Formulation

The goal is to express quantum mechanics in a new way with geometric algebra in order to get an understanding for space and spin.

The wavefunction can be expressed as

$$\Psi = \sum_{i=1}^{n} \frac{1}{\sqrt{2}}(e_i + J f_i)(\Re\psi_i + J\,\Im\psi_i)$$

where e_i, f_i are orthonormal vectors which square to $+1$,

$$J = e_0 f_0$$

is an independent bivector, $\psi_i = \Re\psi_i + i\,\Im\psi_i$ are the complex valued components of the wave vector with a real and imaginary part. In total there are $2n+2$ orthonormal vectors for an n-dimensional original wave vector.

This form is fruitful, because the full quantum mechanical probability to observe one state Ψ_1 in another state Ψ_2 turns out to be

$$\begin{aligned} P(1 \to 2) &= \langle \Psi_1 \Psi_1^\dagger \Psi_2 \Psi_2^\dagger \rangle - 1 \\ &= (\Psi_1 \Psi_1^\dagger) \cdot (\Psi_2 \Psi_2^\dagger) - 1 \end{aligned}$$

which is an inner product between two states and you can apply the same rotor for $\Psi\Psi^\dagger$ and Ψ if needed.

Alternatively, the bivector

$$\boxed{\Psi f_0 = \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \left((e_i f_0 + f_i e_0)\Re\psi_i + (e_i e_0 - f_i f_0)\Im\psi_i\right)}$$

is a more natural choice, but for the derivation I will use the first version. The difference to other treatments of spin with geometric algebra is that here the basis is of the form $ab + cd$ with 2 additional basis vectors, instead of $\sigma_{12} = e_1 e_2$.

There is a possibility that the bivectors $e_i f_0, f_i e_0, e_i e_0, -f_i f_0$ are related to the Dirac matrices and they come into action when they do not share the same coefficient like here (see section later). Maybe this whole expression is related to spinors then.

For a spin-1/2 particle the observable $\Psi\Psi^\dagger$ and the space coordinates will come out as

$$
\begin{aligned}
J\Psi\Psi^\dagger &= J + N + Xx + Yy + Zz \\
X &= \frac{1}{2}(e_1 f_2 + e_2 f_1) \\
Y &= \frac{1}{2}(e_1 e_2 + f_1 f_2) \\
Z &= \frac{1}{2}(e_2 f_2 - e_1 f_1) \\
N &= \frac{1}{2}(e_1 f_1 + e_2 f_2)
\end{aligned}
$$

where coordinates do not depend on e_0, f_0. N belongs to Z somehow. Space rotations can be derived in this basis and the same rotor can be applied to Ψ too. They anti-commute and obey

$$
XY = Z \qquad YZ = X \qquad ZX = Y
$$

which has been derived from the spin wave vector alone.

Observation in quantum mechanics

The probability $P = |\langle\psi_1|\psi_2\rangle|^2$ to measure a state ψ_1 in a state ψ_2 can be calculated from (see appendix)

$$
\begin{aligned}
P(1 \to 2) &= \langle(1 - \Psi_1\Psi_1^\dagger)(1 - \Psi_2\Psi_2^\dagger)\rangle \\
&= \langle\Psi_1\Psi_1^\dagger\Psi_2\Psi_2^\dagger\rangle - 1 \\
&= \Omega_1 \cdot \Omega_2 - 1
\end{aligned}
$$

with the state vector

$$
\Omega = J\Psi\Psi^\dagger
$$

This is an inner product between two state multivectors. The observable state vector can be expanded into a state bivector

$$
\boxed{
\begin{aligned}
\Omega &= J\Psi\Psi^\dagger \\
&= J + \sum_i e_i f_i\,|\psi_i|^2 + \sum_{i<j}(e_i f_j + e_j f_i)\,\Re(\psi_i\psi_j^*) - \sum_{i<j}(e_i e_j + f_i f_j)\,\Im(\psi_i\psi_j^*)
\end{aligned}
}
$$

(see appendix) for an easier translation from the usual representation with a complex wave vector ψ_i. Note that J appears only once at the front. $\Psi\Psi^\dagger$ takes the role of the density matrix.

Unitary transformation

A unitary transformation of the wavefunction can be represented as a rotor in geometric algebra. Note that the same rotor can be applied to Ψ or $\Psi\Psi^\dagger$.

State vector of a single spin

The wavefunction for a single spin-up in a direction given by Euler angles θ, ϕ is usually written as

$$\psi = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

up to an arbitrary phase.

The state vector written in geometric algebra then is

$$\begin{aligned}
\Omega &= J\Psi\Psi^\dagger \\
&= J + \sum_i e_i f_i \, |\psi_i|^2 + \sum_{i<j} (e_i f_j + e_j f_i) \, \Re(\psi_i \psi_j^*) - \sum_{i<j} (e_i e_j + f_i f_j) \, \Im(\psi_i \psi_j^*) \\
&= J + e_1 f_1 \cos^2\frac{\theta}{2} + e_2 f_2 \sin^2\frac{\theta}{2} \\
&\quad + (e_1 f_2 + e_2 f_1) \cos\frac{\theta}{2} \sin\frac{\theta}{2} \cos\phi \\
&\quad + (e_1 e_2 + f_1 f_2) \cos\frac{\theta}{2} \sin\frac{\theta}{2} \sin\phi \\
&= J + e_1 f_1 \frac{1 - \cos\theta}{2} + e_2 f_2 \frac{1 + \cos\theta}{2} \\
&\quad + (e_1 f_2 + e_2 f_1) \frac{1}{2} \sin\theta \cos\phi \\
&\quad + (e_1 e_2 + f_1 f_2) \frac{1}{2} \sin\theta \sin\phi \\
&= J + \frac{1}{2}(e_1 f_1 + e_2 f_2) \\
&\quad + \frac{1}{2}(e_2 f_2 - e_1 f_1) \cos\theta \\
&\quad + \frac{1}{2}(e_1 f_2 + e_2 f_1) \sin\theta \cos\phi \\
&\quad + \frac{1}{2}(e_1 e_2 + f_1 f_2) \sin\theta \sin\phi
\end{aligned}$$

Remembering that we have Euler angles, we can identify the multivectors for space coordinates from this expression for a single spin

$$
\begin{aligned}
J\Psi\Psi^\dagger &= J + N + Xx + Yy + Zz \\
X &= \frac{1}{2}(e_1 f_2 + e_2 f_1) \\
Y &= \frac{1}{2}(e_1 e_2 + f_1 f_2) \\
Z &= \frac{1}{2}(e_2 f_2 - e_1 f_1) \\
N &= \frac{1}{2}(e_1 f_1 + e_2 f_2)
\end{aligned}
$$

Space rotations will be derived soon, but from $XY = Z$ we can already see the rotation.

Interpretation

A vague idea why quantum mechanics is this way, is because due to the rules of probability, a state should be a (multi)vector and probabilities be calculated from an inner product. The probabilites rules are that they should sum to 1 and redoing a measurement yields the same results.

This state is constantly being rotated looking like $\Omega = \cdots R_3 R_2 R_1 \Omega_0 R_1^\dagger R_2^\dagger R_3^\dagger \cdots$ and for some reason we get, that actually Ω should split into $\Omega = \Psi\Psi^\dagger$ - as if all particles start with the same state Ω_0.

Ψ has only roughly the square root number of element from the observable Ω.

Other ideas

Looking at

$$
\Psi f_0 = \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \left((e_i f_0 + f_i e_0) \Re \psi_i + (e_i e_0 - f_i f_0) \Im \psi_i \right)
$$

one may also consider what happens if the terms $e_i f_0, f_i e_0, e_i e_0, f_i f_0$ have independent coefficients. Maybe this is related to Dirac matrices like

$$
\begin{aligned}
\Psi' f_0 &= \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \left(e_i f_0 \alpha_i^{(1)} + f_i e_0 \alpha_i^{(2)} + e_i e_0 \alpha_i^{(3)} - f_i f_0 \alpha_i^{(4)} \right) \\
\Psi' &= \frac{1}{\sqrt{2}} \sum_{i=1}^{n} \left(e_i \left(\alpha_i^{(1)} + J\alpha_i^{(3)} \right) + f_i \left(-\alpha_i^{(4)} + J\alpha_i^{(2)} \right) \right)
\end{aligned}
$$

with real α (and $n = 2$ for spin-1/2). Basically, the system is that all basis vectors are combined with e_0 and f_0, or alternatively vectors multiplied by $\alpha + J\beta$.

Multiple spins

For multiple spins one could take the wavefunction of dimension $n = 2^k$ and apply the above procedure. However, it may also be insightful to generate a new wavefunction from multiplying two single spin wave functions

$$\Psi = \sum_{\substack{i=1,2\\ j=3,4}} \frac{1}{\sqrt{2}}(e_i + Jf_i)\left(\Re\psi_i^{(1)} + J\,\Im\psi_i^{(1)}\right)\frac{1}{\sqrt{2}}(e_j + Jf_j)\left(\Re\psi_j^{(2)} + J\,\Im\psi_j^{(2)}\right)$$

This is for independent spins. For entangled spins, the cross-terms will have separate coefficients.

Appendix

Derivation

$$\Psi = \sum_i \frac{1}{\sqrt{2}}(e_i + Jf_i)(\Re\psi_i + J\,\Im\psi_i)$$

With $z_i = \Re\psi_i + J\,\Im\psi_i$

$$\begin{aligned}
\Psi\Psi^\dagger &= \frac{1}{2}\sum_{ij}(e_i + Jf_i)z_i z_j^\dagger(e_j - Jf_j) \\
&= \sum_i (1 - e_i f_i J) z_i z_i^\dagger \\
&\quad + \frac{1}{2}\sum_{i<j}\left((e_i + Jf_i)(e_j - Jf_j)z_i z_j^\dagger + (e_j + Jf_j)(e_i - Jf_i)z_j z_i^\dagger\right) \\
&= \sum_i (1 - e_i f_i J) z_i z_i^\dagger \\
&\quad + \frac{1}{2}\sum_{i<j}(e_i e_j + f_i f_j - (e_i f_j + e_j f_i)J) z_i z_j^\dagger \\
&\quad + \frac{1}{2}\sum_{i<j}(e_j e_i + f_j f_i - (e_j f_i + e_i f_j)J) z_j z_i^\dagger
\end{aligned}$$

With the real and imaginary parts

$$
\begin{aligned}
R_{ij} &= \frac{1}{2}(z_i z_j^\dagger + z_j z_i^\dagger) \\
JI_{ij} &= \frac{1}{2}(z_i z_j^\dagger - z_j z_i^\dagger) \\
z_i z_j^\dagger &= R_{ij} + J\,I_{ij} \\
z_j z_i^\dagger &= R_{ij} - J\,I_{ij}
\end{aligned}
$$

where R_{ij}, I_{ij} are scalars this becomes

$$
\begin{aligned}
\Psi\Psi^\dagger = &\sum_i (1 - e_i f_i J) z_i z_i^\dagger \\
&+\frac{1}{2}\sum_{i<j}(e_i e_j + f_i f_j - (e_i f_j + e_j f_i)J)(R_{ij} + J\,I_{ij}) \\
&+\frac{1}{2}\sum_{i<j}(-e_i e_j - f_i f_j - (e_i f_j + e_j f_i)J)(R_{ij} - J\,I_{ij}) \\
= &\sum_i z_i z_i^\dagger - \sum_i e_i f_i J z_i z_i^\dagger \\
&-\sum_{i<j}(e_i f_j + e_j f_i) J R_{ij} \\
&+\sum_{i<j}(e_i e_j + f_i f_j) J\,I_{ij}
\end{aligned}
$$

For normalized wave vectors

$$
\sum_i z_i z_i^\dagger = 1
$$

Therefore

$$
J\Psi\Psi^\dagger = J + \sum_i e_i f_i z_i z_i^\dagger + \sum_{i<j}(e_i f_j + e_j f_i) R_{ij} - \sum_{i<j}(e_i e_j + f_i f_j)\,I_{ij}
$$

is a grade-2 multivector.

The probability in quantum mechanics can be calculated from the dot product of two real vectors where states have the components

$$
(\psi_i \psi_i^*, \ldots, \sqrt{2}\Re(\psi_i \psi_j^*), \ldots, \sqrt{2}\Im(\psi_i \psi_j^*), \ldots)
$$

due to

$$
\begin{aligned}
P(\psi \to \phi) &= |\langle \psi | \phi \rangle|^2 \\
&= \sum_i \psi_i \phi_i^* \sum_j \phi_j \psi_j^* \\
&= \sum_i \psi_i \phi_i^* \phi_i \psi_i^* + \sum_{i<j} \left(\psi_i \phi_i^* \phi_j \psi_j^* + \psi_j \phi_j^* \phi_i \psi_i^* \right) \\
&= \sum_i \psi_i \psi_i^* \phi_i \phi_i^* + \sum_{i<j} 2\Re(\psi_i \psi_j^* \phi_i^* \phi_j) \\
&= \sum_i \psi_i \psi_i^* \phi_i \phi_i^* + \sum_{i<j} 2 \left(\Re(\psi_i \psi_j^*) \Re(\phi_i \phi_j^*) + \Im(\psi_i \psi_j^*) \Im(\phi_i \phi_j^*) \right)
\end{aligned}
$$

being a dot product of vectors with components $(\psi_i \psi_i^*, \sqrt{2}\Re(\psi_i \psi_j^*), \sqrt{2}\Im(\psi_i \psi_j^*)), (i < j)$.

Therefore in our case and for normalized wave vectors the probability can also be calculated from

$$
P(1 \to 2) = \langle (1 - \Psi_1 \Psi_1^\dagger)(1 - \Psi_2 \Psi_2^\dagger) \rangle = \langle \Psi_1 \Psi_1^\dagger \Psi_2 \Psi_2^\dagger \rangle - 1
$$

Relations between single spin vectors

$$
\begin{array}{ccc}
XY = Z & YZ = X & ZX = Y \\
XN = 0 & YN = 0 & ZN = 0
\end{array}
$$

$$
\begin{aligned}
\Pi &= e_1 f_1 e_2 f_2 \\
XX &= -\frac{1}{2}(1 + \Pi) \\
YY &= -\frac{1}{2}(1 + \Pi) \\
ZZ &= -\frac{1}{2}(1 + \Pi) \\
NN &= -\frac{1}{2}(1 - \Pi)
\end{aligned}
$$

Wave vector basis

$$
\begin{aligned}
\Psi &= \sum_{i=1}^{n} \frac{1}{\sqrt{2}} (e_i + J f_i)(\Re\psi_i + J\,\Im\psi_i) \\
&= \sum_{i=1}^{n} \Lambda_i \left(\Re\psi_i + J\,\Im\psi_i \right)
\end{aligned}
$$

with

$$
\Lambda_i = \frac{1}{\sqrt{2}} (e_i + J f_i)
$$

$$X_{ij} = \frac{1}{2}(e_i f_j + e_j f_i) \qquad \tilde{X}_{ij} = \frac{1}{2}(e_i f_j - e_j f_i)$$
$$Y_{ij} = \frac{1}{2}(e_i e_j + f_i f_j) \qquad \tilde{Y}_{ij} = \frac{1}{2}(e_i e_j - f_i f_j)$$
$$Z_{ij} = \frac{1}{2}(e_j f_j - e_i f_i) \qquad \tilde{Z}_{ij} = \frac{1}{2}(e_j f_j + e_i f_i)$$
$$T_{ij} = \frac{1}{2}(e_i f_i + e_j f_j) \qquad \tilde{T}_{ij} = \frac{1}{2}(e_i f_i - e_j f_j)$$
$$e_i f_i = T_{ij} - Z_{ij} = \tilde{T}_{ij} + \tilde{Z}_{ij}$$
$$e_j f_j = T_{ij} + Z_{ij} = \tilde{T}_{ij} - \tilde{Z}_{ij}$$

Neither of these $ab + cd$ terms has an inverse.

$$\Lambda_i^2 = 0 \qquad \Lambda_i \Lambda_i^\dagger = 1 - J(T_{ij} - Z_{ij})$$
$$\Lambda_i \Lambda_j =$$

$$e_1 f_1 = T - Z$$
$$e_2 f_2 = T + Z$$
$$TT + XX = -1$$