2 Linear Algebra 2 Lecture 2: Application of Determinants (24/04/2020)

2.1 Properties of the determinant

The determinant is <u>not</u> a linear mapping, but it does behave similarly to one. Let A and B be $n \times n$ matrices, then:

- $|cA| = c^n|A|$ for each $c \in \mathbb{R}$
- $|A^T| = |A|$
- A^{-1} exists if $|A| \neq 0$
- $|A^{-1}| = \frac{1}{|A|}$
- $|AB| = |A| \cdot |B|$

2.2 The determinant as volume and area

When working with 2 or 3 dimensional space the can be interpreted geometrically as follows:

- For a 2×2 matrix $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix}$ this is the area of a parallelogram with vertices $\vec{0}$, $\vec{a} + 1$, $\vec{a}_2$ and $\vec{a}_1 + \vec{a}_2$.
- For a 3×3 matrix $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$ this is the volume of a parralelopiped.

Generalizing this to any n dimensional space we can thus conclude that the determinant in fact represents the n-dimensional volume of the generalized parralelopiped with the origin as vertex and the columns of A as the n-vertices adjacent to the origin.

2.3 Volume, area and linear mappings

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix A. Suppose $S \subset \mathbb{R}^n$ has n-dimensional volume denoted by $\text{Vol}(S)$. Then $T(S)$ has the volume $|\det(A)|\text{Vol}(S)$. What this essentially means is that the n-dimensional volume gets scaled by a factor of the determinant of the standard matrix associated with said linear transformation.