Formulation

The goal is to express quantum mechanics in a new way with geometric algebra in order to get an understanding for space and spin.

The wavefunction can be expressed as

$$\Psi = \sum_i \frac{1}{\sqrt{2}}(e_i + J f_i)(\Re\psi_i + J\,\Im\psi_i)$$

where e_i, f_i are orthonormal vectors which square to $+1$, $J = e_0 f_0$ is an independent bivector, and ψ_i are the complex valued components of the wave vector. This makes the wave function an odd-grade multivector.

For a spin-1/2 particle space coordinates will come out as

$$\begin{aligned} X &= \frac{1}{2}(e_1 f_2 + e_2 f_1) \\ Y &= \frac{1}{2}(e_1 e_2 + f_1 f_2) \\ Z &= \frac{1}{2}(e_2 f_2 - e_1 f_1) \end{aligned}$$

which means that space rotations can be derived in this basis.

Observation in quantum mechanics

The probability $P = |\langle\psi_1|\psi_2\rangle|^2$ to measure a state ψ_1 in a state ψ_2 can be calculated from

$$\begin{aligned} P(1 \to 2) &= \langle(1 - \Psi_1\Psi_1^\dagger)(1 - \Psi_2\Psi_2^\dagger)\rangle \\ &= \langle\Psi_1\Psi_1^\dagger\Psi_2\Psi_2^\dagger\rangle - 1 \end{aligned}$$

which is an inner product between two state multivectors of the form $\Psi\Psi^\dagger$.

Unitary transformation

A unitary transformation of the wavefunction can be represented as a rotor in geometric algebra. Note that the same rotor can be applied to Ψ or $\Psi\Psi^\dagger$.

Single spin

The wavefunction for a single spin-up in a direction given by Euler angles θ, ϕ is usually written as

$$\psi = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

up to an arbitrary phase.

Written in geometric algebra this is

$$\begin{aligned} J\Psi\Psi^\dagger &= J + e_1 f_1 \cos^2\frac{\theta}{2} + e_2 f_2 \sin^2\frac{\theta}{2} \\ &+ (e_1 f_2 + e_2 f_1)\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\phi \\ &+ (e_1 e_2 + f_1 f_2)\cos\frac{\theta}{2}\sin\frac{\theta}{2}\sin\phi \end{aligned}$$

(see derivation in appendix).

$$\begin{aligned} J\Psi\Psi^\dagger &= J + e_1 f_1 \frac{1-\cos\theta}{2} + e_2 f_2 \frac{1+\cos\theta}{2} \\ &+ (e_1 f_2 + e_2 f_1)\frac{1}{2}\sin\theta\cos\phi \\ &+ (e_1 e_2 + f_1 f_2)\frac{1}{2}\sin\theta\sin\phi \\ &= J + \frac{1}{2}(e_1 f_1 + e_2 f_2) \\ &+ \frac{1}{2}(e_2 f_2 - e_1 f_1)\cos\theta \\ &+ \frac{1}{2}(e_1 f_2 + e_2 f_1)\sin\theta\cos\phi \\ &+ \frac{1}{2}(e_1 e_2 + f_1 f_2)\sin\theta\sin\phi \end{aligned}$$

Remembering that we have Euler angles, we can identify the multivectors for space coordinates from this expression for a single spin

$$\begin{aligned} X &= \frac{1}{2}(e_1 f_2 + e_2 f_1) \\ Y &= \frac{1}{2}(e_1 e_2 + f_1 f_2) \\ Z &= \frac{1}{2}(e_2 f_2 - e_1 f_1) \end{aligned}$$

They are orthogonal (e.g. $\langle XY \rangle = 0$) and obey

\begin{align*} XY&=Z & YZ&=X & ZX&=Y \end{align*} $$Note that these multivectors

\Psi=\sum_i \frac{1}{\sqrt{2}}(e_i+Jf_i)(\Re\psi_i+J,\Im\psi_i)

$$With\$z_i = \Re\psi_i + J\Im\psi_i\$$$

\begin{align}
\Psi\Psi^\dagger&=\frac{1}{2}\sum{ij}(e_i+Jf_i)z_iz_j^\dagger(e_j-Jf_j)\
&=\sum_i (1-e_if_iJ)z_iz_i^\dagger\
&\quad+\frac{1}{2}\sum{i<j}\left((ei+Jf_i)(e_j-Jf_j)z_iz_j^\dagger+(e_j+Jf_j)(e_i-Jf_i)z_jz_i^\dagger\right)\
&=\sum_i (1-e_if_iJ)z_iz_i^\dagger\
&\quad+\frac{1}{2}\sum{i<j}\left(eie_j+f_if_j-(e_if_j+e_jf_i)J\right)z_iz_j^\dagger\
&\quad+\frac{1}{2}\sum{i<j}\left(e_je_i+f_jf_i-(e_jf_i+e_if_j)J\right)z_jz_i^\dagger
\end{align}

$$Withtherealandimaginaryparts$$

\begin{align}
R{ij}&=\frac{1}{2}(z_iz_j^\dagger+z_jz_i^\dagger)\
JI{ij}&=\frac{1}{2}(ziz_j^\dagger-z_jz_i^\dagger)\
z_iz_j^\dagger&=R{ij}+J,I{ij}\
z_jz_i^\dagger&=R{ij}-J,I_{ij}
\end{align}

$$where\$R_{ij}, I_{ij}\$arescalarsthisbecomes$$

\begin{align}
\Psi\Psi^\dagger&=\sumi (1-e_if_iJ)z_iz_i^\dagger\
&\quad+\frac{1}{2}\sum{i<j}\left(eie_j+f_if_j-(e_if_j+e_jf_i)J\right)(R{ij}+J,I{ij})\
&\quad+\frac{1}{2}\sum{i<j}\left(-eie_j-f_if_j-(e_if_j+e_jf_i)J\right)(R{ij}-J,I{ij})\
&=\sum_i z_iz_i^\dagger-\sum_i e_if_iJz_iz_i^\dagger\
&\quad-\sum{i<j}(eif_j+e_jf_i)JR{ij}\
&\quad+\sum{i<j}(e_ie_j+f_if_j)J,I{ij}
\end{align}

$$Fornormalizedwavevectors$$

\langle \Psi\Psi^\dagger\rangle=1

$$Therefore$$

\begin{align}
J\Psi\Psi^\dagger&=J+\sumi e_if_iz_iz_i^\dagger+\sum{i<j}(eif_j+e_jf_i)R{ij}-

\sum{i<j}(e_ie_j+f_if_j),I{ij}
\end{align}

$isagrade - 2multivector. Theprobabilityinquantummechanicscanbecalculatedfromth$

(\psi_i\psi_i^, \ldots,\sqrt{2}\Re(\psi_i\psi_j^),\ldots,\sqrt{2}\Im(\psi_i\psi_j^*),\ldots)

$(shownelsewhere). Thereforeinourcaseandfornormalizedwavevectorstheprobability$

P(1\to2)=\langle (1-\Psi_1\Psi_1^\dagger)(1-\Psi_2\Psi_2^\dagger)\rangle=\langle\Psi_1\Psi_1^\dagger\Psi_2\Psi_2^\dagger\rangle-1
$$