Motivation

The goal is to express quantum mechanics in a new way with geometric algebra in order to get an understanding for space and spin.

The wavefunction will be expressed as

$$\Psi = \sum_i \frac{1}{\sqrt{2}}(e_i + J f_i)(\Re\psi_i + J\,\Im\psi_i)$$

where the variables are orthonormal vectors and $J = e_0 f_0$ is another orthogonal bivector.

For a spin-1/2 particle space coordinates will comes out as

$$\begin{aligned} X &= \frac{1}{2}(e_1 f_2 + e_2 f_1) \\ Y &= \frac{1}{2}(e_1 e_2 + f_1 f_2) \\ Z &= \frac{1}{2}(e_2 f_2 - e_1 f_1) \end{aligned}$$

which means that space rotations can be derived in this basis.

The probability for measurement is

$$P = \langle \psi_1 \psi_1^\dagger \, \psi_2^\dagger \psi_2 \rangle$$

The below text currently uses a slightly different (rotated) version, but I will adjust that soon.

Formulation

In order to replicate the tensor product that is needed to do quantum calculations we use the basis

$$\Lambda_i = \frac{1}{2}(e_0 e_i + f_0 f_i)$$

as $\Lambda_i \Lambda_i^\dagger$ has a non-vanishing non-scalar part. The goal is to write measurement probabilities as an inner product of two multivectors $\Delta = \psi\psi^\dagger$.

$$P = \Delta_1 \cdot \Delta_2$$

where Δ_1 encodes the whole state and Δ_2 is the measurement target state.

This is probably the main difference with what is usually done to express quantum mechanics in geometric algebra. Here the wavefunction will be like a grade-3 multivector and the square of the wavefunction a grade-2 vector.

Δ is very similar to the density matrix, but in the language of geometric algebra.

Any finite wavefunction vector will be expressed as

$$\Psi = \sum_i \Lambda_i \Psi_i$$

where the originally complex wavefunction components are expressed in geometric algebra as as

$$\Psi_i = \Re\psi_i + e_0 f_0 \, \Im\psi_i$$

and $e_0 f_0$ anticommutes with Λ_i. All basis vectors square to $+1$.

Calculating measurement probability

To calculate the measurement probability which is usually

$$P(1 \to 2) = |\langle\psi_1|\psi_2\rangle|^2$$

we can use geometric algebra to do

$$\Delta = 1 - \Psi\Psi^\dagger$$
$$P(1 \to 2) = \langle\Delta_1\Delta_2^\dagger\rangle$$

which shows that the probability is an inner product of two grade-4 multivectors (in another version grade-2). You could verify that the math comes out right.

This way of writing quantum mechanics will give correct results for any finite dimensional wavefunction.

Unitary transformation

A unitary transformation of the wavefunction can be represented as a rotor in geometric algebra. Note that the same rotor can be applied to Ψ or Δ.

Single spin

The wavefunction for a single spin-up in a direction given by Euler angles θ, ϕ is usually written as

$$\psi = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

up to an arbitrary phase.

Written in geometric algebra this is

$$\begin{aligned}
\Psi\Psi^\dagger &= -e_0 f_0 + e_1 f_1 \cos^2\frac{\theta}{2} + e_2 f_2 \sin^2\frac{\theta}{2} \\
&\quad + (e_1 f_2 + e_2 f_1) \cos\frac{\theta}{2} \sin\frac{\theta}{2} \cos\phi \\
&\quad + (e_1 e_2 + f_1 f_2) \cos\frac{\theta}{2} \sin\frac{\theta}{2} \sin\phi
\end{aligned}$$

(see derivation in appendix). I chose $\Delta = \Psi\Psi^\dagger$ instead of Ψ, because it makes it easier to compare the proper space directions.

$$\begin{aligned}
\Psi\Psi^\dagger &= -e_0 f_0 + e_1 f_1 \frac{1-\cos\theta}{2} + e_2 f_2 \frac{1+\cos\theta}{2} \\
&\quad + (e_1 f_2 + e_2 f_1) \frac{1}{2} \sin\theta \cos\phi \\
&\quad + (e_1 e_2 + f_1 f_2) \frac{1}{2} \sin\theta \sin\phi \\
&= e_1 f_1 + e_2 f_2 - e_0 f_0 \\
&\quad + \frac{1}{2}(e_2 f_2 - e_1 f_1) \cos\theta \\
&\quad + \frac{1}{2}(e_1 f_2 + e_2 f_1) \sin\theta \cos\phi \\
&\quad + \frac{1}{2}(e_1 e_2 + f_1 f_2) \sin\theta \sin\phi
\end{aligned}$$

Remembering that we have Euler angles, we can identify the multivectors for space coordinates from this expression for a single spin

$$\begin{aligned}
X &= \frac{1}{2}(e_1 f_2 + e_2 f_1) \\
Y &= \frac{1}{2}(e_1 e_2 + f_1 f_2) \\
Z &= \frac{1}{2}(e_2 f_2 - e_1 f_1)
\end{aligned}$$

Note that these multivectors are not invertable. In a way the multivectors Λ_i are the square root of space. Note all this assumed only the spin wavefunction and no time evolution was considered.

What is left to do is finding if there is an interpretation of e_1, e_2, f_1, f_2 in terms of the space multivectors X, Y, Z (if there is one).

Space rotations will be derived soon.

Interpretation

A vague idea why quantum mechanics is this way, is because due to the rules of probability, a state should be a (multi)vector and probabilities be calculated from an inner product. The probabilites rules are that they should sum to 1 and redoing a measurement yields the same results.

This state is constantly being rotated looking like $\Delta = \cdots R_3 R_2 R_1 ? R_1^\dagger R_2^\dagger R_3^\dagger \cdots$ we somehow get, that actually Δ should split into $\Delta = \Psi\Psi^\dagger$ with the consequence that the degrees of freedom are largely reduced. The above construction with geometric algebra demonstrates how to do this "square root".

Appendix

Derivation

$$\begin{aligned}
\Psi\Psi^\dagger &= \sum_{ij} \Lambda_i \Psi_i \Psi_j^\dagger \Lambda_j^\dagger \\
&= \sum_i \Lambda_i \Psi_i \Psi_i^\dagger \Lambda_i^\dagger \\
&\quad + \sum_{i<j} \left(\Lambda_i \Psi_i \Psi_j^\dagger \Lambda_j^\dagger + \Lambda_j \Psi_j \Psi_i^\dagger \Lambda_i^\dagger \right)
\end{aligned}$$

Introducing real and imaginary components $X \pm X^\dagger$ we get

$$\begin{aligned}
R_{ij} &= \frac{1}{2}(\Psi_i \Psi_j^\dagger + \Psi_j \Psi_i^\dagger) \\
e_0 f_0 \, I_{ij} &= \frac{1}{2}(\Psi_i \Psi_j^\dagger - \Psi_j \Psi_i^\dagger) \\
\Psi_i \Psi_j^\dagger &= R_{ij} + e_0 f_0 \, I_{ij} \\
\Psi_j \Psi_i^\dagger &= R_{ij} - e_0 f_0 \, I_{ij}
\end{aligned}$$

where R_{ij}, I_{ij} are scalars. We get

$$\Psi\Psi^\dagger = \sum_i \Lambda_i\Lambda_i^\dagger \Psi_i\Psi_i^\dagger$$
$$+\sum_{i<j}\left(\left(\Lambda_i\Lambda_j^\dagger + \Lambda_j\Lambda_i^\dagger\right)R_{ij} - e_0 f_0\left(\Lambda_i\Lambda_j^\dagger - \Lambda_j\Lambda_i^\dagger\right)I_{ij}\right)$$

With

$$\Lambda_i\Lambda_i^\dagger = 1 + e_0 f_0 e_i f_i$$
$$\Lambda_i\Lambda_j^\dagger + \Lambda_j\Lambda_i^\dagger = e_0 f_0 e_i f_j + e_0 f_0 e_j f_i$$
$$\Lambda_i\Lambda_j^\dagger + \Lambda_j\Lambda_i^\dagger = e_i e_j + f_i f_j$$

this results in

$$\Delta = \Psi\Psi^\dagger = \sum_i \Psi_i\Psi_i^\dagger + e_0 f_0 \sum_i e_i f_i\ \Psi_i\Psi_i^\dagger$$
$$+e_0 f_0 \sum_{i<j}((e_i f_j + e_j f_i)R_{ij} - (e_i e_j + f_i f_j)I_{ij})$$

which for normalized wavefunctions is 1 plus a grade-4 multivector.

Since all that is left to do is an inner product $P = \langle \Delta_i \Delta_j \rangle$ one might as well put these real scalar coefficients into a normal column vector at this point to do a dot product with real vectors.

We can premultiply this expression by $f_0 e_0$ as it does not matter for the inner product.

$$\Delta' = -e_0 f_0 + \sum_i e_i f_i\ \Psi_i\Psi_i^\dagger$$
$$+\sum_{i<j}((e_i f_j + e_j f_i)R_{ij} - (e_i e_j + f_i f_j)I_{ij})$$

for normalized wavefunctions, and this is a grade-2 multivector..