3 Newton-Euler equations

3.1 Linear momentum

3.1.1 Single particle systems

The definition of linear momentum is given as:

$$\vec{p} = m\dot{\vec{r}} \tag{3.1}$$

When we recall Newton's original formulation for the second law we find that:

$$\Sigma\vec{F} = \frac{\mathrm{d}(m\vec{v})}{\mathrm{d}t} = m\ddot{\vec{r}} = \dot{\vec{p}} \tag{3.2}$$

Thus the sum of all forces in a system is nothing but the rate of chage of momentum with respect to time. We can integrate $\Sigma\vec{F}$ with respect to time to find that:

$$\int_{t_0}^{t_1} \Sigma\vec{F}\,\mathrm{d}t = \vec{p}(t_1) - \vec{p}(t_0) \tag{3.3}$$

Which is nothing but the change in momentum over some time interval $[t_0, t_1]$. This quantity is referred to as impulse.

3.1.2 Multi-particle systems

We can extend the definition of equation 3.1 to any systen of N particles. We do this by adding up all the internal and external forces acting on a particle together:

$$\sum_{i=1}^{N}(\vec{F}_i + \sum_{j=1}^{N}\vec{f}_{ij}) = \sum_{i=1}^{N}\dot{\vec{p}}_i \tag{3.4}$$

Since we know all internal forces of a system should cancel out as they can't cause a net change in momentum of the system we get:

$$\sum_{i=1}^{N}\vec{F}_i = \sum_{i=1}^{N}\dot{\vec{p}}_i \tag{3.5}$$

We can simplify this even further. Recall the definition of the center of mass:

$$\vec{r}_C = \frac{\sum_{i=1}^{N} m_i\vec{r}_i}{\sum_{i=1}^{N} m_i} = \frac{\sum_{i=1}^{N} m_i\vec{r}_i}{m} \tag{3.6}$$

Taking the time derrivative of this we find that:

$$\ddot{\vec{r}}_C = \frac{\sum_{i=1}^{N} m_i\ddot{\vec{r}}_i}{m} \tag{3.7}$$

Which gives the following result when subsituting it back in equation 3.4:

$$\Sigma\vec{F} = m\ddot{\vec{r}}_C \tag{3.8}$$

Thus the system of multiple particles can be described as an eqquipolent system with all forces acting on the center of mass of the system C.

3.2 Angular momentum

3.2.1 Single particle systems

Euler's second law of motion states that the angular momentum about some point O and it's tiem derrivative can be given as:

$$\vec{H}_O = \vec{r} \times (m\dot{\vec{r}}) \tag{3.9}$$

$$\dot{\vec{H}}_O = \dot{\vec{r}} \times (m\dot{\vec{r}}) + \vec{r} \times (m\ddot{\vec{r}}) \tag{3.10}$$

The cross product of a vector with itself will always be $\vec{0}$, thus the time derrivative of angular momentum is given as:

$$\dot{\vec{H}}_O = \vec{r} \times (m\ddot{\vec{r}}) = \vec{r} \times \vec{F} = \vec{M}_O \tag{3.11}$$

Which is nothing but the resultant moment vector about point O.

3.2.2 Multi-particle systems

This defention to can be extended to a system of particles by, again, summing all the individual components together:

$$\sum_{i=1}^{N} \vec{r}_i \times (\vec{F}_i + \sum_{j=1}^{N} \vec{f}_{ij}) = \sum_{i=1}^{N} (\vec{r}_i \times \vec{F}_i) + \sum_{i=1}^{N} \sum_{j=1}^{N} (\vec{r}_i \times \vec{f}_{ij}) \tag{3.12}$$

Since Newton's 3rd law states $F_{action} = -F_{reaction}$ for every internal force $\vec{f}_{ij}$ there is another internal force $\vec{f}_{ji}$ which has the same line of action but in the opposite direction. Because of this we find the relation:

$$\vec{\rho}_i \times \vec{f}_{ij} = -\vec{\rho}_j \times \vec{f}_{ji} \tag{3.13}$$

This means the sum of all moments as a result of internal forces will form a moment pair with opposite direction, efficitively canceling them all out. This leaves us with:

$$\dot{\vec{H}}_O = \sum_{i=1}^{N} (\vec{r}_i \times \vec{F}_i) \tag{3.14}$$