Course WI 4221 in the Spring Semester of 2025
Control of Stochastic Systems

Homework Set 7

27 March 2025 (date homework set is issued).
24 April 2025 (due date of solution).

1. *Stochastic controllability of a map to a set of probability measures on a finite set.* Consider an elementary finite stochastic control system described by the map,

$$
\begin{aligned}
E[x_1 |\ F^{x_0,\ u}] &= A(u)x_0, \\
X &= \{e_1,\ e_2, e_3\},\ \forall\ i \in \mathbb{Z}_3,\ e_i \in \mathbb{R}^3 \text{ is the } i\text{-th unit vector}, \\
U &= [0,\ 1]^2 \subset \mathbb{R}^2,\ A : U \to \mathbb{R}_{st}^{3\times 3},\ T = \{0,\ 1\},\ P(X) = \mathbb{R}_{st}^3, \\
& \quad\ x_0,\ x_1 : \Omega \to X,\ u : \Omega \to U,\ E[x_1 |\ F^{x_0,\ u}] \in P(X), \\
A(u) &= \begin{bmatrix} 0.3 - 0.2u_1 & 0.4 & 0.3 \\ 0.3 + 0.2u_1 & 0.5 - 0.4u_2 & 0.2 \\ 0.4 & 0.1 + 0.4u_2 & 0.5 \end{bmatrix}.
\end{aligned}
$$

   For any random variable x_0 and any variable $u \in U$, the expression $p_{x_1} = A(u)x_0$ determines a probability measure $p_{x_1} \in P(X)$ on the state set X for the random variable x_1.

   Call this map *stochastically controllable* for the set of control objective measures $P_{co} \subseteq P(X)$, if for all $p_{co} \in P_{co}$ and for all x_0 there exists an input vector $u \in U$ such that $p_{co} = A(u)x_0$.

   (a) Provide arguments why for any $u \in U$, the matrix $A(u) \in \mathbb{R}_{st}^{n_x \times n_x}$ hence is a stochastic matrix having elements in $\mathbb{R}_+$ and satisfies that of each column the sum of the column elements equals $1 \in \mathbb{R}_+$. Chapter 18 of the lecture notes provides the definition of a stochastic matrix and results for those matrices. Results of Chapter 18 are not needed for this part.

   (b) Describe mathematically the controllable set of probability measures on the next state x_1. See Section 10.4 for the concept of a controllable set of probability measures.

   (c) Can you describe the geometric structure of the controllable set determined in (b)? Is it a line piece, an object in a plane, or a more complicated geometric object?

   (d) Consider next a set of control-objective probability measures $P_{co} \subseteq P(X)$. Can you specify a set theoretic relation between the controllable set of probability measures of the stochastic control system and the set of control-objective probability measures which is equivalent to stochastic controllability?

   Can you formulate a relation of linear algebra which is equivalent to the above requested characterization of stochastic controllability? Do this only for the case in which $x_0 = e_1$, the first unit vector.

2. *Elementary control theory.* This exercise is rather simple for students familiar with control theory. However, the exercise is useful for course participants who are *not* familiar with control theory. Students familiar with control theory may be interested in part (d).

Consider a time-invariant Gaussian stochastic control system representation

$$x(t+1) = Ax(t) + Bu(t) + Mv(t),\ x(t_0).$$

Carry out the following steps of control synthesis and control design.

(a) *Construction of a closed-loop system.* Consider the linear control law $g(x) = Fx$ with a matrix $F \in \mathbb{R}^{n_u \times n_x}$. Construct the closed-loop system consisting of the above defined time-invariant Gaussian stochastic control system and the linear control law. What is the system matrix of the closed-loop Gaussian system?

(b) *Control objective of stability.* Consider first the control objective of an exponentially stable closed-loop system. Determine a condition such that there exists a feedback matrix F such that the system matrix of the closed-loop system is exponentially stable. Refer for the solution to a result of the lecture notes.

(c) *Control objective of eigenvalue assignment.* Consider next the more demanding control objective that the eigenvalues of the closed-loop system should be assigned to prespecified values. In old times, the buyers of airplanes specified these values.

Can you determine a feedback matrix such that the eigenvalues of the system matrix of the closed-loop system are equal to $\{0.6,\ 0.7\}$. Do this only for the case where the system matrices have the form

$$A = \begin{bmatrix} 0 & 1 \\ 0.24 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \ n_x = 2,\ n_u = 1.$$

Once you understand the principle of this method then you can do this for any $n_x \in \mathbb{Z}_+$ with $n_u = 1$.

(d) *Control objective of optimal control.* An even more demanding control objective is to use optimal control theory. But this may be too complicated for a beginner. Therefore a simplication is formulated.

Optimal control is formulated in terms of a cost function. Consider the simplication that only the cost rate is minimized. Thus solve the following optimization problem for the optimal control law g^* and for the value J^*,

$$\inf_{u \in \mathbb{R}^{n_u}} \begin{bmatrix} x \\ u \end{bmatrix}^T Q \begin{bmatrix} x \\ u \end{bmatrix},\ \ \forall\, x \in \mathbb{R}^{n_x},$$

$$Q = \begin{bmatrix} Q_x & Q_{x,u} \\ Q_{x,u}^T & Q_u \end{bmatrix} \in \mathbb{R}_{pds}^{(n_x+n_u)\times(n_x+n_u)},$$

$$Q_x \in \mathbb{R}^{n_x \times n_x},\ Q_u \in \mathbb{R}^{n_u \times n_u},\ Q_{xu} \in \mathbb{R}^{n_x \times n_u},\ \ 0 \prec Q_u.$$

One then finds the optimal input $u^* = g^*(x)$ as a function of the state x. The optimal control law is then g^* and it infimizes the cost rate.

Reading Advice

Lecture 6 Presented on Thursday 27 March 2025.

Read of the lecture notes: of Chapter 10 the Sections 10.1, 10.2, and 10.3, and 10.4 only if you are interested in finite stochastic control systems; and
of Chapter 11 the Sections 11.2, 11.3, 11.4, 11.5, and 11.8.

Lecture 7 To be presented on Thursday 24 April 2025.

Read of Chapter 12 the sections 12.2, 12.3, 12.6, and 12.7. It should be clear that this advice is not required reading.