SC42110
Dynamic Programming and Stochastic Control
Introduction

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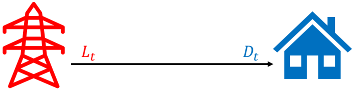
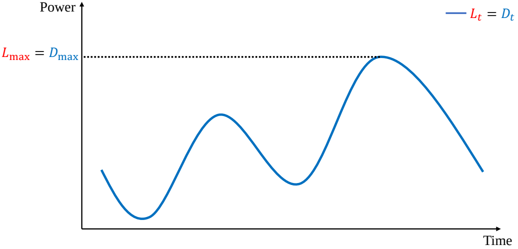
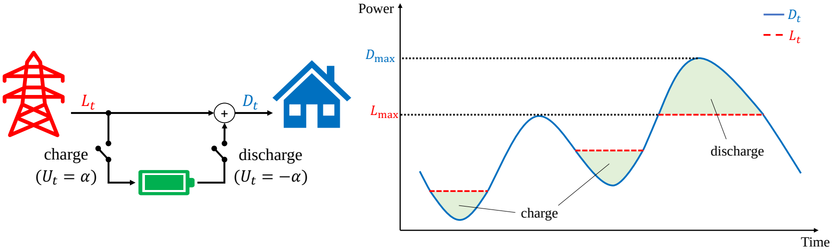
An example: Peak Shaving





- Consumer's electricity cost: $g_{\mathrm{e}} = \sum_{t=1}^{24} L_t \cdot \phi(t) + \left(\max_{t=1}^{24} L_t\right) \phi_{\text{peak}}$

An example: Peak Shaving



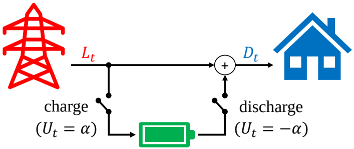
- Consumer's electricity cost: $g_e = \sum_{t=1}^{24} L_t \cdot \phi(t) + \left(\max_{t=1}^{24} L_t\right) \phi_{\text{peak}}$.

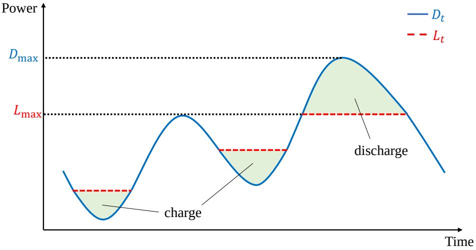
- Dynamics with battery: $\begin{cases} C_{t+1} = \min\{\max\{C_t + U_t, 0\}, C\}, \\ L_{t+1} = (C_{t+1} - C_t) + D_{t+1}, \end{cases}$

  with $U_t \in \mathbb{U} = \{\alpha \text{ (charge)},\ -\alpha \text{ (discharge)},\ 0 \text{ (idle)}\}$.

- Battery usage cost: $g_b = \sum_{t=0}^{23} |U_t|$.

An example: Peak Shaving



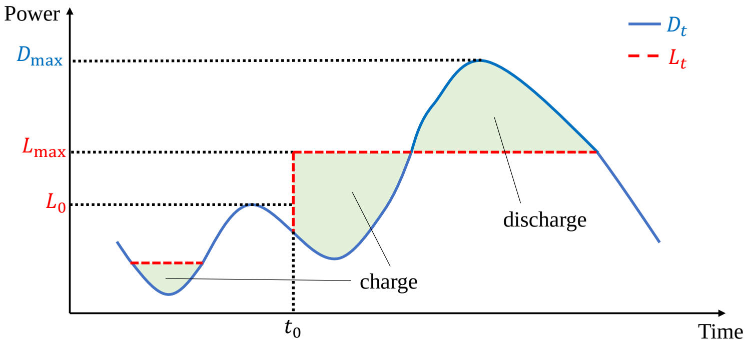


$$\min_{(U_t)_{t=0}^{23}} \; g_{\mathrm{e}} + g_{\mathrm{b}}$$

An example: Peak Shaving

Sequential decision-making under uncertainty:

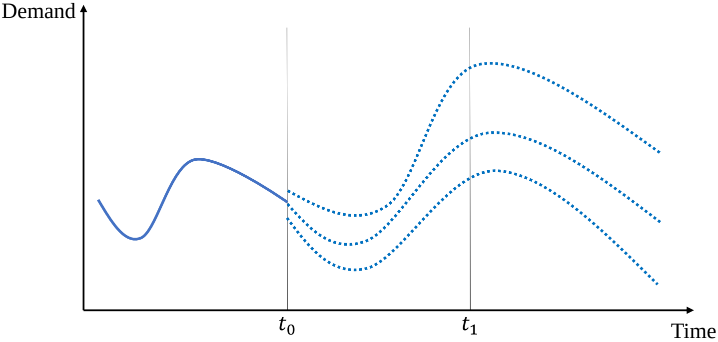
(1) dynamics



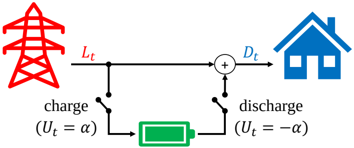
An example: Peak Shaving

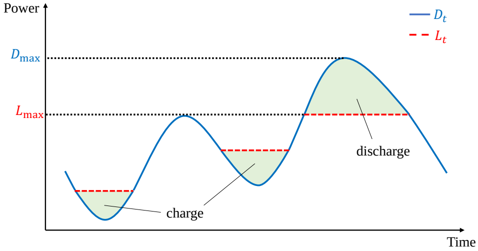
Sequential decision-making under uncertainty:

(2) uncertainty



An example: Peak Shaving

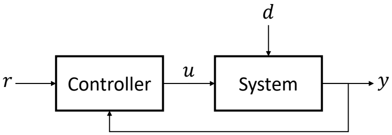




$$\min_{\left(U_t = \mu_t(X_t)\right)_{t=0}^{23}} g_e + g_b$$

We have to look for closed-loop control laws $\mu_t : \mathbb{X} \to \mathbb{U}$!

Stochastic control



Preliminaries: Probability

Random variable (r.v.) $X \in \mathbb{X} \subseteq \mathbf{R}$
$\mathbb{P}(A)$: Probability of the event $A \subseteq \mathbb{X}$

- Discrete r.v.: Fair die

- Continuous r.v.: Uniform$[a, b]$

Preliminaries: Probability

Given discrete r.v.'s $X \in \mathbb{X}$ and $Y \in \mathbb{Y}$ with joint distribution

$$p(x, y) = \mathbb{P}(X = x, Y = y), \quad \forall (x, y) \in \mathbb{X} \times \mathbb{Y}.$$

- Marginalization

- Conditioning

- Independence

Preliminaries: Probability

For discrete r.v.'s $X \in \mathbb{X}$ and $Y \in \mathbb{Y}$:

Lemma (Law of total probability):

$$p_X(x) = \sum_{y \in \mathbb{Y}} p_{X|Y}(x|y) \cdot p_Y(y), \quad \forall x \in \mathbb{X}.$$

Lemma (Bayes' rule):

$$p_{X|Y}(x|y) \cdot p_Y(y) = p_{Y|X}(y|x) \cdot p_X(x), \quad \forall (x,y) \in \mathbb{X} \times \mathbb{Y}.$$

Preliminaries: Probability

Given discrete r.v.'s $X \in \mathbb{X}$ and $Y \in \mathbb{Y}$:

- Expectation
- Variance
- Conditional expectation

Preliminaries: Probability

For two r.v.'s $X \in \mathbb{X}$ and $Y \in \mathbb{Y}$:

Lemma (Linearity of expectation):

$$\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

Lemma (Law of total expectation):

$$\mathbb{E}(X) = \mathbb{E}\big(\mathbb{E}(X|Y)\big).$$

Preliminaries: Optimization

Given a function $f : \mathbf{R}^n \to \mathbf{R}$ and a set $\mathbb{X} \in \mathbf{R}^n$,

$$\text{minimize } f(x) \text{ subject to } x \in \mathbb{X}.$$

Assuming the minimum is attained and the minimizer is unique:

$$f^\star = \min_{x \in \mathbb{X}} f(x) \qquad\qquad x^\star = \operatorname*{argmin}_{x \in \mathbb{X}} f(x)$$

E.g., if $Q \in \mathbf{R}^{n \times n}$ is positive definite,

$$\operatorname*{argmin}_{x \in \mathbf{R}^n} \{x^\top Q x + q^\top x\} = -\frac{1}{2} Q^{-1} q.$$

Preliminaries: Mathematical induction

To show a statement $\mathcal{S}_k$ is true for all $k \in [n]$, it suffices to show

(1) Base case: $\mathcal{S}_k$ is true for $k = 1$.

(2) Induction step: if $\mathcal{S}_k$ is true for some $k < n$ (induction hypothesis), then $\mathcal{S}_{k+1}$ is true.

A classic example:

$$\mathcal{S}_k : \; 1 + 2 + \ldots + k = \frac{k(k+1)}{2}, \quad \forall k \in \mathbf{N}.$$