5 Describing orientation in space

5.1 Rotation matrices

A triad can be thought of as an orthonormal basis to a vector space. This means we can linearly transform a triad to another triad just by applying a matrix transformation. Since the triads are both orthonormal basis the determinant of this rotation matrix should also be equal to 1 as this would otherwise not preserve the fact that the basis vectors are all of length 1. Suppose $\vec{r}$ is just a random vector in $\mathbb{R}^3$ space. Furthermore suppose this space has 2 triads: triad $\mathcal{N}$ with unit vectors $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_2$ and $\hat{\mathbf{n}}_3$ in the X, Y and Z directions. The second triad is triad $\mathcal{B}$ which has the unit vectors $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$ and $\hat{\mathbf{b}}_3$ in the x', y' and z' directions. Since we are dealing with 2 triads we can choose to represent our vector $\vec{r}$ in 2 ways:

$$^{\mathcal{N}}\vec{r} = \begin{bmatrix} r_{n1} \\ r_{n2} \\ r_{n3} \end{bmatrix} \quad \text{or} \quad ^{\mathcal{B}}\vec{r} = \begin{bmatrix} r_{b1} \\ r_{b2} \\ r_{b3} \end{bmatrix} \tag{5.1}$$

Notice that the actual vector $\vec{r}$ stays the same no matter what triad we describe it in. We can translate a vector expressed in one triad to the same vector expressed in a different triad by taking the inner product of the vector and the unitvectors of triad $\mathcal{B}$ expressed in terms of triad $\mathcal{N}$. This looks a bit like:

$$r_{b,i} = {}^{\mathcal{N}}\vec{r} \cdot \hat{\mathbf{b}}_i = |\vec{r}| \cos(\alpha_{ij}) \tag{5.2}$$

Instead of doing this to some random vector $\vec{r}$ we can also describe the unit vectors of different triads in terms of one another. In the case of expressing $\hat{\mathbf{b}}_1$ in terms of triad $\mathcal{N}$ we get:

$$^{\mathcal{N}}\hat{\mathbf{b}}_1 = \begin{bmatrix} \hat{\mathbf{b}}_1 \cdot \hat{\mathbf{n}}_1 \\ \hat{\mathbf{b}}_1 \cdot \hat{\mathbf{n}}_2 \\ \hat{\mathbf{b}}_1 \cdot \hat{\mathbf{n}}_3 \end{bmatrix} = \begin{bmatrix} \cos(\alpha_{x'X}) \\ \cos(\alpha_{x'Y}) \\ \cos(\alpha_{x'Z}) \end{bmatrix} \tag{5.3}$$

We can repeat this for are unit vectors in the triad and package them together in a matrix. When when we then apply this matrix to vectors in triad $\mathcal{N}$ it translates them into vectors expressed in triad $\mathcal{B}$ this means it's nothing but a change of basis matrix. From this we can also deduce that the inverse of the rotation matrix must be the matrix which translates triad $\mathcal{B}$ to triad $\mathcal{N}$, as this is nothing but changing the basis back into the original triad. The rotation matrix is given as:

$$^{\mathcal{B}}C_{\mathcal{N}} = \begin{bmatrix} \cos(\alpha_{x'X}) & \cos(\alpha_{x'Y}) & \cos(\alpha_{x'Z}) \\ \cos(\alpha_{y'X}) & \cos(\alpha_{y'Y}) & \cos(\alpha_{y'Z}) \\ \cos(\alpha_{z'X}) & \cos(\alpha_{z'Y}) & \cos(\alpha_{z'Z}) \end{bmatrix} \tag{5.4}$$

Where the following relations holds:

$$^{\mathcal{N}}C_{\mathcal{B}} = \left({}^{\mathcal{B}}C_{\mathcal{N}}\right)^{-1} \tag{5.5}$$

A shortcut to this is that for symmetric matrices it's inverse and transpose are the same. Thus when we are dealing with a symmetric rotation matrix we find:

$$^{\mathcal{N}}C_{\mathcal{B}} = \left({}^{\mathcal{B}}C_{\mathcal{N}}\right)^{T} \tag{5.6}$$