

0 WB2630-T1 & T2 Math review

0.1 Vectors and Matrices

Vectors are lists of numbers which can graphically be represented as arrows in space. They can be denoted as either row or column vectors:

$$\begin{aligned}\text{Column: } \vec{r} &= \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \\ \text{Row: } \vec{r} &= (r_1 \quad \cdots \quad r_n)^T\end{aligned}$$

Matrices are just multi-dimensional arrays. Or just think of them as a vector of vectors. They can graphically be represented as some linear transformation acting on a vector. Matrices will usually be denoted as:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

0.2 Properties of Vectors

Vectors can be added and subtracted together if they are expressed in terms of the same basis vectors. Vectors can also be scaled by some scalar value $\alpha \in \mathbb{R}$. Multiplication of vectors takes 2 forms. The dot or scalar product and the cross or vector product.

The Dot product of 2 vectors outputs a scalar value and represents roughly how much 2 values are in the same direction. The dot product is given as:

$$c = \vec{a} \cdot \vec{b} \equiv |\vec{a}||\vec{b}|\cos(\theta) \quad c \in \mathbb{R}, \vec{a}, \vec{b} \in \mathbb{R}^n$$

The dot product is commutative:

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

There are several other ways to represent the dot product. Some examples are:

$$\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b} = \sum_{i=1}^n a_i b_i$$

The cross product outputs a vector that is perpendicular to the plane spanned by the other 2 vectors. The magnitude of this 3rd vector represents the area of the plane spanned by the first 2 vectors. The cross product is given as:

$$\begin{aligned}\vec{c} &= \vec{a} \times \vec{b} \\ |\vec{c}| &= |\vec{a}||\vec{b}|\sin(\theta) \quad \theta \in [0, \pi]\end{aligned}$$

Let \vec{a}, \vec{b} and \vec{c} be:

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \quad \vec{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} \quad \vec{c} = \begin{pmatrix} c_x \\ c_y \\ c_z \end{pmatrix}$$

if $\vec{c} = \vec{a} \times \vec{b}$, then:

$$\begin{aligned}c_x &= a_y b_z - b_y a_z \\c_y &= b_x a_z - a_x b_z \\c_z &= a_x b_y - b_x a_y\end{aligned}$$

The cross product is anti-commutative:

$$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$$

0.3 Properties of Matrices

Some matrix A can have any $m \times n$ shape. Matrices can be multiplied with eachother to create new matrices. They can also be multiplied with vectors to map them to different vectors. Let A be an $m \times n$ matrix:

$$A\vec{x} = \vec{y} \quad \vec{x} \in \mathbb{R}^n, \vec{y} \in \mathbb{R}^m$$

Matrices can be transposed. The transpose operation interchanges the rows and columns of a matrix and is denoted with T . If a matrix is symmetric then:

$$A^T = A$$

Square matrices with a non-zero determinant can be inverted. Multiplying by the inverse of a matrix is analogous to multiplying by the inverse of a number:

$$AA^{-1} = I_n$$

Which can then be used to solve vector equations:

$$A\vec{x} = \vec{y} \Rightarrow \vec{x} = A^{-1}\vec{y}$$

0.4 Eigenvalues and eigenvectors

For some special cases the following relation holds for matrix-vector multiplication:

$$A\vec{x} = \lambda\vec{x}$$

Where λ denotes an eigenvalue of A . This expression can then be used to find the eigenvalues of A :

$$(A - \lambda I)\vec{x} = \vec{0}$$

This form can then be used to find eigenvalues of A with the following:

$$\det(A - \lambda I) = 0$$

0.5 Differentiation and integration

Time derivatives of variables will be denoted with a dot on top of the variable name.

$$\dot{r} = \frac{dr}{dt} = \lim_{\Delta t \rightarrow 0} \frac{r(t + \Delta t) - r(t)}{\Delta t}$$

The basic rules for differentiation are:

$$(c \cdot f(x))' = c \cdot f'(x)$$

$$(f(g(x)))' = f'(g(x)) \cdot g'(x) = \frac{df}{dg} \frac{dg}{dx} \quad (\text{Chain Rule})$$

$$(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x) \quad (\text{Product Rule})$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \quad (\text{Quotient Rule})$$

For multivariable functions the derivative is described in terms of partial derivatives:

$$df(x, y, z) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

This can also more compactly be written as:

$$df(x_1 \cdots x_N) = \sum_{i=1}^N \frac{\partial f}{\partial x_i} dx_i$$

In some cases $x = x(t)$. The total derivative is then found applying the chain rule:

$$\frac{df(x_1 \cdots x_N)}{dt} = \sum_{i=1}^N \frac{\partial f}{\partial x_i} \frac{dx_i}{dt}$$

Partial derivatives are also used to express the gradient vector, which is the multi-variable analogy of the derivative of some function.

$$\nabla f(x, y, z) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$