3 Testing series for convergence

3.1 The integral test

let s_n be a series who is defined as follows:

$$s_n = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \tag{3.1}$$

Now let's compare this series to the graph of $f(x) = \frac{1}{x^2}$ on the interval $[1, \infty)$. This can be found in the figure below. Using this figure we can easily see that the total area of the series must be less then

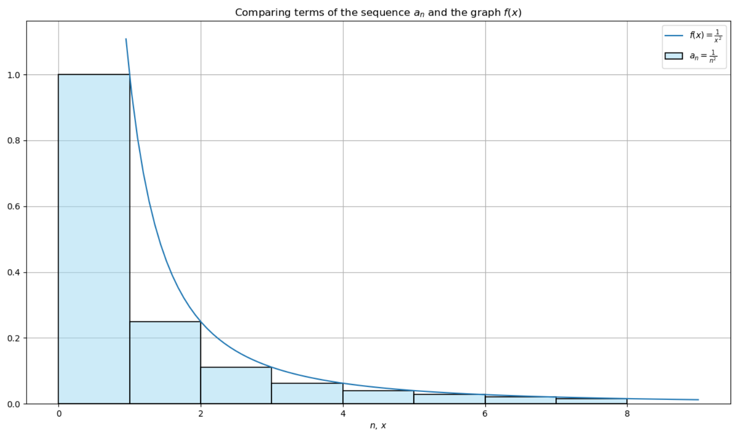


Figure 3.1: The graph $f(x) = \frac{1}{x^2}$ as compared to the series $s_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$

the total area under the graph of $f(x)$. This means we end up with the following relation:

$$s_n < \int_1^{\infty} f(x)\,\mathrm{d}x \tag{3.2}$$

$$1 + \sum_{n=2}^{\infty} \frac{1}{n^2} < 1 + \int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x \tag{3.3}$$

$$< 1 + 1 = 2$$

Thus the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ must converge since the integral of $f(x)$ also converges. We can state that it has a sum greater then 1 but smaller then 2. Computing the exact value of the series is outside of the scope of the course, however the answer is $\frac{\pi^2}{6}$. Look up the Basel problem if interested.

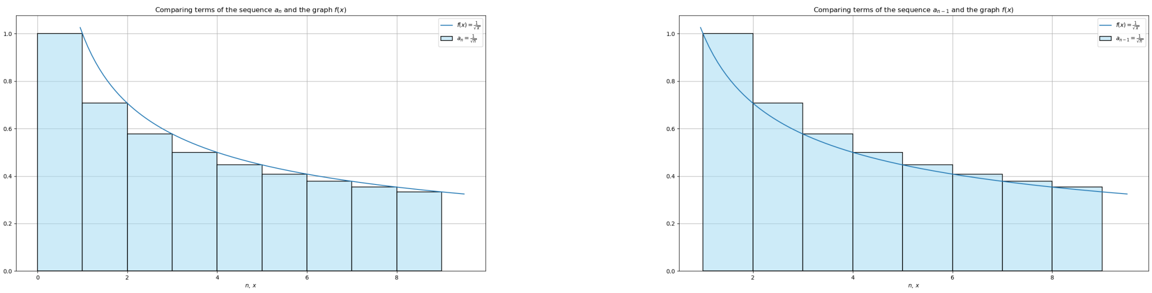
From these graphs we can easily find the following relation:

$$\sum_{n=2}^{\infty} f(n) \leq \int_1^{\infty} f(x)\,\mathrm{d}x \leq \sum_{n=1}^{\infty} f(n) \tag{3.4}$$

Let f be a positive, continuous, (eventually) decreasing function on the interval $[1, \infty)$. Let $a_n = f(n)$. The series $\sum_{n=1}^{\infty} a_n$ converges iff the integral $\int_1^{\infty} f(x)\,\mathrm{d}x$ is convergent.

Example: Find whether the following sum converges or diverges:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$$



(a) $\sum_{n=1}^{\infty} f(n) \geq \int_1^{\infty} f(x)\,\mathrm{d}x$

(b) $\sum_{n=2}^{\infty} f(n) \leq \int_1^{\infty} f(x)\,\mathrm{d}x$

Figure 3.2: The series $f(n)$ and the function $f(x)$ graphed twice for starting the series at 1 and at 2.

We compare this to the integral from 1 to infinity of the graph $f(x) = \frac{1}{\sqrt{x+1}}$. This gives:

$$\begin{aligned}
\int_1^{\infty} \frac{1}{\sqrt{x+1}}\,\mathrm{d}x &= \lim_{t\to\infty} \int_1^{t} \frac{1}{\sqrt{x+1}}\,\mathrm{d}x \\
&= \lim_{t\to\infty} 2\sqrt{x+1}\Big|_1^t \\
&= \infty
\end{aligned}$$

Since $f(x)$, a continuous decreasing function on the interval $[1, \infty)$, diverges we can state that the sum $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ must also diverge since it must be greater then the integral $\int_1^{\infty} f(x)\,\mathrm{d}x$.

3.2 P-series and comparison to p-series

Let's start with a question: for which value of p is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?
Right of the bat we notice 2 things:

- $p = 1$ gives us the harmonic series which we know diverges
- $p \leq 0$ has a general term that doesn't go to 0 and thus can't converge

Let's apply the comparison to an integral to this series:

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^p}\,\mathrm{d}x &= \lim_{t\to\infty} \int_1^{t} x^{-p}\,\mathrm{d}x \\
&= \lim_{t\to\infty} \frac{x^{-p+1}}{-p+1}\Bigg|_0^t
\end{aligned} \tag{3.5}$$

From this we can see that this limit converges if $p > 1$. From this we can conclude that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent iff[1] $p > 1$. This is a fact we can use in our analysis. Whenever we encounter a series which takes the form of the p-series we can conclude whether it converges or diverges based on the value of p alone. Further analysis will then not be required.

3.3 Estimating remainder of a series

Let a_k be some sequence. We can use this sequence to construct a series. Recall that the partial sum of a series is given as the first n terms of the series. This leaves a remainder of the series from the

[1] if and only if

term $n+1$ up untill ∞. We can express this as:

$$\begin{aligned}\sum_{k=1}^{\infty} a_k &= \sum_{k=1}^{n} a_k + \sum_{k=n+1}^{\infty} a_k \\ &= s_n + R_n \end{aligned} \tag{3.6}$$

Where R_n is the remainder term of the series and s_n the partial sum of the first n terms. If f is some continuous, positive, (eventually) decreasing function on the interval $[1, \infty)$ and $a_n = f(n)$:

$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq R_n \leq \int_{n}^{\infty} f(x)\,\mathrm{d}x \tag{3.7}$$

We can add s_n to all term of this expression and since $\sum_{k=1}^{\infty} a_k = s_n + R_n$ we can subsitute this back in to find:

$$s_n + \int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq s \leq s_n + \int_{n}^{\infty} f(x)\,\mathrm{d}x \tag{3.8}$$

Example: How many terms do we need to add up in te series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ for $R_n < 10^{-3}$? We start by applying the theorem we established earlier:

$$\begin{aligned}\int_{n+1}^{\infty} \frac{1}{x^4}\,\mathrm{d}x \leq & R_n \leq \int_{n}^{\infty} \frac{1}{x^4}\,\mathrm{d}x \\ \lim_{t\to\infty} \left.\frac{1}{3(x+1)^3}\right|_{n+1}^{t} \leq & R_n \leq \lim_{t\to\infty} \left.\frac{1}{3x^3}\right|_{n+1}^{t} \\ \frac{1}{3(n+1)^3} \leq & R_n \leq \frac{1}{3n^3}\end{aligned}$$

We now found the expression for the upper and lower bound on the remainder term. From here we can try different values for n either by hand or using a computer to find which value for n statisfies the condition that $R_n < 10^{-3}$. We find that for $n = 7$ we get:

$$\frac{1}{3\cdot 8^3} \leq R_7 \leq \frac{1}{3\cdot 7^3} 0.00065 \leq R_7 \leq 0.00097 < 10^{-3}$$

Thus by adding up the first 7 terms of the series we get an answer which is within 10^{-3} of the exact answer. We can use this to find that:

$$\sum_{k=1}^{7} \frac{1}{k^4} \approx 1.08154$$

From which we can conclude that the exact value of the series when we add up all terms up untill ∞ will be the following:

$$1.08219 \leq \sum_{n=1}^{\infty} \frac{1}{n^4} \leq 1.08251$$

3.4 Comparison testing

To test whether a series that is hard to solve diverges we can compare it to a series which is easier to solve. This can be more formally stated as follows: Let $\{a_n\}$ and $\{b_n\}$ be sequences with positive for which (eventually): $0 \leq a_n \leq b_n$. If $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ is also convergent. Conversely if $\sum_{n=0}^{\infty} a_n$ diverges the series $\sum_{n=0}^{\infty} b_n$ will also diverge. Important: if the infinite series of b_n diverges we can not say for sure that the infinite series of a_n diverges because a_n is smaller then b_n. The 2 most often used series for comparison are the geometric series Σr^n and the p-series $\Sigma \frac{1}{n^p}$. We see if the we can approximate the series we are looking as behaving roughly like one of those 2. The behaviour of the p-series and geometric series is easy to study which is why we wish to use them for comparison testing.

Example: Does the infinite series $\sum_{n=2}^{\infty} \frac{\ln(n)}{\sqrt{n}}$ converge or diverge? Let's first start by noting that $\ln(n)$ will get bigger very slowly compared to the $\sqrt{(n)}$ term. This means we can say the series roughly behaves like the following series:

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$

This is nothing but a p-series for which $p = \frac{1}{2}$. We found earlier that for a p-series with $p < 1$ the series diverges, which means that the series $\sum_{n=2}^{\infty} \frac{\ln(n)}{\sqrt{n}}$ also diverges.

Example: Given are the following sequences: $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ order these sequence by smallest to largest sum.

$$\begin{aligned} a_n &= \frac{2}{n+\sqrt{n}} \quad \text{behaves like} \quad \frac{2}{n} = \frac{2\sqrt{n}}{n\sqrt{n}} \\ b_n &= \frac{2\sqrt{n}}{n^2+1} \quad \text{behaves like} \quad \frac{2}{n\sqrt{n}} \\ c_n &= \frac{\ln(n^2)}{n\sqrt{n}} \quad \text{behaves like} \quad \frac{2\ln(n)}{n\sqrt{n}} \end{aligned}$$

Since all of these have the same denominator we can easily compare these series by looking at the numerators only. From this it's easy to see that $b_n < c_n < a_n$.

3.5 The limit comparison test

Let $\{a_n\}$ and $\{b_n\}$ be sequences with positive terms and let the limit to infinity of the ratio of a_n and b_n be equal to some constant value c: $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ then:

1. if $c > 0$: $\sum_{n=0}^{\infty} a_n$ converges $\Leftrightarrow$ $\sum_{n=0}^{\infty} b_n$ converges. The equivalent is also true. If one diverges so does the other.

2. if $c = 0$: $\sum_{n=0}^{\infty} b_n$ converges $\Rightarrow$ $\sum_{n=0}^{\infty} a_n$ converges. The equivalent is also true if b_n diverges then so does a_n. Be aware that this goes 1 way only, not both ways. The divergence or convergence of a_n in the case where $c = 0$ doesn't give us any information on b_n.