J. Math. Anal. Appl. 552 (2025) 129796

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Preservers of operator commutativity

Gerardo M. Escolano [a], Antonio M. Peralta [a,b,*], Armando R. Villena [a,b]

[a] *Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain*
[b] *Instituto de Matemáticas de la Universidad de Granada (IMAG), Spain*

A R T I C L E I N F O

Article history:
Received 10 September 2024
Available online 11 June 2025
Submitted by M. Mathieu

Keywords:
Preservers
Jordan algebras
JBW*-algebras
Associating traces and linear maps
Functional identities
Preservers of operator commutativity

A B S T R A C T

Let $\mathfrak{M}$ and $\mathfrak{J}$ be JBW*-algebras admitting no central summands of type I_1 and I_2, and let $\Phi : \mathfrak{M} \to \mathfrak{J}$ be a linear bijection preserving operator commutativity in both directions, that is,

$$[x, \mathfrak{M}, y] = 0 \Leftrightarrow [\Phi(x), \mathfrak{J}, \Phi(y)] = 0,$$

for all $x, y \in \mathfrak{M}$, where the associator of three elements a, b, c in $\mathfrak{M}$ is defined by $[a, b, c] := (a \circ b) \circ c - (c \circ b) \circ a$. We prove that under these conditions there exist a unique invertible central element z_0 in $\mathfrak{J}$, a unique Jordan isomorphism $J : \mathfrak{M} \to \mathfrak{J}$, and a unique linear mapping β from $\mathfrak{M}$ to the centre of $\mathfrak{J}$ satisfying

$$\Phi(x) = z_0 \circ J(x) + \beta(x),$$

for all $x \in \mathfrak{M}$. Furthermore, if Φ is a symmetric mapping (i.e., $\Phi(x^*) = \Phi(x)^*$ for all $x \in \mathfrak{M}$), the element z_0 is self-adjoint, J is a Jordan *-isomorphism, and β is a *-symmetric mapping too.

In case that $\mathfrak{J}$ is a JBW*-algebra admitting no central summands of type I_1, we also address the problem of describing the form of all symmetric bilinear mappings $B : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$ whose trace is associating (i.e., $[B(a, a), b, a] = 0$, for all $a, b \in \mathfrak{J}$) providing a complete solution to it. We also determine the form of all associating linear maps on $\mathfrak{J}$.

1. Introduction

In the mathematical formulation of quantum mechanics, (bounded) observables are given by self-adjoint operators on a complex Hilbert space H (cf., for example, [29, page 75], [19]). Some pairs of quantum observables may not be simultaneously measurable, a property referred to as complementarity, what is

* Corresponding author.
E-mail addresses: gemares@ugr.es (G.M. Escolano), aperalta@ugr.es (A.M. Peralta), avillena@ugr.es (A.R. Villena).

https://doi.org/10.1016/j.jmaa.2025.129796

mathematically expressed by non-commutativity of their corresponding self-adjoint operators. Perhaps for this reason, linear maps preserving commutativity are among the most extensively studied operators in the setting of preserver problems on associative algebras (see, for example, [4,7,8,13,15,32,34,36]). The contribution by M. Brešar and C.R. Miers in [13] describing the linear bijections between von Neumann algebras preserving commutativity in both senses constitutes one of the most influencing achievements in this line. The main result in [13] shows that if M and N are von Neumann algebras with no central summands of type I_1 or I_2, and $\Theta : M \to N$ is a bijective additive map which preserves commutativity in both directions, then there exist a central invertible element $c \in N$, a Jordan isomorphism $J : M \to N$, and an additive map f from M to the centre of N, such that

$$\Theta(x) = cJ(x) + f(x), \text{ for all } x \in M.$$

It is additionally proved that if Θ is a symmetric mapping (i.e., $\Theta(x^*) = \Theta(x)^*$, $x \in M$), then J is a Jordan *-isomorphism.

In words of Alfsen and Shultz [1, Introduction of chapter 6] "*When a C*-algebra or a von Neumann algebra is used as an algebraic model of quantum mechanics, then it is only the self-adjoint part of the algebra that represents observables. However, the self-adjoint part of such an algebra is not closed under the given associative product, but only under the Jordan product* $a \circ b = \frac{1}{2}(ab + ba)$. *Therefore it has been proposed to model quantum mechanics on Jordan algebras rather than associative algebras [26,41].*" Jordan norm-closed self-adjoint subalgebras of C*-algebras are called special *JB*-algebras* or *JC*-algebras*. There are examples of Jordan algebras, called *exceptional Jordan algebras*, which cannot be embedded as Jordan subalgebras of associative algebras, an example is given by the Jordan algebra $H_3(\mathbb{O})$, of all Hermitian 3×3 matrices with entries in the complex octonions (cf. [21, Corollary 2.8.5]).

A real or complex Jordan-Banach algebra is a Banach space $\mathfrak{J}$ equipped with a bilinear mapping $(a, b) \to a \circ b$ (called Jordan product) satisfying the following axioms:

(J1) $\|a \circ b\| \leq \|a\|\|b\|$, for all $a, b \in \mathfrak{J}$;
(J2) $a \circ b = b \circ a$, for all $a, b \in \mathfrak{J}$ (commutativity);
(J3) $(a^2 \circ b) \circ a = (a \circ b) \circ a^2$, for all $a, b \in \mathfrak{J}$ (Jordan identity).

There are two closely related types of Jordan-Banach algebras, JB-algebras and JB*-algebras. A *JB-algebra* is a real Jordan-Banach algebra $\mathfrak{J}$ in which the norm satisfies the following two additional conditions:

(JB1) $\|a^2\| = \|a\|^2$, for all $a \in \mathfrak{J}$;
(JB2) $\|a^2\| \leq \|a^2 + b^2\|$, for all $a, b \in \mathfrak{J}$.

If $\mathfrak{J}$ is unital, it is clear from (JB1) that $\|\mathbf{1}\| = 1$.

The Jordan-model analogue to C*-algebras, introduced by I. Kaplansky in 1976, is known as JB*-algebra. A complex Jordan-Banach algebra $\mathfrak{J}$ equipped with an involution * is said to be a *JB*-algebra* if the following condition holds:

(JB*1) $\|a\|^3 = \|U_a(a^*)\|$, for all $a \in \mathfrak{J}$.

If $\mathfrak{J}$ is unital, it clearly follows that $\mathbf{1}^* = \mathbf{1}$. It is known that the involution of each JB*-algebra $\mathfrak{J}$ satisfies $\|a^*\| = \|a\|$ for all $a \in \mathfrak{J}$ (cf. [42, Lemma 4]).

These two Jordan mathematical models are intrinsically connected in the following way: the set, $\mathfrak{J}_{sa}$, of all *self-adjoint* elements in a JB*-algebra $\mathfrak{J}$, i.e. $\mathfrak{J}_{sa} := \{a \in \mathfrak{J} : a^* = a\}$, is a JB-algebra [21, see Proposition

3.8.2]. Conversely, by a deep result due to J.D.M. Wright (cf. [44]), each JB-algebra corresponds to the self-adjoint part of a (unique) JB*-algebra.

A *JBW*-algebra* (resp., a *JBW-algebra*) is a JB*-algebra (resp., a JB-algebra) which is also a dual Banach space. Thus, JBW*-algebras can be considered as the Jordan analogue of von Neumann algebras. Additional details and some structure results are revisited along subsections 1.1, 1.2, 1.3, and 1.4.

The Jordan product of each JB*-algebra $\mathfrak{J}$ is commutative by definition. This property makes indistinguishable the left and right multiplication operators. For each $a \in \mathfrak{J}$ we denote by M_a the Jordan multiplication operator by the element a, that is, $M_a : \mathfrak{J} \to \mathfrak{J}$, $M_a(b) = a \circ b$. We say that elements a and b in a Jordan algebra $\mathfrak{J}$ *operator commute* if the operators M_a, M_b commute, that is, $(a \circ c) \circ b = a \circ (c \circ b)$ for every $c \in \mathfrak{J}$, equivalently, the associator $[a, c, b] = (a \circ c) \circ b - a \circ (c \circ b)$ is zero for all $c \in \mathfrak{J}$. D.M. Topping showed in [40, Proposition 1] that if $\mathfrak{J}$ is a JC*-algebra, regarded as a JB*-subalgebra of a C*-algebra A, two self-adjoint elements a and b in $\mathfrak{J}$ operator commute if, and only if, they commute with respect to the associate product of the C*-algebra A. Subsection 1.2 is devoted to revisit the basic background on operator commutativity. We also add some new result proving that if a and b are elements in a JB*-algebra $\mathfrak{J}$ such that the JB*-subalgebra of $\mathfrak{J}$ that they generate is a JB*-subalgebra of some C*-algebra A, then a and b commute in the usual sense as elements of A as soon as they operator commute in $\mathfrak{J}$. Consequently, two elements a, b in a JC*-algebra $\mathfrak{J}$ acting on a C*-algebra A operator commute in $\mathfrak{J}$ if, and only if, they commute in A (see Proposition 1.2).

M. Brešar, D. Eremita, and the third author of this paper studied linear bijections preserving operator commutativity between certain classes of Jordan algebras in [12]. The result reads as follows: Let $\mathfrak{M}$ and $\mathfrak{J}$ be non-quadratic, unital, centrally closed, prime non-degenerate Jordan algebras over a field $\mathbb{K}$ of characteristic different from 2, 3, and 5. Then for every bijective linear mapping $\Phi : \mathfrak{M} \to \mathfrak{J}$ such that $[\Phi(a^2), \mathfrak{J}, \Phi(a)] = 0$, for all $a \in \mathfrak{J}$, there exist a non-zero $\alpha \in \mathbb{K}$, a Jordan isomorphism $J : \mathfrak{M} \to \mathfrak{J}$, and a linear mapping $\beta : \mathfrak{M} \to \mathbb{K}$ satisfying

$$\Phi(x) = \alpha J(x) + \beta(x)\mathbf{1},$$

for all $x \in \mathfrak{M}$ (cf. [12, Theorem 5.3]).

The main purpose of this study is to extend the just quoted result by Brešar, Eremita and Villena to linear bijections preserving operator commutativity in both directions between JBW*-algebras admitting no central summands of type I_1 and I_2, à la Brešar–Meiers. Our study culminates in Theorem 8.6 where we establish that if $\mathfrak{M}$ and $\mathfrak{J}$ are JBW*-algebras with no central summands of type I_1 and I_2, and $\Phi : \mathfrak{M} \to \mathfrak{J}$ is a linear bijection preserving operator commutativity in both directions, that is,

$$[x, \mathfrak{M}, y] = 0 \Leftrightarrow [\Phi(x), \mathfrak{J}, \Phi(y)] = 0,$$

for all $x, y \in \mathfrak{M}$, then there exist a unique invertible central element z_0 in $\mathfrak{J}$, a unique Jordan isomorphism $J : \mathfrak{M} \to \mathfrak{J}$, and a unique linear mapping β from $\mathfrak{M}$ to the centre of $\mathfrak{J}$ satisfying

$$\Phi(x) = z_0 \circ J(x) + \beta(x),$$

for all $x \in \mathfrak{M}$. Furthermore, if Φ is a symmetric mapping (i.e., $\Phi(x^*) = \Phi(x)^*$ for all $x \in \mathfrak{M}$), the element z_0 is self-adjoint, J is a Jordan *-isomorphism, and β is a *-symmetric mapping too (cf. Corollary 8.9).

In a spin factor V, two elements a, b operator commute if and only if b is a linear combination of a and the unit element (see Remark 1.5). Therefore, any linear mapping on a spin factor preserves operator commutativity. This is essentially the reason to avoid JBW*-algebras containing summands of type I_2 in Theorem 8.6 (see Remark 8.8).

We also obtain a version of our main result for linear preservers of operator commutativity between JBW-algebras in Theorem 8.10, where we prove that if $\mathfrak{M}$ and $\mathfrak{J}$ are JBW*-algebras admitting no central summands of type I_1 and I_2, then for every linear bijection $\Phi : \mathfrak{M}_{sa} \to \mathfrak{J}_{sa}$ preserving operator commutativity in both directions, there exist a unique invertible element z_0 in $Z(\mathfrak{J}_{sa})$, a unique Jordan isomorphism $J : \mathfrak{M}_{sa} \to \mathfrak{J}_{sa}$, and a unique linear mapping $\beta : \mathfrak{M}_{sa} \to Z(\mathfrak{J}_{sa})$ satisfying

$$\Phi(x) = z_0 \circ J(x) + \beta(x),$$

for all $x \in \mathfrak{M}_{sa}$. The conclusion also holds when $\mathfrak{M}$ and $\mathfrak{J}$ are von Neumann algebras without central summands of type I_1 and I_2 and "operator commutativity" means "commutativity" (cf. Corollary 8.11), a result which improves the known conclusions in this setting.

The just presented results have required a time-consuming effort and a great deal of novelty in order to combine tools and techniques from several branches such as C*-algebra theory, Jordan algebra theory and preservers. The arguments are technically complex, but we have tried to make the work self-contained. It is perhaps worth to comment the route map we followed. In words of M. Bresar [9], "It has turned out that for rather large classes of rings all these (and some similar) problems can be solved using a unified approach based on a characterization of commuting traces of biadditive maps." In the Jordan setting, "commuting traces" are replaced by "associating traces".

A mapping q on a linear space X is said to be a *trace* if there exists a symmetric bilinear mapping $B : X \times X \to X$ such that $q(x) = B(x, x)$ for all $x \in X$. Let $F : \mathfrak{J} \to \mathfrak{J}$ be a mapping on a Jordan algebra. We shall say that F is *associating* if for every $x \in \mathfrak{J}$, $F(x)$ and x operator commute as elements in $\mathfrak{J}$, that is, $[F(x), \mathfrak{J}, x] = 0$. In Theorem 6.6 we prove that every linear associating mapping T on a JBW*-algebra $\mathfrak{J}$ admitting no direct summands of type I_1, can be expressed in the form:

$$T(x) = \lambda \circ x + \mu(x), \text{ for all } x \in \mathfrak{J},$$

where $\lambda \in Z(\mathfrak{J})$, and $\mu : \mathfrak{J} \to Z(\mathfrak{J})$ is linear. Furthermore, for every associating trace $q(x) = B(x, x)$ on $\mathfrak{J}$, there exist a unique $\lambda \in Z(\mathfrak{J})$, a unique linear mapping $\mu : \mathfrak{J} \to Z(\mathfrak{J})$, and a unique bilinear map $\nu : \mathfrak{J} \times \mathfrak{J} \to Z(\mathfrak{J})$ satisfying

$$B(x, x) = \lambda \circ x^2 + \mu(x) \circ x + \nu(x, x) \circ \mathbf{1},$$

for all $x \in \mathfrak{J}$. If $\mathfrak{J}$ is a JBW*-algebra of type I_2 the element λ is always zero (cf. Theorem 7.5).

1.1. Background

We say that a Jordan algebra $\mathfrak{J}$ is *unital* if there exists an element $\mathbf{1} \in \mathfrak{J}$ (called the unit of $\mathfrak{J}$) such that $\mathbf{1} \circ a = a$ for all $a \in \mathfrak{J}$. Given elements $a, c \in \mathfrak{J}$, the symbol $U_{a,b}$ will stand for the linear mapping on $\mathfrak{J}$ defined by $U_{a,c}(b) := (a \circ b) \circ c + (b \circ c) \circ a - (a \circ c) \circ b$ $(b \in \mathfrak{J})$. We simply write U_a for $U_{a,a}$.

As we commented in the introduction, a natural example of Jordan algebra is provided by any associative algebra A equipped with the natural Jordan product given by $a \circ b := \frac{1}{2}(ab + ba)$. Any linear subspace of an associative algebra which is closed under the Jordan product is a Jordan algebra. Such Jordan algebras are called *special*. Given an element a in an associative algebra A, we shall write L_a (respectively, R_a) for the left (respectively, right) multiplication operator by the element a, that is, $L_a(x) = ax$ (respectively, $R_a(x) = xa$). Jordan algebras which cannot be embedded as a Jordan subalgebras of an associative algebra are called *exceptional*. A widely known example of exceptional Jordan algebra is the algebra $H_3(\mathbb{O})$ of all Hermitian 3×3 matrices with entries in the complex octonions (see [21, Corollary 2.8.5] for more information).

An element a in a unital Jordan-Banach algebra $\mathfrak{J}$ is called *invertible* if we can find $b \in \mathfrak{J}$ satisfying $a \circ b = \mathbf{1}$ and $a^2 \circ b = a$. The element b is unique, it is called the inverse of a and denoted by a^{-1} (cf. [21, 3.2.9]).

Along this paper, the symbol "$*$" will mainly stand for the involution of a JB*-algebra. For each algebra A equipped with an algebra involution *, we shall write A_{sa} for the set of all symmetric elements for this involution. The reader should be warned that we shall also write X^* for the dual space of a Banach space X, and X_* for a predual of X in case that the latter is a dual Banach space.

The reader should know that every JBW-algebra is unital (see [21, Lemma 4.1.7]). It is also known that a JB*-algebra $\mathfrak{J}$ is a JBW*-algebra if, and only if, $\mathfrak{J}_{sa}$ is a JBW-algebra [17]. It is further established in the just quoted reference (see also [6, Lemma 2.2]) that the following assertions hold:

(i) $\mathfrak{J}_{sa}$ is weak*-closed in $\mathfrak{J}$.
(ii) The operator $\phi : (\mathfrak{J}_*)_{sa} \to (\mathfrak{J}_{sa})_*$ defined by $\phi(\omega) = \omega|_{\mathfrak{J}_{sa}}$ is an onto linear isometry of real Banach spaces, where $(\mathfrak{J}_*)_{sa} = \{\varphi \in \mathfrak{J}_{sa} : \varphi(a^*) = \overline{\varphi(a)},\ \forall a \in \mathfrak{J}\}$ is the self-adjoint part of the predual, $\mathfrak{J}_*$, of $\mathfrak{J}$.
(iii) The operator $\psi : \mathfrak{J}_{sa} \times \mathfrak{J}_{sa} \to \mathfrak{J}$ defined by $\psi(x, y) = x + iy$ is a onto real-linear weak*-to-weak* homeomorphism.

For the basic theory of JB- and JB*-algebras, the reader is referred to the monographs [1,21] and [14].

Let $B(H)$ denote the C*-algebra of all bounded linear operators on a complex Hilbert space H. A *JC-algebra* is a JB-algebra that is isometrically isomorphic to a norm closed Jordan subalgebra of $B(H)_{sa}$ [1, see Proposition 1.35]. There exist JB-algebras which are not JC-algebras, for instance the algebra $H_3(\mathbb{O})$ of all Hermitian 3×3 matrices with entries in the complex octonions $\mathbb{O}$ (see [21, Corollary 2.8.5]). A *JC*-algebra* is a JB*-algebra which materialises as a norm-closed self-adjoint Jordan subalgebra of a C*-algebra, and hence, by the Gelfand-Naimark theorem, a norm-closed self-adjoint Jordan subalgebra of some $B(H)$.

A *JW-algebra* is a weak*-closed real Jordan subalgebra of some $B(H)_{sa}$. JW-algebras were first studied by D.M. Topping [40] and E. Størmer [39]. A *JW*-algebra* is a JC*-algebra which is also a dual Banach space, or equivalently, a weak*-closed JB*-subalgebra of some von Neumann algebra.

1.2. Operator commutativity and the centre

The product of a Jordan algebra $\mathfrak{J}$ is, by definition, commutative, so every pair of elements in $\mathfrak{J}$ commute if we only employ the "usual" sense. However, if we assume that an associative algebra A is equipped with its natural Jordan product $a \circ b = \frac{1}{2}(ab + ba)$, it can be easily checked that if $a, b \in A$ commute with respect to the associative product (i.e., $ab = ba$), then the corresponding Jordan multiplication operators M_a and M_b commute. This is the motivation for the usual notion of operator commutativity in a Jordan algebra. According to the standard sources, elements a and b in a Jordan algebra $\mathfrak{J}$ are said to *operator commute* if the operators M_a, M_b commute (i.e., $(a \circ c) \circ b = a \circ (c \circ b)$ for every $c \in \mathfrak{J}$). For example, the Jordan identity is equivalent to say that, for each element a in a Jordan algebra $\mathfrak{J}$, a^2 and a operator commute. The *associator* of three elements a, b, c in a Jordan algebra $\mathfrak{J}$, defined by $[a, c, b] := (a \circ c) \circ b - (b \circ c) \circ a$, tests the operator commutativity of a and b since a and b operator commute in $\mathfrak{J}$ if, and only if, $[a, \mathfrak{J}, b] = 0$.

On an associative algebra A we shall write $[\cdot, \cdot]$ for the usual Lie bracket given by $[a, b] := ab - ba$, for all $a, b \in A$.

The reader should be warned that, in a general Jordan algebra, operator commutativity of a and b is not always a necessary nor sufficient condition to the property that a and b generate a commutative and associative subalgebra of $\mathfrak{J}$ (cf. [21, 2.5.1 and Example 2.5.2]).

The *centre* of a Jordan algebra $\mathfrak{J}$ consists of all elements $z \in \mathfrak{J}$ such that z operator commutes with every element of $\mathfrak{J}$. The symbol $Z(\mathfrak{J})$ will stand for the centre of $\mathfrak{J}$, and its elements are called central. The centre of a JB*-algebra $\mathfrak{J}$ is a commutative C*-algebra, and contains the identity of $\mathfrak{J}$ if it exists (see [1, Proposition 1.52]. The centre of a JBW*-algebra is a commutative von Neumann (see [1, Proposition 2.36], [17]). If $Z(\mathfrak{J})$ consists of scalar multiples of the identity alone, $\mathfrak{J}$ is called a JBW*-factor.

We recall that a *(Jordan) factor representation* of a JB*-algebra $\mathfrak{J}$ is a Jordan *-homomorphism π from $\mathfrak{J}$ onto a weak*-dense Jordan *-subalgebra of a JBW*-algebra factor. Observe that every JB*-algebra admits a faithful or separating family of Jordan factor representations (cf. [2, Corollary 5.7]).

Remark 1.1. Let a be an element in a JB*-algebra $\mathfrak{J}$. The following statements are equivalent:

(a) $a \in Z(\mathfrak{J})$,
(b) $\pi(a) \in Z(\mathfrak{J}_\pi)$, for every Jordan factor representation $\pi : \mathfrak{J} \to \mathfrak{J}_\pi$,
(c) $\pi_i(a) \in Z(\mathfrak{J}_{\pi_i})$, for every representation π_i in a faithful family of Jordan factor representations $\{\pi_i : \mathfrak{J} \to \mathfrak{J}_{\pi_i}\}_i$.

Namely, the implication $(a) \Rightarrow (b)$ follows by just applying the weak*-density of $\pi(\mathfrak{J})$ in $\mathfrak{J}_\pi$ together with the separate weak*-continuity of the Jordan product of the latter JBW*-algebra. The implication $(b) \Rightarrow (c)$ is clear. To see $(c) \Rightarrow (a)$, we assume the existence of a faithful family of Jordan factor representations $\{\pi_i : \mathfrak{J} \to \mathfrak{J}_{\pi_i}\}_i$ satisfying that $\pi_i(a)$ lies in $Z(\mathfrak{J}_{\pi_i})$ for all i. Define $\pi_0 : \mathfrak{J} \to \bigoplus_i^\infty \mathfrak{J}_{\pi_i}$ by $\pi_0(a) = (\pi_i(a))_i$, which is clearly a Jordan *-monomorphism. The hypothesis implies that $\pi_0(a) \in Z(\bigoplus_i^\infty \mathfrak{J}_{\pi_i})$. Since $\pi_0[a, \mathfrak{J}, \mathfrak{J}] = [\pi_0(a), \pi_0(\mathfrak{J}), \pi_0(\mathfrak{J})] = 0$, we deduce that $[a, \mathfrak{J}, \mathfrak{J}] = 0$, and thus $a \in Z(\mathfrak{J})$.

In connection with the above remark, it is perhaps worth to note that every Jordan *-monomorphism between JB*-algebras is automatically an isometry, while every Jordan *-homomorphism is automatically continuous and non-expansive (see [44, Corollaries 1.4 and 1.5]).

If $\mathfrak{J}$ is a JC*-algebra, regarded as a JB*-subalgebra of a C*-algebra A, a result originally due to Topping, shows that two self-adjoint elements a and b in $\mathfrak{J}$ operator commute if, and only if, they commute with respect to the associate product of the C*-algebra A (cf. [40, Proposition 1], see also the subsequent rediscoveries in [1, Proposition 1.49], [20, Lemma 5.1] and [43, Proposition 3.2]). The study of operator commutativity for self-adjoint elements in a JB*-algebra $\mathfrak{J}$ was revisited by van de Wetering in [43, Theorem 3.13], who showed, via Shirshov-Cohn theorem, that the following statements are equivalent for any $a, b \in \mathfrak{J}_{sa}$:

(a) a and b operator commute.
(b) a and b generate an associative JB*-algebra.
(c) a and b generate an associative JB*-algebra of mutually operator commuting elements.
(d) a and a^2 operator commute with b and b^2.

In Topping's result, as well as in the just quoted characterization obtained by van de Wetering, the elements are required to be self-adjoint. Furthermore, in Topping's result the elements a and b are assumed in a JC*-algebra and we have a C*-algebra dominating the Jordan algebra and providing an associative product. Next we refine the argument in Topping's original paper to improve the previous results by showing that if two (non-necessarily self-adjoint) elements in a JB*-algebra operator commute, then they commute inside any C*-algebra containing the JB*-subalgebra that they generate. The conclusion is new even for JC*-algebras.

Proposition 1.2. *Let a, b be elements in a JB*-algebra $\mathfrak{J}$, and let $\mathfrak{J}_{a,b}$ denote the JB*-subalgebra of $\mathfrak{J}$ generated by a and b. Suppose we can find a C*-algebra A containing $\mathfrak{J}_{a,b}$ as a JB*-subalgebra. Assume that a and b operator commute in $\mathfrak{J}$. Then a and b commute as elements in A. In particular, two elements a, b in a JC*-algebra $\mathfrak{J}$ acting on a C*-algebra A operator commute in $\mathfrak{J}$ if, and only if, they commute in A.*

Proof. Since a and b operator commute in $\mathfrak{J}$ we have $[a, \mathfrak{J}, b] = 0$. Let us denote the product of A by mere juxtaposition. Suppose B denotes the C*-subalgebra of A generated by $\mathfrak{J}_{a,b}$. By rewriting the hypotheses in terms of the associative product of B we arrive to

$$0 = -4[a, \mathfrak{J}_{a,b}, b] = [[a, b], \mathfrak{J}_{a,b}].$$

It is well-known that the mapping $x \mapsto [[a, b], x]$ is a (continuous) derivation on B. Therefore, given $y, z \in \mathfrak{J}_{a,b}$ we have

$$[[a, b], yz] = [[a, b], y]z + y[[a, b], z] = 0,$$

and thus, a basic argument mixing linearity and continuity implies that $[[a, b], B] = 0$. We have therefore shown that $[a, b]$ is a central element in the C*-algebra B. In particular $[[a, b], a] = 0$, and hence Kleinecke's theorem [27] assures that $[a, b]$ is a quasi-nilpotent element, that is, it has zero spectral radius in B, which is equivalent to say that $[a, b] = 0$ since the latter is a central element. Therefore, a and b commute in B (and in A). The rest is clear. □

1.3. Some structure results of JBW*-algebras

An element p in a JB-algebra is called a *projection* if $p^2 = p \circ p = p$. Similarly, an element p in a JBW*-algebra $\mathfrak{J}$ is called a projection if $p^* = p = p \circ p = p^2$, i.e., p is a projection in $\mathfrak{J}$ if, and only if, it is a projection in $\mathfrak{J}_{sa}$. Two projections p, q in $\mathfrak{J}$ are called *orthogonal* ($p \perp q$ in short) if $p \circ q = 0$ (see [21, Lemma 4.2.2] for equivalent conditions).

An element s in a unital JB*-algebra $\mathfrak{J}$ is said to be a *symmetry* if $s = s^*$ and $s^2 = \mathbf{1}$. It can be proved that for each symmetry $s \in \mathfrak{J}$ the map U_s is a Jordan *-automorphism (cf. [1, Proposition 2.34]). The group $\mathrm{Int}(\mathfrak{J})$ generated by all Jordan *-automorphisms of the form U_s with s running in the set of all symmetries in $\mathfrak{J}$ is called the group of all *inner automorphisms* of $\mathfrak{J}$, and its elements are called inner automorphisms of $\mathfrak{J}$.

Two projections p, q in a JBW-algebra $\mathfrak{J}$ are called *equivalent* ($p \sim q$ in short) if there is an inner automorphism α of $\mathfrak{J}$ such that $q = \alpha(p)$. If α can be written as $\alpha = U_{s_1} U_{s_2} \cdots U_{s_n}$ we write $p \sim_n q$. If $n = 1$ we say p and q are *exchanged by a symmetry*.

A projection p in a JBW*-algebra $\mathfrak{J}$ is called *abelian* if the algebra $\mathfrak{J}_p = U_p(\mathfrak{J}) := \{U_p(x) : x \in \mathfrak{J}\}$ is associative. If $\mathfrak{J}$ is a JW*-algebra (resp. a JW-algebra) it follows from [1, 1.49] that p is abelian if, and only if, $\mathfrak{J}_p$ consists of mutually operator commuting elements.

Given an associative algebra A, we denote by A^{op} the *opposite algebra* of A, that is, the algebra formed by reversing the order of the product in A. If A is C*-algebra, A^{op} is a C*-algebra with respect to the same norm and involution.

The next lemma states a property which is probably well known by experts on JB*-algebras. We included it in this paper for completeness reasons. We recall first that given a JBW*-algebra $\mathfrak{J}$, there exists a unique von Neumann algebra $W^*(\mathfrak{J})$ together with a normal Jordan homomorphism $\Psi : \mathfrak{J} \to W^*(\mathfrak{J})$ satisfying:

- (i) $\Psi(\mathfrak{J})$ generates $W^*(\mathfrak{J})$ as a von Neumann algebra.
- (ii) If $\mathscr{N}$ is a von Neumann algebra and $\Phi : \mathfrak{M} \to \mathscr{N}$ is a normal (Jordan) homomorphism, then there is a normal *-homomorphism $\widehat{\Phi} : W^*(\mathfrak{J}) \to \mathscr{N}$ such that $\widehat{\Phi}\Psi = \Phi$.
- (iii) There is a *-anti-automorphism τ of order 2 of $W^*(\mathfrak{J})$ such that $\tau(\Psi(x)) = \Psi(x)$ for all $x \in \mathfrak{J}$, i.e. $\Psi(\mathfrak{J}) \subseteq H(W^*(\mathfrak{J}), \tau) = \{x \in W^*(\mathfrak{J}) : \tau(x) = x\}$.

The von Neumann algebra $W^*(\mathfrak{J})$ is called the *universal von Neumann algebra for* $\mathfrak{J}$ [21, Theorem 7.1.9]. It is further known (see [21, Remark 7.2.8]) that the Jordan *-homomorphism Ψ is an isometric Jordan *-monomorphism when $\mathfrak{J}$ is special (i.e. it is a JW*-algebra). If $\mathfrak{J}$ is a JW*-algebra without spin part, we actually have $\mathfrak{J} = H(W^*(\mathfrak{J}), \tau)$ (cf. [21, Proposition 7.3.3]).

Lemma 1.3. *Let τ be a *-anti-homomorphism of order 2 on a von Neumann algebra W. Then the JW*-algebra $H(W,\tau)=\{x\in W:\tau(x)=x\}$ is a factor if, and only if, one of the next statements holds*

(*a*) *W is a factor;*
(*b*) *$H(W,\tau)$ is *-isomorphic to a factor von Neumann algebra N which is a weak*-closed ideal of W and $W=N\oplus^\infty\tau(N)$.*

Proof. Suppose that $H(W,\tau)$ is a factor. Let Z stand for the centre of W. Clearly $\tau(Z)=Z$, and $\tau|_Z:Z\to Z$ is an order 2 *-automorphism. Lemma 7.3.4 in [21] assures the existence of two projections $q,p\in Z$ such that $p+q+\tau(q)=\mathbf{1}$, and every subprojection of p in Z is τ-invariant. Observe that $p,q,\tau(q)$ are mutually orthogonal. Since the projections $p, q+\tau(q)$ lie in $H(W,\tau)$ and the latter is a factor, we must have $p=0$ or $q=0$.

Suppose first that $p=0$. It is clear that $N=qW$ is a weak*-closed ideal of W, $W=N\oplus^\infty\tau(N)$, and $H(W,\tau)=\{(a,\tau(a)):a\in N\}$. Consider the von Neumann algebra $\tilde{W}=N\oplus^\infty\tau(N)^{op}$. Observe that $H(W,\tau)$ is a weak*-closed C*-subalgebra of $\tilde{W}$. Finally, the mapping $a\mapsto(a,\tau(a))$ is an isometric C*-isomorphism from N onto $H(W,\tau)$ when the latter is regarded as a von Neumann subalgebra of $\tilde{W}$, which implies that N is a factor.

If $q=0$, $p=\mathbf{1}$ and every projection in Z is τ-symmetric, and hence a projection in $H(W,\tau)$. Since the latter is a factor, the von Neumann algebra W is a factor too.

To see the reciprocal implication, we observe that if (*b*) holds, then $H(W,\tau)$ is clearly a factor. If W is a factor, by observing that every central projection in $H(W,\tau)$ is a central projection in W, the desired conclusion is clear (see [21, Lemma 7.3.2]). $\square$

1.4. Spin factors

An important example of JBW*-algebra, which also has an important role in this paper, is given by the so-called (complex) spin factors. The notion of spin factor dates back to the analytic classification of bounded symmetric domains in complex Banach spaces by authors like E. Cartan, W. Kaup, L. Harris (cf. [22,23] and the introduction of [25] or [18, §3]). The construction is as follows. Let H be a complex Hilbert space. A *spin factor* (also known as a *spinor* or *Cartan factor of type IV*) is a norm closed complex subspace V of $B(H)$ such that $\dim(V)>2$, $a^*\in V$, and $a^2\in\mathbb{C}\mathbf{1}$, for all $a\in V$, where $\mathbf{1}$ denotes the unit of $B(H)$. For example, the space $S_2(\mathbb{C})$, of all symmetric 2×2 complex matrices is an spin factor.

It is further known that there exists an inner product $\langle\cdot|\cdot\rangle$ on V satisfying

$$ab^*+b^*a=2\langle a|b\rangle\mathbf{1},\ \text{ for all } a,b\in V.$$

The norm $\|a\|_2^2:=\langle a|a\rangle$ given by the inner product is equivalent to the operator norm $\|\cdot\|$, and both are related by

$$\|a\|^2=\|a\|_2^2+\left(\|a\|_2^4-|\langle a|a^*\rangle|^2\right)^{\frac{1}{2}}.$$

It is also known that V is closed for the triple product given by

$$\{a,b,c\}=\frac{1}{2}(ab^*c+cb^*a)=\langle a|b\rangle c+\langle c|b\rangle a-\langle a|c^*\rangle b^*\quad(a,b,c\in V).$$

For each norm-one element $u\in V$ with $u^*=u$, the Banach space V is a unital JW*-algebra with unit u, Jordan product $a\circ_u b=\{a,u,b\}=\frac{1}{2}(aub+bua)$, and involution $a^{*_u}:=\{u,a,u\}=ua^*u$ (cf. [23, page 358]). Every element a in the JW*-algebra $(V,\circ_u,*_u)$ satisfies

$$a \circ_u a = \{a, u, a\} = 2\langle a|u\rangle a - \langle a|a^*\rangle u,$$

that is, the square of every element a in $(V, \circ_u, *_u)$ is a linear combination of a and the unit element.

Remark 1.4. We recall that a non-necessarily associative algebra $\mathfrak{J}$ is called *quadratic* if it is unital and the square of every element a in $\mathfrak{J}$ lies in the linear hull of $\{1, a\}$ (cf. [21, 2.2.5] or [14, §2.5.9]). It is known that every quadratic JB*-algebra is a spin factor (see [14, Theorem 3.5.5 and Corollary 3.5.7]).

It is worth to note that if V is a spin factor, the self-adjoint part of the associated JW*-algebra, i.e. $V_{sa} = \{a \in V : a^{*_u} = a\}$, is a real spin factor in the sense of [20, §6].

Remark 1.5. It should be also noted that for each element x in a spin factor V, the associator of x, that is the collection of all $y \in V$ operator commuting with x reduces to $\mathbb{C}\mathbf{1} \oplus \mathbb{C}x$. Namely, let V be a spin factor with inner product $\langle \cdot|\cdot\rangle$, involution $a \mapsto \overline{a}$, unit $\mathbf{1}$ (i.e. a norm-one element with $\overline{\mathbf{1}} = \mathbf{1}$), and Jordan product

$$a \circ b = \{a, \mathbf{1}, b\} = \langle a|\mathbf{1}\rangle b + \langle b|\mathbf{1}\rangle a - \langle a|\overline{b}\rangle \mathbf{1}, \text{ and } a^* = \{\mathbf{1}, a, \mathbf{1}\} = 2\langle \mathbf{1}|a\rangle \mathbf{1} - \overline{a},$$

for all $a, b \in V$. Observe that $V = \mathbb{C}\mathbf{1} \oplus^{\ell_2} \{\mathbf{1}\}^{\perp_2}$, where $\{\mathbf{1}\}^{\perp_2}$ stands for the orthogonal complement of $\{\mathbf{1}\}$ in the underlying Hilbert space. Furthermore, for each $a, b \in \{\mathbf{1}\}^{\perp_2}$, we have $a \circ b = -\langle a|\overline{b}\rangle \mathbf{1}$. Pick two operator commuting elements $\lambda\mathbf{1} + a, \mu\mathbf{1} + b$ in V (with $a, b \in \{\mathbf{1}\}^{\perp_2}$). By assumptions, for each $c \in \{\mathbf{1}\}^{\perp_2}$ we have

$$((\lambda\mathbf{1} + a) \circ c) \circ (\mu\mathbf{1} + b) = (\lambda\mathbf{1} + a) \circ (c \circ (\mu\mathbf{1} + b)),$$
$$\lambda\mu c + \lambda c \circ b + \mu a \circ c + (a \circ c) \circ b = \lambda\mu c + \lambda c \circ b + \mu a \circ c + (b \circ c) \circ a,$$
$$-\langle a|\overline{c}\rangle b = -\langle b|\overline{c}\rangle a,$$

and thus the arbitrariness of c gives $b \in \mathbb{C}a$, and consequently $\mu\mathbf{1} + b$ lies in $\mathbb{C}\mathbf{1} + \mathbb{C}(\lambda\mathbf{1} + a)$, as desired.

2. Elementary operators on JB*-algebras

Let $\mathfrak{J}$ be a JB*-algebra. We write $\mathscr{B}(\mathfrak{J})$ for the Banach algebra of all bounded linear operators on the Banach space $\mathfrak{J}$, and we write $\mathscr{E}\ell(\mathfrak{J})$ for the subalgebra of $\mathscr{B}(\mathfrak{J})$ generated by the multiplication operators M_a with a running in $\mathfrak{J}$. The elements of $\mathscr{E}\ell(\mathfrak{J})$ are called *elementary operators* on $\mathfrak{J}$. Furthermore, we will denote by $\mathscr{E}\ell_{\mathfrak{J}_{sa}}(\mathfrak{J})$ the real subalgebra of $\mathscr{E}\ell(\mathfrak{J})$ generated by the multiplication operators M_a with a running in $\mathfrak{J}_{sa}$. It is worth pointing out that each operator $\mathscr{E} \in \mathscr{E}\ell_{\mathfrak{J}_{sa}}(\mathfrak{J})$ satisfies the condition

$$\mathscr{E}(x^*) = \mathscr{E}(x)^*, \quad \forall x \in \mathfrak{J}. \tag{2.1}$$

If $\mathfrak{J}$ is a JBW*-algebra, then the subset of $\mathscr{B}(\mathfrak{J})$ consisting of all weak*-continuous operators on $\mathfrak{J}$ is a subalgebra of $\mathscr{B}(\mathfrak{J})$ which, on account of the separate weak*-continuity of the product of $\mathfrak{J}$, contains the multiplication operators on $\mathfrak{J}$. This implies that each elementary operator on $\mathfrak{J}$ is weak*-continuous.

Remark 2.1. Let a, z be elements in a Jordan algebra $\mathfrak{J}$, with z central. Clearly, $M_a M_z = M_a M_z$, and thus, by linearity, $\mathscr{E} M_z = M_z \mathscr{E}$, for every elementary operator $\mathscr{E} \in \mathscr{E}\ell(\mathfrak{J})$.

Lemma 2.2. *Let $\mathfrak{J}$ and $\mathfrak{H}$ be two Jordan algebras, and let $\phi\colon \mathfrak{J} \to \mathfrak{H}$ be a Jordan homomorphism. The following statements hold.*

(i) *For each $\mathscr{E} \in \mathscr{E}\ell(\mathfrak{J})$ there exists $\mathscr{F} \in \mathscr{E}\ell(\mathfrak{H})$ such that $\phi\mathscr{E} = \mathscr{F}\phi$.*

(ii) *Suppose that* $\mathfrak{H}$ *is a JBW*-algebra and that* $\phi(\mathfrak{J})$ *is weak*-dense in* $\mathfrak{H}$. *Then for each* $\mathscr{E} \in \mathscr{E}\ell(\mathfrak{J})$ *there exists a unique* $\Phi(\mathscr{E}) \in \mathscr{E}\ell(\mathfrak{H})$ *such that* $\phi\mathscr{E} = \Phi(\mathscr{E})\phi$. *Further, the map* $\Phi\colon \mathscr{E}\ell(\mathfrak{J}) \to \mathscr{E}\ell(\mathfrak{H})$, $\mathscr{E} \mapsto \Phi(\mathscr{E})$ *is a homomorphism.*

Proof. (i) It is immediate to check that the set

$$\mathscr{L} = \left\{\mathscr{E} \in \mathscr{E}\ell(\mathfrak{J}) : \phi\mathscr{E} = \mathscr{F}\phi \text{ for some } \mathscr{F} \in \mathscr{E}\ell(\mathfrak{H})\right\}$$

is a subalgebra of $\mathscr{E}\ell(\mathfrak{J})$. Moreover, for each $a \in \mathfrak{J}$, we have $\phi M_a = M_{\phi(a)}\phi$, and therefore $M_a \in \mathscr{L}$. Consequently, $\mathscr{L} = \mathscr{E}\ell(\mathfrak{J})$.

(ii) Suppose that $\mathscr{F}_1, \mathscr{F}_2 \in \mathscr{E}\ell(\mathfrak{H})$ are such that $\mathscr{F}_1\phi = \mathscr{F}_2\phi$. From the weak*-continuity of both maps $\mathscr{F}_1$ and $\mathscr{F}_2$ and the weak*-density of $\phi(\mathfrak{J})$ in $\mathfrak{H}$ we deduce that $\mathscr{F}_1 = \mathscr{F}_2$. We conclude from (i) that, for each $\mathscr{E} \in \mathscr{E}\ell(\mathfrak{J})$, there exists a unique $\Phi(\mathscr{E}) \in \mathscr{E}\ell(\mathfrak{H})$ such that $\phi\mathscr{E} = \Phi(\mathscr{E})\phi$. Hence, we can define a map $\Phi\colon \mathscr{E}\ell(\mathfrak{J}) \to \mathscr{E}\ell(\mathfrak{H})$ through the condition $\phi\mathscr{E} = \Phi(\mathscr{E})\phi$ for each $\mathscr{E} \in \mathscr{E}\ell(\mathfrak{J})$, and routine verifications show that Φ is a homomorphism. □

Our next result is one of the key tools in our arguments. The statement combines non-associative Jordan algebras, and in particular JBW*-algebras with a rich geometric–algebraic structure, and associative algebras regarded as Jordan algebras via their natural Jordan product.

Proposition 2.3. *Let* $\mathfrak{J}$ *be a JBW*-algebra with no direct summands of type* I_1 *or* I_2. *Then there exist* $u, v \in \mathfrak{J}_{sa}$ *and* $\mathscr{E}_0, \mathscr{E}_1, \mathscr{E}_2 \in \mathscr{E}\ell_{\mathfrak{J}_{sa}}(\mathfrak{J})$ *with the following properties:*

(i) $\mathscr{E}_i(u^j) = \delta_{ij}\mathbf{1}$, *for all* $i, j \in \{0, 1, 2\}$,
(ii) *for each unital complex associative algebra* $\mathfrak{A}$ *and each unital Jordan homomorphism* $\phi\colon \mathfrak{J} \to \mathfrak{A}$, *the elements* $[\phi(u)^2, \phi(v)]$, $[\phi(u), \phi(v)^2]$, *and* $[\phi(u), \phi(v)]$ *are linearly independent, where the brackets stand for the usual Lie product on* $\mathfrak{A}$.

We can further assume that $\|\mathscr{E}_i\| \leq 10$ *for every* $i \in \{0, 1, 2\}$.

Proof. It is well known that $\mathfrak{J}_{sa}$ is a JBW-algebra, and further $\mathfrak{J}$ and $\mathfrak{J}_{sa}$ have the same lattice of projections (cf. [17,44]). Therefore, $\mathfrak{J}_{sa}$ is a JBW-algebra with no direct summands of type I_1 or I_2. The proof of the proposition will be divided into a sequence of cases according to the structure of $\mathfrak{J}_{sa}$.

Case 1: We assume first that $\mathfrak{J}_{sa}$ satisfies that there exist pairwise orthogonal projections p_1, p_2, p_3, p_4 which are pairwise exchangeable by symmetries such that $p_1 + p_2 + p_3 + p_4 = \mathbf{1}$. It should be pointed out that this condition holds in each one of the following cases:

(1) $\mathfrak{J}_{sa}$ is a JBW-algebra with no direct summand of type I (see [21, Proposition 5.2.15]),
(2) $\mathfrak{J}_{sa}$ is a JBW-algebra of type I_∞ (see [1, Proposition 3.24]),
(3) $\mathfrak{J} = \mathfrak{J}_1 \oplus^\infty \mathfrak{J}_2$, where $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are JBW*-algebras as those considered in (1) and (2), respectively.

(i) Take symmetries $s, s', s'' \in \mathfrak{J}_{sa}$ such that

$$U_s(p_1) = p_2,\ U_{s'}(p_1) = p_3,\ \text{and } U_{s''}(p_1) = p_4.$$

Define $u, v \in \mathfrak{J}_{sa}$ by

$$u = 2s \circ p_1,\ \text{and } v = 2s' \circ p_1,$$

so that, by the commented Shirshov–Cohn theorem,

$$u^2 = p_1 + p_2, \text{ and } v^2 = p_1 + p_3.$$

Then,

$$U_{p_3}(\mathbf{1}) = p_3, \; U_{p_3}(u) = 0, \text{ and } U_{p_3}(u^2) = 0,$$

where in the middle term we compute the value in the JC-subalgebra generated by $p_1 - p_2$, s and $\mathbf{1}$, and the fact $U_{\mathbf{1}-p_1-p_2} U_{p_3} U_{\mathbf{1}-p_1-p_2} = U_{U_{\mathbf{1}-p_1-p_2}(p_3)} = U_{p_3}$, whence

$$\begin{aligned} U_{s'}U_{p_3}(\mathbf{1}) &= p_1, & U_{s'}U_{p_3}(u) &= 0, & U_{s'}U_{p_3}(u^2) &= 0,\\ U_sU_{s'}U_{p_3}(\mathbf{1}) &= p_2, & U_sU_{s'}U_{p_3}(u) &= 0, & U_sU_{s'}U_{p_3}(u^2) &= 0,\\ U_{s''}U_{s'}U_{p_3}(\mathbf{1}) &= p_4, & U_{s''}U_{s'}U_{p_3}(u) &= 0, & U_{s''}U_{s'}U_{p_3}(u^2) &= 0. \end{aligned}$$

By defining

$$\mathscr{E}_0 = U_{p_3} + U_{s'}U_{p_3} + U_sU_{s'}U_{p_3} + U_{s''}U_{s'}U_{p_3},$$

we have $\mathscr{E}_0 \in \mathscr{E}\ell_{\mathfrak{J}_{sa}}(\mathfrak{J})$ and

$$\mathscr{E}_0(\mathbf{1}) = \mathbf{1}, \quad \mathscr{E}_0(u) = 0, \quad \mathscr{E}_0(u^2) = 0.$$

Observe that

$$(U_{p_1} - U_{s'}U_{p_3})(\mathbf{1}) = 0, \qquad (U_{p_1} - U_{s'}U_{p_3})(u) = 0, \qquad (U_{p_1} - U_{s'}U_{p_3})(u^2) = p_1,$$

and therefore

$$\begin{aligned} U_s(U_{p_1} - U_{s'}U_{p_3})(\mathbf{1}) &= 0, & U_s(U_{p_1} - U_{s'}U_{p_3})(u) &= 0, & U_s(U_{p_1} - U_{s'}U_{p_3})(u^2) &= p_2,\\ U_{s'}(U_{p_1} - U_{s'}U_{p_3})(\mathbf{1}) &= 0, & U_{s'}(U_{p_1} - U_{s'}U_{p_3})(u) &= 0, & U_{s'}(U_{p_1} - U_{s'}U_{p_3})(u^2) &= p_3,\\ U_{s''}(U_{p_1} - U_{s'}U_{p_3})(\mathbf{1}) &= 0, & U_{s''}(U_{p_1} - U_{s'}U_{p_3})(u) &= 0, & U_{s''}(U_{p_1} - U_{s'}U_{p_3})(u^2) &= p_4. \end{aligned}$$

By defining

$$\mathscr{E}_2 = (U_{p_1} - U_{s'}U_{p_3}) + U_s(U_{p_1} - U_{s'}U_{p_3}) + U_{s'}(U_{p_1} - U_{s'}U_{p_3}) + U_{s''}(U_{p_1} - U_{s'}U_{p_3}),$$

we obtain $\mathscr{E}_2 \in \mathscr{E}\ell_{\mathfrak{J}_{sa}}(\mathfrak{J})$ and

$$\mathscr{E}_2(\mathbf{1}) = 0, \quad \mathscr{E}_2(u) = 0, \quad \mathscr{E}_2(u^2) = \mathbf{1}.$$

Finally, note that

$$M_u(\mathbf{1}) = u, \; M_u(u) = u^2, \text{ and } M_u(u^2) = u.$$

Consequently, by defining $\mathscr{E}_1 = \mathscr{E}_2 M_u$, we get

$$\mathscr{E}_1(\mathbf{1}) = 0, \quad \mathscr{E}_1(u) = \mathbf{1}, \text{ and } \mathscr{E}_1(u^2) = 0.$$

(ii) Let $\mathfrak{A}$ be a unital complex associative algebra, and let $\phi\colon \mathfrak{J} \to \mathfrak{A}$ be a unital Jordan homomorphism. Note that, for all $i, j \in \{1,2,3\}$ with $i \neq j$, we have

$$\phi(p_i)^2 = \phi(p_i \circ p_i) = \phi(p_i),$$

$$\begin{aligned}\phi(p_i)\phi(p_j) &= \phi\big(\underbrace{U_{p_j}p_i}_{=0}\big) + \phi(p_i)\phi(p_j) = U_{\phi(p_j)}\phi(p_i) + \phi(p_i)\phi(p_j)^2 \\ &= 2\big(\phi(p_i)\circ\phi(p_j)\big)\phi(p_j) = 2\phi(\underbrace{p_i \circ p_j}_{=0})\phi(p_j) = 2\phi(0)\phi(p_j) = 0,\end{aligned}$$

$$\phi(s)^2 = \phi(s \circ s) = \phi(\mathbf{1}) = \mathbf{1},$$

$$\phi(s')^2 = \phi(s' \circ s') = \phi(\mathbf{1}) = \mathbf{1},$$

$$\phi(s)\phi(p_1) = \phi(s)\phi(p_1)\phi(s)^2 = U_{\phi(s)}\phi(p_1)\phi(s) = \phi\big(U_s p_1\big)\phi(s) = \phi(p_2)\phi(s),$$

$$\phi(p_1)\phi(s) = \phi(s)^2\phi(p_1)\phi(s) = \phi(s)U_{\phi(s)}\phi(p_1) = \phi(s)\phi\big(U_s p_1\big) = \phi(s)\phi(p_2),$$

$$\phi(s')\phi(p_1) = \phi(s')\phi(p_1)\phi(s')^2 = U_{\phi(s')}\phi(p_1)\phi(s') = \phi\big(U_{s'} p_1\big)\phi(s') = \phi(p_3)\phi(s'),$$

$$\phi(p_1)\phi(s') = \phi(s')^2\phi(p_1)\phi(s') = \phi(s')U_{\phi(s')}\phi(p_1) = \phi(s')\phi\big(U_{s'} p_1\big) = \phi(s')\phi(p_3).$$

Therefore

$$\begin{cases}\begin{aligned}[\phi(u)^2, \phi(v)] &= [\phi(p_1) + \phi(p_2), \phi(s')\phi(p_1) + \phi(p_1)\phi(s')] \\ &= \phi(p_1)\phi(s')\phi(p_1) + \phi(p_1)^2\phi(s') + \phi(p_2)\phi(s')\phi(p_1) \\ &\quad + \phi(p_2)\phi(p_1)\phi(s') - \phi(s')\phi(p_1)^2 - \phi(p_1)\phi(s')\phi(p_1) \\ &\quad - \phi(s')\phi(p_1)\phi(p_2) - \phi(p_1)\phi(s')\phi(p_2) \\ &= \phi(p_1)\phi(s') - \phi(s')\phi(p_1) + \underbrace{\phi(p_2)\phi(p_3)}_{=0}\phi(s') \\ &\quad + \underbrace{\phi(p_2)\phi(p_1)}_{=0}\phi(s') - \phi(s')\underbrace{\phi(p_1)\phi(p_2)}_{=0} - \phi(s')\underbrace{\phi(p_3)\phi(p_2)}_{=0} \\ &= \phi(p_1)\phi(s') - \phi(s')\phi(p_1),\end{aligned}\end{cases} \tag{2.2}$$

and similarly

$$[\phi(u), \phi(v)^2] = \phi(s)\phi(p_1) - \phi(p_1)\phi(s). \tag{2.3}$$

On the other hand,

$$\left\{\begin{aligned}[\phi(u),\phi(v)] &= [\phi(s)\phi(p_1)+\phi(p_1)\phi(s),\phi(s')\phi(p_1)+\phi(p_1)\phi(s')]\\ &= \phi(s)\phi(p_1)\phi(s')\phi(p_1)+\phi(s)\phi(p_1)^2\phi(s')\\ &+\phi(p_1)\phi(s)\phi(s')\phi(p_1)+\phi(p_1)\phi(s)\phi(p_1)\phi(s')\\ &-\phi(s')\phi(p_1)\phi(s)\phi(p_1)-\phi(p_1)\phi(s')\phi(s)\phi(p_1)\\ &-\phi(s')\phi(p_1)^2\phi(s)-\phi(p_1)\phi(s')\phi(p_1)\phi(s)\\ &= \phi(s)\underbrace{\phi(p_1)\phi(p_3)}_{=0}\phi(s')+\phi(s)\phi(p_1)\phi(s')\\ &+\phi(s)\underbrace{\phi(p_2)\phi(p_3)}_{=0}\phi(s')+\phi(s)\underbrace{\phi(p_2)\phi(p_1)}_{=0}\phi(s')\\ &-\phi(s')\underbrace{\phi(p_1)\phi(p_2)}_{=0}\phi(s)-\phi(s')\underbrace{\phi(p_3)\phi(p_2)}_{=0}\phi(s)\\ &-\phi(s')\phi(p_1)\phi(s)-\phi(s')\underbrace{\phi(p_3)\phi(p_1)}_{=0}\phi(s)\\ &= \phi(s)\phi(p_1)\phi(s')-\phi(s')\phi(p_1)\phi(s).\end{aligned}\right. \tag{2.4}$$

Observe that

$$\begin{aligned}\mathbf{1} &= \phi(\mathbf{1}) = \phi(p_1)+\phi(p_2)+\phi(p_3)+\phi(p_4)\\ &= \phi(p_1)+\phi\big(U_s p_1\big)+\phi\big(U_{s'}p_1\big)+\phi\big(U_{s''}p_1\big)\\ &= \phi(p_1)+\phi(s)\phi(p_1)\phi(s)+\phi(s')\phi(p_1)\phi(s')+\phi(s'')\phi(p_1)\phi(s'').\end{aligned}$$

The previous identity implies that $\phi(p_1)\neq 0$, and, since

$$\phi(p_1)=\phi(s)\phi(p_2)\phi(s)=\phi(s')\phi(p_3)\phi(s')=\phi(s'')\phi(p_4)\phi(s''),$$

it follows that

$$\phi(p_2),\phi(p_3),\phi(p_4)\neq 0.$$

Suppose that $\lambda,\mu,\nu\in\mathbb{C}$ are such that

$$\lambda[\phi(u)^2,\phi(v)]+\mu[\phi(u),\phi(v)^2]+\nu[\phi(u),\phi(v)]=0.$$

Then, having in mind (2.2), (2.3) and (2.4), we get

$$\begin{aligned}0 &= \big(\lambda[\phi(u)^2,\phi(v)]+\mu[\phi(u),\phi(v)^2]+\nu[\phi(u),\phi(v)]\big)\phi(p_1)\\ &= \lambda\big(\phi(p_1)\phi(s')\phi(p_1)-\phi(s')\phi(p_1)^2\big)\\ &+\mu\big(\phi(s)\phi(p_1)^2-\phi(p_1)\phi(s)\phi(p_1)\big)\\ &+\nu\big(\phi(s)\phi(p_1)\phi(s')\phi(p_1)-\phi(s')\phi(p_1)\phi(s)\phi(p_1)\big)\\ &= \lambda\big(\underbrace{\phi(p_1)\phi(p_3)}_{=0}\phi(s')-\phi(s')\phi(p_1)\big)\\ &+\mu\big(\phi(s)\phi(p_1)-\underbrace{\phi(p_1)\phi(p_2)}_{=0}\phi(s)\big)\end{aligned}$$

$$
\begin{aligned}
&+\nu\big(\phi(s)\underbrace{\phi(p_1)\phi(p_3)}_{=0}\phi(s')-\phi(s')\underbrace{\phi(p_1)\phi(p_2)}_{=0}\phi(s)\big)\\
&=-\lambda\phi(s')\phi(p_1)+\mu\phi(s)\phi(p_1)=-\lambda\phi(p_3)\phi(s')+\mu\phi(p_2)\phi(s),
\end{aligned}
$$

whence

$$0=\phi(p_3)\big(-\lambda\phi(p_3)\phi(s')+\mu\phi(p_2)\phi(s)\big)=-\lambda\phi(p_3)\phi(s'),$$

and

$$0=\big(-\lambda\phi(p_3)\phi(s')\big)\phi(s')=-\lambda\phi(p_3),$$

which gives $\lambda=0$. Therefore

$$\mu\phi(p_2)\phi(s)=0,$$

and

$$\mu\phi(p_2)=\mu\phi(p_2)\phi(s)^2=\mu\big(\phi(p_2)\phi(s)\big)\phi(s)=0,$$

which yields $\mu=0$. We thus obtain

$$\nu[\phi(u),\phi(v)]=0.$$

Hence

$$
\begin{aligned}
0&=\nu[\phi(u),\phi(v)]\phi(p_3)=\nu\big(\phi(s)\phi(p_1)\phi(s')\phi(p_3)-\phi(s')\phi(p_1)\phi(s)\phi(p_3)\big)\\
&=\nu\phi(s)\phi(s')\underbrace{\phi(p_3)^2}_{=\phi(p_3)}-\phi(s')\phi(s)\underbrace{\phi(p_1)\phi(p_3)}_{=0}=\nu\phi(s)\phi(s')\phi(p_3),
\end{aligned}
$$

and therefore

$$0=\phi(s)\phi(s')\big(\nu\phi(s)\phi(s')\phi(p_3)\big)=\nu\phi(p_3),$$

which gives $\nu=0$.

Case II: We assume next that $\mathfrak{J}_{sa}=\bigoplus_{n\in N}^{\infty}J_n$, where $N\subset\{n\in\mathbb{N}\colon n\geq 3\}$ and J_n is a JBW-algebra of type I_n for each $n\in N$ (cf. [21, 5.3.3 and Theorem 5.3.5]). For each $n\in N$ take $j_n\in\mathbb{N}$ and $k_n\in\{0,1,2\}$ such that

$$n=3j_n+k_n,$$

and take $e_{n,1},\ldots,e_{n,n}$ pairwise orthogonal projections in J_n such that

$$e_{n,1}+\cdots+e_{n,n}=\mathbf{1}_{J_n}$$

and such that they are pairwise exchangeable by a symmetry (cf. [1, Definition 3.21] or [21, 5.3.3]). We define pairwise orthogonal projections p_1,p_2,p_3,q_1,q_2 by

$$p_1 = \sum_{n\in N}\sum_{j=0}^{j_n-1} e_{n,1+3j}, \quad p_2 = \sum_{n\in N}\sum_{j=0}^{j_n-1} e_{n,2+3j}, \quad p_3 = \sum_{n\in N}\sum_{j=0}^{j_n-1} e_{n,3+3j},$$
$$q_1 = \sum_{n\in N:\ k_n\in\{1,2\}} e_{n,1+3j_n}, \quad \text{and} \quad q_2 = \sum_{n\in N:\ k_n=2} e_{n,2+3j_n}.$$

Observe that q_1 and q_2 are possibly zero and

$$p_1 + p_2 + p_3 + q_1 + q_2 = \mathbf{1},$$

and, further, p_1, p_2, p_3 are pairwise exchangeable by a symmetry (cf. [21, Lemma 5.2.9]).
(i) Take symmetries $s, s' \in \mathfrak{J}_{sa}$ such that

$$U_s(p_1) = p_2, \text{ and } U_{s'}(p_1) = p_3.$$

Define $u, v \in \mathfrak{J}_{sa}$ by

$$u = 2s \circ p_1, \text{ and } v = 2s' \circ p_1,$$

so that, as in the previous case,

$$u^2 = p_1 + p_2, \text{ and } v^2 = p_1 + p_3.$$

Then

$$U_{p_3}(\mathbf{1}) = p_3, \quad U_{p_3}(u) = 0, \quad \text{and} \quad U_{p_3}(u^2) = 0,$$

which yields

$$\begin{aligned} U_{s'}U_{p_3}(\mathbf{1}) &= p_1, & U_{s'}U_{p_3}(u) &= 0, & U_{s'}U_{p_3}(u^2) &= 0,\\ U_sU_{s'}U_{p_3}(\mathbf{1}) &= p_2, & U_sU_{s'}U_{p_3}(u) &= 0, & U_sU_{s'}U_{p_3}(u^2) &= 0. \end{aligned}$$

Further,

$$U_{q_1+q_2}(\mathbf{1}) = q_1 + q_2, \quad U_{q_1+q_2}(u) = 0, \quad \text{and} \quad U_{q_1+q_2}(u^2) = 0.$$

We define $\mathscr{E}_0 \in \mathscr{E}\ell_{\mathfrak{J}_{sa}}(\mathfrak{J})$ by

$$\mathscr{E}_0 = U_{p_3} + U_{s'}U_{p_3} + U_sU_{s'}U_{p_3} + U_{q_1+q_2},$$

and we check at once that

$$\mathscr{E}_0(\mathbf{1}) = \mathbf{1}, \quad \mathscr{E}_0(u) = 0, \text{ and } \mathscr{E}_0(u^2) = 0.$$

We now define projections $r_1, r_2 \in \mathfrak{J}_{sa}$ by

$$r_1 = \sum_{n\in N:\ k_n\in\{1,2\}} e_{n,2} \le p_2, \text{ and } r_2 = \sum_{n\in N:\ k_n=2} e_{n,2} \le p_2.$$

Note that the projections r_1 and q_1 are exchanged by a symmetry and the projections r_2 and q_2 are also exchanged by a symmetry (cf. [21, Lemma 5.2.9]). Take symmetries $t, t' \in \mathfrak{J}$ such that $U_t(r_1) = q_1$, and $U_{t'}(r_2) = q_2$. Observe that

$$(U_{p_1} - U_{s'}U_{p_3})(\mathbf{1}) = 0,\ (U_{p_1} - U_{s'}U_{p_3})(u) = 0,\ \text{and}\ (U_{p_1} - U_{s'}U_{p_3})(u^2) = p_1,$$

and therefore

$$\begin{aligned} U_s(U_{p_1} - U_{s'}U_{p_3})(\mathbf{1}) &= 0, & U_s(U_{p_1} - U_{s'}U_{p_3})(u) &= 0, & U_s(U_{p_1} - U_{s'}U_{p_3})(u^2) &= p_2,\\ U_{s'}(U_{p_1} - U_{s'}U_{p_3})(\mathbf{1}) &= 0, & U_{s'}(U_{p_1} - U_{s'}U_{p_3})(u) &= 0, & U_{s'}(U_{p_1} - U_{s'}U_{p_3})(u^2) &= p_3. \end{aligned}$$

On the other hand,

$$(U_{r_1} - U_tU_{q_1})(\mathbf{1}) = 0,\ (U_{r_1} - U_tU_{q_1})(u) = 0,\ \text{and}\ (U_{r_1} - U_tU_{q_1})(u^2) = r_1,$$

which gives

$$U_t(U_{r_1} - U_tU_{q_1})(\mathbf{1}) = 0,\ U_t(U_{r_1} - U_tU_{q_1})(u) = 0,\ \text{and}\ U_t(U_{r_1} - U_tU_{q_1})(u^2) = q_1,$$

and furthermore

$$(U_{r_2} - U_{t'}U_{q_2})(\mathbf{1}) = 0,\ (U_{r_2} - U_{t'}U_{q_2})(u) = 0,\ \text{and}\ (U_{r_2} - U_{t'}U_{q_2})(u^2) = r_2,$$

so that

$$U_{t'}(U_{r_2} - U_{t'}U_{q_2})(\mathbf{1}) = 0,\ U_{t'}(U_{r_2} - U_{t'}U_{q_2})(u) = 0,\ \text{and}\ U_{t'}(U_{r_2} - U_{t'}U_{q_2})(u^2) = q_2.$$

By defining

$$\begin{aligned} \mathscr{E}_2 =& (U_{p_1} - U_{s'}U_{p_3}) + U_s(U_{p_1} - U_{s'}U_{p_3}) + U_{s'}(U_{p_1} - U_{s'}U_{p_3})\\ &+ U_t(U_{r_1} - U_tU_{q_1}) + U_{t'}(U_{r_2} - U_{t'}U_{q_2}), \end{aligned}$$

we obtain $\mathscr{E}_2 \in \mathscr{E}\ell_{\mathfrak{J}_{sa}}(\mathfrak{J})$ and

$$\mathscr{E}_2(\mathbf{1}) = 0, \quad \mathscr{E}_2(u) = 0, \quad \mathscr{E}_2(u^2) = \mathbf{1}.$$

Finally, by defining

$$\mathscr{E}_1 = \mathscr{E}_2 M_u$$

we get $\mathscr{E}_1 \in \mathscr{E}\ell_{\mathfrak{J}_{sa}}(\mathfrak{J})$ and

$$\mathscr{E}_1(\mathbf{1}) = 0, \quad \mathscr{E}_1(u) = \mathbf{1}, \quad \mathscr{E}_1(u^2) = 0.$$

(ii) Let $\mathfrak{A}$ be a unital complex associative algebra, and let $\phi\colon \mathfrak{J} \to \mathfrak{A}$ a unital Jordan homomorphism.

A similarly argument to that employed in the previous *Case I* gives

$$\begin{aligned} [\phi(u)^2, \phi(v)] &= \phi(p_1)\phi(s') - \phi(s')\phi(p_1),\\ [\phi(u), \phi(v^2)] &= \phi(s)\phi(p_1) - \phi(p_1)\phi(s),\ \text{and}\\ [\phi(u), \phi(v)] &= \phi(s)\phi(p_1)\phi(s') - \phi(s')\phi(p_1)\phi(s). \end{aligned}$$

Next, in order to prove that $\phi(p_1), \phi(p_2), \phi(p_3)$ are all non-zero, we shall argue in a different way. Let us assume, contrary to the desired statement, that $\phi(p_1) = 0$. Then we have $\phi(u) = 2\phi(s) \circ \phi(p_1) = 0$ and thus Lemma 2.2(i) implies the existence of $\mathscr{F}_1 \in \mathscr{E}\ell(\mathfrak{A})$

$$0 = \mathscr{F}_1(\phi(u)) = \phi\big(\mathscr{E}_1(u)\big) = \phi(\mathbf{1}) = \mathbf{1},$$

which is impossible. Now, as in *Case I*, we have

$$\phi(p_1) = \phi(s)\phi(p_2)\phi(s) = \phi(s')\phi(p_3)\phi(s'),$$

which implies that $\phi(p_2)$, and $\phi(p_3)$ must be both non-zero. The linear independence of $[\phi(u)^2, \phi(v)]$, $[\phi(u), \phi(v)^2]$, and $[\phi(u), \phi(v)]$ follows as in *Case I* too.

General case. By [21, Theorems 5.1.5 and 5.3.5], $\mathfrak{J}_{sa}$ admits a unique decomposition in the form

$$\mathfrak{J}_{sa} = \underbrace{J_{I_1} \oplus J_{I_2} \oplus \ldots}_{J_2} \oplus \underbrace{J_{I_\infty} \oplus J_{II} \oplus J_{III}}_{J_1},$$

where each J_k is either 0 or a JBW-algebra in the class studied in *Case k* above. This implies that $\mathfrak{J}$ splits into a direct sum

$$\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_2,$$

where $\mathfrak{J}_1$ and $\mathfrak{J}_2$ are JBW*-algebras such that $\mathfrak{J}_{1sa} = J_1$ and $\mathfrak{J}_{2sa} = J_2$, where, according to the observation made at the beginning of the discussion of *Case I*, $\mathfrak{J}_1$ fits the condition required in that case. Since $\mathfrak{J}$ contains no direct summands of type I_1 nor I_2, it follows that $J_{I_1} = J_{I_2} = 0$ and therefore $\mathfrak{J}_2$ fits the assumptions in *Case II*. We denote by $\mathbf{1}_1$ and $\mathbf{1}_2$ the units of $\mathfrak{J}_1$ and $\mathfrak{J}_2$, respectively.

Let $u_1, v_1 \in \mathfrak{J}_1$ and $\mathscr{E}_{1,0}, \mathscr{E}_{1,1}$, $\mathscr{E}_{1,2}$ denote the elements and the elementary operators given by *Case I* for $\mathfrak{J}_1$, and let $u_2, v_2 \in \mathfrak{J}_2$ and $\mathscr{E}_{2,0}, \mathscr{E}_{2,1}$, $\mathscr{E}_{2,2}$ the corresponding elements and elementary operators for $\mathfrak{J}_2$ whose existence is guaranteed by *Case II*. Setting

$$u = u_1 + u_2,\ v = v_1 + v_2 \in \mathfrak{J}_{sa},$$

and

$$\mathscr{E}_i = \mathscr{E}_{1,i} U_{\mathbf{1}_1} + \mathscr{E}_{2,i} U_{\mathbf{1}_2} \in \mathscr{E}\ell_{\mathfrak{J}_{sa}}(\mathfrak{J}) \quad (i \in \{0, 1, 2\}),$$

we get the desired conclusion in (i).

We shall finally prove the statement in (ii). For this purpose, let $\mathfrak{A}$ be a unital complex associative algebra, and let $\phi\colon \mathfrak{J} \to \mathfrak{A}$ be a unital Jordan homomorphism. Then

$$\phi(\mathbf{1}_k)^2 = \phi(\mathbf{1}_k^2) = \phi(\mathbf{1}_k), \quad k \in \{1, 2\},$$

and

$$\phi(\mathbf{1}_1)\phi(\mathbf{1}_2) = \phi(\mathbf{1}_2)\phi(\mathbf{1}_1) = 0.$$

We define subalgebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ of $\mathfrak{A}$ by

$$\mathfrak{A}_k = \big\{a \in \mathfrak{A}\colon a\phi(\mathbf{1}_k) = \phi(\mathbf{1}_k)a = a\big\}, \quad k \in \{1, 2\},$$

and we observe that

$$\mathfrak{A}_1\mathfrak{A}_2 = \mathfrak{A}_2\mathfrak{A}_1 = \{0\}. \tag{2.5}$$

Further, for each $k \in \{1,2\}$ and each $x \in \mathfrak{J}_k$,

$$\phi(x) = \phi\big(U_{\mathbf{1}_k} x\big) = U_{\phi(\mathbf{1}_k)}\phi(x) = \phi(\mathbf{1}_k)\phi(x)\phi(\mathbf{1}_k),$$

which immediately yields $\phi(x) \in \mathfrak{A}_k$, and ϕ gives a homomorphism from $\mathfrak{J}_k$ to $\mathfrak{A}_k$. Since

$$\phi(\mathbf{1}_1) + \phi(\mathbf{1}_2) = \phi(\mathbf{1}) = \mathbf{1},$$

it follows that the elements $\phi(\mathbf{1}_1)$ and $\phi(\mathbf{1}_2)$ cannot be both zero. Suppose that $\phi(\mathbf{1}_1), \phi(\mathbf{1}_2) \neq 0$. Then each $\mathfrak{A}_k$ is unital with unit $\phi(\mathbf{1}_k)$, ϕ yields a unital homomorphism from $\mathfrak{J}_k$ to $\mathfrak{A}_k$, and, consequently,

$$[\phi(u_k)^2, \phi(v_k)], [\phi(u_k), \phi(v_k)^2], \text{ and } [\phi(u_k), \phi(v_k)],$$

are linearly independent. Since

$$\begin{aligned}
[\phi(u)^2, \phi(v)] &= [\phi(u_1)^2, \phi(v_1)] + [\phi(u_2)^2, \phi(v_2)],\\
[\phi(u), \phi(v)^2] &= [\phi(u_1), \phi(v_1)^2] + [\phi(u_2), \phi(v_2)^2],\\
[\phi(u), \phi(v)] &= [\phi(u_1), \phi(v_1)] + [\phi(u_2), \phi(v_2)],
\end{aligned}$$

by using (2.5) we obtain the desired linear independence in statement (ii). We can now suppose that $\phi(\mathbf{1}_1) = 0$. Then $\phi(\mathfrak{J}_1) = 0$, $\mathfrak{A}_2$ is a unital algebra, and ϕ gives a unital homomorphism from $\mathfrak{J}_2$ to $\mathfrak{A}_2$. Hence $[\phi(u)^2, \phi(v)] = [\phi(u_2)^2, \phi(v_2)]$, $[\phi(u), \phi(v)^2] = [\phi(u_2), \phi(v_2)^2]$, and $[\phi(u), \phi(v)] = [\phi(u_2), \phi(v_2)]$, are linearly independent. The same reasoning applies to the case $\phi(\mathbf{1}_2) = 0$.

The statement concerning the norm of $\mathscr{E}_i$ is clear from the definitions. □

The next result also considers JBW*-algebras with non-necessarily zero type I_2 part. The conclusion is, of course, weaker than in the previous Proposition 2.3.

Proposition 2.4. *Let $\mathfrak{J}$ be a JBW*-algebra with no direct summands of type I_1. Then there exist an element $u \in \mathfrak{J}_{sa}$ and $\mathscr{E}_0, \mathscr{E}_1 \in \mathscr{E}\ell_{\mathfrak{J}_{sa}}(\mathfrak{J})$ satisfying $\mathscr{E}_i(u^j) = \delta_{ij}\mathbf{1}$, and $\|\mathscr{E}_i\| \leq 10$, for all i, j in $\{0,1\}$.*

Proof. By representation theory, $\mathfrak{J}$ decomposes as the orthogonal sum of two JBW*-subalgebras $\mathfrak{J} = \mathfrak{J}_{spin} \oplus \mathfrak{J}_{nspin}$, where $\mathfrak{J}_{spin}$ is a JBW*-algebra of type I_2 and $\mathfrak{J}_{nspin}$ contains no direct summands of type I_1 or I_2 (cf. [21, Theorems 5.1.5 and 5.3.5]). In the case of $\mathfrak{J}_{nspin}$, the desired conclusion follows from Proposition 2.3. Arguing as in the final part of the proof of the just quoted proposition, and employing the orthogonality of $\mathfrak{J}_{spin}$ and $\mathfrak{J}_{nspin}$, it suffices to show that the result also holds for $\mathfrak{J}_{spin}$.

By structure theory, we can always find two orthogonal projections p_1, p_2 in $\mathfrak{J}_{spin}$ which are exchangeable by a symmetry $s \in \mathfrak{J}_{spin}$ (see [21, 5.3.3]). Take $u = s$ and observe that $2p_1 \circ s = 2p_2 \circ s = s$. By defining $\mathscr{E}_1 = M_s M_{2p_2} M_{2p_1}$ and $\mathscr{E}_0 = \mathscr{E}_1 M_s$, it is easy to check that $\mathscr{E}_i(u^j) = \delta_{ij}\mathbf{1}$, for all i, j in $\{0,1\}$. □

In order to obtain a refinement of Lemma 2.2, we recall some basic results on the strong*-topology of a JBW*-algebra $\mathfrak{J}$. For each positive normal functional φ in the predual, $\mathfrak{J}_*$, of $\mathfrak{J}$, the sesquilinear form $(x, y) \mapsto \varphi(x \circ y^*)$ is positive, and defines a preHilbertian seminorm $\|x\|_\varphi^2 := \varphi(x \circ x^*)$ on $\mathfrak{J}$. The *strong*-topology* of $\mathfrak{J}$, denoted by $S^*(\mathfrak{J}, \mathfrak{J}_*)$, is the topology generated by all the seminorms $\|\cdot\|_\varphi$ with φ running in the set of all normal states of $\mathfrak{J}$ (see, for example, [5]). In case that $\mathfrak{J}$ is a von Neumann algebra regarded as a JBW*-algebra, the strong*-topology defined above is precisely the usual C*-algebra strong*-topology (see [5, §3]). The strong*-topology is stronger than the weak*-topology of $\mathfrak{J}$ (see [5, Theorem 3.2]). When $\mathfrak{J}$ is regarded as an element in the strictly wider class of JBW*-triples, the strong*-topology, as JBW*-triple, as defined in [5], agrees with the strong*-topology we have just defined (see [37, Proposition 3] or [35, §4]).

It is further known that the strong*-topology of $\mathfrak{J}$ is compatible with the duality $(\mathfrak{J}, \mathfrak{J}_*)$ (cf. [35, Corollary 9]). Consequently, a convex subset of $\mathfrak{J}$ has the same closure with respect to the strong*- and the weak*-topologies. Moreover, a linear map between JBW*-algebras is strong*-continuous if, and only if, it is weak*-continuous (compare [37, Corollary 3] or [35, comments in page 621]). A consequence of this fact implies that every elementary operator on a JBW*-algebra is strong*-continuous.

One additional property of the strong*-topology remains to be recalled. The so-called *Kaplansky density theorem* affirms that if $\mathfrak{B}$ is a weak*-dense JB*-subalgebra of a JBW*-algebra $\mathfrak{J}$, then the closed unit ball of $\mathfrak{B}$ is strong*-dense in the closed unit ball of $\mathfrak{J}$ (cf. [5, Corollary 3.3] or [1, Proposition 2.4]).

After gathering the previous basic properties of the strong*-topology, we can now establish the following generalization of Lemma 2.2.

Lemma 2.5. *Let $\mathfrak{J}$ be a JB*-algebra, $\mathfrak{M}$ a JBW*-algebra, and $\pi\colon \mathfrak{J} \to \mathfrak{M}$ be a Jordan *-homomorphism with weak*-dense image. Then for each $\mathscr{E} \in \mathscr{E}\ell(\mathfrak{J})$, there exists a unique $\widetilde{\mathscr{E}} \in \mathscr{E}\ell(\mathfrak{M})$ such that $\pi\mathscr{E} = \widetilde{\mathscr{E}}\pi$ and $\|\widetilde{\mathscr{E}}\| \leq \|\mathscr{E}\|$.*

Proof. The existence and uniqueness are guaranteed by previous Lemma 2.2. To prove the statement concerning the norm of $\widetilde{\mathscr{E}}$, let us fix z in the closed unit ball of $\mathfrak{M}$ and an arbitrary $\varepsilon > 0$. We begin by observing that, by Kaplansky density theorem [5, Corollary 3.3], we can find a net $(\pi(x_j))_j$ in the closed unit ball of $\pi(\mathfrak{J})$ converging to z in the strong*-topology of $\mathfrak{M}$, and thus $(\widetilde{\mathscr{E}}\pi(x_j))_j \to \widetilde{\mathscr{E}}(z)$ in the strong*-topology, and in the weak*-topology of $\mathfrak{M}$. Since $\ker(\pi)$ is a norm-closed Jordan ideal of $\mathfrak{J}$, and the mapping $\widehat{\pi} : \mathfrak{J}/\ker(\pi) \to \mathfrak{M}$, $\widehat{\pi}(x + \ker(\pi)) := \pi(x)$ is an isometric Jordan *-monomorphism for all j, we can find $k_j \in \ker(\pi)$, depending on x_j, such that the inequality $\|x_j + k_j\| \leq (1+\varepsilon)\|\pi(x_j)\| \leq (1+\varepsilon)$ holds. Finally, since

$$\widetilde{\mathscr{E}}(z) = w^*\text{-}\lim_j \widetilde{\mathscr{E}}\pi(x_j + k_j) = w^*\text{-}\lim_j \pi\mathscr{E}(x_j + k_j),$$

with $\|\pi\mathscr{E}(x_j + k_j)\| \leq (1+\varepsilon)\|\mathscr{E}\|$, it follows that $\|\widetilde{\mathscr{E}}(z)\| \leq (1+\varepsilon)\|\mathscr{E}\|$. $\square$

2.1. Linear dependence in prime C-algebras*

We devote this subsection to survey some results on linear independence of elements in a C*-algebra, and stating several variants to be applied in our Jordan setting.

In this section we shall employ the extended centroid of a prime C*-algebra regarded as a prime ring, in several of our arguments. For this purpose, we recall some basic results. Let R be a (possibly non-unital associative) semiprime ring. Following the standard notation (see [3, §2.1]), we say that an *essentially defined double centraliser* on R is a triple (f, g, I), where f and g are mappings from an essential ring two-sided ideal I of R into R, f is a left R-module homomorphism, g is a right R-module homomorphism, and $f(x)y = xg(y)$ for all $x, y \in I$. Two essentially defined double centralisers (f_1, g_1, I_1) and (f_2, g_2, I_2) are equivalent if f_1 and f_2 agree on their common domain $I_1 \cap I_2$ –note that g_1 and g_2 agree on $I_1 \cap I_2$ too. The *symmetric ring of quotients of* R, $Q_s(R)$, is defined as the set of equivalence classes of essentially defined double centralisers on R. Addition and multiplication operations on $Q_s(R)$ are defined by pointwise addition and composition on the appropriate domains, that is,

$$[(f_1, g_1, I_1)] + [(f_2, g_2, I_2)] := [((f_1 + f_2)|_{I_1\cap I_2}, (g_1 + g_2)|_{I_1\cap I_2}, I_1 \cap I_2)],$$

and

$$[(f_1, g_1, I_1)][(f_2, g_2, I_2)] := [(f_2 f_1)|_{I_1 I_2}, (g_1 g_2)|_{I_1 I_2}, I_1 I_2)].$$

Endowed with these operations, $Q_s(R)$ becomes a semiprime ring with identity $[(id, id, R)]$. The ring R embeds into $Q_s(R)$ via the mapping $a \hookrightarrow [(R_a, L_a, R)]$ $(a \in R)$. The *extended centroid of* R is defined as the centre of $Q_s(R)$, and it is usually denoted by $C(R)$ [3, Definition 2.1.1 and Definition 2.1.4]. The extended centroid of R can be identified with the ring of equivalence classes of essentially defined bimodule homomorphisms on R (cf. comments after Definition 2.1.4 in [3]).

For later purposes, we present an argument to check that the extended centroid of a prime C*-algebra regarded as a ring is the complex field.

Remark 2.6. Let A be a prime C*-algebra, regarded as a prime ring with respect to its natural product. Let I be a non-zero (not necessarily closed) two-sided ring ideal of A, and let $f\colon I \to A$ be a ring A-bimodule homomorphism, that is, an additive map such that

$$f(ax) = f(a)x, \ f(xa) = xf(a), \qquad \forall a \in I, \ \forall x \in A.$$

Clearly, the linear span, $\widetilde{I}$, of I is a two-sided algebra ideal of A. We define a mapping $\widetilde{f}\colon \widetilde{I} \to A$ by $\widetilde{f}\Big(\sum_j \alpha_j a_j\Big) := \sum_j \alpha_j f(a_j)$. If $\sum_j \alpha_j a_j = \sum_k \beta_k b_k$, then $x\Big(\sum_j \alpha_j a_j\Big) = x\Big(\sum_k \beta_k b_k\Big)$, for every $x \in A$. Now, by applying that f is a ring A-bimodule homomorphism we get

$$\begin{aligned} x\Big(\sum_j \alpha_j f(a_j) - \sum_k \beta_k f(b_k)\Big) &= \sum_j x\alpha_j f(a_j) - \sum_k x\beta_k f(b_k) \\ &= f\Big(\sum_j x\alpha_j a_j\Big) - f\Big(\sum_k x\beta_k b_k\Big) \\ &= f\Big(x\Big(\sum_j \alpha_j a_j\Big)\Big) - f\Big(x\Big(\sum_k \beta_k b_k\Big)\Big) = 0, \end{aligned}$$

which, by the primeness of A and the arbitrariness of $x \in A$, gives $\sum_j \alpha_j f(a_j) = \sum_k \beta_k f(b_k)$, and thus $\widetilde{f}$ is a well-defined A-bimodule homomorphism. Since A is prime, by [31, Proposition 2.5], there exists $\alpha \in \mathbb{C}$ such that $\widetilde{f}(x) = \alpha x$ for each $x \in \widetilde{I}$, and hence

$$f(a) = \alpha a, \ \text{for all } a \in I.$$

This shows, in particular, that $C(A) = \mathbb{C}$ when A is regarded as a prime ring.

We continue by recalling another useful tool. Let $n \in \mathbb{N}$ with $n \geq 2$. The noncommutative polynomial c_n in the non-necessarily commutative indeterminates $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_{n-1}$ defined by

$$c_n(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_{n-1}) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) \xi_{\sigma(1)} \eta_1 \xi_{\sigma(2)} \eta_2 \cdots \xi_{\sigma(n-1)} \eta_{n-1} \xi_{\sigma(n)},$$

is called the *nth Capelli polynomial* (cf. [10, Definition 6.11]). Here, S_n stands for the symmetric group of order n.

Our next result shows how Capelli's polynomials can be employed to determine the linear independence of a set of vectors in a prime C*-algebra.

Lemma 2.7. *Let A be a prime C*-algebra, and let $a \in A^n$, $n \geq 2$. Then $a_1, \ldots, a_n$ are linearly dependent if and only if*

$$c_n(a_1, \ldots, a_n, x_1, \ldots, x_{n-1}) = 0, \quad \textit{for all } x_1, \ldots, x_{n-1} \textit{ in } A.$$

Proof. The desired conclusion follows from [10, Theorem 7.45] by just recalling the natural embedding of A inside $Q_s(A)$ together with the fact that $C(A) = \mathbb{C}$ (see Remark 2.6). □

We now state some results on linear identities in the line of [30,31,28], which will be required in our arguments.

Lemma 2.8. *Let A be a C*–algebra, and let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ in A^n. Assume that $a_1, \ldots, a_p$ are linearly independent for some $1 \le p \le n$. Then the following statements hold.*

(i) *If A is prime and $\displaystyle\sum_{j=1}^{n} a_j x b_j = 0$, for all $x \in A$, then each b_j, with $j \in \{1, \ldots, p\}$, is a linear combination of $b_{p+1}, \ldots, b_n$ (in case $p = n$, this should be understood as $b_j = 0$, for each $j \in \{1, \ldots, n\}$).*

(ii) *If A is prime, τ is a ring involution on A, $a_1, \ldots, a_p \in H(A, \tau) = \{z \in A : \tau(z) = z\}$ and*

$$\sum_{j=1}^{n} a_j x b_j = 0, \ \textit{for all } x \in H(A, \tau),$$

then each b_j, with $j \in \{1, \ldots, p\}$, is a linear combination of $b_{p+1}, \ldots, b_n$ (in case $p = n$, this should be understood as $b_j = 0$, for each $j \in \{1, \ldots, n\}$).

(iii) *If τ is a *-anti-automorphism of period-2 on A, and A decomposes in the form $A = N \oplus^{\infty} \tau(N)$, where N is a prime C*-algebra and a closed ideal of A, $a_1, \ldots, a_p \in H(A, \tau)$ and*

$$\sum_{j=1}^{n} a_j x b_j = 0, \ \textit{for all } x \in H(A, \tau),$$

then each b_j, with $j \in \{1, \ldots, p\}$, is a linear combination of $b_{p+1}, \ldots, b_n$ (in case $p = n$, this should be understood as $b_j = 0$, for each $j \in \{1, \ldots, n\}$).

Proof. (i) By renumbering, we may suppose that $\{a_1, \ldots, a_m\}$, with $p \le m \le n$, is a maximal linearly independent subset of $\{a_1, \ldots, a_n\}$. Then we apply [31, Theorem 4.1] to the opposite algebra A^{op} of A, to obtain that each b_j, with $j \in \{1, \ldots, m\}$, is a linear combination of $b_{m+1}, \ldots, b_n$.

(ii) The previous Remark 2.6 shows that the *extended centroid* of A regarded as a ring is isomorphic to $\mathbb{C}$, then the desired result is a consequence of [28, Lemma 3].

(iii) As in the proof of Lemma 1.3, $H(A, \tau) = \{a + \tau(a) : a \in N\}$ admits a structure of C*-subalgebra of $N \oplus^{\infty} \tau(N)^{op}$, making it C*-isomorphic to N. Let π denote the canonical projection of A onto N, which is a C*-homomorphism. Since $\displaystyle\sum_{j=1}^{n} a_j x b_j = 0$, for all $x \in H(A, \tau)$, we conclude that $\displaystyle\sum_{j=1}^{n} \pi(a_j)\pi(x)\pi(b_j) = 0$, for all $x \in H(A, \tau)$, and N is a prime C*-algebra, we deduce from (i) that each $\pi(b_j)$, with $j \in \{1, \ldots, p\}$, is a linear combination of $\pi(b_{p+1}), \ldots, \pi(b_n)$, which in turn gives that each $b_j = \pi(b_j) + \tau\pi(b_j)$, with $j \in \{1, \ldots, p\}$, is a linear combination of $b_{p+1} = \pi(b_{p+1}) + \tau\pi(b_{p+1}), \ldots, b_n = \pi(b_n) + \tau\pi(b_n)$. □

3. Associating traces on JW*-algebras without type I_1 and I_2 summands

The main goal of this section is to determine the general form of all associating traces on a JW*-algebra $\mathfrak{J}$ without associative and spin part. The task will be completed in Theorem 3.6, however the arguments will require to develop certain tools and techniques inspired on arguments developed in different branches. We have already justified in the introduction how the existence of associative summands produces counter-examples to the desired result.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. A bounded 2-homogeneous polynomial from $\mathfrak{X}$ to $\mathfrak{Y}$ is a mapping $P : \mathfrak{X} \to \mathfrak{Y}$ for which there exists a (unique) bounded symmetric bilinear mapping $V : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{Y}$ satisfying $P(x) = V(x, x)$ for all $x \in \mathfrak{X}$. It is well known that P and V are mutually determined via polarization identities. Along this paper we shall frequently identify the 2-homogeneous polynomial and the associated symmetric bilinear mapping.

In the next result we focus now on the uniqueness of the "standard form" of a symmetric bilinear map B on a JBW*-algebra $\mathfrak{J}$ whose trace is associating.

Proposition 3.1. *Let $\mathfrak{J}$ and $\mathfrak{M}$ be JBW*-algebras, $B : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{M}$ a symmetric bilinear mapping, and $\pi : \mathfrak{J} \to \mathfrak{M}$ a Jordan homomorphism having weak*-dense image. Assume that there exist an element w in $\mathfrak{J}$ and elementary operators $\mathscr{E}_0, \mathscr{E}_1$, and $\mathscr{E}_2$ on $\mathfrak{J}$ such that $\mathscr{E}_i(w^j) = \delta_{ij}\mathbf{1}$, for all $i, j \in \{0, 1, 2\}$. Suppose additionally that B admits the following standard representation*

$$B(x, x) = \lambda \circ \pi(x)^2 + \mu(x) \circ \pi(x) + \nu(x, x),$$

for every $x \in \mathfrak{J}$, where $\lambda \in Z(\mathfrak{M})$, $\mu : \mathfrak{J} \to Z(\mathfrak{M})$ is a linear mapping, and $\nu : \mathfrak{J} \times \mathfrak{J} \to Z(\mathfrak{M})$ is a symmetric bilinear mapping. Then this representation is unique and satisfies the following statements:

- $\lambda = \widehat{\mathscr{E}_2}(B(w, w))$,
- $\mu(x) = 2\widehat{\mathscr{E}_1}(B(w, x)) - 2\widehat{\mathscr{E}_2}(B(w, w)) \circ \widehat{\mathscr{E}_1}(\pi(w \circ x)) - \widehat{\mathscr{E}_1}(B(w, w)) \circ \widehat{\mathscr{E}_1}(\pi(x))$, *for every $x \in \mathfrak{J}$,*
- $\nu(x, x) := \widehat{\mathscr{E}_0}(B(x, x)) - \lambda \circ \widehat{\mathscr{E}_0}(\pi(x)^2) - \mu(x) \circ \widehat{\mathscr{E}_0}(\pi(x))$, *for every $x \in \mathfrak{J}$,*

where each $\widehat{\mathscr{E}_i}$ is the (unique) elementary operator in $\mathscr{E}\ell(\mathfrak{M})$ satisfying $\widehat{\mathscr{E}_i}\pi = \pi\mathscr{E}_i$, for every $i \in \{0, 1, 2\}$.

Proof. By Lemma 2.2(ii) there exist unique elementary operators $\widehat{\mathscr{E}_i} \in \mathscr{E}\ell(\mathfrak{M})$ such that $\widehat{\mathscr{E}_i}\pi = \pi\mathscr{E}_i$, for all $i \in \{0, 1, 2\}$. Thus, these maps satisfy

$$\widehat{\mathscr{E}_i}(\pi(w)^j) = \widehat{\mathscr{E}_i}(\pi(w^j)) = \pi(\mathscr{E}_i(w^j)) = \delta_{ij}\mathbf{1} \text{ for all } i, j \in \{0, 1, 2\}.$$

Since the elementary operators $\widehat{\mathscr{E}_i}$ are $\mathbb{C}$-linear and actually $Z(\mathfrak{M})$-linear (cf. Remark 2.1), it follows from the assumptions that

$$\widehat{\mathscr{E}_2}(B(w, w)) = \lambda \circ \widehat{\mathscr{E}_2}(\pi(w)^2) + \mu(w) \circ \widehat{\mathscr{E}_2}(\pi(w)) + \nu(w, w) \circ \widehat{\mathscr{E}_2}(\mathbf{1}) = \lambda \circ \mathbf{1} = \lambda.$$

Similarly, by applying $\widehat{\mathscr{E}_1}$ to $B(w, w)$ we obtain from the hypotheses that

$$\begin{aligned} \widehat{\mathscr{E}_1}(B(w, w)) &= \lambda \circ \widehat{\mathscr{E}_1}(\pi(w)^2) + \mu(w) \circ \widehat{\mathscr{E}_1}(\pi(w)) + \nu(w, w) \circ \widehat{\mathscr{E}_1}(\mathbf{1}) \\ &= \mu(w) \circ \mathbf{1} = \mu(w). \end{aligned}$$

To deduce the second identity in the conclusions, let us take an arbitrary $x \in \mathfrak{J}$. Then, by hypotheses, we have

$$B(w + x, w + x) = 2B(w, x) + B(x, x) + B(w, w).$$

Making some computations we obtain

$$2B(w, x) = 2\lambda \circ \pi(w \circ x) + \mu(w) \circ \pi(x) + \mu(x) \circ \pi(w) + \nu(w, x) + \nu(x, w).$$

Now, by applying $\widehat{\mathscr{E}_1}$ on both sides of the previous identity we get

$$
\begin{aligned}
2\widehat{\mathscr{E}_1}(B(w,x)) &= 2\lambda \circ \widehat{\mathscr{E}_1}(\pi(w\circ x)) + \mu(w)\circ \widehat{\mathscr{E}_1}(\pi(x)) + \mu(x)\circ \widehat{\mathscr{E}_1}(\pi(w)) \\
&\quad + (\nu(w,x)+\nu(x,w))\circ \widehat{\mathscr{E}_1}(\mathbf{1}) \\
&= 2\lambda \circ \widehat{\mathscr{E}_1}(\pi(w\circ x)) + \mu(w)\circ \widehat{\mathscr{E}_1}(\pi(x)) + \mu(x),
\end{aligned}
$$

which implies that

$$
\mu(x) = 2\widehat{\mathscr{E}_1}(B(w,x)) - 2\lambda\circ \widehat{\mathscr{E}_1}(\pi(w\circ x)) - \mu(w)\circ \widehat{\mathscr{E}_1}(\pi(x)),\ \forall x \in \mathfrak{J}.
$$

Having in mind that $\lambda = \widehat{\mathscr{E}_2}(B(w,w))$ and $\mu(w) = \widehat{\mathscr{E}_1}(B(w,w))$ we are led to

$$
\mu(x) = 2\widehat{\mathscr{E}_1}(B(w,x)) - 2\widehat{\mathscr{E}_2}(B(w,w))\circ \widehat{\mathscr{E}_1}(\pi(w\circ x)) - \widehat{\mathscr{E}_1}(B(w,w))\circ \widehat{\mathscr{E}_1}(\pi(x))
$$

for every $x \in \mathfrak{J}$, as we claimed. Finally, by applying $\widehat{\mathscr{E}_0}$ to $B(x,x)$ for $x \in \mathfrak{J}$ we obtain the last desired identity for $\nu(x,x)$. $\square$

The arguments in the proof of the previous proposition are also valid to get the following conclusion.

Proposition 3.2. *Let $\mathfrak{J}$ and $\mathfrak{M}$ be JBW*-algebras, $B : \mathfrak{J}\times\mathfrak{J} \to \mathfrak{M}$ a symmetric bilinear mapping, and $\pi : \mathfrak{J} \to \mathfrak{M}$ a Jordan homomorphism having weak*-dense image. Assume that there exist an element w in $\mathfrak{J}$, and elementary operators $\mathscr{E}_0$ and $\mathscr{E}_1$ on $\mathfrak{J}$ such that $\mathscr{E}_i(w^j) = \delta_{ij}\mathbf{1}$, for all $i, j \in \{0, 1\}$. Suppose additionally that B admits the following standard representation*

$$
B(x,x) = \mu(x)\circ\pi(x) + \nu(x,x),
$$

for every $x \in \mathfrak{J}$, where $\mu : \mathfrak{J} \to Z(\mathfrak{M})$ is a linear mapping, and $\nu : \mathfrak{J}\times\mathfrak{J} \to Z(\mathfrak{M})$ is a symmetric bilinear mapping. Then this representation is unique and satisfies the following statements:

- $\mu(x) = 2\widehat{\mathscr{E}_1}(B(w,x)) - \widehat{\mathscr{E}_1}(B(w,w))\circ\widehat{\mathscr{E}_1}(\pi(x))$, *for every* $x \in \mathfrak{J}$,
- $\nu(x,x) := \widehat{\mathscr{E}_0}(B(x,x)) - \mu(x)\circ\widehat{\mathscr{E}_0}(\pi(x))$, *for every* $x \in \mathfrak{J}$,

where each $\widehat{\mathscr{E}_i}$ is the (unique) elementary operator in $\mathscr{E}\ell(\mathfrak{M})$ satisfying $\widehat{\mathscr{E}_i}\pi = \pi\mathscr{E}_i$, $i \in \{0, 1, 2\}$.

The next lemma establishes a kind of selection principle for mappings whose trace at every point is a linear combination of a Jordan representation at that point and the unit element, that is, the "properly quadratic" monomial part vanishes pointwise. The result is also the first step towards the existence of the mappings $\mu(\cdot)$ and $\nu(\cdot,\cdot)$ appearing in the standard form of a symmetric bilinear mapping whose trace is associating.

Lemma 3.3. *Let $\mathfrak{J}, \mathfrak{M}$ be JBW*-algebras where $\mathfrak{J}$ has no direct summands of type I_1 and I_2, $\pi : \mathfrak{J} \to \mathfrak{M}$ a Jordan homomorphism having weak*-dense image, and $B : \mathfrak{J}\times\mathfrak{J} \to \mathfrak{M}$ a symmetric bilinear map. Assume that there exist an element w in $\mathfrak{J}$ and elementary operators $\mathscr{E}_0, \mathscr{E}_1, \mathscr{E}_2 \in \mathscr{E}\ell(\mathfrak{J})$ such that $\mathscr{E}_i(w^j) = \delta_{ij}\mathbf{1}$, for all $i, j \in 0, 1, 2$. Suppose additionally that for each $x \in \mathfrak{J}$ there exist $\alpha_x, \beta_x \in \mathbb{C}$ such that*

$$
B(x,x) = \alpha_x\pi(x) + \beta_x\mathbf{1}.
$$

Then there exist a linear map $\mu : \mathfrak{J} \to \mathbb{C}$ and a symmetric bilinear map $\nu : \mathfrak{J}\times\mathfrak{J} \to \mathbb{C}$ such that

$$
B(x,x) = \mu(x)\pi(x) + \nu(x,x)\mathbf{1},
$$

for every $x \in \mathfrak{J}$.

Proof. By Lemma 2.2(ii) we can find unique elementary operators $\widehat{\mathscr{E}_j} \in \mathscr{E}\ell(\mathfrak{M})$ satisfying $\pi\mathscr{E}_i = \widehat{\mathscr{E}_i}\pi$, for all $i \in \{0,1,2\}$. In particular, $\widehat{\mathscr{E}_j}(\pi(w)^i) = \delta_{ij}\mathbf{1}$, $\forall i, j \in \{0,1,2\}$.

Let us fix an arbitrary $x \in \mathfrak{J}$. The identity

$$B(x+w, x+w) + B(x-w, x-w) = 2B(x,x) + 2B(w,w),$$

holds by hypotheses. Thus, we have

$$\begin{aligned}(\alpha_{x+w} + \alpha_{x-w} - 2\alpha_x)\pi(x) + (\alpha_{x+w} - \alpha_{x-w} - 2\alpha_w)\pi(w)& \\ + (\beta_{x+w} + \beta_{x-w} - \beta_x - 2\beta_t)\mathbf{1} = 0.&\end{aligned}$$

If the vectors $\{\pi(x), \mathbf{1}, \pi(w)\}$ are linearly independent we deduce that

$$\alpha_{x+w} + \alpha_{x-w} = 2\alpha_x, \text{ and } \alpha_{x+w} - \alpha_{x-w} = 2\alpha_w,$$

which implies that

$$\alpha_{x+w} = \alpha_x + \alpha_w, \tag{3.1}$$

for every $x \in \mathfrak{J}$ such that $\pi(x)$ is not a linear combination of $\mathbf{1}$ and $\pi(w)$. Note that by the properties of the elementary operators $\widehat{\mathscr{E}_j} \in \mathscr{E}\ell(\mathfrak{M})$, the vectors $\mathbf{1}$ and $\pi(w)$ are linearly independent.

Let $\mu : \mathfrak{J} \to \mathfrak{M}$ and $\nu : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{M}$ be the maps defined by

$$\mu(x) := 2\widehat{\mathscr{E}_1}(B(x,w)) - \widehat{\mathscr{E}_1}(B(w,w)) \circ \widehat{\mathscr{E}_1}(\pi(x)), \text{ and } \nu(x,x) := B(x,x) - \mu(x) \circ \pi(x).$$

By definition, μ is a linear map while ν is given by a symmetric bilinear map. It only remains to prove that both maps take values in $\mathbb{C}\mathbf{1}$. Indeed,

(1) If $\pi(x) \notin \operatorname{Span}\{\mathbf{1}, \pi(w)\}$, it is clear from (3.1) that

$$B(x+w, x+w) = (\alpha_x + \alpha_w)\pi(x+w) + \beta_{x+w}\mathbf{1},$$

and thus, by combining the properties of B and $\widehat{\mathscr{E}_1}$ with the previous identity we get

$$\begin{aligned}2\widehat{\mathscr{E}_1}(B(x,w)) &= \alpha_x\widehat{\mathscr{E}_1}(\pi(w)) + \alpha_w\widehat{\mathscr{E}_1}(\pi(x)) + (\beta_{x+w} - \beta_w - \beta_x)\widehat{\mathscr{E}_1}(\mathbf{1}) \\ &= \alpha_x\mathbf{1} + \alpha_w\widehat{\mathscr{E}_1}(\pi(x)) = \alpha_x\mathbf{1} + \widehat{\mathscr{E}_1}(B(w,w)) \circ \widehat{\mathscr{E}_1}(\pi(x)),\end{aligned}$$

and thus $\mu(x) = 2\widehat{\mathscr{E}_1}(B(x,w)) - \widehat{\mathscr{E}_1}(B(w,w)) \circ \widehat{\mathscr{E}_1}(\pi(x)) = \alpha_x\mathbf{1} \in \mathbb{C}\mathbf{1}$. Observe that $\nu(x,x) = B(x,x) - \mu(x) \circ \pi(x) = B(x,x) - \alpha_x \circ \pi(x) = \beta_x\mathbf{1}$.

(2) If $\pi(x) \in \operatorname{Span}\{\mathbf{1}, \pi(w)\}$, by the properties of the mappings $\widehat{\mathscr{E}_i}$, $\pi(w^2) \notin \operatorname{Span}\{\mathbf{1}, \pi(w)\}$, and we can thus conclude that for every $\varepsilon > 0$, the element $\pi(x + \varepsilon w^2)$ is not in $\operatorname{Span}\{\mathbf{1}, \pi(w)\}$. By applying (3.1) we obtain

$$B(x + \varepsilon w^2, x + \varepsilon w^2) - \mu(x + \varepsilon w^2)\pi(x + \varepsilon w^2) \in \mathbb{C}\mathbf{1}.$$

Note that, by what we have just proved in (1), $\mu(x + \varepsilon w^2) = \alpha_{x+\varepsilon w^2}\mathbf{1}$. We also know that μ is linear. Thus

$$\begin{aligned}B(x,x) + 2B(x, \varepsilon w^2) + \varepsilon^2 B(w^2, w^2) - (\mu(x) + \varepsilon\mu(w^2)) \circ \pi(x + \varepsilon w^2) \in \mathbb{C}\mathbf{1},& \\ \mu(x + \varepsilon w^2) = \mu(x) + \varepsilon\mu(w^2) \in \mathbb{C}\mathbf{1}.&\end{aligned}$$

Note that μ is linear and continuous on the subspace generated by x and w^2, so taking $\varepsilon \to 0$ we get

$$\nu(x,x) = B(x,x) - \mu(x) \circ \pi(x) \in \mathbb{C}\mathbf{1}, \text{ and } \mu(x) \in \mathbb{C}\mathbf{1}, \tag{3.2}$$

for every $x \in \mathfrak{J}$. □

Remark 3.4. We gather in this remark some useful identities inspired by previous contributions in [7, Theorem 2], [28, pages 2891-2892] and [12, Lemma 3.8.].

Let $\mathfrak{X}$ be a linear space, A an associative algebra, $\pi : \mathfrak{X} \to A$ a linear mapping, and $B : \mathfrak{X} \times \mathfrak{X} \to A$ a symmetric bilinear map. Suppose $[B(x,x), \pi(x)] = 0$, for every $x \in \mathfrak{X}$, where the brackets denote the usual Lie product on A. Then

$$2[B(x,y), \pi(y)] = -[B(y,y), \pi(x)], \text{ for all } x, y \in \mathfrak{X}. \tag{3.3}$$

Namely, the desired identity is just a clear consequence of the identity $0 = [B(x+\alpha y, x+\alpha y), \pi(x+\alpha y)]$, for all $x, y \in \mathfrak{X}$, $\alpha \in \mathbb{R}$, by just expanding the polynomial on α via the properties of B.

Similar arguments building upon identity (3.3) lead to

$$[B(x,y), \pi(z)] + [B(x,z), \pi(y)] + [B(y,z), \pi(x)] = 0, \text{ for all } x, y \in \mathfrak{X}. \tag{3.4}$$

Suppose additionally that $\mathfrak{X}$ is a Jordan algebra and π is a Jordan homomorphism. We combine the associative structure of A with the previous identities to deduce the following:

$$\begin{aligned}
[B(x^2,x^2), \pi(y)] &= -2[B(x^2,y), \pi(x^2)] \\
&= -2[B(x^2,y), \pi(x)]\pi(x) - 2\pi(x)[B(x^2,y), \pi(x)] \\
&= 2\left([B(x^2,x), \pi(y)] + [B(y,x), \pi(x^2)]\right)\pi(x) \\
&\quad + 2\pi(x)\left([B(x^2,x), \pi(y)] + [B(y,x), \pi(x^2)]\right) \\
&= 2\left([B(x^2,x), \pi(y)] + [B(y,x), \pi(x)]\pi(x) + \pi(x)[B(y,x), \pi(x)]\right)\pi(x) \\
&\quad + 2\pi(x)\left([B(x^2,x), \pi(y)] + [B(y,x), \pi(x)]\pi(x) + \pi(x)[B(y,x), \pi(x)]\right) \\
&= 2\pi(x^2)[B(y,x), \pi(x)] + 4\pi(x)[B(y,x), \pi(x)]\pi(x) \\
&\quad + 2[B(y,x), \pi(x)]\pi(x^2) + 2\pi(x)[B(x^2,x), \pi(y)] \\
&\quad + 2[B(x^2,x), \pi(y)]\pi(x) \\
&= -\pi(x^2)[B(x,x), \pi(y)] - 2\pi(x)[B(x,x), \pi(y)]\pi(x) \\
&\quad - [B(x,x), \pi(y)]\pi(x^2) + 2\pi(x)[B(x^2,x), \pi(y)] \\
&\quad + 2[B(x^2,x), \pi(y)]\pi(x),
\end{aligned}$$

where in the first equality we applied (3.3), while in the third and sixth equalities we applied (3.4). If we rewrite the previous identity in terms of the associative product we get

$$\begin{aligned}
&\pi(\mathbf{1})\pi(y)\left(2B(x^2,x)\pi(x) - B(x^2,x^2) - B(x,x)\pi(x^2)\right) \\
&+ \pi(x)\pi(y)\left(2B(x^2,x) - 2B(x,x)\pi(x)\right) - \pi(x^2)\pi(y)B(x,x) \\
&+ \left(B(x^2,x^2) + \pi(x^2)B(x,x) - 2\pi(x)B(x,x^2)\right)\pi(y)\pi(\mathbf{1}) \\
&+ \left(2\pi(x)B(x,x) - 2B(x,x^2)\right)\pi(y)\pi(x) + B(x,x)\pi(y)\pi(x^2) = 0,
\end{aligned} \tag{3.5}$$

for all $x, y \in \mathfrak{X}$.

We finally observe that the arguments above are valid to get the following conclusion in the Jordan setting. Suppose $\mathfrak{X}$ is a linear space, $\pi : \mathfrak{X} \to \mathfrak{J}$ is a linear mapping, where $\mathfrak{J}$ is a Jordan algebra, and $B : \mathfrak{X} \times \mathfrak{X} \to \mathfrak{J}$ is a symmetric bilinear map such that $[B(x,x), \mathfrak{J}, \pi(x)] = 0$, for every $x \in \mathfrak{X}$. Then

$$2[B(x,y), a, \pi(y)] = -[B(y,y), a, \pi(x)], \text{ for all } x, y \in \mathfrak{X}, a \in \mathfrak{J}. \tag{3.6}$$

We deal next with general bilinear maps with associating trace.

Lemma 3.5. *Let $\mathfrak{J}$ be a JBW*-algebra without direct summands of type I_1 and I_2, A a prime von Neumann algebra, $\pi : \mathfrak{J} \to A$ a unital Jordan homomorphism with weak*-dense image, and $B : \mathfrak{J} \times \mathfrak{J} \to A$ a symmetric bilinear map. Suppose that $[B(x,x), \pi(x)] = 0$, for every $x \in \mathfrak{J}$, where the brackets denote the usual Lie product on A. Assume, additionally, that for each $x \in \mathfrak{J}$ there exist $\lambda_x, \mu_x, \nu_x \in \mathbb{C}$ such that*

$$B(x,x) = \lambda_x \pi(x)^2 + \mu_x \pi(x) + \nu_x \mathbf{1}.$$

Then there exists a complex value $\lambda \in \mathbb{C}$ such that $B(x,x) - \lambda \pi(x)^2 \in Span\{\mathbf{1}, \pi(x)\}$ for every $x \in \mathfrak{J}$.

Proof. Let us take $x, y \in \mathfrak{J}$, by the hypotheses we get

$$\begin{aligned} B(x+y, x+y) &= \lambda_{x+y} \pi(x+y)^2 + \mu_{x+y} \pi(x+y) + \nu_{x+y} \mathbf{1} \\ &= \lambda_{x+y} (\pi(x)^2 + \pi(y^2) + 2\pi(x \circ y)) + \mu_{x+y} (\pi(x) + \pi(y)) + \nu_{x+y} \mathbf{1}, \end{aligned}$$

which leads us to the following identity

$$\begin{aligned} 2B(x,y) =& (\lambda_{x+y} - \lambda_x) \pi(x)^2 + (\lambda_{x+y} - \lambda_y) \pi(y)^2 \\ &+ (\mu_{x+y} - \mu_x) \pi(x) + (\mu_{x+y} - \mu_y) \pi(y) \\ &+ (\nu_{x+y} - \nu_x - \nu_y) \mathbf{1} + 2\lambda_{x+y} \pi(x \circ y). \end{aligned}$$

Next, we apply the commutator $[\cdot, \pi(y)]$ on the previous identity to derive that

$$\begin{aligned} [2B(x,y), \pi(y)] =& (\lambda_{x+y} - \lambda_x)[\pi(x)^2, \pi(y)] + (\lambda_{x+y} - \lambda_y) \underbrace{[\pi(y)^2, \pi(y)]}_{=0} \\ &+ (\mu_{x+y} - \mu_x)[\pi(x), \pi(y)] + (\mu_{x+y} - \mu_y) \underbrace{[\pi(y), \pi(y)]}_{=0} \\ &+ (\nu_{x+y} - \nu_x - \nu_y) \underbrace{[\mathbf{1}, \pi(y)]}_{=0} + 2\lambda_{x+y} [\pi(x \circ y), \pi(y)]. \end{aligned}$$

By (3.3) we have $2[B(x,y), \pi(y)] = -[B(y,y), \pi(x)]$. To deal with the remaining summands on the right-hand-side of the previous identity, we allude to the associative structure of A to make the following computations:

$$\begin{aligned} [2\pi(x \circ y)], \pi(y)] &= [\pi(x)\pi(y) + \pi(y)\pi(x), \pi(y)] \\ &= [\pi(x)\pi(y), \pi(y)] + [\pi(y)\pi(x), \pi(y)] \\ &= \pi(x)\pi(y)^2 - \pi(y)\pi(x)\pi(y) + \pi(y)\pi(x)\pi(y) - \pi(y)^2\pi(x) \\ &= \pi(x)\pi(y)^2 - \pi(y)^2\pi(x) = [\pi(x), \pi(y)^2]. \end{aligned}$$

By combining all the previous observations we get

$$-[B(y,y),\pi(x)] = (\lambda_{x+y} - \lambda_x)[\pi(x)^2, \pi(y)] \\ + (\mu_{x+y} - \mu_x)[\pi(x), \pi(y)] + \lambda_{x+y}[\pi(x), \pi(y)^2].$$

Having in mind that, by hypotheses, $B(y,y) = \lambda_y \pi(y)^2 + \mu_y \pi(y) + \nu_y \mathbf{1}$ for some $\lambda_y, \mu_y, \nu_y \in \mathbb{C}$, we arrive to

$$-[B(y,y),\pi(x)] = \lambda_y[\pi(x), \pi(y)^2] + \mu_y[\pi(x), \pi(y)].$$

The last two previous identities together give

$$(\lambda_{x+y} - \lambda_x)[\pi(x)^2, \pi(y)] + (\lambda_{x+y} - \lambda_y)[\pi(x), \pi(y)^2] \\ + (\mu_{x+y} - \mu_x - \mu_y)[\pi(x), \pi(y)] = 0, \tag{3.7}$$

for every $x, y \in \mathfrak{J}$.

Let us take elements $u, v \in \mathfrak{J}$ whose existence is guaranteed by Proposition 2.3. By hypotheses we have

$$B(u,u) = \lambda_u \pi(u)^2 + \mu_u \pi(u) + \nu_u \mathbf{1}.$$

The just quoted Proposition 2.3 assures the existence of elementary operators $\mathscr{E}_j \in \mathscr{E}\ell(\mathfrak{J})$ satisfying $\mathscr{E}_j(u^i) = \delta_{i,j}\mathbf{1}$, for all $i, j \in \{0,1,2\}$. Lemma 2.2(ii) implies the existence of elementary operators $\widehat{\mathscr{E}_j} \in \mathscr{E}\ell(A)$ satisfying $\widehat{\mathscr{E}_j}(\pi(u)^i) = \delta_{i,j}\mathbf{1}$, for all $i, j \in \{0,1,2\}$. Consequently, the elements $\pi(u)^2$, $\pi(u)$ and $\mathbf{1}$ are linearly independent. Set $\lambda := \lambda_u$. We claim that

$$B(x,x) - \lambda\pi(x)^2 \in \operatorname{Span}\{\mathbf{1}, \pi(x)\}, \text{ for every } x \in \mathfrak{J}. \tag{3.8}$$

To prove the claim we shall employ the results on linear independence in terms of Capelli's polynomials. We shall distinguish two cases:

(1) *Case I*: The vectors $\{[\pi(x)^2, \pi(u)], [\pi(x), \pi(u)^2], [\pi(x), \pi(u)]\}$ are linearly independent for some $x \in \mathfrak{J}$, then we deduce from (3.7) that $\lambda_{x+u} - \lambda_x = 0$ and $\lambda_{x+u} - \lambda_u = 0$. This implies that $\lambda_x = \lambda_u = \lambda$ by definition, and by hypotheses

$$B(x,x) = \lambda\pi(x)^2 + \mu_x \pi(x) + \nu_x \mathbf{1},$$

which proves the claim in this case.

(2) *Case II*: There exists $x \in \mathfrak{J}$ such that the vectors $[\pi(x)^2, \pi(u)]$, $[\pi(x), \pi(u)^2]$, and $[\pi(x), \pi(u)]\}$ are linearly dependent. Consider now the third Capelli polynomial $c_3(\cdot,\cdot,\cdot,\cdot,\cdot)$, and consider the polynomials $f, g : \mathfrak{J} \to A$ defined by

$$f_{a,b}(x) := c_3\Big(B(x,x) - \lambda\pi(x)^2, \mathbf{1}, \pi(x), a, b\Big),$$
$$g_{c,d}(x) := c_3\Big([\pi(x)^2, \pi(u)], [\pi(x), \pi(u)^2], [\pi(x), \pi(u)], c, d\Big),$$

where a, b, c, d are fixed but arbitrary elements in A.

The element v plays now its role, since by Proposition 2.3(ii) the operators $[\pi(v)^2, \pi(u)]$, $[\pi(v), \pi(u)^2]$, and $[\pi(v), \pi(u)]$ are linearly independent. Thus, Lemma 2.7 assures that $g_{c,d}(v) \neq 0$ for some $c, d \in A$. As $g_{c,d} : \mathfrak{J} \to A$ is continuous there exists $r > 0$ such that $g_{c,d}(z) \neq 0$ for every $z \in \mathbf{B}(v,r)$. Here $\mathbf{B}(v,r)$ stands for the open unit ball in $\mathfrak{J}$ of centre v and radius r. Since for every $z \in \mathbf{B}(v,r)$ the operators $[\pi(z)^2, \pi(u)], [\pi(z), \pi(u)^2], [\pi(z), \pi(u)]$ must be linearly independent by Lemma 2.7, we can proceed as in *Case I* to deduce that $B(z,z) - \lambda\pi(z)^2 \in \operatorname{Span}\{\mathbf{1}, \pi(z)\}$, and hence $f_{a,b}(z) = 0$, for all $a,b \in A$, $z \in \mathbf{B}(v,r)$ by Lemma 2.7. Fix two arbitrary elements $a,b \in A$.

Observe that the element $x \in \mathfrak{J}$ fixed at the beginning of this *Case II* satisfies $x \neq v$. Let us consider a polynomial mapping from $\mathbb{C}$ to A given by $\zeta \to p_{a,b}(\zeta) = f_{a,b}\Big(v+\zeta(x-v)\Big)$, which is clearly continuous because $f_{a,b}$ is a polynomial. Note that, by what we have just deduced in the previous paragraph, $f_{a,b}\Big(v+\zeta(x-v)\Big) = 0$ for every $\zeta \in \mathbf{B}_{\mathbb{C}}\left(0, \frac{r}{\|x-v\|}\right)$. Therefore the polynomial $p_{a,b}(\zeta)$ must be zero, that is, $f_{a,b}\Big(v+\zeta(x-v)\Big) = 0$, for any $\zeta \in \mathbb{C}$. In particular taking $\zeta = 1$ we get $f_{a,b}(x) = 0$ for every $a,b \in A$, and thus, $B(x,x) - \lambda\pi(x)^2 \in \operatorname{Span}\{\mathbf{1}, \pi(x)\}$ by Lemma 2.7, as we wanted to prove. $\square$

We establish next our main conclusion on the generic form of those symmetric bilinear maps on special JBW*-algebras without spin part whose trace is associating.

Theorem 3.6. *Let $\mathfrak{J}$ be a special JBW*-algebra with no direct summands of type I_1 or I_2, and let $B : \mathfrak{J}\times\mathfrak{J} \to \mathfrak{J}$ be a symmetric bilinear mapping. Suppose that $[B(x,x), \mathfrak{J}, x] = 0$, for all $x \in \mathfrak{J}$, where the brackets denote the associator with respect to the Jordan product of $\mathfrak{J}$. Then B can be uniquely written in the form:*

$$B(x,x) = \lambda \circ x^2 + \mu(x) \circ x + \nu(x,x), \quad \textit{for all } x \in \mathfrak{J},$$

where $\lambda \in Z(\mathfrak{J})$, $\mu : \mathfrak{J} \to Z(\mathfrak{J})$ is a linear mapping and $\nu : \mathfrak{J}\times\mathfrak{J} \to Z(\mathfrak{J})$ is a symmetric bilinear map.

Proof. The proof is divided into 3 steps.

(1) *First Step*: Let us take an arbitrary Jordan factor representation $\pi : \mathfrak{J} \to \mathfrak{J}_\pi$ (here, $\mathfrak{J}_\pi$ is a factor JBW*-algebra). We shall study the composition of B with π, i.e., $\pi \circ B : \mathfrak{J}\times\mathfrak{J} \to \mathfrak{J}_\pi$.

We observe that $\mathfrak{J}_\pi$ must be a special and unital JBW*-algebras since $\mathfrak{J}$ so is (cf. [21, 7.2.1., Theorems 7.2.3 and 7.2.7] and [44]).

We claim that $\mathfrak{J}_\pi$ cannot be a JBW*-algebra factor of type I_1 or I_2, equivalently, it is a non-quadratic factor (see [14, Theorem 3.5.5] and Remark 1.4). Namely, if $\mathfrak{J}_\pi$ were a factor of type I_1 or of type I_2, then it would be quadratic, or equivalently, the square of every element in $\mathfrak{J}_\pi$ would be a linear combination of the element itself and the unit. However, by Proposition 2.3, there exist $u \in \mathfrak{J}$ and elementary operators $\mathscr{E}_0, \mathscr{E}_1, \mathscr{E}_2$ in $\mathfrak{J}$ such that $\mathscr{E}_i(u^j) = \delta_{ij}\mathbf{1}$, for all $i,j \in \{0,1,2\}$, and by Lemma 2.2(ii) there are elementary operators $\widehat{\mathscr{E}_i} \in \mathscr{E}\ell(\mathfrak{J}_\pi)$ such that $\pi\mathscr{E}_i = \widehat{\mathscr{E}_i}\pi$ for $i \in \{0,1,2\}$. This contradicts that $\pi(u^2) = \pi(u)^2$ is a linear combination of $\pi(u)$ and $\mathbf{1}$.

Since π is a Jordan homomorphism, it follows from the hypotheses on $B(\cdot,\cdot)$ that the identity

$$[\pi(B(x,x)), \pi(\mathfrak{J}), \pi(x)] = 0,$$

holds for every $x \in \mathfrak{J}$. In particular, by the weak*-density of $\pi(\mathfrak{J})$ in $\mathfrak{J}_\pi$, and the separate weak*-continuity of the Jordan product of $\mathfrak{J}_\pi$, we can further conclude that

$$[\pi(B(x,x)), J_\pi, \pi(x)] = 0, \tag{3.9}$$

for every $x \in \mathfrak{J}$. Furthermore, having in mind that $\mathfrak{J}_\pi$ is a factor JW*-algebra not of type I_1 or I_2, its universal von Neumann algebra, $W^*(\mathfrak{J}_\pi)$, satisfies that $\mathfrak{J}_\pi = H(W^*(\mathfrak{J}_\pi), \tau)$, where τ is a period-2 *-anti-automorphism on $W^*(\mathfrak{J}_\pi)$ and the latter either is a factor von Neumann algebra, or it can be written in the form $W^*(\mathfrak{J}_\pi) = N \oplus^\infty \tau(N)$, for some weak*-closed ideal N of $W^*(\mathfrak{J}_\pi)$ which is a factor von Neumann algebra (cf. Lemma 1.3 and the comments prior to it or [21, Theorem 7.1.9, Remark 7.2.8 and Proposition 7.3.3]). The product of $W^*(\mathfrak{J}_\pi)$ will be denoted by mere juxtaposition. Thus, according to this notation, (3.9) combined with Proposition 1.2 imply that $\pi(B(x,x))$ and $\pi(x)$ commute in the von Neumann algebra $W^*(\mathfrak{J}_\pi)$ for all $x \in \mathfrak{J}$, that is,

$$[\pi(B(x,x)), \pi(x)] = \pi(B(x,x))\pi(x) - \pi(x)\pi(B(x,x)) = 0, \ \forall x \in \mathfrak{J},$$

identity where we employed the associative product of $W^*(\mathfrak{J}_\pi)$.

(2) *Second step*: In order to simplify the proof in this step, fix an arbitrary $x \in \mathfrak{J}$, and define $B_0 := \pi(B(x,x))$, $B_1 := \pi(B(x,x^2))$ and $B_2 := \pi(B(x^2,x^2))$. Observe that we are in a position to apply Remark 3.4, and more concretely (3.5) to deduce that

$$\begin{aligned}\pi(\mathbf{1})\pi(y)\left(2B_1\pi(x) - B_2 - B_0\pi(x^2)\right) &+ \pi(x)\pi(y)\left(2B_1 - 2B_0\pi(x)\right)\\ &+ \pi(x^2)(-\pi(y)B_0) + \left(B_2 + \pi(x^2)B_0 - 2\pi(x)B_1\right)\pi(y)\pi(\mathbf{1})\\ &+ \left(2\pi(x)B_0 - 2B_1\right)\pi(y)\pi(x) + B_0\pi(y)\pi(x^2) = 0,\end{aligned}$$

for all $y \in \mathfrak{J}$. Furthermore, by weak*-density of $\pi(\mathfrak{J})$ in $\mathfrak{J}_\pi$, and the separate weak*-continuity of the product of $W^*(\mathfrak{J}_\pi)$, the previous identity also holds when $\pi(y)$ is replaced by any element $z \in \mathfrak{J}_\pi = H(W^*(\mathfrak{J}_\pi), \tau)$, that is,

$$\begin{aligned}\pi(\mathbf{1})z\left(2B_1\pi(x) - B_2 - B_0\pi(x^2)\right) &+ \pi(x)z\left(2B_1 - 2B_0\pi(x)\right)\\ &- \pi(x^2)zB_0 + \left(B_2 + \pi(x^2)B_0 - 2\pi(x)B_1\right)z\pi(\mathbf{1})\\ &+ \left(2\pi(x)B_0 - 2B_1\right)z\pi(x) + B_0z\pi(x^2) = 0,\end{aligned} \tag{3.10}$$

for all $z \in \mathfrak{J}_\pi = H(W^*(\mathfrak{J}_\pi), \tau)$.

Three possible cases can be considered here:

(1) <u>*Case 1*</u>: $\{\mathbf{1} = \pi(\mathbf{1}), \pi(x), \pi(x^2)\}$ are $\mathbb{C}$-linearly independent. Having in mind that $\mathfrak{J}_\pi = H(W^*(\mathfrak{J}_\pi), \tau)$ is a factor, we can combine Lemma 1.3 and Lemma 2.8 with the identity in (3.10), to conclude that

$$\pi(B(x,x)) = B_0 = \lambda_{0,x}\mathbf{1} + \lambda_{1,x}\pi(x) + \lambda_{2,x}\pi(x^2),$$

where $\lambda_{i,x} \in \mathbb{C}$ for every $i \in \{0,1,2\}$.

(2) <u>*Case 2*</u>: $\{\mathbf{1}, \pi(x), \pi(x^2)\}$ are $\mathbb{C}$-linearly dependent and $\pi(x) \notin \mathbb{C}\mathbf{1}$. In this case there exist $\alpha, \beta \in \mathbb{C}$ such that

$$\pi(x^2) = \alpha\pi(x) + \beta\pi(1).$$

By making some computations, via (3.3) in Remark 3.4, we obtain

$$\begin{aligned}\pi(x)[B_0, \pi(y)] + [B_0, \pi(y)]\pi(x) &= -2\pi(x)[\pi(B(x,y)), \pi(x)]\\ &\quad - 2[\pi(B(x,y)), \pi(x)]\pi(x)\\ &= -2[\pi(B(x,y)), \pi(x^2)]\\ &= -2[\pi(B(x,y)), \alpha\pi(x) + \beta\pi(\mathbf{1})]\\ &= -2\alpha[\pi(B(x,y)), \pi(x)] = \alpha[B_0, \pi(y)],\end{aligned}$$

for all $y \in \mathfrak{J}$, which leads us to the next identity

$$\begin{gathered}(\pi(x)-\alpha\pi(\mathbf{1}))B_0\pi(y)\pi(\mathbf{1})+B_0\pi(y)\pi(x)+\\+\pi(\mathbf{1})\pi(y)B_0(\alpha\pi(\mathbf{1})-\pi(x))-\pi(x)\pi(y)B_0=0,\end{gathered}\tag{3.11}$$

for all $y\in\mathfrak{J}$ (and thus the conclusion holds when $\pi(y)$ is replaced with any $z\in\mathfrak{J}_\pi=H(W^*(\mathfrak{J}_\pi),\tau)$). Since $\{\pi(\mathbf{1}),\pi(x)\}$ are $\mathbb{C}$-linearly independent, by employing Lemma 2.8 once again (see also Lemma 1.3), applied to (3.11), we conclude that there exist $\lambda_{0,x},\lambda_{1,x}\in\mathbb{C}$ such that

$$B_0=\pi(B(x,x))=\lambda_{0,x}\pi(\mathbf{1})+\lambda_{1,x}\pi(x).$$

(3) <u>*Case 3*</u>: $\{\mathbf{1},\pi(x),\pi(x^2)\}$ are $\mathbb{C}$-linearly dependent and $\pi(x)\in\mathbb{C}\mathbf{1}$. Then it is clear from Remark 3.4, and in particular by (3.3), that

$$[B_0,\pi(y)]=-2[\pi(B(x,y)),\pi(x)]=0,\ \text{for every } y\in\mathfrak{J}.$$

Moreover, due to the weak*-density of $\pi(\mathfrak{J})$ in $\mathfrak{J}_\pi$, and the separate weak*-continuity of the product of $W^*(\mathfrak{J}_\pi)$, the previous identity also holds for any $z\in\mathfrak{J}_\pi$, i.e., $[B_0,z]=0$ for every $z\in\mathfrak{J}_\pi$. This shows that $B_0\in Z(\mathfrak{J}_\pi)=\mathbb{C}\mathbf{1}$, and thus, that there exists $\lambda_{0,x}\in\mathbb{C}$ such that

$$B_0=\pi(B(x,x))=\lambda_{0,x}\pi(\mathbf{1}).$$

In summary, all the previous cases lead to the conclusion that for each $x\in\mathfrak{J}$ there exist complex numbers $\lambda_{0,x},\lambda_{1,x},\lambda_{2,x}$ such that

$$\pi(B(x,x))=\lambda_{0,x}\pi(\mathbf{1})+\lambda_{1,x}\pi(x)+\lambda_{2,x}\pi(x^2).\tag{3.12}$$

(3) *Step 3*: We are now in a position to apply Lemma 3.5 to deduce the existence of $\alpha\in\mathbb{C}$ such that $\pi(B(x,x))-\alpha\pi(x)^2\in\operatorname{Span}\{\mathbf{1},\pi(x)\}$ for every $x\in\mathfrak{J}$. We observe that the symmetric bilinear mapping $\mathfrak{J}\times\mathfrak{J}\to\mathfrak{J}_\pi$, $(x,y)\mapsto\pi(B(x,y))-\alpha\pi(x\circ y)$ satisfies the hypothesis in Lemma 3.3. Therefore, by the just quoted lemma, there exist a linear map $\beta:\mathfrak{J}\to\mathbb{C}$ and a symmetric bilinear map $\gamma:\mathfrak{J}\times\mathfrak{J}\to\mathbb{C}$ such that

$$\pi(B(x,x))=\alpha\pi(x^2)+\beta(x)\pi(x)+\gamma(x,x)\pi(\mathbf{1}),$$

for every $x\in\mathfrak{J}$. Now observe that $\alpha,\beta(x),\gamma(x,x)\in Z(\mathfrak{J}_\pi)=\mathbb{C}\mathbf{1}$, for all $x\in\mathfrak{J}$, which allows us to write the above identity in terms of the natural Jordan product defined in $W^*(\mathfrak{J}_\pi)$ as follows

$$\pi(B(x,x))=\alpha\circ\pi(x^2)+\beta(x)\circ\pi(x)+\gamma(x,x)\circ\pi(\mathbf{1}).\tag{3.13}$$

As commented in *Step 1*, there exist an element $\pi(u)\in\mathfrak{J}_\pi$ and elementary operators $\widehat{\mathscr{E}_i}$ for all $i\in\{0,1,2\}$ such that $\widehat{\mathscr{E}_i}\pi=\pi\mathscr{E}_i$, and $\widehat{\mathscr{E}_i}(\pi(u)^j)=\delta_{ij}\mathbf{1}$, for all $i,j\in\{0,1,2\}$. By Proposition 3.1, applied to the identity in (3.13), we deduce that $\alpha,\beta(x)$, and $\gamma(x,x)$ are uniquely written as follows:

$$\begin{cases}\quad\alpha=\widehat{\mathscr{E}_2}(\pi(B(u,u)))=\pi(\mathscr{E}_2(B(u,u))),\\ \beta(x)=2\widehat{\mathscr{E}_1}(\pi(B(u,x)))-2\widehat{\mathscr{E}_2}(\pi(B(u,u)))\circ\widehat{\mathscr{E}_1}(\pi(t\circ x))\\ \qquad-\widehat{\mathscr{E}_1}(\pi(B(u,u)))\circ\widehat{\mathscr{E}_1}(\pi(x))\\ \qquad=\pi\Big(2\mathscr{E}_1(B(u,x))-2\mathscr{E}_2(B(u,u))\circ\mathscr{E}_1(t\circ x)-\mathscr{E}_1(B(u,u))\circ\mathscr{E}_1(x)\Big),\\ \gamma(x,x)=\widehat{\mathscr{E}_0}(\pi(B(x,x)))-\alpha\circ\widehat{\mathscr{E}_0}(\pi(x^2))-\beta(x)\circ\widehat{\mathscr{E}_0}(\pi(x))\\ \qquad=\pi\Big(\mathscr{E}_0(B(x,x))-\mathscr{E}_2(B(u,u))\circ\mathscr{E}_0(x^2)-\Big(2\mathscr{E}_1(B(u,x))\\ \qquad-2\mathscr{E}_2(B(u,u))\circ\mathscr{E}_1(t\circ x)-\mathscr{E}_1(B(u,u))\circ\mathscr{E}_1(x)\Big)\circ\mathscr{E}_0(x)\Big),\end{cases}\tag{3.14}$$

for all $x \in \mathfrak{J}$. Observe that the Jordan factor representation π was arbitrarily chosen, so by setting $\lambda = \mathscr{E}_2(B(u,u))$, defining $\mu : \mathfrak{J} \to \mathfrak{J}$ and $\nu : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$ by

$$\begin{aligned}&\mu(x) = 2\mathscr{E}_1(B(u,x)) - 2\mathscr{E}_2(B(u,u)) \circ \mathscr{E}_1(t \circ x) - \mathscr{E}_1(B(u,u)) \circ \mathscr{E}_1(x),\\ &\text{and } \nu(x,x) = \mathscr{E}_0(B(x,x)) - \lambda \circ \mathscr{E}_0(x^2) - \mu(x) \circ \mathscr{E}_0(x),\end{aligned}$$

and combining (3.13) and (3.14), we arrive to

$$\pi(B(x,x)) = \pi(\lambda \circ x^2 + \mu(x) \circ x + \nu(x,x) \circ \mathbf{1}), \tag{3.15}$$

for every $x \in \mathfrak{J}$, and every Jordan factor representation π of $\mathfrak{J}$. Since every JB*-algebra admits a separating family of Jordan factor representations of this form (cf. [2, Corollary 5.7]), it follows from (3.15) that

$$B(x,x) = \lambda \circ x^2 + \mu(x) \circ x + \nu(x,x), \tag{3.16}$$

for every $x \in \mathfrak{J}$. Note that $\pi(\lambda), \pi(\mu(x))$ and $\pi(\nu(x,x))$ belong to $Z(\mathfrak{J}_\pi) = \mathbb{C}\mathbf{1}$, for every Jordan factor representation $\pi : \mathfrak{J} \to \mathfrak{J}_\pi$, so by Remark 1.1 we finally deduce that $\lambda, \mu(x)$, and $\nu(x,x)$ belong to $Z(\mathfrak{J})$ for each $x \in \mathfrak{J}$ as we desired. $\square$

4. Associating traces on purely exceptional JBW*-algebras

In this section we study symmetric bilinear maps with associating trace on exceptional JBW*-algebras. Our goal is to show that these maps also have the standard form established above when the domain is a JW*-algebra without associative and spin part established in Theorem 3.6. As we commented in the introduction, the case of exceptional factors was treated by M. Brešar, D. Eremita, and the third author of this paper in [12, §4].

We begin by introducing some notation. Given a set Ω and a Jordan algebra $\mathfrak{J}$, we shall write $\mathscr{F}(\Omega,\mathfrak{J})$ for the space of all functions from Ω to $\mathfrak{J}$, and we equip it with the pointwise Jordan product. We can actually consider the Jordan product of an element in the space $C(\Omega,\mathfrak{J})$, of all continuous functions from Ω to $\mathfrak{J}$, by an element in $\mathscr{F}(\Omega,\mathfrak{J})$. Given $w \in \mathfrak{J}$, we shall denote by $\widehat{w}$ the constant continuous function $\widehat{w} \in C(\Omega,\mathfrak{J})$, $\widehat{w}(t) = w$ $(t \in \Omega)$. Clearly, each elementary operator $\mathscr{E} \in \mathscr{E}\ell(\mathfrak{J})$ extends to an elementary operator $\widetilde{\mathscr{E}}$ on $C(\Omega,\mathfrak{J})$ given by the corresponding extensions of the involved elements in $\mathfrak{J}$ as constant functions. The extended elementary operator satisfies $\widetilde{\mathscr{E}}(u)(s) = \mathscr{E}(u(s))$, and $\widetilde{\mathscr{E}}(\alpha \circ u)(s) = \alpha(s) \circ \widetilde{\mathscr{E}}(u)(s)$, for all $u \in C(\Omega,\mathfrak{J})$, $\alpha \in \mathscr{F}(\Omega,\mathbb{C}\mathbf{1})$, and $s \in \Omega$.

The proof of Proposition 3.1 is essentially algebraic and the arguments remain valid to establish the following variant.

Proposition 4.1. *Let Ω be a compact Hausdorff space, $\mathfrak{J}$ a factor JBW*-algebra containing an element $w \in \mathfrak{J}$ and admitting elementary operators $\mathscr{E}_0, \mathscr{E}_1, \mathscr{E}_2$ on $\mathfrak{J}$ such that $\mathscr{E}_i(w^j) = \delta_{ij}\mathbf{1}$, for all $i,j \in \{0,1,2\}$. Let $B : C(\Omega,\mathfrak{J}) \times C(\Omega,\mathfrak{J}) \to C(\Omega,\mathfrak{J})$ be a symmetric bilinear mapping admitting the following weak standard representation:*

$$B(x,x)(t) = \left(\lambda \circ x^2 + \mu(x) \circ x + \nu(x,x)\right)(t),$$

for all $x \in C(\Omega,\mathfrak{J})$, $t \in \Omega$, where $\lambda \in \mathscr{F}(\Omega,\mathbb{C}\mathbf{1})$, $\mu : C(\Omega,\mathfrak{J}) \to \mathscr{F}(\Omega,\mathbb{C}\mathbf{1})$ is a linear mapping, and $\nu : C(\Omega,\mathfrak{J}) \times C(\Omega,\mathfrak{J}) \to \mathscr{F}(\Omega,\mathbb{C}\mathbf{1})$ is a symmetric bilinear map. Then, this representation is unique and satisfies:

- $\lambda = \widetilde{\mathscr{E}}_2(B(\widehat{w},\widehat{w})) \in C(\Omega,\mathbb{C}\mathbf{1})$,

- $\mu(x) = 2\widetilde{\mathscr{E}_1}(B(\widehat{w}, x)) - 2\widetilde{\mathscr{E}_2}(B(\widehat{w}, \widehat{w})) \circ \widetilde{\mathscr{E}_1}(\widehat{w} \circ x)) - \widetilde{\mathscr{E}_1}(B(\widehat{w}, \widehat{w})) \circ \widetilde{\mathscr{E}_1}(x) \in C(\Omega, \mathbb{C}\mathbf{1})$, *for every* x *in* $C(\Omega, \mathfrak{J})$,
- $\nu(x, x) := \widetilde{\mathscr{E}_0}(B(x, x)) - \lambda \circ \widetilde{\mathscr{E}_0}(x^2) - \mu(x) \circ \widetilde{\mathscr{E}_0}(x) \in C(\Omega, \mathbb{C}\mathbf{1})$, *for every* x *in* $C(\Omega, \mathfrak{J})$,

where each $\widetilde{\mathscr{E}_i}$ *is the natural elementary operator in* $\mathscr{E}\ell(C(\Omega, \mathfrak{J}))$ *extending* $\mathscr{E}_i$ *to* $C(\Omega, \mathfrak{J})$.

Our main result towards determining all associating traces on exceptional JBW*-algebras is stated in the next proposition.

Proposition 4.2. *Let* $\mathfrak{J}$ *denote the JB*-algebra* $C(\Omega, H_3(\mathbb{O}))$, *where* Ω *is a compact Hausdorff space. Let* $B : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$ *be a symmetric bilinear map satisfying*

$$[B(x, x), \mathfrak{J}, x] = 0, \quad \text{for all } x \in \mathfrak{J}. \tag{4.1}$$

Then there exist a unique element λ *in* $Z(\mathfrak{J}) = C(X, \mathbb{C})$, *a unique linear mapping* $\mu : \mathfrak{J} \to Z(\mathfrak{J})$, *and a unique symmetric bilinear map* $\nu : \mathfrak{J} \times \mathfrak{J} \to Z(\mathfrak{J})$ *such that*

$$B(x, x) = \lambda \circ x^2 + \mu(x) \circ x + \nu(x, x), \quad \text{for all } x \in \mathfrak{J}.$$

Proof. Fix an arbitrary $t \in \Omega$. According to the notation of this section, given $a \in H_3(\mathbb{O})$, we shall write $\hat{a}$ for the continuous function in $\mathfrak{J}$ defined by $\hat{a} : \Omega \to H_3(\mathbb{O})$, $s \mapsto \hat{a}(s) := a$. For each $u \in \mathfrak{J}$, we write $u_t := \widehat{u(t)} : \Omega \to H_3(\mathbb{O})$, $s \mapsto u_t(s) := u(t)$.

Note that for every $u \in \mathfrak{J}$ the bilinear mapping B can be decomposed as follows:

$$B(u, u) = B(u - u_t, u - u_t) + 2B(u - u_t, u_t) - B(u_t, u_t). \tag{4.2}$$

Let us see how each summand on the right hand side of the previous identity can be represented.

(1) We begin with the term $B(u_t, u_t)$. Note that for each $s \in \Omega$ we can define a symmetric bilinear mapping $B_s : H_3(\mathbb{O}) \times H_3(\mathbb{O}) \to H_3(\mathbb{O})$, $B_s(a, b) := B(\hat{a}, \hat{b})(s)$. We claim that B_s is associating. Namely, if we take $a, b \in H_3(\mathbb{O})$ we can see from the hypotheses on B that

$$[B_s(a, a), b, a] = [B(\hat{a}, \hat{a}), \hat{b}, \hat{a}](s) = 0.$$

Thus, by applying [12, Lemma 4.1], $B_s(\cdot, \cdot)$ admits the following representation:

$$B_s(a, a) = B(\hat{a}, \hat{a})(s) = \lambda_s \circ a^2 + \mu_s(a) \circ a + \nu_s(a, a), \tag{4.3}$$

where $\lambda_s \in \mathbb{C}\ \mathbf{1}$, $\mu_s : H_3(\mathbb{O}) \to \mathbb{C}\ \mathbf{1}$ is a linear map, and $\nu_s : H_3(\mathbb{O}) \times H_3(\mathbb{O}) \to \mathbb{C}\ \mathbf{1}$ is a symmetric bilinear mapping, where $\mathbf{1}$ stands for the unit of $H_3(\mathbb{O})$, and this procedure can be done for every $s \in \Omega$.

Note that, by Proposition 2.3 there exists an element $w \in H_3(\mathbb{O})_{sa}$ and elementary operators $\mathscr{E}_0, \mathscr{E}_1, \mathscr{E}_2$ such that $\mathscr{E}_i(w^j) = \delta_{ij}\mathbf{1}$ ($i, j \in \{0, 1, 2\}$). We can apply Proposition 3.1, with the Jordan homomorphism $\pi : H_3(\mathbb{O}) \to H_3(\mathbb{O})$ as the identity operator, to deduce that

$$\begin{cases} \lambda_s = \mathscr{E}_2(B_s(w, w)), \\ \mu_s(a) = 2\mathscr{E}_1(B_s(w, a)) - 2\lambda_s \circ \mathscr{E}_1(w \circ a) - \mathscr{E}_1(B_s(w, w)) \circ \mathscr{E}_1(a), \\ \nu_s(a, a) = \mathscr{E}_0(B_s(a, a)) - \lambda_s \circ \mathscr{E}_0(a^2) - \mu_s(a) \circ \mathscr{E}_0(a), \end{cases} \tag{4.4}$$

for all $a \in H_3(\mathbb{O})$ and all $s \in \Omega$.

Let us consider the natural extension of $w \in H_3(\mathbb{O})$ as the constant continuous function $\widehat{w}$ in $\mathfrak{J} = C(\Omega, H_3(\mathbb{O}))$, and the corresponding extension of the elementary operator $\mathscr{E}_i$ ($i \in \{0, 1, 2\}$) as an elementary operator $\widetilde{\mathscr{E}_i}$ on $\mathfrak{J} = C(\Omega, H_3(\mathbb{O}))$ ($i \in \{0,1,2\}$) given by the corresponding extensions of the involved elements as constant functions. By construction, $\widetilde{\mathscr{E}_i}(\widehat{w}^j)(s) = \mathscr{E}_i(\widehat{w}^j(s)) = \mathscr{E}_i(w^j) = \delta_{ij}\mathbf{1}$, for every $s \in \Omega$.

We also set

$$\begin{cases} \lambda := \widetilde{\mathscr{E}_2}(B(\widehat{w}, \widehat{w})) \in \mathfrak{J} = C(\Omega, H_3(\mathbb{O})), \\ \mu(u) := 2\widetilde{\mathscr{E}_1}(B(\widehat{w}, u)) - 2\lambda \circ \widetilde{\mathscr{E}_1}(\widehat{w} \circ u) - \widetilde{\mathscr{E}_1}(B(\widehat{w}, \widehat{w})) \circ \widetilde{\mathscr{E}_1}(u) \in \mathfrak{J}, \\ \nu(u,u) := \widetilde{\mathscr{E}_0}(B(u,u)) - \lambda \circ \widetilde{\mathscr{E}_0}(u^2) - \mu(u) \circ \widetilde{\mathscr{E}_0}(u) \in \mathfrak{J}, \end{cases} \tag{4.5}$$

for all $u \in \mathfrak{J} = C(\Omega, H_3(\mathbb{O}))$. Clearly, $\mu : \mathfrak{J} \to \mathfrak{J}$ is a linear mapping, while ν induces a symmetric bilinear mapping on $\mathfrak{J}$.

Now, by applying $\widetilde{\mathscr{E}_2}$ to $B(\widehat{w}, \widehat{w})$ and having in mind (4.4) we obtain

$$\lambda(s) = \widetilde{\mathscr{E}_2}(B(\widehat{w}, \widehat{w}))(s) = \mathscr{E}_2(B(\widehat{w}, \widehat{w})(s)) = \mathscr{E}_2(B_s(w,w)) = \lambda_s \in \mathbb{C}\mathbf{1}, \tag{4.6}$$

for every $s \in \Omega$. Observe that $\lambda(s) \in \mathbb{C}\mathbf{1}$ for every $s \in \Omega$, and thus $\lambda \in Z(\mathfrak{J})$.

Fix arbitrary elements $s, t \in \Omega$ and $u \in \mathfrak{J}$. By (4.4), (4.5), and (4.6) we also get

$$\begin{cases} \mathbb{C}\mathbf{1} \ni \mu_s(u(t)) = \mathscr{E}_1\Big(2B_s(w, u(t))\Big) - 2\lambda(s) \circ \mathscr{E}_1\Big(w \circ u(t)\Big) \\ \qquad - \mathscr{E}_1\Big(B_s(w,w)\Big) \circ \mathscr{E}_1\Big(u(t)\Big) \\ \qquad = 2\mathscr{E}_1(B(\widehat{w}, u_t)(s)) - 2\lambda(s) \circ \mathscr{E}_1(\widehat{w}(s) \circ u_t(s)) \\ \qquad - \mathscr{E}_1(B(\widehat{w}, \widehat{w})(s)) \circ \mathscr{E}_1(u_t(s)) \\ \qquad = 2\widetilde{\mathscr{E}_1}(B(\widehat{w}, u_t))(s) - 2\lambda(s) \circ \widetilde{\mathscr{E}_1}(\widehat{w} \circ u_t)(s) \\ \qquad - \widetilde{\mathscr{E}_1}(B(\widehat{w}, \widehat{w}))(s) \circ \widetilde{\mathscr{E}_1}(u_t)(s) \\ = \Big(2\widetilde{\mathscr{E}_1}(B(\widehat{w}, u_t)) - 2\lambda \circ \widetilde{\mathscr{E}_1}(\widehat{w} \circ u_t) - \widetilde{\mathscr{E}_1}(B(\widehat{w}, \widehat{w})) \circ \widetilde{\mathscr{E}_1}(u_t)\Big)(s) \\ \qquad = \mu(u_t)(s), \end{cases} \tag{4.7}$$

and consequently, $\mu(u_t) \in Z(\mathfrak{J})$ for all $u \in \mathfrak{J}$, $t \in \Omega$.

Fix $u \in \mathfrak{J}$ and $s, t \in \Omega$. By definition of $\nu(\cdot,\cdot)$ (cf. (4.5)), and having in mind (4.4), (4.5), (4.6), and (4.7), we obtain

$$\begin{cases} \nu(u_t, u_t)(s) = \Big(\widetilde{\mathscr{E}_0}(B(u_t, u_t)) - \lambda \circ \widetilde{\mathscr{E}_0}(u_t^2) - \mu(u_t) \circ \widetilde{\mathscr{E}_0}(u_t)\Big)(s) \\ \qquad = \mathscr{E}_0(B_s(u_t, u_t)) - \lambda(s) \circ \mathscr{E}_0(u(t)^2) - \mu(u_t)(s) \circ \mathscr{E}_0(u(t)) \\ \qquad = \mathscr{E}_0(B_s(u_t, u_t)) - \lambda_s \circ \mathscr{E}_0(u(t)^2) - \mu_s(u(t)) \circ \mathscr{E}_0(u(t)) \\ \qquad = \nu_s(u(t), u(t)) \in \mathbb{C}\mathbf{1}, \text{ for every } s \in X. \end{cases} \tag{4.8}$$

Consequently, $\nu(u_t, u_t) \in Z(\mathfrak{J})$ for all $u \in \mathfrak{J}$, $t \in \Omega$.

The previous conclusions show, via (4.3), (4.6), (4.7), and (4.8), that the term $B(u_t, u_t)$ can be written as follows

$$\begin{cases} B(u_t, u_t)(s) = B(\widehat{u(t)}, \widehat{u(t)})(s) = B_s(u(t), u(t)) \\ \qquad = \lambda_s \circ u(t)^2 + \mu_s(u(t)) \circ u(t) + \nu_s(u(t), u(t)) \circ \mathbf{1} \\ \qquad = \Big(\lambda \circ u_t^2 + \mu(u_t) \circ u_t + \nu(u_t, u_t) \circ \mathbf{1}\Big)(s). \end{cases} \tag{4.9}$$

(2) We deal next with the term $B(u-u_t, u-u_t)$. It is not hard to check that for every $a, b \in H_3(\mathbb{O})$ and $s \in \Omega$, by (3.6) in Remark 3.4, we have

$$\begin{aligned}[B(u-u_t, u-u_t)(s), a, b] &= [B(u-u_t, u-u_t), \widehat{a}, \widehat{b}](s) \\ &= -2[B(u-u_t, \widehat{b}), \widehat{a}, u-u_t](s).\end{aligned}$$

When $s = t$, we obtain that $B(u-u_t, u-u_t)(t) \in \mathbb{C}\mathbf{1} = Z(H_3(\mathbb{O}))$. We can thus define a symmetric bilinear mapping γ from $\mathfrak{J} \times \mathfrak{J} \to \mathscr{F}(\Omega, \mathbb{C}\mathbf{1})$ given by

$$\gamma(u,u)(t) = B(u-u_t, u-u_t)(t) \in \mathbb{C}\mathbf{1}. \tag{4.10}$$

(3) It is now the turn to handle the term $B(u-u_t, u_t)$. For this purpose, for each $s \in \Omega$ and $u \in \mathfrak{J}$, we define a linear mapping $T_{s,u} : H_3(\mathbb{O}) \to H_3(\mathbb{O})$ defined by

$$T_{s,u}(a) := B(u-u_t, \widehat{a})(s).$$

If we pick an arbitrary element $b \in H_3(\mathbb{O})$, it is easy to see, via a new application of (3.6), that

$$\begin{aligned}[T_{s,u}(a), b, a] &= [B(u-u_t, \widehat{a})(s), \widehat{b}(s), \widehat{a}(s)] \\ &= [B(u-u_t, \widehat{a}), \widehat{b}, \widehat{a}](s) \\ &= -\frac{1}{2}[B(\widehat{a}, \widehat{a}), \widehat{b}, u-u_t](s).\end{aligned}$$

For $s = t$, we obtain that $[T_{t,u}(a), b, a] = 0$ for every $a \in H_3(\mathbb{O})$. Thus, by applying [11, Theorem 2.1] we deduce the existence of $\alpha_{t,u} \in \mathbb{C}$ and a linear map $\beta_{t,u} : H_3(\mathbb{O}) \to \mathbb{C}$ such that

$$T_{t,u}(a) = \alpha_{t,u} a + \beta_{t,u}(a)\mathbf{1}, \text{ for all } a \in H_3(\mathbb{O}).$$

When taking $a = w$, we obtain the following identity

$$B(u-u_t, \widehat{w})(t) = T_{t,u}(w) = \alpha_{t,u} w + \beta_{t,u}(w)\mathbf{1}.$$

For each $u \in \mathfrak{J}$, set $\alpha(u), \beta(u,u) : \Omega \to \mathbb{C}\mathbf{1}$ given by

$$\begin{aligned}\alpha(u)(t) &:= \widetilde{\mathscr{E}_1}(B(u-u_t, \widehat{w}))(t) = \mathscr{E}_1(B(u-u_t, \widehat{w})(t)) = \alpha_{t,u}\mathbf{1}, \text{ and} \\ \beta(u,u)(t) &:= \widetilde{\mathscr{E}_0}(B(u-u_t, u_t))(t) - \alpha(u)(t) \circ \mathscr{E}_0(u(t)) = \beta_{t,u}(u(t)), \text{ respectively.}\end{aligned}$$

It is easy to check that the assignment $\alpha : C(\Omega, H_3(\mathbb{O})) \to \mathscr{F}(\Omega, \mathbb{C}\mathbf{1})$, $u \mapsto \alpha(u)$ (respectively, $\beta : C(\Omega, H_3(\mathbb{O})) \times C(\Omega, H_3(\mathbb{O})) \to \mathscr{F}(\Omega, \mathbb{C}\mathbf{1})$, $(u,u) \mapsto \beta(u,u)$) defines a linear (respectively, symmetric bilinear) mapping. According to this notation we can write

$$B(u-u_t, u_t)(t) = T_{t,u}(u(t)) = \alpha(u)(t) \circ u(t) + \beta(u,u)(t). \tag{4.11}$$

Back to (4.2), (4.10), (4.11) and (4.9) we conclude that

$$\begin{aligned}B(u,u)(t) = \lambda(t) \circ u(t)^2 &+ \Big(\mu(u_t) + \alpha(u)\Big)(t) \circ u(t) \\ &+ \Big(\nu(u_t, u_t) + \beta(u,u) + \gamma(u,u)\Big)(t),\end{aligned}$$

for all $u \in \mathfrak{J}$, $t \in \Omega$. Proposition 4.1 asserts that $\mu(u) = \mu(u_t) + \alpha(u)$, and $\nu(u,u) = \nu(u_t,u_t) + \beta(u,u) + \gamma(u,u)$, for all $u \in \mathfrak{J}$, which completes the proof of the existence of the desired decomposition. The uniqueness claimed in the conclusion follows from Proposition 4.1. $\square$

Note that, by [38, Theorem 3.9], every JBW-algebra $\mathfrak{J}$ admits a unique decomposition in the form $\mathfrak{J} = \mathfrak{J}_{sp} \oplus^\infty \mathfrak{J}_{ex}$, where $\mathfrak{J}_{sp}$ is special (i.e. a JW-algebra) and $\mathfrak{J}_{ex}$ is a purely exceptional JBW-algebra. Moreover, $\mathfrak{J}_{ex}$ is isomorphic to $C(\Omega, H_3(\mathbb{O}))_{sa}$, where Ω is a *hyperStonean* compact Hausdorff space. Therefore, each purely exceptional JBW*-algebra is JBW*-isomorphic to $C(\Omega, H_3(\mathbb{O}))$ for some compact Hausdorff space Ω (cf. [44, Theorem 2.8] and [17, Theorem 3.2]). The next result is a straightforward consequence of these comments and Proposition 4.2 above.

Corollary 4.3. *Let $\mathfrak{J}$ be a purely exceptional JBW*-algebra. Let $B : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$ be a symmetric bilinear map satisfying $[B(x,x), \mathfrak{J}, x] = 0$, for all $x \in \mathfrak{J}$. Then there exist a unique element λ in $Z(\mathfrak{J})$, a unique linear mapping $\mu : \mathfrak{J} \to Z(\mathfrak{J})$, and a unique symmetric bilinear map $\nu : \mathfrak{J} \times \mathfrak{J} \to Z(\mathfrak{J})$ such that B admits the following representation:*

$$B(x,x) = \lambda \circ x^2 + \mu(x) \circ x + \nu(x,x), \quad \text{for all } x \in \mathfrak{J}.$$

5. Associating traces on type I_2 JW*-algebras

The main result in Section 3 does not cover the case of associating traces on a JW*-algebra admitting summands of type I_1 and I_2. This section is devoted to fill in the gap in the case of JW*-algebras of type I_2. We recall that every JW*-algebra of type I_2 admits a separating family of Jordan factor representations into spin factors (cf. [21, Propositions 4.6.4, Proposition 5.3.12, and Theorem 6.1.8]).

We begin with a variant of Lemma 3.3. Observe first that every non-associative factor JBW*-algebra must be at least 3-dimensional (cf. [21, 5.3.3] and the fact that the dimension of every spin factor is at least 3). So, if $\pi : \mathfrak{J} \to \mathfrak{J}_\pi$ is a Jordan factor representation, where $\mathfrak{J}$ is a JBW*-algebra without type I_1 summands, then $\dim(\pi(\mathfrak{J})) \geq 3$ (cf. Proposition 2.4).

Lemma 5.1. *Let $\mathfrak{J}$ be a JBW*-algebra admitting no direct summands of type I_1, and let $\pi : \mathfrak{J} \to \mathfrak{J}_\pi$ be a Jordan factor representation. Assume that $B : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}_\pi$ is a symmetric bilinear mapping satisfying that for each $x \in \mathfrak{J}$ there exist $\alpha_x, \beta_x \in \mathbb{C}$ such that*

$$B(x,x) = \alpha_x \pi(x) + \beta_x \mathbf{1}.$$

Then there exist a linear mapping $\mu : \mathfrak{J} \to \mathbb{C}$ and a symmetric bilinear map $\nu : \mathfrak{J} \times \mathfrak{J} \to \mathbb{C}$ such that

$$B(x,x) = \mu(x)\pi(x) + \nu(x,x)\mathbf{1},$$

for every $x \in \mathfrak{J}$.

Proof. Proposition 2.4 implies the existence of an element $w \in \mathfrak{J}$ and elementary operators $\mathscr{E}_0, \mathscr{E}_1 \in \mathscr{E}\ell(\mathfrak{J})$ such that $\mathscr{E}_i(w^j) = \delta_{ij}\mathbf{1}$, for all $i, j \in 0, 1$. By Lemma 2.2(ii) we can find elementary operators $\widehat{\mathscr{E}_j} \in \mathscr{E}\ell(\mathfrak{J}_\pi)$ satisfying $\pi\mathscr{E}_i = \widehat{\mathscr{E}_i}\pi$, for all $i \in \{0,1\}$. Consequently, $\widehat{\mathscr{E}_j}(\pi(w)^i) = \delta_{ij}\mathbf{1}$, $\forall i, j \in \{0,1\}$.

As in the proof of Lemma 3.3, the identity

$$B(x+w, x+w) + B(x-w, x-w) = 2B(x,x) + 2B(w,w), \quad x \in \mathfrak{J},$$

implies that

$$\alpha_{x+w} = \alpha_x + \alpha_w, \tag{5.1}$$

for every $x \in \mathfrak{J}$ such that $\pi(x)$ is not a linear combination of $\mathbf{1}$ and $\pi(w)$ (observe that the vectors $\mathbf{1}$ and $\pi(w)$ are linearly independent). We define the mappings $\mu : \mathfrak{J} \to \mathfrak{J}_\pi$, and $\nu : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}_\pi$ by

$$\begin{aligned}\mu(x) :=& 2\widehat{\mathscr{E}_1}(B(x,w)) - \widehat{\mathscr{E}_1}(B(w,w)) \circ \widehat{\mathscr{E}_1}(\pi(x)),\\ \nu(x,x) :=& B(x,x) - \mu(x) \circ \pi(x).\end{aligned}$$

By definition, μ is a linear map while ν is a bilinear map. It only remains to prove that both maps take values in $\mathbb{C}\mathbf{1}$. Indeed, by mimicking the proof of Lemma 3.3

(1) If $\pi(x) \notin \operatorname{Span}\{\mathbf{1}, \pi(w)\}$, $\mu(x) = 2\widehat{\mathscr{E}_1}(B(x,w)) - \widehat{\mathscr{E}_1}(B(w,w)) \circ \widehat{\mathscr{E}_1}(\pi(x)) = \alpha_x \mathbf{1} \in \mathbb{C}\mathbf{1}$, and $\nu(x,x) = B(x,x) - \mu(x) \circ \pi(x) = B(x,x) - \alpha_x \circ \pi(x) = \beta_x \mathbf{1}$.
(2) If $\pi(x) \in \operatorname{Span}\{\mathbf{1}, \pi(w)\}$, by the comments preceding this lemma, there exists $y \in \mathfrak{J}$ such that $\pi(y) \notin \operatorname{Span}\{\mathbf{1}, \pi(w)\}$, and we can thus conclude that for every $\varepsilon > 0$, the element $\pi(x + \varepsilon y)$ is not in $\operatorname{Span}\{\mathbf{1}, \pi(w)\}$. By applying (5.1) and the conclusion in (1) we obtain

$$B(x + \varepsilon y, x + \varepsilon y) - \mu(x + \varepsilon y)\pi(x + \varepsilon y) \in \mathbb{C}\mathbf{1}.$$

Note that, by what we have just proved in (1), $\mu(x + \varepsilon y) = \alpha_{x+\varepsilon y}\mathbf{1}$. We also know that μ is linear. Thus, it follows that

$$\begin{aligned}B(x,x) + 2B(x,\varepsilon y) + \varepsilon^2 B(y,y) - (\mu(x) + \varepsilon\mu(y)) \circ \pi(x + \varepsilon y) &\in \mathbb{C}\mathbf{1},\\ \mu(x + \varepsilon y) =\mu(x) + \varepsilon\mu(y) &\in \mathbb{C}\mathbf{1}.\end{aligned}$$

Taking $\varepsilon \to 0$ we get $\nu(x,x) = B(x,x) - \mu(x) \circ \pi(x) \in \mathbb{C}\mathbf{1}$, and $\mu(x) \in \mathbb{C}\mathbf{1}$, for every $x \in \mathfrak{J}$. $\square$

The next result determines the general form of all associating traces on a JBW*-algebra of type I_2. Briefly speaking, we shall see that the set of all associating traces on JBW*-algebras of type I_2 constitute a more restrictive set than what we have in other cases.

Theorem 5.2. *Let $\mathfrak{J}$ be a JBW*-algebra of type I_2. Let $B : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$ be a symmetric bilinear map satisfying*

$$[B(x,x), \mathfrak{J}, x] = 0, \quad \textit{for all } x \in \mathfrak{J}.$$

Then B admits the following unique representation:

$$B(x,x) = \mu(x) \circ x + \nu(x,x), \quad \textit{for all } x \in \mathfrak{J},$$

where $\mu : \mathfrak{J} \to Z(\mathfrak{J})$ is linear, and $\nu : \mathfrak{J} \times \mathfrak{J} \to Z(\mathfrak{J})$ is bilinear and symmetric.

Proof. Fix an arbitrary Jordan factor representation $\pi : \mathfrak{J} \to \mathfrak{J}_\pi$, where $\mathfrak{J}_\pi$ must be a spin factor by the hypothesis on $\mathfrak{J}$. Take an arbitrary $x \in \mathfrak{J}$. Since, by assumptions, $B(x,x)$ and x operator commute, the same conclusion holds for $\pi B(x,x)$ and $\pi(x)$. Therefore $\pi B(x,x)$ must be a linear combination of $\pi(x)$ and $\pi(\mathbf{1})$ (cf. Remark 1.5). By applying Lemma 5.1 we prove the existence of a linear mapping $\mu_\pi : \mathfrak{J} \to \mathbb{C}$ and a bilinear mapping $\nu_\pi : \mathfrak{J} \times \mathfrak{J} \to \mathbb{C}$ such that

$$\pi(B(x,x)) = \mu_\pi(x) \circ \pi(x) + \nu_\pi(x,x). \tag{5.2}$$

By Proposition 2.4 and Lemma 2.2(ii) we find an element $w \in \mathfrak{J}$ and elementary operators $\mathscr{E}_j \in \mathscr{E}\ell(\mathfrak{J})$, $\widehat{\mathscr{E}_j} \in \mathscr{E}\ell(\mathfrak{J}_\pi)$ satisfying $\mathscr{E}_i(w^j) = \delta_{ij}\mathbf{1}$, for all i, j in $\{0, 1\}$, and $\pi\mathscr{E}_i = \widehat{\mathscr{E}_i}\pi$, for all $i \in \{0, 1\}$. Due to Proposition 3.2, the previous representation of πB is unique and satisfies

$$\begin{aligned} \mu_\pi(x) &= 2\widehat{\mathscr{E}_1}(\pi(B(w, x))) - \widehat{\mathscr{E}_1}(\pi(B(w, w))) \circ \widehat{\mathscr{E}_1}(\pi(x)), \\ \nu_\pi(x, x) &= \widehat{\mathscr{E}_0}(\pi(B(x, x))) - \mu_\pi(x) \circ \widehat{\mathscr{E}_0}(\pi(x)), \end{aligned} \tag{5.3}$$

for every $x \in \mathfrak{J}$. Observe that in the previous arguments the Jordan representation π was arbitrary. Define now a linear map $\mu : \mathfrak{J} \to \mathfrak{J}$ and a symmetric bilinear mapping $\nu : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$ by

$$\begin{cases} \mu(x) = 2\mathscr{E}_1(B(w, x)) - \mathscr{E}_1(B(w, w)) \circ \mathscr{E}_1(x) \quad (x \in \mathfrak{J}), \\ \nu(x, x) = \mathscr{E}_0(B(x, x)) - \mu(x) \circ \mathscr{E}_0(x) \quad (x \in \mathfrak{J}). \end{cases} \tag{5.4}$$

We combine (5.2), (5.3) and (5.4) to deduce that

$$\pi(B(x, x)) = \pi(\mu(x) \circ x + \nu(x, x)) \tag{5.5}$$

for every $x \in \mathfrak{J}$, and every Jordan factor representation π of $\mathfrak{J}$. Since every JB*-algebra admits a separating family of Jordan factor representations of this form (cf. [2, Corollary 5.7] and the comments at the beginning of this section), it follows from (5.5) that

$$B(x, x) = \mu(x) \circ x + \nu(x, x) \tag{5.6}$$

for every $x \in \mathfrak{J}$. Note that $\pi(\mu(x))$ and $\pi(\nu(x, x))$ belong to $Z(\mathfrak{J}_\pi) = \mathbb{C}\mathbf{1}$ for every Jordan factor representation $\pi : \mathfrak{J} \to \mathfrak{J}_\pi$, so it follows from Remark 1.1 that $\mu(x)$ and $\nu(x, x)$ belong to $Z(\mathfrak{J})$, for each $x \in \mathfrak{J}$, as we desired. $\square$

6. Associating linear maps on JBW*-algebras

It is perhaps of independent interest to determine the structure of those linear maps T on a JBW*-algebra $\mathfrak{J}$ enjoying the property of being associating, i.e. $T(x)$ and x operator commute for all x in $\mathfrak{J}$. By obvious reasons we must avoid JBW*-algebras admitting associative summands –observe that every linear mapping between associative JB*-algebras is associating. In this paper, this type of linear maps will also play a special role in the next section.

Lemma 6.1. *Let $\mathfrak{J}, \mathfrak{M}$ be JBW*-algebras where $\mathfrak{J}$ has no direct summands of type I_1. Let $\pi : \mathfrak{J} \to \mathfrak{M}$ be a Jordan homomorphism whose image is at least 3-dimensional and weak*-dense in $\mathfrak{M}$, and let $T : \mathfrak{J} \to \mathfrak{M}$ be a linear map. Suppose that for each $x \in \mathfrak{J}$ there exist $\alpha_x, \beta_x \in \mathbb{C}$ such that*

$$T(x) = \alpha_x \pi(x) + \beta_x \mathbf{1}.$$

Then there exist a complex number $\lambda \in \mathbb{C}$ and a linear map $\mu : \mathfrak{J} \to \mathbb{C}$ such that

$$T(x) = \lambda \pi(x) + \mu(x)\mathbf{1},$$

for every $x \in \mathfrak{J}$.

Proof. By Proposition 2.4 and Lemma 2.2(ii) we can find an element $u \in \mathfrak{J}$ and elementary operators $\mathscr{E}_j \in \mathscr{E}\ell(\mathfrak{J})$, $\widehat{\mathscr{E}_j} \in \mathscr{E}\ell(\mathfrak{M})$ satisfying $\mathscr{E}_i(u^j) = \delta_{ij}\mathbf{1}$, for all i, j in $\{0, 1\}$, and $\pi\mathscr{E}_i = \widehat{\mathscr{E}_i}\pi$, for all $i \in \{0, 1\}$. In particular, $\widehat{\mathscr{E}_j}(\pi(u)^i) = \delta_{ij}\mathbf{1}$, for all $i, j \in \{0, 1\}$.

Let us fix an arbitrary $x \in \mathfrak{J}$. The identities

$$T(x+u) - T(x-u) = 2T(u), \text{ and } T(x+u) + T(x-u) = 2T(x)$$

hold by hypothesis. Thus we have

$$(\lambda_{x+u} - \lambda_{x-u})\pi(x) + (\lambda_{x+u} + \lambda_{x-u} - 2\lambda_u)\pi(u) + (\mu_{x+u} - \mu_{x-u} - 2\mu_u)\mathbf{1} = 0,$$
$$(\lambda_{x+u} - \lambda_{x-u})\pi(u) + (\lambda_{x+u} + \lambda_{x-u} - 2\lambda_x)\pi(x) + (\mu_{x+u} + \mu_{x-u} - 2\mu_x)\mathbf{1} = 0,$$

for every $x \in \mathfrak{J}$. If the vectors in the set $\{\mathbf{1}, \pi(x), \pi(u)\}$ are linearly independent, we deduce from the previous identities that $\lambda_{x+u} = \lambda_{x-u} = \lambda_u$ and $\mu_{x+u} = \mu_x + \mu_u$. Furthermore, since

$$T(x+u) = \lambda_u(\pi(x) + \pi(u)) + (\mu_x + \mu_u)\mathbf{1},$$

we derive that $\lambda_x = \lambda_u$. Let us define $\lambda := \lambda_u = \widehat{\mathscr{E}_1}(T(u))$. We shall prove that $T(x) - \lambda\pi(x) \in \mathbb{C}\mathbf{1}$ for all $x \in \mathfrak{J}$.

Two possible cases can be considered here:

[(1)] <u>*Case 1*</u>: $\{\mathbf{1}, \pi(x), \pi(u)\}$ are linearly independent. It follows from the previous conclusions and the hypotheses that $T(x) = \lambda\pi(x) + \mu_x\mathbf{1}$, and thus $T(x) - \lambda\pi(x)$ lies in $\mathbb{C}\mathbf{1}$.

[(2)] <u>*Case 2*</u>: $\{\mathbf{1}, \pi(x), \pi(u)\}$ are linearly dependent. By hypotheses, there exists $z \in \mathfrak{J}$ such that the vectors $\pi(z), \mathbf{1} = \pi(\mathbf{1})$, and $\pi(u)$ are linearly independent. Therefore for any $\varepsilon > 0$, the vectors $\{\mathbf{1}, \pi(u), \pi(x + \varepsilon z)\}$ are linearly independent, and thus, by applying the previous *Case 1* we get $T(x + \varepsilon z) - \lambda\pi(x + \varepsilon z) \in \mathbb{C}\mathbf{1}$, or equivalently, $T(x) + \varepsilon T(z) - \lambda\pi(x) - \varepsilon\lambda\pi(u) \in \mathbb{C}\mathbf{1}$ for every $\varepsilon > 0$. Taking $\varepsilon \to 0$ we deduce that $T(x) - \lambda\pi(x) \in \mathbb{C}\mathbf{1}$.

Summarising, $T(x) - \lambda\pi(x) \in \mathbb{C}\mathbf{1}$ for every $x \in \mathfrak{J}$. Finally, observe that the map $\mu : \mathfrak{J} \to \mathfrak{M}$ given by $\mu(x) := T(x) - \lambda\pi(x)$ is a linear map such that $\mu(x) \in \mathbb{C}\mathbf{1}$, and $T(x) = \lambda\pi(x) + \mu(x)\mathbf{1}$ for every $x \in \mathfrak{J}$. $\square$

Lemma 6.2. *Let p be a central projection in a unital JB*-algebra $\mathfrak{J}$. Suppose $T : \mathfrak{J} \to \mathfrak{J}$ is an associating linear map. Then $p \circ T((\mathbf{1} - p) \circ x)$ and $(\mathbf{1} - p) \circ T(p \circ x)$ are central elements for all $x \in \mathfrak{J}$.*

Proof. To simplify the notation, let us denote $q = \mathbf{1} - p$. By hypotheses, $[T(x), \mathfrak{J}, x] = 0$, for all $x \in \mathfrak{J}$. So, by linearizing we get

$$[T(x), a, y] = [x, a, T(y)],$$

for all $x, y, a \in \mathfrak{J}$. Having this identity in mind and the assumption on p we deduce that

$$[q \circ T(p \circ x), \mathfrak{J}, y] = q \circ [T(p \circ x), \mathfrak{J}, y] = q \circ [p \circ x, \mathfrak{J}, T(y)] = [q \circ (p \circ x), \mathfrak{J}, T(y)] = 0,$$

for all $x, y \in \mathfrak{J}$, witnessing that $q \circ T(p \circ x) \in Z(\mathfrak{J})$. The rest is clear since q also is central. $\square$

We begin our study with the case of associating linear maps on special JW*-algebras without type I_1 part.

Theorem 6.3. *Let $\mathfrak{J}$ be a JW*-algebra admitting no direct summands of type I_1, and let $T : \mathfrak{J} \to \mathfrak{J}$ be a linear mapping. Suppose additionally that $[T(x), \mathfrak{J}, x] = 0$, for all $x \in \mathfrak{J}$. Then T can be uniquely written in the form:*

$$T(x) = \lambda \circ x + \mu(x), \quad \textit{for all } x \in \mathfrak{J},$$

where $\lambda \in Z(\mathfrak{J})$*, and* $\mu : \mathfrak{J} \to Z(\mathfrak{J})$ *is linear.*

Proof. Let us take an arbitrary Jordan factor representation $\pi : \mathfrak{J} \to \mathfrak{J}_\pi$, where $\mathfrak{J}_\pi$ is a factor JW*-algebra which is not of type I_1.

Since π is a Jordan homomorphism, it follows from the hypotheses that the identity

$$[\pi(T(x)), \pi(\mathfrak{J}), \pi(x)] = 0,$$

holds for every $x \in \mathfrak{J}$. The weak*-density of $\pi(\mathfrak{J})$ and the separate weak*-continuity of the Jordan product of $\mathfrak{J}_\pi$ lead to

$$[\pi(T(x)), \mathfrak{J}_\pi, \pi(x)] = 0, \tag{6.1}$$

for every $x \in \mathfrak{J}$.

Assume first that $\mathfrak{J}_\pi$ is a spin factor or a factor JW*-algebra of type I_2 (cf. [21, Theorem 6.1.8]), having in mind that $\pi T(x)$ and $\pi(x)$ operator commute in $\mathfrak{J}_\pi$ (cf. (6.1)), Remark 1.5 implies the existence of $\lambda_{1,x}\lambda_{0,x} \in \mathbb{C}$ such that

$$T(x) = \lambda_{1,x}\pi(x) + \lambda_{0,x}\mathbf{1}, \tag{6.2}$$

for all $x \in \mathfrak{J}$.

If $\mathfrak{J}_\pi$ is not a spin factor, by the same arguments employed in Theorem 3.6, $\mathfrak{J}_\pi$ cannot be a JW*-algebra factor of type I_1 or I_2, and its universal von Neumann algebra, $W^*(\mathfrak{J}_\pi)$, satisfies that $\mathfrak{J}_\pi = H(W^*(\mathfrak{J}_\pi), \tau)$, where τ is a period-2 *-anti-automorphism on $W^*(\mathfrak{J}_\pi)$ and the latter either is a factor von Neumann algebra, or it can be written in the form $W^*(\mathfrak{J}_\pi) = N \oplus^\infty \tau(N)$, for some weak*-closed ideal N of $W^*(\mathfrak{J}_\pi)$ which is a factor von Neumann algebra (cf. Lemma 1.3 and the comments prior to it or [21, Theorem 7.1.9, Remark 7.2.8 and Proposition 7.3.3]). As in previous cases, the product of $W^*(\mathfrak{J}_\pi)$ will be denoted by mere juxtaposition.

Proposition 1.2 proves that (6.1) is equivalent to

$$[\pi(T(x)), \pi(x)] = \pi(T(x))\pi(x) - \pi(x)\pi(T(x)) = 0, \text{ for all } x \in \mathfrak{J}, \tag{6.3}$$

in terms of the associative product of $\mathfrak{J}_\pi = W^*(\mathfrak{J}_\pi)$. A linearization argument gives

$$[\pi(T(x+y), \pi(x+y)] = [\pi(T(x)), \pi(y)] + [\pi(T(y)), \pi(x)] = 0,$$

for all $x, y \in \mathfrak{J}$. Therefore,

$$[\pi(T(x)), \pi(y)] = -[\pi(T(y)), \pi(x)] \tag{6.4}$$

for every $x, y \in \mathfrak{J}$.

Now, by combining (6.3) and (6.4) it can be easily checked that the identity

$$\begin{aligned}
[\pi(T(x^2)), \pi(y)] &= -[\pi(T(y)), \pi(x^2)] \\
&= -([\pi(T(y)), \pi(x)]\pi(x) + \pi(x)[\pi(T(y)), \pi(x)]) \\
&= [\pi(T(x)), \pi(y)]\pi(x) + \pi(x)[\pi(T(x)), \pi(y)] \\
&= \pi(T(x))\pi(y)\pi(x) - \pi(y)\pi(T(x))\pi(x) \\
&\quad + \pi(x)\pi(T(x))\pi(y) - \pi(x)\pi(y)\pi(T(x)),
\end{aligned}$$

holds for every $x, y \in \mathfrak{J}$. Obviously, the previous identity holds when $\pi(y)$ is replaced with an arbitrary $z \in \mathfrak{J}_\pi$, and hence

$$\begin{aligned} &\pi(T(x^2))z\pi(\mathbf{1}) - \pi(\mathbf{1})z\pi(T(x^2)) + z\pi(T(x))\pi(x) \\ &+ \pi(x)z\pi(T(x)) - \pi(T(x))z\pi(x) - \pi(x)\pi(T(x))z = 0, \end{aligned} \tag{6.5}$$

for all $x \in \mathfrak{J}$ and any $z \in \mathfrak{J}_\pi$. Two possibilities arise here:

(1) *Case 1*: $\{\mathbf{1} = \pi(\mathbf{1}), \pi(x)\}$ are $\mathbb{C}$-linearly independent. Having in mind that $\mathfrak{J}_\pi = H(W^*(\mathfrak{J}_\pi), \tau)$ is a factor, we can combine Lemma 1.3 and Lemma 2.8 with the identity in (6.5), to conclude that

$$\pi(T(x)) = \lambda_{1,x}\pi(x) + \lambda_{0,x}\mathbf{1},$$

where $\lambda_{i,x} \in \mathbb{C}$ for $i \in \{0, 1\}$.

(2) *Case 2*: $\{\mathbf{1}, \pi(x)\}$ are $\mathbb{C}$-linearly dependent. In this case there exist $\alpha \in \mathbb{C}$ such that

$$\pi(x) = \alpha\pi(1).$$

Then it is clear by (6.4) that

$$[\pi(T(x)), \pi(y)] = -[\pi(T(y)), \pi(x)] = 0, \text{ for every } y \in \mathfrak{J}.$$

Moreover, due to the weak*-density of $\pi(\mathfrak{J})$ in $\mathfrak{J}_\pi$, and the separate weak*-continuity of the product of $W^*(\mathfrak{J}_\pi)$, the previous identity also holds when $\pi(y)$ is replaced by an arbitrary $z \in \mathfrak{J}_\pi$, and hence $T(x) \in Z(\mathfrak{J}_\pi) = \mathbb{C}\mathbf{1}$. Therefore, there exists $\lambda_{0,x} \in \mathbb{C}$ such that

$$\pi(T(x)) = \lambda_{0,x}\mathbf{1}.$$

Summarising, all the previous cases lead to the conclusion that for each $x \in \mathfrak{J}$ there exist complex numbers $\lambda_{0,x}, \lambda_{1,x}$ satisfying

$$\pi(T(x)) = \lambda_{1,x}\pi(x) + \lambda_{0,x}\mathbf{1}. \tag{6.6}$$

We are now in a position to apply Lemma 6.1 to deduce the existence of $\alpha \in \mathbb{C}$ and a linear map $\beta : \mathfrak{J} \to \mathbb{C}$ satisfying

$$\pi(T(x)) = \alpha\pi(x) + \beta(x)\mathbf{1}. \tag{6.7}$$

Now observe that due to Proposition 2.4 and Lemma 2.2 there exist an element $u \in \mathfrak{J}_{sa}$ and elementary operators $\mathscr{E}_i \in \mathscr{E}\ell(\mathfrak{J})$, $\widehat{\mathscr{E}_i} \in \mathscr{E}\ell(\mathfrak{J}_\pi)$ for all $i \in \{0, 1\}$ such that $\widehat{\mathscr{E}_i}\pi = \pi\mathscr{E}_i$, and $\widehat{\mathscr{E}_i}(\pi(u)^j) = \delta_{ij}\mathbf{1}$, for any $i, j \in \{0, 1\}$. Arguing as in the proof of Proposition 3.1, we deduce from (6.7) that α and $\beta(x)$ can be uniquely written as follows:

$$\begin{cases} \alpha\mathbf{1} = \widehat{\mathscr{E}_1}(\pi(T(u))) = \pi(\mathscr{E}_1(T(u)), \\ \beta(x)\mathbf{1} = \widehat{\mathscr{E}_0}(\pi(T(x))) - \widehat{\mathscr{E}_1}(\pi(T(u))) \circ \widehat{\mathscr{E}_0}(\pi(x)) \\ \qquad\quad = \pi\Big(\mathscr{E}_0(T(x)) - \mathscr{E}_1(T(u)) \circ \mathscr{E}_0(x)\Big), \end{cases} \tag{6.8}$$

for all $x \in \mathfrak{J}$. Observe that the Jordan factor representation π was arbitrarily chosen, so by setting $\lambda = \mathscr{E}_1(T(u)) \in \mathfrak{J}$ and defining a linear mapping $\mu : \mathfrak{J} \to \mathfrak{J}$ by

$$\mu(x) = \mathscr{E}_0(T(x)) - \mathscr{E}_1(T(u)) \circ \mathscr{E}_0(x),$$

we arrive, via (6.7) and (6.8), to

$$\pi(T(x)) = \pi(\lambda \circ x + \mu(x)), \tag{6.9}$$

for every $x \in \mathfrak{J}$, and every Jordan factor representation π of $\mathfrak{J}$. Since every JB*-algebra admits a separating family of Jordan factor representations of this form (cf. [2, Corollary 5.7]), it follows from (6.9) that

$$T(x) = \lambda \circ x + \mu(x), \tag{6.10}$$

for every $x \in \mathfrak{J}$. Note that $\pi(\lambda)$ and $\pi(\mu(x))$ belong to $Z(\mathfrak{J}_\pi) = \mathbb{C}\mathbf{1}$ for every Jordan factor representation $\pi : \mathfrak{J} \to \mathfrak{J}_\pi$, so Remark 1.1 implies that λ and $\mu(x)$ belong to $Z(\mathfrak{J})$ for each $x \in \mathfrak{J}$ as we desired.

Observe finally, that the uniqueness of $\lambda \in Z(\mathfrak{J})$ and the linear mapping μ is essentially guaranteed by the identities $\lambda = \mathscr{E}_1(T(u))$, and $\mu(x) = \mathscr{E}_0(T(x)) - \mathscr{E}_1(T(u)) \circ \mathscr{E}_0(x)$. $\square$

Since Theorem 6.3 is only valid for JW*-algebras, linear associating maps on exceptional JBW*-algebras are treated in our next result.

Theorem 6.4. *Let $\mathfrak{J}$ denote the JB*-algebra $C(\Omega, H_3(\mathbb{O}))$, where Ω is a compact Hausdorff space. Suppose $T : \mathfrak{J} \to \mathfrak{J}$ is a linear mapping satisfying*

$$[T(x), \mathfrak{J}, x] = 0, \text{ for all } x \in \mathfrak{J}.$$

Then T can be uniquely written in the form:

$$T(x) = \lambda \circ x + \mu(x), \text{ for all } x \in \mathfrak{J},$$

where $\lambda \in Z(\mathfrak{J})$, $\mu : \mathfrak{J} \to Z(\mathfrak{J})$ is linear.

Proof. We keep the notation employed in Section 4. That is, for each $a \in H_3(\mathbb{O})$, we write $\hat{a}$ for the continuous function on $\mathfrak{J}$ with constant value a, while for $u \in \mathfrak{J}$ and $t \in \Omega$, we write $u_t := \widehat{u(t)} \in \mathfrak{J}$.

The uniqueness can be justified as in the final observation in the proof of the previous Theorem 6.3.

Fix now an arbitrary $t \in \Omega$. Note that the linear map T can be decomposed as follows:

$$T(u) = T(u - u_t) + T(u_t). \tag{6.11}$$

Let us deal with each summand on the right hand side of the previous identity independently.

(1) We begin with the term $T(u_t)$. Note that for each $s \in \Omega$ we can define a linear mapping $T_s : H_3(\mathbb{O}) \to H_3(\mathbb{O})$, $T_s(a) := T(\widehat{a})(s)$. We claim that T_s is associating. Indeed, if we take $a, b \in H_3(\mathbb{O})$ we can see from the hypotheses on T that

$$[T_s(a), b, a] = [T(\widehat{a}), \widehat{b}, \widehat{a}](s) = 0.$$

Thus, by applying [11, Theorem 2.1] we deduce that T_s admits the following representation:

$$T_s(a) = \lambda_s \circ a + \mu_s(a) = \lambda_s \circ a + \mu_s(a) \circ \mathbf{1}, \tag{6.12}$$

where $\lambda_s \in Z(H_3(\mathbb{O})) = \mathbb{C}\mathbf{1}$, and $\mu_s : H_3(\mathbb{O}) \to Z(H_3(\mathbb{O})) = \mathbb{C}\mathbf{1}$ is a linear map. Observe that this procedure can be done for every $s \in \Omega$.

Note now that, by Proposition 2.4, we can find an element $w \in H_3(\mathbb{O})_{sa}$ and elementary operators $\mathscr{E}_0, \mathscr{E}_1 \in \mathscr{E}\ell(H_3(\mathbb{O}))$ such that $\mathscr{E}_i(w^j) = \delta_{ij}\mathbf{1}$ $(i, j \in \{0, 1\})$. Proposition 3.1, applied to T_s and the Jordan homomorphism $\pi : H_3(\mathbb{O}) \to H_3(\mathbb{O})$ given by the identity operator, assures that

$$\begin{cases} \quad \lambda_s = \mathscr{E}_1(T_s(w)) \in Z(H_3(\mathbb{O})) = \mathbb{C}\mathbf{1}, \\ \mu_s(a) = \mathscr{E}_0(T_s(a)) - \lambda_s \circ \mathscr{E}_0(a) \in Z(H_3(\mathbb{O})) = \mathbb{C}\mathbf{1}, \end{cases} \tag{6.13}$$

for all $a \in H_3(\mathbb{O})$ and all $s \in \Omega$.

Let us consider the natural extension of $w \in H_3(\mathbb{O})_{sa}$ as the constant continuous function $\widehat{w} \in \mathfrak{J}_{sa}$, and the corresponding extension of the elementary operator $\mathscr{E}_i$ $(i \in \{0, 1\})$ as an elementary operator $\widetilde{\mathscr{E}_i}$ on $\mathfrak{J} = C(\Omega, H_3(\mathbb{O}))$ $(i \in \{0, 1\})$ given by the corresponding extensions of the involved elements as constant functions. It follows from the definition that $\widetilde{\mathscr{E}_i}(\widehat{w}^j)(s) = \mathscr{E}_i(\widehat{w}^j(s)) = \mathscr{E}_i(w^j) = \delta_{ij}\mathbf{1}$, for every $s \in \Omega$.

We also set

$$\begin{cases} \quad \lambda := \widetilde{\mathscr{E}_1}(T(\widehat{w})) \in \mathfrak{J} = C(\Omega, H_3(\mathbb{O})), \\ \mu(u) := \widetilde{\mathscr{E}_0}(T(u)) - \lambda \circ \widetilde{\mathscr{E}_0}(u) \in \mathfrak{J}, \end{cases} \tag{6.14}$$

for all $u \in \mathfrak{J} = C(\Omega, H_3(\mathbb{O}))$. Clearly, $\mu : \mathfrak{J} \to \mathfrak{J}$ is a linear mapping on $\mathfrak{J}$.

Now by applying $\widetilde{\mathscr{E}_1}$ to $T(\widehat{w})$ and having in mind (6.13) we get

$$\lambda(s) = \widetilde{\mathscr{E}_1}(T(\widehat{w}))(s) = \mathscr{E}_1(T(\widehat{w})(s)) = \mathscr{E}_1(T_s(w)) = \lambda_s \in \mathbb{C}\mathbf{1}, \tag{6.15}$$

for every $s \in \Omega$. Observe that $\lambda(s) \in \mathbb{C}\mathbf{1}$ for every $s \in \Omega$, and thus $\lambda \in Z(\mathfrak{J})$.

Fix now two arbitrary elements $s, t \in \Omega$ and $u \in \mathfrak{J}$. By combining (6.13), (6.14) and (6.15) we derive that

$$\begin{cases} \mathbb{C}\mathbf{1} \ni \mu_s(u(t)) = \mathscr{E}_0\Big(T_s(u(t))\Big) - \lambda(s) \circ \mathscr{E}_0\Big(u(t)\Big) \\ \qquad\qquad\qquad = \mathscr{E}_0(T(u_t)(s)) - \lambda(s) \circ \mathscr{E}_0(u_t(s)) \\ \qquad\qquad\qquad = \widetilde{\mathscr{E}_0}(T(u_t))(s) - \lambda(s) \circ \widetilde{\mathscr{E}_0}(u_t)(s) \\ \qquad\qquad\qquad = \Big(\widetilde{\mathscr{E}_0}(T(u_t)) - \lambda \circ \widetilde{\mathscr{E}_0}(u_t)\Big)(s) \\ \qquad\qquad\qquad = \mu(u_t)(s), \end{cases} \tag{6.16}$$

and consequently, $\mu(u_t) \in Z(\mathfrak{J})$ for all $u \in \mathfrak{J}$, $t \in \Omega$.

This shows, via (6.12), (6.15) and (6.16) that the term $T(u_t)$ can be written as follows

$$\begin{cases} T(u_t)(s) = T(\widehat{u(t)})(s) = T_s(u(t)) \\ \qquad\qquad = \lambda_s \circ u(t) + \mu_s(u(t))\mathbf{1} \\ \qquad\qquad = \Big(\lambda \circ u_t + \mu(u_t)\Big)(s). \end{cases} \tag{6.17}$$

(2) We deal next with the term $T(u - u_t)$ in (6.11). It is not hard to check, from the hypotheses on T, that for every $a, b \in H_3(\mathbb{O})$ we have

$$[T(u - u_t)(s), a, b] = [T(u - u_t), \widehat{a}, \widehat{b}](s) = -[T(\widehat{b}), \widehat{a}, u - u_t](s).$$

When $s = t$, we have $T(u - u_t)(t) \in \mathbb{C}\mathbf{1} = Z(H_3(\mathbb{O}))$. We can thus define a linear mapping $\beta : \mathfrak{J} \to \mathscr{F}(\Omega, \mathbb{C}\mathbf{1})$ given by

$$\beta(u)(t) = T(u - u_t)(t) \in \mathbb{C}\mathbf{1}. \tag{6.18}$$

Back to (6.11), (6.18) and (6.17) we conclude that

$$T(u)(t) = \lambda(t) \circ u(t) + \Big(\mu(u_t) + \beta(u)\Big)(t), \tag{6.19}$$

for all $u \in \mathfrak{J}$, $t \in \Omega$. By (6.15), (6.16), and (6.12) we have

$$\begin{aligned}
\widetilde{\mathscr{E}_0}(T(u))(t) &= \mathscr{E}_0(T(u)(t)) = \mathscr{E}_0\Big(\lambda(t) \circ u(t) + \mu(u_t)(t) + \beta(u)(t)\Big) \\
&= \mathscr{E}_0\Big(\lambda_t \circ u(t) + \mu_t(u(t))\Big) + \beta(u)(t) \\
&= \mathscr{E}_0\Big(T_t(u(t))\Big) + \beta(u)(t) = \mathscr{E}_0\Big(T(u_t)(t)\Big) + \beta(u)(t) \\
&= \widetilde{\mathscr{E}_0}\Big(T(u_t)\Big)(t) + \beta(u)(t),
\end{aligned}$$

for all $u \in \mathfrak{J}$, $t \in \Omega$, which proves that $\beta(u) = \widetilde{\mathscr{E}_0}(T(u)) - \widetilde{\mathscr{E}_0}\Big(T(u_t)\Big) \in Z(\mathfrak{J})$. Furthermore, back to (6.14) and (6.19), we derive that

$$\mu(u) = \widetilde{\mathscr{E}_0}\Big(T(u)\Big) - \lambda \circ \widetilde{\mathscr{E}_0}(T(u)) = \mu(u_t) + \beta(u),$$

for all $u \in \mathfrak{J}$, which concludes the proof. $\square$

By the same observations made before Corollary 4.3, the next result is a straightforward consequence of the representation theory for exceptional JBW*-algebras and the previous Theorem 6.4.

Corollary 6.5. *Let $\mathfrak{J}$ be a purely exceptional JBW*-algebra. Suppose $T : \mathfrak{J} \to \mathfrak{J}$ is a linear map satisfying*

$$[T(x), \mathfrak{J}, x] = 0, \text{ for all } x \in \mathfrak{J}.$$

Then T can be uniquely written in the form:

$$T(x) = \lambda \circ x + \mu(x), \text{ for all } x \in \mathfrak{J},$$

where $\lambda \in Z(\mathfrak{J})$, and $\mu : \mathfrak{J} \to Z(\mathfrak{J})$ is linear.

The main result of this section reads as follows:

Theorem 6.6. *Let $\mathfrak{J}$ be a JBW*-algebra admitting no direct summands of type I_1, and let $T : \mathfrak{J} \to \mathfrak{J}$ be a linear mapping. Suppose, additionally, that $[T(x), \mathfrak{J}, x] = 0$, for all $x \in \mathfrak{J}$. Then T can be uniquely written in the form:*

$$T(x) = \lambda \circ x + \mu(x), \text{ for all } x \in \mathfrak{J},$$

where $\lambda \in Z(\mathfrak{J})$, and $\mu : \mathfrak{J} \to Z(\mathfrak{J})$ is linear.

Proof. The uniqueness follows as in the previous two theorems.

Standard structure theory guarantees that $\mathfrak{J}$ decomposes as the orthogonal sum $\mathfrak{J} = \mathfrak{J}_1 \oplus^\infty \mathfrak{J}_2$ of two weak*-closed ideals $\mathfrak{J}_1$ and $\mathfrak{J}_2$, where $\mathfrak{J}_1$ is a (possibly zero) JW*-algebra admitting no direct summands of type I_1, and $\mathfrak{J}_2$ is a (possibly zero) purely exceptional JBW*-algebra [21, Theorems 5.1.5 and 5.3.5]. Let p denote the projection in $\mathfrak{J}$ corresponding to the unit of $\mathfrak{J}_1$. Clearly, p is a central projection in $\mathfrak{J}$, and $q = \mathbf{1} - p$ is the unit of $\mathfrak{J}_2$ (i.e., $\mathfrak{J}_1 = p \circ \mathfrak{J}$ and $\mathfrak{J}_2 = q \circ \mathfrak{J}$).

Lemma 6.2 implies that the linear mappings $T_{1,2} := M_p T M_q$ and $T_{2,1} := M_q T M_p$ take values in $p \circ Z(\mathfrak{J}) = Z(p \circ \mathfrak{J}) = Z(\mathfrak{J}_1)$ and $q \circ Z(\mathfrak{J}) = Z(q \circ \mathfrak{J}) = Z(\mathfrak{J}_2)$, respectively.

By applying that p and q are central projections, it can be easily checked that the linear mappings $T_1 = M_p T|_{\mathfrak{J}_1} : \mathfrak{J}_1 \to \mathfrak{J}_1$ and $T_2 = M_q T|_{\mathfrak{J}_2} : \mathfrak{J}_2 \to \mathfrak{J}_2$ are associating. We deduce from Theorem 6.3 and Corollary 6.5 the existence of $\lambda_i \in Z(\mathfrak{J}_i)$ and $\mu_i : \mathfrak{J}_i \to Z(\mathfrak{J}_i)$ linear satisfying

$$T_i(x) = \lambda_i \circ x + \mu_i(x) \ \ (x \in \mathfrak{J}_i, \ i = 1, 2).$$

Finally, we can write

$$\begin{aligned} T(x) &= p \circ T(p \circ x) + q \circ T(q \circ x) + q \circ T(p \circ x) + p \circ T(q \circ x) \\ &= T_1(p \circ x) + T_2(q \circ x) + T_{2,1}(x) + T_{1,2}(x) \\ &= \lambda_1 \circ (p \circ x) + \mu_1(p \circ x) + \lambda_2 \circ (q \circ x) + \mu_2(q \circ x) + T_{2,1}(x) + T_{1,2}(x) \\ &= (\lambda_1 + \lambda_2) \circ x + \mu_1(p \circ x) + \mu_2(q \circ x) + T_{2,1}(x) + T_{1,2}(x), \end{aligned}$$

for all $x \in \mathfrak{J}$. Taking $\lambda = \lambda_1 + \lambda_2 \in Z(\mathfrak{J})$, and $\mu : \mathfrak{J} \to Z(\mathfrak{J}) = Z(\mathfrak{J}_1) \oplus^\infty Z(\mathfrak{J}_2)$, $\mu(x) := \mu_1(p \circ x) + \mu_2(q \circ x) + T_{2,1}(x) + T_{1,2}(x)$, we get the desired decomposition for T. $\square$

Determining all associating linear maps on JBW-algebras is a problem of its own importance. We shall see next how we can obtain a general characterization of all associating linear maps on JBW-algebras from Theorem 6.6 above.

Theorem 6.7. *Let $\mathfrak{J}$ be a JBW*-algebra admitting no direct summands of type I_1, and let $T : \mathfrak{J}_{sa} \to \mathfrak{J}_{sa}$ be a linear mapping. Suppose, additionally, that*

$$[T(x), \mathfrak{J}_{sa}, x] = 0, \ \textit{for all } x \in \mathfrak{J}_{sa}.$$

Then T can be uniquely written in the form

$$T(x) = \lambda \circ x + \mu(x), \ \textit{for all } x \in \mathfrak{J}_{sa},$$

where $\lambda \in Z(\mathfrak{J}_{sa})$, and $\mu : \mathfrak{J}_{sa} \to Z(\mathfrak{J}_{sa})$ is linear.

Proof. We can extend T to a complex linear mapping on $\mathfrak{J}$ given by $T_{\mathbb{C}} : \mathfrak{J} \to \mathfrak{J}$, $T_{\mathbb{C}}(a + ib) = T(a) + iT(b)$. We claim that $T_{\mathbb{C}}$ is associating. Namely, since T is associating, we have

$$\begin{aligned} 0 = [T(a+b), \mathfrak{J}_{sa}, a+b] &= [T(a), \mathfrak{J}_{sa}, a] + [T(a), \mathfrak{J}_{sa}, b] \\ &\quad + [T(b), \mathfrak{J}_{sa}, a] + [T(b), \mathfrak{J}_{sa}, b] \\ &= [T(a), \mathfrak{J}_{sa}, b] + [T(b), \mathfrak{J}_{sa}, a], \end{aligned}$$

for all $a, b \in \mathfrak{J}_{sa}$. Therefore, the previous identity shows that

$$\begin{aligned} [T_{\mathbb{C}}(a+ib), \mathfrak{J}_{sa}, a+ib] &= [T(a), \mathfrak{J}_{sa}, a] - [T(b), \mathfrak{J}_{sa}, b] \\ &\quad + i[T(a), \mathfrak{J}_{sa}, b] + i[T(b), \mathfrak{J}_{sa}, a] = 0, \end{aligned}$$

for all $a + ib \in \mathfrak{J}$, and thus $[T_{\mathbb{C}}(a + ib), \mathfrak{J}, a + ib] = 0$, which proves the claim.

Now, Theorem 6.6, applied to $T_{\mathbb{C}}$, assures the existence of a unique $\gamma \in Z(\mathfrak{J})$, and a unique linear mapping $\nu : \mathfrak{J} \to Z(\mathfrak{J})$ such that

$$T_{\mathbb{C}}(a + ib) = \gamma \circ (a + ib) + \nu(a + ib), \text{ for all } a + ib \in \mathfrak{J}.$$

Observe that, by construction, $T_{\mathbb{C}}(a + ib)^* = T_{\mathbb{C}}((a + ib)^*)$ for all $a + ib \in \mathfrak{J}$. It then follows that

$$T_{\mathbb{C}}(a + ib) = T_{\mathbb{C}}((a + ib)^*)^* = \gamma^* \circ (a + ib) + \nu((a + ib)^*)^*, \text{ for all } a + ib \in \mathfrak{J}.$$

The uniqueness of the decomposition implies that $\gamma^* = \gamma$ lies in $\mathfrak{J}_{sa}$ and $\nu(a + ib)^* = \nu((a + ib)^*)$, for all $a + ib \in \mathfrak{J}$. The mapping $\mu := \nu|_{\mathfrak{J}}$ takes values in $Z(\mathfrak{J}_{sa}) = Z(\mathfrak{J})_{sa}$, and taking $\lambda = \gamma$, we get

$$T(x) = \lambda \circ x + \mu(x), \text{ for all } x \in \mathfrak{J}_{sa}. \quad \square$$

7. Conclusions on associating traces on general JBW*-algebras

As the reader can already guess by the title, the goal of this section is to gather all previous results to determine the form of all symmetric bilinear mappings on a general JBW*-algebra whose trace is associating.

Our first technical result is a bilinear version of the previous Lemma 6.2.

Lemma 7.1. *Let p be a central projection in a unital JB*-algebra $\mathfrak{J}$, and let $B : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$ be a symmetric bilinear map satisfying $[B(x, x), \mathfrak{J}, x] = 0$ for all $x \in \mathfrak{J}$. Then the mappings*

$$x \mapsto p \circ B((\mathbf{1} - p) \circ x, (\mathbf{1} - p) \circ x), \quad (x \in \mathfrak{J})$$
$$x \mapsto (\mathbf{1} - p) \circ B(p \circ x, p \circ x), \quad (x \in \mathfrak{J})$$

are centre-valued.

Proof. Set $q = \mathbf{1} - p$, which is also central. Let us take $x, z, y \in \mathfrak{J}$. By (3.6) in Remark 3.4 we have

$$\begin{aligned}[p \circ B(q \circ x, q \circ x), y, z] &= p \circ [B(q \circ x, q \circ x), y, z] \\ &= p \circ (-2[B(q \circ x, z), y, q \circ x]) = 0,\end{aligned}$$

where the last equality holds because p is central. The remaining statement follows by similar arguments. $\square$

The next technical conclusion is perhaps interesting by itself.

Proposition 7.2. *Let $\mathfrak{J}$ be a JBW*-algebra without direct summands of type I_1, let p be a central projection in $\mathfrak{J}$, and set $q = \mathbf{1} - p$. Suppose that $B : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$ is a symmetric bilinear mapping satisfying $[B(x, x), \mathfrak{J}, x] = 0$, for all $x \in \mathfrak{J}$. Then there exist a unique linear mapping $\mu : \mathfrak{J} \to p \circ Z(\mathfrak{J}) = Z(p \circ \mathfrak{J})$, and a unique bilinear mapping $\nu : \mathfrak{J} \times \mathfrak{J} \to p \circ Z(\mathfrak{J}) = Z(p \circ \mathfrak{J})$ satisfying $\mu(x) = \mu(q \circ x)$, $\nu(x, y) = \nu(q \circ x, p \circ y)$, and*

$$p \circ B(p \circ y, q \circ x) = \mu(x) \circ (p \circ y) + \nu(x, y) = \mu(x) \circ y + \nu(x, y),$$

for all $x, y \in \mathfrak{J}$.

Proof. The projection $q = \mathbf{1} - p$ is also central. Let us fix an arbitrary $x \in \mathfrak{J}$ and define a linear mapping $T_{p,x} : p \circ \mathfrak{J} \to p \circ \mathfrak{J}$ given by $T_{p,x}(p \circ a) = p \circ B(p \circ a, q \circ x)$. The assumptions combined with (3.6) in Remark 3.4, give

$$\begin{aligned}[T_{p,x}(p \circ a), b, p \circ a] &= [p \circ B(p \circ a, q \circ x), b, p \circ a] = p \circ [B(p \circ a, q \circ x), b, p \circ a] \\ &= -\frac{1}{2} p \circ [B(p \circ a, p \circ a), b, q \circ x] = 0,\end{aligned}$$

for all $a, b \in \mathfrak{J}$. Consequently, $T_{p,x}$ is an associating linear map on $p \circ \mathfrak{J}$, and the latter cannot contain summands of type I_1. Therefore, by applying Theorem 6.6, there exist a unique element $\lambda_{p,x} \in Z(p \circ \mathfrak{J})$ and a unique linear mapping $\mu_{p,x} : p \circ \mathfrak{J} \to Z(p \circ \mathfrak{J})$ such that

$$p \circ B(p \circ y, q \circ x) = T_{p,x}(p \circ y) = \lambda_{p,x} \circ (p \circ y) + \mu_{p,x}(p \circ y), \tag{7.1}$$

for all $y \in \mathfrak{J}$.

Let $\mathscr{E}_i \in \mathscr{E}\ell(\mathfrak{J})$ $(i = 0, 1)$ and $u \in \mathfrak{J}$ be the elementary operators on $\mathfrak{J}$ and the element whose existence is guaranteed by Proposition 2.4. We define now a linear mapping $\mu : \mathfrak{J} \to p \circ \mathfrak{J}$, and a bilinear map $\nu : \mathfrak{J} \times \mathfrak{J} \to p \circ \mathfrak{J}$ given by

$$\begin{cases} \mu(x) = \mathscr{E}_1(p \circ B(p \circ u, q \circ x)) = p \circ \mathscr{E}_1(B(p \circ u, q \circ x)), \\ \nu(x, y) = \mathscr{E}_0(p \circ B(p \circ y, q \circ x)) - \mu(x) \circ \mathscr{E}_0(p \circ y) \\ \qquad = p \circ \mathscr{E}_0(B(p \circ y, q \circ x)) - \mu(x) \circ \mathscr{E}_0(p \circ y), \end{cases} \tag{7.2}$$

for all $x, y \in \mathfrak{J}$. Fix $x \in \mathfrak{J}$. We observe that, by applying $\mathscr{E}_1$ to $T_{p,x}(p \circ u)$ in (7.1), we obtain

$$\begin{aligned}\mu(x) = \mathscr{E}_1\Big(p \circ B(p \circ u, q \circ x)\Big) &= \mathscr{E}_1(T_{p,x}(p \circ u)) = \mathscr{E}_1(\lambda_{p,x} \circ (p \circ u)) \\ &= p \circ \mathscr{E}_1(\lambda_{p,x} \circ u) = \lambda_{p,x} \in Z(p \circ \mathfrak{J}).\end{aligned} \tag{7.3}$$

Thus, the arbitrariness of $y \in \mathfrak{J}$ implies that μ is a linear mapping whose image is in $Z(p \circ \mathfrak{J})$. It is easy to check from the definition that $\mu(x) = \mu(q \circ x)$, for all $x \in \mathfrak{J}$.

On the other hand, having in mind (7.3) and (7.1) we derive:

$$\begin{aligned}\nu(x, y) &= \mathscr{E}_0\Big(p \circ B(p \circ y, q \circ x)) - \mu(x) \circ (p \circ y)\Big) \\ &= \mathscr{E}_0\Big(T_{p,x}(p \circ y) - \lambda_{p,x} \circ (p \circ y)\Big) = \mu_{p,x}(p \circ y) \in Z(p \circ \mathfrak{J}),\end{aligned}$$

which shows that ν is centre-valued. By construction we have

$$\begin{aligned}p \circ B(p \circ y, q \circ x) &= T_{p,x}(p \circ y) = \lambda_{p,x} \circ (p \circ y) + \mu_{p,x}(p \circ y) \\ &= \mu(x) \circ (p \circ y) + \nu(x, y) = \mu(x) \circ y + \nu(x, y),\end{aligned}$$

for all $x, y \in \mathfrak{J}$.

The uniqueness of μ and ν has been implicitly proved by the uniqueness of the expressions in (7.2) in terms of $\mathscr{E}_i$ and the bilinear mapping $p \circ B(p \circ y, q \circ x)$ $(x, y \in \mathfrak{J})$. Actually, if $\mu_1 : \mathfrak{J} \to p \circ Z(\mathfrak{J}) = Z(p \circ \mathfrak{J})$ is a linear map and $\nu_1 : \mathfrak{J} \times \mathfrak{J} \to p \circ Z(\mathfrak{J}) = Z(p \circ \mathfrak{J})$ is a bilinear mapping satisfying the same properties of μ and ν, it follows from the identity

$$\begin{aligned}\mu(x) = \mathscr{E}_1\Big(p \circ B(p \circ u, q \circ x)\Big) &= \mathscr{E}_1\Big(\mu_1(x) \circ (p \circ u) + \nu_1(x, u)\Big) \\ &= \mathscr{E}_1\Big(\mu_1(x) \circ u + \nu_1(x, u)\Big) = \mu_1(x) \quad (x \in \mathfrak{J}),\end{aligned}$$

that $\mu = \mu_1$. Finally, since

$$\mu(x) \circ y + \nu(x, y) = p \circ B(p \circ y, q \circ x) = \mu_1(x) \circ y + \nu_1(x, y),$$

for all $x, y \in \mathfrak{J}$, we deduce that $\nu = \nu_1$. □

For the sake of simplicity, before stating our next technical tool we introduce some notation. We shall say that a JB*-algebra $\mathfrak{J}$ satisfies the *standard factorization property for associating traces* (SFP in short) if for every symmetric bilinear mapping $B : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$ with associating trace there exist a unique $\lambda \in Z(\mathfrak{J})$, a unique linear mapping $\mu : \mathfrak{J} \to Z(\mathfrak{J})$, and a unique symmetric bilinear mapping $\nu : \mathfrak{J} \times \mathfrak{J} \to Z(\mathfrak{J})$ such that

$$B(x, x) = \lambda \circ x^2 + \mu(x) \circ x + \nu(x, x),$$

for all $x \in \mathfrak{J}$.

Proposition 7.3. *Let p be a central projection in a JBW*-algebra $\mathfrak{J}$ without direct summands of type I_1. Suppose that $p \circ \mathfrak{J}$ and $(\mathbf{1} - p) \circ \mathfrak{J}$ satisfy the SFP. Then $\mathfrak{J}$ satisfies the SFP.*

Proof. Let $B : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$ be a symmetric bilinear map whose trace is associating, that is, $[B(x, x), \mathfrak{J}, x] = 0$, for every $x \in \mathfrak{J}$. Set $q = \mathbf{1} - p$, and let us decompose $B(x, x)$ in the form

$$\begin{cases} B(x, x) = p \circ B(p \circ x, p \circ x) + 2p \circ B(p \circ x, q \circ x) + p \circ B(q \circ x, q \circ x) \\ \qquad\qquad + q \circ B(p \circ x, p \circ x) + 2q \circ B(p \circ x, q \circ x) + q \circ B(q \circ x, q \circ x), \end{cases} \tag{7.4}$$

with $x \in \mathfrak{J}$, and write $B_p : p \circ \mathfrak{J} \times p \circ \mathfrak{J} \to p \circ \mathfrak{J}$, $B_q : q \circ \mathfrak{J} \times q \circ \mathfrak{J} \to q \circ \mathfrak{J}$ for the symmetric bilinear mappings given by

$$B_p(p \circ x, p \circ x) = p \circ B(p \circ x, p \circ x), \text{ and } B_q(q \circ x, q \circ x) = q \circ B(q \circ x, q \circ x),$$

respectively.

It is not hard to see, from the fact that p and q are central projections and the assumptions on B, that the traces of the bilinear maps B_p and B_q are associating. Since $p \circ \mathfrak{J}$ and $(\mathbf{1} - p) \circ \mathfrak{J}$ satisfy the SFP, there exist unique $\lambda_p \in Z(p \circ \mathfrak{J})$, $\lambda_q \in Z(q \circ \mathfrak{J})$, linear mappings $\mu_p : p \circ \mathfrak{J} \to Z(p \circ \mathfrak{J})$, $\mu_q : q \circ \mathfrak{J} \to Z(q \circ \mathfrak{J})$, and bilinear mappings $\nu_p : p \circ \mathfrak{J} \times p \circ \mathfrak{J} \to Z(p \circ \mathfrak{J})$ and $\nu_q : q \circ \mathfrak{J} \times q \circ \mathfrak{J} \to Z(q \circ \mathfrak{J})$ such that

$$B_p(p \circ x, p \circ x) = \lambda_p \circ x^2 + \mu_p(p \circ x) \circ x + \nu_p(p \circ x, p \circ x), \tag{7.5}$$

and

$$B_q(q \circ x, q \circ x) = \lambda_q \circ x^2 + \mu_q(q \circ x) \circ x + \nu_q(q \circ x, q \circ x), \tag{7.6}$$

for all $x \in \mathfrak{J}$.

We shall also employ the following maps:

$$\begin{gathered} \nu_{p,q} : q \circ \mathfrak{J} \times q \circ \mathfrak{J} \longrightarrow p \circ \mathfrak{J}, \qquad \nu_{q,p} : p \circ \mathfrak{J} \times p \circ \mathfrak{J} \longrightarrow q \circ \mathfrak{J} \\ \nu_{p,q}(q \circ x, q \circ x) = p \circ B(q \circ x, q \circ x), \ \nu_{q,p}(p \circ x, p \circ x) = q \circ B(p \circ x, p \circ x). \end{gathered}$$

Lemma 7.1 assures that the above two mappings are centre-valued, that is,

$$\nu_{p,q} (q \circ \mathfrak{J} \times q \circ \mathfrak{J}) \subseteq Z(p \circ \mathfrak{J}) = p \circ Z(\mathfrak{J}), \text{ and } \nu_{q,p} : (p \circ \mathfrak{J} \times p \circ \mathfrak{J}) \subseteq Z(q \circ \mathfrak{J}). \tag{7.7}$$

Consider now the mappings

$$\delta_{p,p,q} : p \circ \mathfrak{J} \times q \circ \mathfrak{J} \longrightarrow p \circ \mathfrak{J}, \qquad \delta_{q,p,q} : p \circ \mathfrak{J} \times q \circ \mathfrak{J} \longrightarrow q \circ \mathfrak{J}$$
$$\delta_{p,p,q}(p \circ x, q \circ x) = 2p \circ B(p \circ x, q \circ x), \;\; \delta_{q,p,q}(p \circ x, q \circ x) = 2q \circ B(p \circ x, q \circ x).$$

Having in mind that $\mathfrak{J}$ admits no type I_1 central summands, we are in a position to apply Proposition 7.2 to deduce the existence of unique linear mappings $\mu_{p,p,q} : \mathfrak{J} \to p \circ Z(\mathfrak{J}) = Z(p \circ \mathfrak{J})$, $\mu_{q,p,q} : \mathfrak{J} \to q \circ Z(\mathfrak{J}) = Z(q \circ \mathfrak{J})$, and unique bilinear mappings $\nu_{p,p,q} : \mathfrak{J} \times \mathfrak{J} \to p \circ Z(\mathfrak{J}) = Z(p \circ \mathfrak{J})$, and $\nu_{q,p,q} : \mathfrak{J} \times \mathfrak{J} \to q \circ Z(\mathfrak{J}) = Z(q \circ \mathfrak{J})$ satisfying $\mu_{p,p,q}(y) = \mu_{p,p,q}(y \circ q)$, $\mu_{q,p,q}(y) = \mu_{q,p,q}(y \circ p)$, $\nu_{p,p,q}(x, y) = \nu_{p,p,q}(q \circ x, p \circ y)$, $\mu_{q,p,q}(y) = \mu_{q,p,q}(y \circ p)$, $\nu_{q,p,q}(x, y) = \nu_{q,p,q}(p \circ x, q \circ y)$,

$$\delta_{p,p,q}(p \circ x, q \circ x) = p \circ B(p \circ x, q \circ x) = \mu_{p,p,q}(q \circ x) \circ x + \nu_{p,p,q}(q \circ x, p \circ x), \tag{7.8}$$

and

$$\delta_{q,p,q}(p \circ x, q \circ x) = q \circ B(p \circ x, q \circ x) = \mu_{q,p,q}(p \circ x) \circ x + \nu_{q,p,q}(p \circ x, q \circ x), \tag{7.9}$$

for all $x \in \mathfrak{J}$. Mixing (7.4), (7.5), (7.6), (7.8), and (7.9) we derive that

$$\begin{aligned} B(x, x) = {} & \lambda_p \circ x^2 + \lambda_q \circ x^2 + \mu_p(p \circ x) \circ x + \mu_q(q \circ x) \circ x + \mu_{p,p,q}(q \circ x) \circ x \\ & + \mu_{q,p,q}(p \circ x) \circ x + \nu_p(p \circ x, p \circ x) + \nu_q(q \circ x, q \circ x) + \nu_{p,q}(q \circ x, q \circ x) \\ & + \nu_{q,p}(p \circ x, p \circ x) + \nu_{p,p,q}(q \circ x, p \circ x) + \nu_{q,p,q}(p \circ x, q \circ x) \\ = {} & (\lambda_p + \lambda_q) \circ x^2 + (\mu_p(p \circ x) + \mu_q(q \circ x) + \mu_{p,p,q}(q \circ x) + \mu_{q,p,q}(p \circ x)) \circ x \\ & + \nu_p(p \circ x, p \circ x) + \nu_q(q \circ x, q \circ x) + \nu_{p,q}(q \circ x, q \circ x) \\ & + \nu_{q,p}(p \circ x, p \circ x) + \nu_{p,p,q}(q \circ x, p \circ x) + \nu_{q,p,q}(p \circ x, q \circ x), \end{aligned}$$

for all $x \in \mathfrak{J}$. Setting $\lambda = \lambda_p + \lambda_q \in Z(\mathfrak{J})$, $\mu : \mathfrak{J} \to Z(\mathfrak{J})$,

$$\mu(x) := \mu_p(p \circ x) + \mu_q(q \circ x) + \mu_{p,p,q}(q \circ x) + \mu_{q,p,q}(p \circ x),$$

and the 2-homogeneous polynomial $P_\nu : \mathfrak{J} \to Z(\mathfrak{J})$ defined by

$$\begin{aligned} P_\nu(x) := \nu(x, x) := {} & \nu_p(p \circ x, p \circ x) + \nu_q(q \circ x, q \circ x) + \nu_{p,q}(q \circ x, q \circ x) \\ & + \nu_{q,p}(p \circ x, p \circ x) + \nu_{p,p,q}(q \circ x, p \circ x) + \nu_{q,p,q}(p \circ x, q \circ x), \end{aligned}$$

we establish the existence of the desired factorization for $B(x, x)$ in the form

$$B(x, x) = \lambda \circ x^2 + \mu(x) \circ x + \nu(x, x) \quad (x \in \mathfrak{J}),$$

where ν is the unique symmetric bilinear map associated with the 2-homogeneous polynomial P_ν.

The uniqueness deserves to be commented at least briefly. It essentially follows from the uniqueness in the SFP and in Proposition 7.2. Namely, suppose we can also write

$$B(x, x) = \widetilde{\lambda} \circ x^2 + \widetilde{\mu}(x) \circ x + \widetilde{\nu}(x, x) \quad (x \in \mathfrak{J}),$$

where $\widetilde{\lambda}$, $\widetilde{\mu}$, and $\widetilde{\nu}$ satisfy the required properties. Clearly

$$
\begin{aligned}
(p \circ \widetilde{\lambda})(p \circ x)^2 + (p \circ \widetilde{\mu}(p \circ x)) \circ (p \circ x) + p \circ \widetilde{\nu}(p \circ x, p \circ x) &= p \circ B(p \circ x, p \circ x) \\
= (p \circ \lambda)(p \circ x)^2 + (p \circ \mu(p \circ x)) \circ (p \circ x) + p \circ \nu(p \circ x, p \circ x)&
\end{aligned},
$$

for all $x \in \mathfrak{J}$, and thus, the SFP of $p \circ \mathfrak{J}$ assures that

$$
\begin{cases} p \circ \widetilde{\lambda} = p \circ \lambda, \ p \circ \widetilde{\mu}(p \circ x) = p \circ \mu(p \circ x), \\ \text{and } p \circ \widetilde{\nu}(p \circ x, p \circ x) = p \circ \nu(p \circ x, p \circ x), \end{cases} \tag{7.10}
$$

for all $x \in \mathfrak{J}$. Similarly,

$$
\begin{cases} q \circ \widetilde{\lambda} = q \circ \lambda, \quad q \circ \widetilde{\mu}(q \circ x) = q \circ \mu(q \circ x), \\ \text{and } q \circ \widetilde{\nu}(q \circ x, q \circ x) = q \circ \nu(q \circ x, q \circ x), \end{cases} \tag{7.11}
$$

for all $x \in \mathfrak{J}$. Consequently, $\lambda = \lambda \circ (p + q) = \widetilde{\lambda} \circ (p + q) = \widetilde{\lambda}$.

On the other hand, having in mind that ν is symmetric we get

$$
\begin{aligned}
p \circ B(p \circ x, q \circ x) &= \frac{1}{2} p \circ \Big(B(x, x) - B(p \circ x, p \circ x) - B(q \circ x, q \circ x) \Big) \\
&= \frac{1}{2} \Big((p \circ \lambda) \circ (p \circ x)^2 + (p \circ \mu(x)) \circ (p \circ x) + p \circ \nu(x, x) \\
&- (p \circ \lambda) \circ (p \circ x)^2 - (p \circ \mu(p \circ x)) \circ (p \circ x) - p \circ \nu(p \circ x, p \circ x) \\
&- p \circ \nu(q \circ x, q \circ x) \Big) \\
&= \frac{1}{2} \Big((p \circ \mu(q \circ x)) \circ (p \circ x) + 2p \circ \nu(p \circ x, q \circ x) \Big),
\end{aligned}
$$

and similarly

$$
p \circ B(p \circ x, q \circ x) = \frac{1}{2} \Big((p \circ \widetilde{\mu}(q \circ x)) \circ (p \circ x) + 2p \circ \widetilde{\nu}(p \circ x, q \circ x) \Big),
$$

for all $x \in \mathfrak{J}$. The uniqueness of the decomposition in Proposition 7.2 implies that

$$
(p \circ \mu(q \circ x)) = (p \circ \widetilde{\mu}(q \circ x)), \text{ and } p \circ \widetilde{\nu}(p \circ x, q \circ x) = p \circ \nu(p \circ x, q \circ x), \tag{7.12}
$$

for all $x \in \mathfrak{J}$.

Since the roles of p and q are clearly symmetric, we also get

$$
(q \circ \mu(p \circ x)) = (q \circ \widetilde{\mu}(p \circ x)), \text{ and } q \circ \widetilde{\nu}(q \circ x, p \circ x) = q \circ \nu(q \circ x, p \circ x), \tag{7.13}
$$

for all $x \in \mathfrak{J}$. We can easily deduce from (7.10), (7.11), (7.12), and (7.13) that $\mu = \widetilde{\mu}$.

Finally, the identities

$$
p \circ \widetilde{\nu}(q \circ x, q \circ x) = p \circ B(q \circ x, q \circ x) = p \circ \nu(q \circ x, q \circ x),
$$

and

$$
q \circ \widetilde{\nu}(p \circ x, p \circ x) = q \circ B(p \circ x, p \circ x) = q \circ \nu(p \circ x, p \circ x), \ (x \in \mathfrak{J}),
$$

combined with the remaining information from (7.10), (7.11), (7.12), and (7.13) give $\nu = \widetilde{\nu}$. $\square$

The following extension of the previous result follows from a simple induction argument.

Proposition 7.4. *Let $p_1, \ldots, p_n$ ($n \in \mathbb{N}$) be central projections in a JBW*-algebra $\mathfrak{J}$ without direct summands of type I_1. Suppose that $p_1 + \ldots + p_n = \mathbf{1}$ and $p_j \circ \mathfrak{J}$ satisfies the SFP for all $j = 1, \ldots, n$. Then $\mathfrak{J}$ satisfies the SFP.*

Proof. In order to give a brief idea of the induction argument, suppose that the statement is true for n, and let us consider $n+1$ central projections $p_1, \ldots, p_n, p_{n+1}$ such that $p_1 + \ldots + p_n + p_{n+1} = \mathbf{1}$ and $p_j \circ \mathfrak{J}$ satisfies the SFP for all j. We deduce, from the induction hypothesis, that taking $p = p_1 + \ldots + p_n$ we get a central projection such that $p \circ \mathfrak{J}$ and $(\mathbf{1} - p) \circ \mathfrak{J} = p_{n+1} \circ \mathfrak{J}$ satisfy the SFP. Proposition 7.3 assures that $\mathfrak{J}$ satisfies the SFP, which concludes the induction argument. $\square$

We can now proceed with the description of all associating traces on a JBW*-algebra without commutative summands.

Theorem 7.5. *Let $\mathfrak{J}$ be a JBW*-algebra with no direct summands of type I_1, and let $B : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$ be a symmetric bilinear map satisfying $[B(x,x), \mathfrak{J}, x] = 0$, for all $x \in \mathfrak{J}$. Then B admits the following (unique) representation*

$$B(x,x) = \lambda \circ x^2 + \mu(x) \circ x + \nu(x,x) \circ \mathbf{1}$$

where $\lambda \in Z(\mathfrak{J})$, $\mu : \mathfrak{J} \to Z(\mathfrak{J})$ is linear and $\nu : \mathfrak{J} \times \mathfrak{J} \to Z(\mathfrak{J})$ is bilinear and symmetric. If $\mathfrak{J}$ is a JBW-algebra of type I_2 the element λ is always zero.*

Proof. We appeal, once again, to the general structure theory of JBW*-algebras to assure the existence of three (possibly zero) central projections p_1, p_2 and p_3 such that $p_1 + p_2 + p_3 = \mathbf{1}$, $p_1 \circ \mathfrak{J}$ is a JW*-algebra with no direct summands of type I_1 and I_2, $p_2 \circ \mathfrak{J}$ is a JW*-algebra of type I_2, and $p_3 \circ \mathfrak{J}$ is a purely exceptional JBW*-algebra (cf. [21, Theorems 5.1.5, 5.3.5 and 7.2.7]). The JBW*-algebras $p_1 \circ \mathfrak{J}$, $p_2 \circ \mathfrak{J}$, and $p_3 \circ \mathfrak{J}$ satisfy the SFP by Theorem 3.6, Theorem 5.2, and Corollary 4.3, respectively. Proposition 7.4 now asserts that $\mathfrak{J}$ also satisfies the SFP, which is equivalent to the desired statement. The final claim is a consequence of Theorem 5.2. $\square$

8. Applications to preservers of operator commutativity

Our previous studies on bilinear maps with associating trace will be now applied to study linear bijections between JBW*-algebras preserving operator commutativity among elements.

The result will be obtained after a series of technical lemmata. Our first stop will be a generalization of [13, Lemma 5], whose proof is new even in the associative setting.

Lemma 8.1. *Let $\mathfrak{J}$ be a JB*-algebra with no one-dimensional Jordan representations. If $c \in Z(\mathfrak{J})$ satisfies $c \circ \mathfrak{J} \subseteq Z(\mathfrak{J})$, then $c = 0$. The conclusion also holds if $\mathfrak{J}$ is a JBW*-algebra with no type I_1 direct summand.*

Proof. Let $\pi : \mathfrak{J} \to \mathfrak{J}_\pi$ be a Jordan factor representation. Since $[\pi(c), \pi(\mathfrak{J}), \pi(a)] = \pi[c, \mathfrak{J}, a] = 0$, for all $a \in \mathfrak{J}$, it follows from the weak*-density of $\pi(\mathfrak{J})$ in the factor JBW*-algebra $\mathfrak{J}_\pi$ and the separate weak*-continuity of the Jordan product of the latter JBW*-algebra that $\pi(c) \in Z(\mathfrak{J}_\pi) = \mathbb{C}\mathbf{1}$. Therefore there exists a complex λ_π such that $\pi(c) = \lambda_\pi \mathbf{1}$, and thus, by hypothesis, $\lambda_\pi \pi(\mathfrak{J}) = \pi(c) \circ \pi(\mathfrak{J}) \subseteq \pi(Z(\mathfrak{J})) \subseteq Z(\mathfrak{J}_\pi) = \mathbb{C}\mathbf{1}$, which combined with the weak*-density of $\pi(\mathfrak{J})$ shows that $\lambda_\pi \mathfrak{J}_\pi \subseteq \mathbb{C}\mathbf{1}$. By observing that $\mathfrak{J}$ admits no one-dimensional Jordan representations, we conclude that $\lambda_\pi = 0$. We have therefore shown that

$\pi(c) = 0$ for every Jordan factor representation $\pi : \mathfrak{J} \to \mathfrak{J}_\pi$. The existence a faithful family of Jordan factor representations for $\mathfrak{J}$ (cf. [2, Corollary 5.7]) assures that $c = 0$, as desired.

For the final statement, note that, by Proposition 2.4 and Lemma 2.2, a JBW*-algebra with no type I_1 direct summand admits no one-dimensional Jordan representations. □

It is known that for each JB*-algebra $\mathfrak{J}$, the centre of the JB-algebra $\mathfrak{J}_{sa}$ coincides with the self-adjoint part of the centre of $\mathfrak{J}$. Therefore the next result is a direct consequence of the previous Lemma 8.1.

Lemma 8.2. *Let $\mathfrak{J}$ be a JB*-algebra with no one-dimensional Jordan representations. If $c \in Z(\mathfrak{J}_{sa})$ satisfies $c \circ \mathfrak{J}_{sa} \subseteq Z(\mathfrak{J}_{sa})$, then $c = 0$. The conclusion also holds if $\mathfrak{J}$ is a JBW*-algebra with no type I_1 direct summands.*

The next result is inspired by [13, Proof of Theorem 3]. The proof, which is included here for completeness reasons, relies on Lemma 8.1 and Theorem 6.6.

Lemma 8.3. *Let $\mathfrak{M}$ and $\mathfrak{J}$ be JBW*-algebras with no central summands of type I_1. Suppose $\Phi : \mathfrak{M} \to \mathfrak{J}$ is a linear bijection preserving operator commutativity in both directions, that is, $[x, \mathfrak{M}, y] = 0$ if, and only if, $[\Phi(x), \mathfrak{J}, \Phi(y)] = 0$, for all $x, y \in \mathfrak{M}$. Then Φ maps $Z(\mathfrak{M})$ onto $Z(\mathfrak{J})$, and there exists an associative isomorphism $\alpha : Z(\mathfrak{M}) \to Z(\mathfrak{J})$ satisfying $\Phi(z \circ a) - \alpha(z) \circ \Phi(a) \in Z(\mathfrak{J})$, for all $z \in Z(\mathfrak{M})$, $a \in \mathfrak{M}$, and $\Phi^{-1}(\tilde{z} \circ b) - \alpha^{-1}(\tilde{z}) \circ \Phi^{-1}(b) \in Z(\mathfrak{M})$, for all $\tilde{z} \in Z(\mathfrak{J})$, $b \in \mathfrak{J}$.*

Proof. Let us fix an arbitrary element $z \in Z(\mathfrak{M})$ and consider the mapping $\Phi_z : \mathfrak{J} \to \mathfrak{J}$ given by $\Phi_z(b) := \Phi(z \circ \Phi^{-1}(b))$. Clearly $z \circ \Phi^{-1}(b)$ operator commutes with $\Phi^{-1}(b)$, and thus, $\Phi(z \circ \Phi^{-1}(b))$ operator commutes with b, that is, $[\Phi_z(b), \mathfrak{J}, b] = [\Phi(z \circ \Phi^{-1}(b)), \mathfrak{J}, b] = 0$, for each b, which shows that Φ_z is associating. Theorem 6.6 implies the existence of a unique element $\alpha(z) \in Z(\mathfrak{J})$ and a unique linear mapping $\mu_z : \mathfrak{J} \to Z(\mathfrak{J})$ (both depending on z) satisfying

$$\Phi(z \circ \Phi^{-1}(b)) = \Phi_z(b) = \alpha(z) \circ b + \mu_z(b), \text{ for all } b \in \mathfrak{J}. \tag{8.1}$$

By defining $\alpha : Z(\mathfrak{M}) \to Z(\mathfrak{J})$, $z \mapsto \alpha(z)$, we deduce from the bijectivity of Φ that

$$\Phi(z \circ a) = \alpha(z) \circ \Phi(a) + \mu_z \Phi(a), \text{ and } \Phi(z \circ a) - \alpha(z) \circ \Phi(a) \in Z(\mathfrak{J})$$

for all $a \in \mathfrak{M}$.

By replacing Φ with Φ^{-1} in the above arguments, we deduce the existence of a mapping $\widetilde{\alpha} : Z(\mathfrak{J}) \to Z(\mathfrak{M})$ with the property that for each $\widetilde{z} \in Z(\mathfrak{J})$, its image, $\widetilde{\alpha}(\widetilde{z})$, is the unique element satisfying

$$\Phi^{-1}(\widetilde{z} \circ \Phi(a)) = \widetilde{\alpha}(\widetilde{z}) \circ a + \mu_{\widetilde{z}}(a), \text{ for all } a \in \mathfrak{M}, \tag{8.2}$$

where $\mu_{\widetilde{z}}$ is a linear mapping from $\mathfrak{M}$ to $Z(\mathfrak{M})$.

By combining (8.1) and (8.2) we arrive to

$$\widetilde{z} \circ \Phi(a) = \Phi\Phi^{-1}(\widetilde{z} \circ \Phi(a)) = \alpha(\widetilde{\alpha}(\widetilde{z})) \circ \Phi(a) + \Phi\mu_{\widetilde{z}}(a),$$

for all $a \in \mathfrak{M}$, which assures that

$$(\widetilde{z} - \alpha(\widetilde{\alpha}(\widetilde{z}))) \circ \Phi(a) = \Phi\mu_{\widetilde{z}}(a) \in Z(\mathfrak{J}), \text{ for each } a \in \mathfrak{M}.$$

The surjectivity of Φ and Lemma 8.1 lead to the conclusion that $\widetilde{z} = \alpha(\widetilde{\alpha}(\widetilde{z}))$, for all $\widetilde{z} \in Z(\mathfrak{J})$. We similarly obtain $z = \widetilde{\alpha}(\alpha(z))$, for each $z \in Z(\mathfrak{M})$. This shows that α is a bijection and $\widetilde{\alpha} = \alpha^{-1}$.

In order to prove that α is an associative homomorphism, we apply the defining properties of α to get

$$\begin{aligned}(\alpha(tz_1+sz_2)-t\alpha(z_1)-s\alpha(z_2))\circ\Phi(a) &= \Phi((tz_1+sz_2)\circ a)-\mu_{tz_1+sz_2}\Phi(a)\\ &\quad - t\Phi(z_1\circ a)-t\mu_{z_1}\Phi(a)\\ &\quad - s\Phi(z_2\circ a)-s\mu_{z_2}\Phi(a),\end{aligned}$$

for all $a\in\mathfrak{M}$ $z_1,z_2\in Z(\mathfrak{M})$, $s,t\in\mathbb{C}$. This implies that

$$\Big(\alpha(tz_1+sz_2)-t\alpha(z_1)-s\alpha(z_2)\Big)\circ\mathfrak{J}\subseteq Z(\mathfrak{J}),$$

and Lemma 8.1 gives $\alpha(tz_1+sz_2)=t\alpha(z_1)+s\alpha(z_2)$.

Finally, by the previous properties,

$$\begin{aligned}\Big(\alpha(z_1)\circ\alpha(z_2)-\alpha(z_1\circ z_2)\Big)\circ\Phi(a) &= \alpha(z_1)\circ\Big(\Phi(z_2\circ a)-\mu_{z_2}\Phi(a)\Big)\\ &\quad -\Phi((z_1\circ z_2)\circ a)-\mu_{z_1\circ z_2}\Phi(a)\\ &= \Phi(z_1\circ(z_2\circ a))-\mu_{z_1}(\Phi(z_2\circ a))\\ &\quad -\Phi((z_1\circ z_2)\circ a)-\mu_{z_1\circ z_2}\Phi(a),\end{aligned}$$

for all $a\in\mathfrak{M}$ $z_1,z_2\in Z(\mathfrak{M})$, proving that $\Big(\alpha(z_1)\circ\alpha(z_2)-\alpha(z_1\circ z_2)\Big)\circ\mathfrak{J}\subseteq Z(\mathfrak{J})$, and Lemma 8.1 concludes that $\alpha(z_1\circ z_2)=\alpha(z_1)\circ\alpha(z_2)$. $\square$

When in the proof of Lemma 8.3, Theorem 6.6 and Lemma 8.1 are replaced with Theorem 6.7 and Lemma 8.2, respectively, the arguments remain valid to get next result.

Lemma 8.4. *Let $\mathfrak{M}$ and $\mathfrak{J}$ be JBW*-algebras with no central summands of type I_1. Suppose $\Phi:\mathfrak{M}_{sa}\to\mathfrak{J}_{sa}$ is a linear bijection satisfying $[x,\mathfrak{M}_{sa},y]=0$ if, and only if, $[\Phi(x),\mathfrak{J}_{sa},\Phi(y)]=0$, for all $x,y\in\mathfrak{M}_{sa}$. Then Φ maps $Z(\mathfrak{M}_{sa})$ onto $Z(\mathfrak{J}_{sa})$, and there exists an associative isomorphism $\alpha:Z(\mathfrak{M}_{sa})\to Z(\mathfrak{J}_{sa})$ satisfying*

$$\Phi(z\circ a)-\alpha(z)\circ\Phi(a)\in Z(\mathfrak{J}_{sa}),\ \text{ and } \Phi^{-1}(\tilde{z}\circ b)-\alpha^{-1}(\tilde{z})\circ\Phi^{-1}(b)\in Z(\mathfrak{M}_{sa}),$$

for all $z\in Z(\mathfrak{M}_{sa})$, $a\in\mathfrak{M}_{sa}$, $\tilde{z}\in Z(\mathfrak{J}_{sa})$, and $b\in\mathfrak{J}_{sa}$.

The next lemma is built upon arguments and ideas from [12].

Lemma 8.5. *Let $\mathfrak{M}$ be a JB*-algebra, and let $\mathfrak{J}$ be a JBW*-algebra with no central summands of type I_1 or I_2. Suppose $T:\mathfrak{M}\to\mathfrak{J}$ is a surjective linear mapping, and $\varepsilon:\mathfrak{M}\times\mathfrak{M}\to\mathfrak{J}$ is a bilinear mapping satisfying*

$$\varepsilon(a,b)\circ\Big[[T(a)^2,c,d],e,[T(a),c,d]\Big]=0, \tag{8.3}$$

for all $a,b\in\mathfrak{M}$, $c,d,e\in\mathfrak{J}$. Then the bilinear mapping ε is zero.

Proof. Let $\pi:\mathfrak{J}\to\mathfrak{J}_\pi$ be a Jordan factor representation. Since $\mathfrak{J}$ does not contain summands of type I_1 or I_2, Proposition 2.3 assures that $\mathfrak{J}_\pi$ is not a spin factor (cf. Remark 1.4). The mappings $\overline{\varepsilon}=\pi\varepsilon$ and $\overline{T}=\pi T$ satisfy the same identity in (8.3) for all $a,b\in\mathfrak{M}$, $e,c,d\in\mathfrak{J}$. Since $Z(\mathfrak{J}_\pi)=\mathbb{C}\mathbf{1}$, it follows that for all $a,b\in\mathfrak{M}$, $c,d,e\in\mathfrak{J}$, we have

$$\overline{\varepsilon}(a,b)=0,\ \text{or}\ \Big[[\overline{T}(a)^2,\pi(c),\pi(d)],\pi(e),[\overline{T}(a),\pi(c),\pi(d)]\Big]=0. \tag{8.4}$$

If $\Big[[\overline{T}(a)^2, \pi(c), \pi(d)], \pi(e), [\overline{T}(a), \pi(c), \pi(d)]\Big] = 0$, for all $a \in \mathfrak{M}$, e, c, $d \in \mathfrak{J}$, we deduce from the surjectivity of T, the strong*-density of the closed unit ball of $\pi(\mathfrak{J})$ in the closed unit ball of $\mathfrak{J}_\pi$, and the joint strong*-continuity of the Jordan product of $\mathfrak{J}_\pi$ on bounded sets (cf [37, Theorem] or [35, Theorem 9]), that

$$\Big[[\overline{a}^2, \overline{c}, \overline{d}], \overline{e}, [\overline{a}, \overline{c}, \overline{d}]\Big] = 0, \text{ for every } \overline{a}, \overline{c}, \overline{d}, \overline{e} \in \mathfrak{J}_\pi.$$

Since $\mathfrak{J}_\pi$ is a prime non-degenerate Jordan algebra, [12, Lemma 5.2] implies that $\mathfrak{J}_\pi$ is a spin factor, which is impossible. Therefore, there exist a_1 in $\mathfrak{M}$, c_1, d_1, and e_1 in $\mathfrak{J}$ such that

$$\Big[[\overline{T}(a_1)^2, \pi(c_1), \pi(d_1)], \pi(e_1), [\overline{T}(a_1), \pi(c_1), \pi(d_1)]\Big] \neq 0.$$

Suppose that $\overline{\varepsilon} \neq 0$. Then there exist $a_2, b_2 \in \mathfrak{J}$ with $\overline{\varepsilon}(a_2, b_2) \neq 0$. Consider the mappings $S : \mathfrak{M}^3 \to \mathfrak{J}_\pi$, $F : \mathfrak{M} \to \mathfrak{J}_\pi$ given by

$$S(x_1, x_2, x_3) := \Big[[\overline{T}(x_1) \circ \overline{T}(x_2), \pi(c_1), \pi(d_1)], \pi(e_1), [\overline{T}(x_3), \pi(c_1), \pi(d_1)]\Big],$$
$$F(x_1) := \overline{\varepsilon}(x_1, b_2).$$

It follows from the hypotheses (see (8.4)) that, for each $a \in \mathfrak{M}$, $F(a) = 0$ or $S(a, a, a) = 0$. Lemma 3.7 in [12] assures that $F(a) = 0$ for all $a \in \mathfrak{M}$, or $S(a, a, a) = 0$ for all $a \in \mathfrak{M}$, but both conclusions are impossible by what we have just seen ($S(a_1, a_1, a_1), F(a_2) \neq 0$).

We have therefore shown that $\overline{\varepsilon} = \pi\varepsilon = 0$ for every Jordan factor representation $\pi : \mathfrak{J} \to \mathfrak{J}_\pi$. Having in mind that $\mathfrak{J}$ admits a faithful family of Jordan factor representations (cf. [2, Corollary 5.7]), we deduce that $\varepsilon = 0$ as desired. □

We are now in a position to establish our main conclusion on linear bijections preserving operator commutativity.

Theorem 8.6. *Let $\mathfrak{M}$ and $\mathfrak{J}$ be JBW*-algebras with no central summands of type I_1 and I_2. Suppose that $\Phi : \mathfrak{M} \to \mathfrak{J}$ is a linear bijection preserving operator commutativity in both directions, that is,*

$$[x, \mathfrak{M}, y] = 0 \Leftrightarrow [\Phi(x), \mathfrak{J}, \Phi(y)] = 0,$$

for all $x, y \in \mathfrak{M}$. Then there exist an invertible element z_0 in $Z(\mathfrak{J})$, a Jordan isomorphism $J : \mathfrak{M} \to \mathfrak{J}$, and a linear mapping $\beta : \mathfrak{M} \to Z(\mathfrak{J})$ satisfying

$$\Phi(x) = z_0 \circ J(x) + \beta(x),$$

for all $x \in \mathfrak{M}$. Furthermore, this decomposition of Φ is unique.

Proof. The proof is divided into five main steps.

(1) *First step:* By hypotheses Φ and Φ^{-1} preserve operator commutativity in both directions, in particular, $\Phi^{-1}(y)^2$ and $\Phi^{-1}(y)$ operator commute for every $y \in \mathfrak{J}$. We define a bilinear mapping $B : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$ given by

$$B(x, y) = \Phi\Big(\Phi^{-1}(x) \circ \Phi^{-1}(y)\Big), \tag{8.5}$$

for every $x, y \in \mathfrak{J}$. The mapping B is well defined and symmetric by definition. Furthermore, by the assumptions on Φ we have

$$0 = [\Phi^{-1}(y), \mathfrak{M}, \Phi^{-1}(y)^2] \Leftrightarrow 0 = [y, \mathfrak{J}, \Phi(\Phi^{-1}(y)^2)] = [y, \mathfrak{J}, B(y, y)],$$

for every $y \in \mathfrak{J}$, that is, B is a symmetric bilinear mapping whose trace is associating. By applying Theorem 7.5 we derive the existence of an element λ in $Z(\mathfrak{J})$, a linear mapping $\mu_1 : \mathfrak{J} \to Z(\mathfrak{J})$, and a symmetric bilinear mapping $\nu_1 : \mathfrak{J} \times \mathfrak{J} \to Z(\mathfrak{J})$ satisfying

$$\Phi(\Phi^{-1}(y)^2) = B(y, y) = \lambda \circ y^2 + \mu_1(y) \circ y + \nu_1(y, y), \tag{8.6}$$

for every $y \in \mathfrak{J}$. Let $\alpha : Z(\mathfrak{M}) \to Z(\mathfrak{J})$ denote the associative isomorphism given by Lemma 8.3. Having in mind the properties of this mapping α and the identity (8.6) we arrive to

$$\begin{cases} \Phi^{-1}(y)^2 = \alpha^{-1}(\lambda) \circ \Phi^{-1}(y^2) + \big(\Phi^{-1}(\lambda \circ y^2) - \alpha^{-1}(\lambda) \circ \Phi^{-1}(y^2)\big) \\ \quad +\alpha^{-1}(\mu_1(y)) \circ \Phi^{-1}(y) + \big(\Phi^{-1}(\mu_1(y) \circ y) - \alpha^{-1}(\mu_1(y)) \circ \Phi^{-1}(y)\big) \\ \quad +\Phi^{-1}\nu_1(y, y) \end{cases}$$

for all $y \in \mathfrak{M}$, and equivalently,

$$\begin{cases} x^2 = \alpha^{-1}(\lambda) \circ \Phi^{-1}(\Phi(x)^2) + \big(\Phi^{-1}(\lambda \circ \Phi(x)^2) - \alpha^{-1}(\lambda) \circ \Phi^{-1}(\Phi(x)^2)\big) \\ \quad + \alpha^{-1}(\mu_1(\Phi(x))) \circ x + \big(\Phi^{-1}(\mu_1(\Phi(x)) \circ \Phi(x)) - \alpha^{-1}(\mu_1(\Phi(x))) \circ x\big) \\ \quad + \Phi^{-1}\nu_1(\Phi(x), \Phi(x)), \end{cases}$$

for every $x \in \mathfrak{M}$. Let us denote by γ the mapping from $\mathfrak{J}$ into $Z(\mathfrak{J})$ given by $\gamma(x) := \Phi^{-1}(\lambda \circ \Phi(x)^2) - \alpha^{-1}(\lambda) \circ \Phi^{-1}(\Phi(x)^2) + \Phi^{-1}(\mu_1(\Phi(x)) \circ \Phi(x)) - \alpha^{-1}(\mu_1(\Phi(x))) \circ x + \Phi^{-1}\nu_1(\Phi(x), \Phi(x))$. According to this notation we can write

$$x^2 = \alpha^{-1}(\lambda) \circ \Phi^{-1}(\Phi(x)^2) + \alpha^{-1}(\mu_1(\Phi(x))) \circ x + \gamma(x), \tag{8.7}$$

for all $x \in \mathfrak{M}$.

(2) *Second step*: For each Jordan factor representation $\pi : \mathfrak{M} \to \mathfrak{M}_\pi$, the element $\pi(\alpha^{-1}(\lambda))$ is invertible in $Z(\mathfrak{M}_\pi) = \mathbb{C}\mathbf{1}_{\mathfrak{M}_\pi}$. Assume on the contrary, that $\pi(\alpha^{-1}(\lambda)) = 0$ for certain Jordan factor representation $\pi : \mathfrak{M} \to \mathfrak{M}_\pi$. Having in mind the assumptions on $\mathfrak{M}$, we can find an element $w \in \mathfrak{M}$ and elementary operators $\mathscr{E}_0, \mathscr{E}_1, \mathscr{E}_2$ on $\mathfrak{M}$, and elementary operators $\widehat{\mathscr{E}_j} \in \mathscr{E}\ell(\mathfrak{M}_\pi)$ satisfying $\mathscr{E}_i(w^j) = \delta_{ij}\mathbf{1}$, and $\widehat{\mathscr{E}_j}(\pi(w)^i) = \delta_{i,j}\mathbf{1}_{\mathfrak{M}_\pi}$, for all $i, j \in \{0, 1, 2\}$ (cf. Proposition 2.3 and Lemma 2.2(ii)). If in (8.7) we replace x with w and we apply π and $\widehat{\mathscr{E}_2}$ consecutively, we arrive to

$$\begin{aligned} \mathbf{1} = \widehat{\mathscr{E}_2}\pi(w^2) &= \pi(\alpha^{-1}(\lambda)) \circ \widehat{\mathscr{E}_2}(\pi(\Phi^{-1}(\Phi(w)^2))) + \pi(\alpha^{-1}(\mu_1(\Phi(x)))) \circ \widehat{\mathscr{E}_2}(\pi(w)) \\ &\quad + \pi(\gamma(x)) \circ \widehat{\mathscr{E}_2}(\mathbf{1}) = 0, \end{aligned}$$

which is impossible.

(3) *Third step*: We claim that $\alpha^{-1}(\lambda)$ (equivalently, λ) is invertible. To prove the statement, let us begin by fixing a faithful family of Jordan factor representations $\{\pi_i : \mathfrak{M} \to \mathfrak{M}_{\pi_i} : i \in \Gamma\}$ whose existence is guaranteed by [2, Corollary 5.7]. The mapping $\pi_0 : \mathfrak{M} \to \bigoplus_{i\in\Gamma}^{\infty} \mathfrak{M}_{\pi_i}$, $\pi_0(a) = (\pi_i(a))_{i\in\Gamma}$ is an isometric unital Jordan *-monomorphism. By (2), $\pi_i(\alpha^{-1}(\lambda))$ is an invertible central element in $\mathfrak{M}_{\pi_i}$ for every $i \in \Gamma$. Thus, it suffices to prove that the set $\{\|\pi_i(\alpha^{-1}(\lambda))\| : i \in \Gamma\}$ is bounded (from below). Note that each π_i is contractive, and thus $\|\pi_i(\alpha^{-1}(\lambda))\| \leq \|\alpha^{-1}(\lambda)\|$ for all $i \in \Gamma$. Arguing by contradiction, we assume the existence of a sequence $(\pi_{i_n}(\alpha^{-1}(\lambda)))_{i_n}$ converging to zero in norm.

For each natural n we can find elementary operators $\widehat{\mathscr{E}}_j^n \in \mathscr{E}\ell(\mathfrak{M}_{\pi_{i_n}})$ satisfying $\widehat{\mathscr{E}}_j^n \pi_{i_n} = \pi_{i_n}\mathscr{E}_j$, and $\|\widehat{\mathscr{E}}_j^n\| \leq 10$, for all $j \in \{0,1,2\}$, $n \in \mathbb{N}$ (see Lemma 2.5), where $w \in \mathfrak{M}_{sa}$ and the elementary operators $\mathscr{E}_j$ are those considered in the previous step.

Now, we deduce from (8.7) that

$$\begin{cases} \pi_{i_n}(x^2) = \pi_{i_n}(\alpha^{-1}(\lambda)) \circ \pi_{i_n}(\Phi^{-1}(\Phi(x)^2)) + \pi_{i_n}(\alpha^{-1}(\mu_1(\Phi(x)))) \circ \pi_{i_n}(x) \\ \qquad\qquad + \pi_{i_n}(\gamma(x)), \end{cases} \tag{8.8}$$

for all $x \in \mathfrak{M}$, $n \in \mathbb{N}$.

Let $\mathscr{U}$ be a free ultrafilter over $\mathbb{N}$, and consider the ultraproduct, $(\mathfrak{M}_{\pi_{i_n}})_{\mathscr{U}}$, of the family $\{\mathfrak{M}_{\pi_{i_n}}\}_{n\in\mathbb{N}}$. Elements in $(\mathfrak{M}_{\pi_n})_{\mathscr{U}}$ are written in the form $[x_n]_{\mathscr{U}}$, and the sequence $(x_n)_n$ is called a *representing family* or a *representative* of $[x_n]_{\mathscr{U}}$. It is known that $\|[x_n]_{\mathscr{U}}\| = \lim_{\mathscr{U}} \|x_n\|$ independently of the chosen representative [24]. The Banach space $(\mathfrak{M}_{\pi_{i_n}})_{\mathscr{U}}$ is a JB*-algebra with product and involution defined by

$$[x_n]_{\mathscr{U}} \circ [y_n]_{\mathscr{U}} = [x_n \circ y_n]_{\mathscr{U}}, \text{ and } [x_n]_{\mathscr{U}}^* = [x_n^*]_{\mathscr{U}},$$

respectively (the argument in [24, Proposition 3. 1] works here, or see [16, Corollary 10] for a more general statement). The mappings

$$\pi_{\mathscr{U}} : \mathfrak{M} \to (\mathfrak{M}_{\pi_{i_n}})_{\mathscr{U}}, \ \pi_{\mathscr{U}}(x) := [\pi_{i_n}(x)]_{\mathscr{U}},$$

and

$$\mathscr{E}_j^{\mathscr{U}} : (\mathfrak{M}_{\pi_{i_n}})_{\mathscr{U}} \to (\mathfrak{M}_{\pi_{i_n}})_{\mathscr{U}}, \ \mathscr{E}_j^{\mathscr{U}}([x_n]_{\mathscr{U}}) := [\widehat{\mathscr{E}}_j^n(x_n)]_{\mathscr{U}},$$

are well-defined bounded linear operators (cf. [24, Definition 2.2]), and $\pi_{\mathscr{U}}$ is a Jordan *-homomorphism. By (8.8) and the assumption made on the sequence $(\pi_{i_n}(\alpha^{-1}(\lambda)))_n$, we derive that

$$\begin{aligned} \pi_{\mathscr{U}}(x)^2 = \pi_{\mathscr{U}}(x^2) &= [\pi_{i_n}(\alpha^{-1}(\lambda))]_{\mathscr{U}} \circ [\pi_{i_n}(\Phi^{-1}(\Phi(x)^2))]_{\mathscr{U}} \\ &\quad + [\pi_{i_n}(\alpha^{-1}(\mu_1(\Phi(x))))]_{\mathscr{U}} \circ [\pi_{i_n}(x)]_{\mathscr{U}} + [\pi_{i_n}(\gamma(x))]_{\mathscr{U}} \\ &= [\pi_{i_n}(\alpha^{-1}(\mu_1(\Phi(x))))]_{\mathscr{U}} \circ [\pi_{i_n}(x)]_{\mathscr{U}} + [\pi_{i_n}(\gamma(x))]_{\mathscr{U}}, \end{aligned}$$

for all $x \in \mathfrak{M}$, where $[\pi_{i_n}(\alpha^{-1}(\mu_1(\Phi(x))))]_{\mathscr{U}}$ and $[\pi_{i_n}(\gamma(x))]_{\mathscr{U}}$ lie in $Z((\mathfrak{M}_{\pi_{i_n}})_{\mathscr{U}})$. By replacing x with w and applying $\mathscr{E}_2^{\mathscr{U}}$ we get

$$\begin{aligned} \mathbf{1}_{(\mathfrak{M}_{\pi_{i_n}})_{\mathscr{U}}} &= [\pi_{i_n}(\mathbf{1})]_{\mathscr{U}} = [\pi_{i_n}\mathscr{E}_2(w^2)]_{\mathscr{U}} = [\widehat{\mathscr{E}}_j^n(\pi_{i_n}(w^2))]_{\mathscr{U}} = \mathscr{E}_2^{\mathscr{U}}(\pi_{\mathscr{U}}(w^2)) \\ &= [\pi_{i_n}(\alpha^{-1}(\mu_1(\Phi(w))))]_{\mathscr{U}} \circ \mathscr{E}_2^{\mathscr{U}}([\pi_{i_n}(w)]_{\mathscr{U}}) + [\pi_{i_n}(\gamma(w))]_{\mathscr{U}} \circ \mathscr{E}_2^{\mathscr{U}}([\mathbf{1}]_{\mathscr{U}}) \\ &= [\pi_{i_n}(\alpha^{-1}(\mu_1(\Phi(w))))]_{\mathscr{U}} \circ [\widehat{\mathscr{E}}_2^n(\pi_{i_n}(w))]_{\mathscr{U}} + [\pi_{i_n}(\gamma(w))]_{\mathscr{U}} \circ [\widehat{\mathscr{E}}_2^n\pi_{i_n}(\mathbf{1})]_{\mathscr{U}} \\ &= [\pi_{i_n}(\alpha^{-1}(\mu_1(\Phi(w))))]_{\mathscr{U}} \circ [\pi_{i_n}(\mathscr{E}_2(w))]_{\mathscr{U}} + [\pi_{i_n}(\gamma(w))]_{\mathscr{U}} \circ [\pi_{i_n}(\mathscr{E}_2(\mathbf{1}))]_{\mathscr{U}} \\ &= 0, \end{aligned}$$

which is impossible.

(4) *Fourth step*: There exist an invertible element z_0 in $Z(\mathfrak{J})$, a Jordan isomorphism $J : \mathfrak{M} \to \mathfrak{J}$, and a linear mapping $\beta : \mathfrak{M} \to Z(\mathfrak{J})$ satisfying

$$\Phi(x) = z_0 \circ J(x) + \beta(x),$$

for all $x \in \mathfrak{M}$.

For this final step we follow the construction in [12]. We begin by observing that, from (8.6) and the bijectivity of Φ, we can write

$$\Phi(x^2) = \lambda \circ \Phi(x)^2 + \mu_1(\Phi(x)) \circ \Phi(x) + \nu_1(\Phi(x), \Phi(x)),$$

for every $x \in \mathfrak{M}$. Set $J : \mathfrak{M} \to \mathfrak{J}$, $J(x) = \lambda \circ \Phi(x) + \frac{1}{2}\mu_1(\Phi(x))$. It is not hard to see from the formula displayed two lines above, and the properties of central elements, that

$$\begin{aligned} J(x^2) - J(x)^2 = \lambda \circ \Phi(x^2) + \frac{1}{2}\mu_1(\Phi(x^2)) - \Big(\lambda \circ \Phi(x) + \frac{1}{2}\mu_1(\Phi(x))\Big)^2 \\ = &\lambda \circ \big(\lambda \circ \Phi(x)^2 + \mu_1(\Phi(x)) \circ \Phi(x) + \nu_1(\Phi(x), \Phi(x))\big) + \frac{1}{2}\mu_1(\Phi(x^2)) \\ &- \lambda^2 \circ \Phi(x)^2 - \frac{1}{4}\mu_1(\Phi(x))^2 - (\lambda \circ \mu_1(\Phi(x))) \circ \Phi(x) \\ = &\lambda \circ \nu_1(\Phi(x), \Phi(x)) + \frac{1}{2}\mu_1(\Phi(x^2)) - \frac{1}{4}\mu_1(\Phi(x))^2 \in Z(\mathfrak{J}), \end{aligned}$$

for all $x \in \mathfrak{M}$, that is,

$$J(x)^2 - J(x^2) = \varepsilon(x, x) \in Z(\mathfrak{J}),$$

is a symmetric bilinear mapping taking values in the centre of $\mathfrak{J}$. It follows that

$$\begin{aligned} J(x \circ y) &= \frac{1}{2}\Big(J((x+y)^2) - J(x^2) - J(y^2)\Big) \\ &= \frac{1}{2}\Big(J(x+y)^2 - \varepsilon(x+y, x+y) - J(x)^2 - \varepsilon(x, x) - J(y)^2 - \varepsilon(y, y)\Big) \\ &= J(x) \circ J(y) - \varepsilon(x, y), \quad x, y \in \mathfrak{M}. \end{aligned}$$

Now, the identities

$$\begin{aligned} J((x^2 \circ y) \circ x) &= J(x^2 \circ y) \circ J(x) - \varepsilon(x^2 \circ y, x) \\ &= \big(J(x^2) \circ J(y)\big) \circ J(x) - \varepsilon(x^2, y) \circ J(x) - \varepsilon(x^2 \circ y, x) \\ &= \big(J(x)^2 \circ J(y)\big) \circ J(x) - (\varepsilon(x, x) \circ J(y)) \circ J(x) - \varepsilon(x^2, y) \circ J(x) \\ &\quad - \varepsilon(x^2 \circ y, x), \end{aligned}$$

and

$$\begin{aligned} J((x \circ y) \circ x^2) &= J(x \circ y) \circ J(x^2) - \varepsilon(x \circ y, x^2) \\ &= (J(x) \circ J(y)) \circ J(x^2) - \varepsilon(x, y) \circ J(x^2) - \varepsilon(x \circ y, x^2) \\ &= (J(x) \circ J(y)) \circ J(x)^2 - (J(x) \circ J(y)) \circ \varepsilon(x, x) - \varepsilon(x, y) \circ J(x^2) \\ &\quad - \varepsilon(x \circ y, x^2), \end{aligned}$$

combined with the Jordan identity $((x \circ y) \circ x^2 = (x^2 \circ y) \circ x)$, assure that

$$
\begin{aligned}
&(\varepsilon(x,y)\circ\lambda)\circ\Big(\lambda\circ\Phi(x)^2+\mu_1(\Phi(x))\circ\Phi(x)+\nu_1(\Phi(x),\Phi(x))\Big)\\
&+\frac{1}{2}\varepsilon(x,y)\circ\mu_1(\Phi(x^2))=\varepsilon(x,y)\circ\Big(\lambda\circ\Phi(x^2)+\frac{1}{2}\mu_1(\Phi(x^2))\Big)\\
&=\varepsilon(x,y)\circ J(x^2)=\varepsilon(x^2,y)\circ J(x)+\varepsilon(x^2\circ y,x)-\varepsilon(x\circ y,x^2)\\
&=\varepsilon(x^2,y)\circ\Big(\lambda\circ\Phi(x)+\frac{1}{2}\mu_1(\Phi(x))\Big)+\varepsilon(x^2\circ y,x)-\varepsilon(x\circ y,x^2),
\end{aligned}
$$

and thus, the invertibility of λ implies that

$$\varepsilon(x,y)\circ\Phi(x)^2\in Z(\mathfrak{J})\circ J(x)+Z(\mathfrak{J}),\ \text{for all } x,y\in\mathfrak{M}.$$

We consequently have

$$0=\Big[[\varepsilon(x,y)\circ\Phi(x)^2,b,c],\mathfrak{J},[\Phi(x),b,c]\Big]=\varepsilon(x,y)\circ\Big[[\Phi(x)^2,b,c],\mathfrak{J},[\Phi(x),b,c]\Big],$$

for all $x,y\in\mathfrak{M}$, $b,c\in\mathfrak{J}$. We are in a position to apply Lemma 8.5, which assures that $\varepsilon(x,y)=0$, for all $x,y\in\mathfrak{M}$, and thus J is a Jordan homomorphism.

Observe finally that, by the invertibility of the central element λ we can write

$$\Phi(x)=\lambda^{-1}\circ J(x)-\frac{1}{2}\lambda^{-1}\circ\mu_1(\Phi(x)),\quad x\in\mathfrak{M}.$$

Taking $z_0=\lambda^{-1}$ and $\beta(x)=-\frac{1}{2}\lambda^{-1}\circ\mu_1(\Phi(x))$ we get the desired factorization for Φ.

It only remains to prove that J is bijective. If $a\in\ker(J)$, we have $\Phi(a)=\beta(a)\in Z(\mathfrak{J})$, and thus $a\in Z(\mathfrak{M})$. We have shown that $\ker(J)\subseteq Z(\mathfrak{M})$. Having in mind that $\ker(J)$ is a Jordan ideal of $\mathfrak{M}$, we have $\ker(J)\circ\mathfrak{M}\subseteq\ker(J)\subseteq Z(\mathfrak{M})$. Lemma 8.1 proves that $\ker(J)=\{0\}$.

Since $J(x)=\lambda\circ\Phi(x)+\frac{1}{2}\mu_1(\Phi(x))$, and Φ maps the centre onto the centre, we easily deduce that $J(Z(\mathfrak{M}))\subseteq Z(\mathfrak{J})$. Let $\alpha:Z(\mathfrak{M})\to Z(\mathfrak{J})$ denote the associative homomorphism given by Lemma 8.3 that we employed in the first step. It is not hard to see from the identities linking α, Φ and J, that $\Phi(z)=\alpha(z)$ for all $z\in Z(\mathfrak{M})$. Namely, since $J(z)$, $\Phi(z)\in Z(\mathfrak{J})$ we have

$$z_0\circ\Big((\alpha(z)-J(z))\circ J(x)\Big)=\Phi(z\circ x)-\alpha(z)\circ\Phi(x)-\beta(z\circ x)+\alpha(z)\circ\beta(x),\quad(\forall x\in\mathfrak{M}),$$

and thus

$$(\alpha(z)-J(z))\circ\Phi(x)=(\alpha(z)-J(z))\circ(z_0\circ J(x)+\beta(x))\in Z(\mathfrak{J}),$$

for all $x\in\mathfrak{M}$, $z\in Z(\mathfrak{M})$. It follows from the surjectivity of Φ and Lemma 8.1 that $\alpha(z)=J(z)$ for all $z\in Z(\mathfrak{M})$. If we write

$$
\begin{aligned}
\Phi(x)&=z_0\circ J(x)+\beta(x)=J(\alpha^{-1}(z_0))\circ J(x)+J(\alpha^{-1}(\beta(x)))\\
&=J(\alpha^{-1}(z_0)\circ x+\alpha^{-1}(\beta(x))),\quad x\in\mathfrak{M},
\end{aligned}
$$

we see that the surjectivity of Φ induces the same property on J.

(5) *Fifth, and last, step*: The Jordan isomorphism J and the mapping β are unique. Suppose we can write $\Phi=z_1\circ J_1+\beta_1$, where z_1 is an invertible element in $Z(\mathfrak{J})$, $J:\mathfrak{M}\to\mathfrak{J}$ is a Jordan isomorphism, and $\beta:\mathfrak{M}\to Z(\mathfrak{J})$ is a linear mapping. We shall show that $z_0=z_1$, $J=J_1$, and $\beta=\beta_1$.

Arguing as in the fourth step we see that $\alpha(z)=J_1(z)$, for all $z\in Z(\mathfrak{M})$. Having this property in mind we get

$$\begin{aligned}\Phi(x) &= z_1 \circ J_1(x) + \beta_1(x) = J_1(\alpha^{-1}(z_1)) \circ J_1(x) + J_1(\alpha^{-1}(\beta_1(x)))\\ &= J_1\Big(\alpha^{-1}(z_1) \circ x + \alpha^{-1}(\beta_1(x))\Big) = J_1\Big(\alpha^{-1}(z_1) \circ x + J_1^{-1}(\beta_1(x))\Big),\end{aligned}$$

for all $x \in \mathfrak{M}$. It is not hard to check from this that

$$\begin{aligned}\Phi^{-1}(y) &= \alpha^{-1}(z_1^{-1}) \circ \Big(J_1^{-1}(y) - J_1^{-1}(\beta_1(\Phi^{-1}(y)))\Big)\\ &= J_1^{-1}(z_1^{-1}) \circ \Big(J_1^{-1}(y) - J_1^{-1}(\beta_1(\Phi^{-1}(y)))\Big)\\ &= J_1^{-1}\Big(z_1^{-1} \circ \Big(y - \beta_1(\Phi^{-1}(y))\Big)\Big),\end{aligned}$$

for all $y \in \mathfrak{J}$.

We turn now our attention to the bilinear mapping $B : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J}$ defined in the first step (see (8.5)). Having in mind the expression for the mapping Φ^{-1} in the previous paragraph we arrive to

$$\begin{aligned}B(y, y) &= \Phi\Big(\Phi^{-1}(y)^2\Big) = \Phi\Big(J_1^{-1}\Big(z_1^{-2} \circ \Big(y - \beta_1(\Phi^{-1}(y))\Big)^2\Big)\Big)\\ &= z_1 \circ \Big(z_1^{-2} \circ \Big(y - \beta_1(\Phi^{-1}(y))\Big)^2\Big) + \beta_1\Big(J_1^{-1}\Big(z_1^{-2} \circ \Big(y - \beta_1(\Phi^{-1}(y))\Big)^2\Big)\Big)\\ &= z_1^{-1} \circ y^2 - 2\left(z_1^{-1} \circ \beta_1(\Phi^{-1}(y))\right) \circ y + z_1^{-1} \circ \beta_1(\Phi^{-1}(y))^2\\ &+ \beta_1\Big(J_1^{-1}\Big(z_1^{-2} \circ \Big(y - \beta_1(\Phi^{-1}(y))\Big)^2\Big)\Big),\end{aligned}$$

for all $y \in \mathfrak{J}$, where $z_1^{-1} \circ \beta_1(\Phi^{-1}(y))^2 + \beta_1\Big(J_1^{-1}\Big(z_1^{-2} \circ \Big(y - \beta_1(\Phi^{-1}(y))\Big)^2\Big)\Big)$ lies in $Z(\mathfrak{J})$. The uniqueness of the decomposition of $B(\cdot, \cdot)$ in (8.6) guaranteed by Theorem 7.5 assures that $z_1^{-1} = \lambda = z_0^{-1}$, and $\mu_1(y) = -2\left(z_1^{-1} \circ \beta_1(\Phi^{-1}(y))\right)$, for all $y \in \mathfrak{J}$. It then follows that

$$\begin{aligned}J(x) &= \lambda \circ \Phi(x) + \frac{1}{2}\mu_1(\Phi(x)) = \lambda \circ (z_1 \circ J_1(x) + \beta_1(x)) - z_1^{-1} \circ \beta_1(x)\\ &= J_1(x) + z_1^{-1}\beta_1(x) - z_1^{-1} \circ \beta_1(x) = J_1(x),\end{aligned}$$

for all $x \in \mathfrak{M}$. This shows that $z_1 = z_0$ and $J_1 = J$, which gives $\beta = \beta_1$ and finishes the proof. $\square$

An alternative argument to derive the claim in the *Third step* of the proof of the previous Theorem 8.6 may be obtained from the next lemma and the *Second step* in the proof. The result is perhaps worthy to be inserted here due to the lacking of explicit references for it.

Lemma 8.7. *Let $\mathfrak{J}$ be a unital JB*-algebra, and let $z \in Z(\mathfrak{J})$. Then the following assertions are equivalent:*

(i) z is invertible.
(ii) For each Jordan factor representation $\pi : \mathfrak{J} \to \mathfrak{J}_\pi$, we have $\pi(z) \neq 0$.

Proof. $(i) \Rightarrow (ii)$ This is more or less clear, and holds under weaker assumptions. Suppose there exists $b \in \mathfrak{J}$ with $z \circ b = \mathbf{1}$. Since π is a (unital) Jordan homomorphism we have $\pi(z) \circ \pi(b) = \pi(z \circ b) = \pi(\mathbf{1}) = \mathbf{1}$, and hence $\pi(z) \neq 0$.

$(ii) \Rightarrow (i)$ Suppose that z is not invertible in $\mathfrak{J}$, and let $\mathfrak{I}$ denote the norm closure of $z \circ \mathfrak{J}$ in $\mathfrak{J}$. It is clear that $\mathfrak{I}$ is a closed Jordan ideal of $\mathfrak{J}$ and $z \in \mathfrak{I}$. We claim that $\mathfrak{I} \neq \mathfrak{J}$. Otherwise, there exists $a \in \mathfrak{J}$ such that $\|z \circ a - \mathbf{1}\| < 1$, which entails

$$\|M_{z\circ a} - Id_{\mathfrak{J}}\| < 1,$$

and hence the operator $M_{z\circ a}$ is invertible on $\mathfrak{J}$. On the other hand, since $z \in Z(\mathfrak{J})$, we have

$$M_z M_a = M_a M_z = M_{z\circ a},$$

and therefore M_z and M_a are both invertible. It is known that this implies that a and z are invertible in $\mathfrak{J}$ (cf. [14, Proposition 4.1.9]), which leads to a contradiction. Let $Q\colon \mathfrak{J} \to \mathfrak{J}/\mathfrak{I}$ be the quotient map onto the unital JB*-algebra $\mathfrak{J}/\mathfrak{I}$, and let π be a Jordan factor representation of $\mathfrak{J}/\mathfrak{I}$. Then πQ is a factor representation of $\mathfrak{J}$ and $(\pi Q)(z) = \pi(0) = 0$, which is impossible. □

Remark 8.8. We analyse now the optimality of the hypotheses in Theorem 8.6 affirming that the involved JBW*-algebras admits no central summands of type I_1 and I_2. Trivially every linear mapping on a commutative C*-algebra preserves operator commutativity. Let V be a spin factor. We have seen in Remark 1.5 that elements a, b in V operator commute if and only if b is a linear combination of a and $\mathbf{1}$. Therefore, any linear mapping on a spin factor preserves operator commutativity. Let $\Phi : V \to V$ be any linear bijection such that $\Phi(\mathbf{1}) \notin \mathbb{C}\mathbf{1}$. We have already justified that Φ preserves operator commutativity in both directions. Recall that $Z(V) = \mathbb{C}\mathbf{1}$. If the mapping Φ were written in the form

$$\Phi(x) = z_0 \circ J(x) + \beta(x),$$

for all $x \in V$, where z_0 is an invertible element in $Z(V) = \mathbb{C}\mathbf{1}$, a Jordan isomorphism $J : V \to V$, and a linear mapping $\beta : V \to Z(V)$, it would follow that $\Phi(\mathbf{1}) = z_0\mathbf{1} + \beta(\mathbf{1}) \in \mathbb{C}\mathbf{1}$, which is impossible.

Let $\mathfrak{M}$ and $\mathfrak{J}$ be two JB*-algebras. Following the standard notation, for each mapping $F : \mathfrak{M} \to \mathfrak{J}$, we write $F^\sharp : \mathfrak{M} \to \mathfrak{J}$ defined by $F^\sharp(x) = F(x^*)^*$ ($x \in \mathfrak{M}$). Clearly, F is linear if, and only if, $F^\sharp$ is, and $F^{\sharp\sharp} = F$. The mapping F is called *symmetric* or *$\sharp$-symmetric* if $F^\sharp = F$. Observe that symmetric Jordan homomorphisms between JB*-algebras are usually called Jordan *-homomorphisms in the literature. Every real linear mapping between the self-adjoint parts of two JBW*-algebras admits a natural extension to a symmetric complex linear mapping between both JBW*-algebras.

Our next result will consist in showing that when the mapping Φ in Theorem 8.6 is symmetric (i.e., $\Phi^\sharp = \Phi$) we can actually assume that J is a Jordan *-isomorphism. The results seem to be new also in the case of JBW*-algebra factors (cf. [12, §5.1]), and the arguments are completely different to those in [13], where the result is proved in the case of von Neumann algebras.

Corollary 8.9. *Let $\mathfrak{M}$ and $\mathfrak{J}$ be JBW*-algebras with no central summands of type I_1 and I_2. Suppose that $\Phi : \mathfrak{M} \to \mathfrak{J}$ is a symmetric linear bijection preserving operator commutativity in both directions, that is, $\Phi^\sharp = \Phi$ and*

$$[x, \mathfrak{M}, y] = 0 \Leftrightarrow [\Phi(x), \mathfrak{J}, \Phi(y)] = 0,$$

*for all $x, y \in \mathfrak{M}$. Then there exist a unique invertible element z_0 in $Z(\mathfrak{J})_{sa}$, a unique Jordan *-isomorphism $J : \mathfrak{M} \to \mathfrak{J}$, and a unique symmetric linear mapping $\beta : \mathfrak{M} \to Z(\mathfrak{J})$ satisfying*

$$\Phi(x) = z_0 \circ J(x) + \beta(x),$$

for all $x \in \mathfrak{M}$.

Proof. By Theorem 8.6 there exist a unique invertible element z_0 in $Z(\mathfrak{J})$, a unique Jordan isomorphism $J : \mathfrak{M} \to \mathfrak{J}$, and a unique linear mapping $\beta : \mathfrak{M} \to Z(\mathfrak{J})$ satisfying

$$\Phi(x) = z_0 \circ J(x) + \beta(x),$$

for all $x \in \mathfrak{M}$. Since $\Phi^\sharp = \Phi$ we get

$$z_0 \circ J(x) + \beta(x) = \Phi(x) = \Phi^\sharp(x) = z_0^* \circ J^\sharp(x) + \beta^\sharp(x),$$

for all $x \in \mathfrak{M}$. The uniqueness of the decomposition in the just quoted Theorem 8.6 implies that $z_0 = z_0^*$, $J^\sharp = J$, and $\beta^\sharp = \beta$, which finishes the proof. □

JBW-algebras constitute a suitable mathematical model in quantum mechanics as their elements describe observable physical quantities. Operator commutativity for elements in a JBW-algebra can be seen as a generalization of commutativity (or compatibility in the language of quantum mechanics) for elements in $B(H)_{sa}$. Therefore, the study of linear bijections between JBW-algebras preserving operator commutativity in both directions is an interesting question, worth to be considered by itself. Unfortunately, the result is not a mere consequence of what we proved for JBW*-algebras and requires some extra arguments.

Theorem 8.10. *Let $\mathfrak{M}$ and $\mathfrak{J}$ be JBW*-algebras with no central summands of type I_1 and I_2. Suppose that $\Phi : \mathfrak{M}_{sa} \to \mathfrak{J}_{sa}$ is a linear bijection preserving operator commutativity in both directions, that is,*

$$[x, \mathfrak{M}_{sa}, y] = 0 \Leftrightarrow [\Phi(x), \mathfrak{J}_{sa}, \Phi(y)] = 0, \text{ for all } x, y \in \mathfrak{M}_{sa}.$$

Then there exist an invertible element z_0 in $Z(\mathfrak{J}_{sa})$, a Jordan isomorphism $J : \mathfrak{M}_{sa} \to \mathfrak{J}_{sa}$, and a linear mapping $\beta : \mathfrak{M}_{sa} \to Z(\mathfrak{J}_{sa})$ satisfying

$$\Phi(x) = z_0 \circ J(x) + \beta(x),$$

for all $x \in \mathfrak{M}_{sa}$. Furthermore, this decomposition of Φ is unique.

Proof. Let us begin with some simple observations. The natural complex linear extension $\Phi_{\mathbb{C}} : \mathfrak{M} \to \mathfrak{J}$, $\Phi_{\mathbb{C}}(a+ib) := \Phi(a) + i\Phi(b)$, $a+ib \in \mathfrak{M}$, is a symmetric linear bijection. However, it is not clear at this stage that $\Phi_{\mathbb{C}}$ preserves operator commutativity. For this reason we must rebuild and modify part of the proof of Theorem 8.6.

Lemma 8.4 assures that Φ maps $Z(\mathfrak{M}_{sa})$ onto $Z(\mathfrak{J}_{sa})$, and the existence an associative isomorphism $\alpha : Z(\mathfrak{M}_{sa}) \to Z(\mathfrak{J}_{sa})$ satisfying $\Phi(z \circ a) - \alpha(z) \circ \Phi(a) \in Z(\mathfrak{J}_{sa})$, for all $z \in Z(\mathfrak{M}_{sa})$, $a \in \mathfrak{M}_{sa}$, and $\Phi^{-1}(\tilde{z} \circ b) - \alpha^{-1}(\tilde{z}) \circ \Phi^{-1}(b) \in Z(\mathfrak{M}_{sa})$, for all $\tilde{z} \in Z(\mathfrak{J}_{sa})$, $b \in \mathfrak{J}_{sa}$. Since $Z(\mathfrak{J}_{sa}) = Z(\mathfrak{J})_{sa}$ and $Z(\mathfrak{M}_{sa}) = Z(\mathfrak{M})_{sa}$, the mapping $\Phi_{\mathbb{C}}$ maps $Z(\mathfrak{M})$ onto $Z(\mathfrak{J})$. The mapping $\alpha_{\mathbb{C}} : Z(\mathfrak{M}) \to Z(\mathfrak{J})$, $\alpha_{\mathbb{C}}(a+ib) = \alpha(a) + i\alpha(b)$, is an associative isomorphism satisfying

$$\begin{cases} \Phi_{\mathbb{C}}(z \circ x) - \alpha_{\mathbb{C}}(z) \circ \Phi_{\mathbb{C}}(x) \in Z(\mathfrak{J}), \text{ and} \\ \Phi_{\mathbb{C}}^{-1}(\tilde{z} \circ y) - \alpha_{\mathbb{C}}^{-1}(\tilde{z}) \circ \Phi_{\mathbb{C}}^{-1}(y) \in Z(\mathfrak{M}), \end{cases} \tag{8.9}$$

for all $z \in Z(\mathfrak{M})$, $x \in \mathfrak{M}$, $\tilde{z} \in Z(\mathfrak{J})$, $y \in \mathfrak{J}$.

We claim that

$$[\Phi_{\mathbb{C}}(x), \mathfrak{J}, \Phi_{\mathbb{C}}(x^2)] = 0 = [\Phi_{\mathbb{C}}^{-1}(y), \mathfrak{M}, \Phi_{\mathbb{C}}^{-1}(y^2)], \tag{8.10}$$

for all $x \in \mathfrak{M}$, $y \in \mathfrak{J}$. Namely, given $x = a + ib \in \mathfrak{M}$ with $a, b \in \mathfrak{M}_{sa}$, the hypothesis on Φ implies that

$$\begin{aligned}
0 = [\Phi(a \pm b), \mathfrak{J}_{sa}, \Phi((a \pm b)^2)] &= [\Phi(a), \mathfrak{J}_{sa}, \Phi(a^2)] \pm [\Phi(b), \mathfrak{J}_{sa}, \Phi(b^2)] \\
&\pm 2[\Phi(a), \mathfrak{J}_{sa}, \Phi(a \circ b)] + 2[\Phi(b), \mathfrak{J}_{sa}, \Phi(a \circ b)] \\
&+ [\Phi(a), \mathfrak{J}_{sa}, \Phi(b^2)] \pm [\Phi(b), \mathfrak{J}_{sa}, \Phi(a^2)] \\
&= \pm 2[\Phi(a), \mathfrak{J}_{sa}, \Phi(a \circ b)] + 2[\Phi(b), \mathfrak{J}_{sa}, \Phi(a \circ b)] \\
&+ [\Phi(a), \mathfrak{J}_{sa}, \Phi(b^2)] \pm [\Phi(b), \mathfrak{J}_{sa}, \Phi(a^2)],
\end{aligned}$$

which assures that

$$2[\Phi(a), \mathfrak{J}_{sa}, \Phi(a \circ b)] + [\Phi(b), \mathfrak{J}_{sa}, \Phi(a^2)] = 0,$$

and

$$2[\Phi(b), \mathfrak{J}_{sa}, \Phi(a \circ b)] + [\Phi(a), \mathfrak{J}_{sa}, \Phi(b^2)] = 0.$$

We therefore deduce that

$$\begin{aligned}
[\Phi_{\mathbb{C}}(a + ib), \mathfrak{J}, \Phi_{\mathbb{C}}((a + ib)^2)] &= [\Phi(a), \mathfrak{J}, \Phi(a^2)] - i[\Phi(b), \mathfrak{J}, \Phi(b^2)] \\
&+ 2i[\Phi(a), \mathfrak{J}, \Phi(a \circ b)] - 2[\Phi(b), \mathfrak{J}, \Phi(a \circ b)] \\
&- [\Phi(a), \mathfrak{J}, \Phi(b^2)] - i[\Phi(b), \mathfrak{J}, \Phi(a^2)] = 0,
\end{aligned}$$

which proves the first identity in (8.10). The second identity in the claim follows via similar arguments.

It is not hard to check, thanks to (8.10), that the trace of the symmetric bilinear mapping

$$B : \mathfrak{J} \times \mathfrak{J} \to \mathfrak{J},\ B(y, b) = \Phi_{\mathbb{C}}\left(\Phi_{\mathbb{C}}^{-1}(y) \circ \Phi_{\mathbb{C}}^{-1}(b)\right) \text{ is associating,} \tag{8.11}$$

that is,

$$\begin{aligned}
[B(y, y), \mathfrak{J}, y] &= [\Phi_{\mathbb{C}}\left(\Phi_{\mathbb{C}}^{-1}(y) \circ \Phi_{\mathbb{C}}^{-1}(y)\right), \mathfrak{J}, y] \\
&= [\Phi_{\mathbb{C}}\left(\Phi_{\mathbb{C}}^{-1}(y) \circ \Phi_{\mathbb{C}}^{-1}(y)\right), \mathfrak{J}, \Phi_{\mathbb{C}}(\Phi_{\mathbb{C}}^{-1}(y))] = 0.
\end{aligned}$$

Let us find $w \in \mathfrak{J}_{sa}$ and elementary operators $\mathscr{E}_j \in \mathscr{E}\ell_{\mathfrak{J}_{sa}}(\mathfrak{J})$ satisfying $\mathscr{E}_j(w^i) = \delta_{ij}\mathbf{1}$, for all $i, j \in \{0, 1, 2\}$ (cf. Proposition 2.3). We have gathered all the necessary ingredients in (8.9) and (8.11) in order to repeat, line by line, the proof of Theorem 8.6 but replacing Φ and α there with $\Phi_{\mathbb{C}}$ and $\alpha_{\mathbb{C}}$, respectively, to deduce the existence of a unique invertible element z_0 in $Z(\mathfrak{J})$, a unique linear mapping $\mu_1 : \mathfrak{J} \to Z(\mathfrak{J})$, and a unique symmetric bilinear mapping $\nu_1 : \mathfrak{J} \times \mathfrak{J} \to Z(\mathfrak{J})$ satisfying

$$\Phi_{\mathbb{C}}(\Phi_{\mathbb{C}}^{-1}(y)^2) = B(y, y) = z_0^{-1} \circ y^2 + \mu_1(y) \circ y + \nu_1(y, y), \tag{8.12}$$

for all $y \in \mathfrak{J}$, (cf. Theorem 7.5), $z_0^{-1} = \mathscr{E}_2(B(w, w))$ (cf. Proposition 3.1), and taking $\tilde{J} : \mathfrak{M} \to \mathfrak{J}$, $\tilde{J}(x) = z_0^{-1} \circ \Phi_{\mathbb{C}}(x) + \frac{1}{2}\mu_1(\Phi_{\mathbb{C}}(x))$ it follows that $\tilde{J}$ is a Jordan isomorphism and

$$\Phi_{\mathbb{C}}(x) = z_0 \circ \tilde{J}(x) + \tilde{\beta}(x), \text{ for all } x \in \mathfrak{M},$$

for a certain linear mapping $\tilde{\beta} : \mathfrak{M} \to Z(\mathfrak{J})$.

By recalling that all elementary operators $\mathscr{E}_j \in \mathscr{E}\ell(\mathfrak{J})$ can be assumed to be symmetric linear mappings (cf. Proposition 2.3) we derive that

$$\begin{aligned}
(z_0^{-1})^* = \mathscr{E}_2(B(w, w))^* = \mathscr{E}_2(B(w, w)^*) &= \mathscr{E}_2\left((\Phi_{\mathbb{C}}(\Phi_{\mathbb{C}}^{-1}(w)^2))^*\right) \\
&= \mathscr{E}_2\left(\Phi_{\mathbb{C}}(\Phi_{\mathbb{C}}^{-1}(w)^2)\right) = z_0^{-1},
\end{aligned}$$

which proves that $z_0 = z_0^* \in Z(\mathfrak{J}_{sa})$.

On the other hand, $\Phi_{\mathbb{C}}$ being a symmetric linear mapping implies that $B(y, y) = B(y^*, y^*)^*$, and thus by (8.12),

$$\begin{aligned} z_0^{-1} \circ y^2 + \mu_1(y) \circ y + \nu_1(y, y) = B(y, y) \\ = B(y^*, y^*)^* = z_0^{-1} \circ y^2 + \mu_1(y^*)^* \circ y + \nu_1(y^*, y^*)^*, \end{aligned}$$

for all $y \in \mathfrak{J}$. The uniqueness of the decomposition in (8.12) asserts that $\mu_1(y^*)^* = \mu_1(y)$ and $\nu_1(y, y) = \nu_1(y^*, y^*)^*$ for all $y \in \mathfrak{J}$. We therefore have

$$\begin{aligned} \tilde{J}(x^*) &= z_0^{-1} \circ \Phi_{\mathbb{C}}(x^*) + \frac{1}{2}\mu_1(\Phi_{\mathbb{C}}(x^*)) = z_0^{-1} \circ \Phi_{\mathbb{C}}(x)^* + \frac{1}{2}\mu_1(\Phi_{\mathbb{C}}(x))^* \\ &= (z_0^{-1} \circ \Phi_{\mathbb{C}}(x) + \frac{1}{2}\mu_1(\Phi_{\mathbb{C}}(x)))^* = \tilde{J}(x)^*, \ x \in \mathfrak{M}, \end{aligned}$$

and hence $\tilde{J}$ is a Jordan *-isomorphism. It trivially holds that $\tilde{\beta}$ is a symmetric linear mapping. By defining $J = \tilde{J}|_{\mathfrak{J}_{sa}}$ and $\beta = \tilde{\beta}|_{\mathfrak{J}_{sa}}$ we obtain the desired decomposition for Φ.

Finally, the uniqueness of the decomposition of Φ essentially follows from the uniqueness of the decomposition in (8.12) as explicitly shown in the final part of the proof of Theorem 8.6. □

We conclude this paper with a consequence of our previous Theorem 6.7 and Proposition 1.2 which is a novelty in the setting of von Neumann algebras. Let us first recall a precedent related to our objective. Let A and B be von Neumann algebras, and suppose additionally that A is a factor not of type I_1 or I_2. L. Molnár established in [33, Theorem 4] that for every linear bijection $\Phi : A_{sa} \to B_{sa}$ preserving commutativity in both directions, there exist a non-zero real number λ, a Jordan *-isomorphism $J : A \to B$ and a real-linear functional $\beta : A_{sa} \to \mathbb{R}$ such that

$$\Phi(a) = \lambda J(a) + \beta(a)\mathbf{1}, \text{ for all } a \in A_{sa}.$$

We can now relax the hypothesis concerning the von Neumann algebra A and optimize the conclusion in the just quoted result by Molnár.

Corollary 8.11. *Let A and B be von Neumann algebras with no central summands of type I_1 and I_2. Suppose that $\Phi : A_{sa} \to B_{sa}$ is a linear bijection preserving commutativity in both directions. Then there exist an invertible element z_0 in $Z(B_{sa})$, a Jordan isomorphism $J : A_{sa} \to B_{sa}$, and a linear mapping $\beta : A_{sa} \to Z(B_{sa})$ satisfying*

$$\Phi(x) = z_0 \, J(x) + \beta(x), \text{ for all } x \in A_{sa}.$$

Furthermore, this decomposition of Φ is unique.

Declaration of competing interest

The authors declare that they have no conflict of interest.

Acknowledgments

The authors would like to thank the anonymous reviewers for their exhaustive reports and constructive comments, which greatly improved the overall manuscript.

Authors supported by MICIU/AEI/10.13039/501100011033 and ERDF/EU grant PID2021-122126NB-C31, by "Maria de Maeztu" Excellence Unit IMAG, reference CEX2020-001105-M funded by MICIU/AEI/10.13039/501100011033, and by Junta de Andalucía grants FQM185 and FQM375. First author supported by grant FPU21/00617 at University of Granada founded by Ministerio de Universidades (Spain). Second author partially supported by MOST China project number G2023125007L.

Funding for open access charge: Universidad de Granada / CBUA

Data availability

Statement Data sharing is not applicable to this article as no datasets were generated or analysed during the preparation of the paper.

References

[1] E.M. Alfsen, F.W. Shultz, Geometry of State Spaces of Operator Algebras, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 2003.
[2] E.M. Alfsen, F.W. Shultz, E. Størmer, A Gelfand-Neumark theorem for Jordan algebras, Adv. Math. 28 (1978) 11–56.
[3] P. Ara, M. Mathieu, Local Multipliers of C*-Algebras, Springer Monographs in Mathematics, Springer, London, 2003.
[4] R. Banning, M. Mathieu, Commutativity preserving mappings on semiprime rings, Commun. Algebra 25 (1) (1997) 247–265.
[5] T. Barton, Y. Friedman, Bounded derivations of JB*-triples, Q. J. Math. Oxf. Ser. (2) 41 (163) (1990) 255–268.
[6] M. Bohata, J. Hamhalter, O. Kalenda, Decompositions of preduals of JBW and JBW* algebras, J. Math. Anal. Appl. 446 (1) (2017) 18–37.
[7] M. Brešar, On a generalization of the notion of centralizing mappings, Proc. Am. Math. Soc. 114 (1992) 541–649.
[8] M. Brešar, Commuting traces of biadditive mappings, commutativity preserving mappings, and Lie mappings, Trans. Am. Math. Soc. 335 (1993) 525–546.
[9] M. Brešar, Review of the article "Commutativity preserving mappings on semiprime rings" by R. Banning and M. Mathieu, zbMATH Open Zbl 0865.16015, 1997.
[10] M. Brešar, Introduction to Noncommutative Algebra, Universitext, Springer, 2014.
[11] M. Brešar, M. Cabrera, A.R. Villena, Functional identities in Jordan algebras: associating maps, Commun. Algebra 30 (2002) 5241–5252.
[12] M. Brešar, D. Eremita, A.R. Villena, Functional identities in Jordan algebras: associating traces and Lie triple isomorphisms, Commun. Algebra 31 (2003) 1207–1234.
[13] M. Brešar, C.R. Miers, Commutativity preserving mappings of von Neumann algebras, Can. J. Math. 45 (4) (1993) 695–708.
[14] M. Cabrera García, A. Rodriguez Palacios, Non-Associative Normed Algebras: Volume 1, The Vidav–Palmer and Gelfand–Naimark Theorems, Cambridge University Press, 2014.
[15] M.D. Choi, A.A. Jafarian, H. Radjavi, Linear maps preserving commutativity, Linear Algebra Appl. 87 (1987) 227–241.
[16] S. Dineen, Complete holomorphic vector fields on the second dual of a Banach space, Math. Scand. 59 (1986) 131–142.
[17] C.M. Edwards, On Jordan W*-algebras, Bull. Sci. Math. 104 (1980) 393–403.
[18] Y. Friedman, Physical Applications of Homogeneous Balls. With the Assistance of Tzvi Scarr, Progress in Mathematical Physics, vol. 40, Birkhäuser, Boston, MA, 2005.
[19] D.J. Griffiths, Introduction to Quantum Mechanics, Prentice Hall, Englewood Cliffs, NJ, 1995.
[20] H. Hanche-Olsen, On the structure and tensor products of JC-algebras, Can. J. Math. 35 (6) (1983) 1059–1074.
[21] H. Hanche-Olsen, E. Størmer, Jordan Operator Algebras, Pitman, London, 1984.
[22] L.A. Harris, Bounded symmetric homogeneous domains in infinite dimensional spaces, in: Proc. Infinite Dim. Holomorphy, Lexington 1973, in: Lect. Notes Math., vol. 364, 1974, pp. 13–40.
[23] L.A. Harris, A generalization of C*-algebras, Proc. Lond. Math. Soc. 42 (1981) 331–361.
[24] S. Heinrich, Ultraproducts in Banach space theory, J. Reine Angew. Math. 313 (1980) 72–104.
[25] F.J. Hervés, J.M. Isidro, Isometries and automorphisms of the spaces of spinors, Rev. Mat. Univ. Complut. Madr. 5 (2–3) (1992) 193–200.
[26] P. Jordan, J. von Neumann, E. Wigner, On an algebraic generalization of the quantum mechanical formalism, Ann. Math. 35 (1934) 29–64.
[27] D.C. Kleinecke, On operator commutators, Proc. Am. Math. Soc. 8 (1957) 535–536.
[28] P.-H. Lee, T.-K. Lee, Linear identities and commuting maps in rings with involution, Commun. Algebra 25 (1997) 2881–2895.
[29] G.W. Mackey, The Mathematical Foundations of Quantum Mechanics. A Lecture-Note Volume, The Mathematical Physics Monograph Series, W.A. Benjamin, Inc., New York, Amsterdam, 1963.
[30] W.S. Martindale 3rd, Lie isomorphisms of prime rings, Trans. Am. Math. Soc. 142 (1969) 437–455.
[31] M. Mathieu, Elementary operators on C*-algebras. I, Math. Ann. 284 (1989) 223–224.
[32] C.R. Miers, Commutativity preserving maps of factors, Can. J. Math. 40 (1) (1988) 248–256.

[33] L. Molnár, Linear maps on observables in von Neumann algebras preserving the maximal deviation, J. Lond. Math. Soc. (2) 81 (1) (2010) 161–174.
[34] M. Omladič, H. Radjavi, P. Šemrl, Preserving commutativity, J. Pure Appl. Algebra 156 (2001) 309–328.
[35] A.M. Peralta, A. Rodríguez Palacios, Grothendieck's inequalities for real and complex JBW*-triples, Proc. Lond. Math. Soc. (3) 83 (3) (2001) 605–625.
[36] T. Petek, Additive mappings preserving commutativity, Linear Multilinear Algebra 42 (3) (1997) 205–211.
[37] A. Rodríguez-Palacios, On the strong*-topology of a JBW*-triple, Q. J. Math. Oxf. Ser. (2) 42 (165) (1991) 99–103.
[38] F.W. Shultz, On normed Jordan algebras which are Banach dual spaces, J. Funct. Anal. 31 (3) (1979) 360–376.
[39] E. Størmer, Jordan algebras of type I, Acta Math. 115 (1966) 165–184.
[40] D.M. Topping, Jordan algebras of self-adjoint operators, Mem. Am. Math. Soc. 53 (1965).
[41] J. von Neumann, On an algebraic generalization of the quantum mechanical formalism, Mat. Sb. 1 (1936) 415–482.
[42] M.A. Youngson, A Vidav theorem for Banach Jordan algebras, Math. Proc. Camb. Philos. Soc. 84 (1978) 263–272.
[43] J. van de Wetering, Commutativity in Jordan operator algebras, J. Pure Appl. Algebra 24 (11) (2020) 106407, 15 pp.
[44] J.D.M. Wright, Jordan C* algebras, Mich. Math. J. 24 (1977) 291–302.