Change Detection Algorithms

Fundamentals

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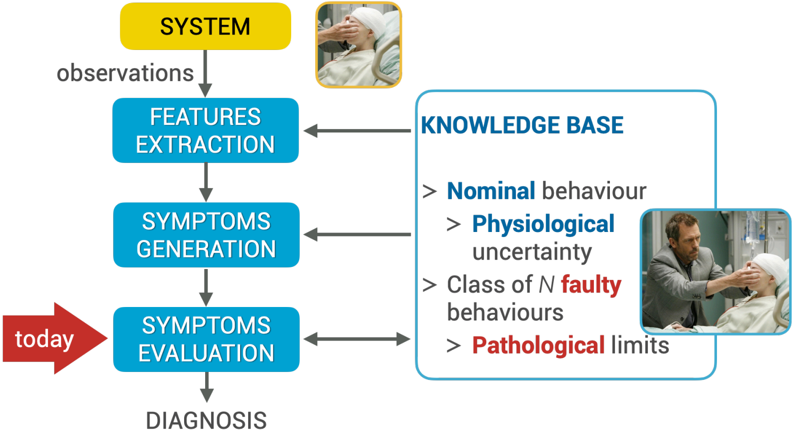
TU Delft

Lecture Summary

Introduction

Introduction

Remember the parallel with medical diagnosis?



INTRODUCTION

DIAGNOSIS $\Rightarrow$ (MULTIPLE) HYPOTHESIS TESTING

1 **DETECTION**

Testing the **null hypothesis**:

$\mathcal{H}_0$: "*Is the system behaving in a* ***nominal*** *way?*"

Introduction

Diagnosis $\Rightarrow$ (multiple) hypothesis testing

1 DETECTION

Testing the **null hypothesis**:

$\mathcal{H}_0$: *"Is the system behaving in a **nominal** way?"*

2 ISOLATION

Testing the N **faulty hypotheses**:

$\mathcal{H}_i$: *"Is the system behaving as if the **i–th fault** is present?"*

Introduction

Diagnosis $\Rightarrow$ (multiple) hypothesis testing

1 **DETECTION**

Testing the **null hypothesis**:

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2 **ISOLATION**

Testing the N **faulty hypotheses**:

$\mathcal{H}_i$: *"Is the system behaving as if the* ***i–th fault*** *is present?"*

3 **IDENTIFICATION AND ESTIMATION**

If every but one $\mathcal{H}_i$ is falsified $\Rightarrow$ **estimate** parameters of i–th fault

If every $\mathcal{H}_i$ is falsified $\Rightarrow$ **identify** model of new fault

Introduction

The boy who cried wolf

Aesop's Fable: The Boy Who Cried Wolf

1. The shepherd boy cries wolf when there is no wolf. Villagers rush in and get mad to find out it was a joke.

[*Repeat 1. for other* $N-1$ *times*]

2. Finally the wolf comes, the boy cries wolf but the villagers do not come. The wolf eats the boy.

https://developers.google.com/machine-learning/crash-course/classification/true-false-positive-negative

Introduction

The boy as a wolf detector: confusion matrix

Here we assume $\mathcal{H}_0 =$ “No wolf” = **healthy**.

		Reality	
		Wolf	No Wolf
Boy	Wolf	**True Positive** (TP) Boy is a hero!	**False positive** (FP) Villagers are mad!
	No Wolf	**False negative** (FN) Boy is eaten	**True negative** (TN) Everyone is fine

An FP is also called Type-I error, while a FN is also called Type-II error.

Introduction

Detection performance

Definition (Detection errors)

True Positive	$TP \triangleq \#\neg\mathcal{H}_0(\mathrm{F})$
True Negative	$TN \triangleq \#\mathcal{H}_0(\mathrm{H})$
False Positive	$FP \triangleq \#\neg\mathcal{H}_0(\mathrm{H})$
False Negative	$FN \triangleq \#\mathcal{H}_0(\mathrm{F})$

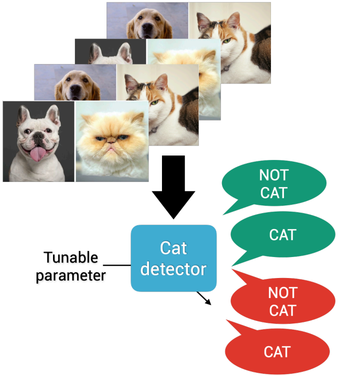
Definition (Detection performance)

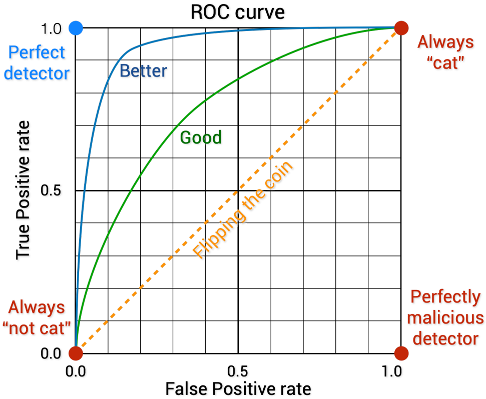
False Positive Rate	$FPR \triangleq \frac{FP}{FP+TN}$
True Negative Rate	$TNR \triangleq 1 - FPR$
True Positive Rate	$TPR \triangleq \frac{TP}{TP+FN}$
False Negative Rate	$FNR \triangleq 1 - TPR$
Accuracy	$ACC \triangleq \frac{TP+TN}{TP+FP+TN+FN}$
Recall	$RE \triangleq \frac{TP}{TP+FN} = TPR$

Tom Fawcett. "An introduction to ROC analysis". In: *Pattern recognition letters* 27.8 (2006), pp. 861–874

Introduction

Detection performance





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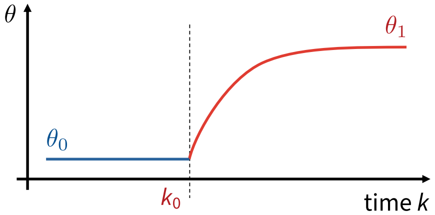
Introduction

The change detection problem

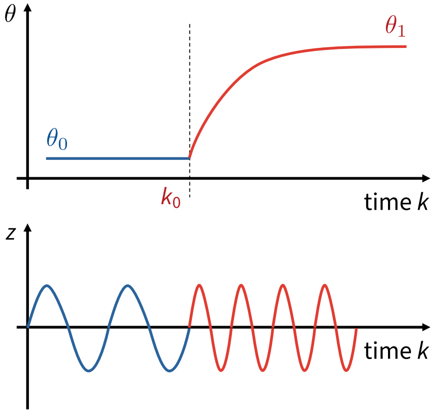
- Behaviour of system is described by a parameter θ
- It changes from a known **nominal** value θ_0 to a **faulty** θ_1 at some time instant k_0



INTRODUCTION

THE CHANGE DETECTION PROBLEM

- Behaviour of system is described by a parameter θ
- It changes from a known **nominal** value θ_0 to a **faulty** θ_1 at some time instant k_0
- You can only measure a **symptom** $z(k)$



INTRODUCTION

THE CHANGE DETECTION PROBLEM

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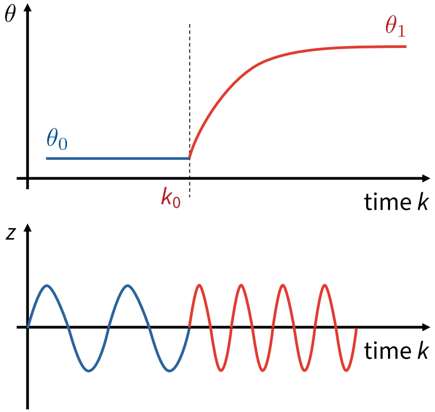


Change detection problem

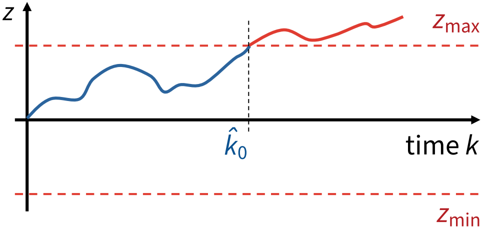
Estimate k_0 and θ_1 by looking at $z(k)$

Deterministic tests

Deterministic tests

Limit check: scalar variables

- In any process you know which are the **minimum** and **maximum allowed** values
  - E.g.: you heart rate at rest should be between 60 and 100 bpm



Intuition: is $z(k)$ inside a known range?

Limit check: $z(k) \notin [z_{min}\ z_{max}] \Rightarrow$ **ALARM**

PROS	CONS

Deterministic tests

Limit check: scalar variables

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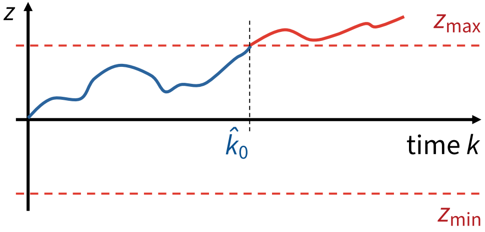

Intuition: is $z(k)$ inside a known range?

Limit check: $z(k) \notin [z_{min} \; z_{max}] \Rightarrow$ **ALARM**

PROS

- ✚ It holds for sure!
- ✚ Simple to define and check

CONS

- ▬ Over conservative
- ▬ Works in steady state only

Deterministic tests

Limit check: vector variables

- Assume $z(k) \in \mathbb{R}^n$

Intuition: evaluation function

Define a scalar valued **evaluation function** and reduce the problem to scalar limit-checking.

$$Z \triangleq f_{\text{ev}}(z(k)), \;\; f_{\text{ev}} : \mathbb{R}^n \mapsto \mathbb{R}$$

Most of the times you want $f_{\text{ev}} : \mathbb{R}^n \mapsto \mathbb{R}^+$

DETERMINISTIC TESTS

LIMIT CHECK: VECTOR VARIABLES

- Assume $z(k) \in \mathbb{R}^n$

Intuition: evaluation function

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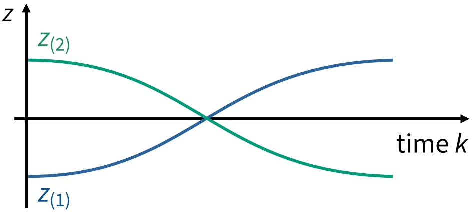

Example: $z \in \mathbb{R}^2$, $f_{ev}(z(k)) = \|z(k)\|_2$

Most of the times you want $f_{ev} : \mathbb{R}^n \mapsto \mathbb{R}^+$

Deterministic tests

Limit check: vector variables

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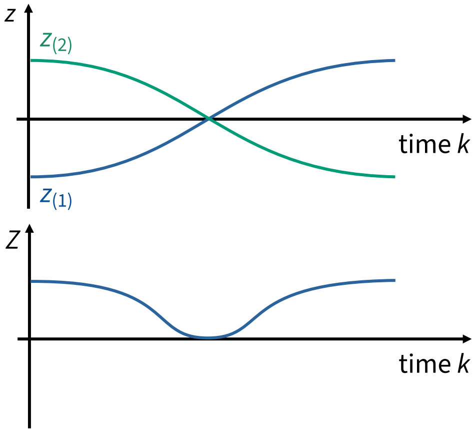
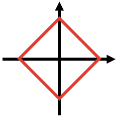


Most of the times you want $f_{\mathrm{ev}} : \mathbb{R}^n \mapsto \mathbb{R}^+$

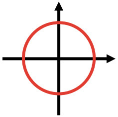
Deterministic tests

Limit check: vector variables – examples

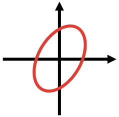
$\|z\|_1 = c$



$\|z\|_2 = c$

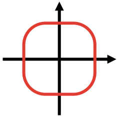


$z^\top A z = c$



$A \succeq 0$

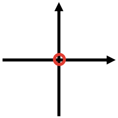
$\|z\|_p = c$



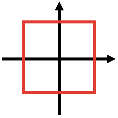
$p > 2$

$c \in \mathbb{R}^+$

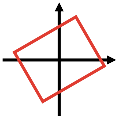
$\|z\|_0 = c = 0$



$\|z\|_\infty = c$

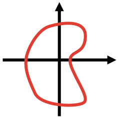


$\|Az\|_\infty = c$



$A \in \mathbb{R}^{2\times 2}$

$poly(z) = c$



DETERMINISTIC TESTS

LIMIT CHECK: SEQUENCES OVER TIME

- In previous examples we applied limit checks only to instantaneous value $z(k)$
- How to check a vector-valued sequence $\{z(k)\}_{k_1}^{k_2}$ over time?

1. First over all the components, then over time

Examples: $Z = \|z(k)\|_{p,[k_1,k_2]} = \sum_{k=k_1}^{k_2} \|z(k)\|_p$ (sum of L^p norms)

$Z = \| \{\|z(k)\|_p\}_{k_1}^{k_2} \|_{\ell^q}$ (ℓ^q norm of L^p norms)

Deterministic tests

Limit check: sequences over time

- In previous examples we applied limit checks only to instantaneous value $z(k)$
- How to check a vector-valued sequence $\{z(k)\}_{k_1}^{k_2}$ over time?

2. First over time, then over all the components

Examples: $$Z = \|\sum_{k=k_1}^{k_2} z(k)/(k_2 - k_1 + 1)\|_p \qquad (L^p \text{ norm of averages})$$

$$Z = \|\mathrm{FFT}(\{z(k)\}_{k_1}^{k_2})\|_p \qquad (L^p \text{ norm of FFT components})$$

Deterministic tests

Limit check: improvements

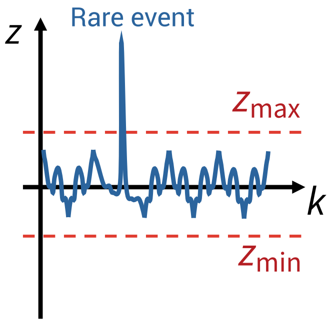
- Time-varying limits
  - Dependent on operating conditions
  - Based on models $\Rightarrow$ valid also in non-steady-state conditions
  - ☞ Lectures L4.x and L5.x
- Probabilistic checks
  - ☞ next section and next lecture

Basic Probabilistic Tests

Basic Probabilistic Tests

Motivations

- ✚ Deterministic checks leads to zero-FPR by construction
- − They can have very poor TPR (no-free-lunch principle)
  - ☞ see slide on ▸ ROC
- ▶ Evident when z can have large but rare values
- ▶ When data is generated by a random process (i.e. always) with infinite support, you cannot define $[z_{min}\ z_{max}]$

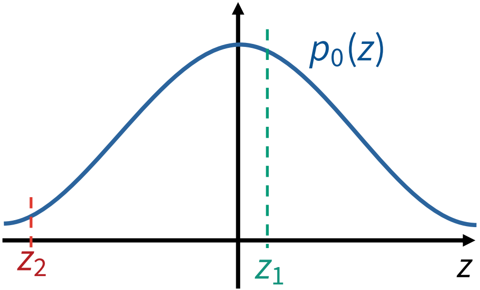


Basic Probabilistic Tests

Statistical Hypothesis testing

Intuition: check if the PDF changed

1. **Assume** that under $\mathcal{H}_0$ data is generated by a random process with given PDF $p_0(z)$
2. **Collect** actual data
3. **Answer** the question: is the data **likely** to have been drawn from $p_0(z)$ or from another PDF ?



Rolf Isermann. *Fault-diagnosis systems: an introduction from fault detection to fault tolerance*. Springer Science & Business Media, 2006, Ch. 7

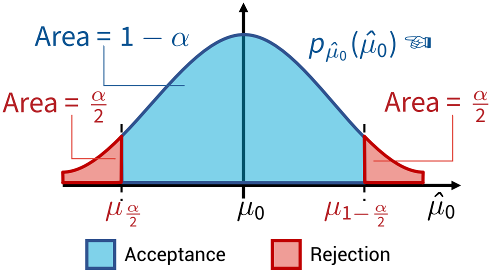
Basic Probabilistic Tests

Statistical Hypothesis testing à-la Fisher – example

Intuition: check p-value of parameters

1. **Assume** that you know one parameter of $p_0(z)$, e.g. the **true mean** μ_0
2. **Compute** the **empirical mean** $\hat{\mu}_0$ from data (its PDF is $p_{\hat{\mu}_0}$)
3. **Check** if its p**–value** $< \alpha$, where $0 \leq \alpha \ll 1$ is the **significance**[a]

[a] The confidence level of this test is $1 - \alpha$

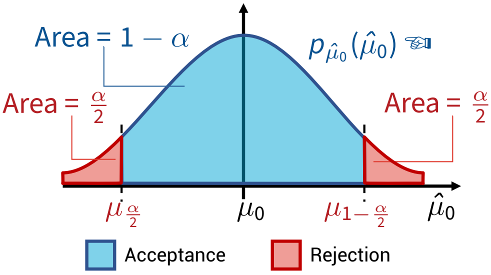


What is the FPR of this test?

Basic Probabilistic Tests

Statistical Hypothesis testing à-la Fisher – example

μ_0	True mean of data (known)
$\hat{\mu}_0$	Empirical mean of data
$p_{\hat{\mu}_0}$	PDF of $\hat{\mu}_0$ (not of data!)
$\mu_{\frac{\alpha}{2}}$:	$\int_{-\infty}^{\mu_{\frac{\alpha}{2}}} p_{\hat{\mu}_0}(\mu)\mathrm{d}\mu = \frac{\alpha}{2}$
$\mu_{1-\frac{\alpha}{2}}$:	$\int_{\mu_{1-\frac{\alpha}{2}}}^{\infty} p_{\hat{\mu}_0}(\mu)\mathrm{d}\mu = \frac{\alpha}{2}$



- Did you know? $\mathbb{E}[\hat{\mu}_0] = \mu_0$, because $\hat{\mu}_0$ is an unbiased estimator

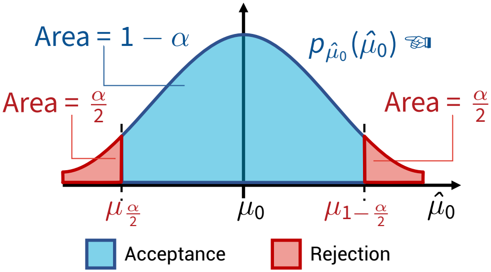
Basic Probabilistic Tests

Statistical Hypothesis testing à-la Fisher – example

Definition (p-value)

The **p-value** of a given realization $x^\star$ of a random process X is the **probability** that X can take values more **extreme** than $x^\star$.

🐒 Do you know how many tails has your distribution?

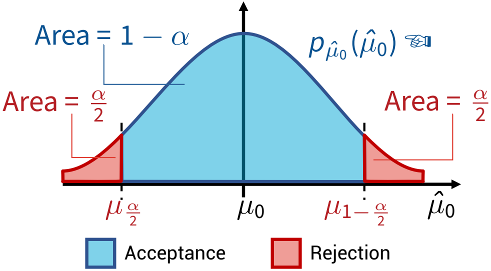


Basic Probabilistic Tests

Statistical Hypothesis testing à-la Fisher – example

The Perils of the P-value

- One of most-abused concept in science
- It tells you how likely is the data given $\mathcal{H}_0$, not the other way around!
- Completely nonsense if your hypothesis on PDF of data is wrong



https://www.scientificamerican.com/article/the-significant-problem-of-p-values/

Basic Probabilistic Tests

Empirical moments

- In the following we will make use of empirical mean and variance computed over a moving window

Mean
$$\hat{\mu}_{[k-N+1,k]} \triangleq \frac{1}{N} \sum_{j=k-N+1}^{k} z(j)$$

Variance
$$\hat{\sigma}^2_{[k-N+1,k]} \triangleq \frac{1}{N-1} \sum_{j=k-N+1}^{k} (z(j) - \mu_0)^2$$

- We may use a shortened notation such as $\hat{\mu}(k) \equiv \hat{\mu}_{[k-N+1,k]}$
- In case μ_0 is not known, we may replace it with $\hat{\mu}$ in the computation of $\hat{\sigma}^2$

Basic Probabilistic Tests

Testing the mean – t-Student

PROBLEM

- Determine if the mean of a normal random variable changed
- Nominal mean μ_0 is known
- Nominal variance is not known, but constant
- Empirical moments $\hat{\mu}(k)$ and $\hat{\sigma}^2(k)$ are estimated from last N samples

SOLUTION

Basic Probabilistic Tests

Testing the mean – t-Student

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- Determine if the mean of a normal random variable changed
- Nominal mean μ_0 is known
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SOLUTION

- The following random variable

$$t(k) = \frac{\hat{\mu}(k) - \mu_0}{\hat{\sigma}(k)/\sqrt{N}}$$

is **t–Student** with $N-1$ degrees of freedom

- test if $|t| > t_{\alpha,N-1}$

Basic Probabilistic Tests

Testing the mean – more t-Student

PROBLEM

- Same as before, but ... nominal mean μ_0 is **unknown**

SOLUTION

Basic Probabilistic Tests

Testing the mean – more t-Student

PROBLEM

- Same as before, but ... nominal mean μ_0 is **unknown**

SOLUTION

- Estimate true mean before and after change as $\hat{\mu}_0(k) = \hat{\mu}_{[k-N_0+1,k]}$ and, respectively, $\hat{\mu}_1(k) = \hat{\mu}_{[k,k+N_1-1]}$
- Similarly estimate $\hat{\sigma}_0^2$ and $\hat{\sigma}_1^2$
- The following random variable is **t–Student** with $N_0 + N_1 - 2$ degrees of freedom

$$t(k) = \frac{\hat{\mu}_0(k) - \hat{\mu}_1(k)}{\sqrt{(N_0 - 1)\hat{\sigma}_0^2(k) + (N_1 - 1)\hat{\sigma}_1^2(k)}} \sqrt{\frac{N_0 N_1 (N_0 + N_1 - 2)}{N_0 + N_1}}$$

Basic Probabilistic Tests

Testing the variance – χ^2

PROBLEM

- Determine if variance of a normal random variables changed from a known value σ_0^2

SOLUTION

Basic Probabilistic Tests

Testing the variance – χ^2

PROBLEM

- Determine if variance of a normal random variables changed from a known value σ_0^2

SOLUTION

- Compute the empirical variance $\hat{\sigma}^2(k)$ from previous N samples
- The following random variable is χ^2 with $N-1$ degrees of freedom

$$\chi^2(k) = \frac{(N-1)\hat{\sigma}^2}{\sigma_0^2}$$

- if $\chi^2(k) > \chi^2_{N-1,1-\alpha}$ then reject $\mathcal{H}_0$ with significance α

🐒 how many tails has χ^2 ?

Basic Probabilistic Tests

Testing the variance – F-test

PROBLEM

- Same as before, but true σ_0 is unknown

SOLUTION

Basic Probabilistic Tests

Testing the variance – F-test

PROBLEM

- Same as before, but true σ_0 is unknown

SOLUTION

- Compute the empirical variance $\hat{\sigma}_0^2(k)$ from previous N_0 samples, and $\hat{\sigma}_1^2(k)$ from next N_1 samples
- The following random variable is ***F*** with $(N_0 - 1,\ N_1 - 1)$ degrees of freedom

$$F(k) = \frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}$$

- if $F > F_{N_0-1,\,N_1-1,\,1-\alpha}$ then reject $\mathcal{H}_0$ with significance α

🐒 how many tails has F ?

Basic Probabilistic Tests

Limitations

- To **falsify** the null hypothesis with **good significance** you need a **high number of samples** before and after the (hypothetical) change
- this **delays detection**
- ☞ **likelihood based algorithms** (such as CUSUM), next lecture

Conclusions

CONCLUSIONS

IN THIS LECTURE WE COVERED

- **Fundamentals of change detection**
- **Deterministic tests**
- **Basic probabilistic tests**

Next lecture: **Advanced probabilistic tests**

CONCLUSIONS

THANK YOU FOR YOUR ATTENTION!

For further information:

Course page on **Brightspace**

or

our **MS Team**