

2 Lecture 2: Spans and vector equations (14/02/2020)

2.1 vectors in \mathbb{R}^n space

Any vector in \mathbb{R}^n space is denoted with the following form:

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

The important algebraic properties of any vector in \mathbb{R}^n space will be listed below.

$$\vec{u} + \vec{v} = \vec{v} + \vec{u} \quad (1)$$

$$\vec{u} + -\vec{u} = -\vec{u} + \vec{u} = 0 \quad (2)$$

$$c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v} \quad (3)$$

$$(c + d)\vec{u} = c\vec{u} + d\vec{u} \quad (4)$$

$$c(d\vec{u}) = cd\vec{u} \quad (5)$$

It is also worth knowing that a vector can be divided by its own magnitude in which case all of its components become smaller than 1. This process is called normalization and creates a vector of size 1 which only represents the direction of said vector.

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|}$$

2.2 vector equations

Vector equations are a different way of representing a given linear system. They take the following form:

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$$

$$\vec{y} = \sum_{i=1}^n c_i\vec{v}_i$$

\vec{y} is referred to as the linear combination of $\vec{v}_1 \dots \vec{v}_n$. The c_i terms are referred to as the weight.

The vector equation for \vec{y} can also be represented as a $1 \times n$ matrix as follows:

$$[\vec{v}_1 \quad \vec{v}_2 \quad \dots \quad \vec{v}_n \mid \vec{y}]$$

As mentioned before linear systems can also be expressed as vector equations. This looks like the following:

$$\begin{cases} x_1 + 5x_2 + 3x_3 &= 1 \\ 2x_1 + x_2 + 15x_3 &= 8 \end{cases} \Leftrightarrow \begin{pmatrix} 2x_1 + 3x_2 \\ -x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

Which can be rewritten as:

$$x_1 \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

The advantage of vector equations is that they are easily graphically interpretable in 2 or 3 dimensional space. They can also give information on the possible solution of a given system of equations. When lines are parallel they have no solution. When lines cross in a singular point the system has a unique solution.

2.3 Vector spans

given a set of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ in \mathbb{R}^n . The set of all linear combinations is denoted by:

$$\text{Span} \{ \vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \}$$

The span is the subset of all vectors that can be written as $\sum_{i=1}^n c_i \vec{v}_i$. This means the span is just the collection of any given point in an \mathbb{R}^n space. that can be reached with the given vectors.

$$\text{Span} \{ \vec{v}_1, \vec{v}_2 \} \quad \text{where} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \vec{v}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\text{All possible vectors are of the form: } \begin{pmatrix} a \\ 0 \end{pmatrix}, a \in \mathbb{R}$$

Graphically this means that all possible values of the linear combination of these vectors are somewhere on a horizontal line through the origin. When instead the following vectors were given:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The span would be all possible points on a 2 dimensional plane, or represented as a vector:

$$\begin{pmatrix} a \\ b \end{pmatrix}, a, b \in \mathbb{R}$$

The span of a vector can either be an entire plane, a line or a single point in any \mathbb{R}^n space¹. To reach any given point in \mathbb{R}^n space atleast n vectors are required.

¹Or something more abstract when considering a vector space with more than 3 dimensions since this cannot graphically interpreted.