1 Linear Algebra 2 Lecture 1: Determinants (21/04/2020)

1.1 Definition of determinants

The determinant of a 2×2 matrix A is defined as:

$$\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{1}$$

The determinant of a 3×3 matrix can be written in terms of the triple scalar product, where the columns of A form the vectors $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$:

$$\det(A) = \begin{vmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{vmatrix} = \vec{a}_1 \cdot (\vec{a}_2 \times \vec{a}_3) \tag{2}$$

The determinant can be used to determin whether a matrix is invertible or not. A random $n \times n$ matrix A is invertible iff $\det(A) \neq 0$. In other words: The matrix A is row equivalent to I_n iff it has a non-zero determinant.

$$\begin{bmatrix} A \mid I_n \end{bmatrix} \sim \begin{bmatrix} I_n \mid A^{-1} \end{bmatrix} \text{ iff } \det(A) \neq 0 \tag{3}$$

Quick proof for this statement for a 2×2 matrix:
If a matrix is not invertible it's columns must be linearly dependent.

$$A = \begin{bmatrix} a & \lambda a \\ c & \lambda c \end{bmatrix} \rightarrow \det(A) = \lambda ac - \lambda ac = 0 \quad \square \tag{4}$$

1.2 Cofactors and cofactor expansion

A cofactor is a number which can be associated with an element of a square matrix. For example: let A be an $n \times n$ matrix. The (i, j)-cofactor is defined as:

$$C_{ij} = (-1)^{i+j} \det(A_{ij}) \tag{5}$$

Where A_{ij} is the matrix obtained from A by deleting row i and column j. Cofactors can be used to compute determinants. Let A be an $n \times n$ matrix. Pick any random row r is A. Then:

$$\det(A) = a_{r1}C_{r1} + \cdots + a_{rn}C_{rn} = \sum_{i=1}^{n} a_{ri}C_{ri} \tag{6}$$

Where a_{ij} are the elements of A on row i and column j. This equation holds for any random row or column in A. Pick a random column k in A, then also:

$$\det(A) = \sum_{i=1}^{n} a_{ik}C_{ik} \tag{7}$$

1.3 Triangular matrices

Let A be an $n \times n$ matrix:

- The main diagonal is the diagonal from top left to bottom right in the matrix. These are the entries with equal row and column number (a_{nn}).
- A is upper triangular if all the entries below the main diagonal are 0.
- A is lower triangular if all the entries above the main diagonal are 0.

For triangular matrices the following holds: The determinant is equal to the product of the main entries:

$$\det(A) = a_{11}a_{22} \cdots a_{nn} = \prod_{i=1}^{n} a_{ii} \tag{8}$$

1.4 Determinants and row operations

The determinant of a matrix changes with row reduction. Luckily it changes in a very predictable way. Let B be a matrix obtained by performing one elementry row operation on matrix A.

- Interchange 2 rows $\Rightarrow |B| = -|A|$
- Multiply one row by a constant $\lambda \Rightarrow |B| = \lambda|A|$
- Add a multiple of one row to another $\Rightarrow |B| = |A|$