4 Alternating series

4.1 Defining alternating series

Let $\{a_n\}$ be a sequence with positive terms. A series constructed from a_n of the form $\sum_{n=1}^{\infty}(-1)^n a_n$ is called an alternating series because the terms alternate between positive and negative.
There are 3 conditions which need to be statisfied for an alternating series to be convergent. Let $\{b_n\}$ be a sequence which statisfies the following conditions:

1. $b_n > 0$ all terms are greater then 0.
2. $\lim_{n\to\infty} b_n = 0$, b_n goes to 0 as n goes to infinity.
3. $b_n > b_{n+1}$, the sequence is strictly decreasing

If all 3 conditions are met the series $\sum_{n=1}^{\infty}(-1)^n b_n$ converges.

Example: Let b_n be a sequence given as:

$$b_n = \frac{1}{n+1}$$

Does the series $\sum_{n=1}^{\infty}(-1)^n b_n$ converge?

1. $b_n = \frac{1}{n+1} > 0$ for all $n \geq 0$.
2. $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n+1} = 0$
3. $b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}$

All conditions are statisfies, thus the series converges.

4.2 Limits of alternating series

Let s_n be an alternating series for which $s_1 > s_3 > s_5 > \cdots > 0$ and $s_2 < s_4 < s_6 < \cdots < 1$. This means the sequence s_{2n+1} is strictly decrasing and lower bounded by 0. The sequence s_{2n} is strictly increasing and bounded by 1. These sequences are both convergent as per the monotomic convergence theorem which states that sequence which are stricly increasing/decreasing and upper/lower bounded always converge. This means:

$$\lim_{n\to\infty} s_{2n+1} = L \tag{4.1}$$

$$\lim_{n\to\infty} s_{2n} = M \tag{4.2}$$

Since s_{2n+1} and s_{2n} describe the same sequence we know that:

$$L = M \tag{4.3}$$

4.3 Estimating remainder terms of alternating series

The remainder term for an alternating term is found the same as for a regular series. The sum to infinity is the same as the partial sum plus the remainder:

$$\sum_{n=0}^{\infty}(-1)^n b_n = \sum_{k=0}^{n}(-1)^k b_k + R_n \tag{4.4}$$

If $s = \sum_{n=0}^{\infty}(-1)^n b_n$ is the sum of a sequence which statisfies $b_n > 0$, $b_{n+1} > b_n$ and $\lim_{n\to\infty} b_n = 0$ then the remainder terms is given as:

$$|R_n| = |s - s_n| \leq b_{n+1} \tag{4.5}$$

Example: The following series is given:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$

Find for which value of n $R_n < 0.005 = \frac{1}{200}$.

$$\sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k^2} \Rightarrow |R_n| \leq b_{n+1} = \frac{1}{(n+1)^2}$$

$$\frac{1}{(n+1)^2} = \frac{1}{200}$$

$$n \approx 14$$

Thus the remainder of the series will be smaller then 0.005 for $n \geq 14$.

4.4 Absolute and relative convergence

A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges. If the series $\sum_{n=1}^{\infty} a_n$ converges but the series $\sum_{n=1}^{\infty} |a_n|$ diverges the series is considered to be a condtionally convergent series. This means that if the function $\sum_{n=1}^{\infty} |a_n|$ converges it automatically implies that the series $\sum_{n=1}^{\infty} a_n$. Any series which is absolutely convergent implies convergence. This does not work the other way around. A convergent series does not automatically implie that the series is absolutely convergent.

4.5 The ratio test

Consider the series $\sum_{n=0}^{\infty} a_n$. Let $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$. We can conclude the following for different values of L:

- $L < 1$, $\sum_{n=0}^{\infty} a_n$ is an absolutely convergent series
- $L > 1$, $\sum_{n=0}^{\infty} a_n$ is a divergent series
- $L = 0$, nothing of significance can be concluded

Example: Use the ratio test to determine if the following series converges:

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$$

We apply the theorem:

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{(-3)^{n+1}}{(n+1)!} \Big/ \frac{(-3)^n}{n!} \\
&= \lim_{n\to\infty} \frac{(-3)^{n+1}}{(n+1)n!} \Big/ \frac{(-3)^n}{n!} \\
&= \lim_{n\to\infty} \frac{3}{n+1} = 0 = L < 1
\end{aligned}$$

Thus since $L > 1$ we may conclude that the series is absolutely convergent.

4.6 The root test

The root test kinda sucks. $\sum_{n=0}^{\infty} a_n$ let $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, then:

- $L < 1$, $\sum_{n=0}^{\infty} a_n$ absolutely converges
- $L > 1$, $\sum_{n=0}^{\infty} a_n$ diverges
- $L = 0$, nothing of significance can be concluded

An important limit that often comes up when dealing with n-th roots is:

$$\lim_{n\to\infty} \sqrt[n]{n} = 1 \Rightarrow \lim_{n\to\infty} \sqrt[n]{n^k} = \lim_{n\to\infty} (\sqrt[n]{n})^k = 1^k = 1 \tag{4.6}$$