

Formulation

The goal is to express quantum mechanics in a new way with geometric algebra in order to get an understanding for space and spin.

The wavefunction can be expressed as

$$\Psi = \sum_{i=1}^n \frac{1}{\sqrt{2}} (e_i + J f_i) (\Re \psi_i + J \Im \psi_i)$$

where e_i, f_i are orthonormal vectors which square to $+1$,

$$J = e_0 f_0$$

is an independent bivector, and ψ_i are the complex valued components of the wave vector. In total there are $2n + 2$ orthonormal vectors for an n -dimensional original wave vector.

Alternatively, the bivector

$$\Psi f_0 = \frac{1}{\sqrt{2}} \sum_{i=1}^n ((e_i f_0 + f_i e_0) \Re \psi_i + (e_i e_0 - f_i f_0) \Im \psi_i)$$

is a more natural choice, but for the (classical) derivation I will use the first version. The difference to other treatments of spin with geometric algebra is that here the basis is of the form $ab + cd$ with 2 additional basis vectors, instead of $\sigma_{12} = e_1 e_2$.

One advantage is that the full probability calculation becomes

$$P(1 \rightarrow 2) = \langle \Psi_1 \Psi_1^\dagger \Psi_2 \Psi_2^\dagger \rangle - 1$$

which can take the same rotor for $\Psi \Psi^\dagger$ and Ψ .

For a spin-1/2 particle coordinates will come out as

$$J\Psi\Psi^\dagger = J + T + Xx + Yy + Zz$$

$$X = \frac{1}{2}(e_1f_2 + e_2f_1)$$

$$Y = \frac{1}{2}(e_1e_2 + f_1f_2)$$

$$Z = \frac{1}{2}(e_2f_2 - e_1f_1)$$

$$T = \frac{1}{2}(e_1f_1 + e_2f_2)$$

where coordinates do not depend on e_0, f_0 . Space rotations can be derived in this basis and the same rotor can be applied to Ψ too. T may be related to time, but for now it is only an expression.

They anti-commute and obey

$$\begin{array}{lll} XY = Z & YZ = X & ZX = Y \\ XT = 0 & YT = 0 & ZT = 0 \end{array}$$

$$\Pi = e_1f_1e_2f_2$$

$$XX = -\frac{1}{2}(1 + \Pi)$$

$$YY = -\frac{1}{2}(1 + \Pi)$$

$$ZZ = -\frac{1}{2}(1 + \Pi)$$

$$TT = -\frac{1}{2}(1 - \Pi)$$

which has been derived from the spin wave vector alone without actually considering time and for a positive signature vector space. Note that these multivectors are not invertable.

Observation in quantum mechanics

The probability $P = |\langle\psi_1|\psi_2\rangle|^2$ to measure a state ψ_1 in a state ψ_2 can be calculated from

$$\begin{aligned} P(1 \rightarrow 2) &= \langle(1 - \Psi_1\Psi_1^\dagger)(1 - \Psi_2\Psi_2^\dagger)\rangle \\ &= \langle\Psi_1\Psi_1^\dagger\Psi_2\Psi_2^\dagger\rangle - 1 \\ &= \Psi_1\Psi_1^\dagger \cdot \Psi_2\Psi_2^\dagger - 1 \end{aligned}$$

which is an inner product between two state multivectors of the form $\Psi\Psi^\dagger$ (see appendix).

Unitary transformation

A unitary transformation of the wavefunction can be represented as a rotor in geometric algebra. Note that the same rotor can be applied to Ψ or $\Psi\Psi^\dagger$.

State vector

$$\Psi = \sum_{i=1}^n \frac{1}{\sqrt{2}} (e_i + J f_i) (\Re \psi_i + J \Im \psi_i)$$

The wave function squared can be expanded into a bivector

$$\begin{aligned} \Omega &= J\Psi\Psi^\dagger \\ &= J + \sum_i e_i f_i |\psi_i|^2 + \sum_{i<j} (e_i f_j + e_j f_i) \Re(\psi_i \psi_j^*) - \sum_{i<j} (e_i e_j + f_i f_j) \Im(\psi_i \psi_j^*) \end{aligned}$$

(see appendix) for an easier translation from the usual representation with a complex wave vector ψ_i .

Single spin

The wavefunction for a single spin-up in a direction given by Euler angles θ, ϕ is usually written as

$$\psi = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\phi} \end{pmatrix}$$

up to an arbitrary phase.

Written in geometric algebra this is

$$\begin{aligned}
J\Psi\Psi^\dagger &= J + e_1 f_1 \cos^2 \frac{\theta}{2} + e_2 f_2 \sin^2 \frac{\theta}{2} \\
&\quad + (e_1 f_2 + e_2 f_1) \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \phi \\
&\quad + (e_1 e_2 + f_1 f_2) \cos \frac{\theta}{2} \sin \frac{\theta}{2} \sin \phi \\
&= J + e_1 f_1 \frac{1 - \cos \theta}{2} + e_2 f_2 \frac{1 + \cos \theta}{2} \\
&\quad + (e_1 f_2 + e_2 f_1) \frac{1}{2} \sin \theta \cos \phi \\
&\quad + (e_1 e_2 + f_1 f_2) \frac{1}{2} \sin \theta \sin \phi \\
&= J + \frac{1}{2} (e_1 f_1 + e_2 f_2) \\
&\quad + \frac{1}{2} (e_2 f_2 - e_1 f_1) \cos \theta \\
&\quad + \frac{1}{2} (e_1 f_2 + e_2 f_1) \sin \theta \cos \phi \\
&\quad + \frac{1}{2} (e_1 e_2 + f_1 f_2) \sin \theta \sin \phi
\end{aligned}$$

Remembering that we have Euler angles, we can identify the multivectors for space coordinates from this expression for a single spin

$$\begin{aligned}
J\Psi\Psi^\dagger &= J + T + Xx + Yy + Zz \\
X &= \frac{1}{2} (e_1 f_2 + e_2 f_1) \\
Y &= \frac{1}{2} (e_1 e_2 + f_1 f_2) \\
Z &= \frac{1}{2} (e_2 f_2 - e_1 f_1) \\
T &= \frac{1}{2} (e_1 f_1 + e_2 f_2)
\end{aligned}$$

The expression for T is just the remaining term.

Space rotations will be derived soon.

$$\begin{aligned}
e_1 f_1 &= T - Z \\
e_2 f_2 &= T + Z \\
TT + XX &= -1
\end{aligned}$$

$$\Omega = J + \sum_i e_i f_i |\psi_i|^2 + 2 \sum_{i < j} X_{ij} \Re(\psi_i \psi_j^*) - 2 \sum_{i < j} Y_{ij} \Im(\psi_i \psi_j^*)$$

Interpretation

A vague idea why quantum mechanics is this way, is because due to the rules of probability, a state should be a (multi)vector and probabilities be calculated from an inner product. The probabilities rules are that they should sum to 1 and redoing a measurement yields the same results.

This state is constantly being rotated looking like $\Delta = \cdots R_3 R_2 R_1 R_1^\dagger R_2^\dagger R_3^\dagger \cdots$ we somehow get, that actually Δ should split into $\Delta = \Psi \Psi^\dagger$ with the consequence that the degrees of freedom are largely reduced. The above construction with geometric algebra demonstrates how to do this "square root".

Other ideas

Looking at

$$\Psi f_0 = \frac{1}{\sqrt{2}} \sum_{i=1}^n ((e_i f_0 + f_i e_0) \Re \psi_i + (e_i e_0 - f_i f_0) \Im \psi_i)$$

one may also consider what happens if the terms $e_i f_0, f_i e_0, e_i e_0, f_i f_0$ have independent coefficients. Maybe this is related to Dirac matrices.

Also, one may wonder if a term $e_i e_j$ is existing.

Appendix

Derivation

$$\Psi = \sum_i \frac{1}{\sqrt{2}} (e_i + J f_i) (\Re \psi_i + J \Im \psi_i)$$

With $z_i = \Re \psi_i + J \Im \psi_i$

$$\begin{aligned}
\Psi\Psi^\dagger &= \frac{1}{2} \sum_{ij} (e_i + Jf_i) z_i z_j^\dagger (e_j - Jf_j) \\
&= \sum_i (1 - e_i f_i J) z_i z_i^\dagger \\
&\quad + \frac{1}{2} \sum_{i < j} \left((e_i + Jf_i)(e_j - Jf_j) z_i z_j^\dagger + (e_j + Jf_j)(e_i - Jf_i) z_j z_i^\dagger \right) \\
&= \sum_i (1 - e_i f_i J) z_i z_i^\dagger \\
&\quad + \frac{1}{2} \sum_{i < j} (e_i e_j + f_i f_j - (e_i f_j + e_j f_i) J) z_i z_j^\dagger \\
&\quad + \frac{1}{2} \sum_{i < j} (e_j e_i + f_j f_i - (e_j f_i + e_i f_j) J) z_j z_i^\dagger
\end{aligned}$$

With the real and imaginary parts

$$\begin{aligned}
R_{ij} &= \frac{1}{2} (z_i z_j^\dagger + z_j z_i^\dagger) \\
JI_{ij} &= \frac{1}{2} (z_i z_j^\dagger - z_j z_i^\dagger) \\
z_i z_j^\dagger &= R_{ij} + JI_{ij} \\
z_j z_i^\dagger &= R_{ij} - JI_{ij}
\end{aligned}$$

where R_{ij}, I_{ij} are scalars this becomes

$$\begin{aligned}
\Psi\Psi^\dagger &= \sum_i (1 - e_i f_i J) z_i z_i^\dagger \\
&\quad + \frac{1}{2} \sum_{i < j} (e_i e_j + f_i f_j - (e_i f_j + e_j f_i) J) (R_{ij} + JI_{ij}) \\
&\quad + \frac{1}{2} \sum_{i < j} (-e_i e_j - f_i f_j - (e_i f_j + e_j f_i) J) (R_{ij} - JI_{ij}) \\
&= \sum_i z_i z_i^\dagger - \sum_i e_i f_i J z_i z_i^\dagger \\
&\quad - \sum_{i < j} (e_i f_j + e_j f_i) J R_{ij} \\
&\quad + \sum_{i < j} (e_i e_j + f_i f_j) J I_{ij}
\end{aligned}$$

For normalized wave vectors

$$\langle \Psi\Psi^\dagger \rangle = 1$$

Therefore

$$J\Psi\Psi^\dagger = J + \sum_i e_i f_i z_i z_i^\dagger + \sum_{i<j} (e_i f_j + e_j f_i) R_{ij} - \sum_{i<j} (e_i e_j + f_i f_j) I_{ij}$$

is a grade-2 multivector.

The probability in quantum mechanics can be calculated from the dot product of two real vectors where states have the components

$$(\psi_i \psi_i^*, \dots, \sqrt{2}\Re(\psi_i \psi_j^*), \dots, \sqrt{2}\Im(\psi_i \psi_j^*), \dots)$$

due to

$$\begin{aligned} P(\psi \rightarrow \phi) &= |\langle \psi | \phi \rangle|^2 \\ &= \sum_i \psi_i \phi_i^* \sum_j \phi_j \psi_j^* \\ &= \sum_i \psi_i \phi_i^* \phi_i \psi_i^* + \sum_{i<j} (\psi_i \phi_i^* \phi_j \psi_j^* + \psi_j \phi_j^* \phi_i \psi_i^*) \\ &= \sum_i \psi_i \psi_i^* \phi_i \phi_i^* + \sum_{i<j} 2\Re(\psi_i \psi_j^* \phi_i^* \phi_j) \\ &= \sum_i \psi_i \psi_i^* \phi_i \phi_i^* + \sum_{i<j} 2 (\Re(\psi_i \psi_j^*) \Re(\phi_i \phi_j^*) + \Im(\psi_i \psi_j^*) \Im(\phi_i \phi_j^*)) \end{aligned}$$

being a dot product of vectors with components $\psi_i \psi_i^*, \sqrt{2}\psi_i \psi_j^* (i < j)$.

Therefore in our case and for normalized wave vectors the probability can also be calculated from

$$P(1 \rightarrow 2) = \langle (1 - \Psi_1 \Psi_1^\dagger)(1 - \Psi_2 \Psi_2^\dagger) \rangle = \langle \Psi_1 \Psi_1^\dagger \Psi_2 \Psi_2^\dagger \rangle - 1$$