2 Lecture 2 (13/02/2020)

2.1 Dynamics of pulley mechanisms

Analyzing the dynamics of pulley mechanisms will raise some interesting questions. Say there is a pulley mechanism as shown in figure 1 below. At what velocicty would mass A move upwards as a result of pulling mass B downwards. Intuition tells us that these velocities are indeed different but by what factor? Setting up just the equations of motion does not solve this problem since there are 2 equations with 3 variables. This is an unsolveable system. The following example will discuss techniques that can be applied in analyzing the geometry of these kinds of systems to create an extra constraint equation to solve the system. The constraint equation represents a boundary under which the system can excist as described.

2.2 Example problem involving pulleys

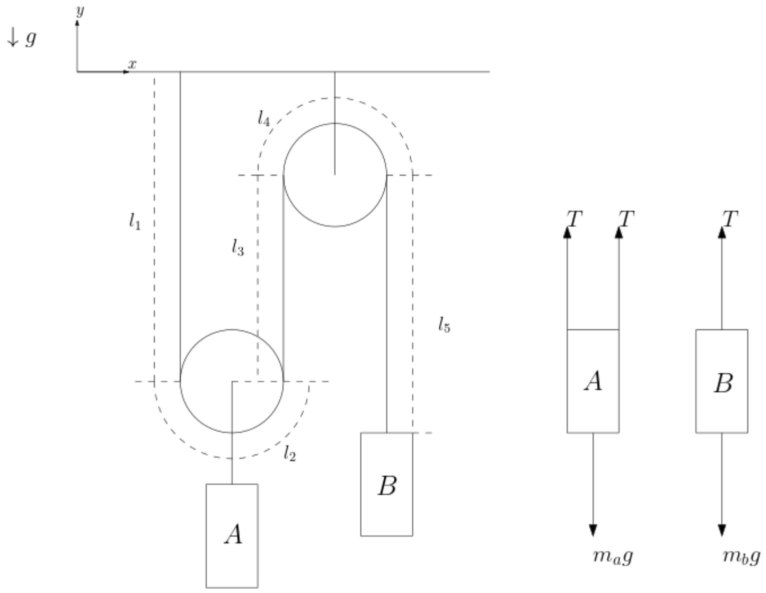

Figure 1: Free Body Diagram of the Pulley mechanism to be analyzed for the example

$$\Sigma F_{ay} = 2T - m_a g = m_a \ddot{y}_a \tag{1}$$
$$\Sigma F_{by} = T - m_b g = m_b \ddot{y}_b \tag{2}$$

Note that the problem has 3 unknown variables (T, $\ddot{y}_a$ and $\ddot{y}_b$). There are only 2 equations of motion, which would make this system of equations unsolveable. To correct for this a third eqaution is introduced. The rope is divided into 5 lengths. When mass b is moved downwards l_5 gets longer. mass a moves upwards which means l_3 and l_1 both get shorter. Since the total length of the cable does not change[1].

$$l_t = l_1 + \cdots + l_5 \Leftrightarrow l_t = \sum_{i=1}^{5} l_i \tag{3}$$

We can conclude that the length of l_2 and l_4 does not change since the size of the pulleys does not magically change. The total length must also stay constant since it's still the same rope. The

[1]Mechanics of materials tells us that the length does in fact change as a result of tension in the rope, but since this deformation is much smaller then the total length of the rope it's not worth considering this elongation.

rate of displacement and rate of change in velocity of the parts of the ropes are also equal to one another because the equality holds when both sides of an equation are derrived. It is also know that the change in velocity of l_1 and l_3 must be equal to $\ddot{y}_a$ but in opposite direction. This also holds for l_5 and $\ddot{y}_b$.

$$f(x) = g(x) \tag{4}$$
$$\frac{d}{dx}\Big(f(x) - g(x)\Big) = 0 \Rightarrow f'(x) = g'(x) \tag{5}$$

From this we can conclude that:

$$l_t = \sum_{i=1}^{5} l_i \Leftrightarrow \ddot{l}_t = \sum_{i=1}^{5} \ddot{l}_i \tag{6}$$
$$\ddot{l}_t = 0 \qquad \ddot{l}_1 = -\ddot{y}_a \qquad \ddot{l}_2 = 0 \qquad \ddot{l}_3 = -\ddot{y}_a \qquad \ddot{l}_4 = 0 \qquad \ddot{l}_5 = -\ddot{y}_b \tag{7}$$

subsituting equation (7) into equation (6) will give the following:

$$0 = -\ddot{y}_a + -\ddot{y}_a + -\ddot{y}_b \tag{8}$$
$$\ddot{y}_b = -2\ddot{y}_a \tag{9}$$

Equation (9) is the constraint equation. This equation determins the condition under which the system is consistent with itself. There are now 3 variables and 3 equations making this system solveable.

2.3 Vectors and their properties

Much like calculus, linear algebra proves to be a powerfull tool for solving dynamics problems. Examples of this are expressing forces as a vector and equations of motion as systems of equations. Some important properties of vectors will be listed below. These include vector addition, vector multiplication with cross- and dotproduct and some general algebraic rules for manipulating vectors. All defenition are given in 3D space. Since this is the most common use for use of vectors in mechanics, since physical real space has 3 dimensions.

$$\vec{r}_b = \vec{r}_a + \vec{r}_{b/a} \tag{1}$$
$$c\vec{a} = ca_1\hat{i} + ca_2\hat{j} + ca_3\hat{k} \tag{2}$$
$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\vec{a}||\vec{b}|\cos(\theta) \tag{3}$$
$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k} \tag{4}$$
$$|\vec{a} \times \vec{b}| = |\vec{a}|\vec{b}||\sin(\theta) \tag{5}$$

Also note that the accelaration vector ($\vec{a}$) has some interesting properties when considering circular motion. The vector can be split up is 2 components. 1 tangential to the circular motion which could be considered the instantaneous accelaration and 1 perpendicular to that towards the center of the circle. This is the apparant cetrifugal force in circular motions. More on this topic will discussed in a later lecture.

2.4 State dependent accelaration

Consider the velocity and accelaration as described by a derrivative. Note that both of these are time dependent. In some cases a dynamical system is described by it's current state[2] rather then by the passage of time (example: $a = kv^2$, $a = ks$, where k is some random constant). It is usefull for cases like this to have an equation which describes the accelaration in terms of velocity rather then time. Note that a mathematician would probably shoot me for the "derrivation"[3] I am about to give but it works for all intents of purposes.

$$a = \frac{dv}{dt} \qquad v = \frac{ds}{dt} \tag{6}$$

$$dt = \frac{dv}{a} \qquad dt = \frac{ds}{v} \tag{7}$$

$$\frac{dv}{a} = \frac{ds}{v} \tag{8}$$

$$ads = vdv \Leftrightarrow a = v\frac{dv}{ds} \tag{9}$$

Note that in equation (9) $\frac{dv}{ds}$ is the derrivative of velocity with respect to displacement. If v is constant equation (9) can be further reduced as follows:

$$\int a\, ds = \int v\, dv \tag{10}$$

$$\int a\, ds = \frac{1}{2}v^2 \tag{11}$$

[2]The state of any dynamical system is it's posistion and velocity, it's accelaration is not part of the state.

[3]Treating infitesimals as fraction is mathematically considered bad practice since it can lead to nonsensical math such as $\left(\frac{dy}{dx}\right)^2 = \frac{(dy)^2}{(dx)^2}$. That being said the notation does allow for fraction-like manipulations that are in fact correct (such as $f'(x) = \frac{df(x)}{dx} \Leftrightarrow f(x) = \int f'(x)\, dx$). The real derrivation for $ads = vdv$ would be far more rigoureus.