6 Describing rotation using Euler angles

6.1 Euler angles and rotation matrices

Rather then describing rotation in terms of angles between unit vectors in triads we can also specify 3 angles for rotation to fully describe orientation. In the case of Euler angle we describe rotation using the yaw-pitch-roll convention. Yaw describes rotation about the Z axis of the $\mathcal{N}$ tiad by an angle of ψ where $\psi \in [0, 2\pi)$. We create an intermediate triad $\mathcal{F}$ after the yaw rotation with the axis x', y' and z'. Pitch happens about the y' axis of this intermediate triad by an angle of θ, where $\theta \in [0, 2\pi)$. This creates another intermediate triad that we call $\mathcal{G}$ with the axis x'', y'' and z''. Finally we roll about the x'' axis of this second intermediate triad by an angle of φ where $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. This creates our final triad $\mathcal{B}$. We can describe these 3 succesive rotations in terms of matrices. Since the rotation only happens about one axis at a time we can simplify the rotation matrix which gives the following:

$$
\begin{aligned}
{}^{\mathcal{B}}C_{\mathcal{N}} &= {}^{\mathcal{B}}C_{\mathcal{G}}(\varphi)\, {}^{\mathcal{G}}C_{\mathcal{F}}(\theta)\, {}^{\mathcal{F}}C_{\mathcal{N}}(\psi) && (6.1) \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi) & \sin(\varphi) \\ 0 & -\sin(\varphi) & \cos(\varphi) \end{bmatrix} \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} && (6.2)
\end{aligned}
$$

6.2 Principal action of rotation

Euler's rotation theorem states that every chain of 3 succesive rotation can also be described by a single rotation about the principal axis of rotation. This principal axis of rotation implies that there exists a subspace which under the transformation of the rotation matrix stays the same and unscaled. From this we can conclude 2 things:

1. The rotation matrix has an eigenvalue of 1
2. The eigenvector corresponding to this eigenvalue is the direction for the principal axis of rotation.