1 Introduction to continuum mechanics

1.1 The continuum theory

Steel when observed from afar looks like one homogeneous medium, however if we zoom in we find that the material, which initally looked smooth, has a crystalline structure. Zooming in a bit further then that we find that dislocations occur in the crystalline structure. Zooming in even further we find that the material is made up of individual atoms arranged in a lattice. When analysing a material we usually want to ignore details like crysltalline structure as this would make models very computationally expensive and too complex. So instead for the analysis we choose a volume element of the material which is sufficiently large, such that we interpret variables of interest as some mean value over a small discrete volume element. Describing materials this way is referred to as a **continuum description**.
Because of the continuum description variables such as strain, stress, temperature, displacement, etc. can be described as a continuous differentiable function. The continuous function represent the mean values of these variables for a small volume of material. This description is only valid if the fluctuation of these variables takes place at a scale which is large as compared to the material length scale.

1.2 The basic equations in statics

In statics we generally consider three distinct sets of equations:

1. The requirements for continuity, a set of equations ensuring all material stays connected to itself without any discontinuities arising.
2. Equilibrium, a set of equations which relates stress distributions and externally applied loads.
3. Relating deformation and stress.

This is often called a constitutive model. In general the computationally easiest way to go about this is the following order:

$$\text{Displacements} \Rightarrow \text{Deformations} \Rightarrow \text{Stresses} \Rightarrow \text{Extrernal Loads}$$

The problem is: in statics we usually start with external loads, and we want to figure out what displacement these loads are causing.

1.3 Continuity

Consider a 3 dimensional body in undeformed and in deformed configuration as show in figure 1.1. The position vector from the origin to the material point P can be given as:

$$\vec{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z \tag{1.1}$$

When deformed the point P will be transformed to the point P'. The vector from the origin to this new point P' will be $\vec{r}\,'$. The vector $\vec{u}$ which is the position vector to point P' with respect to point P can then be given as follows:

$$\vec{u} = u_x\hat{\mathbf{e}}_x + u_y\hat{\mathbf{e}}_y + u_z\hat{\mathbf{e}}_z = \vec{r}\,' - \vec{r} \tag{1.2}$$

Note that both the vectors $\vec{r}$ and $\vec{u}$ are vector valued functions of spatial coordinates:

$$\vec{u} = \vec{u}(x, y, z) \tag{1.3}$$
$$\vec{r} = \vec{r}(x, y, z) \tag{1.4}$$

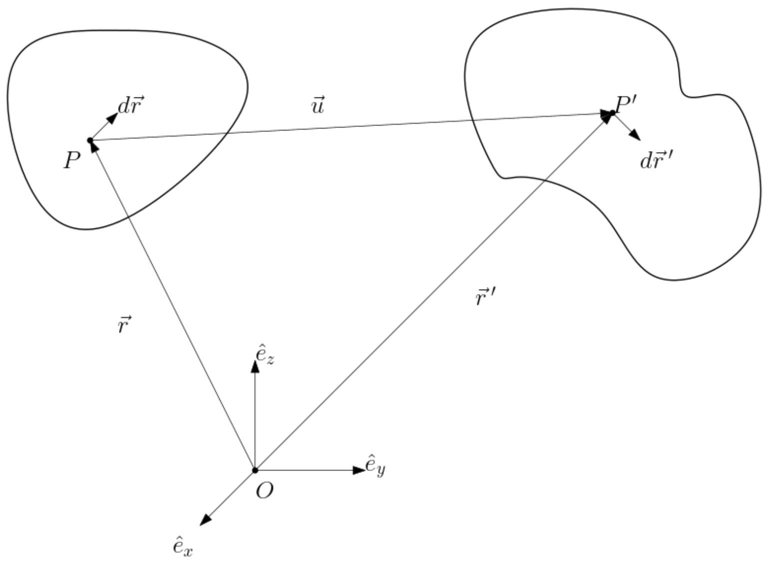

Figure 1: Representation of some patato shaped continuum in udeformed and in deformed configuration

Now consider a differential element of the vector $\vec{r}$:

$$\mathrm{d}\vec{r} = \mathrm{d}x\hat{\mathbf{e}}_x + \mathrm{d}y\hat{\mathbf{e}}_y + \mathrm{d}z\hat{\mathbf{e}}_z \tag{1.5}$$

This vector represents a material element which is infinitesimally close to the original point P. We consider the same point infinitesimally close to P' with the unit vector $\vec{r}\,'$:

$$\mathrm{d}\vec{r}\,' = \mathrm{d}(\vec{r} + \vec{u}) = \mathrm{d}\vec{r} + \mathrm{d}\vec{u} \tag{1.6}$$

We can take a differential element of the vector $\vec{u}$ as:

$$\mathrm{d}\vec{u} = \mathrm{d}u_x\hat{\mathbf{e}}_x + \mathrm{d}u_y\hat{\mathbf{e}}_y + \mathrm{d}u_z\hat{\mathbf{e}}_z \tag{1.7}$$

Where the terms $\mathrm{d}u_x$, $\mathrm{d}u_y$ and $\mathrm{d}u_z$ represent the deformation in the x, y and z direction. If we let this deformation 'act' on some position vector $\vec{r}$ to some point on the continuum we get:

$$\mathrm{d}u_x = u_x(\vec{r} + \mathrm{d}\vec{r}) - u_x(\vec{r}) \tag{1.8}$$

$$= \cancel{u_x} + \frac{\partial u_x}{\partial x}\mathrm{d}x + \frac{\partial u_x}{\partial y}\mathrm{d}y + \frac{\partial u_x}{\partial z}\mathrm{d}z - \cancel{u_x} \tag{1.9}$$

$$= \frac{\partial u_x}{\partial x}\mathrm{d}x + \frac{\partial u_x}{\partial y}\mathrm{d}y + \frac{\partial u_x}{\partial z}\mathrm{d}z \tag{1.10}$$