7 Lecture 7: Transposed matrices and inverted matrices (03/03/2020)

7.1 Transposed matrices

A transpose is a type of matrix operation where the rows and columns of a matrix are interchanged. This takes the following form for a 2×2 matrix, but can be extenden to any $m \times n$ matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{Then} \quad A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

Some algebraic rules for matrix transposition are:

$$(A^T)^T = A \tag{1}$$
$$(A+B)^T = A^T + B^T \tag{2}$$
$$(rA)^T = rA^T \tag{3}$$
$$(AB)^T = B^T A^T \tag{4}$$

Quick proof for equation (1):

$$(A^T)^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A \quad \square$$

It's worth noting that equation (4) may not intuitivly make sensen based on regular algebraic rules, however when we consider that the columns in B must be equal to the rows in A for matrix multiplication to make sense it's easy to see that the order B and then A makes sense. if A,B are $m \times n$ and $n \times m$ matrices respectivly, the transposed matrices A^T and B^T must be of the form $n \times m$ and $p \times n$.

7.2 Inverse matrices

Theorem: if A is an invertible $m \times n$ matrix, then for each $\vec{b}$ in $\mathbb{R}^n$ the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.

(a) A^{-1} doesn't exist

(b) A^{-1} does exist

Figure 1: Representation of the existence of an inverse matrix in 2-dimensional space

Note figure 1 (a). There is no single unique vector $\vec{v}$ that will project to the x-axis under the given matrix transformation, rather there is an entire line of vectors which will give the same solution, making the solution non-unique and by extension the matrix non-invertable. Figure 1 (b) represents a rotation of space by $\frac{\pi}{2}$ rad. The transfomation of vector will always be unique, making the matrix invertible.

Some rules to consider for inverting matrices:

$$(A^{-1})^{-1} = A \tag{5}$$
$$(AB)^{-1} = B^{-1}A^{-1} \tag{6}$$
$$(A^T)^{-1} = (A^{-1})^T \tag{7}$$

Proof for equation (6) and (7):

$$(AB)^{-1} \cdot AB = I_n$$
$$B^{-1}A^{-1} \cdot AB = B^{-1}I_nB = B^{-1}B = I_n \quad \square$$

$$(A^T)^{-1}A^T = I_n$$
$$(A^{-1})A^T = (AA^{-1})^T = I_n^T = I_n \quad \square$$

Theorem: A matrix $A : n \times n$ is invertible if and only if A is row equivelant to I_n.

This means the matrix A should have exactly n pivots since there can be no free variables in an invertable matrix.

7.3 Special case of a 2×2 matrix

Computing the inverse matrix of any given 2×2 matrix is much faster then a larger $n \times n$ matrix due to a computational shortcut. This looks like the following:

$$\text{let } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$
$$\text{Matrix is invertible, thus:}$$
$$ad - cb \neq 0$$
$$A^{-1} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$A^{-1}A = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - cb} \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \quad \square$$

7.4 Example algebraic manipultion of matrices

let A, B, C be invertible $n \times n$ matrices.
Does the $C^{-1}(A + x)B^{-1} = I_n$ have a solution?

$$CC^{-1}(A + x)B^{-1}B = CI_nB$$
$$I_n(A + x)I_n = CB$$
$$A + x = CB$$
$$x = CB - A$$

Note that the order of multiplication is important since $AB \neq BA$.

Compute the inverted matrix A^{-1}

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\text{To prevent having to do Gaussian elemination twice:}$$
$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$$

Note that this is of the form: $\begin{bmatrix} A \mid I_n \end{bmatrix} \sim \begin{bmatrix} I_n \mid A^{-1} \end{bmatrix}$