Lecture 07: Stochastic Control Systems and Stochastic Control Problems

Stochastic Control Systems

Definition (Stochastic Control System)**.** *Formally, we define a stochastic control system as a collection satisfying the relation*

$$\begin{aligned} &\textit{cpm}(\ (x(t+1), y(t)) \mid F_t^{x-} \vee F_{t-1}^{y-} \vee F_t^{u-}) \\ &= \textit{cpm}(\ (x(t+1), y(t)) \mid F^{x(t)} \vee F^{u(t)})\ \forall t \in T \end{aligned}$$

which implies that we have that the next state and current measurement and the previous measurements, states and control inputs are conditionally independent given the current state and input:

$$(F^{x(t+1)} \vee F^{y(t)},\ F_t^{x-} \vee F_{t-1}^{y-} \vee F_t^{u-} \mid F^{x(t),u(t)}) \in CI \quad \forall t \in T$$

Where $u : \Omega \times T \to U$ is the input process, $c : \Omega \times T \to X$ the state process and $y : \Omega \times T \to Y$ the output process. If the above does not explicitly depend on time it is time-invariant and if $x_0 \in G$ and cpm[1] *is conditionally Gaussian, the system in Gaussian. We denote*

$$\{\Omega, F, P, T, Y, B_Y, X, B_X, U, B_U, y, x, u\} \in \textit{StocCS}$$

Less formally, we can represent the systems defined above in terms of a recursive system:

$$x(t+1) = f(t, x(t), u(t), v(t)),\ x(0) = x_0$$

where F^{x_0}, F_∞^v are independent for all $t \in T$. We also have that $F^{v(t)}, F_t^u$ also independent. This system is time invariant if f is not explicitly dependent on time. In the Gaussian (linear) case we can write this recursion as

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) + M(t)v(t),\ x(t_0) = x_0, \\ y(t) &= C(t)x(t) + D(t)u(t) + N(t)v(t), \end{aligned}$$

Where (A, M) is a supportable pair and (A, B) a controllable pair. Additionally, we often have $n_y \leq n_v$ and $\text{Rank}(N) = n_y$, $\text{Rank}([M^T, N^T]) = n_v$.

One can use the characteristic equation to prove that this Gaussian representation conforms to the abstract definition above.

Definition (Controlled Output)**.** *Define the controlled output on a finite horizon of a Gaussian control system representation*

$$\begin{aligned} x(t) &= C_z(t)x(t) + D_z(t)u(t),\ \forall t \in T \setminus \{t_1\} \\ z(t_1) &= C_z(t_1)x(t_1) \end{aligned}$$

Stochastic Controllability

Informally, this concept is needed to define the set of a reachable states in finite time t_1. The idea is to go from input process on an interval and initial conditions to a conditional measure on the state at time t_1. See slide 21/62 of lecture 07 for more details on notation. Intuitively we know that the set of probability measures in generally strictly smaller than the set of all probability measures on the set X. We consider the set P_{co} of control-objective probability measures, where

$$P_{co}(X, B(X)) \subseteq P_c(t_1, X, B(X))$$

Where P_c is the set of reachable measures and P_{co} the set of measures that our control objective states we *want* to reach.

We consider a system $\Sigma \in$ StocCS. This is consdier stochasticalyl controllable in the control interval T_c with respect to the control objective probability measure P_{co} if

$$P_{co}(X, B(X)) \subseteq P_c(t_1, X, B(x)),\ T_c = \{t_0 : t_1\} \subseteq T$$

We also define stochastic co-controllability via a similar argument but this is practically not been applied in literature so far.

It can be shown that the set of reachable measures depends on $F^{x(t_0)}$, $F_{t_1-1}^u$ and the conditional covariance of x given the mentioned σ-algebras. We can control the mean, *but not the variance, as the variance does not depend on $u(t)$.*

Practically speaking, we can check whether the pair (A, B) is controllable using

$$\text{Rank}(\mathcal{C}(A, B)) = \text{Rank}\left(\begin{bmatrix} B & AB & \cdots & A^{n_x-1}B \end{bmatrix}\right) = n_x$$

Note that we need the property that (A, M) is a supportable pair. If this is not the case, then part of the stochastic system is deterministic and we need to check other properties. As before, if the system is *not* stochastically controllable, we can do a Kalman decomposition

$$x(t+1) = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} v(t)$$
$$\text{s.t. } (A_{11}, B_1) \text{ a controllable pair}$$

Control Laws

For control we distinguish:

- Input process $y : \Omega \times T \to U$
- Control law $g : X \times T \to U$

In general

$$u(t) = g(t, x(t))$$

In general a control law is more useful than an input trajectory. A control law is a mapping which specifies the control input for each state.

We specify the information structure as a σ-algebra family $\{G_t, t \in T\}$ such that for all $t \in T$. G_t specifies all the information available for the input $u(t)$. These are very useful for decentralized/distributed control.

Definition (Special Information structuresc). *Past-state information structure*

$$\{F_t^{x-}, \forall t \in T\},\ F_t^{x-} = \sigma(\{x(s), t_0 \leq s \leq t\}$$

Markov information structure

$$\{F^{x(t)}, \forall t \in T\},\ F^{x(t)} = \sigma(\{x(t)\})$$

Past-state information structure

$$\{F_{t-1}^{y-}, \forall t \in T\},\ F_{t-1}^{y-} = \sigma(\{y(s), t_0 \leq s \leq t-1\})$$

Classical information structure

$$\{H_t, \forall t \in T\}$$

if there is only one controller with one information structure and 2 satisfies perfect recall:

$$\forall t \in T,\ H_t \subseteq H_{t+1}$$

We will generally work with past-state control laws (depend on past states) and Markov control laws (depend on current state).

Closed-Loop systems

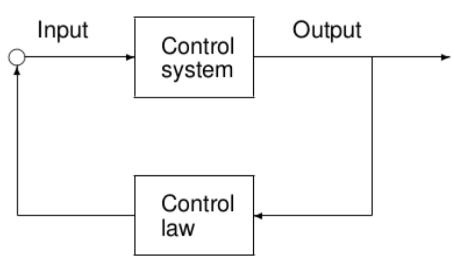

Figure 1: *Classical Closed-loop system*

Given Gaussian stochastic control system

$$\begin{cases} x(t+1) = A(t)x(t) + B(t)u(t) + M(t)v(t) \\ z(t) = C_z x(t) + D_z(t)u(t) \end{cases}$$

with control law $g_t : X^{t+1} \to U$, define recursively the closed-loop Gaussian stochastic system

$$\begin{cases} x^g(t+1) = A(t)x^g(t) + B(t)g_t(x^g(0:t)) + M(t)v(t) \\ z(t) = C_z x^g(t) + D_z(t)g_t(x^g(0:t)) \end{cases}$$

Note that the closed loop system x^g is a Markov process under a Markov control law. Proof is on slide 49/62 lecture 7.

Control Objectives

A control objective is a property that a control system can have, and that an engineer strives to attain. Important ones are:

- **Stability**, finite and bounded variance asymptotically
- **Assignment of Dynamics**, pole placement
- **Optimal Control**, minimize the cost function over all control laws
- **Robustness under uncertainty**, satisfactory dynamics under different operating conditions, unmodelled dynamics and exogenous pertubations.
- **Adaption**, satisfactory performance under slow variations over time (e.g. power systems where load varies over 24h each day).

Note. The general stochastic control problem is then, given a stochastic control system, an information structure, a set of admissible control laws and a set of control objectives, synthesize a control law such that the closed-loop system satisfies the control objectives as well as possible.

More on optimal control later, but generally the goal is to minimize a cost function $J(g)$ over all control laws g such that the expected value of the (quadratic) cost function is minimized. More formally:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Mv(t) \\ y(t) &= Cx(t) + Du(t) + Nv(t) \\ z(t) &= C_z x(t) + D_z u(t) + Nv(t) \\ z(t_1) &= C_z x(t_1); \\ J &: G \to \mathbb{R}_+, \\ J(g) &= \mathbb{E}\left[z(t_1)^T z(t_1) + \sum_{s=0}^{t_1-1} z(s)^T z(s) \right] \end{aligned}$$

Then we solve

$$J^* = \inf_{g \in G} J(g) = J(g^*)$$

Distinguish:

- Control Synthesis: Develop control theory and design procedures. Develop approaches and procedures for finding control laws
- Control Design: Develop and compute actual control laws. Domain dependent and use simulation and testing