Course WI4221 in the Spring Semester of 2025
Control of Discrete-Time Stochastic Systems

Homework Set 5

13 March 2025 (date homework set is issued).
20 March 2025 (date solution due).

1. *Strong Gaussian stochastic realization of a tuple of jointly Gaussian random variables of finite dimensions.*

   Consider a tuple of Gaussian random variables,

$$
\begin{aligned}
& y^+ : \Omega \rightarrow \mathbb{R}^{n_{y^+}}, \; y^- : \Omega \rightarrow \mathbb{R}^{n_{y^-}}, \; (y^+, \; y^-) \in G(0, Q_{(y^+,y^-)}), \\
Q_{(y^+,y^-)} & = \left(\begin{array}{cc} Q_{y^+} & Q_{y^+,y^-} \\ Q^T_{y^+,y^-} & Q_{y^-} \end{array} \right) \in \mathbb{R}^{(n_{y^+}+n_{y^-}) \times (n_{y^+}+n_{y^-})}, \;\; n_{y^+}, \; n_{y^-} \in \mathbb{Z}_+, \\
& \text{assume that } 0 \prec Q_{y^-}, \; Q_{y^+,y^-} \neq 0, \text{ and } n_{y^+} = n_{y^-}.
\end{aligned}
$$

   (a) Construction of a state vector. Define the random variable,

$$
\begin{aligned}
x & = E[y^+ | \; F^{y^-}], \; x : \Omega \rightarrow \mathbb{R}^{y^+}; \\
& \text{prove that } (F^{y^+}, \; F^{y^-} \mid F^x) \in CI.
\end{aligned}
$$

   Hint. First calculate an expression for x in terms of y^- and, with that expression, prove the conditional independence. However, there is an alternative proof.

   (b) The dimension of the state x constructed in (a) may be too large. Consider the special case where the covariance matrix has the form,

$$
Q_{(y^+, \; y^-)} = \left(\begin{array}{ccc|ccc} I & 0 & 0 & I & 0 & 0 \\ 0 & I & 0 & 0 & D & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ \hline I & 0 & 0 & I & 0 & 0 \\ 0 & D & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array} \right), \;\; \left(\begin{array}{c} y_1^+ \\ y_2^+ \\ y_3^+ \\ \hline y_1^- \\ y_2^- \\ y_3^- \end{array} \right),
$$

$$
D = \text{Diag}(d_1, \; d_2, \; \ldots, \; d_{n_D}) \in \mathbb{R}^{n_D \times n_D}_{s+,diag}, \;\; d_1 \geq d_2 \geq \ldots \geq d_{n_D} > 0.
$$

   Calculate the random variable x defined in (a) with respect to this representation. Based on that formula, construct a random variable $z : \Omega \rightarrow \mathbb{R}^{n_z}$, $z \in G(0, \; Q_z)$, with $n_z < n_x$ and $n_z = \text{rank}(Q_z)$ such that $F^z = F^x$.

   If the value of d_{n_D} is very small and you would not include the last component of x into z, would that still be a good approximation for the random variable x?

Reading Advice

Lecture 5 Presented on 13 March 2025.
Read of the book: Primarily the Sections 6.3 – 6.5, and 6.7 – 6.8. If you are interested in the background of realization, then read the Sections 6.1 and 6.13.

Lecture 6 To be presented on 20 March 2025.
Read parts of the Chapters 21, in particular Section 21.8, and parts of the Chapters 23 and 24. Read Section 6.6 with the proof of the weak Gaussian stochastic realization theorem. This reading advice is optional, it is not required reading.