7 Linear Algebra 2 Lecture 7: Discrete Dynamical Systems (15/04/2020)

7.1 Definition of a discrete dynamical system (DDS)

A (linear) discrete dynamical system on $\mathbb{R}^n$ is a squence of vectors $\vec{x}_1, \vec{x}_2, \cdots \in \mathbb{R}^n$ such that $\vec{x}_{k+1} = A\vec{x}_k$ for all $k \geq 0$ and fixed $n \times n$ matrix A. The vectors $\vec{x}_k$ are referred to as the state vectors of the system at the discrete time instants $k = 0, 1, 2, \cdots$. The vector $\vec{x}_0$ is called the initial state vector.

7.2 General solutions if A is diagonalizable

Let A be an $n \times n$ diagonalizable matrix with the real eigenvalues $\lambda_1, \cdots, \lambda_n$ and the corresponding eigenvectors $\vec{v}_1, \cdots, \vec{v}_n$. Then the discrete dynamical system defined by $\vec{x}_{k+1} = A\vec{x}_k$ has the following general solution:

$$\vec{x}_k = c_1\lambda_1^k\vec{v}_1 + \cdots + c_n\lambda_n^k\vec{v}_n = \sum_{i=1}^{n} c_i\lambda_i^k\vec{v}_i \tag{1}$$

If the initial state vector $\vec{x}_0$ of the system is given the coefficients c_j can be determined by solving:

$$\vec{x}_0 = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \sum_{i=1}^{n} c_i\vec{v}_i \tag{2}$$

7.3 Graphical description of real solutions

$\vec{x}_{k+1} = A\vec{x}_k$ can geometrically be interpreted as what happens to the $\vec{x}_0$ after repeadetly applying the linear transformation $\vec{x} \mapsto A\vec{x}$. The graph of the vectors $\vec{x}_0, \vec{x}_1, \cdots$ is called the trajectory of the DDS. There are several different trajectories which can be observed depending on whether the origin is an attractor, repeller or saddle point.

a Attractor, This happens when $\lambda_j < 1$ for all j. All initial state vectors tend towards 0.

b Repeller, this happens when $\lambda_j > 1$ for all j. All initial state vectors tend away from 0.

c Saddle point, this happens when $\lambda_j < 1$ for some j and $\lambda_k > 1$ for some k. Whether the initial state vector tends away from or towards the origin varies on a case by case basis.

7.4 graphical description of complex solutions

There are 2 notable cases for DDS with complex eigenvalues. The case where the transformation applied is some scale-rotation matrix C:

$$\vec{x}_{k+1} = C\vec{x}_k \tag{3}$$

Or the more general case where with some matrix A which is similar to a scale-rotation matrix C, which gives:

$$\vec{x}_{k+1} = A\vec{x} \quad \text{where} \quad A = PCP^{-1} \tag{4}$$

Given a 2×2 matrix with the complex eigenvalues $\lambda = r(\cos(\phi) \pm i\sin(\phi))$, the trajectory of the points $\vec{x}_0, \vec{x}_1, \vec{x}_2, \cdots$ (where $\vec{x}_0 \neq 0$) of the DDS $\vec{x}_{k+1} = A\vec{x}_k$ is:

a A spiral towards the 0 if $|\lambda| < 1$

b elliptical around the 0 if $|\lambda| = 1$

c a spiral away from 0 if $|\lambda| > 1$