Formulation

The goal is to express quantum mechanics in a new way with geometric algebra in order to get an understanding for space and spin.

The wavefunction can be expressed as

$$\Psi = \sum_{i=1}^{n} \frac{1}{\sqrt{2}}(e_i + Jf_i)(\Re\psi_i + J\,\Im\psi_i)$$

where e_i, f_i are orthonormal vectors which square to $+1$,

$$J = e_0 f_0$$

is an independent bivector, and ψ_i are the complex valued components of the wave vector. This makes the wave function an odd-grade multivector.

For a spin-1/2 particle coordinates will come out as

$$\begin{aligned}
J\Psi\Psi^\dagger &= J + Tt + Xx + Yy + Zz \\
X &= \frac{1}{2}(e_1 f_2 + e_2 f_1) \\
Y &= \frac{1}{2}(e_1 e_2 + f_1 f_2) \\
Z &= \frac{1}{2}(e_2 f_2 - e_1 f_1) \\
T &= \frac{1}{2}(e_1 f_1 + e_2 f_2)
\end{aligned}$$

which means that space rotations can be derived in this basis and the same rotor can be applied to Ψ too.

They anti-commute and obey

$$\begin{array}{ccc}
XY = Z & YZ = X & ZX = Y \\
XT = 0 & YT = 0 & ZT = 0
\end{array}$$

$$
\begin{aligned}
\Pi &= e_1 f_1 e_2 f_2 \\
XX &= -\frac{1}{2}(1+\Pi) \\
YY &= -\frac{1}{2}(1+\Pi) \\
ZZ &= -\frac{1}{2}(1+\Pi) \\
TT &= -\frac{1}{2}(1-\Pi)
\end{aligned}
$$

which has been derived from the spin wave vector alone without actually considering time and for a positive signature vector space. Note that these multivectors are not invertable.

Observation in quantum mechanics

The probability $P = |\langle\psi_1|\psi_2\rangle|^2$ to measure a state ψ_1 in a state ψ_2 can be calculated from

$$
\begin{aligned}
P(1 \to 2) &= \langle (1 - \Psi_1 \Psi_1^\dagger)(1 - \Psi_2 \Psi_2^\dagger) \rangle \\
&= \langle \Psi_1 \Psi_1^\dagger \Psi_2 \Psi_2^\dagger \rangle - 1
\end{aligned}
$$

which is an inner product between two state multivectors of the form $\Psi\Psi^\dagger$.

Unitary transformation

A unitary transformation of the wavefunction can be represented as a rotor in geometric algebra. Note that the same rotor can be applied to Ψ or $\Psi\Psi^\dagger$.

State vector

$$
\Psi = \sum_{i=1}^{n} \frac{1}{\sqrt{2}} (e_i + J f_i)(\Re\psi_i + J\,\Im\psi_i)
$$

The wave function squared can be expanded into a bivector

$$
\boxed{
\begin{aligned}
\Omega &= J\Psi\Psi^\dagger \\
&= J + \sum_i e_i f_i \, |\psi_i|^2 + \sum_{i<j} (e_i f_j + e_j f_i)\, \Re(\psi_i \psi_j^*) - \sum_{i<j} (e_i e_j + f_i f_j)\, \Im(\psi_i \psi_j^*)
\end{aligned}
}
$$

(see appendix) such that the probability is

$$
P = \Omega_1 \cdot \Omega_2 - 1
$$

Single spin

The wavefunction for a single spin-up in a direction given by Euler angles θ, ϕ is usually written as

$$\psi = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix}$$

up to an arbitrary phase.

Written in geometric algebra this is

$$\begin{aligned} J\Psi\Psi^\dagger &= J + e_1f_1\cos^2\frac{\theta}{2} + e_2f_2\sin^2\frac{\theta}{2} \\ &\quad + (e_1f_2 + e_2f_1)\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\phi \\ &\quad + (e_1e_2 + f_1f_2)\cos\frac{\theta}{2}\sin\frac{\theta}{2}\sin\phi \\ &= J + e_1f_1\frac{1-\cos\theta}{2} + e_2f_2\frac{1+\cos\theta}{2} \\ &\quad + (e_1f_2 + e_2f_1)\frac{1}{2}\sin\theta\cos\phi \\ &\quad + (e_1e_2 + f_1f_2)\frac{1}{2}\sin\theta\sin\phi \\ &= J + \frac{1}{2}(e_1f_1 + e_2f_2) \\ &\quad + \frac{1}{2}(e_2f_2 - e_1f_1)\cos\theta \\ &\quad + \frac{1}{2}(e_1f_2 + e_2f_1)\sin\theta\cos\phi \\ &\quad + \frac{1}{2}(e_1e_2 + f_1f_2)\sin\theta\sin\phi \end{aligned}$$

Remembering that we have Euler angles, we can identify the multivectors for space coordinates from this expression for a single spin

$$\begin{aligned} J\Psi\Psi^\dagger &= J + Tt + Xx + Yy + Zz \\ X &= \frac{1}{2}(e_1f_2 + e_2f_1) \\ Y &= \frac{1}{2}(e_1e_2 + f_1f_2) \\ Z &= \frac{1}{2}(e_2f_2 - e_1f_1) \\ T &= \frac{1}{2}(e_1f_1 + e_2f_2) \end{aligned}$$

The expression for T is a guess from the remaining term.

Space rotations will be derived soon.

$$\begin{aligned} e_1 f_1 &= T - Z \\ e_2 f_2 &= T + Z \\ TT + XX &= -1 \end{aligned}$$

$$\Omega = J + \sum_i e_i f_i \, |\psi_i|^2 + 2 \sum_{i<j} X_{ij} \, \Re(\psi_i \psi_j^*) - 2 \sum_{i<j} Y_{ij} \, \Im(\psi_i \psi_j^*)$$

Interpretation

A vague idea why quantum mechanics is this way, is because due to the rules of probability, a state should be a (multi)vector and probabilities be calculated from an inner product. The probabilites rules are that they should sum to 1 and redoing a measurement yields the same results.

This state is constantly being rotated looking like $\Delta = \cdots R_3 R_2 R_1 ? R_1^\dagger R_2^\dagger R_3^\dagger \cdots$ we somehow get, that actually Δ should split into $\Delta = \Psi \Psi^\dagger$ with the consequence that the degrees of freedom are largely reduced. The above construction with geometric algebra demonstrates how to do this "square root".

Appendix

Derivation

$$\Psi = \sum_i \frac{1}{\sqrt{2}} (e_i + J f_i)(\Re\psi_i + J \, \Im\psi_i)$$

With $z_i = \Re\psi_i + J \, \Im\psi_i$

$$
\begin{aligned}
\Psi\Psi^\dagger &= \frac{1}{2}\sum_{ij}(e_i + Jf_i)z_i z_j^\dagger(e_j - Jf_j) \\
&= \sum_i (1 - e_i f_i J) z_i z_i^\dagger \\
&\quad + \frac{1}{2}\sum_{i<j}\Big((e_i + Jf_i)(e_j - Jf_j)z_i z_j^\dagger + (e_j + Jf_j)(e_i - Jf_i)z_j z_i^\dagger\Big) \\
&= \sum_i (1 - e_i f_i J) z_i z_i^\dagger \\
&\quad + \frac{1}{2}\sum_{i<j}(e_i e_j + f_i f_j - (e_i f_j + e_j f_i)J)z_i z_j^\dagger \\
&\quad + \frac{1}{2}\sum_{i<j}(e_j e_i + f_j f_i - (e_j f_i + e_i f_j)J)z_j z_i^\dagger
\end{aligned}
$$

With the real and imaginary parts

$$
\begin{aligned}
R_{ij} &= \frac{1}{2}(z_i z_j^\dagger + z_j z_i^\dagger) \\
JI_{ij} &= \frac{1}{2}(z_i z_j^\dagger - z_j z_i^\dagger) \\
z_i z_j^\dagger &= R_{ij} + J\,I_{ij} \\
z_j z_i^\dagger &= R_{ij} - J\,I_{ij}
\end{aligned}
$$

where R_{ij}, I_{ij} are scalars this becomes

$$
\begin{aligned}
\Psi\Psi^\dagger &= \sum_i (1 - e_i f_i J) z_i z_i^\dagger \\
&\quad + \frac{1}{2}\sum_{i<j}(e_i e_j + f_i f_j - (e_i f_j + e_j f_i)J)(R_{ij} + J\,I_{ij}) \\
&\quad + \frac{1}{2}\sum_{i<j}(-e_i e_j - f_i f_j - (e_i f_j + e_j f_i)J)(R_{ij} - J\,I_{ij}) \\
&= \sum_i z_i z_i^\dagger - \sum_i e_i f_i J z_i z_i^\dagger \\
&\quad - \sum_{i<j}(e_i f_j + e_j f_i)JR_{ij} \\
&\quad + \sum_{i<j}(e_i e_j + f_i f_j)J\,I_{ij}
\end{aligned}
$$

For normalized wave vectors

$$
\langle \Psi\Psi^\dagger \rangle = 1
$$

Therefore

$$J\Psi\Psi^\dagger = J + \sum_i e_i f_i z_i z_i^\dagger + \sum_{i<j}(e_i f_j + e_j f_i) R_{ij} - \sum_{i<j}(e_i e_j + f_i f_j)\, I_{ij}$$

is a grade-2 multivector.

The probability in quantum mechanics can be calculated from the dot product of two real vectors where states have the components

$$(\psi_i \psi_i^*, \ldots, \sqrt{2}\Re(\psi_i \psi_j^*), \ldots, \sqrt{2}\Im(\psi_i \psi_j^*), \ldots)$$

(shown elsewhere). Therefore in our case and for normalized wave vectors the probability can also be calculated from

$$P(1 \to 2) = \langle (1 - \Psi_1 \Psi_1^\dagger)(1 - \Psi_2 \Psi_2^\dagger) \rangle = \langle \Psi_1 \Psi_1^\dagger \Psi_2 \Psi_2^\dagger \rangle - 1$$