Course WI4221 in the Spring Semester of 2025
Control of Discrete-Time Stochastic Systems

Homeworkset 6

20 March 2025 (date homework set is issued).
27 March 2025 (date solution due).

1. *The observable canonical form of a minimal single-output time-invariant linear system.*

   Consider a single-output time-invariant linear system without input, with system representation,

$$
\begin{aligned}
x(t+1) &= Ax(t),\ x(0) = x_0, \\
y(t) &= Cx(t),\ n_x \in \mathbb{Z}_+,\ n_y = 1, \\
\mathrm{LSP}(n_y,\ n_x) &= \left\{(A,\ C) \in \mathbb{R}^{n_x \times n_x} \times \mathbb{R}^{n_y \times n_x}\right\}, \\
\mathrm{LSP}_{\min}(n_y,\ n_x) &= \left\{(A,\ C) \in \mathbb{R}^{n_x \times n_x} \times \mathbb{R}^{n_y \times n_x} |\ (A,\ C) \text{ an observable pair}\right\}.
\end{aligned}
$$

   Define the *observable canonical form* of this system by the formulas,

$$
\begin{aligned}
A_{ocf} &= \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n_x-2} & -a_{n_x-1} \end{pmatrix} \in \mathbb{R}^{n_x \times n_x}, \\
C_{ocf} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{1 \times n_x}, \\
\mathrm{LSP}_{\min ocf}(n_y, n_x) &= \{(A,\ C) \in \mathrm{LSP}|\ (A,\ C) = (A_{ocf},\ C_{ocf})\}.
\end{aligned}
$$

   The understanding is that any element of $\mathrm{LSP}_{\min ocf}$ has the structured system matrices as displayed above where the values of the a_i coefficients are dependent on the particular tuple considered.

   Call the elements $(A_a,\ C_a),\ (A_b,\ C_b) \in \mathrm{LSP}_{\min}$ *similar* if

$$\exists\ L \in \mathbb{R}^{n_x \times n_x}_{nsng} \text{ such that } A_b = LA_aL^{-1} \text{ and } C_b = C_aL^{-1}.$$

   Establish by the following steps that the observable canonical form is a true canonical form as defined in Def. 17.1.4 of the lecture notes.

   (a) Prove that any element $(A,\ C) \in \mathrm{LSP}_{\min ocf}$ is an observable pair so that $\mathrm{LSP}_{\min ocf} \subseteq \mathrm{LSP}_{\min}$ holds.

   (b) Prove that any tuple $(A,\ C) \in \mathrm{LSP}_{\min}$ is similar to an element of $\mathrm{LSP}_{\min ocf}$. This is done by construction of a nonsingular matrix $L \in \mathbb{R}^{n_x \times n_x}_{nsng}$ such that $(LAL^{-1}, CL^{-1}) \in \mathrm{LSP}_{\min ocf}$.

       Hint. Use the interpretation of the result of (a) to construct L.

   (c) Prove that if $(A_a,\ C_a),\ (A_b,\ C_b) \in \mathrm{LSP}_{\min ocf}$ are similar then they are identical. It is sufficient to prove this only for $n_x = 3$. The general case is then easily proven.

   (d) Argue convincingly that the items (a), (b), and (c) together imply that $\mathrm{LSP}_{\min ocf}$ is a canonical form for $\mathrm{LSP}_{\min}$.

2. *Construction of a Kalman realization.*

   (a) Suppose one is given a covariance realization described by the matrices $(F,\ G,\ H,\ J)$ as described in Chapter 6.

   Which matrix in the set of state variance matrices characterizes a Kalman realization? Please refer to the lecture notes for your answer.

   (b) Argue how the matrix described in (a) can be calculated or computed on the basis of the matrices $(F,\ G,\ H,\ J)$.

   (c) Construct next the matrices $(A,\ C,\ M,\ N)$ of the Kalman realization according to a procedure of Chapter 6.

   (d) Can you determine from the matrices constructed in (b), properties of the matrices $(A,\ C,\ M,\ N)$ of a Kalman realization. Determine as many conditions as you can. You may use results of Chapter 24, but, if you do, refer to the results used.

Reading advice

Lecture 6 Presented on 20 March 2025.
Please read of the book or the lecture notes: Section 21.8 with the problem statement and Theorem 21.8.9 on realization theory of a time-invariant linear system.

Chapter 23 and Chapter 24 provide background on covariance functions, dissipative systems, and the dissipation matrix inequality. Reading selected definitions and results will help you to understand stochastic realization theory.

Read in Section 6.6 the proof of Theorem 6.4.3. The proof makes use of the concepts and results of Section 21.8 and of the Chapters 23 and 24.

Lecture 7 To be presented on 27 March 2025.
Please read parts of the Chapters 10 and 11 of the book if you want to learn what the lecture is about. This reading advice is not required reading.