11 Lecture 11: Inner product and orthogonality (17/03/2020)

11.1 The inner product

let $\vec{u}, \vec{v} \in \mathbb{R}^n$, where:

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \ \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix},$$

The matrix-vector product $\vec{u}^T\vec{v}$ outputs a 1×1 matrix, which is just a single number (or scalar in Linear algebra terms). This operation between to column vectors is referred to as the dot-poduct or inner-product, and is usually denoted as follows:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i \tag{1}$$

The algebraic rules for the dot product are:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u} \tag{2}$$

$$(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w} \tag{3}$$

$$(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{u}) \tag{4}$$

$$\vec{u} \cdot \vec{u} \geq 0, \text{ Where } \vec{u} \cdot \vec{u} = 0 \text{ iff } \vec{u} = 0 \tag{5}$$

The length of any vector in $\mathbb{R}^n$ space is defined as:

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{\sum_{i=1}^{n} v_i^2} \tag{6}$$

$$||\vec{v}||^2 = \vec{v} \cdot \vec{v} \tag{7}$$

Notice that the lenght of a vector has linear properties just like the vector itself:

$$||c\vec{v}|| = |c| \, ||\vec{v}|| \tag{8}$$

11.2 normalizing a vector

Normalizing a vector will output a vecotr referred to as a unit vector. This vector (denoted with a hat rather then a regular vector arrow) has a length of 1 and represents nothing but the direction of the original vector.

$$\hat{u} = \frac{\vec{u}}{||\vec{u}||} \tag{9}$$

11.3 Distance between points in $\mathbb{R}^n$

The distance between points in any $\mathbb{R}^n$ space is defined as follows: let $\vec{u}, \vec{v} \in \mathbb{R}^n$. The distance between $\vec{u}$ and $\vec{v}$, denoted as: $\operatorname{dist}(\vec{u}, \vec{v})$ is the length of the vector $\vec{u} - \vec{v}$.

$$\operatorname{dist}(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}|| \tag{10}$$

$$= \sqrt{\sum_{j=1}^{n} (\sum_{i=1}^{n} (u_i - v_i)_j)^2} \tag{11}$$

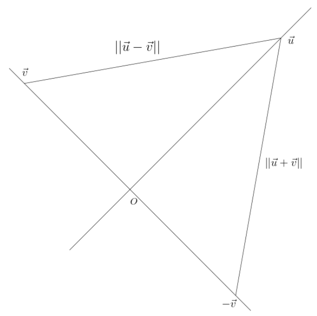

Figure 1: Representation of 2 perpendicular vectors $\vec{u}$ and $\vec{v}$ in 2D space.

11.4 orthogonal vectors

The vectors $\vec{u}$ and $\vec{v}$ (in any $\mathbb{R}^n$ space) are perpendicular (or orthogonal in more then 2 dimensional space) iff $||\vec{u}-\vec{v}|| = ||\vec{u}-(-\vec{v})||$.

$$\begin{aligned}(\text{dist } (\vec{u}, -\vec{v})^2) &= ||\vec{u}+\vec{v}||^2 \qquad (12)\\ &= ||\vec{u}||^2 + ||\vec{v}||^2 + 2\vec{u}\cdot\vec{v} \qquad (13)\end{aligned}$$

$$\begin{aligned}(\text{dist } (\vec{u}, \vec{v})^2) &= ||\vec{u}-\vec{v}||^2 \qquad (14)\\ &= ||\vec{u}||^2 + ||\vec{v}||^2 - 2\vec{u}\cdot\vec{v} \qquad (15)\end{aligned}$$

These distances are equal iff $2\vec{u}\cdot\vec{v} = -2\vec{u}\cdot\vec{v}$. The only case where this can happen is if $2\vec{u}\cdot\vec{v} = 0$, thus: 2 vectors in any $\mathbb{R}^n$ space are orthogonal iff $\vec{u}\cdot\vec{v} = 0$[1].

11.5 Orthogonal complements

let W be a subspace in $\mathbb{R}^n$. If a vector $\vec{z}$ is orthagonal to every vector in W, then $\vec{z}$ is referred to as the orthogonal compliment to W, and denoted as $W^\perp$. Notes:

1. $\vec{x} \in W^\perp$ iff $\vec{x}$ is orthogonal to every single vector in W.
2. $W^\perp$ is a subspace of $\mathbb{R}^n$.

Theorem 1 *let A be an $m \times n$ matrix. The orthogonal complement of the row space of A os the null space of A and the orthagonal complement of the column space of A is the null space of A^T.*

$$(Row\ A)^\perp = Nul\ A \qquad (16)$$
$$(Col\ A)^\perp = Nul\ A^T \qquad (17)$$

11.6 Orthogonal and orthonormal sets

A set of vectors $\{\vec{u}_1, \cdots, \vec{u}_p\} \in \mathbb{R}^n$ is said to be orthagonal if $\vec{u}_i \cdot \vec{v}_j = 0$ whenever $i \neq j$. An orthogonal basis is a basis to a subspace that is also an orthogonal set.

Theorem 2 *Let $\{\vec{u}_1, \cdots, \vec{u}_p\}$ be an orthagonal basis to the subsapce W of $\mathbb{R}^n$. For each $\vec{y}$ in W the weights in the linear combination:*

$$\vec{y} = \sum_{i=1}^{p} c_i \vec{u}_i \qquad (18)$$

are given by:

$$c_j = \frac{\vec{y}\cdot\vec{u}_j}{\vec{u}_j\cdot\vec{u}_j} \qquad (j = 1, \cdots, p) \qquad (19)$$

[1] $\vec{0}^T\vec{v} = 0$, $\vec{u}^T\vec{0} = 0$ thus every vector is considered to be orthogonal with the zero vector.

A set of vectors $\{\ \vec{u}_1, \cdots, \vec{u}_p\}$ is orthonormal if it's an orthagonal set of unit vectors. The easiest example of this is the set of unit vectors describing $\mathbb{R}^n$ cartesian space:

$$\{\hat{\boldsymbol{e}}_{\mathbf{1}}, \cdots, \hat{\boldsymbol{e}}_{\boldsymbol{n}}\} \in \mathbb{R}^n \tag{20}$$

Theorem 3 *An $m \times n$ matrix has orthonormal columns iff $U^T U = I$*

Theorem 4 *Let U be an $m \times n$ matrix with orthonormal columns. let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then:*

1. $||U\vec{x}|| = ||\vec{x}||$
2. $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$
3. $(U\vec{x}) \cdot (U\vec{y}) = 0$ *iff* $\vec{x} \cdot \vec{y} = 0$

Note that the mapping $\vec{x} \to U\vec{x}$ preserves lengths and orthogonality of vectors.