Formulation

The goal is to express quantum mechanics in a new way with geometric algebra in order to get an understanding for space and spin.

The wavefunction can be expressed as

$$\Psi = \sum_{i=1}^{n} \frac{1}{\sqrt{2}}(e_i + J f_i)(\Re\psi_i + J\,\Im\psi_i)$$

where e_i, f_i are orthonormal vectors which square to $+1$,

$$J = e_0 f_0$$

is an independent bivector, and ψ_i are the complex valued components of the wave vector. In total there are $2n+2$ orthonormal vectors for an n-dimensional original wave vector.

Alternatively, the bivector

$$\boxed{\Psi f_0 = \frac{1}{\sqrt{2}} \sum_{i=1}^{n} ((e_i f_0 + f_i e_0)\Re\psi_i + (e_i e_0 - f_i f_0)\Im\psi_i)}$$

is a more natural choice, but for the (classical) derivation I will use the first version. The difference to other treatments of spin with geometric algebra is that here the basis is of the form $ab + cd$ with 2 additional basis vectors, instead of $\sigma_{12} = e_1 e_2$.

One advantage is that the full probability calculation becomes

$$P(1 \to 2) = \langle \Psi_1 \Psi_1^\dagger \Psi_2 \Psi_2^\dagger \rangle - 1$$

which can take the same rotor for $\Psi\Psi^\dagger$ and Ψ.

For a spin-1/2 particle coordinates will come out as

$$\begin{aligned}
J\Psi\Psi^\dagger &= J + T + Xx + Yy + Zz \\
X &= \frac{1}{2}(e_1 f_2 + e_2 f_1) \\
Y &= \frac{1}{2}(e_1 e_2 + f_1 f_2) \\
Z &= \frac{1}{2}(e_2 f_2 - e_1 f_1) \\
T &= \frac{1}{2}(e_1 f_1 + e_2 f_2)
\end{aligned}$$

where coordinates do not depend on e_0, f_0. Space rotations can be derived in this basis and the same rotor can be applied to Ψ too. T may be related to time, but for now it is only an expression.

They anti-commute and obey

$$\begin{array}{ccc}
XY = Z & YZ = X & ZX = Y \\
XT = 0 & YT = 0 & ZT = 0
\end{array}$$

$$\begin{aligned}
\Pi &= e_1 f_1 e_2 f_2 \\
XX &= -\frac{1}{2}(1 + \Pi) \\
YY &= -\frac{1}{2}(1 + \Pi) \\
ZZ &= -\frac{1}{2}(1 + \Pi) \\
TT &= -\frac{1}{2}(1 - \Pi)
\end{aligned}$$

which has been derived from the spin wave vector alone without actually considering time and for a positive signature vector space. Note that these multivectors are not invertable.

Observation in quantum mechanics

The probability $P = |\langle\psi_1|\psi_2\rangle|^2$ to measure a state ψ_1 in a state ψ_2 can be calculated from

$$\begin{aligned}
P(1 \to 2) &= \langle(1 - \Psi_1\Psi_1^\dagger)(1 - \Psi_2\Psi_2^\dagger)\rangle \\
&= \langle\Psi_1\Psi_1^\dagger\Psi_2\Psi_2^\dagger\rangle - 1 \\
&= \Psi_1\Psi_1^\dagger \cdot \Psi_2\Psi_2^\dagger - 1
\end{aligned}$$

which is an inner product between two state multivectors of the form $\Psi\Psi^\dagger$ (see appendix).

Unitary transformation

A unitary transformation of the wavefunction can be represented as a rotor in geometric algebra. Note that the same rotor can be applied to Ψ or $\Psi\Psi^\dagger$.

Observable state vector

$$\Psi = \sum_{i=1}^{n} \frac{1}{\sqrt{2}}(e_i + J f_i)(\Re\psi_i + J\,\Im\psi_i)$$

The observable state vector which is used in the dot product to calculate probabilities is the wave function squared and can be expanded into a bivector

$$\boxed{\begin{aligned}\Omega &= J\Psi\Psi^\dagger \\ &= J + \sum_i e_i f_i\,|\psi_i|^2 + \sum_{i<j}(e_i f_j + e_j f_i)\,\Re(\psi_i\psi_j^*) - \sum_{i<j}(e_i e_j + f_i f_j)\,\Im(\psi_i\psi_j^*)\end{aligned}}$$

(see appendix) for an easier translation from the usual representation with a complex wave vector ψ_i.

Single spin

The wavefunction for a single spin-up in a direction given by Euler angles θ, ϕ is usually written as

$$\psi = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

up to an arbitrary phase.

Written in geometric algebra this is

$$
\begin{aligned}
J\Psi\Psi^\dagger &= J + e_1 f_1 \cos^2\frac{\theta}{2} + e_2 f_2 \sin^2\frac{\theta}{2} \\
&\quad + (e_1 f_2 + e_2 f_1)\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\phi \\
&\quad + (e_1 e_2 + f_1 f_2)\cos\frac{\theta}{2}\sin\frac{\theta}{2}\sin\phi \\
&= J + e_1 f_1 \frac{1-\cos\theta}{2} + e_2 f_2 \frac{1+\cos\theta}{2} \\
&\quad + (e_1 f_2 + e_2 f_1)\frac{1}{2}\sin\theta\cos\phi \\
&\quad + (e_1 e_2 + f_1 f_2)\frac{1}{2}\sin\theta\sin\phi \\
&= J + \frac{1}{2}(e_1 f_1 + e_2 f_2) \\
&\quad + \frac{1}{2}(e_2 f_2 - e_1 f_1)\cos\theta \\
&\quad + \frac{1}{2}(e_1 f_2 + e_2 f_1)\sin\theta\cos\phi \\
&\quad + \frac{1}{2}(e_1 e_2 + f_1 f_2)\sin\theta\sin\phi
\end{aligned}
$$

Remembering that we have Euler angles, we can identify the multivectors for space coordinates from this expression for a single spin

$$
\begin{aligned}
J\Psi\Psi^\dagger &= J + T + Xx + Yy + Zz \\
X &= \frac{1}{2}(e_1 f_2 + e_2 f_1) \\
Y &= \frac{1}{2}(e_1 e_2 + f_1 f_2) \\
Z &= \frac{1}{2}(e_2 f_2 - e_1 f_1) \\
T &= \frac{1}{2}(e_1 f_1 + e_2 f_2)
\end{aligned}
$$

The expression for T is just the remaining term.

Space rotations will be derived soon.

$$
\begin{aligned}
e_1 f_1 &= T - Z \\
e_2 f_2 &= T + Z \\
TT + XX &= -1
\end{aligned}
$$

$$
\Omega = J + \sum_i e_i f_i \, |\psi_i|^2 + 2\sum_{i<j} X_{ij}\,\Re(\psi_i \psi_j^*) - 2\sum_{i<j} Y_{ij}\,\Im(\psi_i \psi_j^*)
$$

Interpretation

A vague idea why quantum mechanics is this way, is because due to the rules of probability, a state should be a (multi)vector and probabilities be calculated from an inner product. The probabilites rules are that they should sum to 1 and redoing a measurement yields the same results.

This state is constantly being rotated looking like $\Omega = \cdots R_3 R_2 R_1 \Omega_0 R_1^\dagger R_2^\dagger R_3^\dagger \cdots$ and for some reason we get, that actually Ω should split into $\Omega = \Psi \Psi^\dagger$ - as if all particles always start with the same state $\Omega_0 = 1$. Ψ has only roughly the square root number of parameters than the "observable" Ω.

Therefore we have quantum mechanics, because all particles have the same Ω_0. The wave function is not a state; it is the rotor applied to a unique state Ω_0.

$$\Omega = \Psi \Omega_0 \Psi^\dagger$$

Other ideas

Looking at

$$\Psi f_0 = \frac{1}{\sqrt{2}} \sum_{i=1}^{n} ((e_i f_0 + f_i e_0) \Re\psi_i + (e_i e_0 - f_i f_0) \Im\psi_i)$$

one may also consider what happens if the terms $e_i f_0, f_i e_0, e_i e_0, f_i f_0$ have independent coefficients. Maybe this is related to Dirac matrices.

Also, one may wonder if a term $e_i e_j$ is existing. This may provide an extension of quantum mechanics.

Appendix

Derivation

$$\Psi = \sum_i \frac{1}{\sqrt{2}} (e_i + J f_i)(\Re\psi_i + J\, \Im\psi_i)$$

With $z_i = \Re\psi_i + J\, \Im\psi_i$

$$
\begin{aligned}
\Psi\Psi^\dagger &= \frac{1}{2}\sum_{ij}(e_i + Jf_i)z_i z_j^\dagger(e_j - Jf_j) \\
&= \sum_i (1 - e_i f_i J) z_i z_i^\dagger \\
&\quad + \frac{1}{2}\sum_{i<j}\Big((e_i + Jf_i)(e_j - Jf_j)z_i z_j^\dagger + (e_j + Jf_j)(e_i - Jf_i)z_j z_i^\dagger\Big) \\
&= \sum_i (1 - e_i f_i J) z_i z_i^\dagger \\
&\quad + \frac{1}{2}\sum_{i<j}(e_i e_j + f_i f_j - (e_i f_j + e_j f_i)J)z_i z_j^\dagger \\
&\quad + \frac{1}{2}\sum_{i<j}(e_j e_i + f_j f_i - (e_j f_i + e_i f_j)J)z_j z_i^\dagger
\end{aligned}
$$

With the real and imaginary parts

$$
\begin{aligned}
R_{ij} &= \frac{1}{2}(z_i z_j^\dagger + z_j z_i^\dagger) \\
J I_{ij} &= \frac{1}{2}(z_i z_j^\dagger - z_j z_i^\dagger) \\
z_i z_j^\dagger &= R_{ij} + J\,I_{ij} \\
z_j z_i^\dagger &= R_{ij} - J\,I_{ij}
\end{aligned}
$$

where R_{ij}, I_{ij} are scalars this becomes

$$
\begin{aligned}
\Psi\Psi^\dagger &= \sum_i (1 - e_i f_i J) z_i z_i^\dagger \\
&\quad + \frac{1}{2}\sum_{i<j}(e_i e_j + f_i f_j - (e_i f_j + e_j f_i)J)(R_{ij} + J\,I_{ij}) \\
&\quad + \frac{1}{2}\sum_{i<j}(-e_i e_j - f_i f_j - (e_i f_j + e_j f_i)J)(R_{ij} - J\,I_{ij}) \\
&= \sum_i z_i z_i^\dagger - \sum_i e_i f_i J z_i z_i^\dagger \\
&\quad - \sum_{i<j}(e_i f_j + e_j f_i)JR_{ij} \\
&\quad + \sum_{i<j}(e_i e_j + f_i f_j)J\,I_{ij}
\end{aligned}
$$

For normalized wave vectors

$$
\sum_i z_i z_i^\dagger = 1
$$

Therefore

$$J\Psi\Psi^\dagger = J + \sum_i e_i f_i z_i z_i^\dagger + \sum_{i<j}(e_i f_j + e_j f_i)R_{ij} - \sum_{i<j}(e_i e_j + f_i f_j)\, I_{ij}$$

is a grade-2 multivector.

The probability in quantum mechanics can be calculated from the dot product of two real vectors where states have the components

$$(\psi_i\psi_i^*, \ldots, \sqrt{2}\Re(\psi_i\psi_j^*), \ldots, \sqrt{2}\Im(\psi_i\psi_j^*), \ldots)$$

due to

$$\begin{aligned}
P(\psi \to \phi) &= |\langle\psi|\phi\rangle|^2 \\
&= \sum_i \psi_i\phi_i^* \sum_j \phi_j\psi_j^* \\
&= \sum_i \psi_i\phi_i^*\phi_i\psi_i^* + \sum_{i<j} \left(\psi_i\phi_i^*\phi_j\psi_j^* + \psi_j\phi_j^*\phi_i\psi_i^*\right) \\
&= \sum_i \psi_i\psi_i^*\phi_i\phi_i^* + \sum_{i<j} 2\Re(\psi_i\psi_j^*\phi_i^*\phi_j) \\
&= \sum_i \psi_i\psi_i^*\phi_i\phi_i^* + \sum_{i<j} 2\left(\Re(\psi_i\psi_j^*)\Re(\phi_i\phi_j^*) + \Im(\psi_i\psi_j^*)\Im(\phi_i\phi_j^*)\right)
\end{aligned}$$

being a dot product of vectors with components $\psi_i\psi_i^*, \sqrt{2}\psi_i\psi_j^*(i<j)$.

Therefore in our case and for normalized wave vectors the probability can also be calculated from

$$P(1 \to 2) = \langle(1 - \Psi_1\Psi_1^\dagger)(1 - \Psi_2\Psi_2^\dagger)\rangle = \langle\Psi_1\Psi_1^\dagger\Psi_2\Psi_2^\dagger\rangle - 1$$