Control of Stochastic Systems
Lecture 7
Stochastic Control Systems and Stochastic Control Problems

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Outline

Outline

Example Course Keeping

Example. Motivation

- A ship needs to keep its course as determined by the captain.
- The course is the direction the ship should direct to.
  The staff member in charge of steering, controls the rudder of the ship.
- The wind and the sea force the ship to different directions.
- How to steer the ship so that it keeps its course or direction?
- Originally a helmsman set the direction of the ship.
  Later mechanical machines controlled the rudder.
  Later a computer program
  controlled the mechanical steering machine.
- How to design a controller for a ship
  so that the ship keeps its course?
- Here course keeping. Another problem is course changing.
- Project at TUDelft by J. van Amerongen around 1970.

Example Course Keeping

Modeling of ship dynamics

- Consider a ship on the ocean or on a large and deep sea.
  A separate case is that if the ship is in a harbor.
- Ship dynamics is modelled by
  the laws of mechanics in a plane ($\mathbb{R}^2$),
  by a second-order differential equation.
- Sea dynamics also may be modelled.
  Several models exist. Below a simple model.
- Adaptive model has been proposed.
  It adapts the control system based on
  the identification of the sea dynamics and on the ship loading.

Example Course Keeping

Modeling by a control system

The ship dynamics at sea
are described by the laws of mechanics according to

$$\tau_1 \frac{d^2\phi(t)}{dt^2} + \frac{d\phi(t)}{dt} = K_1 u(t).$$

ϕ denotes the ship heading with respect to a reference direction,
u denotes the angle of rudder with respect to a reference direction, and
$K_1,\ \tau_1 \in (0,\infty)$ denote constants depending of the ship.

- Rudder is activated by a hydraulic machine.
  Dynamics of rudder not useful for model because it is relatively fast.
- Rudder angle and rudder speed are limited in magnitude.
  Dynamics of rudder not useful for model.
- The sea dynamics is modelled as a disturbance process.

Example Course Keeping

Def. Deterministic control system

Transformation to a state-space control system,

$$
\begin{aligned}
x_1(t) &= \phi(t),\ x_2(t) = d\phi(t)/dt, \\
dx_1(t)/dt &= d\phi(t)/dt = x_2(t), \\
dx_2(t)/dt &= d^2\phi(t)/dt^2 \\
&= -\frac{1}{\tau_1} d\phi(t)/dt + \frac{K_1}{\tau_1} u(t) = a_{22} x_1(t) + b_2 u(t), \\
\frac{dx(t)}{dt} &= Ax(t) + Bu(t),\ x(0), \\
A &= \begin{bmatrix} 0 & 1 \\ 0 & a_{22} \end{bmatrix},\ B = \begin{bmatrix} 0 \\ b_2 \end{bmatrix}, \\
a_{22} &= -1/\tau_1,\ b_2 = K_1/\tau_1.
\end{aligned}
$$

Example Course Keeping

Def. Stochastic control system

Gaussian stochastic control system in continuous time

$$dx(t) = Ax(t)dt + Bu(t)dt + Mdv(t),\ x(0) = x_0.$$

The process v denotes a Brownian motion process.
A stochastic differential equation is used above.

Example Course Keeping

Problem. Control for course keeping

- Synthesize a control law $g \in G$
- such that the closed-loop system satisfies the control objectives:
  (1) minimize course deviations and
  (2) minimize fuel costs.
- These objectives are conflicting!
  A control engineer has
  to set a compromise between these conflicting objectives,
  based on the numerical values of the cost function.

Comments on control problem

- Relative weights of the two control objectives
  have to be set in control design.
- On the ocean both (1) and (2) are important.
  In a harbor (1) is more important than (2).

Example Course Keeping

Def. Cost function

$$
\begin{aligned}
J(g) &= E[\int_0^{t_1} [e(s)^2 + c\, u(s)^2] ds], \\
e(t) &= x_1(t) - \phi_0 = \phi(t) - \phi_0, \\
\phi_0 & \quad \text{reference angle of course,} \\
c \in (0, \infty) & \quad \text{a constant,} \\
e(t) & \quad \text{angle difference from a specified direction,} \\
u(t) & \quad \text{rudder angle, representing energy cost of rudder.}
\end{aligned}
$$

Problem. Optimal control

Solve for the value J^* and the optimal control law $g^* \in G$

$$J^* = \inf_{g \in G} J(g) = J(g^*).$$

Example Course Keeping

Optimal control law for course keeping

Optimal control law $g^* \in G$ is provided by the formula

$$u(t) = g^*(x(t), \phi_0)) = Fx(t) - F_0\phi_0.$$

Comments

- A linear control law is optimal.
- Formulas which specify how to compute the matrix F and the constant $F_0 \in \mathbb{R}$.
- But the state $x(t)$ is not available for the use in the control law. One needs to estimate the state, estimate $\hat{x}(t)$ of $x(t)$. This is done by the Kalman filter.
- Optimal control law then becomes

$$u(t) = F\hat{x}(t) - F_0\phi_0.$$

Example Course Keeping

Control design

Control engineering suggests to modify the optimal control law to

$$u(t) = F\hat{x}(t) - F_0\phi_0 + k_i(t),$$
$$k_i(t) = \frac{1}{t_2} \int_{t-t_2}^{t} u(s)ds.$$

The additional term compensates slow variations of the rudder.

Testing and conclusions

- Control law was implemented by a computer program.
- Control law was tested on ships.
- Performance of the controller for a ship was very good.
- Computer program available from a company at a cost.

Outline

Stochastic Control Systems

Def. Stochastic control system

Define a stochastic control system as a collection satisfying,

$$
\begin{aligned}
& \mathrm{cpm}(\ (x(t+1), y(t)) \mid F_t^{x-} \vee F_{t-1}^{y-} \vee F_t^{u-}\ ) \\
& = \mathrm{cpm}(\ (x(t+1), y(t)) \mid F^{x(t)} \vee F^{u(t)}), \ \ \forall\, t \in T; \\
\Leftrightarrow \quad & (F^{x(t+1)} \vee F^{y(t)},\ F_t^{x-} \vee F_{t-1}^{y-} \vee F_t^{u-} \mid F^{x(t),u(t)}) \in CI, \ \ \forall\, t \in T, \\
\Leftrightarrow \quad & (F_{t+1}^{x} \vee F_t^{y},\ F_t^{x-} \vee F_{t-1}^{y-} \vee F_t^{u-} \mid F^{x(t),u(t)}) \in CI, \ \ \forall\, t \in T; \\
& \{\Omega, F, P, T, Y, B_Y, X, B_X, U, B_U,\ y,\ x,\ u\} \in \mathrm{StocCS}.
\end{aligned}
$$

Input process $u : \Omega \times T \to U$,
state process $x : \Omega \times T \to X$, output process $y : \Omega \times T \to Y$.
Call the system time-invariant
if the above map does not depend on time explicitly.
Call the system Gaussian
if the transition cpm is conditionally Gaussian and if $x_0 \in G$.
The abbreviation cpm denotes conditional probability measure.

Stochastic Control Systems

Def. Recursive stochastic control system

Define a recursive stochastic control system
as a stochastic control system
in which the state and input process are related by

$$\{\Omega, F, P, T, X, B_X, U, B_U,\ x,\ u\},$$
$$x(t+1) = f(t, x(t), u(t), v(t)), x(0) = x_0.$$

The stochastic process v is an independent sequence,
F^{x_0} and F^{v}_{∞} are independent,
for all $t \in T$, $F^{v(t)}$ is independent of F^{u}_{t}.
Call this system time-invariant
if the function f does not depend on the explicit time variable.

Stochastic Control Systems

Def. Gaussian stochastic control system

Define a Gaussian stochastic control system representation
as a special case of a recursive stochastic control system such that

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) + M(t)v(t),\ x(t_0) = x_0, \\ y(t) &= C(t)x(t) + D(t)u(t) + N(t)v(t), \\ & \quad (U, B(U)) = (\mathbb{R}^{n_u}, B(\mathbb{R}^{n_u})), \\ & \quad \{\Omega, F, P,\ T,\ \mathbb{R}^{n_y}, B(\mathbb{R}^{n_y}),\ \mathbb{R}^{n_x}, B(\mathbb{R}^{n_x}),\ \mathbb{R}^{n_u}, B(\mathbb{R}^{n_u}),\ y,\ x,\ u\}. \end{aligned}$$

Call it a time-invariant Gaussian stochastic control system representation
if the functions $A,\ B,\ M,\ C,\ D,\ N$ do not depend on time.
Conditions often used,

$$n_y \leq n_v,\ \mathrm{rank}(N) = n_y,\ \mathrm{rank}\begin{bmatrix} M \\ N \end{bmatrix} = n_v;$$

$(A,\ M)$ is a supportable pair, $(A,\ B)$ is a controllable pair.

Stochastic Control Systems

Proposition. Gaussian stochastic control system

The processes $(x,\ y,\ u)$
of a Gaussian stochastic control system representation
satisfy the definition of a stochastic control system.

Proof sketch

By induction $F^{v(t)}$ is independent of $F^x_t \vee F^y_{t-1} \vee F^u_t \vee F^v_{t-1}$ for all $t \in T$.

$$\begin{aligned}
& E\left[\exp\left(i\begin{bmatrix} w_x \\ w_y \end{bmatrix}^T \begin{bmatrix} x(t+1) \\ y(t) \end{bmatrix}\right) \mid F^{x-}_t \vee F^{y-}_{t-1} \vee F^u_t\right] \\
& = \exp\left(i\begin{bmatrix} w_x \\ w_y \end{bmatrix}^T \begin{bmatrix} A(t)x(t)+B(t)u(t) \\ C(t)x(t)+D(t)u(t) \end{bmatrix} - \frac{1}{2}\begin{bmatrix} w_x \\ w_y \end{bmatrix}^T Q_r(t)\begin{bmatrix} w_x \\ w_y \end{bmatrix}\right) \\
& = E[\ldots \mid F^{x(t)} \vee F^{u(t)}],
\end{aligned}$$

$$Q_r(t) = \begin{bmatrix} M(t) \\ N(t) \end{bmatrix}\begin{bmatrix} M(t) \\ N(t) \end{bmatrix}^T \in \mathbb{R}^{(n_x+n_y)\times(n_x+n_y)}_{pds}.$$

Stochastic Control Systems

Def. Controlled output

Define the controlled output on a finite horizon
of a Gaussian stochastic control system representation

$$\begin{aligned} z(t) &= C_z(t)x(t) + D_z(t)u(t), \ \ \forall\, t \in T\backslash\{t_1\}, \\ z(t_1) &= C_z(t_1)x(t_1), \\ &\quad z : \Omega \times T \to \mathbb{R}^{n_z}, \ n_z \in \mathbb{Z}_+, \\ &\quad C_z : T \to \mathbb{R}^{n_z \times n_x}, \ D_z : T(0 : t_1 - 1) \to \mathbb{R}^{n_z \times n_u}. \end{aligned}$$

It is called a time-invariant controlled output if

$$\begin{aligned} C_z(t) &= C_z \in \mathbb{R}^{n_z \times n_x}, \\ D_z(t) &= D_z \in \mathbb{R}^{n_z \times n_u}, \ \ \forall\, t \in T. \end{aligned}$$

Outline

Stochastic Controllability

Introduction

- Needed is a concept of controllability of a stochastic control system.
- Controllability of a stochastic control system
  differs from controllability of a deterministic control system.
- The stochastic input-to-state-measure map
  will be used to define stochastic controllability.

Stochastic Controllability

Informal introduction (1)

For a stochastic control system and an input
define the controllable set of state measures at the terminal time
of a considered interval,

$$\begin{aligned}
&\forall\ t_0,\ t_1 \in T,\ \text{such that} \\
&T_c = \{t_0,\ t_0+1,\ \ldots, t_1-1,\ t_1\} \subseteq T; \\
&\forall\ u(t_0 : t_1-1) \in U^{t_1-t_0}\ \text{define} \\
&u(t_0 : t_1-1) \mapsto \text{cpm}\left(x(t_1) |\ F^{x(t_0)} \vee F^{u(t_0:t_1-1)}\right),\ \text{define the subset,} \\
&P_c(t_1,\ X,\ B(X)) = \left\{ \begin{array}{l} \text{cpm}\left(x(t_1) |\ F^{x(t_0)} \vee F^{u(t_0:t_1-1)}\right) \in P(X, B(X))\ | \\ \forall\ u(t_0 : t_1-1) \in U^{t_1-t_0} \end{array} \right\}.
\end{aligned}$$

T_c denotes the considered control interval.
cpm denotes a conditional probability measure.
P_c denotes the controllable set of state probability measures on $x(t_1)$.
Stochastic future-input-to-state-measure map.

Stochastic Controllability

Informal introduction (2)

- Control of a stochastic control system determines a probability measure on the terminal state $x(t_1)$ at the end of the considered interval.
- The controllable set of probability measures is in general strictly smaller than the set of all probability measures on the set X. See the case of Gaussian stochastic control systems below.
- Consider the set P_{co} of control-objective probability measures which control is to attain.
- Stochastic controllability then requires that

$$P_{co}(X,\ B(X)) \subseteq P_c(t_1,\ X,\ B(X)).$$

Stochastic Controllability

Def. Stochastic controllability

Consider a stochastic control system

$$\{\Omega, F, P, T, \mathbb{R}^{n_y}, B(\mathbb{R}^{n_y}), \mathbb{R}^{n_x}, B(\mathbb{R}^{n_x}), \mathbb{R}^{n_u}, B(\mathbb{R}^{n_u}), y, x, u\} \in \text{StocCS}.$$

Call this system stochastically controllable in the control interval T_c with respect to the control objective probability measures P_{co} if

$$P_{co}(X, B(X)) \subseteq P_c(t_1,\ X,\ B(X)),\ \text{where}$$
$$T_c = \{t_0,\ t_0 + 1,\ \ldots,\ t_1\} \subseteq T.$$

Consider a time-invariant stochastic control system.
Call it stochastically controllable with respect to P_{co}
if there exist $t_0,\ t_1 \in T$ such that $T_c \subseteq T$ and
the system is stochastically controllable on the interval T_c
with respect to P_{co} as defined above.
Stochastic co-controllability is defined correspondingly,
with respect to a backward control interval.

Stochastic Controllability

Introduction. Stochastic controllability of a Gaussian control system

Consider
a time-invariant Gaussian stochastic control system representation,

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Mv(t), x(t_0) = x_0, \\ y(t) &= Cx(t) + Du(t) + Nv(t); \\ & u : T \to \mathbb{R}^{n_u}; \text{ then on an interval,} \end{aligned}$$

$$x(t_1) = A^{t_1 - t_0} x(t_0) + \sum_{s=t_0}^{t_1 - 1} A^{t_1 - s - 1} Bu(s) + \sum_{s=t_0}^{t_1 - 1} A^{t_1 - s - 1} Mv(s);$$

Stochastic Controllability

Introduction. Stochastic controllability of a Gaussian control system

$$
\begin{aligned}
& E[\exp(iw^T x(t_1)) | F^{x(t_0)} \vee F^u_{t_1-1}] \\
& = E[\exp(iw^T m_c(t_1) + \sum_{s=t_0}^{t_1-1} A^{t_1-1-s} M v(s)) | F^{x(t_0)} \vee F^u_{t_1-1}] \\
& = \exp\left(\; iw^T \; m_c(t_1) - \tfrac{1}{2} w^T Q_c(t_1) w \; \right),
\end{aligned}
$$

$$
m_c(t_1) = [A^{t_1-t_0} x(t_0) + \sum_{s=t_0}^{t_1-1} A^{t_1-1-s} B u(s)],
$$

$$
Q_c(t_1) = \sum_{s=t_0}^{t_1-1} A^{t_1-1-s} M M^T (A^T)^{t_1-1-s}.
$$

The cpm is a Gaussian measure.
The conditional mean $m_c(t_1)$ depends on the input,
the conditional variance $Q_c(t_1)$ does not depend on the input.

Stochastic Controllability

Notation

$$
\begin{aligned}
T_c &= \{t_0, t_0+1, \ldots, t_1-1\} \subset T, \\
P_c(t_1,\ X,\ B(X)) &= \left\{ \begin{array}{l} G(m_c(t_1),\ Q_c(t_1)) \in P(X, B(X)) | \\ m_c(t_1) \in \mathbb{R}^{n_x},\ Q_c(t_1) \in \mathbb{R}^{n_x \times n_x}_{pds} \\ \text{as specified below} \end{array} \right\}, \\
m_c(t_1) &= A^{t_1-t_0} x(t_0) + \sum_{s=t_0}^{t_1-1} A^{t_1-1-s} B u(s), \\
Q_c(t_1) &= \sum_{s=t_0}^{t_1-1} A^{t_1-1-s} M M^T (A^T)^{t_1-1-s}, \\
P_{co}(X, B(X)) &= \{G(m_{co}, Q_c(t_1)) \in P(X, B(X)) |\ \forall\ m_{co} \in \mathbb{R}^{n_x}\}.
\end{aligned}
$$

Stochastic Controllability

Theorem. Characterization stochastic controllability

Consider a time-invariant Gaussian stochastic control system and assume that $(A,\ M)$ is a supportable pair.
This system is stochastically controllable
with respect to the defined set of control objective measures P_{co}
if and only if
the matrix tuple $(A,\ B)$ is a controllable pair.

Proof

$$\forall\ m_{co} \in \mathbb{R}^{n_x}\ \exists\ u(t_0 : t_1 - 1) \text{ such that,}$$

$$\begin{aligned} m_{co} - A^{t_1 - t_0} x(t_0) &= \sum_{s=t_0}^{t_1 - 1} A^{t_1 - 1 - s} B u(s) \\ &= \text{conmat}_{t_1 - t_0 - 1}(A,\ B) \ \times \ u(t_1 - 1 : t_0), \\ &\Leftrightarrow \ \text{rank}(\text{conmat}_{n_x}(A,\ B)) = n_x. \end{aligned}$$

Stochastic Controllability

Proposition

Consider a time-invariant Gaussian stochastic control system.
If the system is not stochastically controllable
then there exists a state-space transformation
resulting in the system represenation,

$$x(t+1) = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} v(t),$$

$(A_{11},\ B_1)$ a controllable pair.

State component x_1 can be controlled.
State component x_2 is not useful for control,
but models the effect of the disturbance on the state.

Proof

Use system theory of linear systems.

Stochastic Controllability

Comments

- Stochastic controllability
  of a time-varying Gaussian stochastic control system on an interval
  can be defined and characterized correspondingly.
- Stochastic co-controllability characterized correspondingly.
  So far not used.
- Stochastic controllability of a finite stochastic control system
  has a higher complexity than
  that of stochastic controllability of a Gaussian stochastic control system.
  See the book, Section 10.4.

Outline

Example. Inventory control (1)

Example. Inventory control formulation

- Consider a shop that sells radios to customers.
  Or cars, or airplaines, or kitchen machines.
- The shop manager can order radios from a company.
- Customers may come in daily to buy radios.
  How many customers will arrive tomorrow and buy a radio?
  Uncertainty is in the number of radios sold tomorrow.
  There is a need for a realistic model. Data of customers are useful.
- If the shop has no more radios and a customer arrives to buy a radio then customer departs and may never return.
- Radios in the shop at the end of the day
  are stored in a storage space at a cost.
- Aim of the shop operator is to make a profit.
  Profit is income from sales of radios
  minus the costs of operating the shop.
- Stochastic control problem. How to make the largest profit?

Example. Inventory control (2)

Example. Control system (1)

$$
\begin{aligned}
x(t) &= \text{number of radios at start of day } t, \\
u(t) &= \text{number of radios ordered and delivered at start of day } t, \\
w(t) &= \text{demand of customers for radios on day } t, \\
T &= \mathbb{N},\ X = U = W = \mathbb{N}, \\
& x : \Omega \rightarrow X,\ u : \Omega \rightarrow U, \\
& \{w(t) \in W,\ \forall\, t \in T\} \text{ independent sequence}, \\
p_w(k) &= P_w(\{w(t) = k\}),\ \forall\, k \in W = \mathbb{N}.
\end{aligned}
$$

A more detailed model is needed
if the customer demand process w
has a pattern of a week and of a year.

Example. Inventory control (3)

Example. Control system (2)

The stochastic control system is specified by

$$
\begin{aligned}
x(t+1) = \max\{0,\ x(t) + u(t) - w(t)\}; \\
&\text{note that } x(t) + u(t) \text{ equals} \\
&\text{number of radios in the shop at start of day;} \\
&w(t) \text{ customer demand for radios;}
\end{aligned}
$$

$$
x(t) + u(t) - w(t) = \begin{cases} \geq 0, & \text{radios left at end of day,} \\ < 0, & \text{there were customers} \\ & \text{who could not buy a radio.} \end{cases}
$$

Information structure past state $\{F_t^x,\ \forall\, t \in T\}$.
If a Markov control law g is used then the closed-loop system is

$$
x(t+1) = \max\{0,\ x(t) + g(x(t)) - w(t)\}.
$$

Example. Inventory control (3)

Example. Profit

Profit equals income minus costs.

$$\begin{aligned}
\text{income} &= p \ \min\{w(t),\ x(t)+u(t)\}, \\
& p \in (0,\infty) \text{ sale price per radio}, \\
\text{cost 1} &= r\ u(t), \quad r \in (0,\infty) \text{ cost of ordering a radio}, \\
\text{cost 2} &= h \ \max\{x(t)+u(t)-w(t),\ 0\}, \\
& h \in (0,\infty) \text{ costs of holding radios not sold}, \\
\text{cost 3} &= dl \max\{w(t)-(x(t)+u(t)),\ 0\}, \\
& dl \in (0,\infty) \text{ costs of loss of demand not met}, \\
J(g) &= E[\sum_{t=0}^{t_1-1} (\text{income} - (\text{cost 1 + cost 2 + cost 3}))\,; \\
\sup_{g \in G} J(g) &= J^* = J(g^*).
\end{aligned}$$

Example. Inventory control (3)

Example. Optimal control

Define the threshold control law by the formula,

$$g^*(x) = \begin{cases} x_{ub} - x, & \text{if } x \le x_{lb}, \\ 0, & \text{if } x_{lb} < x, \end{cases}$$

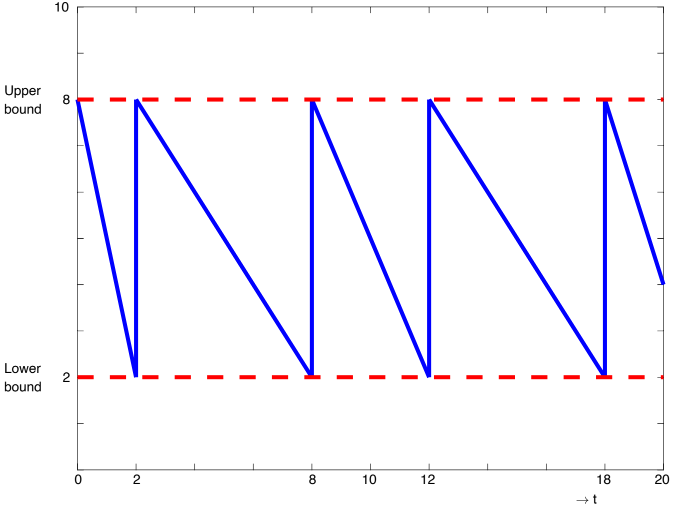
$$x_{lb},\ x_{ub} \in \mathbb{R}_+,\ x_{lb} < x_{ub};$$

$$\text{if } g^*(x) = x_{ub} - x$$

$$\text{then } x + g^*(x) = x + (x_{ub} - x) = x_{ub}.$$

x_{lb} called the lower bound and
x_{ub} called the upper bound of the optimal control law.
Values of lower bound and upper bound
depend on cost function parameters, $p,\ r,\ h,\ dl$.
In case of specific assumptions one can prove that,
a threshold control law is an optimal control law.

Behavior of Closed-Loop Control System



Outline

Control Laws

Remark. Input function and control law

Distinguish for control:

- input process or input trajectory $u : \Omega \times T \to U$.
  Input is a stochastic process in general;
- control law $g : X \times T \to U$, hence input $u(t) = g(t, x(t))$.
  Control law is a function, for any state $x(t)$,
  the input is specified by $u(t) = g(t, x(t))$.

Comments

- In stochastic control theory,
  a control law is more useful than an input trajectory.
  Example. Use the measurements of the dry paper weight
  to determine the amount of pulp from the reservoir.
- Input process useful in open-loop control, when the state is not needed.
  Example. Boil the rice for 8 minutes (input function).
  Example. Boil the potatoes till they are sufficiently soft (control law).

Control Laws

Def. Information structure

Consider a stochastic control system.
Define an information structure
as a σ-algebra family $\{G_t, t \in T\}$ such that, for all $t \in T$,
the σ-algebra G_t specifies all the information
available to the controller for the input value $u(t)$.

Comments

- In general an information structure is not a filtration, meaning that it is not true that $\forall\ s,\ t \in T$ with $s < t$ implies that $G_s \subseteq G_t$.
- Information structure very useful for decentralized control and for control of distributed systems.
- Example. Local telephone switch has information of all its substations but no information from other telephone switches of the same level.

Control Laws

Def. Special information structures

Define:

- Past-state information structure
  $\{F_t^{x-},\ \forall\ t \in T\}$ where $F_t^{x-} = \sigma(\{x(0),\ x(1),\ \ldots,\ x(t)\})$.
- Markov information structure
  $\{F^{x(t)},\ \forall\ t \in T\}$ where $F^{x(t)} = \sigma(\{x(t)\})$.
- Past-output information structure
  $\{F_{t-1}^{y-},\ \forall\ t \in T\}$ where $F_{t-1}^{y-} = \sigma(\{y(0),\ y(1),\ \ldots,\ y(t-1)\})$.
- Classical information structure $\{H_t,\ \forall\ t \in T\}$ if
  (1) there is only one controller with one information structure and
  (2) the information structure satisfies
  perfect recall: $\forall\ t \in T$, $H_t \subseteq H_{t+1}$.
  Called otherwise, a non-classical information structure.

Control Laws

Comments. Special information structures

- The past-state information structure has perfect recall because $F_t^{x-} \subseteq F_{t+1}^{x-}$ for all $t \in T$.
- The past-output information structure has perfect recall because $F_t^{y-} \subseteq F_{t+1}^{y-}$ for all $t \in T$.
- The Markov information structure does not satisfy perfect recall because in general $F^{x(t)} \not\subseteq F^{x(t+1)}$ for all $t \in T$.

Control Laws

Def. Sets of control laws

- Define a past-state control law
  $g = \{g_0, g_1, g_2, \ldots\}$ as a collection of measurable maps
  such that for all $t \in T$, $g_t : X^{t+1} \to U$.
  Hence $u(t) = g_t(x(0),\ x(1),\ \ldots,\ x(t))$.
  The past-state information structure is used here.
- Define a time-varying Markov control law
  as a measurable map $g : X \times T \to U$. Hence $u(t) = g(t,\ x(t))$.
  The Markov information structure is used here.
- Define a time-invariant Markov control law
  as a measurable map $g : X \to U$. Hence $u(t) = g(x(t))$.
- Define a past-output control law as a collection of measurable maps,

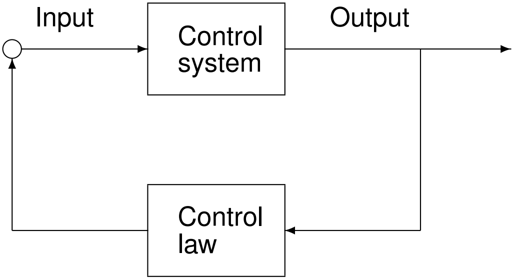
$$g = \{g_t,\ t \in T | g_0 \in U,\ \forall\ t \in T,\ g_{t+1} : Y^t \to U\},$$
$$u(t) = g_t(y(0),\ y(1),\ \ldots,\ y(t-1)).$$

Past-output information structure is used.

Outline

Closed-Loop Systems

Closed-loop stochastic control system



Closed-Loop Systems

Def. Closed-loop Gaussian stochastic control system

Consider a time-varying Gaussian stochastic control system representation, with a controlled output,

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) + M(t)v(t), \\ z(t) &= C_z(t)x(t) + D_z(t)u(t). \end{aligned}$$

Consider a past-state information structure $\{F_t^{x-},\ t \in T\}$ and consider a past-state control law $g \in G$, for all $t \in T$, $g_t : X^{t+1} \to U$. Define recursively the closed-loop Gaussian stochastic control system

$$\begin{aligned} x^g : \Omega \times T \to \mathbb{R}^{n_x},\ & u^g : \Omega \times T \to \mathbb{R}^{n_u}, \\ x^g(t+1) &= A(t)x^g(t) + B(t)g_t(x^g(0:t)) + M(t)v(t),\ x^g(0) = x_0, \\ z^g(t) &= C_z(t)x^g(t) + D_z(t)g_t(x^g(0:t)), \\ u^g(t) &= g_t(x^g(0:t)), \\ x^g(0:t) &= (x^g(0), x^g(1), x^g(2), \ldots, x^g(t)). \end{aligned}$$

Closed-Loop Systems

Comments. Closed-loop Gaussian stochastic control system

- Discussion below restricted to
  control with complete observations,
  $\{u(t),\ F_t^{x-},\ \forall\ t \in T\}$ adapted process.
- Control with partial observations
  $\{u(t),\ F_t^{y-},\ \forall\ t \in T\}$ adapted process.
  Treated in Lectures 12 and 13.
- The state and the controlled output
  have a well defined probability measure
  only if either the input process or the control law g is specified.
- Superindex g attached to $x^g(t)$
  to remind ourselves that the probability measure of $x^g(t)$
  depends on the control law g used.
- The processes u^g and v are not independent in general.
  But for all $t \in T$, $F^{v(t)}$ is independent of $F_t^{x-} \vee F_{t-1}^{y}$ hence of $u^g(t)$.

Closed-Loop Systems

Comments. Closed-loop Gaussian stochastic control system

Note the logical order of the operations within each time step

$$
\begin{aligned}
x^g(t) & \quad \text{assumed available,} \\
x^g(0:t) &= (x^g(0:t-1),\ x^g(t)), \\
u^g(t) &= g_t(x^g(0:t)), \\
x^g(t+1) &= A(t)x^g(t) + B(t)u^g(t) + M(t)v(t), \\
&= A(t)x^g(t) + B(t)g_t(x^g(0:t)) + M(t)v(t), \\
x^g(t+1) & \quad \text{is observed in complete observations.}
\end{aligned}
$$

Closed-Loop Systems

Def. Closed-loop recursive stochastic control system

Consider a time-varying recursive stochastic control system representation, with a controlled output,

$$\begin{aligned} x(t+1) &= f(t, x(t), u(t), v(t)),\ x(0) = x_0, \\ z(t) &= h(t, x(t), u(t)). \end{aligned}$$

Consider a past-state control law.
Define the closed-loop time-varying recursive stochastic control system

$$\begin{aligned} g &= \{g_0,\ g_1,\ \ldots,\ \},\ g_t : X^{t+1} \rightarrow U, \\ x^g(t+1) &= f(t, x^g(t), g_t(x^g(0:t)), v(t)),\ x^g(0) = x_0, \\ z^g(t) &= h(t, x^g(t), g_t(x^g(0:t))). \end{aligned}$$

Closed-Loop Systems

Proposition. State of closed-loop system is a Markov process

Consider an abstract stochastic control system and a Markov control law. Then the state of the closed-loop system is a Markov process

$$\begin{aligned}
(x(t), u(t)) &\mapsto \text{cpm}(x(t+1)|F_t^{x-} \vee F^{u(t)}), \\
u^g(t) = g(t, x^g(t)), &\text{ by Markov control law}, \\
&\Rightarrow \ (F_t^{x^g+}, \ F_t^{x^g-}| \ F^{x^g(t)}) \in CI, \ \forall \ t \in T.
\end{aligned}$$

Proof

$$\begin{aligned}
&(x^g(t), g_t(x^g(t))) \mapsto \text{cpm}(x^g(t+1)|F_t^{x^g-} \vee F_t^{u^g-}), \\
&\text{cpm}(x^g(t+1)| \ F_t^{x^g-} \vee F_t^{u^g-}) \\
= &\text{cpm}(x^g(t+1)| \ F^{x^g(t)} \vee F^{u^g(t)}), \ \text{ by the above map}, \\
= &\text{cpm}(x^g(t+1)| \ F^{x^g(t)}), \ \text{ because } F^{u^g(s)} \subseteq F^{x^g(s)}, \ \forall \ s \in T.
\end{aligned}$$

Hence x^g is a Markov process.

Outline

Stochastic Control Problems

Def. Engineering control problems

- Set point control
  Keep the controlled output close to a particular value
  called the set point.
  Example. Keep temperature at 21 degree Celsius.
- Minimum variance control
  Minimize the variance of the controlled output.
  Example. Minimize the deviations of a ship from its set course.
- Optimal control of the controlled output
  Infimize the performance criterion.
  Example. Minimize jointly the course deviations and the fuel used.
- Tracking control
  The controlled output should track or follow closely
  a reference signal.
  Example. A rocket should follow
  a pre-computed trajectory from Earth to Mars.

Stochastic Control Problems

Def. Control objectives

A control objective is a property of the closed-loop system
that an engineer may strive to attain.

- Stability
  The long term behavior of the state of the closed-loop system
  should be bounded or have finite variance.
- Assignment of dynamics
  The closed-loop system should have prespecified dynamics.
- Optimal control
  Infimize the cost function over all control laws.
- Control in case of uncertain dynamics; robustness
  Satisfactory dynamics in case of uncertain dynamics;
  satisfactory dynamics in case of unmodelled dynamics.
- Adaptation
  Satisfactory performance if the dynamics slowly changes over time.
  Example. Power system for which
  the power demand changes during 24 hours, each day is different.

Stochastic Control Problems

Problem. Control of a stochastic control system

Consider

- a stochastic control system,
- an information structure,
- a set of control laws, and
- a set of control objectives.

Synthesize a control law
such that the closed-loop system
satisfies the predefined control objectives as well as possible.

This is the main problem of stochastic control theory!

Stochastic Control Problems

Problem. Optimal control of a time-invariant Gaussian control system on a finite horizon

Consider the system and the performance criterion,

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Mv(t), \\ y(t) &= Cx(t) + Du(t) + Nv(t), \\ z(t) &= C_z x(t) + D_z u(t), \ \forall\ t = 0,\ 1,\ \ldots,\ t_1 - 1, \\ z(t_1) &= C_z x(t_1); \ \ J : G \to \mathbb{R}_+, \\ J(g) &= E\left[z(t_1)^T z(t_1) + \sum_{s=0}^{t_1-1} z(s)^T z(s) \right]. \end{aligned}$$

The problem is to solve

$$\inf_{g \in G} J(g),$$

$$J^* = \inf_{g \in G} J(g) = J(g^*), \ \ g^* \in G.$$

Stochastic Control Problems

Problem. Optimal control (continued)

Call $J^* \in \mathbb{R}_+$ the value of the control problem and
call $g^* \in G$ an optimal control law.

Comment on robust control (1)

A closed-loop control system is said to be robust
if the performance of the closed-loop system is satisfactory
even when the real system is a little different from the model system.

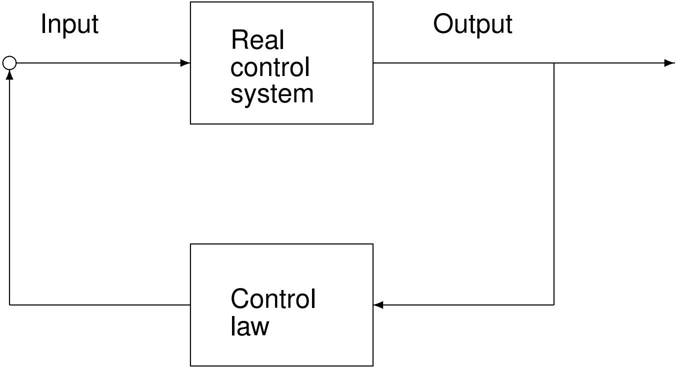
Comments

- The performance will diminish or degrade
  when the model system is far away from the real system.
- Effective is often an analytic approach with a ball of uncertainties.
- Figures explain robustness, see next slides.

Stochastic Control Problems

Comment on robust control (2)

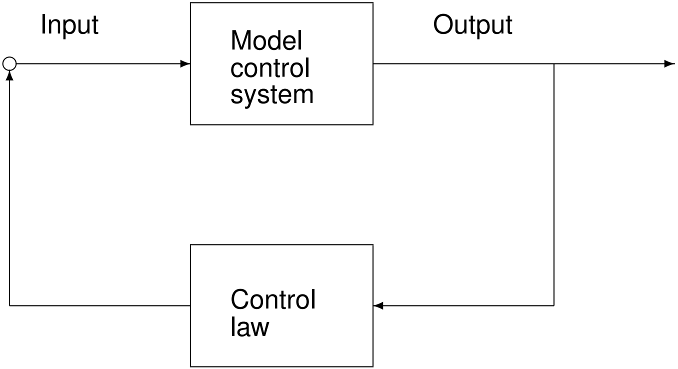
The real control system in closed-loop with a control law.
Example. Airplane Boeing 747.



Stochastic Control Problems

Comment on robust control (3)

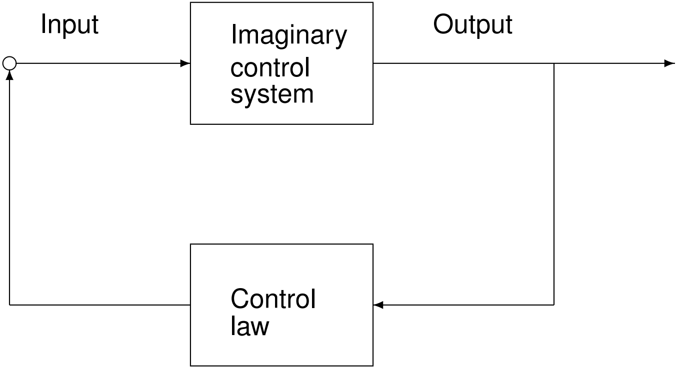
The model control system in closed-loop with a control law.
The model control system is made by a control engineer.
The control law is synthesized using the model control system.



Stochastic Control Problems

Comment on robust control (4)

The imaginary control system in closed-loop with a control law.
The imaginary control systems differs a little from the model control system.
Is the performance of this closed-loop system still satisfactory?



Stochastic Control Problems

Comment on robust control (5)

- Example of control of an airplane.
- The real system is the actual air plane, say a Boeing 747.
- The model system is
  a mathematical model in the form of a control system
  for example a linear system.
- Control objective: Follow the commands of the pilot,
  or increase the level to a particular altitude.
- Robustness: the performance of the closed-loop system
  should also be satisfactory
  when the model system is not quite close to the real system.
  For example,
  if only half the number of maximal passengers is on board.

Outline

Control Synthesis

Distinguish

- Control synthesis
  Develop control theory.
- Control design
  Develop and compute the actual control laws.
  Domain dependent, use simulation and testing.

In this course the emphasis is on control synthesis.

Concluding Remarks

Overview of Lecture 7

- Concept of a stochastic control system.
- Stochastic controllability.
- Control law.
- Closed-loop control system.
- Stochastic control problem.

Lecture 8, next course meeting

Optimal stochastic control
with complete observations on a finite horizon.