Course WI4221 Control of Discrete-Time Stochastic Systems in the Spring Semester of 2025

Homeworkset 1

13 February 2025 (Date Homework Set issued)
20 February 2025 (Date solution due)
Contact the lecturer on 20 February at the latest, if you have trouble meeting this due date. It is possible to extend the due date in case of special circumstances.

If you have trouble understanding an exercise of a homework set then contact the lecturer for a hint. You may also come to the office hours of the course to discuss the exercise. It is important for your participation in the course that you learn from the exercises. Misunderstandings should not block your progress on an exercise.

You may write your answer for an exercise on paper and hand the paper in the class room to the lecturer. You may also scan your answer and then send that to the lecturer by email. In the past, several students have typeset their answers. This is acceptable but not required. It is not necessary to copy the text of the exercise as listed on this sheet, start directly with your answer.

Write on the first page of your answers, your name and TUD student number if you have one.

1. *From a conditional Gaussian measure to a random variable representation.*
   Exercise 2.11.3 of the lecture notes.

2. *The concept of a sufficient statistic and conditional independence.*
   Exercise 2.11.6.

3. *Conditional expectation for a tuple of finite-valued random variables.*

   Consider a tuple of finite-valued random variables described by,

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\begin{aligned}
X &= \{1,\ 3\} = \{a_1,\ a_2\} \subset \mathbb{R},\ n_x = 1,\ n_{i_x} = 2, x : \Omega \rightarrow X, \\
Y &= \{2,\ 4,\ 6\} = \{b_1,\ b_2,\ b_3\} \subset \mathbb{R},\ n_y = 3,\ n_{i_y} = 3,\ y : \Omega \rightarrow Y, \\
1/12 &= P(\{\omega \in \Omega |\ x(\omega) = a_1,\ y(\omega) = b_1\}), \\
2/12 &= P(\{\omega \in \Omega |\ x(\omega) = a_1,\ y(\omega) = b_2\}), \\
1/12 &= P(\{\omega \in \Omega |\ x(\omega) = a_1,\ y(\omega) = b_3\}), \\
3/12 &= P(\{\omega \in \Omega |\ x(\omega) = a_2,\ y(\omega) = b_1\}), \\
2/12 &= P(\{\omega \in \Omega |\ x(\omega) = a_2,\ y(\omega) = b_2\}), \\
3/12 &= P(\{\omega \in \Omega |\ x(\omega) = a_2,\ y(\omega) = b_3\}).
\end{aligned}
$$

   (a) State the simple variable representations of x and of y and compute their parameters.

   (b) Calculate m_{i_x}, m_{i_y}, $E[x]$, and $E[y]$. The calculations are simple but take a little time.

   (c) Are x and y independent random variables?

   (d) Calculate and compute the conditional expectation $E[x|\ F^y]$ in terms of a simple variable represenation. Use for this part the results of part (b).

4. *Conditional expectation and measurability.*

   Consider an integrable real-valued random variable $x : \Omega \to \mathbb{R}$, hence $E|x| < \infty$. Consider two sub-σ-algebras $G,\ H \subset F$ satisfying that $H \subseteq G$.

   Prove that, if $E[x|\ G]$ is an H measurable random variable,
   then $E[x|\ G] = E[x|\ H]$.

   This result will be used much during the course.

Reading Advice

The lecture notes in pdf format will be emailed to you if you have provided your email address to the lecturer by email or participated in the first lecture.

Lecture 1 Presented on 13th February 2025.
For this lecture you are advised to read of the lecture notes:

- Chapter 1, Section 1.1 or 1.3; and
- Chapter 2, primarily:
  Section 2.7 Gaussian Random Variables and
  Section 2.8 Conditional Expectation.

A reader who is interested in more probability theory is advised to read other sections of Chapter 2. Chapter 19 (Appendix C) contains additional theory on probability but the topics presented in that appendix are of an advanced level.

Lecture 2 To be presented on 20 February 2025. If you like to read about the topic to be discussed during Lecture 2 before the lecture is to be presented, then you could read parts of the Sections 3.1–3.3 of Chapter 3 of the lecture notes.
It should be clear that this reading is not required.