9 Lecture 9: Subspaces

9.1 Definition of a subspace

A subspace in $\mathbb{R}^n$ is a set of vectors W where:

1. $\vec{0} \in W$
2. if $\vec{u}$, $\vec{v} \in W$, then $\vec{u} + \vec{v} \in W$
3. if $\vec{u} \in W$ and $c \in \mathbb{R}$, then $c\vec{u} \in W$

Items 2 in the list can also be formulated differently as: All linear combinations of the vectors that span W will also be in W.

Theorem 1 *The span of a set of vectors* $\{\vec{v}_1, \cdots, \vec{v}_n\} \in \mathbb{R}^m$ *is a subspace of* $\mathbb{R}^m$.

Because of theorem 1 we can conclude that $\text{Span}\{W\} \in W$.

9.2 Null space and column space

The Null space Nul A is the set of all solutions to the homogeneous equation $A\vec{x} = \vec{0}$. To prove that the Nul space is indeed a subspace we have to prove that it statisfy the 3 criteria of a subspace noted earlier.

$$\text{let: } \vec{v}, \vec{w} \in \text{Nul } A, c \in \mathbb{R}$$
$$1)\ A\vec{0} = \vec{0} \tag{1}$$
$$2)\ A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = A\vec{0} + A\vec{0} = \vec{0} \tag{2}$$
$$3)\ A(c\vec{v}) = cA\vec{v} = c \cdot \vec{0} = \vec{0} \quad \square \tag{3}$$

The column space Col A of a matrix A is the span of the columns of A.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{then Col } A \text{ will be:}$$
$$\text{Span}\left\{ \begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \right\}$$

This can be written down more generally as:

$$\text{Col } A = \text{Span } \{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\} = \{x_1\vec{a}_1 + \cdots + x_n\vec{a}_n | x_1, \cdots, x_n \in \mathbb{R}\}$$

Note that $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$ or written in terms of a matrix vector product $A\vec{x} = \vec{b}$ is consistent if $\vec{b} \in \text{Col } A$.

Theorem 2 *Nul A of an $m \times n$ matrix A is a subspace of* $\mathbb{R}^n$

Theorem 3 *Col A of an $m \times n$ matrix is a subspace of* $\mathbb{R}^m$.

9.3 Null Space example

let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$A\vec{x} = \vec{0}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right]$$

$$\begin{cases} x_1 = -2x_3 \\ x_2 = -\frac{1}{2}x_3 \\ x_3 = \text{free} \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -2 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$$

$$\text{Thus: Nul } A = \text{Span}\left\{ \begin{pmatrix} -2 \\ -\frac{1}{2} \\ 1 \end{pmatrix} \right\}$$

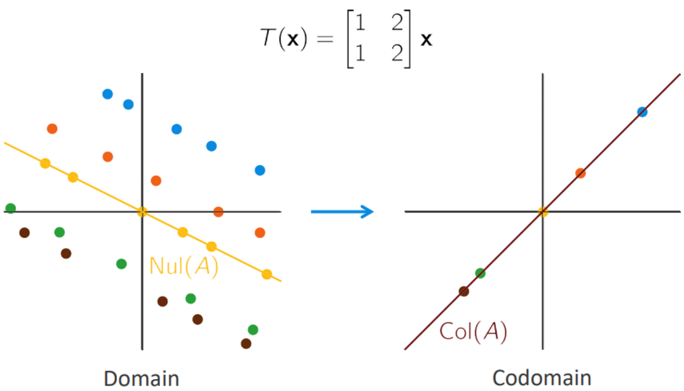

Figure 1: A visual example of column and null spaces

9.4 Basis of a subspace

A basis of a subspace W is defined as the set of vectors $\{\vec{v}_1, \cdots, \vec{v}_p\}$ in W which is both linearly independent and spans W. This is not a unique set of vectors; A subspace can have many different bases.

Theorem 4 *Different bases of the same subspace W of $\mathbb{R}^n$ will have the same number of vectors.*