6 Taylor and Maclaurin series

6.1 Introducing the Taylor and Maclaurin series

Taylor series are a result of how power series function. Let $f(x)$ be some power series centered at the point a.

$$\begin{aligned} f(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \\ &= \sum_{n=0}^{\infty} c_n (x-a)^n \ , \ |x-a| < R \end{aligned} \tag{6.1}$$

When we evaluate this function at $x = a$ we find that $f(a) = c_0$. Now let's take the derrivative of $f(x)$ and evaluate it again at $x = a$:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

Which gives $f'(a) = c_1$. We can keep repeating this for any n-th derrivative of $f(x)$. The pattern that starts to emerge when we do this for all terms of $f(x)$ up untill any k-th term will then be:

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)c_n(x-a)^{n-k} \tag{6.2}$$

Which we can evaluate at a to give:

$$f^{(k)}(a) = k!c_k \tag{6.3}$$

Rearranging this we find that:

$$c_k = \frac{f^{(k)}(a)}{k!} \tag{6.4}$$

Which means any k-th component of the power series $f(x)$ can be given as the k-th derrivative of the function eveluated at a divided by $k!$. Subsituting this back into our original power series $f(x)$ with $k = n$ we get:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{6.5}$$

In the special case that $a = 0$ we call this a Maclaurin series, which is nothing but a Taylor series centered at $x = 0$.

Example: Determine the Maclaurin series of $f(x) = \exp(x)$.

$$\forall\, n \in \mathbb{N}\ f^{(n)}(x) = \exp(x) \Rightarrow f^{(n)}(0) = 1$$

$$\therefore \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \forall\, x \in \mathbb{R}$$

When we evaluate this polynomial at $x = 1$ we find that this polynomial gets closer and closer to a specific value for larger choices. When $n \to \infty$ we find that this eveluates exactly to:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

Where e is Euler's number ($\approx 2.71828\cdots$). This means Euler's number is nothing but the exponetial function $\exp(x)$ eveluated at 1.
We can also eveluate this series with a more complex function: let $f(x) = \exp(2x)$. We tehn get:

$$f^{(n)}(x) = 2^n \exp(2x) \Rightarrow f^{(n)}(0) = 2^n \ \forall\, n \in \mathbb{N}$$

$$\therefore \exp(2x) = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$$

Which is the same result as what we would find for subsituting $2x$ for x in the Maclaurin series for $\exp(x)$.

6.2 Remainder terms of Taylor series and Taylor's theorem

A complete Taylor series from 0 up untill ∞ can be split up in a partial sum of the Taylor series and the remainder terms of the series. This looks a bit like:

$$\sum_{n=0}^{\infty} T_n = \sum_{n=0}^{N} T_n + \sum_{n=N+1}^{\infty} T_n \tag{6.6}$$

Where $\sum_{n=N+1}^{\infty} T_n = R_n$ which is the remainder term of the Taylor series. Subistuting back in the definition of a Taylor series we get:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + \sum_{n=N+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{6.7}$$

Taylor's theorem is a way of quantifying the remainder term of a Taylor series:

$$\begin{aligned} &|f^{(n+1)}(x)| \leq M \ , (a-d \leq x \leq a+d) \to \ x \in [a-d, \, a+d] : \\ &|R_n(x)| = |f(x) - T(x)| \leq \frac{M}{(n+1)!}|x-a|^{(n+1)} \leq \frac{Md^{(n+1)}}{(n+1)!} \end{aligned} \tag{6.8}$$

6.3 Taylor series of trigonometric functions

In this section we will analyse the Taylor series of a trigonometric function by studying $f(x) = \sin(x)$:

$$\begin{cases} f(x) = \sin(x) & \Rightarrow f(0) = 0 \\ f'(x) = \cos(x) & \Rightarrow f'(0) = 1 \\ f''(x) = -\sin(x) & \Rightarrow f''(0) = 0 \\ f'''(x) = -\cos(x) & \Rightarrow f'''(0) = -1 \\ f''''(x) = \sin(x) & \Rightarrow f''''(0) = 0 \\ \vdots \end{cases} \tag{6.9}$$

By filling this into the Maclaurin series of $f(x)$ we find that:

$$\begin{aligned} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ \therefore \sin(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \end{aligned} \tag{6.10}$$

From this we can also derrive the Taylor series of $\cos(x)$ since it's nothing but the first derriavtive of the Taylor series of $\sin(x)$ we find that:

$$\begin{aligned} \forall x \in \mathbb{R}, \, \cos(x) &= \frac{\mathrm{d}(\sin(x))}{\mathrm{d}x} \\ &= \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left((-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \end{aligned} \tag{6.11}$$

6.4 List of common Taylor series

$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad x \in \mathbb{R}$

$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad x \in \mathbb{R}$

$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad x \in \mathbb{R}$

$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad x \in (-1,1]$

$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad x \in (-1,1)$