

3 Lecture 3 (17/02/2020)

3.1 Choice of coordinate system

The choice of coordinate system should not affect the motion of an object. The coordinate system should be thought off as a geometrical way of expressing a motion. The way in which the motion is expressed may differ but the motion is still the same motion no matter how it is described. This is important since it builds intuition for describing motion along a curve. This concept will be covered with an example below.

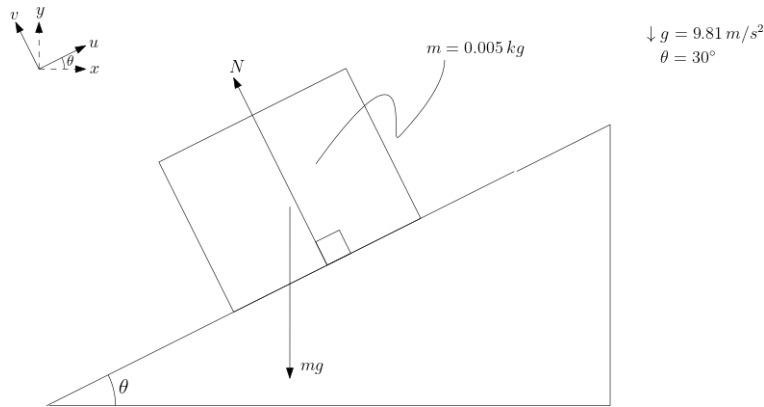


Figure 1: The Free Body Diagram of the cup on a plane

Recall the example from lecture 1 with a cup on a table. The coordiante system for that problem was solved using the u- and v-axis which where offset from the x- and y-axis respectively by an angle of θ . The solution below will solve the problem using the x- and y-axis.

$$\Sigma F_x = N \cdot \sin(\theta) = m\ddot{x} \quad (1)$$

$$\Sigma F_y = N \cdot \cos(\theta) - mg = m\ddot{y} \quad (2)$$

There are now 3 variables in 2 equation making this system currently unsolvable. There needs to be another equation. This equation based on some geometrical properties of the surface which can be seen in figure 2. Since the cup is moving along the surface we know the following:

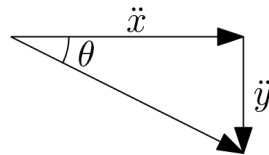


Figure 2: The vector sum describing the relation between horizontal and vertical acceleration

$$\tan(\theta) = \frac{\ddot{y}}{\ddot{x}} \Rightarrow \ddot{y} = \ddot{x} \tan(\theta) \quad (3)$$

Equation (3) being the constraint equation for this system introduces the 3rd and final equation needed for solving this system.

3.2 n,t-coordinates

n,t-coordinates refers to normal,tangential coordinates. This is a coordinate system commonly used for describing motion along curved surfaces. It's important to note that this is in fact an inertial frame of reference. The total displacement is always going to be 0 because of this since the coordinates move along with the object that is being described.

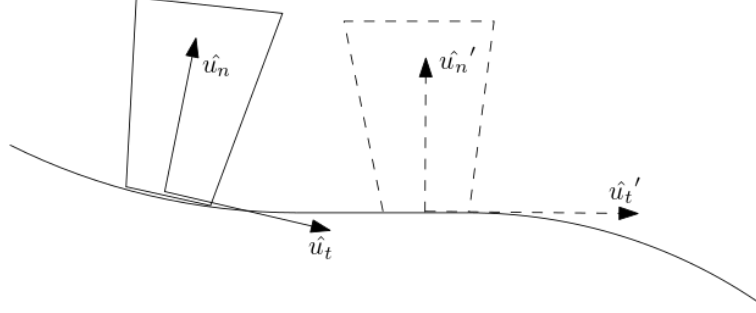


Figure 3: The movement of a cup along a curved surface

$$\vec{v} = v_t \cdot \hat{u}_t + 0 \cdot \hat{u}_n \quad (4)$$

Equation (4) describes the actual velocity of the object along the surface. Only the tangential component of the force contributes to changes in velocity, and only the tangential component of the velocity vector describes the velocity along the surface. It's worth knowing that this is all still expressed in a cartesian coordinate system. the velocity vector \vec{v} can also be described as $\vec{v} = v_x \hat{i} + v_y \hat{j}$. By extension this means that the unit vector \hat{u}_t and \hat{u}_n are also vectors expressed in cartesian coordinates.

3.3 derrivation for motion along a curved path

The acceleration vector \vec{a} is considered a type of constraint equation for the movement, since the object has to move along the curved path. Applying the product rule gives the following:

$$a = \frac{dv}{dt} \Rightarrow \vec{a} = \frac{d(v_t \cdot \hat{u})}{dt} = \frac{dv}{dt} \cdot \hat{u} + v_t \frac{d\hat{u}}{dt} \quad (5)$$

Consider the following: a point mass moves an infetesimal distance ds^1 along a curved path. This means the angle ϕ increases with $d\phi$. The radius of curvature will be ρ . Since the increases in angle are small, which is to say, way less then 1 a first order Taylor expansion can be used to approximate the value:

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \quad \text{for } f(x) = \sin(x) \quad \text{at } a=0 \quad (6)$$

$$T_1(x) = 0 + \frac{\cos(0)}{1} (x-0) = x \quad (7)$$

$$\text{Thus : } \sin(x) \approx x \quad (8)$$

Because of this property we can determine the following:

¹Since all values for the differential element are very small ds is approximated to do be straight eventhough it is technically curved very slightly.

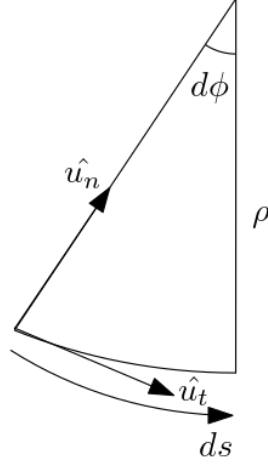


Figure 4: A differential element of an object moving along a curved path

$$ds = \rho d\phi \quad (9)$$

$$d\phi = \frac{ds}{\rho} \quad (10)$$

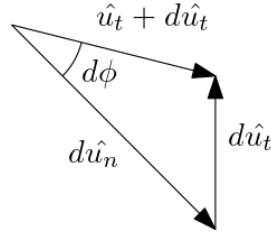


Figure 5: The geometrical relation between $d\hat{\mathbf{u}}_n$, $d\hat{\mathbf{u}}_t$ and $d\phi$

From figure 5 we can see that:

$$d\hat{\mathbf{u}}_t = d\phi \cdot \hat{\mathbf{u}}_n \quad (11)$$

Using equation (10) and (11) and taking the derivative with respect to time we can see that:

$$\frac{\hat{\mathbf{u}}_t}{dt} = \frac{\frac{ds}{\rho} \cdot \hat{\mathbf{u}}_n}{dt} = \frac{ds}{dt} \cdot \frac{1}{\rho} \cdot \hat{\mathbf{u}}_n \quad (12)$$

Equation (12) can then be simplified further to the following:

$$\vec{a} = a_t \cdot \hat{\mathbf{u}}_t + \frac{v_t^2}{\rho} \cdot \hat{\mathbf{u}}_n \quad (13)$$

Equation (13) is now in practical, usable form for calculations.

3.4 Examples with n,t-coordinates 1

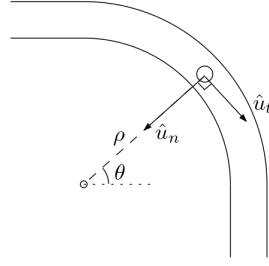


Figure 6: Representation of the problem, let $\rho = 100\text{ m}$, $v_t = 5\text{ m/s}$ and $a_t = 1\text{ m/s}^2$

$$\vec{a} = a_t \hat{u}_t + a_n \hat{u}_n \quad (14)$$

$$\hat{u}_t = \begin{pmatrix} \cos(45^\circ) \\ -\sin(45^\circ) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} \end{pmatrix} = \frac{1}{2}\sqrt{2}\hat{i} - \frac{1}{2}\sqrt{2}\hat{j} \quad (15)$$

$$\hat{u}_n = \begin{pmatrix} -\cos(45^\circ) \\ -\sin(45^\circ) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} \end{pmatrix} = -\frac{1}{2}\sqrt{2}\hat{i} - \frac{1}{2}\sqrt{2}\hat{j} \quad (16)$$

The unit vectors \hat{u}_t and \hat{u}_n represent the direction of the components of the acceleration and velocity vector. \hat{u}_t is tangential to the rotation and \hat{u}_n is towards the center of the curve.

Using this information:

$$a_n = \frac{v_t^2}{\rho} = \frac{(5\text{ m/s})^2}{100\text{ m}} = 0.25\text{ m/s}^2 \quad (17)$$

3.5 Example with n,t-coordiantes 2

In 3D the n,t-coordinates gain an extra component. This component is perpendicular to both the normal and tangential vector, and is referred to as the binomial vector. The following example deals with a problem in 3D which is reduced to two 2-dimensional problems to solve. 3-dimensional dynamics is outside the scope of the course and will be covered in advanced Dynamics in year 2 quarter 1.

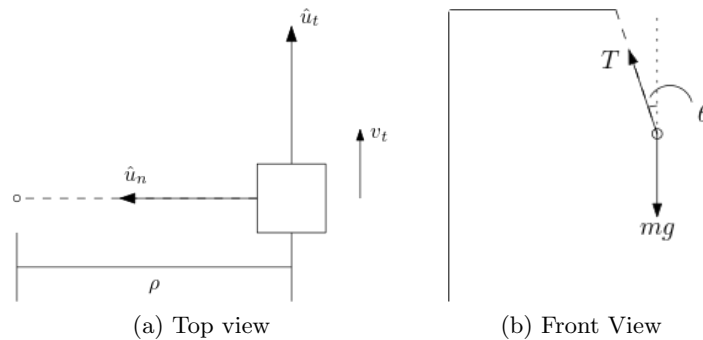


Figure 7: The problem reduced to two 2-dimensional, easier to solve situations

Looking at the 2D simplification of the problem from above shows a very typical circular motion. The distance ρ is the distance to the circle around which the mass rotates, and perpendicular to that the tangential component pointing in the direction of rotation. Figure [x] Shows the Free body diagram of the problem when observed from the front. From here it is possible to determine the equation of motion.

$$\Sigma F_x = -\sin(\theta) \cdot T = m\ddot{x} \quad (18)$$

$$\ddot{x} = -a_n = -\frac{v_t^2}{\rho} \quad (19)$$

After this it's a fairly typical solution solving for the value of T.

3.6 Example with unknown radius of curvature

In some case a problem presents a curved surface, where the radius of curvature ρ is not know. Since the surface is often curved as some function of x the radius of curvature can also be computed as a function of x . This can be done with the following equation:

$$\rho = \frac{(1 + (\frac{dy}{dx})^2)^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad (20)$$

The following example follows a problem where the the curved surface is a function of x .

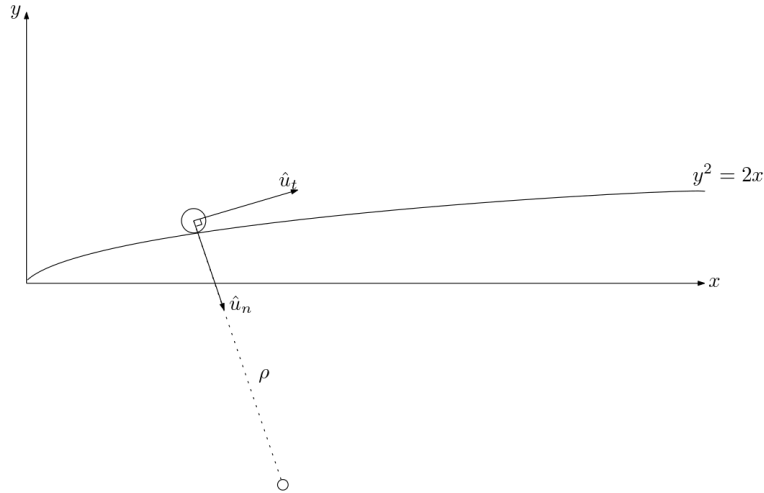


Figure 8: Curved surface y as a function of x

$$y^2 = 2x \Rightarrow y = \sqrt{2x} \quad (21)$$

$$\frac{dy}{dx} = \frac{\sqrt{2}}{2\sqrt{x}} \quad (22)$$

$$\frac{d^2y}{dx^2} = \frac{-\sqrt{2}}{4x\sqrt{x}} \quad (23)$$

Substituting equations (22) and (23) into (20) and simplifying gives the following:

$$\rho = \frac{-4x\sqrt{x} - 2\sqrt{x}\sqrt{1 + \frac{1}{2x}}}{\sqrt{2}} \quad (24)$$

This equation determines the radius of curvature ρ at any point x along the curve.