4 Lecture 3 (20/02/2020)

4.1 Primer on polar coordinates

Describing rotation in cartesian coordinates can be tedious and time consumig. Luckily, polar coordinates are a thing. Using polar coordinates makes describing motion along a curved path and rotational motion much easier and faster. Some basic properties and conversion between polar and cartesian coordinates will be covered below.

$$(x, y) = (r\cos(\theta), r\sin(\theta)) \tag{1}$$

$$r = \sqrt{x^2 + y^2} \tag{2}$$

$$dA = dxdy = rdrd\theta \tag{3}$$

$$\iint_D dA = \int_a^b \int_c^d dxdy = \int_\alpha^\beta \int_0^r r\,drd\theta \tag{4}$$

4.2 Known path in polar coordinates

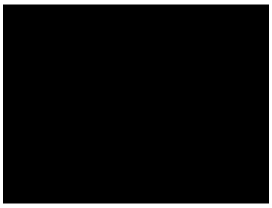

Figure 1: Figure placeholder

$$\vec{x} = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix} = r \cdot \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \tag{5}$$

$$\text{let } \theta = \theta(t),\ r = r(t) \tag{6}$$

$$\dot{\vec{x}} = \frac{d\vec{x}}{dt} = \begin{pmatrix} \dot{r}\cos(\theta) - r\sin(\theta)\cdot\dot{\theta} \\ \dot{r}\sin(\theta) + r\cos(\theta)\cdot\dot{\theta} \end{pmatrix} = \dot{r}\cdot\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} + r\dot{\theta}\cdot\begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \tag{7}$$

It's worth noting that the unit vectors $\hat{\boldsymbol{u}}_{\boldsymbol{t}}$ and $\hat{\boldsymbol{u}}_{\boldsymbol{n}}$ for the tangential and perpendicular components can be recognized in the definition of $\dot{\vec{x}}$. The following definition follows:

$$\hat{\boldsymbol{u}}_{\boldsymbol{r}} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \Rightarrow \text{Perpendicular component} \tag{8}$$

$$\hat{\boldsymbol{u}}_{\boldsymbol{\theta}} = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} \Rightarrow \text{Tangential component} \tag{9}$$

$$\tag{10}$$

Derrivation for $\ddot{\vec{x}}$ can be found in the Hibbeler Dynamics book, I am currently to lazy to look it up and write it out so you'll just have to believe what I wrote. This gives the following:

$$\begin{cases} \vec{x} = r\hat{\boldsymbol{u}}_{\boldsymbol{r}} \\ \dot{\vec{x}} = \dot{r}\hat{\boldsymbol{u}}_{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{u}}_{\boldsymbol{\theta}} \\ \ddot{\vec{x}} = (\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{u}}_{\boldsymbol{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{u}}_{\boldsymbol{\theta}} \end{cases} \tag{11}$$

Side note for equation (11), $\ddot{r} - r\dot{\theta}^2$ and $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ are the components a_n and a_t respectivly

4.3 Example Problem using polar coordinates with a known path of motion

A point mass with $m = 0,5\,kg$ moves along the curved path $r = (0,1\theta)\,m/rad$ on the horizontal plane. let $\theta = \pi\,rad$, $\dot{\theta} = 4\,rad/s$, $\ddot{\theta} = 0\,rad/s^2$

Figure 2: Figure placeholder

$$\Sigma F_r = -N\cos(\phi) = ma_r \tag{1}$$
$$\Sigma F_\theta = F + N\sin(\phi) = ma_\theta \tag{2}$$
$$a_r = \ddot{r} - r\dot{\theta}^2 \tag{3}$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \tag{4}$$

There are now 4 equations, 2 equations of motion and 2 constraint equation. The system has 5 variables so a new equation needs to be introduced to solve for the angle ϕ. Looking at the point mass

Figure 3: Figure placeholder

from really close reveals some interesting properties if the angle θ becomes infitesimally small. The line offset from r by $d\theta$ is approximately parrallel to r. This implies that the distance traversed by the particle is $rd\theta$. Using the triangle in figure 3 gives the following:

$$\tan(\phi) = \frac{dr}{rd\theta} = \frac{1}{r}\cdot\frac{dr}{d\theta} \Rightarrow \phi = \arctan\left(\frac{1}{r}\cdot\frac{dr}{d\theta}\right) \tag{5}$$

This makes equation (5) this fifth and final equation required to solve the problem. It is also possible to determin the value of angle ϕ using a complemantary angle ψ. This method is used in the Hibbeler Dynamics book.

Using the given information and equation (3) and (4) gives:

$$a_r = -5,0625\,m/s^2 \tag{6}$$
$$a_\theta = 3,2\,m/s^2 \tag{7}$$

For the angle ϕ we know the following:

$$\frac{dr}{d\theta} = \frac{d(0,1\theta)}{d\theta} = 0,1 \tag{8}$$

$$\phi = \arctan\left(\frac{1}{\pi} \cdot 0.1\right) = 0,03182\, rad \tag{9}$$

Subsituting the found values from equation (6),(7) and (9) into equation (1) and (2) gives the following values for N and F:

$$N = \frac{-ma_r}{\cos\phi} = 2,53\, N \tag{10}$$

$$F = ma_\theta - N\sin(\theta) = 1,52\, N \tag{11}$$

Note: Add figures then this is done.