Change Detection Algorithms

Advanced

Riccardo M.G. Ferrari

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TU Delft

Lecture Summary

1. Advanced probabilistic tests

2. Conclusions

Advanced probabilistic tests

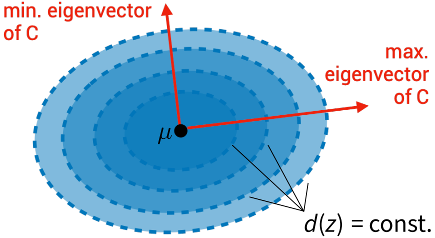
ADVANCED PROBABILISTIC TESTS

MULTIVARIATE CASE: MAHALANOBIS MEETS CHEBYCHEV

- $\mathcal{H}_0 =: z \in \mathbb{R}^n$ is a **random vector** with **known mean** μ and **covariance** C
- The **Mahalanobis distance (MD)** $d(z)$ tells us **how far** a sample z is from its nominal distribution

Definition (Mahalanobis distance)

$$d(z) \triangleq \sqrt{(z-\mu)^\top C^{-1}(z-\mu)}$$



Xinjia Chen. "A new generalization of Chebyshev inequality for random vectors". In: *arXiv:0707.0805* (2007)

ADVANCED PROBABILISTIC TESTS

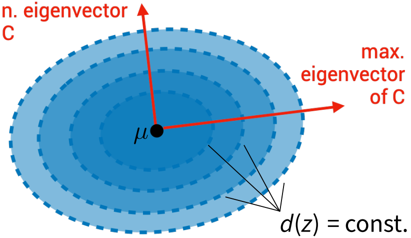
MULTIVARIATE CASE: MAHALANOBIS MEETS CHEBYCHEV

- The Chebychev inequality allows to test $\mathcal{H}_0$ with a desired FPR α, whatever is the distribution that generated z

Definition (Chebychev inequality)

$$\mathbb{P}[d(z) \geq \bar{d}] \leq \alpha \,,$$

$$\text{with } \bar{d} = \sqrt{\frac{n}{\alpha}} \quad \begin{array}{l} \rightarrow \text{n. of dimensions} \\ \rightarrow \text{desired FPR} \end{array}$$



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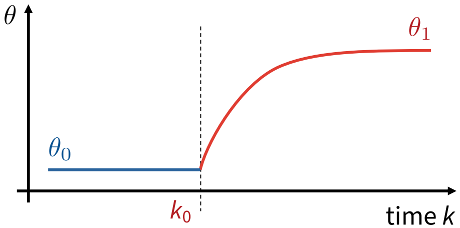
Advanced probabilistic tests

Statistical hypothesis testing à-la Neyman-Pearson

- **Two** hypotheses are considered

null $\mathcal{H}_0 \,:\, z \sim p_{\theta_0}$

alternate $\mathcal{H}_1 \,:\, z \sim p_{\theta_1}$



Problem

Design a decision function

$$d(z(k)) \,:\, \mathbb{R} \mapsto \{\mathcal{H}_0,\, \mathcal{H}_1\}$$

with a given FPR α and the highest possible TPR

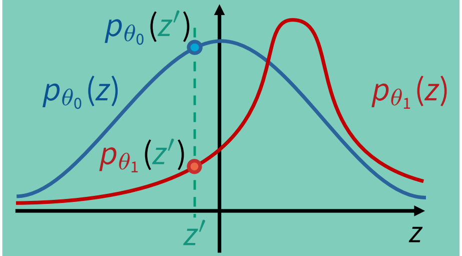
$\Rightarrow$ d is said to be **Most Powerful** (MP)

Michele Basseville, Igor V Nikiforov, et al. *Detection of abrupt changes: theory and application*. Vol. 104. prentice Hall Englewood Cliffs, 1993, Ch. 4

Advanced probabilistic tests

Statistical hypothesis testing à-la Neyman-Pearson

Intuition: likelihood matters



$$\mathbb{E}[p_{\theta_i}|\mathcal{H}_i] \geq \mathbb{E}[p_{\theta_i}|\mathcal{H}_j]$$

Neyman-Pearson Lemma (simplified)

- Let the decision function g be defined as

$$d(z) = \frac{p_{\theta_1}(z)}{p_{\theta_0}(z)} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \lambda$$

where λ is chosen such that $\text{FPR} = \alpha$.

- Then d is the MP detector with $\text{FPR} \leq \alpha$.

Michele Basseville, Igor V Nikiforov, et al. *Detection of abrupt changes: theory and application*. Vol. 104. prentice Hall Englewood Cliffs, 1993, Ch. 4

ADVANCED PROBABILISTIC TESTS

THE LOG-LIKELIHOOD RATIO

Definition (LLR)

$$s(z) \triangleq \ln \frac{p_{\theta_1}(z)}{p_{\theta_0}(z)}$$

- The expectation of $s(z)$ under **null** and **alternate** hypothesis has opposite sign!

$$\mathbb{E}[s|\mathcal{H}_0] = \int_{-\infty}^{\infty} s(z)p_{\theta_0}(z)\mathrm{d}z < 0$$

$$\mathbb{E}[s|\mathcal{H}_1] = \int_{-\infty}^{\infty} s(z)p_{\theta_1}(z)\mathrm{d}z > 0$$

Mogens Blanke et al. *Diagnosis and fault-tolerant control.* Vol. 2. Springer, 2006, Ch. 6

Advanced probabilistic tests

The CUSUM test

Intuition

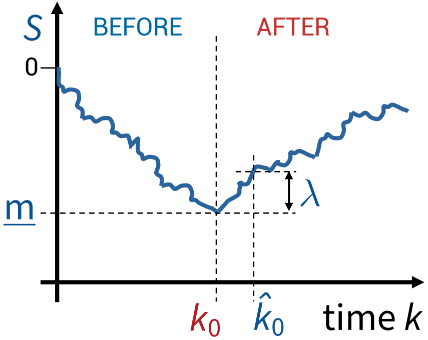
Approximate online the expectation $\mathbb{E}[s]$ with a **cumulative sum**, then look for **drift** changes

Definition (CUSUM test)

$$S(k) \triangleq \sum_{i=1}^{k} s(z(i)) = \sum_{i=1}^{k} \ln \frac{p_{\theta_1}(z(i))}{p_{\theta_0}(z(i))}$$

$$g(k) \triangleq S(k) - m(k) \quad m(k) \triangleq \min_{i=1:k} S(i)$$

$$d(k) \triangleq g(k) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \lambda$$



Mogens Blanke et al. *Diagnosis and fault-tolerant control*. Vol. 2. Springer, 2006, Ch. 6

ADVANCED PROBABILISTIC TESTS

THE CUSUM TEST – EXAMPLES

Example 6.12 *Change in the mean of a Gaussian sequence*

Remember that the Gaussian probability density function for a random variable with mean μ and variance σ is

$$p_\mu(z) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(z-\mu)^2}{2\sigma^2}\right). \tag{6.121}$$

The resulting likelihood ratio for detecting a change in the mean from μ_0 to μ_1 is

$$\frac{p_{\mu_1}(z)}{p_{\mu_0}(z)} = \exp\left(-\frac{(z-\mu_1)^2}{2\sigma^2} + \frac{(z-\mu_0)^2}{2\sigma^2}\right).$$

Hence straightforward computations yield the following expression for the log-likelihood ratio $s(z)$:

$$s(z) = \frac{2(\mu_1-\mu_0)z + (\mu_0^2-\mu_1^2)}{2\sigma^2} = \frac{\mu_1-\mu_0}{\sigma^2}\left(z - \frac{\mu_0+\mu_1}{2}\right). \tag{6.122}$$

ADVANCED PROBABILISTIC TESTS

THE CUSUM TEST – EXAMPLES

Example 6.13 *Change in the mean and variance*

If both mean and variance change after a fault, the following relation

$$\frac{p_{\mu_1}(z)}{p_{\mu_0}(z)} = \frac{\sigma_0}{\sigma_1} \exp\left(-\frac{(z-\mu_1)^2}{2\sigma_1^2} + \frac{(z-\mu_0)^2}{2\sigma_0^2}\right)$$

holds and the log-likelihood ratio is

$$s(z) = \ln \frac{\sigma_0}{\sigma_1} + \frac{(z-\mu_0)^2}{2\sigma_0^2} - \frac{(z-\mu_1)^2}{2\sigma_1^2}.$$

ADVANCED PROBABILISTIC TESTS

THE CUSUM TEST – PARAMETER TUNING

- Parameter θ_0 (i.e. $\theta_0 = (\mu_0, \sigma_0^2)$) is assumed to be known

Problem

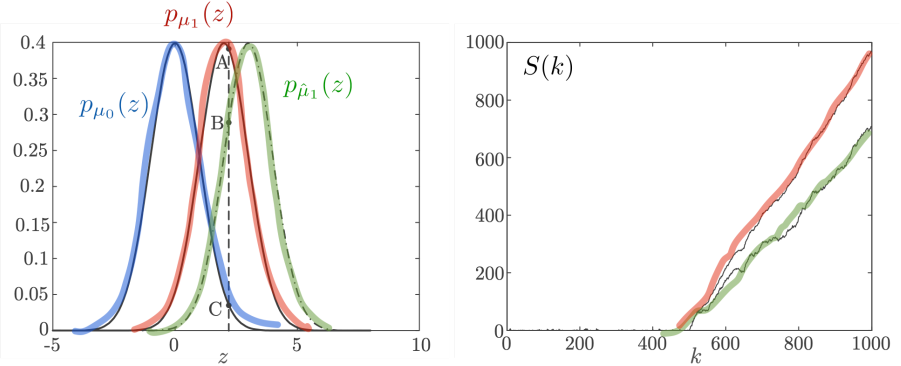
- What values should we choose for θ_1?
- What value should we choose for λ?

- some options next

Advanced probabilistic tests

The CUSUM test – tuning μ_1

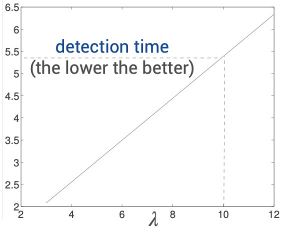
- **replace** by **minimum change** you want to detect (both $\pm \Rightarrow$ two sided CUSUM)
- **estimate** it from **data** (pro: ideal CUSUM conditions; con: detection delay!)

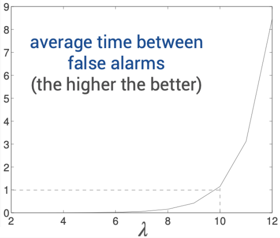


ADVANCED PROBABILISTIC TESTS

THE CUSUM TEST – TUNING λ

- **exact methods** exist for some distributions (i.e. Gaussian)
  - book by Blanke, page 244-246
- otherwise, tuning of λ is **empirical**
  - $\lambda = desired - detection - time \times slope - of - S - after - a - change$
  - run a series of Monte Carlo simulations and compute detection time, FPR and TPR as a function of λ





ADVANCED PROBABILISTIC TESTS

THE GENERALIZED LIKELIHOOD RATIO (GLR) TEST

Intuition

Estimate both the change time k_0 and the new parameter θ_1

Definition (GLR test)

$$S_j^k(\theta_1) \triangleq \sum_{i=j}^{k} s(z(i)) = \sum_{i=1}^{k} \ln \frac{p_{\theta_1}(z(i))}{p_{\theta_0}(z(i))}$$

$$(\hat{k}_0, \hat{\theta}_1) = \arg\left\{ \max_{1 \le k_0 \le k} \max_{\theta_1} S_j^k(\theta_1) \right\}$$

$$g(k) = \max_{1 \le k_0 \le k} \max_{\theta_1} S_j^k(\theta_1)$$

- Decision $d(k)$ as usual
- The solution is a **maximum likelihood** estimate
- ⚠ Solution involves a **double maximization**

ADVANCED PROBABILISTIC TESTS

THE GLR TEST – EXAMPLE

Example 6.12 (cont.) *Change in the mean of a Gaussian sequence*

In this particular case, it is possible to find an explicit expression for $\hat{\mu}_1(k, j)$, the maximum likelihood estimate of μ_1 at the present time instant k, assuming that the fault occurred at time instant j. Indeed, from (6.122), $S_j^k(\mu_1)$ takes the following form:

$$S_j^k(\mu_1) = \frac{\mu_1 - \mu_0}{\sigma^2} \sum_{i=j}^{k} \left(z(i) - \frac{\mu_0 + \mu_1}{2} \right) \tag{6.140}$$

In order to maximise this expression with respect to μ_1, one has to take the derivative of $S_j^k(\mu_1)$ with respect to μ_1 and equate that expression to zero:

$$\frac{\partial S_j^k(\mu_1)}{\partial \mu_1} = \frac{1}{\sigma^2} \sum_{i=j}^{k} \left(z(i) - \frac{\mu_0 + \mu_1}{2} \right) - \frac{k - j + 1}{2} \frac{(\mu_1 - \mu_0)}{\sigma^2} = 0. \tag{6.141}$$

Equation (6.141) yields:

$$\hat{\mu}_1(k, j) = \frac{1}{k - j + 1} \sum_{i=j}^{k} z(i). \tag{6.142}$$

ADVANCED PROBABILISTIC TESTS

THE GLR TEST – EXAMPLE

Substituting this expression for μ_1 in (6.140) results, after straightforward computations, in:

$$S_j^k(\hat{\mu}_1(k,\, j)) = \frac{1}{2\sigma^2}\frac{1}{k-j+1}\left[\sum_{i=j}^{k}(z(i)-\mu_0)\right]^2. \tag{6.143}$$

Hence the GLR decision function can be written:

$$g(k) = \frac{1}{2\sigma^2}\max_{k-M+1\leq j\leq k}\frac{1}{k-j+1}\left[\sum_{i=j}^{k}(z(i)-\mu_0)\right]^2. \tag{6.144}$$

If $\mathcal{H}_1$ is accepted in the above GLR algorithm, at the alarm time k_a, the estimated change occurrence time is given as:

$$\hat{k}_0 = \arg\{\frac{1}{2\sigma^2}\max_{k_a-M+1\leq j\leq k_a}\frac{1}{k_a-j+1}\left[\sum_{i=j}^{k_a}(z(i)-\mu_0)\right]^2. \; \square \tag{6.145}$$

Conclusions

Conclusions

In this lecture we covered

- **Mahalanobis distance**
- **Neyman-Pearson approach to statistical hypothesis test**
- **CUSUM**
- **GLR**

Next lecture: **Signal-based diagnosis**

Conclusions

Thank you for your attention!

For further information:

Course page on **Brightspace**

or

our **MS Team**