5 Power series

5.1 The Geometric series as a power series

Recall that the geometric series was given as:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^2 + \cdots \tag{5.1}$$

The series converges for $-1 < x < 1$, in which case the series is given as:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \tag{5.2}$$

The terms of the polynomial when thought of as power series will start approximating the graph of $\frac{1}{1-x}$ closer and closer for larger choices of the partial sum n. This is visualised in the figure below.

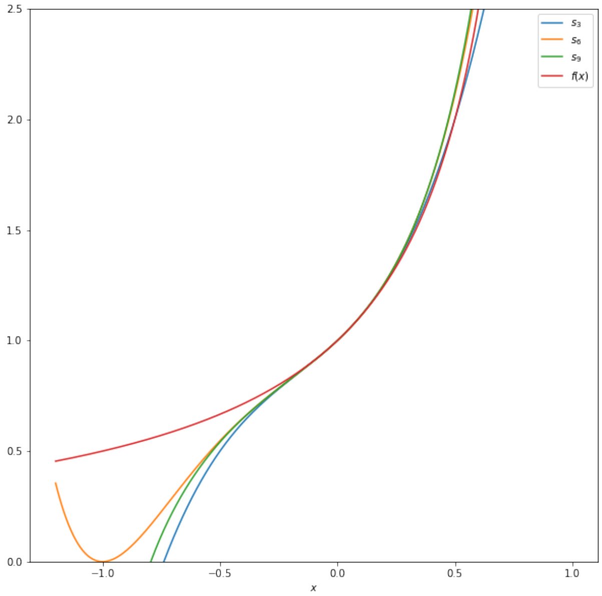

Figure 5.1: The terms of the polynomial $s_n = \sum_{n=0}^{n} x^n$ approximating the graph of $f(x) = \frac{1}{1-x}$ closer and closer for larger choices of n.

We already know from studying the geometric series in an earlier lecture that the limit only exists when:

$$\lim_{n\to\infty} s_n = \frac{1}{1-x} \quad \text{if} \quad -1 < x < 1 \tag{5.3}$$

Thus we are lead to the conclusion that if $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, then $f(x) \approx s_n$ where $s_n = c + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$.

Example:

$$\frac{1}{5-x} = \frac{1}{1-(x-4)} = \sum_{n=0}^{\infty} (x-4)^n$$

The condition for the polynomial to exists is:

$$-1 < (x-4) < 1$$
$$3 < x < 5$$

Thus:

$$\frac{1}{5-x} \approx s_6 = \sum_{n=0}^{6} (x-4)^i$$

5.2 Defining a power series

Let $x \in \mathbb{R}$. A power series is then given as:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \tag{5.4}$$

The c_n terms are called the coefficients and a is the center of the polynomial. Every power series will satisfy 1 of the following 3 conditions:

1. The series converges (absolutely) for all $x \in \mathbb{R}$
2. The series converges only for $x = a$
3. There exists a number $R > 0$ such that the series converges (absolutely) if $|x-a| < R$ and diverges for $|x-a| > R$.

In case 3 the number R is referred to as the radius of convergence. The best way to determine R is by using the root or ratio test.

Example:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{Let} \quad a_n = \frac{x^n}{n!}$$

Applying the ratio test we find that:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)!} \Big/ \frac{x^n}{n!}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 \quad \text{for all } x \in \mathbb{R}$$

Thus the series converges absolutely for all $x \in \mathbb{R}$.

$$\sum_{n=0}^{\infty} \frac{(x-1)^{2n}}{n2^n} \quad \text{Let} \quad a_n = \frac{(x-1)^{2n}}{n2^n}$$

Applying the ratio test here we find:

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(x-1)^{2(n+1)}}{(n+1)2^{n+1}} \Big/ \frac{(x-1)^{2n}}{n2^n}\right| \\ &= \lim_{n\to\infty} \frac{(x-1)^2}{2} \cdot \frac{n}{n+1} = \frac{(x-1)^2}{2}\end{aligned}$$

Thus this series converges absolutely for $(x-1)^2 < 2$. This means the series diverges for $x = 1 \pm \sqrt{2}$. This give the convergence interval $(1-\sqrt{2}, 1+\sqrt{2})$.

5.3 Integrating and differentiating power series

One big advantage of power series is the fact that all terms are polynomial terms. We like those because they are easy to integrate and differentiate. If $p(x)$ is some n-th degree polynomial we get:

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n \tag{5.5}$$

$$\frac{\mathrm{d}p(x)}{\mathrm{d}x} = c_1 + 2c_2(x-a) + 3c_3(x-a)^3 + \cdots + nc_n(x-a)^{n-1} \tag{5.6}$$

$$\int p(x)\,\mathrm{d}x = C + c_0(x-a) + \frac{1}{2}c_1(x-a)^2 + \cdots + \frac{1}{n-1}c_n(x-a)^{n+1} \tag{5.7}$$

Applying this to the power series we can write the derrivative $s'(x)$ of some power series $s(x) \sum_{n=0}^{\infty} x_n(x-a)^n$ and the integral $S(x)$ of this power series as:

$$s'(x) = \frac{\mathrm{d}s(x)}{\mathrm{d}x} = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1} \tag{5.8}$$

$$S(x) = \int s(x)\,\mathrm{d}x = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} \tag{5.9}$$