9 Linear Algebra 2 Lecture 9: Symmetric Matrices (26/05/2020)

9.1 Symmetric matrices

A matrix A is considered to be symmetric if it's equal to it's own transpose:

$$A = A^T \tag{1}$$

The eigenvectors of a symmetric matrix that correspond to distinct eigenvalues are always orthogonal. This can be proven as follows: Let λ_1 and λ_2 be distinct eigenvalues of A with corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$. Then:

$$\begin{aligned}
\lambda_1 \vec{v}_1 \cdot \vec{v}_2 &= (\lambda_1 \vec{v}_1^T)\vec{v}_2 = (A\vec{v}_1)^T \vec{v}_2 \\
&= (\vec{v}_1^T A^T)\vec{v}_2 = \vec{v}_1^T (A\vec{v}_2) \\
&= \vec{v}_1^T (\lambda_2 \vec{v}_2) \\
&= \lambda \vec{v}_1^T \vec{v}_2 = \lambda_2 \vec{v}_1 \cdot \vec{v}_2
\end{aligned}$$

Hence we end up with the expression $\lambda_1 \vec{v}_1 \cdot \vec{v}_2 = \lambda_2 \vec{v}_1 \cdot \vec{v}_2$. This implies:

$$(\lambda_1 - \lambda_2)\vec{v}_1 \cdot \vec{v}_2 = 0$$

Since $\lambda_1 \neq \lambda_2$ as we defined these to be distinct eigenvalues of A we can assert that $\lambda_1 - \lambda_2 \neq 0$. This means that for the previously found expression to equal 0 the inner product of $\vec{v}_1$ and $\vec{v}_2$ must be 0. This means that the vectors must be orthogonal.
Furthermore a symmetric matrix will always only have real eigenvalues. This can be proven using a hermitian transpose[1] but I'm not going to do that here since Hermitian transposes are not covered in the course.

9.2 Orthogonally diagonalizable matrices

An $n \times n$ matrix is orthogonally diagonalizable if there exists an orthogonal matrix P and a diagonalizable matrix D such that $A = PDP^{-1}$. Since P is defined to be orthogonal this theorem is the same as saying:

$$A = PDP^T$$

Note that the columns of P are the eigenvectors of A and form a basis for $\mathbb{R}^n$ space. Such a diagonalization requires n linearly independent orthonormal eigenvectors. Using this information we can state that A is symmetric iff A is orthogonally diagonalizable, since:

$$A^T = (PDP^T)^T = P^{TT}DP^T = PDP^T = A$$

9.3 The spectral theorem

The set of eigenvalues of a matrix A is sometimes called the spectrum of A. The following description of the eigenvalues is sometimes called the spectral theorem:
An $n \times n$ matrix A has the following properties:

- *A has n eigenvalues$\in \mathbb{R}^n$, counting multiplicities*
- *$dim(E_\lambda)$ for eacht λ equals the multiplicity of λ as a root of the characteristic equation*
- *The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal*
- *A is orthogonally diagonalizable*

[1]Taking the transpose of a matrix and the complex conjugate of all of it;s entries

9.4 Algorithm for orthogonal diagonalization

1. Compute the eigenvalues of A
2. For each eigenvalue: construct a basis of the corresponding eigenspace
3. For each eigenvalue: construct an orthonormal basis of the corresponding eigenspace (use the Gram-Schmidt process to orthogonalize if neccasary. Use scaling to normalize the basis vectors)
4. Construct a matrix P using the basis vectors constructed is step 3 as columns
5. Construct a matric D using the corresponding eigenvalues on the diagonal