

9 Lecture 9 (09/03/2020)

9.1 The moment of Inertia

The moment of inertia is a mathematical description of resistance to accelerating rotation. This is not to be confused with the area moment of inertia¹ which is the same physical quantity and the same integral but over an area rather than a mass-density function. The integral describing the moment of Inertia of a rigid body with 1 or 2 dimensions are as follows:

$$\int \rho(x)r^2 dx \quad (1)$$

$$\iint \rho(x,y)r^2 dA \quad (2)$$

This resistance to rotation term appears quite naturally when analyzing a rotating body.

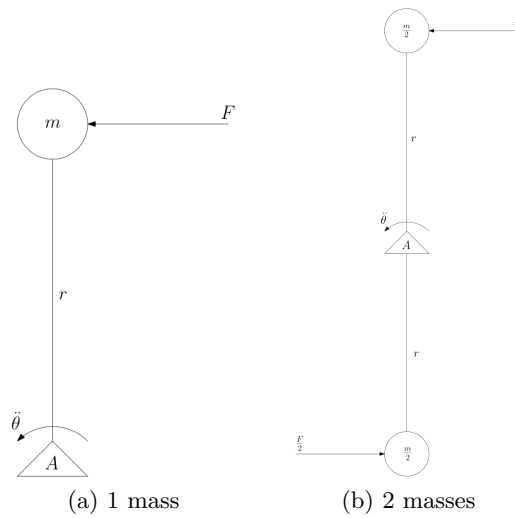


Figure 1: Rigid body with mass at the end of a stick

$$\Sigma r \cdot F = rma \quad (3)$$

$$a = r\alpha = r\ddot{\theta} \quad (4)$$

$$M_A = rF \quad (5)$$

$$\Sigma M_A - mr^2\ddot{\theta} = I_A\ddot{\theta} \quad (6)$$

Looking at Figure 1 (b) it's easy to see that it does not matter if there is a single mass with a force F acting on it or 2 masses with a force $\frac{F}{2}$ acting on them. This process of adding more and more masses can be repeated indefinitely. This creates a circle with radius r . This circle is referred to as the radius of gyration. Only mass and radius of Gyration are needed to compute the moment of inertia. Any given object will always want to rotate about its center of gravity. Orientation and location of the object in space do not matter. When an object is not attached at the center of gravity such as figure 2 shows an extra term needs to be accounted for in the moment of inertia. This term is referred to as the Steiner term. It takes the following form:

$$\Sigma M_O = (mr^2 + md^2)\ddot{\theta} \quad (7)$$

where $I_G = mr^2$ and md^2 the Steiner term. Note that I would add an image to visualize, but LaTeX is being a prick about it so you can google it or look at the book or something.

¹as seen in WB1631-15 Mechanics of Materials

Moment of inertia of a 1D body:

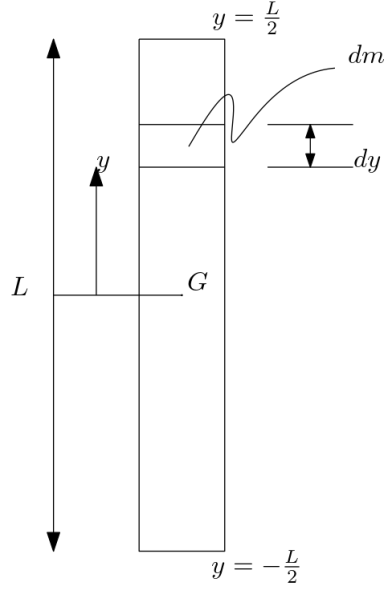


Figure 2: 1 dimensional bar for determining the moment of inertia

Assuming the width is much less than the length reduces this to a 1D problem.

$$\frac{d\rho}{dL} = 0 \Rightarrow \rho = \frac{m}{L} \quad (8)$$

$$dm = \rho dy \quad (9)$$

$$dI_G = y dm = \rho y^2 dy \quad (10)$$

$$I_G = \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho y^2 dy = \frac{1}{3} \rho y^3 \Big|_{-\frac{L}{2}}^{\frac{L}{2}} \quad (11)$$

$$I_G = \frac{1}{12} \rho L^3 = \frac{1}{12} m L^2 \quad (12)$$

When not rotating about the center of gravity:

$$I_A = I_G + m d^2, \text{ where } m d^2 \text{ is the Steiner term} \quad (13)$$

$$I_A = \frac{1}{12} m L^2 + \frac{1}{4} m L^2 = \frac{1}{3} m L^2 \quad (14)$$

Moment of inertia of a 2D body: Might add this later

9.2 Equations of motion

Since we are now considering rigid bodies in a 2D space rather than just point masses we should also consider the effects of rotation². The new set of equations of motions will be the following:

$$\begin{cases} \Sigma F_x = m \ddot{x} \\ \Sigma F_y = m \ddot{y} \\ \Sigma M_G = I_G \ddot{\theta} \end{cases} \quad (15)$$

²Some people prefer to define point masses as not being able to rotate others as define them as having no inertia, it doesn't really matter

It's important to note that in Statics we were free to choose any given point to calculate the moment sum, since the entire body was in static equilibrium. This concept does NOT hold for dynamics. The center of rotation matters and cannot be chosen freely due to the effect of Steiner terms.

9.3 Example of a pendulum type motion as a rigid body

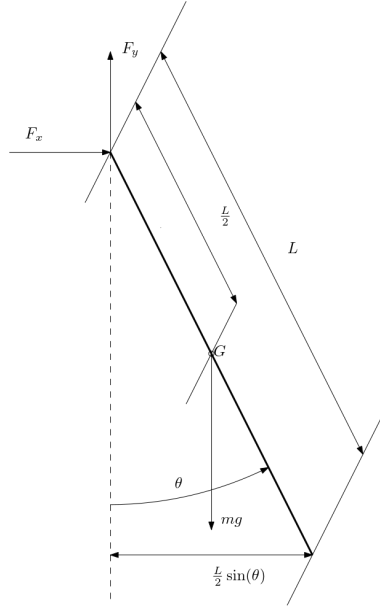


Figure 3: An object which will swing in a pendulum type motion

$$\Sigma M_A = \frac{L}{2} mg \sin(\theta) = I_A \ddot{\theta} \quad (16)$$

This is a non-linear second order differential equation

To make solving this easier we assume small angles

$$\theta \ll 1 \Rightarrow \sin(\theta) \approx \theta \quad (17)$$

$$\ddot{\theta} + \frac{Lmg}{2I_A} \theta = 0 \quad (18)$$

$$r^2 + \frac{Lmg}{2I_A} r = 0 \quad (19)$$

$$r = 0 \pm i \sqrt{\frac{Lmg}{2I_A}} \quad (20)$$

$$\alpha = 0 \quad \beta = \omega_n = \sqrt{\frac{Lmg}{2I_A}} \quad (21)$$

$$y_c = e^{\alpha t} (A \sin(\beta t) + B \cos(\beta t)) \quad (22)$$