Dynamic Programming & Stochastic Control (SC42110)
Exercise Set 1

Thursday 8^{th} May, 2025

Exercise 1. Suppose that 9 balls are distributed among 8 baskets such that each distribution of the balls is equally likely. This means that, for example, having 9 balls in one basket is as likely as having 4 balls in the first basket and 5 in the last one.

a) What is the probability that there will be at least 3 nonempty baskets after all the balls have been distributed?

b) On average, how many baskets will contain balls?

Suppose now that 9 balls are "successively" uniformly distributed among 8 baskets. Namely, at each step, one ball is added to one of these baskets with a uniform probability. We ask the same questions as before (*Hint:* Try to model this process as a Markov chain):

c) What is the probability that there will be at least 3 nonempty baskets after all the balls have been distributed?

d) On average, how many baskets will contain balls?

Exercise 2. Three white and three black balls are distributed in two urns in such a way that each contains three balls. We say that the system is in state $i \in \{0, 1, 2, 3\}$, if the first urn contains i white balls. At each step, we draw one ball from each urn and swap them: the ball from the first urn goes to the second urn, and vice versa. Let X_t stand for the state after t draws. Show that $(X_t)_{t\in\mathbf{N}_0}$ is a Markov chain and find its transition probability matrix.

Exercise 3. Suppose that the weather forecast – which only tells you whether it rains today or not – depends on the previous weather conditions through the last three days. Namely,

- if it was raining during the past three days, it will rain today with probability 0.8;
- if it was not raining during the past three days, then it will rain with probability 0.2;

- in all other cases, the weather is going to be the same as yesterday with probability 0.6.

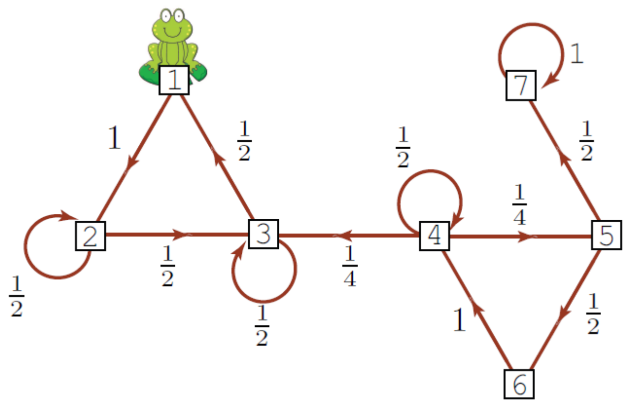
Show that this system may be analyzed by using a Markov chain by specifying its state space, transition probability matrix, and graphical representation. Try to use as few states as possible.

Exercise 4. Consider a virus with $n > 1$ possible strains. In each period, the virus *mutates* with probability α, in which case it changes randomly to any of the remaining $n - 1$ strains.

a) Construct a Markov chain for the process of virus mutations.

b) What is the probability of the virus strain being the same after $t \in \mathbf{N}$ periods?

Exercise 5. A frog hops about on 7 lily pads according to the topology shown in Figure 1. The frog can only jump to a neighboring lily pad or stay on the same pad.



$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 1: Topology of lily pads and the corresponding transition probability matrix in Exercise 5.

a) Suppose that the frog starts in state 1. What is the probability that it is in state 1 after taking 1 hop, 3 hops, and 16 hops?

b) Suppose next that the frog starts in state 4. What is the probability that the frog ever reaches state 7? (*Hint:* What if state 4 is replaced by state 5 or state 6? Are these probabilities related somehow?)

Exercise 6 (Stochastic matrix*)**.** Prove Lemma 2.12. (*Hint:* Use the *Gershgorin circle theorem* for the third property.)

Exercise 7 (Hitting times*)**.** Consider a Markov chain with $n \in \mathbf{N}$ states and transition probability matrix P. Let $p_{ij} := P(i,j)$ for $i, j \in [n]$. We define T_{ij} to be the (random) time the chain takes to hit the state j starting from state i *for the first time*, i.e.,

$$T_{ij} := \min\{s \in \mathbf{N}_0 : \ X_s = j \text{ given } X_0 = i\}.$$

Let

- $q_{ij}(s) := \mathbb{P}(T_{ij} = s)$ for $s \in \mathbf{N}_0$, that is, the *probability* of hitting state j from state i for the first time at step s, and,
- $t_{ij} := \mathbb{E}(T_{ij})$, that is, the *expected* time of hitting state j from state i for the first time.

Observe that, by definition, we have $T_{ii} = 0$ with probability 1 (i.e., $q_{ii}(0) = 1$) and hence $t_{ii} = 0$ for any state i.

Assume that state $j \neq i$ is reachable from state i in the given Markov chain. Show that

$$q_{ij}(s) = \sum_{k \in [n] \setminus \{j\}} p_{ik} \cdot q_{kj}(s-1), \quad \forall s \in \{2, 3, \ldots\}; \tag{1}$$

$$t_{ij} = 1 + \sum_{k \in [n] \setminus \{j\}} p_{ik} \cdot t_{kj}. \tag{2}$$

Exercise 8 (Limiting distribution*)**.** Prove Lemma 2.22. For simplicity assume that the eigenvalues of P are real and distinct.

Exercise 9. Consider a finite-state Markov chain with state space $\mathbb{X} = [n] = \{1, 2, \ldots, n\}$ with $n \in \mathbf{N}$ and transition probability matrix P. Assume $\pi \in \Delta(\mathbb{X})$ is the *limiting* distribution of the MC. Show that the average probability to find the chain in state $i \in \mathbb{X}$ equals $\pi(i)$, i.e.,

$$\lim_{t \to \infty} \frac{\mathbb{P}(X_1 = i) + \mathbb{P}(X_2 = i) + \ldots + \mathbb{P}(X_t = i)}{t} = \pi(i), \quad \forall i \in \mathbb{X}.$$

(*Hint:* Use the *Stolz–Cesàro theorem.*)

Exercise 10. A net surfer has three favorite webpages and checks one of them every hour, starting from a randomly chosen page. Reading webpage $i \in \{1, 2, 3\}$, the surfer finds its content boring with probability $q_i \in (0, 1)$ and interesting with probability $1 - q_i$. If the surfer is interested, (s)he will open the same webpage in the next hour. Otherwise, the surfer will switch to the next webpage in the cyclic order $1 \mapsto 2 \mapsto 3 \mapsto 1$. Analyzing the statistics of web surfing, it has been discovered that in the long run, the surfer always visits each of the webpages 1 and 2 twice more often than webpage 3. Denoting the index of webpage opened at step t by X_t, the Markov chain X_t thus should have the *limiting* distribution $\pi = (2/5, 2/5, 1/5)$, i.e.,

$$\lim_{t \to \infty} \mathbb{P}(X_t = i) = \pi(i), \quad \forall i \in \{1, 2, 3\}. \tag{3}$$

a) Find all triples (q_1, q_2, q_3) for which the convergence in (3) takes place.

b) Find all triples (q_1, q_2, q_3) for which the *convergence rate* in (3) is faster than 2^{-t}, that is,

$$\lim_{t \to \infty} \left(2^t \cdot |\mathbb{P}(X_t = i) - \pi(i)|\right) = 0, \quad \forall i \in \{1, 2, 3\}.$$