# 6 Lecture 6: Matrix Operations (28/02/2020)

## 6.1 Types of matrices

There are several different types of matrices, described by there entries and rank. These are as follows:

- Zero matrix, all entries are 0
- Square matrix, $m \times n$ matrix where $m = n$
- Diagonal matrix, matrix containing only zeros and main entries
- Identity matrix, diagonal matrix where all main entries are 1
- Lower/Upper triangular matrix, a square matrix in echelon form.

Examples of these are[1]:

$$\text{Zero Matrix: } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{Diagonal matrix: } \begin{bmatrix} \blacksquare & 0 & 0 \\ 0 & \blacksquare & 0 \\ 0 & 0 & \blacksquare \end{bmatrix}$$

$$\text{Identity: } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Upper triangular matrix: } \begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \end{bmatrix}$$

## 6.2 Matrix multiplication

Theorem: Given two linear transformations $T : \mathbb{R}^p \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^m$, then the composite $S \circ T : \mathbb{R}^p \to \mathbb{R}^m$, defined by $(S \circ T)(\vec{x}) = S(T(\vec{x}))$ is also a linear transformation.

Proof that $S \circ T$ is indeed linear:
For a transformation to be linear it needs to statisfy the following 2 properties:

1. $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
2. $T(c\vec{u}) = cT(\vec{u})$

let $T : \mathbb{R}^p \to \mathbb{R}^n$ and $S : \mathbb{R}^n \to \mathbb{R}^m$ be linear transformations, let $\vec{u}$ and $\vec{v}$ be vectors and let $c$ be a scalar.

$$\begin{aligned} (S \circ T)(\vec{u} + \vec{v}) &= S(T(\vec{u}) + T(\vec{v})) \\ &= S(T(\vec{u})) + S(T(\vec{v})) \\ &= (S \circ T)(\vec{u}) + (S \circ T)(\vec{v}) \quad \square \end{aligned}$$

$$\begin{aligned} (S \circ T)(c\vec{u}) = S(cT(\vec{u})) &= cS(T(\vec{u})) \\ &= c(S \circ T)(\vec{u}) \quad \square \end{aligned}$$

The matrix multiplication of the matrices $A : m \times n$ and $B : n \times p$ is equal to the standard matrix of the composite mapping $S \circ T$, where $S(\vec{x}) = A\vec{x}$ and $T(\vec{x}) = B\vec{x}$. Note that $AB$ will be an $m \times p$

[1] These examples use a $3 \times 3$ matrix since it's easy to fit on the page for an example, but the matrices can be any $m \times n$ dimension.

matrix. In general matrix multiplication takes the following form:

$$
\begin{aligned}
&S:\mathbb{R}^n \to \mathbb{R}^m \quad \text{Where} \quad S(\vec{x}) = A\vec{x} \quad \text{and} \quad A: m\times n \\
&T:\mathbb{R}^p \to \mathbb{R}^n \quad \text{Where} \quad T(\vec{y}) = B\vec{y} \quad \text{and} \quad B: n\times p \\
&S\circ T:\mathbb{R}^p \to \mathbb{R}^m \quad \text{and} \quad A\cdot B: [m\times n][n\times p]
\end{aligned}
$$

What this means is that $S$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $T$ a linear transformation from $\mathbb{R}^p$ to $\mathbb{R}^m$. The matrix product associated with that linear transformation will be a matrix product of $A$ with dimension $n \times m$ and a matrix $B$ with dimensions $n \times p$, which will output a single matrix of dimensions $m \times p$.

Computing a specific entry of a matrix $AB$ can be done with the row column rule. The most general form of matrix multiplication looks like the following: entry $(i, j)$ of $AB$ equals $\text{row}_i(A)b_j$.

$$
\begin{aligned}
(AB)_{ij} &= a_{1i}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \\
&= \sum_{k=1}^{n} a_{ik}b_{kj}
\end{aligned}
$$

$$
\begin{bmatrix}
a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\
\vdots & & \vdots & & \vdots \\
a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\
\vdots & & \vdots & & \vdots \\
a_{m1} & \cdots & a_{mj} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\
\vdots & & \vdots & & \vdots \\
b_{i1} & \cdots & b_{ij} & \cdots & b_{ip} \\
\vdots & & \vdots & & \vdots \\
b_{n1} & \cdots & b_{nj} & \cdots & b_{np}
\end{bmatrix}
$$

There are some important algebraic properties to consider for matrix multiplication. Most of these will seem familiar since they look a lot like regular algebraic expressions, however there is 1 key difference, mainly that matrices are non-communative. The rules for manipulating matrices algebraicly are as follow:

$$AB \neq BA \tag{1}$$
$$A(BC) = (AB)C \tag{2}$$
$$A(B + C) = AB + AC \tag{3}$$
$$(B + C)A = BA + CA \tag{4}$$
$$I_m A = A = AI_n \tag{5}$$
$$r(AB) = (rA)B = A(rB) \tag{6}$$
$$A^k = AA\cdots A\text{where A is repeated } k \text{ times} \tag{7}$$
$$A^0 = I \tag{8}$$

Some common mistakes that are made:

$$AB = AC \quad \text{does not imply that } B = C. \tag{9}$$
$$AB = 0 \quad \text{does not imply that either } A \text{ or } B \text{ is always a zero-matrix, thought they could be.} \tag{10}$$
$$(A + B)^2 = A^2 + AB + BA + B^2 \quad \text{note that } AB \neq BA \tag{11}$$

## 6.3 Numerical example of matrix multiplication

let $A$ be an $2 \times 3$ matrix and let $B$ be a $3 \times 2$ matrix of the following form:

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0+2+0 & 1-2+6 \\ 0+0+0 & -1+0+4 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 0 & 3 \end{bmatrix}$$

Recall that this is the same as repeated matrix-vector multiplication, and again the same as the dot product between a column of the second matrix and a row of the first matrix.

## 6.4 Transpose of a matrix

Might write this here, or might write it in Lecture 7 Notes haven't decided yet.