# Control of Stochastic Systems
# Lecture 5
# Weak Stochastic Realization of Gaussian Systems

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# Outline

# Outline

# Motivation

### Example. Paper Machine (1)

Motivation of stochastic realization problem.

- Recall the model of a paper machine, presented in Lecture 3.
- Measurements were available from dry basis weight.
- Needed is a Gaussian system representation which is a realistic model of these measurement.
- Suppose that the measurements can be approximated by a stationary Gaussian process.
- How to obtain a time-invariant Gaussian system from measurements of the output process?

# Motivation

## Example. Paper Machine (2)

- Distinguish:
  - system identification and
  - stochastic realization.
- System identification
  is the subject of control and system theory in which one goes from measurements
  to a system with its parameter values.
  System identification is a problem of approximation.
- Stochastic realization
  Stochastic realization is a problem of exact representation.
  Stochastic realization is used in system identification.

# Motivation

## Example. Paper Machine (3)

Consider observations, without inputs, and compute,

$$\{z(s) \in \mathbb{R}^{n_y},\ \forall\ t \in T_1 = \{1,\ 2,\ \ldots,\ t_1\}\},$$

$$z_a = \frac{1}{t_1} \sum_{s=1}^{t_1} z(s),\ \text{estimate of average of time series},$$

$$\widehat{W}(t) = \frac{1}{(t_1 - t) - 1} \sum_{s=1}^{t_1 - t} (z(t+s) - z_a)(z(s) - z_a)^T,$$

$$\forall\ t \in T_2 = \{0,\ 1,\ 2,\ \ldots, t_2\},\ t_2 < t_1,$$

$$\widehat{W}(-t) = \widehat{W}(t)^T,\ \forall\ t \in T_2.$$

$\widehat{W}$ is an estimate of a covariance function.
The function $\widehat{W}$ is a covariance function
if and only if it is a positive definite function,
if and only if it satisfies a block Toeplitz matrix condition for all times.

# Motivation

### Example. Paper Machine (4)

**Problem**

Does there exists
a time-invariant Gaussian system representation of the form,

$$\begin{aligned} x(t+1) &= Ax(t) + Mv(t),\ x(0) = x_0 \in G(0, Q_x), \\ y(t) &= Cx(t) + Nv(t),\ v(t) \in G(0, I_{n_v}), \\ &\ \mathrm{spec}(A) \subset \mathbb{D}_o, \\ Q_x &= AQ_xA^T + MM^T, \end{aligned}$$

# Motivation

## Example. Paper Machine (5)

such that

- the probability measure of the output process $y$ is equal to the probability measure associated with the stochastic process obtained from the measurements?
- equivalently, if for all $t \in T$,

$$\widehat{W}(t) = W_y(t) = \begin{cases} CA^{t-1} Q_{x+,y}, & 0 < t, \\ CQ_x C^T + NN^T, & 0 = t, \end{cases}$$

- the equivalence follows because the probability measure of a stationary Gaussian process is characterized by that of its finite-dimensional Gaussian probability distributions, equivalently by equality of the two covariance functions, if the mean value function is assumed to be zero.

# Motivation

### Example. Paper Machine (6)

- Problem reformulated.
  How to determine from $\widehat{W}$ on $T_2$,
  the dimensions $n_x \in \mathbb{N}$ and $n_v \in \mathbb{Z}_+$, and
  the system matrices $(A,\ C,\ M,\ N)$
  such that

$$\widehat{W}(t) = W_y(t) = \begin{cases} CA^{t-1}Q_{x+,y}, & 0 < t, \\ CQ_xC^T + NN^T, & 0 = t, \end{cases} \quad \forall\, t \in T.$$

# Motivation

## Motivation of stochastic realization

- There is a need for modeling of a time series
  by a time-invariant Gaussian system representation.
  Modeling and system identification.
- There is a need for characterization of minimality
  of a stochastic realization,
  in terms of stochastic observabilty and stochastic co-observability.
- There is a need for identifiability conditions
  of a time-invariant Gaussian system representation.
- There is a need for a procedure
  to obtain an estimate of the parameters of a Gaussian system.

R.E. Kalman focused attention on
system identification and stochastic realization.

# Outline

# Problem

**Problem. Weak Gaussian Stochastic Realization Problem (1)**

Consider a stationary Gaussian process,

$$\begin{aligned} &\text{taking values in } (\mathbb{R}^{n_y},\ B(\mathbb{R}^{n_y})),\\ m(t) = 0,\ \forall\, t \in T,\ &\text{mean value function},\\ W : T \to \mathbb{R}^{n_y \times n_y},\ &\text{covariance function}. \end{aligned}$$

Comment. The mean value function may be modeled separately by a deterministic system.

# Problem

## Problem. Weak Gaussian Stochastic Realization Problem (2)

**(a)** Does there exist
a time-invariant Gaussian system representation of the form,

$$
\begin{aligned}
& n_x \in \mathbb{N},\ n_v \in \mathbb{Z}_+,\ x : \Omega \times T \to \mathbb{R}^{n_x},\ \text{such that,} \\
x(t+1) &= Ax(t) + Mv(t),\ x(0) = x_0 \in G(0, Q_x), \\
y(t) &= Cx(t) + Nv(t),\ v(t) \in G(0, I_{n_v}), \\
& \mathrm{spec}(A) \subset \mathbb{D}_o, \\
Q_x &= AQ_xA^T + MM^T,\ \text{ such that for all } t \in T = \mathbb{N}, \\
W(t) = W_y(t) &= \begin{cases} CA^{t-1}Q_{x+,y}, & 0 < t, \\ CQ_xC^T + NN^T, & 0 = t, \end{cases}
\end{aligned}
$$

Equivalently, the considered stationary Gaussian process and
the output process $y$ of the system representation,
have the same probability distributions.
Call such a system representation
a weak Gaussian stochastic realization of the considered process.

# Problem

**Problem. Weak Gaussian Stochastic Realization Problem (3)**

**(b)** Characterize those weak Gaussian stochastic realizations which are of minimal state-space dimension.
Call a weak Gaussian stochastic realization minimal if the state-space dimension $n_x$ is minimal over all such weak Gaussian stochastic realizations.

**(c)** Classify or describe all minimal weak Gaussian stochastic realizations of the considered process.
In general, there is no unique minimal realization.
Relate any two minimal weak stochastic realizations.

**(d)** Formulate a procedure by which one can construct all minimal weak Gaussian stochastic realizations.

Statement of theorem and of proof in lecture notes and in book.
Lecture 6 provides additional information on proof.

# Outline

# Theory

### Def. Hankel Matrix (1)

Consider a covariance function of a stationary process,
$W : T \to \mathbb{R}^{n_y \times n_y}$ on $T = \mathbb{N}$.
Define by the following formula
the finite Hankel matrix of $W$ with $k$ row blocks and $m$ column blocks,

$$H_W(k,m) = \begin{bmatrix} W(1) & W(2) & \dots & W(m-1) & W(m) \\ W(2) & W(3) & \dots & W(m) & W(m+1) \\ W(3) & W(4) & \dots & W(m+1) & W(m+2) \\ \vdots & & \ddots & & \vdots \\ W(k) & W(k+1) & \dots & W(k+m-2) & W(k+m-1) \end{bmatrix} \in \mathbb{R}^{kn_y \times mn_y}.$$

Define the infinite Hankel matrix $H_W \in \mathbb{R}^{\infty \times \infty}$ as an infinite matrix
of which each upper-left block
is a finite Hankel matrix $H_W(k,\ m)$ as defined above.

# Theory

**Def. Hankel Matrix(2)**

Define the rank of the infinite Hankel matrix by the formula,

$$\mathrm{rank}(H_W) = \sup_{k,\ m\ \in\ \mathbb{Z}_+} \mathrm{rank}(H_W(k,m)) \in \mathbb{Z}_+ \cup \{+\infty\}.$$

Call $H_W$ of finite rank if $\mathrm{rank}(H_W) < \infty$.
This is a theoretical condition, it is never computed.
The computation of the rank of the infinite Hankel,
is in general undecidable.

Hermann Hankel (1839 – 1873) was a German mathematician.

# Theory

### Def. Set of State Variance Matrices

Define the set of state variances matrices
associated with a parametrization of a covariance function by the formula,

$$
\begin{aligned}
& (n_y,\ n_x,\ n_y,\ F,\ G,\ H,\ J) \in \mathrm{LSP}, \\
& n_y,\ n_x \in \mathbb{Z}_+, \\
& F \in \mathbb{R}^{n_x \times n_x},\ G \in \mathbb{R}^{n_x \times n_y},\ H \in \mathbb{R}^{n_y \times n_x},\ J \in \mathbb{R}^{n_y \times n_y},
\end{aligned}
$$

$$
\begin{aligned}
& \mathbf{Q_{lsdp}}(F,\ G,\ H,\ J) \\
& = \{Q \in \mathbb{R}_{pds}^{n_x \times n_x} \mid 0 \preceq Q_{v,lsdp}(Q)\}, \\
& Q_{v,lsdp}(Q) \\
& = \begin{bmatrix} Q - FQF^T & G - FQH^T \\ (G - FQH^T)^T & J + J^T - HQH^T \end{bmatrix} \in \mathbb{R}_{pds}^{(n_x+n_y)\times(n_x+n_y)}.
\end{aligned}
$$

Call $0 \preceq Q_{v,lsdp}(Q)$ a linear matrix inequality (LMI) for the matrix $Q$.
See for details Chapters 23 and 24 of the book.

# Theory

### Theorem. Weak Gaussian Stochastic Realization (1)

Consider the weak Gaussian stochastic realization problem and assume that,

$$T = \mathbb{Z};\ m(t) = 0,\ \forall\ t \in T;\ 0 \prec W(0);\ \lim_{t \to \infty} W(t) = 0.$$

**(a)** There exists a time-invariant Gaussian system representation

$$\begin{aligned} x(t+1) &= Ax(t) + Mv(t),\ x(0) = x_0 \in G(0, Q_x), \\ y(t) &= Cx(t) + Nv(t),\ v(t) \in G(0,\ I_{n_v}), \\ & \text{spec}(A) \subset \mathbb{D}_o, \\ Q_x &= AQ_xA^T + MM^T; \\ & \text{such that the output process } y \\ & \text{equals the considered process in probability;} \\ & \text{equivalently, if } W(t) = W_y(t),\ \forall\ t \in T; \\ \Leftrightarrow\ & \operatorname{rank}(H_W) < \infty, \\ \Leftrightarrow\ & \text{the rank of the infinite Hankel matrix is finite.} \end{aligned}$$

# Theory

## Theorem. Weak Gaussian Stochastic Realization (2)

**(a)**
- Call this system
  a weak Gaussian stochastic realization of the considered process.
- For any weak Gaussian stochastic realization
  there exists a state variance matrix such that $Q_x \in \mathbf{Q}_{\mathbf{lsdp}}$.
- If a weak realization exists
  then there exists of the covariance function $W$
  a covariance realization of the form,

$$(n_y,\ n_x,\ n_y,\ F,\ G,\ H,\ J) \in \mathrm{LSP};$$

$$W(t) = \begin{cases} HF^{t-1}G, & t > 0, \\ J + J^T, & t = 0, \\ (HF^{-t-1}G)^T = G^T(F^T)^{-t-1}H^T, & t < 0. \end{cases}$$

### Notation

$(n_y,\ n_x,\ n_v,\ A,\ C,\ M,\ N) \in \mathrm{WGSRP}$
for the set of parameters of a weak Gaussian stochastic realization.

# Theory

### Theorem. Weak Gaussian Stochastic Realization (3)

**(b)** A weak Gaussian stochastic realization with $\text{spec}(A) \subset \mathbb{D}_o$ is of minimal state-space dimension over all such realizations, if and only if the following conditions all hold:

- **(b.a)** support of the Gaussian measure $G(0, Q_x)$ of the state process equals $\mathbb{R}^{n_x}$;
  if and only if $0 \prec Q_x$
  if and only if $(A,\ M)$ is a supportable pair;
- **(b.b)** the system representation is stochastically observable;
  if and only if $(A_f,\ C_f)$ is an observable pair; and
- **(b.c)** the system representation is stochastically co-observable;
  if and only if $(A_b,\ C_b)$ is an observable pair.

### Notation

$(n_y,\ n_x,\ n_v,\ A,\ C,\ M,\ N) \in \mathrm{WGSRP}_{\mathrm{min}}$
for the set of parameters of a minimal weak stochastic realization.

# Theory

## Theorem. Weak Gaussian Stochastic Realization (4)

**(c.a)** Classification. The description.
Fix a covariance realization
$\mathrm{lsp} = (n_y,\ n_x,\ n_y,\ F,\ G,\ H,\ J) \in \mathrm{LSP}_{\min}$.
Define the classification map as the function,

$$
\begin{aligned}
c_{lsp} &: \mathbf{Q}_{\mathbf{lsdp}} \to \mathrm{WGSRP}_{\min} \text{ is a bijection,} \\
c_{lsp}(Q) &= (n_y,\ n_x,\ n_v,\ A,\ C,\ M,\ N), \\
A &= F,\ C = H,\ n_v = n_x + n_y \in \mathbb{Z}_+, \\
M &\in \mathbb{R}^{n_x \times n_v},\ N \in \mathbb{R}^{n_y \times n_v}, \\
\begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix}^T &= \begin{bmatrix} Q - FQF^T & G - FQH^T \\ (G - FQH^T)^T & J + J^T - HQH^T \end{bmatrix} = Q_{v,lsdp}(Q).
\end{aligned}
$$

The map $c_{lsp}$ describes all realizations in $\mathrm{WGSRP}_{\min}$,
for fixed $(n_y,\ n_x,\ n_y,\ F,\ G,\ H,\ J)$.
The classification is by the set of state variance matrices $\mathbf{Q}_{\mathbf{lsdp}}$.

# Theory

## Theorem. Weak Gaussian Stochastic Realization (5)

**(c.a)** Classification. The set $\mathbf{Q}_{\mathbf{lsdp}}$ parametrizing all realizations.
Fix a covariance realization
$\mathrm{lsp} = (n_y,\ n_x,\ n_y,\ F,\ G,\ H,\ J) \in \mathrm{LSP}_{\min}$.
There exists
a unique minimal element and a unique maximal element
of the set of state variance matrices according to,

$$\begin{aligned} & \exists\ Q^-,\ Q^+ \in \mathbf{Q}_{\mathbf{lsdp}}(F,\ G,\ H,\ J), \\ & \text{such that } \forall\ Q \in \mathbf{Q}_{\mathbf{lsdp}}(F,\ G,\ H,\ J) \\ Q^- \preceq Q \preceq Q^+. \quad & (Q \preceq Q^+ \ \Leftrightarrow\ w^T Q w \leq w^T Q^+ w,\ \forall\ w \in \mathbb{R}^{n_x}). \end{aligned}$$

The set $\mathbf{Q}_{\mathbf{lsdp}}$ is partially ordered by $\preceq$.
See Chapter 24 of the lecture notes
for a more detailed description of $\mathbf{Q}_{\mathbf{lsdp}}$.

# Theory

### Theorem. Weak Gaussian Stochastic Realization (6)

**(c.b)** Classification. The relation of minimal realizations.

**(c.b.1)** The following transformation transforms the matrices of any minimal weak Gaussian stochastic realization to those of another minimal weak Gaussian stochastic realization.

$$
\begin{aligned}
& (n_y,\ n_x,\ n_v,\ A,\ C,\ M,\ N) \in \mathrm{WGSRP}_{\min}, \\
& \forall\ L_x \in \mathbb{R}_{nsng}^{n_x \times n_x},\ U_v \in \mathbb{R}_{ortg}^{n_v \times n_v}\ (\Leftrightarrow\ U_v^T U_v = I_{n_v} = U_v U_v^T), \\
\Rightarrow \quad & (n_y,\ n_x,\ n_v,\ L_x A L_x^{-1},\ C L_x^{-1},\ L_x M U_v,\ N U_v) \in \mathrm{WGSRP}_{\min}.
\end{aligned}
$$

Note that $L_x M U_v (L_x M U_v)^T = L_x M M^T L_x^T$.

# Theory

## Theorem. Weak Gaussian Stochastic Realization (7)

**(c.b)** Classification. The relation of minimal realizations.

**(c.b.2)** If there exist two minimal weak Gaussian stochastic realizations of the same considered process then there exists transformation matrices such that the system matrices of these realizations are related according to,

$$(n_y,\ n_x,\ n_v,\ A_1,\ C_1,\ M_1,\ N_1) \in \mathrm{WGSRP}_{\min},$$
$$(n_y,\ n_x,\ n_v,\ A_2,\ C_2,\ M_2,\ N_2) \in \mathrm{WGSRP}_{\min},$$
$$\Rightarrow\ \exists\ L_x \in \mathbb{R}_{nsng}^{n_x \times n_x},\ \exists\ U_v \in \mathbb{R}_{ortg}^{n_v \times n_v} \text{ such that}$$
$$\begin{aligned} A_1 &= L_x A_2 L_x^{-1}, \\ C_1 &= C_2 L_x^{-1}, \\ M_1 &= L_x M_2 U_v, \\ N_1 &= N_2 U_v. \end{aligned}$$

**(d)** The realization procedure defined below is well defined.

# Theory

### Procedure. Weak Gaussian Stochastic Realization (1)

Consider a stationary Gaussian process with zero mean value function and with covariance function $W : T \to \mathbb{R}^{n_y \times n_y}$.
Assume the conditions of the theorem hold.

**(1)** Determine a minimal covariance realization of the covariance function $W$ which exists by the finite-rank condition for the infinite-Hankel matrix.

$$\exists\, (n_y,\ n_x,\ n_y,\ F,\ G,\ H,\ J) \in \mathrm{LSP}_{\min},$$
$$W(t) = \begin{cases} HF^{t-1}G, & t > 0, \\ J + J^T, & t = 0, \\ G^T(F^T)^{-t-1}H^T = W(-t)^T, & t < 0. \end{cases}$$

Procedures for this step follow from realization of a time-invariant linear system, see Section 21.8.

**(2)** Determine a matrix $Q \in \mathbf{Q_{lsdp}}(F,\ G,\ H,\ J)$, hence $Q \in \mathbb{R}^{n_x \times n_x}_{pds}$.
Chapter 23 of the lecture notes provides an existence result.

# Theory

## Procedure. Weak Gaussian Stochastic Realization (2)

**(3)** Construct,

$$
\begin{aligned}
& A = F,\ C = H,\ n_v = n_x + n_y, \\
& M \in \mathbb{R}^{n_x \times n_v},\ N \in \mathbb{R}^{n_y \times n_v}, \\
& \begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix}^T = Q_{v,lsdp}(Q) = \begin{bmatrix} Q - FQF^T & G - FQH^T \\ (G - FQH^T)^T & J + J^T - HQH^T \end{bmatrix} \succeq 0; \\
& \Omega = \mathbb{R}^{n_x} \times (\mathbb{R}^{n_v})^T,\ \omega = (\omega_1,\ \omega_2), \\
& x_0(\omega) = \omega_1, \\
& v(\omega, t) = \omega_2(t),\ \text{such that,} \\
& \quad x_0 \in G(0, Q), \\
& \quad v(t) \in G(0,\ I_{n_v}),\quad v \text{ standard Gaussian white noise,} \\
& \quad F^{x_0},\ F^v_\infty \text{ independent; then,} \\
& \quad (n_y,\ n_x,\ n_y,\ A,\ C,\ M,\ N) \in \mathrm{WGSRP}_{\min}.
\end{aligned}
$$

# Theory

**Procedure. Weak Gaussian Stochastic Realization (3)**

**(4)** Then the following time-invariant Gaussian system,

$$x(t+1) = Ax(t) + Mv(t),\ x(0) = x_0 \in G(0, Q),$$
$$y(t) = Cx(t) + Nv(t),\ v(t) \in G(0, I_{n_v}).$$

is a minimal weak Gaussian stochastic realization of the considered stationary Gaussian process.

# Theory

### Def. Kalman realization

Define the Kalman realization of a stationary Gaussian process as the weak Gaussian stochastic realization satisfying,

$$
\begin{aligned}
x(t+1) &= Ax(t) + Mv(t),\ x(0) = x_0 \in G(0, Q_x),\\
y(t) &= Cx(t) + Nv(t),\ v(t) \in G(0,\ I_{n_v}),\\
& \text{spec}(A) \subset \mathbb{D}_o,\\
& (A,\ M) \text{ supportable pair } (\Rightarrow\ 0 \prec Q_x),\\
& (A,\ C) \text{ observable pair},\\
& (A_b,\ C_b) \text{ observable pair};\\
& n_v = n_y,\ N \in \mathbb{R}^{n_y \times n_y},\ \mathrm{rank}(N) = n_y,\\
& \text{spec}(A - MN^{-1}C) \subset \mathbb{D}_o.
\end{aligned}
$$

### Remarks

Term Kalman realization introduced by speaker.
Literature may not include condition of minimality.

# Theory

### Theorem. Kalman realization and Kalman filter (1)

**(a)** Consider a Kalman realization and define

$$Q_x = AQ_xA^T + MM^T,$$

then $Q_x \in \mathbb{R}^{n_x \times n_x}_{spds}$ (meaning $0 \prec Q_x$);

and then,

$$Q_x = Q^- \in \mathbf{Q_{lsdp}}(F,\ G,\ H,\ J).$$

The state variance matrix $Q_x$ of the Kalman realization equals the minimal state variance matrix $Q_x = Q^-$.

# Theory

### Theorem. Kalman realization and Kalman filter (2)

**(b)** From a Kalman realization

$$
\begin{aligned}
x(t+1) &= Ax(t) + Mv(t),\; x(0) = x_0 \in G(0, Q_x), \\
y(t) &= Cx(t) + Nv(t),
\end{aligned}
$$

follows the Kalman filter system according to,

$$
\begin{aligned}
x(t+1) &= Ax(t) + MN^{-1}[y(t) - Cx(t)],\; x(0) = x_0, \\
v(t) &= N^{-1}[y(t) - Cx(t)] = N^{-1}y(t) - N^{-1}Cx(t).
\end{aligned}
$$

Note also that $F^{x(t+1)} \subseteq F_t^{y-} \vee F^{x_0}$, for all $t \in T$.

### Remarks

- Kalman filter system to be derived in a future lecture.
- Backward Kalman realization defined correspondingly.
  Is related to $Q^+ \in \mathbf{Q}_{\mathbf{lsdp}}$.

# Outline

# Explanation

### Example. Non-supportable Gaussian system

- Consider the time-invariant Gaussian system
  for which $(A,\ M)$ is not a supportable pair
  but all other conditions of Theorem 6.4.3 hold.
- After a state-space transformation, one obtains the representation,

$$x(t+1) = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} M_1 \\ 0 \end{bmatrix} v(t),\ x(0) = x_0;$$

$$(A_{11},\ M_1) \text{ supportable pair;}$$

$$\text{spec}(A) \subset \mathbb{D}_o \ \Rightarrow\ a.s. - \lim_{t\to\infty} x_2(t) = 0.$$

- State component $x_2$ not useful in the long run.
  Attention is best restricted to
  a Gaussian system representation with only the first state component.
- These comments illustrate a necessary condition of minimality
  of a stochastic realization.

# Explanation

## Example. Non Stochastically Observable Gaussian system (1)

- Suppose that a stochastic realization is not stochastically observable.
- Assume that $(A,\ M)$ is a supportable pair hence $0 \prec Q_x$.
  The stochastic realization not being stochastically observable implies that the matrix tuple $(A,\ C)$ is not an observable pair.
- Then there exists a linear state-space transformation such that the system representation has the form,

$$
\begin{aligned}
x(t+1) &= \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Mv(t),\ x(0) = x_0, \\
y(t) &= \begin{bmatrix} C_1 & 0 \end{bmatrix} x(t) + Nv(t), \\
& (A_{11},\ C_1) \text{ an observable pair.}
\end{aligned}
$$

- The second component of the state, $x_2$, does not influence the output $y$ at all.

# Explanation

### Example. Non-Stochastically Observable Gaussian system (2)

- Therefore the system representation can be reduced to,

$$\begin{aligned} x_1(t+1) &= A_{11} x_1(t) + M_1 v(t),\ x_1(0) = x_{0,1}, \\ y(t) &= C_1 x_1(t) + N v(t), \\ & (A_{11},\ C_1) \text{ an observable pair.} \end{aligned}$$

- This explains the necessity condition
  of the stochastic observability for minimality.
- A corresponding conclusion holds for stochastic co-observability.

# Explanation

### Example. False conjecture (1)

A mistake which is made by students is
to claim that (a) and (b) below
imply that the corresponding Gaussian system representation
is a minimal weak Gaussian stochastic realization.

$$\begin{aligned} x(t+1) &= a\,x(t) + m\,v(t),\ x(0) = x_0, \\ y(t) &= x(t) + n\,v(t),\ v(t) \in G(0,1), \\ & a \in (-1,\ +1),\ a \neq 0,\ m = (a^2-1)/a \neq 0. \end{aligned}$$

**(a)** The system is supportable, because $(a, m)$ is a supportable pair.

**(b)** The system is stochastically observable, because $(a, 1)$ is an observable pair.

**(c)** The system is not a minimal weak Gaussian stochastic realization of its output process.

# Explanation

### Example. False conjecture (2)

**Proof**

(a) Because of the conditions on $a$, $m \neq 0$.
(b) $(a,\ 1)$ is an observable pair.
(c) Note that,

$$q = a^2\, q + m^2 \ \Rightarrow \ q = (1-a^2)/a^2,\ a\, q + m = 0,$$
$$E[y(t)y(0)] = a^{t-1}\, [a\, q + m] = 0,\ 0 < t.$$

Thus the output process $y$ is Gaussian white noise.
The state-space dimension of
its minimal weak Gaussian stochastic realization is zero,
strictly less than one.
Gaussian white noise is always a stochastic realization
of dimension zero.

# Explanation

### Example. Kalman filter is another stochastic realization

Consider the time-invariant Gaussian system representation,

$$\begin{aligned} x(t+1) &= Ax(t) + Mv(t),\ x(0) = x_0, \\ y(t) &= Cx(t) + Nv(t). \end{aligned}$$

The time-invariant Kalman filter of the above system (Chapter 8) has the system representation

$$\begin{aligned} \hat{x}(t+1) &= A\hat{x}(t) + K[y(t) - C\hat{x}(t)],\ \hat{x}(0) = E[x_0], \\ \overline{v}(t) &= y(t) - C\hat{x}(t),\ \overline{v}(t) \in G(0, Q_{\overline{v}}); \\ & \overline{v} \text{ Gaussian white noise; rewrite system represenation,} \\ \hat{x}(t+1) &= A\hat{x}(t) + K\overline{v}(t), \\ y(t) &= C\hat{x}(t) + \overline{v}(t). \end{aligned}$$

Note: two stochastic realizations of the same output process $y$!
The Kalman filter is also a weak Gaussian stochastic realization!

# Outline

# Transformation

### Procedure. Transformation to a Minimal Realization (1)

Consider a time-invariant Gaussian system representation with,

$$(n_y,\ n_x,\ n_v,\ A,\ C,\ M,\ N) \in \mathrm{WGSRP},\ \ \mathrm{spec}(A) \subset \mathbb{D}_o.$$

**(1)** If $(A,\ M)$ not a supportable pair
then there exists a linear state-space transformation to the form,

$$\begin{aligned}
x(t+1) &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} x(t) + \begin{bmatrix} M_1 \\ 0 \end{bmatrix} v(t), \\
&\quad (A_{11},\ M_1) \text{ supportable pair}, \\
y(t) &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} x(t) + Nv(t); \\
&\quad a.s. - \lim_{t\to\infty} x_2(t) = 0;\ \Rightarrow \\
x_1(t+1) &= A_{11}x_1(t) + M_1 v(t),\ x_1(t) \in \mathbb{R}^{n_{x_1}},\ n_{x_1} < n_x, \\
y(t) &= C_1 x_1(t) + Nv(t), \text{ is a reduced realization}, \\
&\quad (n_y,\ n_x,\ n_v,\ A,\ C,\ M,\ N) \\
&\quad \Rightarrow\ (n_y,\ n_{x_1},\ n_v,\ A_{11},\ C_1,\ M_1,\ N).
\end{aligned}$$

# Transformation

## Procedure. Transformation to a Minimal Realization (2)

**(2)** If the tuple $(A,\ C)$ is not an observable pair,
then there exists a linear state-space transformation to the form,

$$
\begin{aligned}
x(t+1) &= \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} x(t) + \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} v(t), \\
y(t) &= \begin{bmatrix} C_1 & 0 \end{bmatrix} x(t) + Nv(t); \\
& (A_{11},\ C_1) \text{ observable pair}; \\
& x_2 \text{ never affects } y; \ \Rightarrow \\
x_1(t+1) &= A_{11} x_1(t) + M_1 v(t), \\
y(t) &= C_1 x_1(t) + Nv(t), \ \ n_{x_1} < n_x, \\
& \text{is a weak Gaussian realization of dimension } n_{x_1}; \\
& (n_y,\ n_x,\ n_v,\ A,\ C,\ M,\ N) \\
& \Rightarrow \ (n_y,\ n_{x_1},\ n_v,\ A_{11},\ C_1,\ M_1,\ N).
\end{aligned}
$$

# Transformation

### Procedure. Transformation to a Minimal Realization (3)

**(3)** If the Gaussian system representation
is not stochastically co-observable then

- **(3.1)** construct the backward representation,
hence $(A_b,\ C_b)$ is not an observable pair;
- **(3.2)** apply the state-space transformation
as in Step (2) of the procedure and
obtain a weak Gaussian stochastic realization of reduced dimension;
- **(3.3)** transform the backward representation
into a forward representation,
$(n_y,\ n_{x_3},\ n_v,\ A_3,\ C_3,\ M_3,\ N_3) \in \mathrm{WGSRP}_{\min}$ with $n_{x_3} < n_x$.

**(4)** Output $(n_y,\ n_{x_3},\ n_v,\ A_3,\ C_3,\ M_3,\ N_3) \in \mathrm{WGSRP}_{\min}$.
This representation is
supportable, stochastically observable, and stochastically co-observable,
hence a minimal weak Gaussian stochastic realization
of its output process.

# Outline

# Strong Stochastic Realization

**Def. Strong Gaussian stochastic realization**

- Consider a stationary Gaussian stochastic process $\overline{y}$.
- Call a Gaussian system
  a strong Gaussian stochastic realization of $\overline{y}$,
  if there exists a time-invariant Gaussian system
  with state processes $(x,\ y)$ and $F^{x(t)} \subseteq F^{y}_{\infty}$
  such that $\overline{y}(t) = y(t)$ a.s. for all $t \in T$.
- See the book A. Lindquist, G. Picci (2015)
  for strong Gaussian stochastic realizations
  of stationary Gaussian processes.

# Strong Stochastic Realization

### Procedure. Strong stochastic realization in $\sigma$-algebraic setting (1)

1. Construct the state. Fix $t \in T$.
   Construct random variable $x(t) \in X$ such that,

$$(F_t^{y+},\ F_{t-1}^{y-} \mid F^{x(t)}) \in CI, \quad F^{x(t)} \subseteq F_{t-1}^{y-};$$

particular case,

$y_+,\ y_-$ are jointly Gaussian finite-dim. vectors,

$$x = E[y_+ \mid F^{y_-}] = Q_{y_+,y_-} Q_{y_-}^{-1} y_-,$$

$$(F^{y_+},\ F^{y_-} \mid F^x) \in CI, \quad F^x \subseteq F^{y-}.$$

# Strong Stochastic Realization

**Procedure. Strong stochastic realization in $\sigma$-algebraic setting(2)**

**2.** Note that,

$$
\begin{aligned}
& (F_t^{y+},\ F_{t-1}^{y-} |\ F^{x(t)}) \in CI,\ \ \forall\ t \in T, \\
\Leftrightarrow\ & (F_t^{y+} \vee F^{x(t)},\ F_{t-1}^{y-} \vee F^{x(t)} |\ F^{x(t)}) \in CI,\ \ \forall\ t \in T,\ \ \text{by } CI, \\
\Leftrightarrow\ & (F_t^{y+} \vee F_t^{x+},\ F_{t-1}^{y-} \vee F_t^{x-} |\ F^{x(t)}) \in CI,\ \ \forall\ t \in T, \\
& \text{if the family is transitive, a restrictive condition, Chapter 7,} \\
\Leftrightarrow\ & (x,\ y) \text{ are state and output process of a stochastic system.}
\end{aligned}
$$

# Outline

# System Identification

## System identification with stochastic realization – Overview

- Identifiability conditions
  for system identification of a Gaussian system
  follow directly from weak Gaussian stochastic realization:
  1. A characterization of minimality of the Gaussian system
     in terms of supportability, stoc. observability, and stoc. co-observability.
  2. The description of the equivalence class of stochastic realizations.
- A procedure to determine
  an approximate weak stochastic realization, called
  the subspace identification algorithm.
  Due to H. Akaike, R. Mehra, W. Larimore, etc.
  Explanation follows.
- Construction of a canonical form
  of a weak Gaussian stochastic realization.
  For example, the observable canonical form.

# System Identification

### Procedure. Subspace identification (1; sketch)

Consider a stationary Gaussian process $y$ on a finite horizon.

1. Fix time $t_0 \in T$. Restrict attention from infinite past and infinite future to finite past and finite future. Construct,

$$
\begin{aligned}
F^{y-}_{t_0-1} \;&\Rightarrow\; y_-(t_0-t_1 : t_0-1),\; t_1 \in \mathbb{Z}_+, \\
F^{y+}_{t_0} \;&\Rightarrow\; y_+(t_0 : t_0+t_1-1);\;\; (y_+,\, y_-) \in G, \\
x(t_0) = E[y_+ |\, F^{y_-}] &= L\, y_-(t_0-t_1 : t_0-1),\ \text{using } \widehat{W}, \\
&\text{then } (F^{y_+},\, F^{y_-} |\, F^{x(t_0)}) \in CI,\;\; F^{x(t_0)} \subseteq F^{y_-}; \\
&\text{compute} \\
x(t_0+1) = L\, y_-(t_0-t_1+1 : t_0).
\end{aligned}
$$

# System Identification

### Procedure. Subspace identification (2)

2. Construct the system matrices and the noise process.

$$\begin{aligned}
&(x_o(t_0),\ x_o(t_0+1),\ y(t_0)) \in G \ \Rightarrow \\
\begin{bmatrix} A \\ C \end{bmatrix} x(t_0) = E&\left[\begin{bmatrix} x(t_0+1) \\ y(t_0) \end{bmatrix} \mid F^{x(t_0)}\right], \\
&v : \Omega \times T \to \mathbb{R}^{n_x+n_y}, \text{Gaussian white noise}, \\
v(t) = &\begin{bmatrix} x(t+1) - Ax(t) \\ y(t) - Cx(t) \end{bmatrix} \in G(0, Q_v), \quad Q_v \neq I_{n_v}; \\
M = &\begin{bmatrix} I_{n_x} & 0 \end{bmatrix}, \ N = \begin{bmatrix} 0 & I_{n_y} \end{bmatrix}.
\end{aligned}$$

# System Identification

## Procedure. Subspace identification (3)

3. One obtains a Gaussian system representation of which the output process $y_a$ is an approximation of the considered Gaussian process,

$$\begin{aligned} x(t+1) &= Ax(t) + Mv(t),\ x(0) = x(t_0), \\ y_a(t) &= Cx(t) + Nv(t), \\ &\quad \forall\ t \in T,\ v(t) \in G(0,\ Q_v). \end{aligned}$$

# Outline

# Concluding Remarks

## Weak Gaussian stochastic realization

- P. Faurre, with R.E. Kalman as advisor, and colleagues developed weak Gaussian stochastic realization theory.
  Stanford University, CA, USA; and INRIA, France, 1965 – 1979.
- Theory used for
  equivalent conditions of a minimal stochastic realization.
- Set of equivalent weak Gaussian stochastic realizations.
  Defined is a canonical form for weak Gaussian realizations.
  A canonical form is based on:
  (1) the minimal Kalman realization and
  (2) the observable canonical form of a linear system.
  Used in system identification.
- Theory used for system identification.
  Subspace identification procedure is based on stochastic realization.
- Read Sections 6.1 and 7.1 for a text on realization theory.
  See also the Further Reading sections of Chapters 6 and 7.

# Concluding Remarks

### Specific Gaussian stochastic realizations

- Defined are output-based strong stochastic realizations where $F^{x(t)} \subset F^{y}_{\infty}$ for all $t \in T$.
- The realization for which $F^{x(t)} \subset F^{y-}_{t-1}$, for all $t \in T$.
  The minimal Kalman realization satisfies this condition, if $F^{x_0} \subseteq F_0$ equals the trivial $\sigma$-algebra.
- The set of weak Gaussian stochastic realizations of a time-reversible output process.
- The set of balanced weak Gaussian stochastic realizations.

See Sections 6.9 and 6.10.

# Concluding Remarks

### Perspectives of stochastic realization

Extensions of stochastic realization theory
to other subsets of stochastic systems,
outside Gaussian systems.

- Stochastic realization of a Gaussian stochastic control system.
  Underdeveloped.
- Stochastic realization of finite-valued processes.
  Formulated by D. Blackwell and L. Koopmans (1957).
  Not yet satisfactorily solved.
- Stochastic realization of stochastic systems in Hilbert spaces.
  See book A. Lindquist, G. Picci (2015).
- Stochastic realization of $\sigma$-algebraic systems,
  see Section 7.4 of the lecture notes.

# Concluding Remarks

## Overview of Lecture 5

- Theorem of weak Gaussian stochastic realization.
- Existence of a weak Gaussian stochastic realization in terms of the rank of the infinite Hankel matrix.
- Characterization of minimality.
- Classification of all minimal weak Gaussian stochastic realizations.
- System identification with stochastic realization.
- Subspace identification procedure for system identification of a Gaussian system.