5 Lecture 5: Linear transformations and standard matrices (25/02/2020)

5.1 Matrix transformations

$$A\vec{x} = \vec{b} \tag{1}$$

$$\sum_{i=1}^{n} c_i \vec{x}_i = \vec{b} \tag{2}$$

The difference between equation (1) and (2) is just a matter of notation. However, a matrix equation does not have to be related to a linear combination of vectors. The matrix A can also be thought of as an object that 'acts' on a vector $\vec{x}$ to produce a new vector $A\vec{x}$. From this perspective, $A\vec{x} = \vec{b}$ amounts to finding all the vectors $\vec{x}$ in $\mathbb{R}^n$ space which are transformed to the vector $\vec{b}$ (or $T(\vec{x})$) in $\mathbb{R}^m$ space. The set $\mathbb{R}^n$ is called the domain and the set $\mathbb{R}^m$ is called the codomain. $T(\vec{x})$ is the range.

Theorem: let $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(\vec{x}) = A\vec{x}$. A vector $\vec{b}$ lies in the range of T if and only if the system $A\vec{x} = \vec{b}$ is consistent.

5.2 Linear transformation

A transformation is defined as a linear transformation if:

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \text{ for all vectors } \vec{u},\vec{v} \text{ in } \mathbb{R}^n. \tag{3}$$

$$T(c\vec{u}) = cT(\vec{u}) \text{ for all scalars } c \text{ in } \mathbb{R} \text{ and } \vec{u} \text{ in } \mathbb{R}^n. \tag{4}$$

Theorem: For any $m \times n$ matrix A, the matrix transformation $T(\vec{x}) = A\vec{x}$ is a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$.

Note: The zero vector will always map to another zero-vecotr with different dimensions. Equation (4) can be generalized as follows:

$$T(\sum_{i=1}^{n} c_i \vec{v}_i) = \sum_{i=1}^{n} c_i T(\vec{v}_i) \tag{5}$$

Geometrically, any linear transformation can be viewed as a mapping from any space $\mathbb{R}^n$ to any other space $\mathbb{R}^m$ where $n > m$ and the space will not curve in any way. The space can however translate, rotate, contract, expand, shear or be projected on a single line[1]

5.3 Numerical example of a linear Transformation and a non-linear transformation

Linear Transformation:

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - 3x_2 \\ 3x_1 - 5x_2 \\ -x_1 + 7x_2 \end{pmatrix}$$

Note that the vector $\vec{x}$ is now mapped from a 2 dimensional vector $\langle x_1, x_2 \rangle$ to a 3 dimensional vector $\langle x_1 - 3x_2, 3x_1 - 5x_2, -x_1 + 7x_2 \rangle$.

Non-linear transformation:

$$S\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 \cdot x_2 \\ x_1 - x_2 \end{pmatrix}$$

$$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 2 + x_1 \\ 3 - x_1 - x_2 \end{pmatrix}$$

[1]This is related to eigenvalues and eigenvectors, but that will be covered Linear Algebra 2

Note the multiplication in the first example and the extra constants not multiplied by a variable make these transformation non-linear. in the first example the space would be rotated because of the multiplication. The second example would map the zero-vector $\vec{0} = \langle 0, 0 \rangle$ to the point $\langle 2, 3 \rangle$. If we recall the definition, the zero-vector should always map to another 0 vector.

5.4 Standard matrices

Theorem: let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a unique $m \times n$ matrix A such that $T(\vec{x}) = A\vec{x}$. The columns of matrix A are the images under T of the standard unit vectors:

$$M_T = \begin{bmatrix} T(\hat{\boldsymbol{e}}_{\boldsymbol{1}}) & \cdots & T(\hat{\boldsymbol{e}}_{\boldsymbol{n}}) \end{bmatrix}$$

A is called the standard matrix of T.

A rotation about the origin with the angle ϕ is a linear transformation with the standard matrix:

$$M_T = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$