10 Lecture 10: Coordinates and Dimensions

10.1 Bases

Let H be a subspace. Every point in H can be written as a linear combination of vectors in the set $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_p\}$. Since $\mathcal{B}$ is a basis we know it is linearly independent.

Suppose $\vec{x}$ can be written in 2 ways.

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_p\vec{b}_p \quad \text{or} \quad \vec{x} = d_1\vec{b}_1 + \cdots + d_p\vec{b}_p \tag{1}$$

then:

$$\vec{0} = \vec{x} - \vec{x} = (c_1 - d_1)\vec{b}_1 + \cdots + (c_p - d_p)\vec{x} \tag{2}$$

Since $\mathcal{B}$ is linearly independent we know that all the weights in equation (2) must be zero., thus: $c_j = d_j$ for $1 \leq j \leq p$, This means that the 2 ways of writing the linear combination for $\vec{x}$ in equation (1) must in fact be the same.

$\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_p\}$ is a basis for the subspace H. For each $\vec{x}$ in H the coordinates of $\vec{x}$ relative to the basis $\mathcal{B}$ are the weights $c_1, \cdots, c_p$ such that $\vec{x} = c_1\vec{b}_1 + \cdots + c_p\vec{b}_p$. The vector:

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} \quad \text{Where} \quad [\vec{x}]_{\mathcal{B}} \in \mathbb{R}^p \tag{3}$$

is called the coordinate vector of $\vec{x}$, relative to $\mathcal{B}$ or the $\mathcal{B}$-coordinate vector of $\vec{x}$.

10.2 example problem bases

$$\text{let: } \vec{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \vec{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}, \mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$$

Determine if $\vec{x}$ is in H, if it is determine $[\vec{x}]_{\mathcal{B}}$.

$$c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix} \quad \text{must be consistent}$$

$$\left[\begin{array}{cc|c} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

$$(c_1, c_2) = (2, 3) \quad \text{and} \quad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

It's important to note that $[\vec{x}]_{\mathcal{B}} \in \mathbb{R}^2$ but outputs vectors in an $\mathbb{R}^3$ space. It can be thought of as a 2 dimensional plane oriented in a 3 dimensional space. The correspondance $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is a one-to-one correspondance between the subspace H and $\mathbb{R}^2$ that preserves linear combinations. This is what is referred to as isomorphism. In general if $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_p\}$ is a basis for H then the mapping $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ makes H look and act like an $\mathbb{R}^P$ space eventhough vectors in H may have more than p entries.

10.3 Dimensions of a subspace

In Linear algebra the concept is a bit different. A dimension is defined as the amount of vectors contained in the basis of a space. This means $\mathbb{R}^2$ contains 2 basis vectors, $\mathbb{R}^3$ contains 3 basis vectors, and so on. The rank of a matrix denoted by $\text{Rank}\, A$ is the dimension of the column space of A. Since the pivot columns of A form the basis for $\text{Col}\, A$ the rank of a matrix A is nothing other then the number of pivot columns in A.

Theorem 1 *The dimension of a nonzero subspace H, denoted by $Dim\, H$, is the number of vectors in the basis of H. The dimension of the null space $\{\vec{0}\}$ is defined to be 0, since the set $\{\vec{0}\}$ is defined to be a linearly dependent set.*

Theorem 2 *if a matrix A has n columns, then: $rank\, A + dim\, Nul\, A = n$.*

Theorem 3 *Let H be a p-dimensional subspace of $\mathbb{R}^n$ any linearly independent set of exactly p elements in H is automatically a basis for H. Also, any set of p elements of H that spans H is automatically a basis for H.*

Theorem 4 *if A is an $n \times n$ matrix, then all the following statements imply that A is an invertible matrix:*

- *The columns of A form the basis of $\mathbb{R}^n$*
- $Col\, A = \mathbb{R}^n$
- $dim\, Col\, A = \mathbb{R}^n$
- $rank\, A = n$
- $Nul\, A = \{\vec{0}\}$
- $dim\, Nul\, A = 0$