_1, V_2.

e.g.:

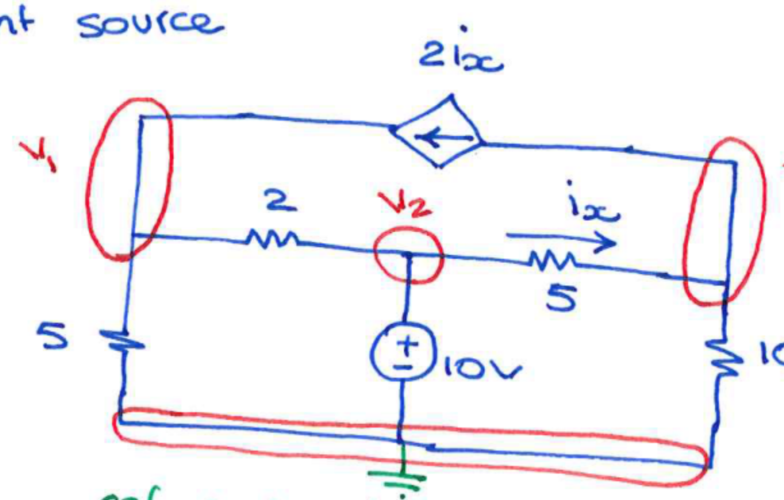
(iii) x 8 : $56V_1 - \cancel{40V_2} = 240$

(iv) x 5 : $-25V_1 + \cancel{40V_2} = 50$ +

$31V_1 = 290$ ∴ $V_1 = 9.35\ V$

sub into (iii) to get $V_2 = 7.1\ V$

Ex 2: with dependent current source



with this choice of ref node:

$V_2 = 10$ V

ref node. (Connected to Voltage Src)

KCL at node 1:

$$\frac{V_1 - 0}{5} + \frac{V_1 - \overset{10V}{V_2}}{2} - 2i_x = 0 \qquad (i)$$

current from dep source going into the node

KCL at node 3:

voltage at node of interest

voltage of adj node

$$\frac{V_3 - 0}{10} + \underbrace{\frac{V_3 - \overset{10V}{V_2}}{5}}_{} + 2i_x = 0 \qquad (ii)$$

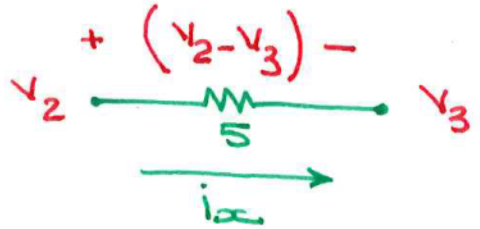
connecting R

ignored i_x when writing this term.

Want to keep eq's in terms of node voltages.

> For dependent sources, we always want to write the controlling element in terms of unknowns (node voltages)

In this case, we need to write i_x in terms of node voltages:



Ohm's Law:

$$i_x = \frac{V_2 - V_3}{5} = \frac{10 - V_3}{5}$$

Plug this into (i) and simplify : $7V_1 + 4V_3 = 90$

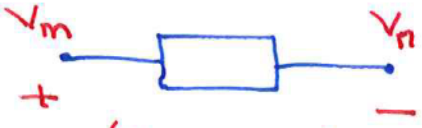
also, plug it into (ii) and simplify : $V_3 = 20$ V

can now find $V_1 = \frac{10}{7}$ V

• How do we use node voltages to find voltage & current for individual elements?

Voltages across an element ("branch voltage")

• Choose a polarity, subtract node voltages accordingly.

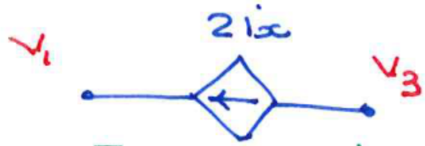


$(v_m - v_n)$

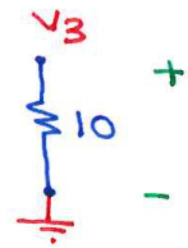
node labelled as +

node labelled as −

• For example, in previous circuit:



$v_3 - v_1 = 18.6\ V$



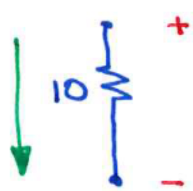
$v_3 - 0 = 20\ V$

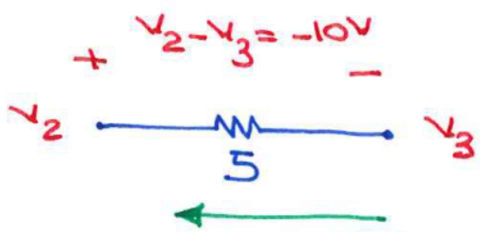
Current through resistors

• Choose a direction, apply Ohm's Law accordingly

For example,

$i = \frac{20V}{10\Omega}$





$i = - \frac{-10V}{5\Omega} = 2A$

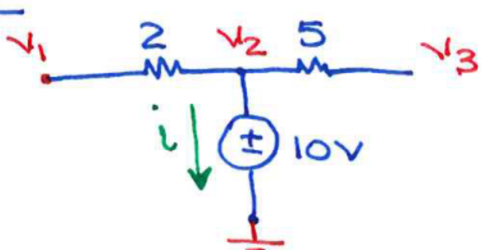
current sees − terminal first

Current through a voltage source

• Use KCL

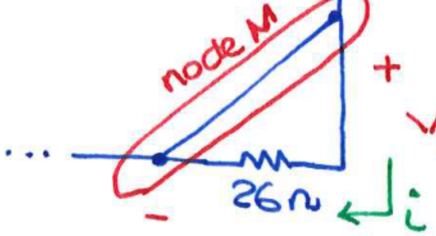
In our example:

$$\frac{v_2 - v_1}{2} + \frac{v_2 - v_3}{5} + i = 0$$



the only unknown is i. We can solve this to find i through 10V source aside:

- From Assignment 2 :

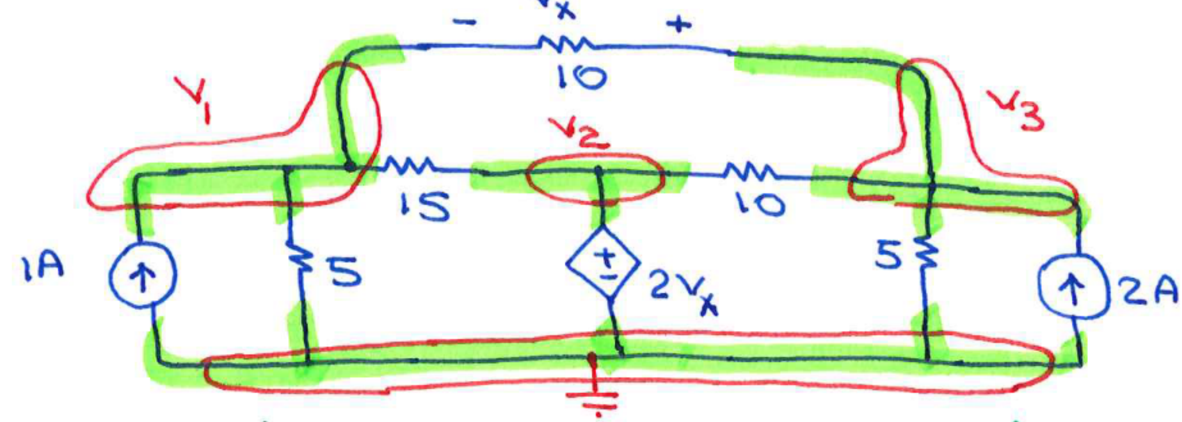


$V_M - V_M = 0$

$$i = \frac{V_M - V_M}{0} = 0 \text{ A}$$

26Ω resistor is connected to the same node at both ends. It is "shorted out" so we can remove it from our analysis

Ex 3 : Find node voltages



chose one terminal of the voltage source as ref.
Doesn't matter that it's dependent.

Because of ref node; $V_2 = 2V_x$

but we also know :

$$V_x = V_3 - V_1$$

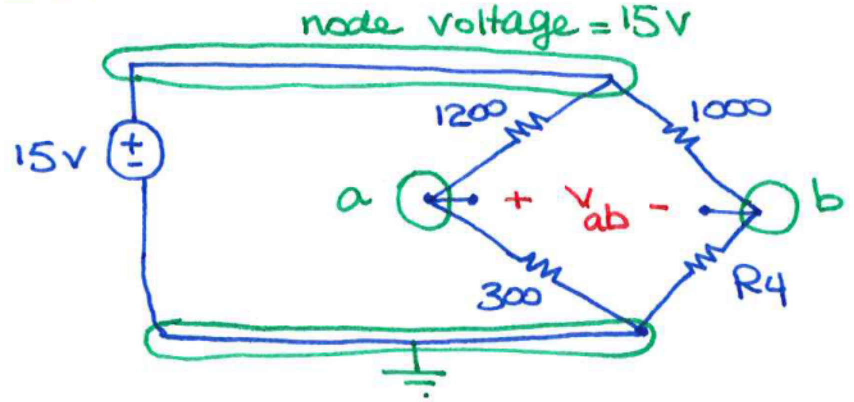
$$\therefore \quad V_2 = 2(V_3 - V_1) \qquad (i)$$

KCL at node 1 : $-1 + \underbrace{\frac{V_1 - V_3}{10}}_{\text{ignored } V_x \text{ when writing this}} + \frac{V_1 - V_2}{15} + \frac{V_1 - 0}{5} = 0$ (ii)

KCL at node 3 : $-2 + \frac{V_3 - V_1}{10} + \frac{V_3 - V_2}{10} + \frac{V_3 - 0}{5} = 0$ (iii)

Solve (i), (ii), (iii) to get $V_1 = 5.4$ V, $V_2 = 3.8$ V, $V_3 = 7.3$ V

Ex: Wheatstone Bridge. Find R_4 such that $V_{ab} = 0$.



Node eq (KCL) at a:

$$\frac{V_a - 15}{1200} + \frac{V_a - 0}{300} = 0$$

$$\therefore V_a = 3 \text{ V}$$

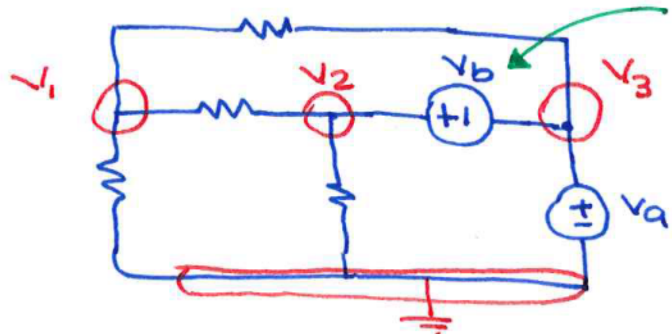
For $V_{ab} = 0$, $V_a = V_b \quad \therefore V_b = 3\text{ V}$

Node eq at b: $\dfrac{\overset{3}{V_b} - 15}{1000} + \dfrac{\overset{3}{V_b} - 0}{R_4} = 0 \qquad \therefore R_4 = 250\,\Omega$

Node Voltage Method - Special case

. There is only one special case to handle when we have a voltage source connected between two regular (non-ref) nodes

. For the easy case where voltage sources are directly connected to other voltage sources:



voltage source between reg nodes

$$V_3 = V_a$$

but we also know that

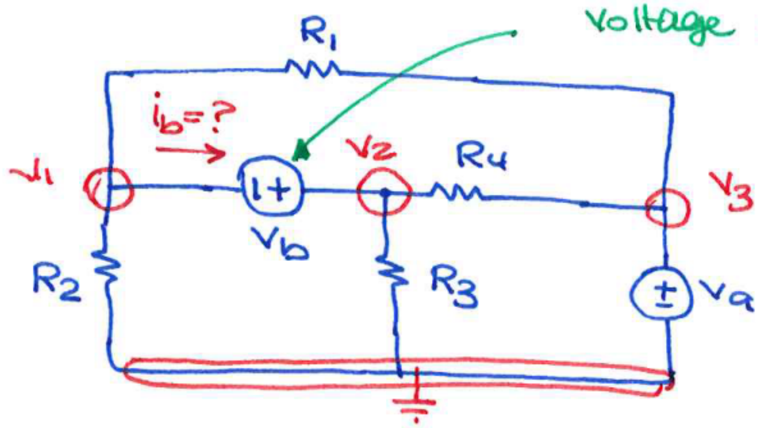
$$V_2 - V_3 = V_b$$

$$\therefore V_2 = V_3 + V_b = V_a + V_b$$

i.e. V_2 & V_3 are known immediately. Leaves one unknown: V_1

Write KCL at node 1 to find V_1

. In the trickier case, voltage sources are not directly connected to each other:



From choice of ref node:

$V_3 = V_a$

~~$V_2 = V_b$~~

Recall that in writing node equations, we sum currents leaving the nodes.

another unknown along with V_1, V_2

At node 1: $\dfrac{V_1 - V_3}{R_1} + \dfrac{V_1 - 0}{R_2} + i_b = 0$ (i)

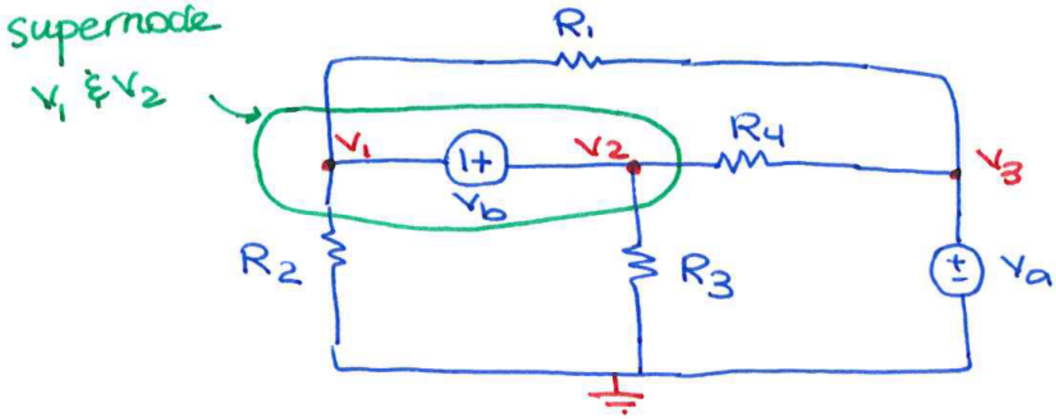
At node 2: $\dfrac{V_2 - V_3}{R_4} + \dfrac{V_2 - 0}{R_3} - i_b = 0$ (ii)

Now, let's eliminate i_b by adding (i) and (ii):

$$\frac{V_1 - V_3}{R_1} + \frac{V_1}{R_2} + \cancel{i_b} + \frac{V_2 - V_3}{R_4} + \frac{V_2}{R_3} - \cancel{i_b} = 0 \qquad \text{(iii)}$$

now have 1 eq & 2 unknowns V_1 & V_2

This problem is handled by the concept of supernode:



$V_3 = V_a$

i) KCL for all currents leaving supernode (green box):

$$\underbrace{\frac{V_1 - V_3}{R_1} + \frac{V_1 - 0}{R_2}}_{\text{left side (Node 1)}} + \underbrace{\frac{V_2 - V_3}{R_4} + \frac{V_2 - 0}{R_3}}_{\text{right side (Node 2)}} = 0$$

i.e. arrived at eq (iii) in one shot.

ii) A <u>dependence equation</u> for the two nodes within the supernode:

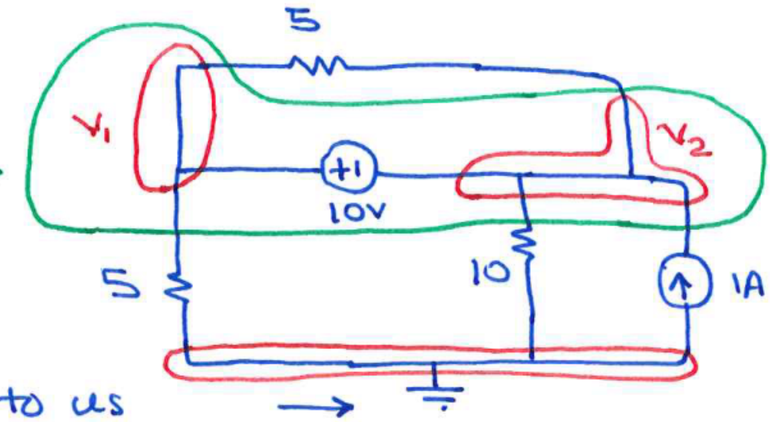
$$V_2 - V_1 = V_b$$

. This results in 2 equations (supernode KCL & dep eq'n) to solve for 2 unknowns V_1 & V_2.

Ex: Find Node voltages

supernode V_1 & V_2.

Voltage source between 2 regular nodes.



ref node given to us
(bad choice for ref node!)

supernode KCL: $$\underbrace{\cancel{\frac{V_1-V_2}{5}} + \frac{V_1-0}{5}}_{\text{node 1 side}} + \underbrace{\cancel{\frac{V_2-V_1}{5}} + \frac{V_2-0}{10}}_{\text{node 2 side}} - 1 = 0$$

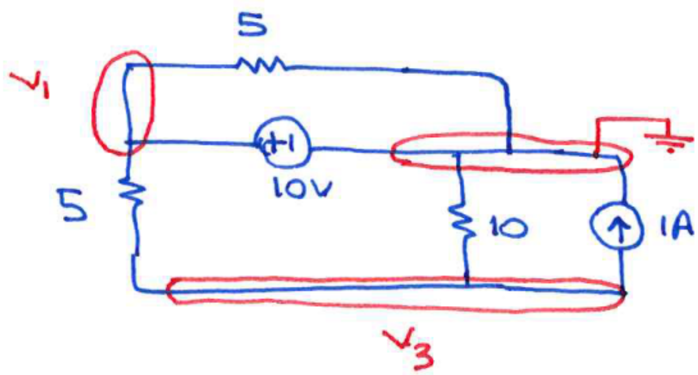
$\times 10$: $\quad 2V_1 + V_2 = 10 \qquad$ (i)

Dependence Eq: $\quad V_1 - V_2 = 10 \qquad$ (ii)

the (+) terminal $\quad$ the (−) terminal

solving (i) & (ii) gives: $V_1 = 6.\overline{6}$ V, $V_2 = -3.3$ V

. what if we used a different ref node?



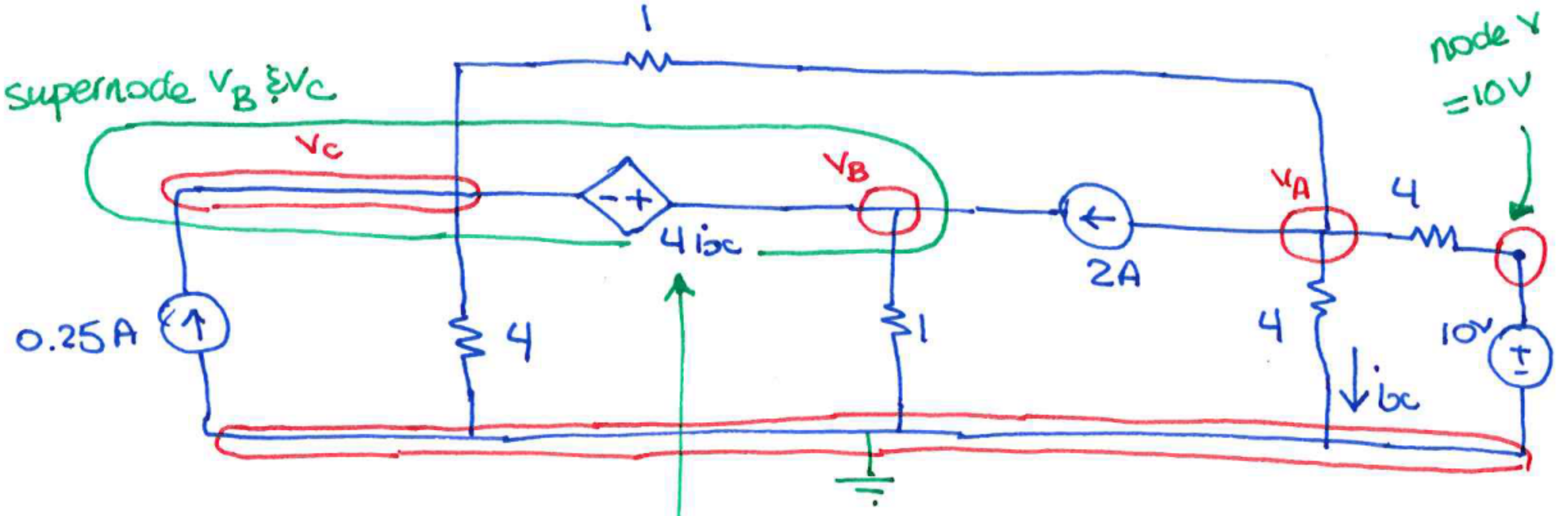
$V_1 = 10$ V

One unknown node voltage, V_3. Write KCL at node 3 to solve for V_3

$V_3 = 3.\overline{3}$ V

. Node voltages change based on ref node location <u>but</u> voltages & currents for individual elements remain the same.

Ex: 2014 MT. Determine node voltages



CCVS (voltage source). Don't know the current thru it.

KCL at node A: $\dfrac{V_A - 10}{4} + \underbrace{\dfrac{V_A - 0}{4}}_{\text{this is } i_x} + \dfrac{V_A - V_C}{1} + 2 = 0$

simplify: $6V_A - 4V_C = 2$ (i)

Supernode KCL: $\underbrace{-0.25 + \dfrac{V_C - 0}{4} + \dfrac{V_C - V_A}{1}}_{\text{node C side}} \underbrace{- 2 + \dfrac{V_B - 0}{1}}_{\text{node B side}} = 0$

simplify: $-4V_A + 4V_B + 5V_C = 9$ (ii)

Dep Eq'n from supernode: $V_B - V_C = 4 i_x$

but we also know $i_x = \dfrac{V_A}{4}$

$\therefore V_B - V_C = V_A$ or $-V_A + V_B - V_C = 0$ (iii)

Solve (i), (ii), (iii). e.g (ii) + $-4 \times$ (iii) gives $9V_C = 9$

$\therefore V_C = 1$

then from (i), $V_A = 1$ V

from (iii), $V_B = 2$ V