Proof of picture from class

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Theorem 1. *Let V, W be finite dimensional vector spaces, and let $A : V \to W$ be a linear mapping with adjoint $A^* : W \to V$. Then:*

1. $N(A) \perp R(A^*)$;
2. $V = N(A) \oplus R(A^*)$.

Proof. 1) Let $x \in N(A)$, $y \in R(A^*)$. Then, since $y \in R(A^*$, there exists a $z \in W$ such that $y = A^*z$. Then:

$$(x, y) = (x, A^*z) = (Ax, z) = 0$$

This proves 1.

For 2, we first observe that since V is finite dimensional, $V = N(A) \oplus N(A)^\perp$. Thus, it suffices to show $N(A)^\perp = R(A^*)$. Let $x \in R(A^*)$. Then, there is a $z \in W$ such that $Az^* = x$. Let $y \in N(A)$. Then:

$$(x, y) = (A^*z, y) = (z, Ay) = 0.$$

Thus, $x \in N(A)^\perp$, and $R(A^*) \subset N(A)^\perp$.

Now let $x \in N(A)^\perp$, but suppose $x \notin R(A^*)$. Since V is finite dimensional, this implies that $x \in R(A^*)^\perp$ (observe $x \neq 0$ since $x \notin R(A^*)$). Now:

$$(Ax, y) = (x, A^*y) = 0 \quad \forall y \in W$$

Hence $Ax = 0$, and $x \in N(A)$. But then $x \in N(A) \cap N(A)^\perp$, which implies that $x = 0$, a contradiction. Thus $N(A)^\perp \subset R(A^*)$, and the proof is complete. $\square$