1 Sequences and limits

1.1 Definition of a sequence

A sequence can be thought of as a list of numbers or as a type of function which maps an input from $\mathbb{N}$ to $\mathbb{R}$. Sequences can be defined either explicitly, or recursively. An explicit definition is a sequence which is defined at any point as some function of n. A recursive definition is a sequence in which the value for a_n is defined in terms of a_{n-1}[1]. Some sequences can be defined only explicitly. Others can be defined exclusivly with recursion. Some sequences both work.

- Explicit: $a_n = f(n)$
- Recursively: $a_n = g(a_{n-1})$ or $a_n = g(a_{n-1}, a_{n-2})$. Note that recursive definitions require initial values to be computed.

Sequences can be denoted in multiple ways, for example:

$$a_1, a_2, a_3, \cdots, a_n, \cdots$$

This is more compactly denoted as:

$$\{a_n\} \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}$$

An example of a sequence is as follows:

$$\{a_0, a_1, a_2, a_3, \cdots\} = \{1, 2, 4, 8, 16, \cdots\}$$

Defining this sequence explicitly gives:

$$a_n = 2^n$$

defining it recursively gives:

$$a_n = 2a_{n-1},\ a_0 = 1$$

Another example:

$$\{a_0, a_1, a_2, a_3, \cdots\} = \{2, 0, 2, 0, 2, \cdots\}$$

This sequence can be given with the following explicit form:

$$a_n = 1 - (-1)^n$$

1.2 Limits and sequences

Some sequences blow up to infinity with sufficiently large choice of n, other converge to 1 single value. We can use a limit to find whether a sequence is convergent, divergent or neither. The limit of sequence can be formally defined as such:
A sequence $\{a_n\}$ has a limit L if for every $\varepsilon > 0$ there is a corresponding integer value $N \in \mathbb{N}$ such that $n \geq N$ and $|a_n - L| < \varepsilon$.
What this means informaly that the limit of a sequences denoted as $\lim_{n\to\infty} a_n = L$ exists if we can get arbitrarily close to L for a sufficiently large n. If the limit exists a sequence is convergent.
Let $\lim_{n\to\infty} a_n = L \in \mathbb{R}$, $\lim_{n\to\infty} b_n = M \in \mathbb{R}$, $c \in \mathbb{R}$. Some algebraic rules for working with limits are:

- $\lim_{n\to\infty} c \cdot a_n = c \cdot L$
- $\lim_{n\to\infty} a_n \pm b_n = L \pm M$
- $\lim_{n\to\infty} a_n \cdot b_n = L \cdot M$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$, iff $M \neq 0$

Furthermore if $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n = L$.

[1] or when $n + 1$ is defined in terms of n, it's the same

1.3 Evaluating limits

Evaluating limits can often give undefined forms such as $\infty \pm \infty$, $\frac{0}{0}$ and $\frac{\infty}{\infty}$. In these cases usual procedure is to divide every term by the highest polynomial power. For example:

$$\begin{aligned}\lim_{x\to\infty}\sqrt{2+6x+4x^2} &= \lim_{x\to\infty}\sqrt{\frac{2}{x^2}+\frac{6x}{x^2}+\frac{4x^2}{x^2}}\\ &= \lim_{x\to\infty}\sqrt{0+0+4} = 2\end{aligned}$$

In some cases the limit can be difficult to actually calculate. In such cases the squeeze theorem can be a helpfull tool. The theorem states that: if $a_n \le b_n \le c_n$ for $n \ge N$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then the $\lim_{n\to\infty} b_n = L$. This is best illustrated with an example. We want to find out whether the following sequence converges or diverges:

$$\{a_n\} = \left\{\frac{\sin(n)}{n}\right\}$$

Recall that $-1 \le \sin(n) \le 1$. Therefore:

$$\frac{-1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n},\ n > 0$$

If we then choose $\{b_n\} = \left\{\frac{-1}{n}\right\}$ and $\{c_n\} = \left\{\frac{1}{n}\right\}$ we find that:

$$\lim_{n\to\infty}\{b_n\} = \lim_{n\to\infty}\{c_n\} = 0$$

This means that by Squeeze theorem the limit of $\{a_n\}$ must also be equal to 0. This is visualized in figure 1.1. Evaluating limits of sequences defined recursively rather then explicitly is a bit different.

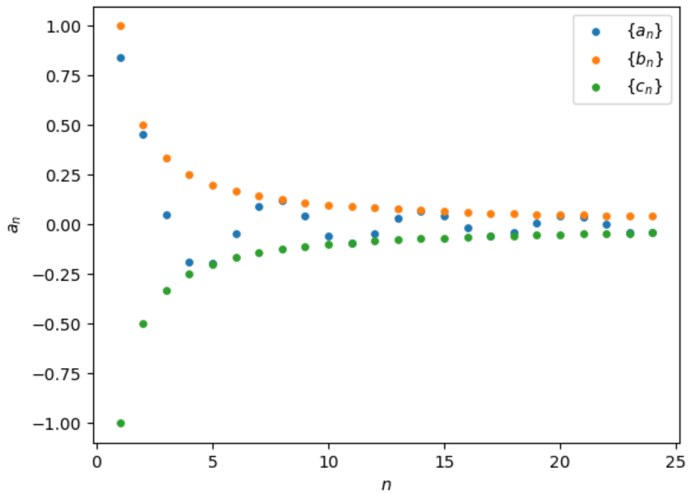

Figure 1: Squeeze theorem visualized. $\{b_n\}$ converges to 0, $\{c_n\}$ also converges to 0, and since $\{a_n\}$ is in between both of them it must also converge to 0. $\{a_n\} = \frac{\sin(n)}{n}$, $\{b_n\} = \frac{-1}{n}$ and $\{a_n\} = \frac{1}{n}$.

let $\{a_n\}_{n=1}^{\infty}$ be defined as follows:

$$a_{n+1} = g(a_n)$$

For a continuous function $g : \mathbb{R} \to \mathbb{R}$. If $\lim_{n\to\infty} a_n = L$ exists then $L = g(L)$.

1.4 L' Hospital's rule

In some cases the following can happen when eveluating limits:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{or} \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

Where either $a \in \mathbb{R}$ or $a = \pm\infty$. L' Hospital'sis another way to evelaute these limits and states that:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = L$$

An example of the application of this is eveluating the limit of $\frac{\sin(x)}{x}$.

$$\begin{aligned} \lim_{x \to 0} \frac{\sin(x)}{x} &= \lim_{x \to 0} \frac{\cos(x)}{1} \\ &= \frac{1}{1} = 1 \end{aligned}$$

1.5 Monotome and bounded sequences

Sequences can either be increasing or decreasing. A sequence is defined to be increasing if for all $n \geq 1$ $a_n < a_{n+1}$. Logically a sequence is decreasing if for all $n \geq 1$ $a_n > a_{n+1}$. Sometimes the terms not-decreasing and not-increasing are also as less strict versions of increasing and decreasing. Even weaker still the term eventually increasing/decreasing is sometimes used for sequences which will increase/decrease first and start decreasing/increasing again for a sufficiently large n. Furthermore a sequence is monotome if it is either increasing or decreasing. Thus we end up with the following list:

- $\{a_n\}$ is increasing if $a_n < a_{n+1}$ for all $n \geq 1$
- $\{a_n\}$ is not dencreasing if $a_n \leq a_{n+1}$ for all $n \geq 1$
- $\{a_n\}$ is decreasing if $a_n > a_{n+1}$ for all $n \geq 1$
- $\{a_n\}$ is not increasing if $a_n \geq a_{n+1}$ for all $n \geq 1$
- $\{a_n\}$ is monotome if it is either increasing or decreasing

Sequences can be bounded. The bound refers to some maximum or minimum value which the sequence will not exceed. A set is bounded if it is both upper and lower bounded.

- $\{a_n\}_{n=1}^{\infty}$ is upper bounded if a value M exists such that $a_n \leq M$ for all $n \geq 1$
- $\{a_n\}_{n=1}^{\infty}$ is upper bounded if a value m exists such that $a_n \geq M$ for all $n \geq 1$
- $\{a_n\}_{n=1}^{\infty}$ is bounded if it is both upper and lower bounded

When a set is both bounded and monotome it will always be convergent.