# Course WI4221 in the Spring Semester of 2025
# Control of Stochastic Systems

## Homework Set 8

29 April 2025 (date homework set issued)
8 May 2025 (date solutions due).

1. *Preventive maintenance.* Consider a finite stochastic control system with $X = \{0, 1\}$, $U = \{0, 1, 2\}$, and $T = \{0, 1, 2\}$.

   Consider a machine that over a finite horizon may be in an operating state, $x(t) = 1$, or in the down state, $x(t) = 0$. The inputs represent: $u = 0$ that no maintenance is performed, $u = 1$ that preventive maintenance is performed, and $u = 2$ that the machine is repaired. The input space is state dependent, if $x = 0$, thus the machine is in the down state, then $U(0) = \{2\}$, hence the action is to repair the machine; if $x = 1$, hence the machine is in the operating state, then $U(1) = \{0, 1\}$ hence there is a choice between no maintenance or preventive maintenance. The state transition function from state $x$ and input $u$ at a particular time to the probability of going to state $x^+$ at the next time is denoted by $p(x^+, x, u)$. The values of this function are displayed in the next table.

   | $(x^+, x, u)$ | $p(x^+, x, u)$ | $(x^+, x, u)$ | $p(x^+, x, u)$ |
   |---|---|---|---|
   | $(0, 0, 2)$ | $p_1$ | $(1, 0, 2)$ | $1 - p_1$ |
   | $(0, 1, 0)$ | $p_2$ | $(1, 1, 0)$ | $1 - p_2$ |
   | $(0, 1, 1)$ | $p_3$ | $(1, 1, 1)$ | $1 - p_3$ |

   The cost function is defined as $b : X \times U \to \mathbb{R}$, $b(0, 0) = 0$, $b(0, 1) = 5$, $b(0, 2) = 7$, $b(1, 0) = 0$, $b(1, 1) = 5$, and $b(1, 2) = 9$. Suppose that the terminal cost is,

   $$V(t_1, x) = b_1(x) = \begin{cases} 11, & \text{if } x = 0, \\ 3, & \text{if } x = 1. \end{cases}$$

   Take $p_1 = 0.3$, $p_2 = 0.6$, and $p_3 = 0.4$.

   (a) Compute the value function for all times and states, and the optimal control law at $t = 0$ and $t = 1$.

   (b) Next treat the values $V(0, 1)$ and $V(1, 1)$ as variables and no longer as numbers. Determine a relation between $V(1, 0)$ and $V(1, 1)$ such that preventive maintenance is better than no maintenance at time zero in state one, thus at $(t, x) = (0, 1)$ for the choice of $u \in U(1) = \{0, 1\}$.

2. *Invariance of a subset of value functions.*
Consider a finite stochastic control system with state set $X = \{1, 2\}$, with input set $U = \{0, 1\}$, and with time index set $T = \{0, 1, 2\}$. The transition probabilities are indicated in the following table where the notation used is,

$$p(x^+, x, u) = P(\{x(t+1) = x^+\}|\, \{x(t) = x,\ u(t) = u\}),$$
$$\forall\, t \in T \text{ such that } t,\ t+1 \in T.$$

| $(x^+, x, u)$ | $p(x^+, x, u)$ | $(x^+, x, u)$ | $p(x^+, x, u)$ |
|---|---|---|---|
| $(2, 1, 0)$ | $p_1$ | $(1, 1, 0)$ | $1 - p_1$ |
| $(2, 1, 1)$ | $p_2$ | $(1, 1, 1)$ | $1 - p_2$ |
| $(1, 2, 0)$ | $p_3$ | $(2, 2, 0)$ | $1 - p_3$ |
| $(1, 2, 1)$ | $p_4$ | $(2, 2, 1)$ | $1 - p_4$ |

Assume that $p_2 < p_1$, $p_4 < p_3$, and $p_2 + p_3 \leq 1$.

The cost rate is denoted by $b : X \times U \to \mathbb{R}_+$. Assume that

$$b(1, 1) < b(1, 0) < b(2, 0) < b(2, 1).$$

Define the subclass of value functions,

$$V_s = \{V : X \to \mathbb{R}_+ | V(1) \leq V(2)\}.$$

(a) Prove that if for the time $t = 1$ the function $V(1, .) : X \to \mathbb{R}_+$ belongs to the class $V_s$ that then also the value function at time $t = 0$, $V(0, .) : X \to \mathbb{R}_+$, belongs to $V_s$.

(b) If $V(1, .) \in V_s$ then, by (a), $V(0, .) \in V_s$. Calculate the optimal control law at time $t = 0$, $g^*(0, x)$, for all values of $x \in X$.

This exercise is motivated by Section 12.9 of the book.

3. Consider a Gaussian stochastic control system representation with complete observations, the past-states and the past-inputs information pattern, a set of control laws $G$ based on the information pattern, the closed-loop system, the cost function, and the optimal control problem according to,

$$\begin{aligned}
x(t+1) &= Ax(t) + Bu(t) + Mv(t),\ x(0) = x_0,\ g \in G, \\
x^g(t+1) &= Ax^g(t) + Bg(t, x^g(0:t)) + Mv(t), \\
x^g(0) &= x_0, \\
u^g(t) &= g(t, x^g(0:t)), \\
J(g) &= E\left[\sum_{s=0}^{t_1-1} \begin{bmatrix} x^g(s) \\ u^g(s) \end{bmatrix}^T Q^c \begin{bmatrix} x^g(s) \\ u^g(s) \end{bmatrix} + x^g(t_1)^T Q_1^c x^g(t_1)\right], \\
& \quad Q_c \in \mathbb{R}_{pds}^{(n_x+n_u)\times(n_x+n_u)},\ Q_{c,uu} \in \mathbb{R}_{spds}^{n_u \times n_u},\ Q_1^c \in \mathbb{R}_{pds}^{n_x \times n_x}; \\
& \quad \inf_{g \in G} J(g).
\end{aligned}$$

However, the input set is a finite set of the form $U = \{\overline{u}_a, \overline{u}_b\} \subset \mathbb{R}^{n_u}$.

Prove by the two followings steps (a) and (b) that the problem can in principle be solved by dynamic programming as explained in Chapter 12 of the lecture notes.

(a) Calculate the value function and the optimal control law for the indicated time moments,

$$V(t_1, x_V);\ \ V(t_1 - 1, x_V);\ g^*(t_1 - 1, x_V).$$

Write out the formula of the control law as explicit as you can. The main task of the exercise is to discover the analytic formulas of $V(t_1 - 1, x_V)$ and of $g^*(t_1 - 1,\ x_V)$.

(b) Can you formulate a conjecture about the analytic form of the value function $V(t, .)$ for arbitrary time $t \in T$, thus for $t < t_1 - 1$? You do not have to derive that formula. Is such a control law easy to implement?

**Reading Advice**

**Lecture 8** Presented on 29th of April 2025.

Please read of the book, Chapter 12, in particular the Sections 12.2, 12.3, 12.6 – 12.9, and 12.15.

The topic of this week is dynamic programming on a finite horizon. The exercises require the student to apply dynamic programming and to show how dynamic programming can be used to prove properties of optimal control laws.

**Lecture 9** Scheduled for presentation on 1st of May 2025.

You may read of Chapter 13 the Sections 13.1 and 13.2. It should be clear that this is not required reading.