4 Lecture 4: Linear Independence (21/02/2020)

4.1 Defenition of linear depency in $\mathbb{R}^n$ space

A set of vectors $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is linearly independent only if the homogenous equation has a non-trivial solution for $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}$. If there is no non-trivial solution the set is linearly dependent. Thus, the set of a single vector $\vec{v}$ is only linearly dependent if the vector $\vec{v}$ is the zero-vector. This can be extended to higher dimensions by recognizing that any set containing the zero-vector will always be linearly dependent.

This means that in practice a set of vectors is linearly dependent if at least one of the vectors in the set can be written as a linear combination of the other vectors in the set. Otherwise the set is linearly independent. This does not imply in any way that any vector $\vec{v}_k$ in a random set can be written as a linear combination of other vectors in the set. A set only needs to contain a single vector which can be written as a linear combination of the other vectors in the set to be considered a linearly dependent set. Through example the following relation can also be found: if there is a set of p vectors in $\mathbb{R}^n$ space where $p > n$, then the set will always be linearly dependent.

4.2 Some examples

let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \in \mathbb{R}^4$. Prove that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ are both linearly dependent if $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

Let's assume the linear depency of the first set is of the form $\vec{v}_1 = c_1\vec{v}_2 + c_2\vec{v}_3$
If this extended to the second set in $\mathbb{R}^4$ it can be found that the second set also has to be linearly dependent since the fourth vector can be scaled by 0 as follows: $\vec{v}_1 = c_1\vec{v}_2 + c_1\vec{v}_3 + 0 \cdot \vec{v}_4$

let $\vec{v}_1 = \langle 1, -1, 4\rangle$, $\vec{v}_2 = \langle -3, 9, -6\rangle$ and $\vec{v}_3 = \langle 5, -7, h\rangle$. For which value of h is this set linearly independent?
The set is linearly dependent if there is a non-trivial solution for $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}$. Thus:

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ h \end{pmatrix} \right\} \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ -1 & -5 & 5 & 0 \\ 4 & 7 & h & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & h-6 & 0 \end{array}\right]$$

From this we can see that the third column will have a pivot when $h - 6 \neq 0$. Thus: The set is linearly dependent when $h = 6$ since this only leaves trivial solutions, and linearly independent for $h \neq 6$, since this leaves an unique of non-trivial solutions.