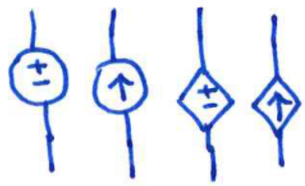
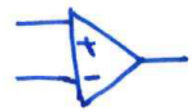
Inductors & Capacitors

. So far, we have considered the following circuit elements:



sources

resistor



op amp

Inductors & capacitors are dependent on electromagnetic fields:

- capacitor: separation of charge produces electric field
- Inductor: Motion of charge produces magnetic field

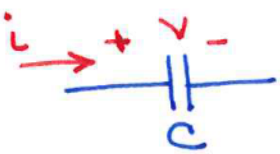
. Unlike resistors, these devices can <u>store</u> energy & <u>return</u> stored energy, but are not producers of energy.

Capacitor

. The circuit symbol is  or 

C — circuit parameter for capacitance

. Voltage - current relationship :



$$\boxed{i(t) = C \frac{dv(t)}{dt}}$$

v: Voltages in Volts (V)

i: Current in Amps (A)

t: time in seconds (s)

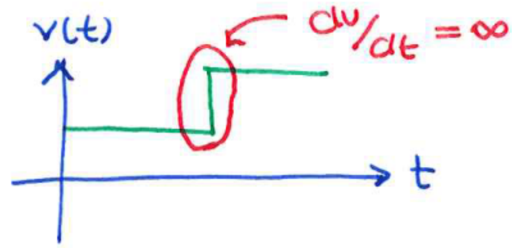
C: Capacitance in Farads (F)

Capacitor Properties

1) Constant voltage across capacitor terminals results in zero current flow. Capacitor looks like open circuit in DC

$$i(t) = C\frac{dV(t)}{dt} = 0 \quad \text{when } V(t) \text{ is constant, as in DC}$$

2) Voltage cannot change instantaneously; current would be infinite



capacitor won't allow this as it would require ∞ current

. Capacitor voltage in terms of current:

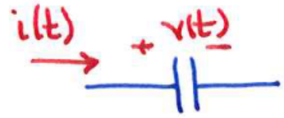
$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau + v(t_0)$$

τ is just an integration variable. The current will be expressed as a function of time, $i(t)$. In the integral above, we are using t as the end point so we will write current as a function of τ for the integral only.

e.g: $i(t) = 10t$ then $v(t) = \frac{1}{C}\int_{t_0}^{t} 10\tau \cdot d\tau + v(t_0)$

More on this later!

Power & Energy in capacitor



. We can write an expression for power:

$$p(t) = v(t) \cdot i(t)$$

if current is labelled in dir of voltage drop (sees + terminal first)

$$= v(t) \cdot C \cdot \frac{dv(t)}{dt}$$

$$= i(t) \left[\frac{1}{C} \int_{t_0}^{t} i(\tau) \cdot d\tau + v(t_0) \right]$$

. For energy, recall $p(t) = \dfrac{dw(t)}{dt}$

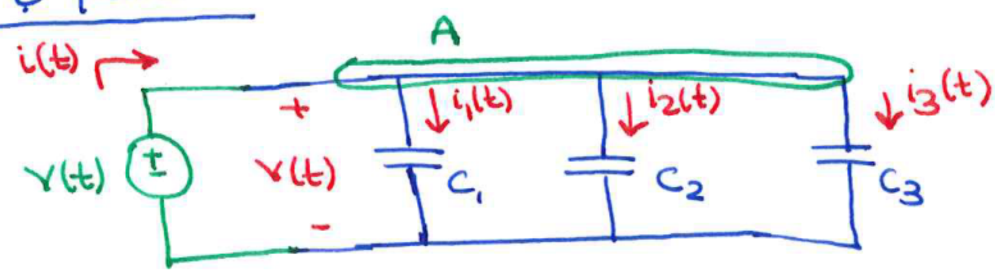
for capacitors. $p(t) = \dfrac{dw(t)}{dt} = C \cdot v(t) \cdot \dfrac{dv(t)}{dt}$

$$\therefore \quad dw(t) = C \cdot v(t) \cdot dv(t)$$

integrate both sides : $\boxed{w(t) = \frac{1}{2} C\, v(t)^2}$

Capacitors in Series & parallel

. In parallel:

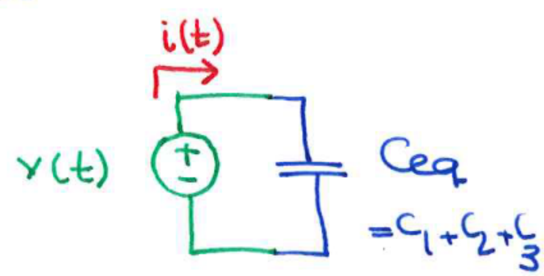


KCL at node A:

$$i = i_1(t) + i_2(t) + i_3(t) = C_1 \frac{dv(t)}{dt} + C_2 \frac{dv(t)}{dt} + C_3 \frac{dv(t)}{dt}$$

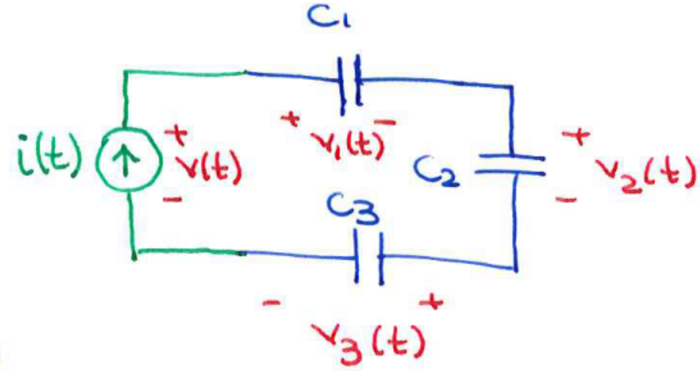
$$= (C_1 + C_2 + C_3) \frac{dv(t)}{dt}$$

$$= C_{eq} \cdot \frac{dv(t)}{dt}$$



We can write: $\boxed{C_{eq} = C_1 + C_2 + C_3}$ caps in parallel add

. In Series :



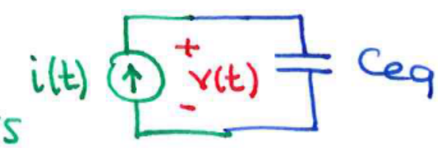
From KVL :

$$v(t) = v_1(t) + v_2(t) + v_3(t)$$

$$= \frac{1}{C_1}\int_0^t i(\tau)d\tau + \frac{1}{C_2}\int_0^t i(\tau)d\tau + \frac{1}{C_3}\int_0^t i(\tau)d\tau$$

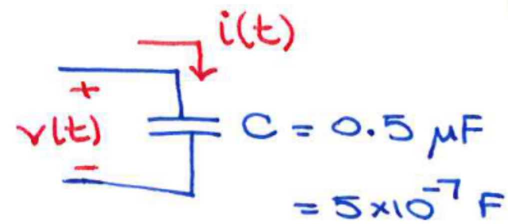
$$= \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)\int_0^t i(\tau)d\tau = \frac{1}{C_{eq}}\int_0^t i(\tau)d\tau$$

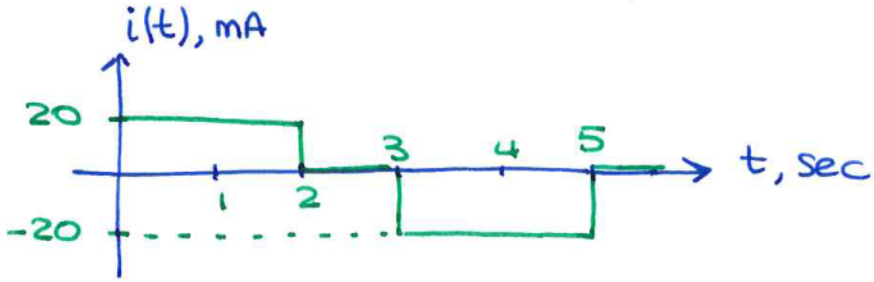
$$\therefore \boxed{C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)^{-1}}$$



Add series caps like parallel resistors

Ex: Find cap voltage for given current. Assume no initial voltage on capacitor.





For a capacitor, $v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau + v(t_0)$

For this capacitor, $i(t) = \begin{cases} 20\ \text{mA} & 0 < t < 2 \\ 0 & 2 < t < 3 \\ -20\ \text{mA} & 3 < t < 5 \\ 0 & t > 5 \end{cases}$

$0 < t < 2$: $\quad i(t) = 0.02$

$$v(t) = \frac{1}{5\times10^{-7}} \int_0^t (0.02)\, d\tau + \overset{0}{\cancel{v(0)}}$$

(Given.)

start of interval

$$= \frac{0.02}{5\times10^{-7}} \cdot \tau \Big|_0^t = 40\,000\,(t-0) = 40\,000\,t$$

$2 < t < 3$: $\quad i(t) = 0$

$$v(t) = \overset{0}{\cancel{\frac{1}{5\times10^{-7}} \int_2^t 0 \cdot d\tau}} + \overbrace{\underbrace{v(2)}}$$

voltage across cap at start of this interval

from previous interval:

$v(2) = 40\,000 \times 2 = 80\,000\ \text{V}$

$\therefore\ v(t) = 80\,000\ \text{V}$

$3 < t < 5$ $\quad i(t) = -0.02$

voltage at the end of last interval $= 80000\ \text{V}$

$$v(t) = \frac{1}{5\times10^{-7}} \int_3^t (-0.02)\, d\tau + v(3)$$

$$= \frac{-0.02}{5\times10^{-7}}\ \tau \Big|_3^t + 80\,000$$

$$= -40\,000\,(t-3) + 80\,000$$

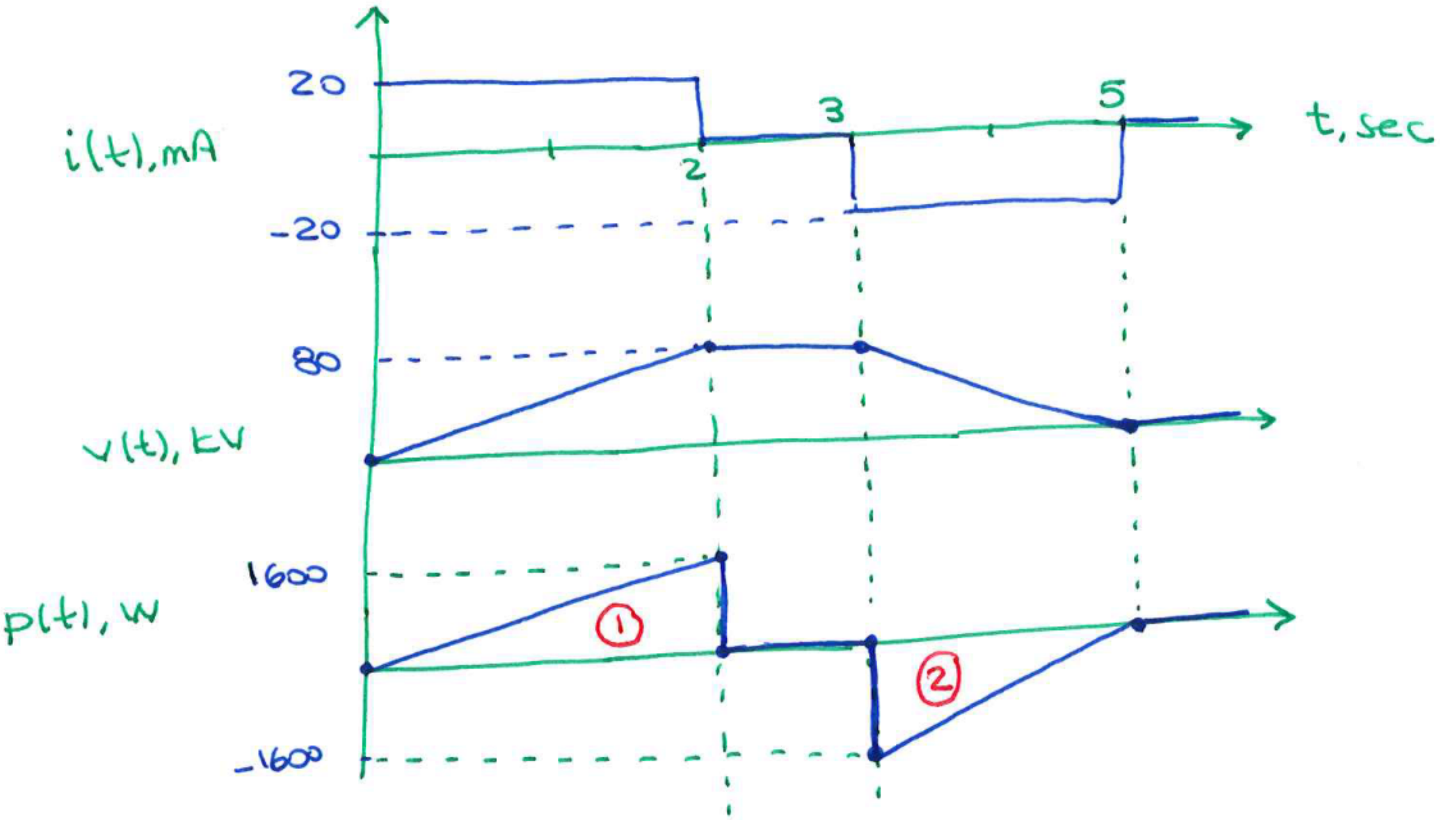
$$= -40\,000\,t + 200\,000$$

$t > 5$ $\quad i(t) = 0$

$v(t)$ remains constant at initial value of $v(5)$

From prev. interval, $v(5) = -40\,000 \times 5 + 200\,000$
$= 0$

Let's sketch $v(t)$, $p(t)$



for $\underline{0<t<2}$: $p(t) = v(t).i(t) = 0.02 \times 40000t = 800t$

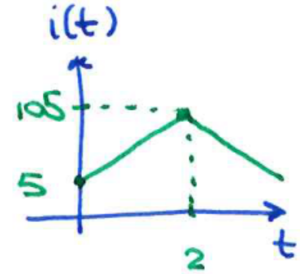
$\underline{3<t<5}$: $p(t) = v(t).i(t) = -0.02\,(-40000t + 200000)$

$= 800t - 4000$

interval ① : $p(t) > 0$ ∴ capacitor is absorbing power during $0<t<2$; i.e. it is storing power

interval ② $p(t) < 0$ ∴ capacitor is supplying power during $3<t<5$; i.e. it is returning stored power

· What if $i(t)$ is a 1^{st} order function? e.g:



To write an expression for $v(t)$, we need an expression for $i(t)$:

$$i(t) = mt + b$$

slope — i(t) axis intercept

use 2 points from each interval (e.g start & end points) to find m & b for that interval

· Suppose $i(t) = 50t + 5$ for $0 < t < 2$, then:

$$v(t) = \frac{1}{c}\int_0^t (50\tau + 5)\,d\tau + v(0)$$

$$= \frac{1}{c}\left(\frac{50\tau^2}{2} + 5\tau\right)\Bigg|_{\tau=0}^{\tau=t} + v(0)$$

$$= \frac{1}{c}\left(25t^2 + 5t\right) + v(0)$$

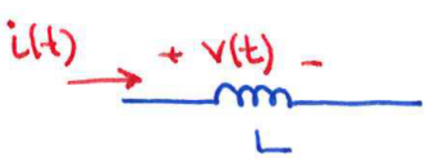
Inductor

· The circuit symbol is

← circuit parameter for inductor

· Voltage-current relationship is:

$$\boxed{v(t) = L\,\frac{di(t)}{dt}}$$



where:

V: voltage in Volts (V)

L: inductance in Henrys (H)

i: current in Amps (A)

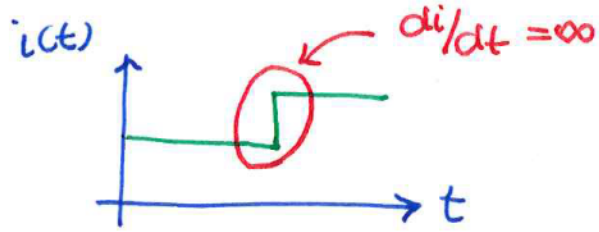
t: time in seconds (s)

- Inductor Properties:

1) A constant current creates zero voltage drop. So, the inductor looks like a short circuit in DC.

$$v(t) = L \underbrace{\frac{di(t)}{dt}}_{=0 \text{ for constant } i}$$

2) Current through an inductor cannot change instantaneously. This would produce an infinite voltage



- Inductor current in terms of voltage:

$$\boxed{i(t) = \frac{1}{L}\int_{t_0}^{t} v(\tau)\,d\tau + i(t_0)}$$

Power & Energy in Inductors

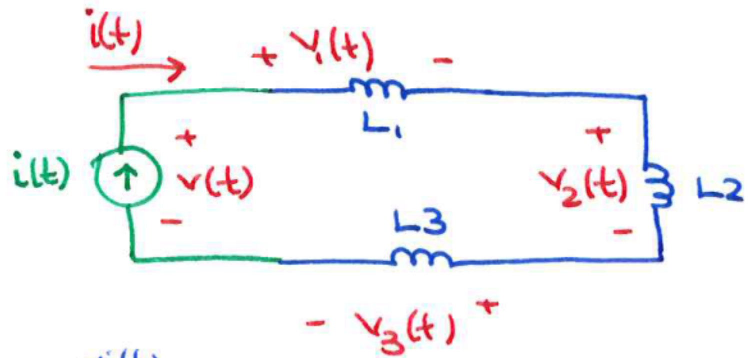
$$p(t) = v(t) \cdot i(t) = L\frac{di(t)}{dt} \cdot i(t)$$

defined as

Also $p \triangleq \dfrac{dw}{dt}$ $\quad\therefore\quad$ $dw = L \cdot di(t) \cdot i(t)$

integrate both sides : $\boxed{w(t) = \frac{1}{2} L\, i(t)^2}$

Inductors in series

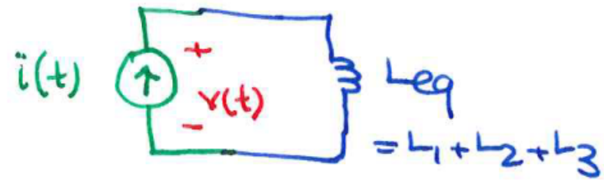


KVL:

$$v(t) = v_1(t) + v_2(t) + v_3(t)$$

$$= L_1 \frac{di(t)}{dt} + L_2 \frac{di(t)}{dt} + L_3 \frac{di(t)}{dt}$$

$$= (L_1 + L_2 + L_3)\frac{di(t)}{dt}$$

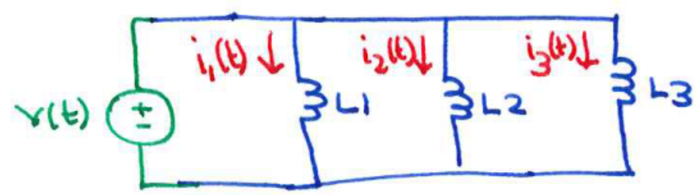


$$\therefore \boxed{L_{eq} = L_1 + L_2 + L_3}$$

inductors in series add

Inductors in parallel

- Using KCL & $i_L(t) = \frac{1}{L}\int_{t_0}^{t} v(\tau)d\tau$

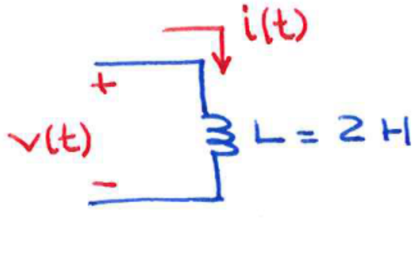


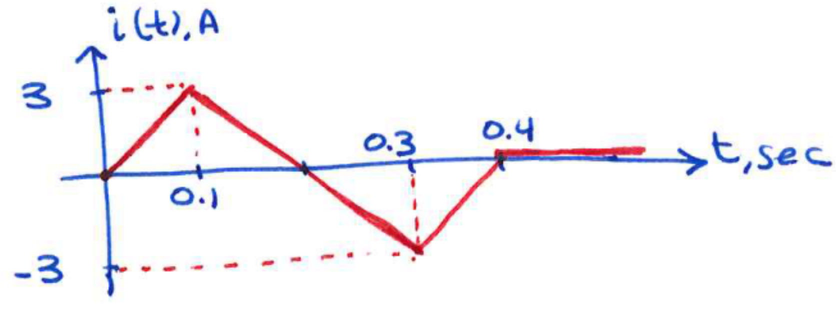
we can show that:

$$\boxed{L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)^{-1}}$$

similar to parallel resistors

Ex: Given $i(t)$ below, find $v(t)$, $p(t)$, $w(t)$





For an inductor, $v(t) = L \dfrac{di(t)}{dt}$

<u>$0 < t < 0.1$</u>

$i(t) = m.t + b$

slope   intercept

$b = 0$ can be seen from the graph.

$$m = \text{slope} = \frac{\Delta i}{\Delta t} = \frac{3-0}{0.1-0} = 30$$

$\therefore \quad i(t) = 30t \quad A$

$v(t) = L \times 30 = 60 \quad V$

$p(t) = v(t).\,i(t) = 60 \times 30t = 1800t$

$w(t) = \frac{1}{2} L\, i(t)^2 = \frac{1}{2} \times 2 \times (30t)^2 = 900t^2$

<u>$0.1 < t < 0.3$</u>

$i(t) = m.t + b$

$$m = \frac{\Delta i}{\Delta t} = \frac{-3-3}{0.3-0.1} = -30 \qquad \therefore\ i(t) = -30t + b$$

use any point from the interval to find b:

using $i(0.2) = 0$ we can get $b = 6$

$\therefore \quad i(t) = -30t + 6 \quad A$

$v(t) = L \times -30 = -60 \quad V$

$p(t) = v(t).\,i(t) = 1800t - 360 \quad W$

$w(t) = \frac{1}{\cancel{2}} (\cancel{2}) (-30t + 6)^2$

$0.3 < t < 0.4$ $\quad m = 30$ again $\quad \therefore i(t) = 30t + b$

using $i(0.4) = 0$ we can get $b = -12$

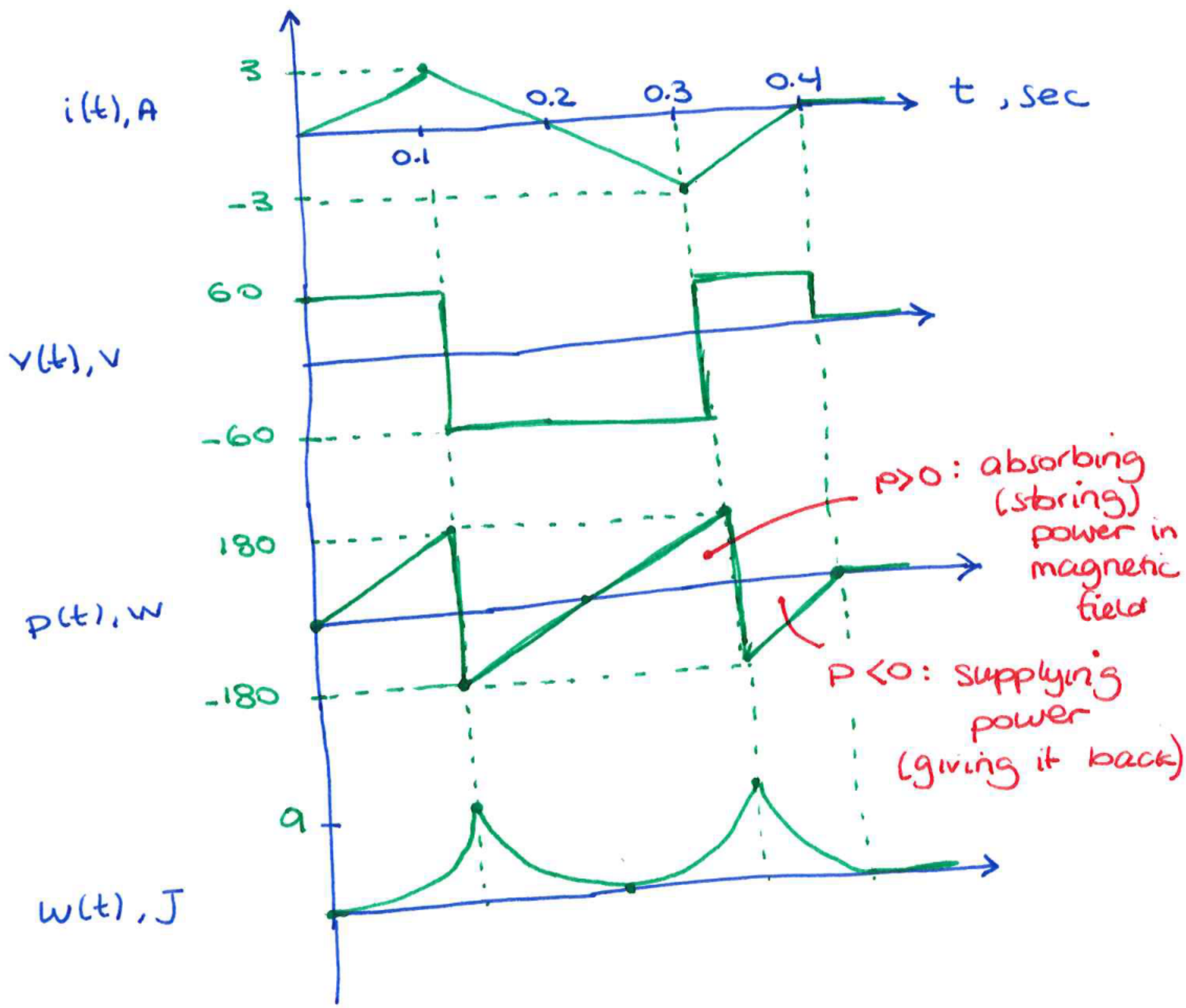
$$\therefore \; i(t) = 30t - 12$$

$$v(t) = L \times 30 = 60 \text{ V}$$

$$p(t) = v(t) \cdot i(t) = 1800t - 720 \text{ W}$$

$$w(t) = \frac{1}{\cancel{2}}(\cancel{2})(30t - 12)^2$$

$t > 0.4$ $\quad i(t) = 0 \quad \therefore \; v(t) = p(t) = w(t) = 0$



Steady-State Sinusoidal Analysis

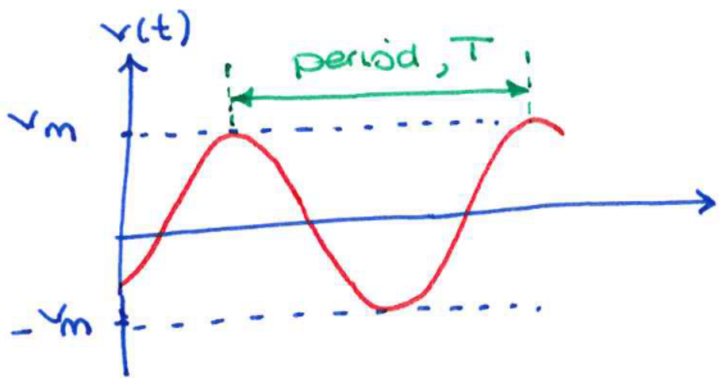
- So far, we have investigated circuits with DC sources. We now investigate circuits where sources deliver sinusoidal (AC) voltages & currents.

Sinusoidal voltages & currents

phase angle (radians)

Let $v(t) = \underbrace{V_m}_{\text{peak}} \cos(\omega t + \theta_v)$

ω: angular frequency (radians/sec)



This sinusoid is periodic with period T. We have one complete period when phase angle changes by 2π.

- Frequency is defined as the number of complete periods (cycles) per second $\quad f = \frac{1}{T}$ frequency in Hertz (Hz)

We also have $\omega = 2\pi f$ (rad/s)

- We will cosine to express voltages & currents, not sine. They are related by:

$$\sin(\omega t) = \cos(\omega t - \pi/2)$$
$$= \cos(\omega t - 90^\circ)$$

We say sine has a phase angle of -90°