2 Series

2.1 Defining a series

A series is constructed by taking some sequence and summing all the terms of said sequence together. For example: let $\{a_n\}$ be a sequence. Then:

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
s_n &= a_1 + a_2 + \cdots + a_n
\end{aligned}$$

Which is then usually written more compactly as:

$$s_n = \sum_{i=1}^{n} a_i$$

Summing the first n terms of a series is called a **partial sum**. We are usually interested in what happens when $n \to \infty$. We can extrapolate how a series behaves when n goes to ∞ by studying the partial sum. If the partial sum is convergent/divergent, then when $n \to \infty$ the series will also be convergent/divergent. When a series is convergent we call the limit of the series s.

$$s = \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

2.2 Geometric Series

An important example of an infinite sum is a geometric sequence. For a geometric series all the terms in the series share a common ratio r:

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{k=0}^{\infty} ar^k$$

if $r \neq 1$ we get the following partial sum:

$$s_n = a + ar + ar^2 + ar^3 + \cdots + ar^n$$

Additionally we can multiply this series with r to get:

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^{n+1}$$

We can now take the difference of these 2 series and we end up with:

$$\begin{aligned}
s_n - rs_n = (1-r)s_n &= (a + \cancel{ar} + \cancel{ar^2} + \cancel{\cdots} + \cancel{ar^n}) - (\cancel{ar} + \cancel{ar^2} + + \cancel{ar^3} + \cancel{\cdots} + ar^{n+1}) \\
&= a(1 - r^{n+1})
\end{aligned}$$

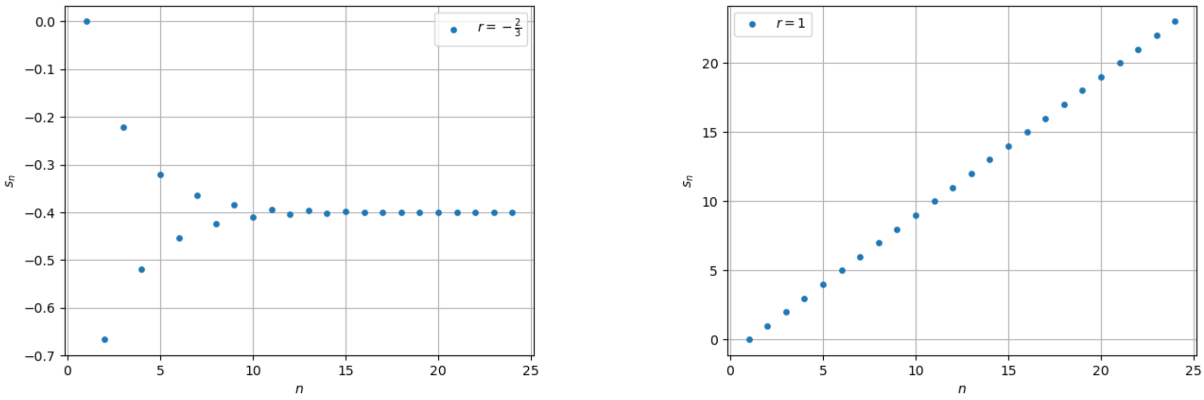
We can simplify this to find an expression for s_n:

$$s_n = \frac{a(1 - r^{n+1})}{1 - r}$$

From this we can see that the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ is convergent for $|r| < 1$ and will be equal to:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad \text{for } |r| < 1$$

if $|r| \geq 1$ the series is divergent and will go to either plus or minus ∞.



(a) $|r| < 1$

(b) $|r| \geq 1$

Figure 1: Visualization of the geometric series for a value of $|r|$ smaller then one and for $|r|$ greater or equal to 1. Note that the series where $r = 1$ the value goes of to infinity.

2.3 Telescoping series

A telescoping series is a series of the form:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots$$

The partial sum will be the first k terms of the series:

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$$

We can use partial fraction decomposition to find that:

$$\frac{1}{k(k+1)} = \frac{1}{k} + \frac{1}{k+1}$$

The partial sum s_n will then be:

$$\begin{aligned} s_n &= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \cancel{\cdots} + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

When we take the limit of this partial sum as n goes to infinity we find:

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} 1 - \frac{1}{n+1} = 1$$

Thus this series converges to 1.

2.4 Harmonic series

The harmonic series is given as:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \cdots$$

We recognize that the sum forms a rough estimation of the area of the curve given as $y = \frac{1}{x}$. Since it's an approximation we know that the series will always be larger then the area under te curve. Hence:

$$\sum_{n=1}^{\infty} \frac{1}{n} > \int_{0}^{\infty} \frac{1}{x}\,\mathrm{d}x$$

We can solve this integral easily as follows:

$$\begin{aligned}\int_{0}^{\infty} \frac{1}{x}\,\mathrm{d}x &= \lim_{t\to\infty} \int_{0}^{t} \frac{1}{x}\,\mathrm{d}x \\ &= \lim_{t\to\infty} \ln(t) = \infty\end{aligned}$$

Since $\sum_{n=1}^{\infty} 1/n > \int_{0}^{\infty} 1/x\,\mathrm{d}x$ we can assert for sure that the limit of the series must also be infinity.

2.5 The general term test

If $\{a_n\}$ is some sequence and s_n is a series constructed from said sequence that converges, then $\lim_{n\to\infty} a_n = 0$. This can also be formulated the other way around as: if $\lim_{n\to\infty} a_n$ does not exists or if $\lim_{n\to\infty} a_n \neq 0$ then the series s_n is divergent. This is because the sum will keep getting larger and larger if the sequence does not converge to 0. Important side note: A geenral term of 0 of some sequence $\{a_n\}$ does not tell you anythin about whether the series converges or diverges.

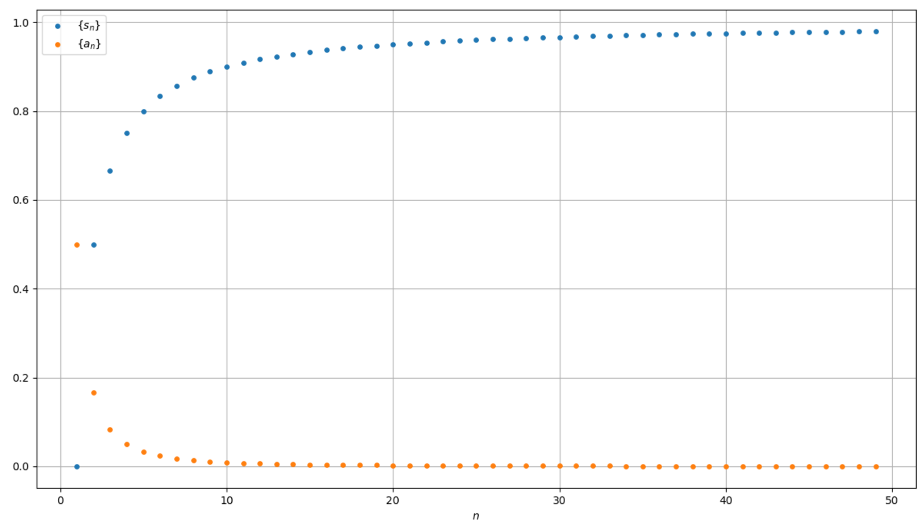


Figure 2: The relation between the convergence of the sequence $\{a_n\} = \frac{1}{n(n+1)}$ and the convergence of the series $\{s_n\} = \sum_{i=1}^{n} \frac{1}{i(i+1)}$ visualized.

2.6 Algebraic rules for manipulating series

Let $\sum_{n=1}^{\infty} a_n = S \in \mathbb{R}$, $\sum_{n=1}^{\infty} b_n = T \in \mathbb{R}$ and $c_1, c_2 \in \mathbb{R}$, then the following rules apply:

- $\sum_{n=1}^{\infty} c_1 a_n = c_1 \sum_{n=1}^{\infty} a_n = c_1 S$
- $\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n = S \pm T$
- $\sum_{n=1}^{\infty} (c_1 a_n \pm c_2 b_n) = c_1 \sum_{n=1}^{\infty} a_n \pm c_2 \sum_{n=1}^{\infty} b_n = c_1 S \pm c_2 T$

Note that these always apply when a_n and b_n both converge.