3 Lecture 3: Matrix-vector products and solution sets (18/02/2020)

3.1 general form of vectors and matrices

Matrices can be thought of an $m \times n$ grid of numbers. A column vector is thus by extension of the form $1 \times n$. let the $\vec{x}$ be a column vector of n entries and A a $m \times 1$ matrix. Then:

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \end{bmatrix} \quad \text{and} \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

The matrix vector product is defined as the following:

$$\vec{y} = A\vec{x} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ a_{21}x_1 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix}$$

It is thus possible to compute the matrix-vector product using this definition, it's worth noting that $A\vec{x}$ of the form $m \times n$ will always be of the rank $\vec{x} \in \mathbb{R}^n$ and $\vec{y} = \mathbb{R}^m$. Which is to say, The matrix vector product of an $m \times n$ matrix and vector of n entries will output a vector with m entries.

A quick numerical example will follow:

$$\begin{bmatrix} 2 & 0 \\ 3 & 2 \\ 1 & 2 \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 1 \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2+0 \\ 3+6 \\ 1+6 \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \\ 7 \end{pmatrix}$$

3.2 Algebraic rules for matrix vector multiplication

There are some important algebraic rules to consider for matrix-vector multiplication. All properties relating to manipulation of matrix-vector multiplication can be derived using these 2 definitions. let A be an $m \times n$ matrix, $\vec{v}$, $\vec{u}$ $1 \times n$ vectors and c, d a scalar.

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} \tag{1}$$

$$A(c\vec{u}) = c(A\vec{u}) \tag{2}$$

Combining equation (1) and (2) gives the following, which is also sometimes given as a third rule:

$$A(c\vec{u} + d\vec{v}) = c(A\vec{u}) + d(A\vec{v}) \tag{3}$$

It's worth noting that vector-matrix multiplication can alternetivly be interpreted as the dot product between a row in the matrix and the column vector $\vec{v}$. Thinking about it this way can save time when computing matrix-vector products.

3.3 Notations for matrix-vector products

All the following forms are equivalant:

$$\begin{cases} x + 2y + 3z = 3 \\ 2x - y - 2z = 4 \\ -x + y + 3z = 5 \end{cases} \Leftrightarrow x \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + y \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + z \cdot \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix} \Leftrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & -2 \\ -1 & 1 & 3 \end{bmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$$

3.4 Homogenous and Non-homogenous systems

A system is defined to be homogenous if $A\vec{x} = \vec{0}$. Conversly a system is defined to be non-homogenous if $A\vec{x} = \vec{y}$ where $\vec{y} \neq 0$. Homogenous systems have some interesting properties. These are as follows:

- Homogenous systems are always consistent, and thus always have a solution
- Homogenous systems have a non-trivial[1] solution if and only if it has 1 free variable.

Some examples of homogenous and non-homogenous systems will follow:

$$\left[\begin{array}{ccc|c} 3 & 4 & -5 & 0 \\ -3 & -2 & 1 & 0 \\ 6 & 1 & 4 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ 2x_3 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 4 & -5 & 1 \\ -3 & -2 & 1 & 1 \\ 6 & 1 & 4 & -5 \end{array}\right] \sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 - x_3 \\ 1 + 2x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Note that the part depending on the free variable does not change between homogenous and non-homogenous systems. Graphically this can be interpreted as shifting the line of infinite solutions away from the origin by some vector (illustrated in figure 1 below). For this example that vector is $\langle -1, 1, 0\rangle$. Because of this general solution to a non-homogenous system can be written in the form $\vec{x} = \vec{x}_p + \vec{x}_h$, where $\vec{x}_p$ is the particular solution and $\vec{x}_h$ the homogenous solution[2]. It can be proven that these are any and all solutions to the system. (maybe insert proof later idk yet)

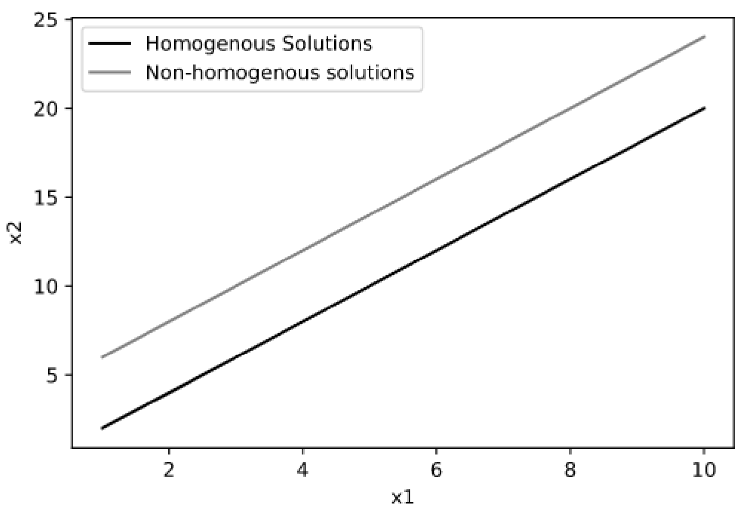

Figure 1: Comparison between the solution sets $\vec{v}_1 = 2\vec{v}_2$ and $\vec{v}_1 = 2\vec{v}_2 + \langle 0, 4\rangle$. Note that the second solution set is offset from the origin by the vector $\vec{b} = \langle 0, 4\rangle$

[1] a trvial solution is a solution where all variables are 0. It's technically true but doesn't tell us anything about the system.

[2] This looks very similar to the solution of a non-homogenous differential equation (Analyse 1)