Course WI4221 in the Spring Semester of 2025
Control of Discrete-Time Stochastic Systems

Homework Set 9

1 May 2025, date homework set issued,
15 May 2025, date solution due.

1. *Computation of an LQG optimal control law for the average cost criterion.*

   Consider a time-invariant Gaussian stochastic control system representation,

   $$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Mv(t),\ x(0) = x_0, \\ z(t) &= C_z x(t) + D_z u(t),\ v(t) \in G(0, I). \end{aligned}$$

   Consider the optimal stochastic control problem with complete observations, on a half-infinite horizon, with the average cost criterion. Compute the associated optimal LQG control law by carrying out the following steps with the Matlab computer program or an equivalent program for the system matrices listed below.

   (a) Compute the spectrum of the system matrix A and note that there is at least one element of the spectrum which does not belong to the open unit disc $\mathbb{D}_o$. The open-loop system is thus unstable!

   (b) Check whether the following properties hold for the system matrices listed below. (b.1) $(A,\ M)$ is a supportable pair. (b.2) $(A,\ B)$ is a controllable pair. (b.3) $(A_c,\ C_c)$ is an observable pair, where (A_c, C_c) are as defined in Def. 13.2.9 of the book. You may want to compute the singular values of the controllability and of the observability matrices to evaluate how well the properties hold.

   (c) Compute the solution Q_c^* of the control algebraic Riccati equation with side conditions for the average cost. The solution can be computed using one of the Matlab commands *idare* (use $E = I$).

   Check carefully whether you use the proper input for these commands. For the input of the Matlab program there are two sets of matrices possible, one starting with the matrix A and the other starting with the matrix A^T. You may want to check the computed solution Q_c^* by computing also $f_{CARE}(Q_c^*)$ and then comparing it with Q_c^*.

   (d) Compute the optimal feedback matrix $F(Q_c^*)$ and the optimal control law for the problem described in (c).

   (e) Compute the spectrum of the closed-loop system with system matrix $A + BF(Q_c^*)$. Note that all eigenvalues are located inside the open unit disc hence the closed-loop system is exponentially stable. Note that this is achieved with a one-dimensional input for a system with a three-dimensional state space. Do you see how these eigenvalues are distributed over the complex plane?

   (f) Compute the value of the optimal stochastic control problem.

The system matrices are specified by

$$\begin{aligned} & n_x = 3,\ n_u = 1,\ n_v = 3,\ n_z = 3; \\ A &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.7560 & +1.2900 & +0.4000 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ M = I_3 \in \mathbb{R}^{3\times 3},\ Q_{x_0} = I_3, \\ C_z &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix},\ D_z = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}. \end{aligned}$$

2. *Optimal stochastic control for a finite stochastic control system with complete observations on an infinite horizon with average cost.*

   Consider the time-invariant finite stochastic control system in the indicator representation with following system representation and an optimal control problem with average cost,

$$\begin{aligned} X &= \{1,\ 2,\ 3\},\ U = [-0.3,\ +0.3] \subset \mathbb{R}_+, \\ E[x(t+1)|\ F_t^{x,u}] &= \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.4-u & 0.7 & 0 \\ 0.4+u & 0 & 0.7 \end{bmatrix} x(t),\ x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \\ G &= \{g : X \to U\},\ J : G \to \mathbb{R}_+, \\ J(g) &= \lim_{t\to\infty} \frac{1}{t}\, E[\sum_{s=0}^{t-1} (2 \times x_2(s) + x_3(s))], \\ & \inf_{g\in G} J(g). \end{aligned}$$

   (a) Can you formulate a conjecture about the optimal control law at the state x_1 based on the problem formulation?

   (b) Please formulate the dynamic programming equation for the average cost, in the formulation of Def. 13.2.20 on p. 497 and Procedure 13.2.21 on p. 498 of the lecture notes.

   (c) Compute the explicit numerical values of the value function $V : X \to \mathbb{R}_+$ and of the optimal control law $g^* : X \to U$ which are the solutions of the dynamic programing equation.

Reading Advice

Lecture 9 Presented on Thursday 1 May 2025.

Please read of the lecture notes of Chapter 13: the Sections 13.1, 13.2 on the average cost case, and 13.6. The topic of this lecture is stochastic control with complete observations on an infinite horizon.

Lecture 10 To be presented on Thursday 8 May 2025 at the usual time.

If you like you may read of the lecture notes of Chapter 8, the Sections 8.1 – 8.4. This is not required reading.