6 Using potential energy

6.1 The spring system

Consider the following spring system. When we would analyze this only using external virtual work we would end up with only 1 equation. We can include the equilibrium between internal and external work in our analysis to get 3 equations rather then 1 when analyzing this system.

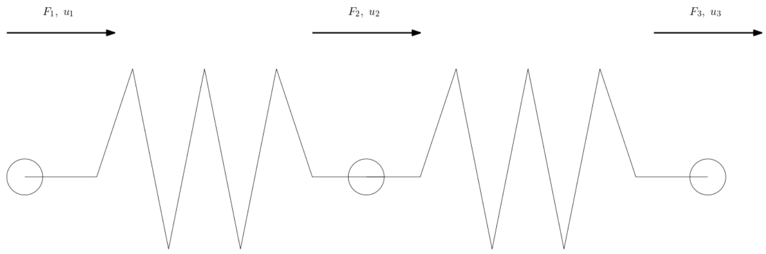

Figure 6.1: The spring system being analyzed for internal and external balance of virtual work

We can easily see from the figure that:

$$\Delta L_I = u_2 - u_1 \tag{6.1}$$
$$\Delta L_{II} = u_3 - u_2 \tag{6.2}$$

When considering an infinitesimal virtual displacement δ we get:

$$\delta(\Delta L_I) = \delta u_2 - \delta u_1 \tag{6.3}$$
$$\delta(\Delta L_{II}) = \delta u_3 - \delta u_2 \tag{6.4}$$

From this we can derrive the following equation by substitution:

$$\delta(\Delta L_I) = \frac{\partial \Delta L_I}{\partial u_1}\delta u_1 + \frac{\partial \Delta L_{II}}{\partial u_2}\delta u_2 \tag{6.5}$$

We now consider the internal and external work of the system:

$$\delta W_i = F_{v,I}\delta(\Delta L_I) + F_{v,II}\delta(\Delta L_{II}) \tag{6.6}$$
$$\delta W_u = F_1\delta u_1 + F_2\delta u_2 + F_3\delta u_3 \tag{6.7}$$

Applying the condition that $\delta W_i = \delta W_u$ since energy can not be created we can equate these to equations and find a system of 3 equations:

$$F_{v,I}(\delta u_2 - \delta u_1) + F_{v,II}(\delta u_3 - \delta u_2) = F_1\delta u_1 + F_2\delta u_2 + F_3\delta u_3 \tag{6.8}$$
$$\begin{cases} F_1 = -F_{v,I} \\ F_2 = F_{v,I} - F_{v,II} \\ F_3 = F_{v,II} \end{cases} \tag{6.9}$$

Thus we now derrived a system of 3 equations rather then 1 equation by also including internal virtual work in out equations.

6.2 Minimum of potential energy

Internal work is nothing but stored energy in the form of elasticity. This is a form of potential energy. The total potential of a system can be given by the following expression:

$$\mathcal{P}[\vec{u}_0] = \mathcal{E}[\vec{u}_0] + \mathcal{B}[\vec{u}_0]$$

Where $\mathcal{P}$ is the total potential, $\mathcal{E}$ the elastic potential and $\mathcal{B}$ the potential energy due to externally applied loads. $\vec{u}_0$ is a random displacement vector which is kinematically allowed. This means the displacement cannot break the continuity of the material. The following 2 additional relations exist:

$$\mathcal{P}[\vec{u}_0] = \text{Stationary if there is equilibrium} \tag{6.10}$$
$$\mathcal{P}[\vec{u}_0] = \text{Local minimum if the equilibrium is stable} \tag{6.11}$$

There are 3 conditions which need to apply for analysis using potential energy to be valid.

1. System must be elastic, though non-lineare behaviour is allowed
2. No damping on the system
3. System is conservative[1]

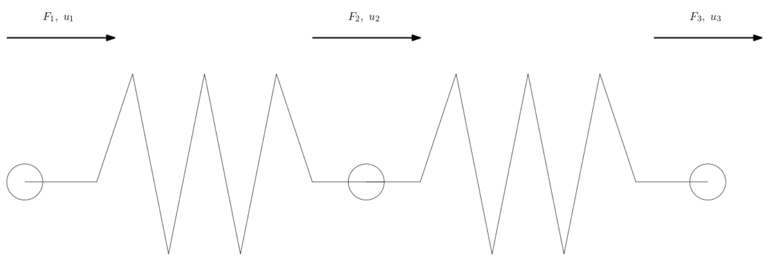

Figure 6.2: Teh same spring system but this time applyign potential enrgy analysis

We can now again analyse the spring system, but this time using potential energy. We start again by defining displacements:

$$\Delta L_I = u_2 \tag{6.12}$$
$$\Delta L_{II} = u_3 - u_2 \tag{6.13}$$
$$\tag{6.14}$$

The elastic potential of this system is then given as:

$$\mathcal{E} = \Sigma\mathcal{E}_i = \frac{1}{2}ku_2^2 + \frac{1}{2}k(u_3 - u_2)^2 \tag{6.15}$$

The load potential for this system is given as:

$$\mathcal{B} = -F_3u_3 \tag{6.16}$$

Which is just the equilibrium condition. We can now derrive the total potential of the system.

$$\mathcal{P} = \mathcal{E} + \mathcal{B} = \frac{1}{2}ku_2^2 + \frac{1}{2}k(u_3 - u_2)^2 - F_3u_3 \tag{6.17}$$

$\mathcal{P}$ must be a stationary value for equilibrium to exist. This implies that:

$$\begin{cases} \frac{\partial\mathcal{P}}{\partial u_1} = 0 \\ \frac{\partial\mathcal{P}}{\partial u_2} = 0 \end{cases} \tag{6.18}$$

Thus for 2 variables we found a corresponding system of 2 equations.

[1]This means that energy is conserved, not that the system supports right-wing politics