13 Lecture 13: The Gram-Schmidt process (24/03/2020)

13.1 The Gram-Schmidt process

The Gram-Schmidt process is a simple process that can take any random basis for a subspace W, and produces and orthogonal or orthonormal basis for any random subspace of $\mathbb{R}^n$.

Theorem 1 *Given a basis $\{\vec{x}_1, \cdots, \vec{x}_p\}$ for a non-zero subspace W of $\mathbb{R}^n$ define:*

$$\begin{aligned}
\vec{v}_1 &= \vec{x}_1 \\
\vec{v}_2 &= \vec{x}_2 - \frac{\langle \vec{x}_2 | \vec{v}_1 \rangle}{\langle \vec{v}_1 | \vec{v}_1 \rangle}\vec{v}_1 \\
\vec{v}_3 &= \vec{x}_3 - \frac{\langle \vec{x}_3 | \vec{v}_1 \rangle}{\langle \vec{v}_1 | \vec{v}_1 \rangle}\vec{v}_1 - \frac{\langle \vec{x}_3 | \vec{v}_2 \rangle}{\langle \vec{v}_2 | \vec{v}_2 \rangle}\vec{v}_2 \\
&\vdots \\
\vec{v}_p &= \vec{x}_p - \frac{\langle \vec{x}_p | \vec{v}_1 \rangle}{\langle \vec{v}_1 | \vec{v}_1 \rangle}\vec{v}_1 - \frac{\langle \vec{x}_p | \vec{v}_2 \rangle}{\langle \vec{v}_2 | \vec{v}_2 \rangle}\vec{v}_2 - \cdots - \frac{\langle \vec{x}_p | \vec{v}_{p-1} \rangle}{\langle \vec{v}_{p-1} | \vec{v}_{p-1} \rangle}\vec{v}_{p-1}
\end{aligned}$$

Then $\{\vec{v}_1, \cdots, \vec{v}_p\}$ is an orthogonal basis for W. In addition Span$\{\vec{v}_1, \cdots, \vec{v}_k\}$ = Span$\{\vec{x}_1, \cdots, \vec{x}_k\}$ for $1 \leq k \leq p$

13.2 Orthonormal Basis from the Gram-Schmidt process

The basis found from the Gram-Schmidt process in theorem 1 can be normalized to obtain an orthonormal basis.

$$\begin{gathered}
\text{orthogonal basis: } \{\vec{v}_1, \cdots, \vec{v}_p\} \\
\text{orthonormal basis: } \{\vec{u}_1, \cdots, \vec{u}_p\} \\
\vec{u}_i = \frac{\vec{v}_i}{||\vec{v}_i||} \qquad (1)
\end{gathered}$$