Formulation

The goal is to express quantum mechanics in a new way with geometric algebra in order to get an understanding for space and spin.

The wavefunction can be expressed as

$$\Psi = \sum_{i=1}^{n} \frac{1}{\sqrt{2}} (e_i + J f_i)(\Re\psi_i + J\,\Im\psi_i)$$

where e_i, f_i are orthonormal vectors which square to $+1$,

$$J = e_0 f_0$$

is an independent bivector, and ψ_i are the complex valued components of the wave vector. In total there are $2n+2$ orthonormal vectors for an n-dimensional original wave vector.

Alternatively, the bivector

$$\boxed{\Psi f_0 = \frac{1}{\sqrt{2}} \sum_{i=1}^{n} ((e_i f_0 + f_i e_0)\Re\psi_i + (e_i e_0 - f_i f_0)\Im\psi_i)}$$

is a more natural choice, but for the (classical) derivation I will use the first version. The difference to other treatments of spin with geometric algebra is that here the basis is of the form $ab + cd$ with 2 additional basis vectors, instead of $\sigma_{12} = e_1 e_2$.

One advantage is that the full probability calculation becomes

$$P(1 \to 2) = \langle \Psi_1 \Psi_1^\dagger \Psi_2 \Psi_2^\dagger \rangle - 1$$

which can take the same rotor for $\Psi\Psi^\dagger$ and Ψ.

For a spin-1/2 particle coordinates will come out as

$$\begin{aligned}
J\Psi\Psi^\dagger &= J + N + Xx + Yy + Zz \\
X &= \frac{1}{2}(e_1 f_2 + e_2 f_1) \\
Y &= \frac{1}{2}(e_1 e_2 + f_1 f_2) \\
Z &= \frac{1}{2}(e_2 f_2 - e_1 f_1) \\
N &= \frac{1}{2}(e_1 f_1 + e_2 f_2)
\end{aligned}$$

where coordinates do not depend on e_0, f_0. Space rotations can be derived in this basis and the same rotor can be applied to Ψ too.

They anti-commute and obey

$$XY = Z \qquad YZ = X \qquad ZX = Y$$

which has been derived from the spin wave vector alone. Note that these multivectors are not invertable.

Observation in quantum mechanics

The probability $P = |\langle\psi_1|\psi_2\rangle|^2$ to measure a state ψ_1 in a state ψ_2 can be calculated from (see appendix)

$$\begin{aligned}
P(1 \to 2) &= \langle(1 - \Psi_1\Psi_1^\dagger)(1 - \Psi_2\Psi_2^\dagger)\rangle \\
&= \langle\Psi_1\Psi_1^\dagger\Psi_2\Psi_2^\dagger\rangle - 1 \\
&= \Omega_1 \cdot \Omega_2 - 1
\end{aligned}$$

with the state vector

$$\Omega = J\Psi\Psi^\dagger$$

This is an inner product between two state multivectors. The observable state vector can be expanded into a state bivector

$$\boxed{\begin{aligned}
\Omega &= J\Psi\Psi^\dagger \\
&= J + \sum_i e_i f_i\,|\psi_i|^2 + \sum_{i<j}(e_i f_j + e_j f_i)\,\Re(\psi_i\psi_j^*) - \sum_{i<j}(e_i e_j + f_i f_j)\,\Im(\psi_i\psi_j^*)
\end{aligned}}$$

(see appendix) for an easier translation from the usual representation with a complex wave vector ψ_i.

Unitary transformation

A unitary transformation of the wavefunction can be represented as a rotor in geometric algebra. Note that the same rotor can be applied to Ψ or $\Psi\Psi^\dagger$.

State vector of a single spin

The wavefunction for a single spin-up in a direction given by Euler angles θ, ϕ is usually written as

$$\psi = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

up to an arbitrary phase.

The state vector written in geometric algebra this is

$$\begin{aligned}
\Omega &= J\Psi\Psi^\dagger \\
&= J + \sum_i e_i f_i \, |\psi_i|^2 + \sum_{i<j} (e_i f_j + e_j f_i) \, \Re(\psi_i \psi_j^*) - \sum_{i<j} (e_i e_j + f_i f_j) \, \Im(\psi_i \psi_j^*) \\
&= J + e_1 f_1 \cos^2\frac{\theta}{2} + e_2 f_2 \sin^2\frac{\theta}{2} \\
&\quad + (e_1 f_2 + e_2 f_1) \cos\frac{\theta}{2} \sin\frac{\theta}{2} \cos\phi \\
&\quad + (e_1 e_2 + f_1 f_2) \cos\frac{\theta}{2} \sin\frac{\theta}{2} \sin\phi \\
&= J + e_1 f_1 \frac{1 - \cos\theta}{2} + e_2 f_2 \frac{1 + \cos\theta}{2} \\
&\quad + (e_1 f_2 + e_2 f_1) \frac{1}{2} \sin\theta \cos\phi \\
&\quad + (e_1 e_2 + f_1 f_2) \frac{1}{2} \sin\theta \sin\phi \\
&= J + \frac{1}{2}(e_1 f_1 + e_2 f_2) \\
&\quad + \frac{1}{2}(e_2 f_2 - e_1 f_1) \cos\theta \\
&\quad + \frac{1}{2}(e_1 f_2 + e_2 f_1) \sin\theta \cos\phi \\
&\quad + \frac{1}{2}(e_1 e_2 + f_1 f_2) \sin\theta \sin\phi
\end{aligned}$$

Remembering that we have Euler angles, we can identify the multivectors for space coordinates from this expression for a single spin

$$
\begin{aligned}
J\Psi\Psi^\dagger &= J + N + Xx + Yy + Zz \\
X &= \frac{1}{2}(e_1 f_2 + e_2 f_1) \\
Y &= \frac{1}{2}(e_1 e_2 + f_1 f_2) \\
Z &= \frac{1}{2}(e_2 f_2 - e_1 f_1) \\
N &= \frac{1}{2}(e_1 f_1 + e_2 f_2)
\end{aligned}
$$

Space rotations will be derived soon, but from $XY = Z$ we can already see the rotation.

Interpretation

A vague idea why quantum mechanics is this way, is because due to the rules of probability, a state should be a (multi)vector and probabilities be calculated from an inner product. The probabilites rules are that they should sum to 1 and redoing a measurement yields the same results.

This state is constantly being rotated looking like $\Omega = \cdots R_3 R_2 R_1 \Omega_0 R_1^\dagger R_2^\dagger R_3^\dagger \cdots$ and for some reason we get, that actually Ω should split into $\Omega = \Psi\Psi^\dagger$ - as if all particles start with the same state Ω_0.

Ψ has only roughly the square root number of parameters than the observable Ω .

Other ideas

Looking at

$$\Psi f_0 = \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \left((e_i f_0 + f_i e_0)\Re\psi_i + (e_i e_0 - f_i f_0)\Im\psi_i \right)$$

one may also consider what happens if the terms $e_i f_0, f_i e_0, e_i e_0, f_i f_0$ have independent coefficients. Maybe this is related to Dirac matrices.

Also, one may wonder if a term $e_i e_j$ is existing. This may provide an extension of quantum mechanics.

Multiple spins

For multiple spins one could take the wavefunction of dimension $n = 2^k$ and apply the above procedure. However, it may also be insightful to generate a new wavefunction from multiplying two single spin wave functions

$$\Psi = \sum_{i=1,2} \frac{1}{\sqrt{2}}(e_i + Jf_i)\left(\Re\psi_i^{(1)} + J\,\Im\psi_i^{(1)}\right) \sum_{j=3,4} \frac{1}{\sqrt{2}}(e_i + Jf_i)\left(\Re\psi_i^{(2)} + J\,\Im\psi_i^{(2)}\right)$$

This is for independent spins. For entangled spins, the cross-terms will have separate coefficients.

Appendix

Derivation

$$\Psi = \sum_i \frac{1}{\sqrt{2}}(e_i + Jf_i)(\Re\psi_i + J\,\Im\psi_i)$$

With $z_i = \Re\psi_i + J\,\Im\psi_i$

$$\begin{aligned}
\Psi\Psi^\dagger &= \frac{1}{2}\sum_{ij}(e_i + Jf_i)z_i z_j^\dagger(e_j - Jf_j) \\
&= \sum_i (1 - e_i f_i J) z_i z_i^\dagger \\
&\quad + \frac{1}{2}\sum_{i<j}\left((e_i + Jf_i)(e_j - Jf_j)z_i z_j^\dagger + (e_j + Jf_j)(e_i - Jf_i)z_j z_i^\dagger\right) \\
&= \sum_i (1 - e_i f_i J) z_i z_i^\dagger \\
&\quad + \frac{1}{2}\sum_{i<j}(e_i e_j + f_i f_j - (e_i f_j + e_j f_i)J) z_i z_j^\dagger \\
&\quad + \frac{1}{2}\sum_{i<j}(e_j e_i + f_j f_i - (e_j f_i + e_i f_j)J) z_j z_i^\dagger
\end{aligned}$$

With the real and imaginary parts

$$\begin{aligned}
R_{ij} &= \frac{1}{2}(z_i z_j^\dagger + z_j z_i^\dagger) \\
JI_{ij} &= \frac{1}{2}(z_i z_j^\dagger - z_j z_i^\dagger) \\
z_i z_j^\dagger &= R_{ij} + J\,I_{ij} \\
z_j z_i^\dagger &= R_{ij} - J\,I_{ij}
\end{aligned}$$

where R_{ij}, I_{ij} are scalars this becomes

$$
\begin{aligned}
\Psi\Psi^\dagger = &\sum_i (1 - e_i f_i J) z_i z_i^\dagger \\
&+\frac{1}{2}\sum_{i<j} (e_i e_j + f_i f_j - (e_i f_j + e_j f_i)J)(R_{ij} + J\, I_{ij}) \\
&+\frac{1}{2}\sum_{i<j} (-e_i e_j - f_i f_j - (e_i f_j + e_j f_i)J)(R_{ij} - J\, I_{ij}) \\
= &\sum_i z_i z_i^\dagger - \sum_i e_i f_i J z_i z_i^\dagger \\
&-\sum_{i<j} (e_i f_j + e_j f_i) J R_{ij} \\
&+\sum_{i<j} (e_i e_j + f_i f_j) J\, I_{ij}
\end{aligned}
$$

For normalized wave vectors

$$\sum_i z_i z_i^\dagger = 1$$

Therefore

$$J\Psi\Psi^\dagger = J + \sum_i e_i f_i z_i z_i^\dagger + \sum_{i<j} (e_i f_j + e_j f_i) R_{ij} - \sum_{i<j} (e_i e_j + f_i f_j)\, I_{ij}$$

is a grade-2 multivector.

The probability in quantum mechanics can be calculated from the dot product of two real vectors where states have the components

$$(\psi_i \psi_i^*, \ldots, \sqrt{2}\Re(\psi_i \psi_j^*), \ldots, \sqrt{2}\Im(\psi_i \psi_j^*), \ldots)$$

due to

$$
\begin{aligned}
P(\psi \to \phi) &= |\langle \psi | \phi \rangle|^2 \\
&= \sum_i \psi_i \phi_i^* \sum_j \phi_j \psi_j^* \\
&= \sum_i \psi_i \phi_i^* \phi_i \psi_i^* + \sum_{i<j} \left(\psi_i \phi_i^* \phi_j \psi_j^* + \psi_j \phi_j^* \phi_i \psi_i^* \right) \\
&= \sum_i \psi_i \psi_i^* \phi_i \phi_i^* + \sum_{i<j} 2\Re(\psi_i \psi_j^* \phi_i^* \phi_j) \\
&= \sum_i \psi_i \psi_i^* \phi_i \phi_i^* + \sum_{i<j} 2 \left(\Re(\psi_i \psi_j^*) \Re(\phi_i \phi_j^*) + \Im(\psi_i \psi_j^*) \Im(\phi_i \phi_j^*) \right)
\end{aligned}
$$

being a dot product of vectors with components $\psi_i \psi_i^*, \sqrt{2}\psi_i \psi_j^* (i < j)$.

Therefore in our case and for normalized wave vectors the probability can also be calculated from

$$P(1 \to 2) = \langle (1 - \Psi_1 \Psi_1^\dagger)(1 - \Psi_2 \Psi_2^\dagger) \rangle = \langle \Psi_1 \Psi_1^\dagger \Psi_2 \Psi_2^\dagger \rangle - 1$$

Relations between single spin vectors

$$\begin{aligned} XY &= Z \qquad & YZ &= X \qquad & ZX &= Y \\ XN &= 0 \qquad & YN &= 0 \qquad & ZN &= 0 \end{aligned}$$

$$\begin{aligned} \Pi &= e_1 f_1 e_2 f_2 \\ XX &= -\frac{1}{2}(1 + \Pi) \\ YY &= -\frac{1}{2}(1 + \Pi) \\ ZZ &= -\frac{1}{2}(1 + \Pi) \\ NN &= -\frac{1}{2}(1 - \Pi) \end{aligned}$$

Wave vector basis

$$\begin{aligned} \Psi &= \sum_{i=1}^{n} \frac{1}{\sqrt{2}} (e_i + J f_i)(\Re \psi_i + J\, \Im \psi_i) \\ &= \sum_{i=1}^{n} \Lambda_i \left(\Re \psi_i + J\, \Im \psi_i\right) \end{aligned}$$

with

$$\Lambda_i = \frac{1}{\sqrt{2}} (e_i + J f_i)$$

$$\begin{aligned} X_{ij} &= \frac{1}{2}(e_i f_j + e_j f_i) \qquad & \tilde{X}_{ij} &= \frac{1}{2}(e_i f_j - e_j f_i) \\ Y_{ij} &= \frac{1}{2}(e_i e_j + f_i f_j) \qquad & \tilde{Y}_{ij} &= \frac{1}{2}(e_i e_j - f_i f_j) \\ Z_{ij} &= \frac{1}{2}(e_j f_j - e_i f_i) \qquad & \tilde{Z}_{ij} &= \frac{1}{2}(e_j f_j + e_i f_i) \\ T_{ij} &= \frac{1}{2}(e_i f_i + e_j f_j) \qquad & \tilde{T}_{ij} &= \frac{1}{2}(e_i f_i - e_j f_j) \\ e_i f_i &= T_{ij} - Z_{ij} = \tilde{T}_{ij} + \tilde{Z}_{ij} \\ e_j f_j &= T_{ij} + Z_{ij} = \tilde{T}_{ij} - \tilde{Z}_{ij} \end{aligned}$$

$$\begin{aligned} \Lambda_i^2 &= 0 \qquad \Lambda_i \Lambda_i^\dagger = 1 - J(T_{ij} - Z_{ij}) \\ \Lambda_i \Lambda_j &= \end{aligned}$$

$$
\begin{aligned}
e_1 f_1 &= T - Z \\
e_2 f_2 &= T + Z \\
TT + XX &= -1
\end{aligned}
$$