3 Linear Algebra 2 Lecture 3: Eigenvectors and eigenvalues (28/04/2020)

3.1 Eigenvectors and eigenvalues

An eigenvector of an $n \times n$ matrix A is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. For these special cases λ is callen an eigenvalue of A. If there is a nontrivial solution $\vec{x}$ for $A\vec{x} = \lambda\vec{x}$ then such an $\vec{x}$ is called an eigenvector corresponding to λ.
Checking if a given vector is an eigenvector and what it's corresponding eigenvalue is , is a very easy process. Take the matrix-vector product $A\vec{x} = \vec{y}$. If there is a value λ for which $\vec{y} = \lambda\vec{x}$ then $\vec{x}$ is an eigenvector and λ is a corresponding eigenvalue. Note that because of this relation eigenvectors are never unique. If there is 1, there are infinitly many. Finding an eigenvector for a given eigenvalue is a bit more tricky but nonetheless fairly easy.

$$\begin{aligned} A\vec{x} &= \lambda\vec{x} \\ A\vec{x} - \lambda\vec{x} &= \vec{0} \\ (A - \lambda I)\vec{x} &= 0 \qquad (1) \end{aligned}$$

if (1) has a nontrivial solution for λ then λ is an eigenvalue. Also note that an invertible matrix cannot have 0 as an eigenvalue. Proof for this statemnent:

Assume A is an invertible matrix with the nonzero eigenvector $\vec{v}$ corresponding to the eigenvalue 0. Then:

$$\vec{v} = A^{-1}A\vec{v} = A^{-1}\lambda\vec{v} = A^{-1}(0\vec{v}) = 0$$

But $\vec{v}$ should be nonzero. Thus if 0 is an eigenvalue of A, then A cannot be invertible. □

3.2 Properties of eigenvectors and eigenvalues

Some properties of note for eigenvalues are as follows:

$$\begin{gathered} A\vec{x} = \lambda\vec{x} \\ A^2\vec{x} = A\lambda\vec{x} = \lambda A\vec{x} = \lambda^2\vec{x} \\ \vdots \\ A^n\vec{x} = \lambda^n\vec{x} \end{gathered}$$

$$\begin{gathered} A\vec{x} = \lambda\vec{x} \\ A^{-1}A\vec{x} = \vec{x} = A^{-1}\lambda\vec{x} \\ \text{if } \lambda \neq 0\text{: } A^{-1}\vec{x} = \lambda^{-1}\vec{x} \end{gathered}$$

We know for a fact that λ cannot be 0 since we have proven that an invertible matrix does not have 0 as an eigenvalue. From this we can make the following list of statements which are all equivalent:

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A &\Leftrightarrow A\vec{x} = \lambda\vec{x} \text{ for some } \vec{x} \neq 0 \\ &\Leftrightarrow (A - \lambda I)\vec{x} = \vec{0} \text{ for some } \vec{x} \neq 0 \\ &\Leftrightarrow A - \lambda I \text{ is not invertible} \\ &\Leftrightarrow |A - \lambda I| = 0 \end{aligned}$$

Because of this relation the eigenvalues of a diagonal matrix correspond to it's main entries:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \quad \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)$$

$$\text{Which is 0 iff: } \lambda = a_{11} \vee \lambda = a_{22} \vee \lambda = a_{33} \quad \square$$

Note that an $n \times n$ matrix will at most give an nth power polynomial. The fundamental theorem of algebra states that an nth power polynomial will have at most n roots. A direct consequence of this is that an $n \times n$ matrix will have at most n eigenvalues. Another property of eigenvalues is as follows: Eigenvectors corresponding to a distinct eigenvalue are always linearly independent, since these live in a different eigenspace by definition.

3.3 Eigenspaces

Definition of an eigenspace:
Let A be an $n \times n$ matrix and λ a scalar. The eigenspace E_λ of A corresponding to λ consist of:

- All eigenvectors with the eigenvalue λ
- The zero vector

In other words, E_λ is the null space of the matrix $A - \lambda I$ and by extension the set of all solutions to the equation $A\vec{x} = \lambda\vec{x}$. Thus $E_\lambda \in \mathbb{R}^n$ and $E_\lambda = \text{Nul}(A - \lambda I)$.

3.4 Characteristic equations

$|A - \lambda I| = 0$ is called the characteristic equation of A. A scalar λ is an eigenvalue if and only if it statisfies the characteristic equation. $|A - \lambda I|$ is called the characteristic polynomial. These characteristic polynomials are important for algebraic multiplicity. The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic equation. Thus if an $n \times n$ matrix has n eigenvalues all these eigenvalues have a multiplicity of 1. An example of multiplicity will follow to help illustrate the concept:
The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the eigenvalues of this matrix and their multiplicities.

$$\begin{aligned} \lambda^6 - 4\lambda^5 - 12\lambda^4 &= \lambda^4(\lambda^2 - 4\lambda - 12) \\ &= \lambda^4(\lambda - 6)(\lambda + 2) \end{aligned}$$

Which implies:

$$\begin{gathered} \lambda^4 = 0 \vee \lambda - 6 = 0 \vee \lambda + 2 = 0 \\ \lambda = 0,\ \text{multiplicity} = 4 \\ \lambda = 6,\ \text{multiplicity} = 1 \\ \lambda = -2,\ \text{multiplicity} = 1 \end{gathered}$$

The sum of the multiplicities of an nth root polynomial will always add up to n.