2 Lecture 2: Spans and vector equations (14/02/2020)

2.1 vectors in $\mathbb{R}^n$ space

Any vector in $\mathbb{R}^n$ space is usually denoted with the following form:

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

The important algebraic properties of any vector in $\mathbb{R}^n$ space will be listed below.

$$\vec{u} + \vec{v} = \vec{v} + \vec{u} \tag{1}$$
$$\vec{u} + -\vec{u} = -\vec{u} + \vec{u} = 0 \tag{2}$$
$$c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v} \tag{3}$$
$$(c + d)\vec{u} = c\vec{u} + d\vec{u} \tag{4}$$
$$c(d\vec{u}) = cd\vec{u} \tag{5}$$

It is also worth knowing that a vector can be divided by it's own magnitude in which case all of it's components become smaller then 1. This proces is called normalizing and creates a vector of size 1 which effectivly only the direction of said vector.

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|}$$

2.2 vector equations

Vector equations are a different way of representing a given linear system. they take the following form:

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$
$$\vec{y} = \sum_{i=1}^{n} c_i\vec{v}_i$$

$\vec{y}$ is refered to as the linear combination of $\vec{v}_1 \cdots \vec{v}_n$. The c_i terms are referred to as the weight.
The vector equation for $\vec{y}$ can also be representad as a $1 \times n$ matrix as follows:

$$\left[\begin{array}{cccc|c} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v_n} & \vec{y} \end{array}\right]$$

As mentioned before linear systems can also be expressed as vector equations. This looks like the following:

$$\begin{cases} x_1 + 5x_2 + 3x_3 & = 1 \\ 2x_1 + x_2 + 15x_3 & = 8 \end{cases} \quad \Leftrightarrow \quad \begin{pmatrix} 2x_1 + 3x_2 \\ -x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

Which can be rewritten as:

$$x_1 \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

The advantage of vector equations is that they are easily graphically interpretable in 2 or 3 dimensional space. They also can give information on the possible solution of a given system of equation. When lines are parrallel they have no solution. When lines cross in a singular point the system has a unique solution.

2.3 Vector spans

given a set of vectors $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ in $\mathbb{R}^n$. The set of all linear combinations is denoted by:

$$\text{Span}\left\{\vec{v}_1, \quad \vec{v}_2, \quad \cdots, \quad \vec{v}_n\right\}$$

The span is the subset of all vectors that can be written as $\sum_{i=1}^{n} c_i\vec{v}_i$. This means the span is just the collection of any given point in an $\mathbb{R}^n$.that can be reached with the given vectors.

$$\text{Span}\left\{\vec{v}_1, \quad \vec{v}_2\right\} \quad \text{where} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \vec{v}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\text{All possible vectors are of the form: } \begin{pmatrix} a \\ 0 \end{pmatrix}, \ a \in \mathbb{R}$$

Graphically this means that all possible values of the linear combination of these vectors are somewhere on a horizontal line through the origin. When instead the following vectors where given:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The span would be all possible points on a 2 dimensional play, or represented as a vector:

$$\begin{pmatrix} a \\ b \end{pmatrix}, \ a, b \in \mathbb{R}$$

The span of a vector can either be an entire plane, a line or a single point in any $\mathbb{R}^n$ space. To reach any given point in $\mathbb{R}^n$ space atleast n vectors are required.