

4 Lecture 3 (20/02/2020)

4.1 Primer on polar coordinates

Describing rotation in cartesian coordinates can be tedious and time consuming. Luckily, polar coordinates are a thing. Using polar coordinates makes describing motion along a curved path and rotational motion much easier and faster. Some basic properties and conversion between polar and cartesian coordinates will be covered below.

$$(x, y) = (r \cos(\theta), r \sin(\theta)) \quad (1)$$

$$r = \sqrt{x^2 + y^2} \quad (2)$$

$$dA = dxdy = r dr d\theta \quad (3)$$

$$\iint_D dA = \int_a^b \int_c^d dxdy = \int_\alpha^\beta \int_0^r r dr d\theta \quad (4)$$

4.2 Known path in polar coordinates

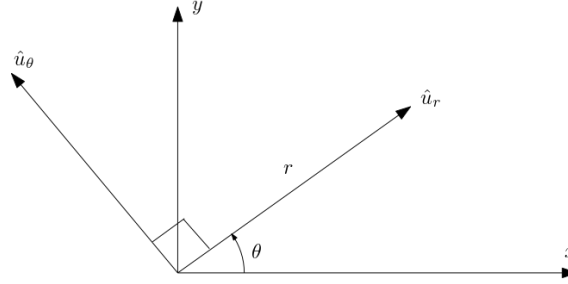


Figure 1: Polar coordinates on cartesian x,y plane

$$\vec{x} = \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \end{pmatrix} = r \cdot \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \quad (5)$$

$$\text{let } \theta = \theta(t), r = r(t) \quad (6)$$

$$\dot{\vec{x}} = \frac{d\vec{x}}{dt} = \begin{pmatrix} \dot{r} \cos(\theta) - r \sin(\theta) \cdot \dot{\theta} \\ \dot{r} \sin(\theta) + r \cos(\theta) \cdot \dot{\theta} \end{pmatrix} = \dot{r} \cdot \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} + r\dot{\theta} \cdot \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \quad (7)$$

It's worth noting that the unit vectors $\hat{\mathbf{u}}_t$ and $\hat{\mathbf{u}}_n$ for the tangential and perpendicular components can be recognized in the definition of $\dot{\vec{x}}$. The following definition follows:

$$\hat{\mathbf{u}}_r = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \Rightarrow \text{Perpendicular component} \quad (8)$$

$$\hat{\mathbf{u}}_\theta = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \Rightarrow \text{Tangential component} \quad (9)$$

$$(10)$$

Derrivation for $\ddot{\vec{x}}$ can be found in the Hibbeler Dynamics book, I am currently too lazy to look it up and write it out so you'll just have to believe what I wrote. This gives the following:

$$\begin{cases} \vec{x} = r\hat{\mathbf{u}}_r \\ \dot{\vec{x}} = \dot{r}\hat{\mathbf{u}}_r + r\dot{\theta}\hat{\mathbf{u}}_\theta \\ \ddot{\vec{x}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{u}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{u}}_\theta \end{cases} \quad (11)$$

Side note for equation (11), $\ddot{r} - r\dot{\theta}^2$ and $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ are the components a_n and a_t respectively

4.3 Example Problem using polar coordinates with a known path of motion

A point mass with $m = 0,5 \text{ kg}$ moves along the curved path $r = (0,1\theta) \text{ m/rad}$ on the horizontal plane. let $\theta = \pi \text{ rad}$, $\dot{\theta} = 4 \text{ rad/s}$, $\ddot{\theta} = 0 \text{ rad/s}^2$

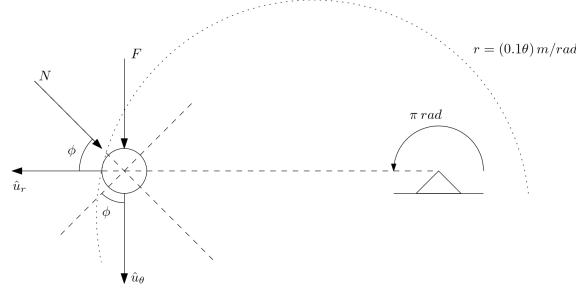


Figure 2: Free body diagram of the problem to be solved

$$\Sigma F_r = -N \cos(\phi) = ma_r \quad (1)$$

$$\Sigma F_\theta = F + N \sin(\phi) = ma_\theta \quad (2)$$

$$a_r = \ddot{r} - r\dot{\theta}^2 \quad (3)$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \quad (4)$$

There are now 4 equations, 2 equations of motion and 2 constraint equation. The system has 5 variables so a new equation needs to be introduced to solve for the angle ϕ . Looking at the point mass

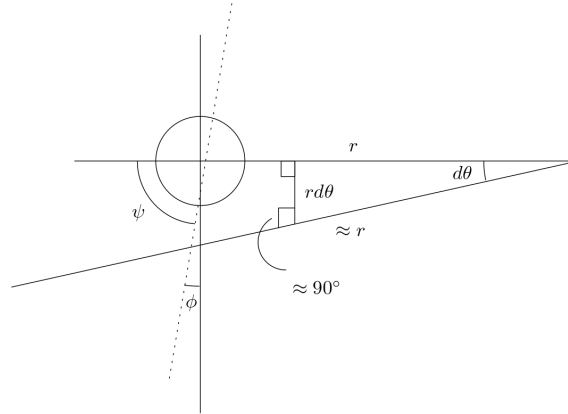


Figure 3: Relations relating to the angle ϕ and ψ

from really close reveals some interesting properties if the angle θ becomes infinitesimally small. The line offset from r by $d\theta$ is approximately parallel to r . This implies that the distance traversed by the particle is $rd\theta$. Using the triangle in figure 3 gives the following:

$$\tan(\phi) = \frac{dr}{rd\theta} = \frac{1}{r} \cdot \frac{dr}{d\theta} \Rightarrow \phi = \arctan\left(\frac{1}{r} \cdot \frac{dr}{d\theta}\right) \quad (5)$$

This makes equation (5) the 5th and final equation required to solve the problem. It is also possible to determine the value of angle ϕ using a complementary angle ψ . This method is used in the Hibbeler Dynamics book.

Using the given information and equation (3) and (4) gives:

$$a_r = -5,0625 \text{ m/s}^2 \quad (6)$$

$$a_\theta = 3,2 \text{ m/s}^2 \quad (7)$$

For the angle ϕ we know the following:

$$\frac{dr}{d\theta} = \frac{d(0,1\theta)}{d\theta} = 0,1 \quad (8)$$

$$\phi = \arctan\left(\frac{1}{\pi} \cdot 0.1\right) = 0,03182 \text{ rad} \quad (9)$$

Substituting the found values from equation (6),(7) and (9) into equation (1) and (2) gives the following values for N and F :

$$N = \frac{-ma_r}{\cos \phi} = 2,53 \text{ N} \quad (10)$$

$$F = ma_\theta - N \sin(\theta) = 1,52 \text{ N} \quad (11)$$