Course Control of Stochastic Systems
Lecture 6
Weak Stochastic Realization of Gaussian Systems (2)

Jan H. van Schuppen

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Delft University of Technology
Delft, The Netherlands

Outline

Outline

Linear Systems

Def. Time-invariant linear control system

Define the impulse response function
of a time-invariant linear deterministic system according to,

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t),\ x(0) = x_0, \\
y(t) &= Cx(t) + Du(t); \\
W_s(t) &= \begin{cases} D, & t = 0, \\ CA^{t-1}B, & t \geq 1, \end{cases} \quad W_s : \mathbb{N} \to \mathbb{R}^{n_y \times n_u};\ \text{if} \\
u_a(t) &= \begin{cases} e_a \in \mathbb{R}^{n_u}, & t = 0, \\ 0, & t \geq 1, \end{cases} \quad e_a \text{ is a unit vector};\ \text{then} \\
y_a(t) &= \sum_{s=0}^{t} W_s(t-s)u_a(s) = W_s(t)e_a,\ \forall\ a \in \mathbb{Z}_{n_u},
\end{aligned}
$$

the output y_a is called
the impulse response of the input u_a.

Linear Systems

Problem. Realization of a linear system

Consider an impulse response function $W : \mathbb{N} \to \mathbb{R}^{n_y \times n_u}$, $n_y, \ n_u \in \mathbb{Z}_+$.

(a) Does there exist a time-invariant linear control system,

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \ x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \\ & (n_y, \ n_x, \ n_u, \ A, \ B, \ C, \ D) \in \mathrm{LSP}, \ \text{such that}, \\ W(t) &= W_s(t) = \begin{cases} D, & t = 0, \\ CA^{t-1}B, & t \geq 1. \end{cases} \end{aligned}$$

If so, call this system a realization of the impulse response function.

(b) Characterize minimal realizations.
A realization is called minimal
if the state-space dimension $n_x \in \mathbb{N}$ is minimal over all realizations.

(c) Classify or parametrize the set of all minimal realizations.
How are two minimal realizations related?

Linear Systems

Terminology

- external description of a linear system:
  the impulse response function, $(n_y,\ n_u,\ W)$;
- internal description of a linear system:
  the system matrices $(A,\ B,\ C,\ D)$ and $(n_x,\ n_y,\ n_u \in \mathbb{Z}_+)$.

The realization problem
goes from the external description to the internal description.
The definition of an impulse response function
goes from an internal description to an external description.

Linear Systems

Def. Hankel matrix

Consider an impulse response function $W : \mathbb{N} \to \mathbb{R}^{n_y \times n_u}$.
Define the block-Hankel matrix with k block-rows and m block-columns of W as the matrix,

$$H_W(k,m) = \begin{bmatrix} W(1) & W(2) & W(3) & \dots & W(m) \\ W(2) & W(3) & W(4) & \dots & W(m+1) \\ W(3) & W(4) & W(5) & \dots & W(m+2) \\ \vdots & & & & \vdots \\ W(k) & W(k+1) & W(k+2) & \dots & W(k+m-1) \end{bmatrix};$$

$$\mathrm{rank}(H_W) = \sup_{k,\ m\ \in \mathbb{Z}_+} \mathrm{rank}(H_W(k,m)) \in \mathbb{N} \cup \{\infty\}.$$

Define the infinite Hankel matrix H_W
as an infinite matrix of which each left-upper block is a $(k,\ m)$ Hankel matrix.
Call the infinite Hankel matrix H_W of finite rank if $\mathrm{rank}(H_W) < \infty$.
This definition is a repeat from Lecture 5.

Linear Systems

Def. A controllable tuple and an observable tuple

Consider a time-invariant linear system.
Call the tuple $(A,\ B)$ a controllable pair and
call the tuple $(A,\ C)$ an observable pair if, respectively,

$$\begin{aligned} n_x &= \mathrm{rank}(\mathrm{conmat}(A,\ B)), \\ n_x &= \mathrm{rank}(\mathrm{obsmat}(A,\ C)),\ \text{where,} \\ \mathrm{conmat}(A,\ B) &= \begin{bmatrix} B & A\,B & A^2B & \dots & A^{n_x-1}B \end{bmatrix}, \\ \mathrm{obsmat}(A,\ C) &= \begin{bmatrix} C \\ C\,A \\ C\,A^2 \\ \vdots \\ C\,A^{n_x-1} \end{bmatrix}. \end{aligned}$$

Linear Systems

Theorem. Realization of a Linear System (1)

Due to R. Kalman (1963).
Consider an impulse response function $W : T = \mathbb{N} \rightarrow \mathbb{R}^{n_y \times n_u}$.

(a) Existence of a realization.
There exists a time-invariant linear control system, with finite state-space dimension $n_x \in \mathbb{N}$,

$$
\begin{aligned}
x(t+1) &= Ax(t) + Bu(t),\ x(0) = x_0, \\
y(t) &= Cx(t) + Du(t),\ \text{such that,} \\
W(t) = W_s(t) &= \begin{cases} CA^{t-1}B, & \text{if } t \geq 1, \\ D, & \text{if } t = 0, \end{cases} \\
&\Leftrightarrow\ \mathrm{rank}(H_W) < \infty \text{ and } W(0) = D.
\end{aligned}
$$

In words, a finite rank of the infinite Hankel matrix is necessary and sufficient for existence of a realization.
See Section 21.8 how to go from W to $(n_x,\ A,\ B,\ C,\ D)$.

Linear Systems

Theorem. Realization of a Linear System (2)

(b) Characterization of minimality
Assume there exists a realization. Equivalence:

(b.1) the realization is a minimal realization (of minimal state-space dimension);

(b.2) $n_x = \mathrm{rank}(H_W) < \infty$ (external characterization).

(b.3) $(A,\ B)$ is a controllable pair and $(A,\ C)$ is an observable pair (internal characterization).

(c.a) Classification.

$$LSP_{min}(W) = \left\{ \begin{array}{l} (n_y,\ n_x,\ n_u,\ A,\ B,\ C,\ D) \in LSP| \\ (A,\ B) \text{ con. pair},\ (A,\ C) \text{ obs. pair}, \\ W(0) = D,\ W(t) = CA^{t-1}B,\ \forall\ t \in \mathbb{Z}_+ \end{array} \right\}.$$

Linear Systems

Theorem. Realization of a Linear System (3)

(c.b) If there exists two minimal realizations
of the same impulse response function with the parameters,

$$(A_1,\ B_1,\ C_1,\ D_1),\ (A_2,\ B_2,\ C_2,\ D_2) \in \mathrm{LSP}_{\min}(n_y,\ n_x,\ n_u),$$

$\Rightarrow$ then these system matrices are similar,

defined as $\exists\ L \in \mathbb{R}_{nsng}^{n_x \times n_x}$ (a nonsingular matrix), such that

$$A_2 = LA_1L^{-1},\ B_2 = LB_1,\ C_2 = C_1L^{-1},\ D_2 = D_1.$$

Conversely, if $(A_1,\ B_1,\ C_1,\ D_1)$
are the system matrices of a minimal realization and if

$$L \in \mathbb{R}_{nsng}^{n_x \times n_x},$$
$$A_2 = LA_1L^{-1},\ B_2 = LB_1,\ C_2 = C_1L^{-1},\ D_2 = D_1.$$

then the second system, with $(A_2,\ B_2,\ C_2,\ D_2)$,
is also a minimal realization of the same response function.

Linear Systems

Theorem. Realization of a Linear System (4)

(d) Procedure. From an arbitrary realization to a minimal realization.
Consider a linear system that is not a minimal realization.
(c.1) Reduce the system representation to a controllable system; and
(c.2) Reduce the system representation to an observable system.
Then one obtains a minimal realization.
See illustration on slides 16 – 17.

Linear Systems

Def. Sylvester's inequality

$$
\begin{aligned}
&\operatorname{rank}(Obs) + \operatorname{rank}(Con) - n \\
\leq &\operatorname{rank}(Obs \times Con) \leq \min\{\operatorname{rank}(Obs),\ \operatorname{rank}(Con)\}, \\
&\forall\ (\ Obs \in \mathbb{R}^{p \times n},\ \ Con \in \mathbb{R}^{n \times m},\ n,\ m,\ p \in \mathbb{Z}_+\ ).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\operatorname{rank}(Obs \times Con) = n, \qquad n \leq p,\ n \leq m, \\
\Rightarrow &\operatorname{rank}(Obs) = n \ \text{ and } \ \operatorname{rank}(Con) = n.
\end{aligned}
$$

(Horn, Johnson, 2nd Ed., 2007, p. 13).
Sylvester's inequality is used in the proof of the theorem.

Linear Systems

Proof of Theorem (1)

(a) ($\Rightarrow$) If there exists a realization then,

$$
\begin{aligned}
W(0) &= D, \quad W(t) = CA^{t-1}B, \ \forall\, t \in \mathbb{Z}_+, \\
H_W(k,m) &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^{m-1}B \end{bmatrix} \\
&= \mathrm{obsmat}_k(A,\ C) \times \mathrm{conmat}_m(A,\ B) \in \mathbb{R}^{kn_y \times mn_u}, \\
\operatorname{rank}(H_W(k,m)) &\le n_x, \quad \forall\, k,\ m \in \mathbb{Z}_+, \\
\operatorname{rank}(H_W) &= \sup_{k,\ m \in \mathbb{Z}_+} \operatorname{rank}(H_W(k,m)) \le n_x < \infty.
\end{aligned}
$$

Linear Systems

Proof of Theorem (2)

(a) ($\Leftarrow$) Use the rank condition $n_x = \mathrm{rank}(H_W) < \infty$
to relate the subspace generated by the infinite Hankel matrix
to a finite-dimensional subspace.
Then a finite Hankel matrix
can be factorized as the product of two matrices over the space $\mathbb{R}^{n_x}$.
Then:

1. construct the C matrix,
2. construct the B matrix,
   see the factorization of the previous slide, and
3. construct the A matrix using the recursion.

Details in book and in lecture notes.

Linear Systems

Proof of Theorem (3)

(b) (b.2) $\Leftrightarrow$ (b.3). For $k,\ m\ \in \mathbb{Z}_+$ sufficiently large,

$$
\begin{aligned}
H_W(k,m) &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} \begin{bmatrix} B & AB & \dots & A^{m-1}B \end{bmatrix} \\
&= \text{obsmat}_k(A,\ C)\ \text{conmat}_m(A,\ B), \\
n_x &= \operatorname{rank}(H_W(k,m)) \\
&\Leftrightarrow^{(1)} \quad n_x = \operatorname{rank}(\text{obsmat}_k(A,\ C)),\ n_x = \operatorname{rank}(\text{conmat}_m(A,\ B)), \\
&\Leftrightarrow^{(2)} \quad n_x = \operatorname{rank}(\text{obsmat}(A,\ C)),\ n_x = \operatorname{rank}(\text{conmat}(A,\ B)), \\
&\Leftrightarrow \quad (A,\ C) \text{ observable pair and } (A,\ B) \text{ controllable pair.}
\end{aligned}
$$

Used above are:
(1) Sylvester's inequality, and (2) the Cayley-Hamilton theorem.

Linear Systems

Proof of Theorem (4)

(c.b) ($\Leftarrow$) This is a direct verification, note,

$$\begin{aligned} W(k) &= C_2 A_2^{k-1} B_2 = C_1 L^{-1} (L A_1 L^{-1})^{k-1} L B_1 = C_1 A_1^{k-1} B_1, \\ &\quad \forall\, k \in \mathbb{Z}_+, \\ W(0) &= D_2 = D_1. \end{aligned}$$

(c.b) ($\Rightarrow$) One has to construct the transformation matrix L
from the factorization of the block Hankel matrix
which is possible due to the assumed minimality of the realization.
(d) The proof is simple. See Slides 18 – 19.

Linear Systems

Minimality illustrated (1)

Consider the time-invariant linear system representation,

$$x(t+1) = \begin{bmatrix} A_{11} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} C_1 & 0 & C_3 & 0 \end{bmatrix} x(t) + Du(t),$$

$$\left(\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right) \text{ a controllable pair,}$$

$$\left(\begin{bmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{bmatrix}, \begin{bmatrix} C_1 & C_3 \end{bmatrix} \right) \text{ an observable pair.}$$

Corresponds to a Kalman decomposition of a linear system,
which is in general neither controllable nor observable,
see Def. 21.3.7 of the book.

Linear Systems

Minimality illustrated (2)

- Consider a linear system
  which is not a minimal realization of its impulse response.
- Reduction of state space to a controllable system
  by elimination of state components $(x_3,\ x_4)$.
  There remain the state components $(x_1,\ x_2)$.
- Reduction of state space to an observable system
  by elimination of state component x_2.
- There remains the subsystem with state component x_1,

$$\begin{aligned} x_1(t+1) &= A_{11}x_1(t) + B_1 u(t),\ x_1(0) = x_{1,0}, \\ y(t) &= C_1 x_1(t) + Du(t), \\ & (A_{11},\ B_1) \text{ controllable pair,} \\ & (A_{11},\ C_1) \text{ observable pair.} \end{aligned}$$

The latter system is a minimal realization of its impulse function.

Linear Systems

Use of Realization Theory of Linear Systems

- Characterization of minimality of a realization of a linear system in terms of controllability and of observability.
- Similarity of minimal realizations by a linear state-space transformation.
- Canonical forms of minimal realizations. Used in system identification.

Linear Systems

Remark on Finite Rank of a Hankel Matrix

It is an undecidable problem to determine whether
an infinite Hankel matrix of an impulse response function,
has a finite rank.

Def. Undecidable Problem

A problem is called undecidable
if there does not exist
a Turing machine (an abstract model of a computer)
which, for any input, will stop after a finite number of steps
with the outcome of the computation of the problem.

See book
M. Sipser, Introduction to the theory of computation,
PWS Publishing Company, Boston, 1997.

Outline

Covariance Functions and Dissipative Systems

Comments

Framework proposed by J.C. Willems (1972– 20xx).
Motivating questions:
(1) When is a linear control system dissipative?
(2) When is the impulse response function of a linear system, a covariance function?

Note the relation between:

- the external behavior of a system, for example dissipativeness,
- to the associated internal behavior of a system, the requested characterization in terms of system matrices.

Details below.

Covariance Functions and Dissipative Systems

Def. Dissipative linear control system

$$
\begin{aligned}
& (n_y,\ n_x,\ n_u,\ F,\ G,\ H,\ J) \in \mathrm{LSP},\ n_y = n_u, \\
x(t+1) &= Fx(t) + Gu(t),\ x(0) = x_0, \\
y(t) &= Hx(t) + Ju(t); \\
& \text{define supply rate}, \quad h : \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \to \mathbb{R},
\end{aligned}
$$

$$
h(u(t),\ y(t)) = u(t)^T y(t) = \frac{1}{2} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}^T J_s \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \quad J_s = \begin{bmatrix} 0 & I_{n_y} \\ I_{n_y} & 0 \end{bmatrix}.
$$

Call this system a dissipative system with supply rate h
if there exists a storage function, for example $S(x) = x^T Qx/2$,
satisfying the dissipation inequality,

$$
\begin{aligned}
& S(x(t)) - S(x(s)) - \sum_{r=s}^{t-1} h(u(r),\ y(r)) \leq 0, \\
& S : \mathbb{R}^{n_x} \to \mathbb{R}_+,\ \forall\, s,\ t \in T,\ \forall\, x(s) \in \mathbb{R}^{n_x},\ \forall\, u \in F(T_d, U).
\end{aligned}
$$

Covariance Functions and Dissipative Systems

Comments. Dissipative Linear Control System

- Time interval of interest $T_d = \{s,\ s+1,\ \ldots,\ t\} \subseteq T$.
- Denote the set of input functions by $F(T_d,\ U) = \{u : T_d \rightarrow U\}$.
- Interpretation of the dissipation inequality:
  over the time interval T_d,
  $S(x(t)) - S(x(s))$, the change in storage,
  $-\sum_{r=s}^{t-1} h(u(r),\ y(r))$, minus the energy supplied,
  the sum is negative, hence has dissipated. Alternatively,

$$S(x(t)) \leq S(x(s)) + \sum_{r=s}^{t-1} h(u(r),\ y(r)).$$

- Example of dissipativeness.
  Think of an electric circuit with only a resistor.
  The heat produced by the resistor is not accounted for in the model, hence the energy of the heat has dissipated.

Covariance Functions and Dissipative Systems

Problem. Characterization of dissipativity

(a) When is a linear control system dissipative?

(b) If a linear control system is dissipative,
classify or describe all storage functions.

Covariance Functions and Dissipative Systems

Def. Available Storage

Define the available storage as the function,

$$S^- : \mathbb{R}^{n_x} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \ +\infty\},$$

$$S^-(x) = \sup_{(t,\ x_1),\ u \in F(T_d, U;\ 0, x, t, x_1)} [-\sum_{r=0}^{t-1} h(u(r),\ y(r))],$$

$$F(T_d, U;\ t_0,\ x_0,\ t_1,\ x_1) = \left\{ \begin{array}{l} u \in F(T_d, U) |\ \exists\ x(t_0) = x_0, \\ (x_0,\ u) \text{ transfers system to } x_1 = x(t_1;\ x_0, u), \\ (t_0,\ x_0)\ \Rightarrow\ (t_1,\ x_1) \end{array} \right\}.$$

Interpretation,
the available storage is the maximal amount of energy which can be extracted from the considered system over the considered future interval.

Covariance Functions and Dissipative Systems

Proposition. Available Storage

(a) The system is dissipative
if and only if
the available storage is a finite valued function;
equivalently, $\forall\, x \in \mathbb{R}^{n_x}$, $S^-(x) < \infty$.

(b) If the system is dissipative
then the available storage is a storage function.

(c) If the system is dissipative and if S is any storage function,
then $0 \leq S^- \leq S$.

Covariance Functions and Dissipative Systems

Def. Required Supply

Consider a linear control system and supply rate h.
Define the required supply as the function,

$$S^{+} : \mathbb{R}^{n_x} \to \mathbb{R}_{+} \cup \{\infty\},$$

$$S^{+}(x) = \inf_{(s<0,\ x_s),\ u \in F(T_d, U;\ s, x_s, 0, x)} \sum_{r=s}^{-1} h(u(r),\ y(t)).$$

Interpretation,
required supply is the supply necessary to transfer the system from the initial tuple $(s,\ x_s)$ to the terminal tuple $(0,\ x)$.
Note the use of the backward interval at $t = 0$.

Covariance Functions and Dissipative Systems

Proposition. Required Supply

(a) Assume that the linear control system is controllable. The system is dissipative if and only if,

$$\exists\ c \in \mathbb{R} \text{ such that } c \leq \sum_{r=s}^{-1} u(r)^T\ y(r); \ \ \text{then,}$$

$$S^+(x) = S^-(x_s) + \inf_{(s<0, x_s),\ u \in F(T_d, U;\ s, x_s,\ 0, x)} \sum_{r=s}^{-1} u(r)^T y(r).$$

is a storage function.

(b) Assume that the system is dissipative with storage function S, and that $S(0) = 0$. Then $S^+(0) = 0$ and $0 \leq S^- \leq S \leq S^+$.

(c) Assume that the system is dissipative and controllable from state 0. Then $S^+ < \infty$ and S^+ is a storage function.

Covariance Functions and Dissipative Systems

Proposition. Convexity and Storage Functions

Assume that the system is dissipative.

(a) The set of storage functions is convex;
equivalently,
if $S_1,\ S_2$ are storage functions and $c \in (0,\ 1)$
then $S_c = cS_1 + (1-c)S_2$ is a storage function.

(b) Assume in addition that
the system is controllable from state $0 \in \mathbb{R}^{n_x}$.
For any constant $c \in (0,\ 1)$,
the following function is a storage function,

$$S_c = cS^- + (1-c)S^+.$$

Covariance Functions and Dissipative Systems

Proposition. Relation Covariance Functions and Dissipative Systems

Assume that the system is controllable from state $x_0 = 0 \in \mathbb{R}^{n_x}$.

$$
\begin{aligned}
x(t+1) &= Fx(t) + Gu(t),\ x(0) = x_0, \\
y(t) &= Hx(t) + Ju(t),\ \ J = J^T, \\
W &: T = \mathbb{Z} \to \mathbb{R}^{n_y \times n_y}, \\
W(t) &= \begin{cases} HF^{t-1}G, & t \geq 1, \\ J + J^T, & t = 0, \\ G^T(F^T)^{-t-1}H^T = W(-t)^T, & t \leq -1. \end{cases}
\end{aligned}
$$

The system is dissipative
if and only if W is a positive-definite function
if and only if W is a covariance function.

Remark

Note relation of system being dissipativity and
the function W being a covariance function.

Covariance Functions and Dissipative Systems

Def. Set of State Variance Matrices (1)

Consider a linear control system with the assumptions,

$$\begin{aligned} lsp &= (n_y,\ n_x,\ n_y,\ F,\ G,\ H,\ J) \in \mathrm{LSP}, \\ &0 \prec J + J^T,\ \mathrm{spec}(F) \subset \mathbb{D}_o; \\ lsdp &= (n_y,\ n_x,\ n_y,\ F^T,\ H^T,\ G^T,\ J^T) \in \mathrm{LSP}, \\ &0 \prec J + J^T,\ \mathrm{spec}(F^T) \subset \mathbb{D}_o. \end{aligned}$$

Call lsdp the tuple of the dual parameters
of the tuple of parameters lsp.
Define the functions,

$$\begin{aligned} Q_{v,lsp} &: \mathbb{R}^{n_x \times n_x} \to \mathbb{R}^{(n_x+n_y)\times(n_x+n_y)}, \\ Q_{v,lsdp} &: \mathbb{R}^{n_x \times n_x} \to \mathbb{R}^{(n_x+n_y)\times(n_x+n_y)}. \end{aligned}$$

Covariance Functions and Dissipative Systems

Def. Set of State Variance Matrices (2)

Define the set of matrices of storage functions
of the linear system $(n_y,\ n_x,\ n_y,\ F,\ G,\ H,\ J) \in \mathrm{LSP}$ by the formula,

$$\mathbf{Q}_{\mathbf{lsp}} = \left\{ Q \in \mathbb{R}_{pds}^{n_x \times n_x} |\ 0 \preceq Q_{v,lsp}(Q) \right\},$$

$$Q_{v,lsp}(Q) = \begin{bmatrix} Q - F^T QF & H^T - F^T QG \\ H - G^T QF & J + J^T - G^T QG \end{bmatrix}.$$

Define the set of state variance matrices
of the covariance function of a Gaussian system
$(n_y,\ n_x,\ n_y,\ F,\ G,\ H,\ J) \in \mathrm{LSP}$, note the duality, by the formula,

$$\mathbf{Q}_{\mathbf{lsdp}} = \left\{ Q \in \mathbb{R}_{pds}^{n_x \times n_x} |\ 0 \preceq Q_{v,lsdp}(Q) \right\},$$

$$Q_{v,lsdp}(Q) = \begin{bmatrix} Q - FQF^T & G - FQH^T \\ G^T - HQF^T & J + J^T - HQH^T \end{bmatrix}.$$

Covariance Functions and Dissipative Systems

Theorem. Algebraic Characterization of Dissipativeness (1)

Due to J.C. Willems (1972).
Consider the linear control system which is
a minimal realization of its impulse response function,

$$
\begin{aligned}
x(t+1) &= Fx(t) + Gu(t),\ x(0) = x_0, \\
y(t) &= Hx(t) + Ju(t),\ 0 \prec J + J^T,\ \mathrm{spec}(F) \subset \mathbb{D}_o, \\
h(u,\ y) &= u^T y, \\
W(t) &= \begin{cases} HF^{t-1}G, & t \geq 1, \\ J + J^T, & t = 0, \\ G^T(F^T)^{-t-1}H^T, & t \leq -1. \end{cases}
\end{aligned}
$$

(a) The following statements are equivalent:

- **(a.1)** the system is dissipative with supply rate h;
- **(a.2)** the function W is positive-definite;
- **(a.3)** there exists a matrix $Q \in \mathbf{Q}_{\mathbf{lsp}}$; and
- **(a.4)** there exists a matrix $Q_d \in \mathbf{Q}_{\mathbf{lsdp}}$.

Covariance Functions and Dissipative Systems

Theorem. Algebraic Characterization of Dissipativeness (2)

(b) If the system is dissipative
then there exists a minimal state-variance matrix which is a solution of the algebraic Riccati equation of stochastic realization,

$$
\begin{aligned}
0 &= D_d(Q), \\
D_d(Q) &= Q - FQF^T + \\
&\quad - (G - FQH^T)(J + J^T - HQH^T)^{-1}(G - FQH^T)^T,
\end{aligned}
$$

with the side conditions,

$$
\begin{aligned}
&(1) \quad Q \in \mathbb{R}_{pds}^{n_x \times n_x}, \\
&(2) \quad 0 \prec J + J^T - HQH^T, \\
&(3) \quad \mathrm{spec}(F - (G - FQH^T)(J + J^T - HQH^T)^{-1} H) \subset \mathbb{D}_o.
\end{aligned}
$$

Define the matrix $Q^- = Q$ to be the solution of the above equation with the three conditions.

Covariance Functions and Dissipative Systems

Theorem. Algebraic Characterization of Dissipativeness (3)

(c) Let $Q \in \mathbb{R}_{pds}^{n_x \times n_x}$. The function $S(x) = \frac{1}{2}\, x^T Q x$ is a storage function if and only if $Q \in \mathbf{Q_{lsp}}$.

(d) Let $Q_d \in \mathbb{R}_{pds}^{n_x \times n_x}$. The function Q_d is a state variance matrix of a weak Gaussian stochastic realization if and only if $Q_d \in \mathbf{Q_{lsdp}}$.

(e) The following relations hold.

$$
\begin{aligned}
Q^- \preceq Q \preceq Q^+, \ \forall\, Q \in \mathbf{Q_{lsp}}, \\
Q_d^- \preceq Q_d \preceq Q_d^+, \ \forall\, Q_d \in \mathbf{Q_{lsdp}}.
\end{aligned}
$$

Remarks

The equation $D_d(Q) = 0$ does not have a unique solution.
But the equation $D_d(Q) = 0$ has a unique solution
if the conditions (1), (2), and (3) of (b) are all required to hold.

Covariance Functions and Dissipative Systems

Proposition. Relation Dissipation Inequality and Noise Variance Matrix
Consider,

$$
\begin{aligned}
& Q \in \mathbb{R}^{n_x \times n_x},\ Q = Q^T,\ s,\ t \in T,\ s < t;\ \text{then,} \\
& S(x(t)) - S(x(s)) - \sum_{r=s}^{t-1} h(u(r),\ y(r)) \\
= \quad & \frac{1}{2} x(t)^T Q x(t) - \frac{1}{2} x(s)^T Q x(s) - \sum_{r=s}^{t-1} u(s)^T y(s)), \\
= - & \sum_{r=s}^{t-1} \frac{1}{2} \begin{bmatrix} x(r) \\ u(r) \end{bmatrix}^T Q_{v,lsp}(Q) \begin{bmatrix} x(r) \\ u(r) \end{bmatrix}.
\end{aligned}
$$

The proof is a simple algebraic calculation.
Consequently, if Q belongs to $\mathbf{Q}_{\mathbf{lsdp}}$, then $0 \preceq Q_{v,lsp}(Q)$, and then the system is dissipative.

Covariance Functions and Dissipative Systems

Theorem. Description of state variance matrices (Thm. 24.7.1)

Consider $(n_y,\ n_x,\ n_y,\ F,\ G,\ H,\ J) \in \mathrm{LSP}_{\min}$ is regular.
Assume that $\mathbf{Q_{lsp}} \neq \emptyset$, there exist $Q^-,\ Q^+ \in \mathbf{Q_{lsp}}$,
$0 \prec Q^-$, $0 \prec (Q^+ - Q^-)$, $\mathrm{spec}(F^-) \subset \mathbb{D}_o$, $\mathrm{spec}(F^+) \subset \mathbb{D}_o$.
Then

$$\mathbf{Q_{lsp}} \subset \mathbf{Q_{lsp}}^+(Q^-) \cap \mathbf{Q_{lsp}}^-(Q^+);\ \text{where,}$$

$$\mathbf{Q_{lsp}}^+(Q^-) = \left\{ \begin{array}{l} Q^- + \Delta Q \in \mathbb{R}^{n_x \times n_x}_{spds} \,| \\ \text{conditions (1) and (2) both hold;} \end{array} \right\},$$

$$\mathbf{Q_{lsp}}^-(Q^+) = \left\{ \begin{array}{l} Q^+ - \Delta P \in \mathbb{R}^{n_x \times n_x}_{spds} \,| \\ \text{corresponding conditions hold} \end{array} \right\};$$

$$(1) \quad \Delta Q \in \mathbb{R}^{n_x \times n_x}_{spds},$$

$$(2) \quad (\Delta Q)^{-1} - F^-(\Delta Q)^{-1}(F^-)^T + \\ - G(J + J^T - G^T Q^- G)^{-1} G^T \succeq 0;$$

$$F^- = F - G(J + J^T - G^T Q^- G)^{-1}(H^T - F^T Q^- G)^T.$$

Covariance Functions and Dissipative Systems

Comments on theorem

- The set of state variances matrices
  is described as contained in the intersection of
  an upward cone of matrices originating at Q^{-} and
  a downward cone of matrices originating at Q^{+}.
  Is a geometric description.
- The upward cone is a polyhedral set of matrices,
  the downward cone is also a polyhedral set of matrices.
- The extremal matrices of these cones
  are described by algebraic Riccati equations.
- Theorem due to J.C. Willems, P. Faurre, and others.
- Extensions to systems in Hilbert spaces and
  to σ-algebraic systems.

Covariance Functions and Dissipative Systems

Comment. Use of Dissipative Systems

- Existence of a state variance matrix $Q_x \in \mathbf{Q_{lsdp}}$.
- The geometric structure of the set state variance matrices.
- Concept of a dissipative system and its relation with stability.
- There is also theory
  for dissipativity of nonlinear deterministic systems.
  Useful for stability of nonlinear systems.

Outline

Proof of Theorem of Weak Gaussian Stochastic Realizaton (1)

(1) Assume there exists a realization,

$$\begin{aligned}
x(t+1) &= Ax(t) + Mv(t),\ x(0) = x_0 \in G(0, Q_{x_0}),\\
y(t) &= Cx(t) + Nv(t),\ \mathrm{spec}(A) \subset \mathbb{D}_o;\\
&\exists\ Q_x \in \mathbb{R}_{pds}^{n_x \times n_x} \text{ such that}\\
Q_x &= AQ_xA^T + MM^T;\ \text{define}\\
Q_{x^+,y} &= AQ_xC^T + MN^T,\\
Q_y &= CQ_xC^T + NN^T;\\
F &= A,\ H = C,\ G = Q_{x^+,y},\ J + J^T = Q_y,\\
W(t) &= W_s(t) = CA^{t-1}Q_{x^+,y} = HF^{t-1}G,\ \forall\ t \geq 1,\\
W(0) &= W_s(0) = Q_y = CQ_xC^T + NN^T = HQ_xH^T + NN^T.
\end{aligned}$$

Proof of Theorem (2)

(1, continued)

$$\begin{aligned} Q_{v,lsdp}(Q_x) &= \begin{bmatrix} Q_x - FQ_xF^T & G - FQ_xH^T \\ (.)^T & J + J^T - HQ_xH^T \end{bmatrix} \\ &= \begin{bmatrix} Q_x - AQ_xA^T & Q_{x^+,y} - AQ_xC^T \\ (.)^T & Q_y - CQ_xC^T \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix}^T \succeq 0, \\ &\Rightarrow \; Q_x \in \mathbf{Q_{lsdp}}. \end{aligned}$$

Proof of Theorem (3)

(1, continued)

$$H_W(k, m) = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^k \end{bmatrix} \begin{bmatrix} G & FG & \dots & F^m G \end{bmatrix} \in \mathbb{R}^{kn_y \times n_x} \times \mathbb{R}^{n_x \times mn_y},$$

$$\mathrm{rank}(H_W(k, m)) \leq \; n_x,$$

$$\mathrm{rank}(H_W) = \sup_{k,\ m \in \mathbb{Z}_+} \mathrm{rank}(H_W(k, m)) \leq n_x.$$

Thus, the infinite Hankel matrix has finite rank.

Proof of Theorem (4)

(2) Consider covariance function W and assumptions of theorem.
Step (1),

$$\begin{aligned} & \mathrm{rank}(H_W) < \infty \\ & \Rightarrow \ \exists \, (n_y, \ n_x, \ n_y, \ F, \ G, \ H, \ J) \in \mathrm{LSP}_{\min}, \\ W(t) = & \left\{ \begin{array}{ll} HF^{t-1}G, & t > 0, \\ J + J^T, & t = 0, \end{array} \right. \\ & (F, \ G) \text{ controllable pair, } (F, \ H) \text{ observable pair.} \end{aligned}$$

Above result follows from
the realization theorem of time-invariant linear systems.
Step (2). Theorem of dissipative systems implies
that the covariance function W yields
existence of a state variance matrix $Q_W \in \mathbf{Q_{lsdp}}$, hence,

$$0 \preceq Q_{v,lsdp}(Q_W).$$

Proof of Theorem (5)

(3) Procedure. Define and note that,

$$
\begin{aligned}
A &= F,\ C = H, \\
\begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix}^T &= Q_{v,lsdp}(Q_W) = \begin{bmatrix} Q_W - FQ_WF^T & G - FQ_WH^T \\ (.)^T & J + J^T - HQ_WH^T \end{bmatrix} \succeq 0, \\
x(t+1) &= Ax(t) + Mv(t),\ x(0) = x_0 \in G(0, Q_W), \\
y(t) &= Cx(t) + Nv(t), v(t) \in G(0, I_{n_v}),\ n_v = n_x + n_y, \\
Q_x &= AQ_xA^T + MM^T, \\
Q_W &= FQ_WF^T + MM^T = AQ_WA^T + MM^T \ \Rightarrow\ Q_x = Q_W, \\
W_s(t) &= CA^{t-1}Q_{x^+,y} = HF^{t-1}G = W(t), \forall\ t \in T, \\
W_s(0) &= CQ_xC^T + NN^T = HQ_WH^T + NN^T = W(0).
\end{aligned}
$$

Thus the constructed system is a weak Gaussian stochastic realization.

Proof of Theorem (6)

(4) Minimality.

$$
\begin{aligned}
& \mathrm{spec}(A) \subset \mathbb{D}_o, \ A = F; \\
W(t) = {} & HF^{t-1}G, \ \text{minimal covariance realization}, \\
\Leftrightarrow & \left\{ \begin{array}{l} (F, \ G) \text{ controllable pair}, \\ (F, \ H) = (A, \ C) \text{ observable pair}, \end{array} \right\}; \\
& 0 \prec Q_W = Q_x, \ \text{by conditions}, \\
Q_x = {} & AQ_xA^T + MM^T \ \Rightarrow \ (A, \ M) \text{ supportable pair}, \\
& \text{using results of the Lyapunov equation}; \\
& (F, \ G) \text{ controllable pair}, \\
\Leftrightarrow & \ (F, \ G) = (A, Q_{x^+,y}) = (A, \ AQ_xC^T + MN^T) \\
= {} & (Q_xA_b^TQ_x^{-1}, \ Q_xC_b^T) \\
& \text{controllable pair of backward representation}, \\
\Leftrightarrow & \ (A_b^T, \ C_b^T) \ \text{controllable pair}, \\
\Leftrightarrow & \ (A_b, \ C_b) \ \text{observable pair}.
\end{aligned}
$$

Outline

Canonical Form

Motivation

- For a considered covariance function,
  there exists a set of minimal weak Gaussian stochastic realizations.
  This set has many elements.
- In system identification one needs for identifiability
  a unique element of this set.
- One mostly selects the Kalman realization as explained next.
- Concept of a canonical form is needed.

Canonical Form

Def. Canonical form

Consider a set X with an equivalence relation E defined on it,

$$
\begin{array}{ll}
X & \text{a set,} \\
E \subseteq & X \times X \text{ equivalence relation, hence such that} \\
(1) & (x, x) \in E, \\
(2) & (x, y) \in E \ \Rightarrow (y, x) \in E, \\
(3) & (x, y) \in E \text{ and } (y, z) \in E \ \Rightarrow \ (x, z) \in E.
\end{array}
$$

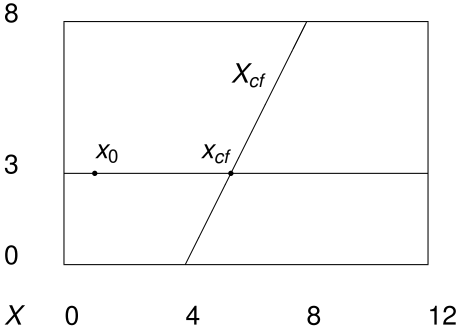
Define a canonical form of $(X,\ E)$ as

$$
\begin{array}{l}
X_{cf} \subseteq X \text{ such that} \\
\qquad \forall\, x \in X,\ \exists \text{ unique } x_{cf} \in X_{cf},\ \text{ such that } (x, x_{cf}) \in E.
\end{array}
$$

Remark. A canonical form is not unique in general.
See Subsection 17.1.1 of the lecture notes.

Canonical Form

Figure canonical form



$$
\begin{aligned}
X &= [0,12] \times [0,8] \subset \mathbb{R}^2, \\
E &= \{(x_1,\ x_2),(y_1,\ y_2) \in X \times X |\ x_2 = y_2\}, \\
X_{cf} &= \{(x_1,\ x_2) \in X |\ x_2 = 2(x_1 - 4),\ 4 \le x_1 \le 8\}, \\
x_0 &= (1,3) \in X \mapsto x_{cf} = (5.5,3) \in X_{cf}.
\end{aligned}
$$

Canonical Form

Towards a canonical form for WGSR

- Consider a stationary Gaussian process
  satisfying the assumptions of Theorem 6.4.3.
- There exists in general a set of
  minimal weak Gaussian stochastic realizations.
- Call two minimal time-invariant Gaussian stochastic systems
  equivalent if they have the same covariance function.
  This defines an equivalence relation on $\mathrm{WGSRP}_{\min}$.
- Needed is a canonical form.
  No satisfactory solution yet.
- Below the observable canonical form
  for a time-invariant linear system with output only.

Canonical Form

Def. Observable canonical form of a time-invariant linear system (1)

Consider a time-invariant linear system without input,
which is a minimal realization of the output.

$$
\begin{aligned}
x(t+1) &= Ax(t),\ x(0) = x_0 \in \mathbb{R}^{n_x}, \\
y(t) &= Cx(t), \\
&\quad (A,\ C) \text{ observable pair.}
\end{aligned}
$$

Canonical Form

Def. Observable canonical form of a time-invariant linear system (2)

Define the observable canonical form of such a system
in terms of the tuple of system matrices $(A,\ C)$
for the single-output case,

$$A_{cf} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n_x-2} & -a_{n_x-1} \end{bmatrix} \in \mathbb{R}^{n_x \times n_x},$$

$$C_{cf} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{1 \times n_x},\ \text{where,}$$

$$A^{n_x} = -\sum_{i=0}^{n_x-1} a_i A^i,\ \text{by the Cayley-Hamilton theorem.}$$

Canonical Form

Remark
Observable canonical form is also known for
a time-invariant linear system with a multivariable output ($n_y > 1$).
Requires extensive notation.

Theorem
The observable canonical form
of a time-invariant linear system,
is a well defined canonical form.
Thus satisfies the conditions of a canonical form stated before.

Proof. Course participants have to construct the proof.
See Homework Set 6.

Canonical Form

Example. Observable canonical form

Consider the following special case of
the single-output observable canonical form
of a time-invariant linear system.

$$
\begin{aligned}
x(t+1) &= A_{cf}x(t),\ x(0) = x_0 \in \mathbb{R}^{n_x},\\
y(t) &= C_{cf}x(t),\\
&n_x = 4,\ n_y = 1,\\
A_{cf} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \in \mathbb{R}^{4\times 4},\\
C_{cf} &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{1\times 4},
\end{aligned}
$$

$(a_0,\ a_1,\ a_2,\ a_3)$ are the parameters of this canonical form.

Outline

Concluding Remarks

Contributions of Lecture 6

- Realization theory of time-invariant linear systems.
  Existence, minimality characterization,
  classification of all realizations and
  relations of minimal realizations.
- Dissipativity theory of a control system and
  the impulse response function being a covariance function.
  Algebraic characterization by a set of state variance matrices
  using a linear matrix inequality.
- The proof of
  the theorem of weak Gaussian stochastic realization.
- Introduction to a canonical form of a time-invariant linear system.