Course WI4221 in the Spring Semester of 2025
Control of Discrete-Time Stochastic Systems

Homework Set 4

6 March 2025 (Date Homework Set 4 is issued).
13 March 2025 (Date solutions due).

1. *Computations for a time-invariant Gaussian systems.*

   Solve Exercise 4.11.2 of the lecture notes for the Gaussian system representation stated there. It may be assumed in part (c) that the system is started in the invariant probability distribution calculated in part (b).

2. *A time-invariant Gaussian system which is stochastically observable and not supportable.*

   Consider the time-invariant Gaussian system, as defined in the lecture notes, which is partly decomposed as described below.

$$
\begin{aligned}
x(t+1) &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} x(t) + \begin{bmatrix} M_1 \\ 0 \end{bmatrix} v(t),\ x(t_0) = x_0, \\
y(t) &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} x(t) + N v(t), \\
& \text{the decomposition of the system displayed above} \\
& \text{corresponds to the specification} \\
& n_1,\ n_2 \in \mathbb{Z}_+,\ n = n_1 + n_2,\ \mathbb{R}^{n_x} = \mathbb{R}^{n_1} \oplus \mathbb{R}^{n_2}, \\
x(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\ x_1(t) \in \mathbb{R}^{n_1},\ x_2(t) \in \mathbb{R}^{n_2},\ \ \mathsf{spec}(A) \subset \mathbb{D}_o, \\
& (A_{11},\ M_1)\ \text{ is a supportable pair,} \\
& \left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},\ \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right) \text{ is an observable pair.}
\end{aligned}
$$

   (a) Explain what is the behavior of the stochastic process x_2. By behavior we mean its properties as a stochastic process in particular when time goes to infinity.

   (b) What is the effect of the process x_2 on the output y?

   (c) Calculate the invariant distribution in terms of the system matrices displayed above, as explicit as possible.

   (d) Assume that the initial state has the invariant distribution of the system state. Calculate the covariance function of the output process.

3. *A time-invariant Gaussian system which is stochastically observable and not stochastically co-observable.*

   Prove by the following steps that a time-invariant Gaussian system may at the same time be stochastically observable and not be stochastically co-observable.

   (a) Consider the time-invariant Gaussian system

   $$\sigma_1 = \{\Omega, F, P, T, \mathbb{R}, B, \mathbb{R}, B, y, x\} \in \text{GStocS}$$

   on $T = \mathbb{Z}$ with backward difference representation

   $$\begin{aligned} x(t-1) &= a_1 x(t) + \begin{bmatrix} 1 & 0 \end{bmatrix} w(t), \\ y(t-1) &= c_1 x(t) + \begin{bmatrix} 0 & 1 \end{bmatrix} w(t), \\ & w(t) \in G(0, Q_w), \; w : \Omega \times T \to \mathbb{R}^2, \end{aligned}$$

   satisfying $a_1 \in (-1, +1)$, $Q_w = Q_w^T > 0$, non-diagonal, or, equivalently, $Q_{w_{21}} \neq 0$, and $c_1 = 0$. Provide arguments why this system is not stochastic co-observable. Note that, because $c_1 = 0$, the output process y is Gaussian white noise.

   (b) Construct the forward representation associated with the Gaussian system of (a) above, say

   $$\begin{aligned} x(t+1) &= a_2 x(t) + \begin{bmatrix} 1 & 0 \end{bmatrix} v(t), \\ y(t) &= c_2 x(t) + \begin{bmatrix} 0 & 1 \end{bmatrix} v(t), \\ & v(t) \in G(0, Q_v). \end{aligned}$$

   Determine expressions for a_2 and c_2 in terms of a_1 and Q_w.

   (c) Provide arguments why the forward representation is stochastic observable.

Reading Advice

Lecture 4 Presented on Thursday 6 March 2025.
Read of the lecture notes: Of Chapter 4, the Sections 4.4, 4.5, and 4.6. Read also Section 4.12 with information of further reading.

Read of Chapter 22, Section 22.1 on the discrete-time Lyapunov equation for a matrix, in particular Theorem 22.1.2, its proof, and lightly the rest of Section 22.1.

Lecture 5 To be presented on Thursday 13 March 2025.
Read of the lecture notes of Chapter 6, the sections 6.2 – 6.5. It should be clear that this advice is not required reading.