10 Linear Algebra 2 Lecture 10: Quadratic forms (29/05/2020)

10.1 Definition of a quadratic form

A function $Q(\vec{x}) : \mathbb{R}^n \to \mathbb{R}$ is called a quadratic form if it can be written in the form $Q(\vec{x}) = \vec{x}^T A\vec{x}$, where A is a symmetric $n \times n$ matrix. The simplest example of this is the situation where $A = I_n$:

$$\begin{aligned} Q(\vec{x}) = \vec{x}^T I\vec{x} = \vec{x}^T\vec{x} = ||\vec{x}||^2 \quad &(1)\\ = x_1 + x_2 + \cdots + x_n \quad &(2) \end{aligned}$$

Every quadratic form can be expressed as:

$$Q(\vec{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_i x_k \quad (3)$$

There exists a symmetric $n \times n$ matrix such that:

$$Q(\vec{x}) = \vec{x}^T A\vec{x} \quad (4)$$

for every vector $\vec{x} \in \mathbb{R}^n$.

10.2 Finding a quadratic form for a given matrix A

let $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then:

$$\begin{aligned} Q(\vec{x}) &= \vec{x}^T A\vec{x} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3x_1 - 2x_2 \\ -2x_1 + 7x_2 \end{bmatrix} \\ &= x_1(3x_1 - 2x_2) + x_2(-2x_1 + 7x_2) \\ &= 3x_1^2 + 7x_2^2 - 4x_1x_2 \end{aligned}$$

Where the term $4x_1x_2$ is referred to as the cross-product term. There are methods such as a change of variables to get rid of this cross-product term whcih will be discussed later.

10.3 Finding the matrix A for a given quadratic form

Let $Q(\vec{x}) = 5x_1^2 + 3x_2^2 + 2x_3^2 - 2x_1x_2 + 8x_2x_3 = \vec{x}^T A\vec{x}$. The vector $\vec{x}$ has 3 entries, and the matrix A is symmetric. Thus A should be a 3×3 matrix. It's main entries will be the squared terms of $Q(\vec{x})$. Since A is a symmetric matrix the cross-product terms need to be split evenly between a_{ij} and a_{ji}. Thus a_{12} and a_{21} should be $\frac{1}{2} \cdot -2 = -1$, a_{23} and a_{32} should be $\frac{1}{2} \cdot 4 = 2$ and a_{13} and a_{31} should both be 0. This leaves us with the following matrix A:

$$A = \begin{bmatrix} 5 & -1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

10.4 The principal axes theorem

Let a quadratic form $Q(\vec{x})$ be given for some symmetric $n \times n$ matrix A. A being symmetric guarantees that there exists an orthogonal matrix P such that $\vec{x} = P\vec{y}$. This transforms the quadratic form $Q(\vec{x})$

to a different quadratic form denoted as $\bar{Q}(\vec{y})$. The diagonal matrix D can be expressed as $D = P^T AP$, thus:

$$\bar{Q}(\vec{y}) = \vec{y}^T D\vec{y} \tag{5}$$

Since D is a diagonal matrix we can know for certain that the quadratic form $\bar{Q}(\vec{y})$ will not have any cross-product terms. We also know that D was constructed to have the eigenvalues of A as it's main entries. The expression for $\bar{Q}(\vec{y})$ then reduces to:

$$\bar{Q}(\vec{y}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 \tag{6}$$

The principal axes of the quadratic form Q are the lines generated by the columns of P. It can be thought of as the geometrical counterpart of the spectral theorem. It has application in dynamics such as when computing the moment of inertia tensor.

10.5 Algorithm for removing cross-product terms

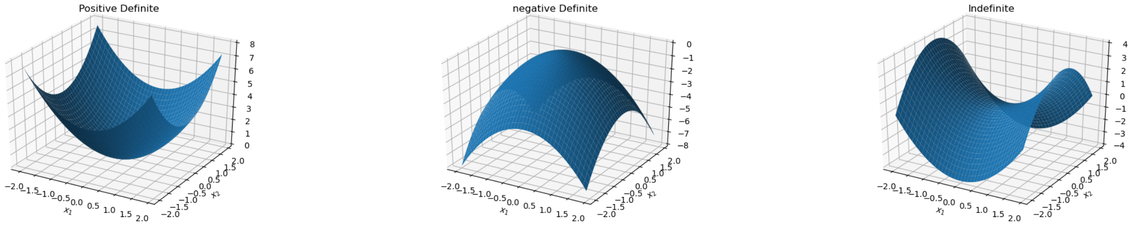
Let $Q(\vec{x})$ be a quadratic form with symmetric matrix A:

1. orthogonal diagonalization: $A = PDP^T$. Define a basis $\mathcal{B} = \{\vec{u}_1, \cdots \vec{u}_n\}$ to be orthonormal and consisting of the columns of P.
2. Define a new variable $\vec{y} = [\vec{x}]_{\mathcal{B}} = P^T\vec{x}$. Note that the matrix P is orthogonal and thus $P^{-1} = P^T$. This implies $\vec{x} = P\vec{y}$
3. Then: $Q(\vec{x}) = \vec{x}^T A\vec{x} = \vec{y}^T D\vec{y}$.

10.6 Classification of quadratic forms

The quadratic form $Q(\vec{x})$ is:

- positive semidefinite if $Q(\vec{x}) \geq 0$ for all $\vec{x}$. This happens when all $\lambda \geq 0$
- positive definite if $Q(\vec{x}) > 0$ for all $\vec{x} \neq 0$. This happens when all $\lambda > 0$
- negative semidefinite if $Q(\vec{x}) \leq 0$ for all $\vec{x}$. This happens when all $\lambda \leq 0$
- negative definite if $Q(\vec{x}) < 0$ for all $\vec{x} \neq 0$. This happens when all $\lambda < 0$.
- indefinite if $Q(\vec{x})$ assumes both positive and negative values. This happens when λ has both positive and negative values.



(a) Positive definite (b) Negative definite (c) Indefinite

Figure 1: Surface plots of the classifications of quadratic forms using matplotlib and Python