

Formulation

The goal is to express quantum mechanics in a new way with geometric algebra in order to get an understanding for space and spin.

The wavefunction can be expressed as

$$\Psi = \sum_{i=1}^n \frac{1}{\sqrt{2}} (e_i + J f_i) (\Re \psi_i + J \Im \psi_i)$$

where e_i, f_i are orthonormal vectors which square to $+1$,

$$J = e_0 f_0$$

is an independent bivector, and ψ_i are the complex valued components of the wave vector. This makes the wave function an odd-grade multivector.

For a spin-1/2 particle coordinates will come out as

$$\begin{aligned} J\Psi\Psi^\dagger &= J + Tt + Xx + Yy + Zz \\ X &= \frac{1}{2}(e_1 f_2 + e_2 f_1) \\ Y &= \frac{1}{2}(e_1 e_2 + f_1 f_2) \\ Z &= \frac{1}{2}(e_2 f_2 - e_1 f_1) \\ T &= \frac{1}{2}(e_1 f_1 + e_2 f_2) \end{aligned}$$

which means that space rotations can be derived in this basis and the same rotor can be applied to Ψ too.

They anti-commute and obey

$$\begin{array}{lll} XY = Z & YZ = X & ZX = Y \\ XT = 0 & YT = 0 & ZT = 0 \end{array}$$

$$\begin{aligned}
\Pi &= e_1 f_1 e_2 f_2 \\
XX &= -\frac{1}{2}(1 + \Pi) \\
YY &= -\frac{1}{2}(1 + \Pi) \\
ZZ &= -\frac{1}{2}(1 + \Pi) \\
TT &= -\frac{1}{2}(1 - \Pi)
\end{aligned}$$

which has been derived from the spin wave vector alone without actually considering time and for a positive signature vector space. Note that these multivectors are not invertable.

Observation in quantum mechanics

The probability $P = |\langle \psi_1 | \psi_2 \rangle|^2$ to measure a state ψ_1 in a state ψ_2 can be calculated from

$$\begin{aligned}
P(1 \rightarrow 2) &= \langle (1 - \Psi_1 \Psi_1^\dagger)(1 - \Psi_2 \Psi_2^\dagger) \rangle \\
&= \langle \Psi_1 \Psi_1^\dagger \Psi_2 \Psi_2^\dagger \rangle - 1
\end{aligned}$$

which is an inner product between two state multivectors of the form $\Psi \Psi^\dagger$.

Unitary transformation

A unitary transformation of the wavefunction can be represented as a rotor in geometric algebra. Note that the same rotor can be applied to Ψ or $\Psi \Psi^\dagger$.

State vector

$$\Psi = \sum_{i=1}^n \frac{1}{\sqrt{2}} (e_i + J f_i) (\Re \psi_i + J \Im \psi_i)$$

The wave function squared can be expanded into a bivector

$$\begin{aligned}
\Omega &= J \Psi \Psi^\dagger \\
&= J + \sum_i e_i f_i |\psi_i|^2 + \sum_{i < j} (e_i f_j + e_j f_i) \Re(\psi_i \psi_j^*) - \sum_{i < j} (e_i e_j + f_i f_j) \Im(\psi_i \psi_j^*)
\end{aligned}$$

(see appendix) such that the probability is

$$P = \Omega_1 \cdot \Omega_2 - 1$$

Single spin

The wavefunction for a single spin-up in a direction given by Euler angles θ, ϕ is usually written as

$$\psi = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

up to an arbitrary phase.

Written in geometric algebra this is

$$\begin{aligned} J\Psi\Psi^\dagger &= J + e_1 f_1 \cos^2 \frac{\theta}{2} + e_2 f_2 \sin^2 \frac{\theta}{2} \\ &\quad + (e_1 f_2 + e_2 f_1) \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \phi \\ &\quad + (e_1 e_2 + f_1 f_2) \cos \frac{\theta}{2} \sin \frac{\theta}{2} \sin \phi \\ &= J + e_1 f_1 \frac{1 - \cos \theta}{2} + e_2 f_2 \frac{1 + \cos \theta}{2} \\ &\quad + (e_1 f_2 + e_2 f_1) \frac{1}{2} \sin \theta \cos \phi \\ &\quad + (e_1 e_2 + f_1 f_2) \frac{1}{2} \sin \theta \sin \phi \\ &= J + \frac{1}{2} (e_1 f_1 + e_2 f_2) \\ &\quad + \frac{1}{2} (e_2 f_2 - e_1 f_1) \cos \theta \\ &\quad + \frac{1}{2} (e_1 f_2 + e_2 f_1) \sin \theta \cos \phi \\ &\quad + \frac{1}{2} (e_1 e_2 + f_1 f_2) \sin \theta \sin \phi \end{aligned}$$

Remembering that we have Euler angles, we can identify the multivectors for space coordinates from this expression for a single spin

$$\begin{aligned} J\Psi\Psi^\dagger &= J + Tt + Xx + Yy + Zz \\ X &= \frac{1}{2} (e_1 f_2 + e_2 f_1) \\ Y &= \frac{1}{2} (e_1 e_2 + f_1 f_2) \\ Z &= \frac{1}{2} (e_2 f_2 - e_1 f_1) \\ T &= \frac{1}{2} (e_1 f_1 + e_2 f_2) \end{aligned}$$

The expression for T is a guess from the remaining term.

Space rotations will be derived soon.

$$\begin{aligned}e_1 f_1 &= T - Z \\e_2 f_2 &= T + Z \\TT + X_i X_i &= -1\end{aligned}$$

Interpretation

A vague idea why quantum mechanics is this way, is because due to the rules of probability, a state should be a (multi)vector and probabilities be calculated from an inner product. The probabilities rules are that they should sum to 1 and redoing a measurement yields the same results.

This state is constantly being rotated looking like $\Delta = \cdots R_3 R_2 R_1 \cdots R_1^\dagger R_2^\dagger R_3^\dagger \cdots$ we somehow get, that actually Δ should split into $\Delta = \Psi \Psi^\dagger$ with the consequence that the degrees of freedom are largely reduced. The above construction with geometric algebra demonstrates how to do this "square root".

Appendix

Derivation

$$\Psi = \sum_i \frac{1}{\sqrt{2}} (e_i + J f_i) (\Re \psi_i + J \Im \psi_i)$$

With $z_i = \Re \psi_i + J \Im \psi_i$

$$\begin{aligned}\Psi \Psi^\dagger &= \frac{1}{2} \sum_{ij} (e_i + J f_i) z_i z_j^\dagger (e_j - J f_j) \\&= \sum_i (1 - e_i f_i J) z_i z_i^\dagger \\&\quad + \frac{1}{2} \sum_{i < j} \left((e_i + J f_i) (e_j - J f_j) z_i z_j^\dagger + (e_j + J f_j) (e_i - J f_i) z_j z_i^\dagger \right) \\&= \sum_i (1 - e_i f_i J) z_i z_i^\dagger \\&\quad + \frac{1}{2} \sum_{i < j} (e_i e_j + f_i f_j - (e_i f_j + e_j f_i) J) z_i z_j^\dagger \\&\quad + \frac{1}{2} \sum_{i < j} (e_j e_i + f_j f_i - (e_j f_i + e_i f_j) J) z_j z_i^\dagger\end{aligned}$$

With the real and imaginary parts

$$\begin{aligned}
R_{ij} &= \frac{1}{2}(z_i z_j^\dagger + z_j z_i^\dagger) \\
JI_{ij} &= \frac{1}{2}(z_i z_j^\dagger - z_j z_i^\dagger) \\
z_i z_j^\dagger &= R_{ij} + J I_{ij} \\
z_j z_i^\dagger &= R_{ij} - J I_{ij}
\end{aligned}$$

where R_{ij}, I_{ij} are scalars this becomes

$$\begin{aligned}
\Psi\Psi^\dagger &= \sum_i (1 - e_i f_i J) z_i z_i^\dagger \\
&\quad + \frac{1}{2} \sum_{i < j} (e_i e_j + f_i f_j - (e_i f_j + e_j f_i) J) (R_{ij} + J I_{ij}) \\
&\quad + \frac{1}{2} \sum_{i < j} (-e_i e_j - f_i f_j - (e_i f_j + e_j f_i) J) (R_{ij} - J I_{ij}) \\
&= \sum_i z_i z_i^\dagger - \sum_i e_i f_i J z_i z_i^\dagger \\
&\quad - \sum_{i < j} (e_i f_j + e_j f_i) J R_{ij} \\
&\quad + \sum_{i < j} (e_i e_j + f_i f_j) J I_{ij}
\end{aligned}$$

For normalized wave vectors

$$\langle \Psi\Psi^\dagger \rangle = 1$$

Therefore

$$J\Psi\Psi^\dagger = J + \sum_i e_i f_i z_i z_i^\dagger + \sum_{i < j} (e_i f_j + e_j f_i) R_{ij} - \sum_{i < j} (e_i e_j + f_i f_j) I_{ij}$$

is a grade-2 multivector.

The probability in quantum mechanics can be calculated from the dot product of two real vectors where states have the components

$$(\psi_i \psi_i^*, \dots, \sqrt{2}\Re(\psi_i \psi_j^*), \dots, \sqrt{2}\Im(\psi_i \psi_j^*), \dots)$$

(shown elsewhere). Therefore in our case and for normalized wave vectors the probability can also be calculated from

$$P(1 \rightarrow 2) = \langle (1 - \Psi_1 \Psi_1^\dagger)(1 - \Psi_2 \Psi_2^\dagger) \rangle = \langle \Psi_1 \Psi_1^\dagger \Psi_2 \Psi_2^\dagger \rangle - 1$$

