Control of Stochastic Systems
Lecture 4
Time-Invariant Stochastic Systems

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Outline

Outline

Controllability

Motivation

- Controllability is a major concept of system theory.
- Controllability is a necessary and sufficient condition for existence of a control law.
- Controllability has been defined for:
  (1) sets and maps,
  (2) a deterministic linear system,
  (3) a stochastic system,
  (4) a Gaussian system.
- Literature has the terms reachability and controllability. The latter term is used in these lectures.

Controllability

Def. Controllable map

Consider a tuple of sets and maps $(U,\ X,\ Y,\ g : U \to X,\ h : X \to Y)$.
Call $g : U \to X$ the input-to-state map.
Call the tuple controllable with respect to the subset $X_{co} \subseteq X$ if

$$\begin{aligned} & g : U \to X_{co} \subseteq X \text{ is surjective,} \\ \Leftrightarrow\ & \forall x_c \in X_{co} \subseteq X,\ \exists\ u_c \in U \text{ such that } x_c = g(u_c); \\ & \text{if in addition } X_{co} = X \\ & \text{then call } g \text{ completely surjective and} \\ & \text{the tuple completely controllable.} \end{aligned}$$

Remark. Interpretation in words

For all states $x_c \in X_{co}$
there exists an input $u_c \in U$ such that $x_c = g(u_c)$;
or, equivalently,
the input value $u_c \in U$ brings the system to state $x_c \in X_{co}$.

Controllability

Proposition

Consider a linear input-to-state map,

$$g(u) = Gu, \quad G \in \mathbb{R}^{n_x \times n_u}, \; X = \mathbb{R}^{n_x}, \quad U = \mathbb{R}^{n_u}, \quad X_{co} = X.$$

This map is completely surjective ($X_{co} = X$),
if and only if

$$\mathrm{rank}(G) = \dim(\mathrm{Im}(G)) = \dim(X_{co}) = \dim(X) = n_x.$$

Proof

($\Rightarrow$) Use the definition of rank and the definition of image,

$$\begin{aligned}\mathrm{Im}(G) &= \{x \in X | \; \exists \; u \in U \text{ such that } x = Gu\} \\ &= X_{co}, \text{ by surjectivity of } g.\end{aligned}$$

Controllability

Def. Controllability of a time-varying linear system

Consider a time-varying deterministic linear system without output

$$x(t+1) = A(t)x(t) + B(t)u(t), \; x(t_0) = x_0.$$

Call the state $x_a \in X_{co} \subseteq X$ controllable on the interval
$\{t_0, \; t_0+1, \; \ldots, \; t_0+t_1-1, \; t_0+t_1\}$
if the following input-to-state map is surjective,

$$\{u(t_0), \; u(t_0+1), \; \ldots, u(t_0+t_1-1)\} \; \mapsto \; x(t_0+t_1) = x_a.$$

Call the linear system controllable with respect to the subset $X_{co} \subseteq X$, if

$$\begin{aligned} & \forall\; t_0 \in T, \; \forall\; t_1 \in \mathbb{Z}_+ \text{ such that } \{t_0, \; t_0+1, \; \ldots, \; t_0+t_1\} \subseteq T, \\ & \forall\; x(t_0) \in X, \; \forall\; x_a \in X_{co}, \text{ the state } x_a \text{ is controllable on this interval} \\ \Leftrightarrow & \{u(t_0), \; u(t_0+1), \; \ldots, u(t_0+t_1-1)\} \; \mapsto \; x_a \text{ is surjective;} \end{aligned}$$

linear system is called completely controllable if $X_{co} = X$.

Controllability

Def. Controllability matrix of an interval

Consider a time-varying deterministic linear system.
Define the controllability matrix of an interval by the formula

$$
\begin{aligned}
&\{t_0,\ t_0+1,\ \ldots,\ t_0+t_1-1\} \subseteq T,\\
&\mathrm{conmat}(A, B, t_0 : t_0+t_1-1)\\
&= \begin{pmatrix} B(t_0+t_1-1) & \ldots & \Phi(t_0+t_1-1, t_0)B(t_0) \end{pmatrix};\\
&\text{note the formulas,}\\
&x(t_0+t_1)\\
&= A(t_0-t_1-1)x(t_0+t_1-1) + B(t_0+t_1-1)u(t_0+t_1-1),\\
&x(t_0+t_1) - \Phi(t_0+t_1 : t_0)x(t_0)\\
&= \sum_{s=t_0}^{t_0+t_1-1} \Phi(t_0+t_1-1, s)\, B(s)\, u(s)\\
&= \mathrm{conmat}(A,\ B,\ t_0 : t_0+t_1-1)\, u(t_0+t_1-1 : t_0).
\end{aligned}
$$

Controllability

Theorem. Controllability of a time-varying linear system

Consider a time-varying linear system.
The system is completely controllable on the interval considered (completely implying that $X_{co} = \mathbb{R}^{n_x}$)
if and only if,

$$n_x = \mathrm{rank}(\text{conmat}(A, B, t_0 : t_0 + t_1 - 1)).$$

Proof

$$x(t_0 + t_1) - \Phi(t_0 + t_1, t_0)x(t_0)$$
$$= \text{conmat}(A,\ B, t_0 : t_0 + t_1 - 1)\ u(t_0 : t_0 + t_1 - 1).$$

and use the proposition
which characterizes when a linear function is completely surjective.

Controllability

Def. Time-invariant linear system

Define a time-invariant linear system as a control system with representation,

$$
\begin{aligned}
x(t+1) = Ax(t) + Bu(t),\ x(t_0) = x_0,& \\
x_0 \in \mathbb{R}^{n_x},\ u : T \to \mathbb{R}^{n_u},\ x : T \to \mathbb{R}^{n_x},& \\
A \in \mathbb{R}^{n_x \times n_x},\ B \in \mathbb{R}^{n_x \times n_u},\ n_x,\ n_u \in \mathbb{Z}_+.&
\end{aligned}
$$

Define the controllability matrix of the matrix tuple $(A,\ B)$ by the formula,

$$
\text{conmat}(A,\ B) = \begin{bmatrix} B & AB & \ldots & A^{n_x-1}B \end{bmatrix} \in \mathbb{R}^{n_x \times n_x n_u}.
$$

Call $(A,\ B)$ a controllable pair if $n_x = \operatorname{rank}(\text{conmat}(A,\ B))$.

Controllability

Remark

By the Cayley-Hamilton theorem,
one may restrict attention to the first n_x matrix powers,

$$A^{n_x} = \sum_{i=0}^{n_x-1} a_i\ A^i \ \Rightarrow\ A^{n_x}B = \sum_{i=0}^{n_x-1} a_i\ A^i B,$$

$$\Rightarrow\ \mathrm{Im}(A^{n_x}B) \subseteq \mathrm{Im}(\mathrm{conmat}(A,\ B));$$

$$\Rightarrow\ \forall\ k \in \mathbb{Z}_+,\ k \geq n_x,$$

$$\mathrm{Im}(A^k B) \subseteq \mathrm{Im}(\mathrm{conmat}(A,\ B)).$$

Proof by recursion.

Controllability

Theorem. Controllability of a time-invariant linear system

A time-invariant linear system
is controllable with respect to $X_{co} = \mathbb{R}^{n_x}$, hence completely controllable,
if and only if $(A,\ B)$ is a controllable pair,
if and only if $n_x = \mathrm{rank}(\text{conmat}(A,\ B))$.

Comments

- Matlab has a command to check
  whether the tuple $(A,\ B)$ is a controllable pair.
- Advice.
  Compute the singular values of the controllability matrix and
  check the locations of these values.

Controllability

Proposition. Control form

Consider a time-invariant linear system.

$$x(t+1) = Ax(t) + Bu(t),\ x(0) = x_0.$$

There exists a linear state-space transformation $z(t) = Sx(t)$ such that the system is transformed to the representation,

$$\begin{aligned} z(t) &= Sx(t),\ S \in \mathbb{R}^{n_x \times n_x}_{nsng}, \\ z(t+1) &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t),\ z(0) = z_0, \\ n_z &= n_x = n_{z_1} + n_{z_2}, \\ &\quad (A_{11},\ B_1) \text{ controllable pair.} \end{aligned}$$

Call this system representation, the Kalman controllable form.
For a particular system, either the part of z_1 or that of z_2 may be missing.
Proof see (R.E. Kalman (1963)); LN, Chapter 21, Section 2.

Controllability

Def. Stabilizability

Consider a time-invariant linear system.
$(A,\ B)$ is called a stabilizable tuple
if one of the following equivalent conditions holds:

(a) $\text{spec}(A_{22}) \subset \mathbb{D}_o$ after transformation to Kalman controllable form;

(b) The eigenvalue $\lambda \in \text{spec}(A)$ is called
$(A,\ B)$-stabilizable if either $\lambda \in \mathbb{D}_o$
or λ is $(A,\ B)$-controllable ($n_x = \operatorname{rank}\left(\begin{bmatrix} A - \lambda I & B \end{bmatrix}\right)$).
$(A,\ B)$ is a stabilizable tuple
if all eigenvalues of A are $(A,\ B)$-stabilizable.

(c) The eigenvalue $\lambda \in \text{spec}(A)$ is called
$(A,\ B)$-spectrally assignable
if $|\lambda| \geq 1$ and $x \in \mathbb{C}^{n_x}$ with $x^T A = \lambda x^T$ and $x^T B = 0$ imply that $x = 0$.
$(A,\ B)$ is called spectrally stabilizable pair
if for all $\lambda \in \text{spec}(A)$ either $\lambda \in \mathbb{D}_o$ or λ is $(A,\ B)$-spectrally assignable.

Controllability

Theorem. Stabilizability

(a) The stabilizability conditions of the above definition, are equivalent.

(b) Equivalence of:

- **(b.a)** $(A,\ B)$ is a stabilizable tuple.
- **(b.b)** There exists a feedback matrix $F \in \mathbb{R}^{n_u \times n_x}$ such that $\mathrm{spec}(A + BF) \subset \mathbb{D}_o$.

Example

The following time-invariant linear control system is stabilizable.

$$x(t+1) = \begin{bmatrix} 1.3 & 1 \\ 0 & 0.9 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),\ \text{because } 0.9 \in \mathbb{D}_o.$$

The following time-invariant linear control system is not stabilizable.

$$x(t+1) = \begin{bmatrix} 0.9 & 1 \\ 0 & 1.3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),\ \text{because } 1.3 \notin \mathbb{D}_o.$$

Outline

Time-Invariant Gaussian System

Comments

- A time-invariant Gaussian system will be proven to have a stationary state process and a stationary output process.
- Why are stationary stochastic processes useful models for control engineering?
- In control engineering, the processes of control systems run for a long horizon. Then a stationary output process is a useful model and with a lower system complexity than a time-varying control system.
- A special case is the model of a stationary Gaussian output process in the form of a time-invariant Gaussian system.
- Besides stationary processes, non-stationary processes are also useful. Both cases are treated in this course.

Time-Invariant Gaussian System

Def. Time-Invariant Gaussian System (1; recall from Lecture 3)

Define a time-invariant Gaussian system representation
by the sets and relations,

$$\begin{aligned}
x(t+1) &= Ax(t) + Mv(t),\ x(0) = x_0, \\
y(t) &= Cx(t) + Nv(t), \\
&\quad (\Omega,\ F,\ P),\ T = \mathbb{N}, \\
&\quad x_0 : \Omega \to \mathbb{R}^{n_x},\ x_0 \in G(m_{x_0}, Q_{x_0}),\quad F^{x_0},\ F_\infty^v \text{ independent}, \\
&\quad v : \Omega \times T \to \mathbb{R}^{n_v},\ \text{Gaussian white noise, } v(t) \in G(0, I_{n_v}), \\
&\quad A \in \mathbb{R}^{n_x \times n_x},\ C \in \mathbb{R}^{n_y \times n_x},\ M \in \mathbb{R}^{n_x \times n_v},\ N \in \mathbb{R}^{n_y \times n_v}, \\
\text{spec}(A) &= \{\lambda \in \mathbb{C} |\ \det(\lambda I - A) = 0\},\ \text{spectrum of matrix } A, \\
\mathbb{D}_o &= \{c \in \mathbb{C} |\ |c| < 1\},\ \text{called the open unit disc;} \\
&\quad \text{call } A \text{ exponentially stable if } \text{spec}(A) \subset \mathbb{D}_o.
\end{aligned}$$

Time-Invariant Gaussian System

Def. Supportable pair

Consider a time-invariant Gaussian system representation,

$$x(t+1) = Ax(t) + Mv(t),\ x(0) = x_0.$$

Call the matrix tuple $(A,\ M)$ a supportable pair if

$$\begin{aligned} n_x &= \mathrm{rank}(\text{conmat}(A,\ M)) \\ &= \mathrm{rank}\left(\begin{bmatrix} M & AM & A^2M & \dots & A^{n_x-1}M \end{bmatrix}\right). \end{aligned}$$

A supportable time-invariant Gaussian system representation is defined as a representation with $(A,\ M)$ a supportable pair.

Comments

Condition of (A, M) a supportable pair
is identical to $(A,\ M)$ a controllable pair.
In a Gaussian stochastic control system,
both $(A,\ B)$ and $(A,\ M)$ occur, hence the need to distinguish.

Outline

Lyapunov Equation

Def. Lyapunov equation of a time-invariant Gaussian system

Consider a time-invariant Gaussian system.
Define the recursion of the state variance function and the discrete-time Lyapunov equation by the respective formulas,

$$
\begin{aligned}
Q_x(t+1) &= AQ_x(t)A^T + MM^T, \; Q_x(0) = Q_{x_0}, \\
Q &= AQA^T + MM^T, \;\; Q \in \mathbb{R}^{n_x \times n_x}, \\
&\quad \text{Lyapunov (matrix) equation}; \\
m_x(t) &= E[x(t)], \; \text{mean value function}, \\
Q_x(t) &= E[(x(t) - m_x(t))(x(t) - m_x(t))^T], \; \text{state variance function}.
\end{aligned}
$$

Remark

The above Lyapunov matrix equation is an equation for the matrix Q where the matrices A, M are considered as known.
A.M. Lyapunov was a Russian mathematician of the late 19th century. His paper on stability was published in 1892.

Lyapunov Equation

Theorem 22.1.2. Lyapunov equation (1)

Consider a time-invariant Gaussian system representation.

(a) If the matrix A is exponentially stable ($\Leftrightarrow$ $\text{spec}(A) \subset \mathbb{D}_o$) then

$$Q = \lim_{t \to \infty} Q_x(t), \quad Q \in \mathbb{R}^{n_x \times n_x},$$
$$Q = AQA^T + MM^T.$$

thus the limit exists and
the limit matrix Q is a solution of the Lyapunov matrix equation.
Hence there exists a solution of the Lyapunov equation.

Lyapunov Equation

Theorem 22.1.2. Lyapunov equation (2)

(b) If the matrix A is exponentially stable
then the Lyapunov equation has a unique solution.
Then also $Q \in \mathbb{R}^{n_x \times n_x}_{pds}$.

Proof outline of 'Then also'.

$$
\begin{aligned}
Q &= AQA^T + MM^T, \; \Rightarrow \\
Q^T &= AQ^TA^T + MM^T, \text{ uniqueness } \Rightarrow \; Q = Q^T; \\
& \quad \forall\, \lambda_i \in \text{spec}(A) \cap \mathbb{R}^1 \;\Rightarrow\; (1 - |\lambda_i|^2) > 0, \\
& \quad \forall\, w_i \in \mathbb{C}^{n_x}, \; w_i^T A = \lambda_i w_i^T, \\
w_i^T Q w_i &= w_i^T AQA^T w_i + w_i^T MM^T w_i \\
0 &\leq \|w_i^T M\|^2 = w_i^T Q w_i \; \underbrace{(1 - |\lambda_i|^2)}_{>0}, \\
0 &\leq \; w_i^T Q w_i.
\end{aligned}
$$

Lyapunov Equation

Theorem. Lyapunov equation (3)

(c) See book. Uses concept of a stabilizable tuple $(A,\ B)$.

(d) Assume that $Q \in \mathbb{R}^{n_x \times n_x}_{pds}$ is a solution of the Lyapunov equation. Any two of the following three statements implies the third:

- **(d.1)** $\mathrm{spec}(A) \subset \mathbb{D}_o$;
- **(d.2)** $(A,\ M)$ is a controllable pair; and
- **(d.3)** $0 \prec Q$; in words, Q is strictly positive definite.

Lyapunov Equation

Theorem. Lyapunov equation (4)

(e) If the matrix A is exponentially stable
then the asymptotic convergence rate
of the state variance sequences, is exponential,

$$\exists\ c \in \mathbb{R}_+,\ \exists\ r \in (0,1),$$

such that for t sufficiently large

$$\|Q_x(t) - Q_x(\infty)\|_2 \le c|r|^t; \text{ and,}$$

$$Q_x(\infty) = \lim_{t\to\infty} \frac{1}{t} \sum_{s=0}^{t-1} Q_x(s),$$

$$Q_x(\infty) = AQ_x(\infty)A^T + MM^T.$$

Lyapunov Equation

Theorem. Lyapunov equation (5)

(f) If A is exponentially stable
then there exists matrices Q_o and Q_c such that,

$$
\begin{aligned}
Q_c &= AQ_cA^T + MM^T, \\
Q_o &= A^TQ_oA + C^TC, \\
\mathrm{tr}(CQ_cC^T) &= \mathrm{tr}(M^TQ_oM).
\end{aligned}
$$

Call Q_o the observability Grammian of $(A,\ C)$ and
call Q_c the controllability Grammian of $(A,\ M)$.
Relation $(A,\ M)$ to $(A^T,\ C^T)$.

Lyapunov Equation

Theorem. Lyapunov equation (6)

(g) See book.

(h) If A is exponentially stable, and if,

$$\begin{aligned}
& W_{min},\ W,\ W_{max} \in \mathbb{R}_s^{n_x \times n_x}, \\
Q_{min} &= A Q_{min} A^T + W_{min}, \\
Q &= A Q A^T + W, \\
Q_{max} &= A Q_{max} A^T + W_{max}, \\
& \text{and the following inequalities both hold,} \\
W_{min} &\preceq W \preceq W_{max}, \\
& \text{then,} \\
Q_{min} &\preceq Q \preceq Q_{max}.
\end{aligned}$$

Proof

Proof of Theorem 22.1.2 in the book and in the lecture notes.

Outline

Invariant Measure

Def. Probability measure on an image space

Consider a stochastic system with state and output process $(x,\ y)$.
For time $t \in T$ consider the map,

$$\left(\begin{array}{l} x(\omega, t+1) \\ y(\omega, t) \end{array} \right) : \Omega \rightarrow \mathbb{R}^{n_x+n_y}.$$

This map induces a probability measure on the image space according to,

$$\begin{aligned} & (X \times Y, B(X \times Y)) = (\mathbb{R}^{n_x+n_y},\ B(\mathbb{R}^{n_x+n_y})), \\ & A \in B(\mathbb{R}^{n_x+n_y}), \\ P_{(x^+,y),t}(A) = {} & P(\{\omega \in \Omega |\ (x(\omega, t+1),\ y(\omega, t)) \in A\}). \end{aligned}$$

Then $P_{(x^+,y),t}$ is a probability measure on $X \times Y = \mathbb{R}^{n_x+n_y}$.
If this concerns a Gaussian system then that probability measure is a Gaussian probability measure.

This definition is a standard result of probability theory, see Lecture 1.

Invariant Measure

Def. Invariant probability measure

A stochastic system admits
an invariant probability measure for the joint state and output process if

$$
\begin{aligned}
& \exists\ P_{(x^+,y)} : B(X) \otimes B(Y) \to [0,\ 1],\ \text{such that,} \\
P_{(x^+,y),t} = {} & P_{(x^+,y)},\ \forall\ t \in T; \\
& \text{thus, invariance of the pdf over time } t \in T.
\end{aligned}
$$

Define
$P_x = P_{(x^+,y)}|_{B(X)}$ as the invariant state probability measure and
$P_y = P_{(x^+,y)}|_{B(Y)}$ as the invariant output probability measure.

Comment

One defines correspondingly
an invariant probability distribution function and
an invariant probability density function.

Invariant Measure

Theorem. Invariant Measure of a Time-Invariant Gaussian System (1)

Consider a time-invariant Gaussian system representation,

$$\begin{aligned} x(t+1) &= Ax(t) + Mv(t),\ x(0) = x_0 \in G(0, Q_{x_0}), \\ y(t) &= Cx(t) + Nv(t), \quad v(t) \in G(0, I_{n_v}), \quad \text{spec}(A) \subset \mathbb{D}_o. \end{aligned}$$

(a) There exists an invariant measure of the system which is a Gaussian measure and which may be constructed as defined below,

$$\begin{aligned} Q_x &= AQ_xA^T + MM^T, \quad \exists\ Q_x \text{ by Th. Lyapunov equation,} \\ Q_y &= CQ_xC^T + NN^T, \\ Q_{x^+,y} &= AQ_xC^T + MN^T, \\ Q_{(x^+,y)} &= \begin{pmatrix} Q_x & Q_{x^+,y} \\ Q_{x^+,y}^T & Q_y \end{pmatrix}, \\ & G(0, Q_{(x^+,y)}) \text{ is the invariant probability measure.} \end{aligned}$$

Invariant Measure

Theorem. Invariant Measure of a Time-Invariant Gaussian System (2)

(b)

$$
\begin{aligned}
& \text{if } x_0 \in G(0,\ Q_x) \text{ equals invariant measure then} \\
\Rightarrow & \ \forall\ t \in T,\ (x(t+1),\ y(t)) \in G(0, Q_{(x^+,y)}); \\
& \text{in particular } \forall\ t \in T,\ x(t) \in G(0, Q_x), \\
& \text{is the invariant state probability measure;} \\
& (x,\ y) \text{ are jointly stationary Gaussian processes,}
\end{aligned}
$$

$$
\begin{aligned}
W_x(t) &= A^t Q_x, \\
W_y(t) &= \begin{cases} CA^{t-1} Q_{x^+,y}, & 0 < t, \\ Q_y, & 0 = t, \end{cases} \\
W_{y,x}(t) &= E[y(t)x(0)^T] = CA^t Q_x.
\end{aligned}
$$

Invariant Measure

Theorem. Invariant Measure of a Time-Invariant Gaussian System (3)

(c) If the initial state x_0
does not have the invariant state probability measure then

$$
\begin{bmatrix} x(t+1) \\ y(t) \end{bmatrix} \in G,
$$

$$
\mathrm{D} - \lim_{t \to \infty} G\left(\begin{bmatrix} m_x(t+1) \\ m_y(t) \end{bmatrix}, \begin{bmatrix} Q_{x^+}(t+1) & Q_{x^+,y}(t) \\ Q_{x^+,y}(t)^T & Q_y(t) \end{bmatrix} \right)
$$

$$
= G(0,\ Q_{(x^+,y)}).
$$

Convergence in distribution to the invariant measure.
For a time-invariant Gaussian system all distributions are Gaussian
hence convergence in distribution is equivalent to
convergence of the mean value function and of the covariance function.

Invariant Measure

Theorem. Invariant Measure of a Time-Invariant Gaussian System (4)

(d) Equivalence of:

(d.1) support of the invariant state pdf $x(t)$ equals the state space $\mathbb{R}^{n_x}$;

(d.2) $0 \prec Q_x$;

(d.3) $(A,\ M)$ is a supportable pair.

Invariant Measure

Proof of Theorem, partly (1)

Proven by induction on time.
Note that by assumption $x_0 \in G(0,\ Q_{x_0})$. Let $t \in T$.
If $x(t) \in G(0, Q_x)$ then,

$$
\begin{aligned}
&E\left[\exp\left(i\begin{bmatrix} w_x \\ w_y \end{bmatrix}^T \begin{bmatrix} x(t+1) \\ y(t) \end{bmatrix}\right) \mid F_t^x\right] \\
&= \exp\left(i\begin{bmatrix} w_x \\ w_y \end{bmatrix}^T \begin{bmatrix} A \\ C \end{bmatrix} x(t) - \frac{1}{2}\begin{bmatrix} w_x \\ w_y \end{bmatrix}^T \begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix}^T \begin{bmatrix} w_x \\ w_y \end{bmatrix}\right),
\end{aligned}
$$

Continued

Invariant Measure

Proof of Theorem, partly (2)

$$
\begin{aligned}
& E\left[\exp\left(i\begin{bmatrix} w_x \\ w_y \end{bmatrix}^T \begin{bmatrix} x(t+1) \\ y(t) \end{bmatrix}\right)\right] \\
&= E\left[E\left[\exp\left(i\begin{bmatrix} w_x \\ w_y \end{bmatrix}^T \begin{bmatrix} x(t+1) \\ y(t) \end{bmatrix}\right) \mid F_t^x\right]\right] \\
&= \exp\left(-\frac{1}{2}\begin{bmatrix} w_x \\ w_y \end{bmatrix}^T \begin{bmatrix} AQ_xA^T + MM^T & AQ_xC^T + MN^T \\ (AQ_xC^T + MN^T)^T & CQ_xC^T + NN^T \end{bmatrix} \begin{bmatrix} w_x \\ w_y \end{bmatrix}\right) \\
&= \exp\left(-\frac{1}{2}\begin{bmatrix} w_x \\ w_y \end{bmatrix}^T \begin{bmatrix} Q_x & Q_{x^+,y} \\ Q_{x^+,y}^T & Q_y \end{bmatrix} \begin{bmatrix} w_x \\ w_y \end{bmatrix}\right),
\end{aligned}
$$

hence the measure is invariant. The remainder of the proof is simple.

Invariant Measure

Proof of Theorem, partly (3)

(d) This follows from a result of the Lyapunov equation.

Th. Lyapunov equation (repeat)

(d) Assume that $Q \in \mathbb{R}_{pds}^{n_x \times n_x}$ is a solution of the Lyapunov equation. Any two of the following three statements implies the third:

- **(d.1)** $\text{spec}(A) \subset \mathbb{D}_o$;
- **(d.2)** $(A,\ M)$ is a supportable pair; and
- **(d.3)** $0 \prec Q$.

Invariant Measure

Def. Backward Supportable Pair

Consider a backward time-invariant Gaussian system representation,

$$x(t-1) = A_b x(t) + M_b v_b(t),\ x(0) = x_0.$$

Call the matrix tuple $(A_b,\ M_b)$ a backward-supportable pair if

$$\begin{aligned} n_x &= \mathrm{rank}(\text{conmat}(A_b,\ M_b)) \\ &= \mathrm{rank}\left(\begin{array}{ccccc} M_b & A_b M_b & A_b^2 M_b & \ldots & A_b^{n_x-1} M_b \end{array}\right). \end{aligned}$$

Call the system representation
a backward supportable system representation
if $(A_b,\ M_b)$ is a backward-supportable pair.

See Theorem 4.4.7 of the book
for the existence of an invariant measure
of a backward Gaussian system representation.

Outline

Forward and Backward Gaussian Systems

Theorem. Relation of forward and backward representations

Consider a time-invariant Gaussian system
having both a forward and a backward Gaussian system representation.
Assume that $\text{spec}(A_f) \subset \mathbb{D}_o$.
Then there exists a unique solution of the Lyapunov equation,

$$Q_x = A_f Q_x A_f^T + M_f M_f^T;$$

$$\text{assume that } 0 \prec Q_x \;\Leftrightarrow\; (A_f,\ M_f) \text{ supportable pair.}$$

The relations between
the matrices of the forward and of the backward represenation are

$$\begin{aligned} A_f &= Q_x A_b^T Q_x^{-1}, \\ C_f &= C_b Q_x A_b^T Q_x^{-1} + N_b Q_{v_b} M_b^T Q_x^{-1}, \\ A_b &= Q_x A_f^T Q_x^{-1}, \\ C_b &= C_f Q_x A_f^T Q_x^{-1} + N_f Q_{v_f} M_f^T Q_x^{-1}; \end{aligned}$$

$$\text{spec}(A_f) = \text{spec}(A_b).$$

Forward and Backward Gaussian Systems

Proof, partly

Proof of equality of spectra,

$$
\begin{aligned}
&\det(sI - A_b) \\
&= \det(sI - Q_x A_f^T Q_x^{-1}) \\
&= \det(Q_x(sI - A_f^T)Q_x^{-1}) \\
&= \det(Q_x)\det(sI - A_f^T)\det(Q_x^{-1}) \\
&= \det(sI - A_f); \\
&\quad \text{using that } \det(Q_x^{-1}) = 1/\det(Q_x).
\end{aligned}
$$

Remainder of proof uses theorem for the time-varying case.

Outline

Stoc. Observability and Stoc. Co-Observability

Def. Stochastic observability and stochastic co-observability (1)

- Consider a time-invariant stochastic system and assume that there exists an invariant probability measure of the system, and that the state and output have the invariant measure.
- Call the state $x(t_0) \in X$ of this stochastic system stochastically observable if $\exists\ t_0,\ t_1$ such that $\{t_0,\ t_0+1,\ \ldots,\ t_0+t_1\} \subseteq T$ and the stochastic state-to-output map

$$x(t_0) \mapsto \text{cpdf}(\{y(t_0),\ y(t_0+1),\ \ldots,\ y(t_0+t_1-1)\}|\ F^{x(t_0)}),$$

  is injective on the support $x(t_0) \in X$.
- Call the system stochastically observable if this holds for all states $x(t_0) \in X$.
- Because of the invariant measure, this then holds for all times $t_0 \in T$ such that etc.

Stoc. Observability and Stoc. Co-Observability

Def. Stochastic observability and stochastic co-observability (2)

- Call the state $x(t_0) \in X$ of this stochastic system
  stochastically co-observable
  if $\exists\ t_0,\ t_1$ such that $\{t_0 - 1,\ t_0 - 2,\ \ldots,\ t_0 - t_1\} \subseteq T$ and
  the stochastic state-to-past-output map

  $$x(t_0) \mapsto \text{cpdf}(\{y(t_0 - 1),\ y(t_0 - 2),\ \ldots,\ y(t_0 - t_1)\} |\ F^{x(t_0)}),$$

  is injective on the support $x(t_0) \in X$.
- Call the system stochastically co-observable
  if this holds for all states $x(t_0) \in X$.
- Because of the invariant measure,
  this then holds for all times $t_0 \in T$ such that etc.

Stoc. Observability and Stoc. Co-Observability

Def. Observable pair

Consider a (forward) time-invariant Gaussian system representation,

$$\begin{aligned} x(t+1) &= Ax(t) + Mv(t),\ x(0) = x_0, \\ y(t) &= Cx(t) + Nv(t). \end{aligned}$$

Call the matrix tuple $(A,\ C)$ an observable pair if

$$n_x = \mathrm{rank}(O_f(A,\ C)),\ \text{where}$$

$$O_f(A,\ C) = \mathrm{obsmat}(A,\ C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n_x-1} \end{bmatrix}.$$

Stoc. Observability and Stoc. Co-Observability

Theorem. Characterization of stochastically observability of a forward Gaussian system representation (1)

Consider a forward time-invariant Gaussian system representation,

$$
\begin{aligned}
x(t+1) &= A_f x(t) + M_f v_f(t),\ x(0) = x_0, \\
y(t) &= C_f x(t) + N_f v_f(t),\ \ v_f(t) \in G(0, I_{n_v}), \\
& \quad \text{spec}(A_f) \subset \mathbb{D}_o; \\
& \quad \text{then } \exists\ Q_x \in \mathbb{R}_{pds}^{n_x \times n_x} \text{such that,} \\
Q_x &= A_f Q_x A_f^T + M_f M_f^T;\ \text{and} \\
& \quad x_0 \in G(0,\ Q_x)\ \Rightarrow\ \forall\ t \in T,\ x(t) \in G(0, Q_x).
\end{aligned}
$$

Stoc. Observability and Stoc. Co-Observability

Theorem. Characterization of stochastically observability of a forward Gaussian system representation (2)

(a) This system is stochastically observable if and only if

$$\ker(Q_x) = \ker(O_f \; Q_x).$$

(b) Assume in addition that
$(A_f, \; M_f)$ is a supportable pair ($\Leftrightarrow \; 0 \prec Q_x$).
Then the system is stochastically observable
if and only if
$(A_f, \; C_f)$ is an observable pair.

Stoc. Observability and Stoc. Co-Observability

Proof of Theorem.

Characterization of a stoc. observable Gaussian system

(a) The result follows from the time-varying case using that,

$$\begin{aligned}
\ker(Q_x), \; & = \ker(O_f(A,\ C)Q_x), \ \text{using},\\
A^{n_x} & = -\sum_{i=0}^{n_x-1} a_i A^i, \ \text{by the Cayley-Hamilton theorem.}
\end{aligned}$$

(b)

$$\begin{aligned}
& (A_f, M_f) \ \text{supportable pair},\\
\Leftrightarrow \ & 0 \prec Q_x, \ \text{by the Lyapunov equation;}\\
\{0\} = \ker(Q_x) = \ker(O_f Q_x),\\
\Leftrightarrow \ & n_x = \mathrm{rank}(O_f)\\
\Leftrightarrow \ & (A_f,\ C_f) \ \text{observable pair.}
\end{aligned}$$

Stoc. Observability and Stoc. Co-Observability

Proposition

For any time-invariant Gaussian system representation there exists a linear transformation such that one obtains the Kalman observable form,

$$x(t+1) = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} v(t), \; x(0) = x_0,$$

$$y(t) = \begin{bmatrix} C_1 & 0 \end{bmatrix} x(t) + Nv(t), \; v(t) \in G(0, \; I_{n_v}),$$

$$\text{spec}(A) \subset \mathbb{D}_o, \; (A_{11}, \; C_1) \text{ an observable pair.}$$

Note that x_2 is excited by the noise v.
Note also that x_2 neither influences y directly nor influences y via x_1.

Stoc. Observability and Stoc. Co-Observability

Theorem. Characterization of stoc. co-observability

Consider a backward time-invariant Gaussian system representation,

$$\begin{aligned}
x(t-1) &= A_b x(t) + M_b v(t), \ x(0) = x_0, \\
y(t-1) &= C_b x(t) + N_b v(t), \ \ v(t) \in G(0, I_{n_v}), \\
& \quad \text{spec}(A_b) \subset \mathbb{D}_o; \ \text{ then } \exists \ Q_x \in \mathbb{R}_{pds}^{n_x \times n_x} \text{ such that} \\
Q_x &= A_b Q_x A_b^T + M_b M_b^T; \text{ and} \\
& \quad x_0 \in G(0, \ Q_x) \ \Rightarrow \ \forall \ t \in T, \ x(t) \in G(0, Q_x).
\end{aligned}$$

(a) This system is stochastically co-observable if and only if

$$\ker(O_b(A_b, \ C_b) \ Q_x) = \ker(Q_x).$$

(b) Assume in addition that $(A_b, \ M_b)$ is a supportable pair. Then the system is stochastically co-observable if and only if $(A_b, \ C_b)$ is an observable pair.

Stoc. Observability and Stoc. Co-Observability

Proposition. Decomposition of a Gaussian System Representation (1)

For any time-invariant Gaussian system representation
there exists a linear state-space transformation
such that with respect to the new representation one obtains
the Kalman decomposition of a linear system (Def. 21.3.7 of book),

$$\begin{aligned}
x(t+1) &= \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} M_1 \\ M_2 \\ 0 \\ 0 \end{bmatrix} v(t), \;\; x(0) = x_0, \\
y(t) &= \begin{bmatrix} C_1 & 0 & C_3 & 0 \end{bmatrix} x(t) + Nv(t), \\
& \left(\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \right) \text{ a supportable pair}, \\
& \left(\begin{bmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{bmatrix}, \begin{bmatrix} C_1 & C_3 \end{bmatrix} \right) \text{ an observable pair.}
\end{aligned}$$

For a particular system, components may or may not be present.

Stoc. Observability and Stoc. Co-Observability

Example. Decomposition of a Gaussian System Representation (2)

Note that,

$$\mathrm{spec}(A) \subset \mathbb{D}_o \;\Rightarrow\; a.s. - \lim_{t \to \infty} \left(\begin{array}{c} x_3(t) \\ x_4(t) \end{array} \right) = 0.$$

In the long run, x_3 and x_4 are not of interest.
Note that x_2 is not observed at all.
Of interest is only the subsystem,

$$\begin{aligned} x_1(t+1) &= A_{11} x_1(t) + M_1 v(t),\; x_1(0) = x_{0,1}, \\ y(t) &= C_1 x_1(t) + N v(t), \\ & (A_{11},\; M_1) \text{ supportable pair}, \\ & (A_{11},\; C_1) \text{ observable pair}. \end{aligned}$$

This examples motivates Lecture 5.

Stoc. Observability and Stoc. Co-Observability

Def. Time-invariant Gaussian system - Special case

Consider the special time-invariant Gaussian system representation,

$$
\begin{aligned}
x(t+1) &= Ax(t) + Mv(t),\ x(0) = x_0 \in G(0,\ Q_{x_0}),\\
y(t) &= Cx(t) + Nv(t),\\
& \quad v \text{ standard Gaussian white noise } v(t) \in G(0,\ I),\\
& \quad F^{x_0},\ F^{v}_{\infty} \text{ independent, } \operatorname{spec}(A) \subset \mathbb{D}_o,\\
Q_x &= AQ_xA^T + MM^T,\\
Q_{x_0} &= Q_x \ \Rightarrow\ \forall\, t \in T,\ x(t) \in G(0,\ Q_x),\\
& \quad (A,\ M) \text{ supportable pair } (\Leftrightarrow\ 0 \prec Q_x),\\
& \quad (A,\ C) \text{ observable pair},\\
& \quad (A_b,\ C_b) \text{ observable pair}.
\end{aligned}
$$

A minimal stochastic realization of the output process y will be the term used in Lecture 5 for this system.

Outline

Concluding Remarks

Def. A time-invariant finite stochastic system

System representation,

$$E\left[\begin{bmatrix} i_x(t+1) \\ i_y(t) \end{bmatrix} \mid F_t^x \vee F_{t-1}^y\right] = \begin{bmatrix} A \\ C \end{bmatrix} i_x(t), \ \forall\, t \in T, \ A \in \mathbb{R}_{st}^{n_x \times n_x},$$
$$i_x : \Omega \times T \to \mathbb{R}^{n_x}, \ \ i_y : \Omega \times T \to \mathbb{R}^{n_y}, \ \ C \in \mathbb{R}_{st}^{n_y \times n_x}.$$

Concepts and results

- Probability measures $p_x(t) = E[i_x(t)]$ and $p_y(t) = E[i_y(t)]$.
- When does a stochastic matrix leave a polyhedral cone invariant?
- Concept of an irreducible and nonperiodic subsystem.
- A decomposition of the system matrix A in terms of subsets of X.
- Existence and uniqueness of steady state equation $p_{x_s} = Ap_{x_s}$.
- Stochastic observability and stochastic co-observability. Characterization in terms of system matrices.
- Read in lecture notes Sections 5.7 and 18.8; also Chapter 18.

Concluding Remarks

Overview

- Time-invariant Gaussian system representations.
- Controllability of a linear system.
- Lyapunov equation.
- Existence of an invariant probability measure.
- Relation of forward and backward Gaussian system representations.
- Stochastic observability and stochastic co-observability of a time-invariant Gaussian system representation.
- Outlook to Lecture 5.

Concept of stochastic controllability of a stochastic system treated in Lectures 7 and 9.