SC42110
Dynamic Programming and Stochastic Control
Markov Chain

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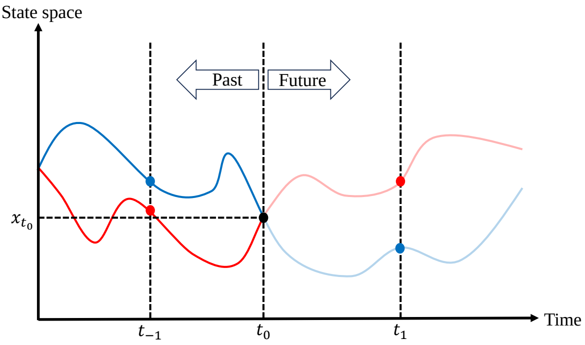
Stochastic process

Definition (Discrete-time stochastic process): A family $(X_t)_{t \in \mathbf{N}_0}$ of random variables $X_0, X_1, X_2, \dots$ with values in state space $\mathbb{X}$.

Example (Discrete-time stochastic process):

- The daily closing price of a stock with $\mathbb{X} = \mathbf{R}_+ \coloneqq [0, +\infty)$
- The demand in a power grid every minute with $\mathbb{X} = \mathbf{R}_+$
- The weekly temperature of a lake with $\mathbb{X} = \mathbf{R}$
- The yearly rate of unemployment with $\mathbb{X} = [0, 100]$
- The number of hits of a website every minute with $\mathbb{X} = \mathbf{N}_0 \coloneqq \{0, 1, 2, \dots\}$
- The hourly occupancy level of a building with $\mathbb{X} = \mathbf{N}_0$

Markov property



- Future depends on the past only through the present.
- The only part of the history of the process that affects its future evolution is its current state.
- There is no additional information in knowing the past history of the process.

Markov chain (MC)

Definition (Markov chain): A discrete-time stochastic process $(X_t)_{t \in \mathbf{N}_0}$ with Markov property, i.e.,

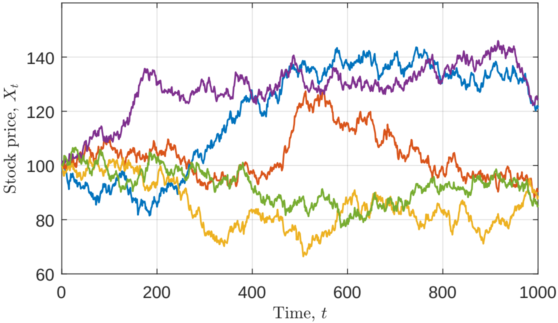
$$\mathbb{P}(X_{t+1} \in A \mid X_0 = x_0, X_1 = x_1, \ldots, X_t = x_t) = \mathbb{P}(X_{t+1} \in A \mid X_t = x_t),$$

$$\forall \ t \in \mathbf{N}_0, \ A \subset \mathbb{X}, \ (x_0, \ldots, x_t) \in \mathbb{X}^{t+1}.$$

Example: Stock price

- X_t: closing price of a stock on day $t \in \mathbf{N}$ with $X_0 = 100$
- $R_{t+1} \sim \mathcal{N}(0, 1)$: rate of return of the stock from day t to day $t+1$ assumed to be i.i.d. and normally distributed

$$X_{t+1} = X_t(1 + R_{t+1}), \quad t \in \mathbf{N}_0$$



Example: Stock price (cont'd)

$$X_{t+1} = X_t(1 + R_{t+1}), \quad t \in \mathbf{N}_0$$

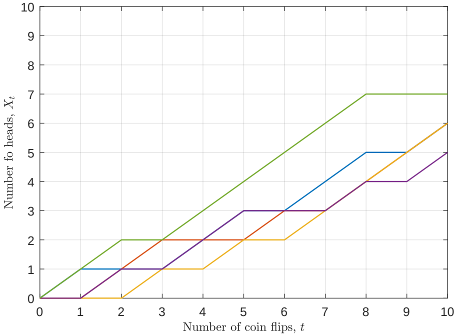
Question: In which case the recursion is not an MC?

1. $R_{t+1} \sim \mathcal{N}(\frac{1}{t+1}, 1)$
2. $R_{t+1} \sim \mathcal{N}(\frac{1}{X_t+1}, 1)$
3. $R_{t+1} \sim \mathcal{N}(\frac{1}{R_t+1}, 1)$

Example: Coin flips

- X_t: number of heads observed in $t \in \mathbf{N}$ flips with $X_0 = 0$
- W_t: i.i.d. Bernoulli process with $p = \frac{1}{2}$ (outcome of t-th coin flip)

$$X_t = W_1 + W_2 + \ldots + W_t = X_{t-1} + W_t, \quad t \in \mathbf{N}$$

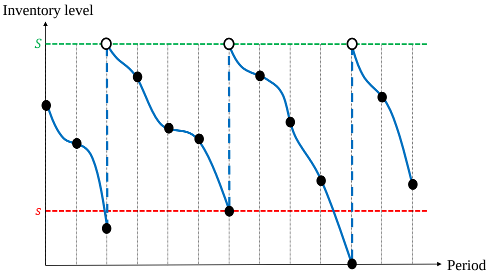


Example: (s, S) inventory model

- X_t: the inventory level at the end of period t
- D_t: the demand during period t

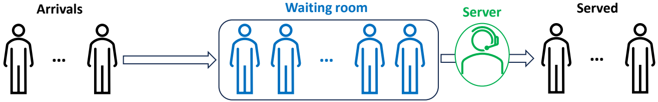
Ordering policy: At the end of each period, order nothing as long as the inventory exceeds a level $s \geq 0$; otherwise, increase the inventory to a level $S > s$.[1]

$$X_{t+1} = \begin{cases} [S - D_{t+1}]_+ & \text{if } X_t \leq s, \\ [X_t - D_{t+1}]_+ & \text{if } X_t > s. \end{cases}$$



[1] $[c]_+ = \max\{0, c\}$ for $c \in \mathbf{R}$.

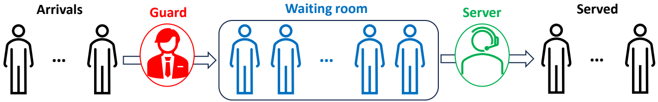
Example: Queuing model



Consider a servicing system involving servers (e.g., a number of receptionists) and customers. The customers arrive at the waiting room according to an i.i.d. stochastic process $(A_t)_{t\in\mathbf{N}}$, where A_t represents the number of customers arriving during period t. The servers can process D_t customers during period t.

Question: Derive the recursion formula for the total number of customers in the waiting room. Is this an MC?

Example: Queuing model (cont'd)



Now, assume that a security guard controls the access to the waiting room: The guard observes the number of people in the room with a delay of 1 period (i.e., at time $t-1$) and if more than K people are in the waiting room, then any new arrivals during period $t+1$ are turned away.

Question: Derive the recursion formula for the total number of customers in the waiting room in the presence of the guard. Is this an MC?

Summary

1. The recursion

$$X_{t+1} = f(X_t, W_{t+1}), \quad t \in \mathbf{N}_0,$$

describes an MC $(X_t)_{t \in \mathbf{N}_0}$ if the disturbance W_{t+1} is conditionally independent given X_t. This characterization is commonly used for MC's with a continuous state space.

2. Many stochastic processes can be converted to Markov chains by enlarging the state space:

$$X_{t+1} = f(X_t, X_{t-1}, \ldots, X_{t-m}) \overset{X'_t := (X_t, \ldots, X_{t-m})}{\Longrightarrow} X'_{t+1} = f'(X'_t).$$

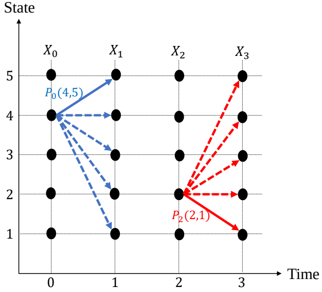
The problem is the state space explosion. So, the key to a good MC model is to control the size of the state space!

Transition (probability) matrix

Hereafter, we only consider countable-state MC's with $\mathbb{X} = [n] \coloneqq \{1, 2, \ldots, n\}$ with $n \in \mathbf{N}^\infty \coloneqq \mathbf{N} \cup \{\infty\}$.[2]

Definition (Transition matrix): $P_t \in [0,1]^{n \times n}$ as the collection of the probabilities of transitioning from state i at time t to state j at time $t+1$, i.e.,

$$P_t(i,j) = \mathbb{P}(X_{t+1} = j \mid X_t = i), \quad i, j \in \mathbb{X} = [n].$$



[2] $n = \infty$ implies $\mathbb{X} = \mathbf{N}$.

Time-homogeneous MC

Hereafter, we only consider time-homogeneous countable-state MC's.

Definition (Time-homogeneous MC): An MC with time-independent transition probability matrices $P_t = P \in [0,1]^{n \times n}$ for all $t \in \mathbf{N}_0$, i.e.,

$$P(i,j) = \mathbb{P}(X_{t+1} = j \mid X_t = i), \quad \forall t \in \mathbf{N}_0,\ i,j \in \mathbb{X} = [n],$$

P is called a "matrix" even though $\mathbb{X}$ might be countably infinite.

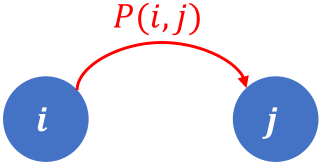
States

States	1	2	$\cdots$	j	$\cdots$
1	$P(1,1)$	$P(1,2)$	$\cdots$	$P(1,j)$	$\cdots$
2	$P(2,1)$	$P(2,2)$	$\cdots$	$P(2,j)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$
i	$P(i,1)$	$P(i,2)$	$\cdots$	$P(i,j)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$

Graphical representation of MC

A countable-state MC can also be characterized by a graph with

1. one node for each element $i \in \mathbb{X}$ of the state space;
2. a directed edge from node i to node j with weight $P(i,j)$.



The graph representation provides the exact same information as the transition matrix, but can be sometimes helpful in identifying specific properties of the MC, e.g., symmetry in transitions.

Example: Coin flips (cont'd)

- $X_t \in \mathbb{X} = \mathbf{N}_0 = \{0, 1, 2, \dots\}$ is the number of heads observed in $t \in \mathbf{N}$ flips with $X_0 = 0$.

One-step transition probabilities:

$$\mathbb{P}(X_{t+1} = y \mid X_t = x) =$$

Stochastic matrix

The transition matrix P of an MC is a (row) stochastic matrix, i.e.,

1. $P(i,j) \geq 0$ for all $i, j \in \mathbb{X} = [n]$.
2. $\sum_{j \in \mathbb{X}} P(i,j) = 1$ for all $i \in \mathbb{X} = [n]$.

Lemma (Stochastic matrix): For a a stochastic matrix $P \in \mathbf{R}^{n \times n}$,

1. $Pe = e$ where $e \coloneqq (1, \ldots, 1)^\top$ is the all-one vector;
2. all eigenvalues of P reside in the unit disc;
3. P^s is a stochastic matrix for all $s \in \mathbf{N}$.

Powers of P

Lemma (Multi-step transition probability): For each $t \in \mathbf{N}_0$,

$$\mathbb{P}(X_{t+s} = j \mid X_t = i) = P^s(i,j), \quad \forall\ i,j \in \mathbb{X},\ s \in \mathbf{N}.$$

Question: Prove the lemma above.

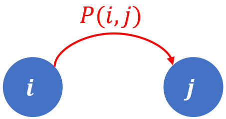
Summary

Three possible ways to characterize a countable-state MC:

1. The recursion
$$X_{t+1} = f(X_t, W_{t+1}), \quad \forall t \in \mathbf{N}_0$$

2. The transition probability matrix
$$P(i,j) = \mathbb{P}(X_{t+1} = j \mid X_t = i), \quad \forall t \in \mathbf{N}_0,\ i,j \in \mathbb{X} = [n]$$

3. The graphical representation



Countable-state MC

Definition (Countable-state MC): The tuple $(\mathbb{X}, P, p_0)$ describing the stochastic process $(X_t)_{t \in \mathbf{N}_0}$ with

1. countable state space $\mathbb{X} = [n]$ where $n \in \mathbf{N}^\infty$;
2. transition probability matrix $P \in [0,1]^{n \times n}$, i.e.,

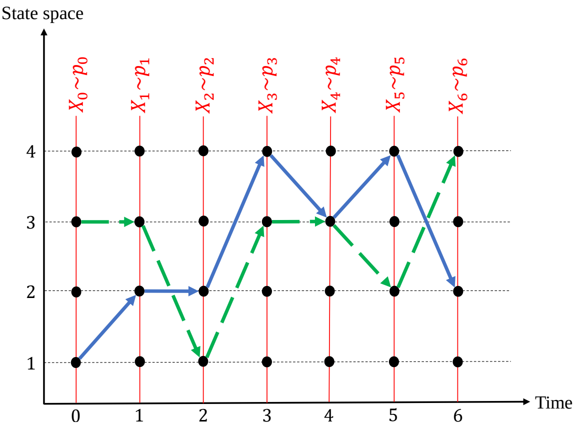
$$P(i,j) = \mathbb{P}(X_{t+1} = j \mid X_t = i), \quad \forall t \in \mathbf{N}_0,\ i, j \in \mathbb{X};$$

3. initial distribution $p_0 \in \Delta(\mathbb{X})$ i.e.,

$$p_0(i) = \mathbb{P}(X_0 = i), \quad \forall i \in \mathbb{X}.$$

Trajectory and state distribution

Example: An MC with four states ($\mathbb{X} = [4]$) over $t = 6$ steps and $p_t \in \Delta([4])$ being distribution of the states at time t, i.e., $p_t(i) = \mathbb{P}(X_t = i)$ for each $i \in \mathbb{X}$.



Trajectory and state distribution

Lemma: Consider a countable-state MC $(\mathbb{X} = [n], P, p_0)$.

1. (Joint state distribution) For each trajectory $(i_0, i_1, \ldots, i_t) \in \mathbb{X}^{t+1}$, we have

$$\mathbb{P}(X_t = i_t, \ldots, X_0 = i_0) = p_0(i_0)\ P(i_0, i_1) \cdots P(i_{t-1}, i_t).$$

2. (State distribution) Let $p_t \in \Delta(\mathbb{X})$ be the state distribution at time t, i.e., $p_t(i) = \mathbb{P}(X_t = i)$ for each $i \in \mathbb{X}$. For each $t \in \mathbf{N}_0$, we have[3]

$$p_t = p_0 P^t.$$

[3] Distributions are treated as row vectors.

Trajectory and state distribution

Proof:

Trajectory and state distribution

Proof:

Example: Coin flips (cont'd)

$X_t \in \mathbb{X} = \mathbf{N}_0 = \{0, 1, 2, \ldots\}$ is the number of heads observed in $t \in \mathbf{N}$ flips with $X_0 = 0$.

- Transition probability matrix (i-th row/column of $P \equiv$ state $i - 1 \in \mathbb{X}$):

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- Initial distribution (the i-th entry of $p_0 \equiv$ state $i - 1 \in \mathbb{X}$):

$$p_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \end{pmatrix}$$

Example: Coin flips (cont'd)

$X_t \in \mathbb{X} = \mathbf{N}_0 = \{0, 1, 2, \ldots\}$ is the number of heads observed in $t \in \mathbf{N}$ flips with $X_0 = 0$.

Question: What is the probability of x heads in t coin flips?

Summary

A countable-state MC is fully characterized by the tuple $(\mathbb{X}, P, p_0)$. Using the initial distribution p_0 and the transition probability matrix P, we can explicitly compute

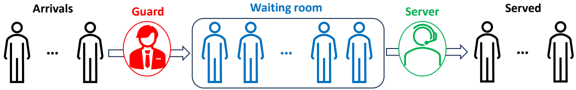
1. the probability distribution of any trajectory (i.e. the joint distribution of states), and,
2. the probability distribution of the states at any time.

Limiting distribution

Hereafter, we only consider time-homogeneous finite-state MC's characterized by the tuple $(\mathbb{X} = [n], P, p_0)$ with $n \in \mathbf{N}$.[4]

We now focus on the long-term behavior of the MC, that is, the asymptotic behavior of the state distribution $p_t \in \Delta(\mathbb{X})$ as $t \to \infty$.

Example (Queuing model): Do we have enough receptionists? Do we have enough space in the waiting room? Should we hire a guard?



Definition (Limiting distribution): $p_\infty = \lim_{t \to \infty} p_t$ for any $p_0 \in \Delta(\mathbb{X})$.

1. Does p_∞ always exist?
2. If yes, how can we compute it?

[4] The result provided hereafter do not hold for countable-state MC's with $\mathbb{X} = \mathbf{N}$.

Limiting distribution & powers of P

Recall the distribution dynamics

$$p_t = p_0 P^t$$

Lemma (Limiting distribution I):[5]

$$p_\infty = \lim_{t\to\infty} p_t,\ \forall p_0 \in \Delta(\mathbb{X}) \iff \lim_{t\to\infty} P^t = e \cdot p_\infty$$

Proof:

[5] $e \coloneqq (1, \dots, 1)^\top$ is the all-one (column) vector.

Powers of a diagonalizable matrix P

Assume $P \in \mathbf{R}^{n \times n}$ is diagonalizable:

$$P = R\Lambda R^{-1} = \begin{pmatrix} | & & | \\ r_1 & \cdots & r_n \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} — & q_1 & — \\ & \vdots & \\ — & q_n & — \end{pmatrix}.$$

Then,

$$P^t = R \begin{pmatrix} \lambda_1^t & & \\ & \ddots & \\ & & \lambda_n^t \end{pmatrix} R^{-1} = \sum_{i \in [n]} \lambda_i^t \underbrace{(r_i \cdot q_i)}_{=: A_i}.$$

Remark (Computation of R and R^{-1}): By explicitly calculating P^t for $t = 0, \ldots, n-1$, we can form a system of n^3 linear equations for the n^3 entries of the matrices $A_1, \ldots, A_n$ by using the equality

$$P^t = A_1 \lambda_1^t + \cdots A_n \lambda_n^t, \quad \forall t \in \mathbf{N}_0.$$

Limiting distribution & invariant distribution

Definition (Invariant distribution): $\pi = \pi P \in \Delta(\mathbb{X})$.

- If $p_{t_0} = \pi$ for some $t_0 \in \mathbf{N}_0$, then $p_t = \pi$ for all $t \geq t_0$.
- Invariant distribution always exists but may not be unique!

Lemma (Limiting distribution II): The limiting distribution is an invariant distribution.

- The reverse statement does not necessarily hold.

Proof:

Limiting distribution & invariant distribution

Lemma (Invariant & limiting distr.): For a finite-state MC $(\mathbb{X} = [n], P, p_0)$, let

- $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of P such that $1 = \lambda_1 \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$;
- $\pi = \pi P \in \Delta(\mathbb{X})$ be an invariant distribution of the MC.

Then,

1. Uniqueness of invariant distribution:

$$\pi \text{ is unique} \iff \lambda_i \neq 1, \forall i \in \{2, \ldots, n\}.$$

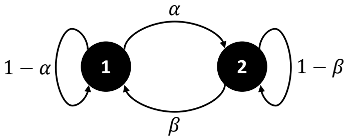
2. Existence of limiting distribution:

$$\pi \text{ is the limiting distribution} \iff |\lambda_i| < 1, \ \forall i \in \{2, \ldots, n\},$$

in which case, there exist $m \in [n-1]$ and constant $C > 0$ such that

$$\lim_{t \to \infty} |P^t(i,j) - \pi(j)| \leq C \cdot t^m \cdot |\lambda_2|^t, \quad \forall i, j \in \mathbb{X}.$$

Example: The simplest MC



Question: Determine the eigenvalues of the transition matrix for different values of $\alpha, \beta \in [0, 1]$.

Example: The simplest MC (cont'd)

Question: Find the values of $\alpha, \beta \in [0, 1]$ for which the invariant distribution is unique. Compute the unique invariant distribution.

Example: The simplest MC (cont'd)

Question: Find the values of $\alpha, \beta \in [0, 1]$ for which the limiting distribution exists. Compute the limiting distribution.

Markov reward process (MRP)

- An extension of Markov chains with a reward function $r : \mathbb{X} \to \mathbf{R}$ (equivalently, a reward column vector $r \in \mathbf{R}^{\mathbb{X}}$) associating scalar "values" to different states.
- Characterized by the tuple $(\mathbb{X}, P, r, p_0)$ with the extra element being the reward function.

Example (Queuing model): Consider the queuing model with state X_t being the number of customers in a queue at time t. By assigning a reward 1 to states $x > 0$ and a reward 0 to the state $x = 0$, we can use the reward signal to determine if the server is "busy."

Lemma (Expected reward): For a finite-state MRP $(\mathbb{X} = [n], P, r, p_0)$,

1. the expected reward at time t is $\mathbb{E}(r(X_t)) = p_0 P^t r$;
2. the limiting expected reward is[6] $\lim_{t\to\infty} \mathbb{E}(r(X_t)) = p_\infty \cdot r$.

[6] Assuming the limiting distribution p_∞ exists.

Summary

For a finite-state MC $(\mathbb{X} = [n], P, p_0)$:

- limiting distribution: $p_\infty = \lim_{t\to\infty} p_t, \ \forall p_0 \in \Delta(\mathbb{X})$;
- invariant distribution: $\pi = \pi P \in \Delta(\mathbb{X})$.

1. π always exists.
2. π is unique $\iff$ $\lambda_1 = 1$ is a simple eigenvalue of P with multiplicity 1 $\iff$ MC has only one **recurrent** class (and possibly some transient states)
3. $\pi = p_\infty$ $\iff$ $\lambda_1 = 1$ is the only eigenvalue of P on the unit circle $\iff$ MC has only one **aperiodic recurrent** class (and possibly some transient states)