4 Linear Algebra 2 Lecture 4: Diagonalization (01/05/2020)

4.1 Similarity

If A and B are $n \times n$ matrices, then A is similar to be of there excists an $n \times n$ invertible matrix P such that $A = PBP^{-1}$. When 2 matrices are similar they have the same characteristic polynomial. By extension this also implies that A and B have the same eigenvalues with the same algebraic multiplicities. Note that this does not imply that 2 matrices with the same eigenvalues are always similar. Consider the following 2 matrices:

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

These matrices both have the same eigenvalue of 2 with the same algebraic multiplicities. However these matrices are not in fact similar since the equality $A = PBP^{-1}$ does not hold for these 2 matrices. The equality $A = PBP^{-1}$ also implies the following: if $\vec{v}$ is an eigenvector of B with the eigenvalue λ, then $P\vec{v}$ is an eigenvector of A with the same eigenvalue λ.

4.2 Diagonalization

An $n \times n$ matrix is diagonal if all of it's elements except for the main entries are 0:

$$\begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} \tag{1}$$

There are several reasons as to why we are interested in diagonal matrices. The first convenient property is a result of matrix multiplication. A diagonal matrix D to any power k will always be equal to all of it's main entries to the power k (this can be proven with matrix multiplication but I can't be asked to type it out...):

$$D^k = \begin{bmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{bmatrix} \tag{2}$$

Furthermore all of the main entries on a diagonal matrix are it's own eigenvalues[1]:

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{3}$$

A matrix A is diagonalizable if A is similar to a diagonal matrix D. This implies that A is diagonalizable if it can be expressed as $A = PDP^{-1}$, where D is a diagonalizable matrix and P and invertible matrix. It can be proven that (but I'm not going to) $A = PDP^{-1}$ only holds if:

$$D = \begin{bmatrix} \lambda_{A,1} & 0 & \cdots & 0 \\ 0 & \lambda_{A,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{A,n} \end{bmatrix} \tag{4}$$

[1]Can be proven with either the formula for the 2×2 matrix, the triple-scalar product for a 3×3 matrix or with cofactor expansion for the most general case

Where $\lambda_{A,1}, \cdots, \lambda_{A,n}$ are the eigenvalues of A and:

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \tag{5}$$

Where the set $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ is a linearly independent set of eigenvectors corresponding to the eigenvalues of A. Diagonalizing A makes it easy to compute higher powers of the matrix-product since:

$$\begin{aligned} A^2 &= (PDP^{-1})(PDP^{-1}) = PD^2P^{-1} \\ A^3 &= (PDP^{-1})(PD^2P^{-1}) = PD^3P^{-1} \\ &\vdots \\ A^k &= (PDP^-1)(PD^{k-1}P^{-1}) = PD^kP^{-1} \end{aligned}$$

Since the matrix P has to be constructed from a set of linearly independent eigenvectors of A we know that A is diagonalizable iff it has n linearly independent eigenvectors, and thus n distinct eigenvalues all with a algebraic multiplicitie of 1. This set of of n linearly independent eigenvectors form what is called an eigenbasis of A.

4.3 Algebraic and geometric multiplicity

Let λ_0 be an eigenvalue corresponding to the matrix A. Then: The algebraic multiplicity, denoted as a.m.(λ_0), is the number of factors $(\lambda - \lambda_0)$ in the characteristic polynomial of A. The geomtric multiplicity of λ_0, denoted as g.m.(λ_0), is the dimension of the subspace E_{λ_0}, or in terms of an equality: g.m.$(\lambda_0) = \dim(\lambda_0)$. Note that the following inequality always holds:

$$1 \leq \text{g.m.}(\lambda_0) \leq \text{a.m.}(\lambda_0) \tag{6}$$

4.4 Second diagonalization theorem

Let A be an $n \times n$ matrix with the eigenvalues $\lambda_1, \cdots, \lambda_p$. Then A is diagonalizable iff for each eigenvalue λ_i the following equality holds:

$$\text{a.m.}(\lambda_i) = \text{g.m.}(\lambda_i) \tag{7}$$

4.5 Algorithm for diagonalizing a matrix

1. Find the eigenvalues $\lambda_0, \lambda_1, \cdots, \lambda_n$ of A
2. Find n linearly independent eigenvectors corresponding to the eigenvalues of A. If these cannot be found the algortihm ends here.
3. Construct P with the eigenvectors found in step 2.
4. Construct D from the eigenvalues found in step 1.