Expectation value of the highest number of d s-sided dice

G.E.J. Koops

1 Introduction

We would like to study the following question: when we have d number of dice and each die has s sides, what is the expectation value $E_+(s,d)$ for the highest number on one of the dice when we roll all d dice at once? On the Youtube channel *Stand-up Maths*, a very instructive video is shown where this problem is treated in a very visual geometric way [1]. Based on some trends and geometrical arguments (hypercubes in d-dimentional space) he concludes that the formula for the expectation value $E_+(s,d)$ is:

$$E_+(s,d) = \frac{d}{d+1} \cdot s + 0.5 \tag{1}$$

this is however a good approximation but not the full equation. We shall give a short deriation of the full formula and discuss the differences with equation 1.

2 Derivation of $E_+(s,d)$

In general, the expectation value for a random variable with finitely many outcomes is defined by:

$$E[X] = \sum_{i=1}^{n} x_i p_i \tag{2}$$

where $x_1, ..., x_n$ are the possible outcomes, each with a probability $p_1, ..., p_n$ of occurring. In order to calculate the expectation value we are looking for, $E_+(s,d)$, we will have to calculate for each combination of s and d the chance p_n that n, $(n \leq s)$ is the highest number of all of the d dice in the roll. We then have to multiply that number by the face-value n and divide by the total number of possible rolls (s^d). Summing all of these numbers over s will give us the expecation value.

[1] https://youtu.be/X_DdGRjtwAo?si=MKNRdQWPp16oCkeo/

We first look at an example. The ways of rolling a maximum of 2 with 3 s-sided dice are:

$$(2,1,1),(1,2,1),(1,1,2),(2,2,1),(2,1,2),(1,2,2),(2,2,2)$$

. With the roll $(1,1,1)$, that has 1 as highest number, these are all combinations for rolling 3 different 2-sided dice. The value of s in not important in this reasoning.
What is now the number of rolls that have n as maximum value? The total number of different rolls is: s^d. The amount of rolls where n is the maximum number is n^d. This is however exactly the number of rolls for d n-sided dice as shown in the example above. From the total of n^d we still have to substract the rolls that only contain numbers smaller than n. When we roll with d n-sided dice, the chance of having a roll containing an n is $p(n,d)_n$. This is the same as $1 - p(n,d)_{\neg n}$ (the chance of not rolling an n). This change is given by:

$$p(n,d)_n = \left(1 - \left(\frac{n-1}{n}\right)^d\right) \tag{3}$$

The total chance of rolling an n as maximum value is now:

$$p(s,d)_n = p(n,d)_n \frac{n^d}{s^d} = \left(1 - \left(\frac{n-1}{n}\right)^d\right) \frac{n^d}{s^d} \tag{4}$$

When we plug this into equation 2, we get the formula for the expectation value for the highest value when rolling d s-sided dice:

$$E_+(s,d) = \sum_{n=1}^{s} n \left(1 - \left(\frac{n-1}{n}\right)^d\right) \frac{n^d}{s^d} \tag{5}$$

We can rework this expression to:

$$E_+(s,d) = \sum_{n=1}^{s} n(n^d - (n-1)^d)/s^d \tag{6}$$

The terms $n^d - (n-1)^d$ are related to the hypercubes as mentioned in the *Stand-up Maths* video. In the summation we can work-out the n^d and $(n-1)^d$ terms and notice that all $n \cdot n^d$ terms reduce to $-n^d$ except for the last term in the summation: $s \cdot s^d$. This brings us to the following formula:

$$E_+(s,d) = s - \frac{1}{s^d} \sum_{n=1}^{s-1} n^d \tag{7}$$

3 Approximation of $E_+(s,d)$

We now want to approximate equation 7 in order to derive the very simple approximation of equation 1. The sum of the d-th power of n can be expressed as a polynomial in n by Faulhabers formula:[2]

$$\sum_{n=1}^{s-1} n^d = \frac{1}{d+1}\sum_{r=0}^{d}\binom{p+1}{r}B_r(s-1)^{p-r+1} \tag{8}$$

Here $\binom{p+1}{r}$ is the binomial coefficient and B_r are the Bernoulli numbers. We can approximate this by taking the two highest order terms:[3]

$$\sum_{n=1}^{(s-1)} n^d = \frac{1}{d+1}(s-1)^{(d+1)} + \frac{1}{2}(s-1)^d + (\text{lower order terms}) \tag{9}$$

Plugging this into equation 7, we find:

$$E_+(s,d) \approx s - \frac{1}{s^d}\left((s-1)^d\left[\left(\frac{s-1}{d+1}+\frac{1}{2}\right)\right]\right) \tag{10}$$

This approximation of the Faulhabers formula is not enough to derive equation 1. We still have to take one extra step. In the original video from *Stand-up Maths* the final formula of equation 1 was obtained for values of $s \to \infty$. When we let $(s-1) \approx s$ in equation 10 we get:

$$E_+(s,d) \approx s - \frac{1}{s^d}\left(s^d\left(\frac{s}{d+1}\right)+\frac{1}{2}\right) = s - \left(\frac{s}{d+1}+\frac{1}{2}\right) = \frac{d}{d+1}s + \frac{1}{2} \tag{11}$$

This shows that equation 1 is an approximation of the full equation 7.

4 Quality of the approximation

In order to derive the approximation 11, we have made two differen approximations:

1. We approximated Faulhabers formula by the two highest order terms.
2. We let $s \to \infty$ for the two highest order terms.

Although these approximations seem rather crude, the approximation is a very accurate description of the real value for $E_+(s,d)$. In Figure 1 we have plotted the value $100\% * (approx - real)/approx$ for $s = 6$ and $s = 20$ for the number of dice d ranging from 1 to 20. This shows indeed that

[2] en.wikipedia.org/wiki/Faulhaber%27s_formula

[3] Sum of n, n^2, or n^3. Brilliant.org. Retrieved 13:22, July 1, 2024, from https://brilliant.org/wiki/sum-of-n-n2-or-n3/

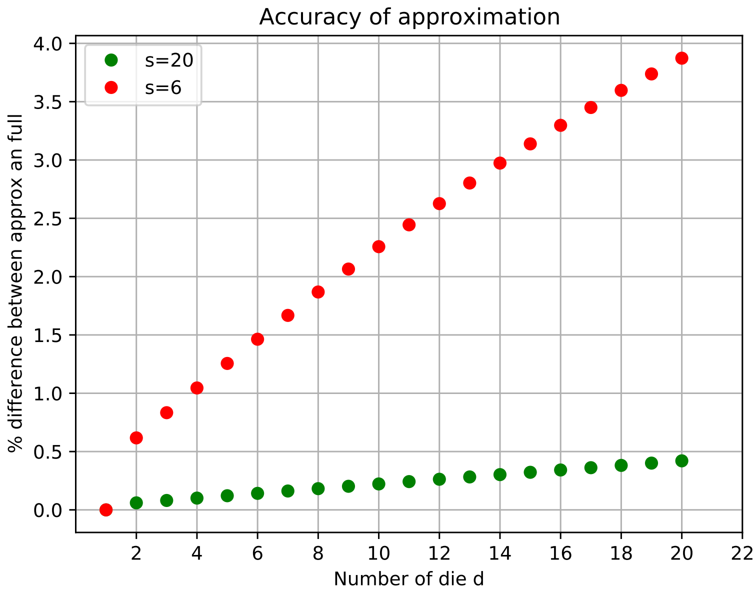

Figure 1: Accuracy of the approximation

the approximation is much better ($< 0.5\%$) for larger s ($s = 20$) than for smaller s (max 4% error for $s = 6$). For all practical purposes, the error of the approximation value remains within a few percent of the real expectation value $E_+(s, d)$. This accuracy of the approximation might be caused by the fact that both approximations counter-act eachother. By only taking the leading order terms of the Faulhaber equation, the approximation of the summation part is less than the full sum. By taking $(s - 1) \approx s$ we increase the value of the summation. A further analysis of the error-terms of the approximation might reveal a more detailed understaning of the sensitivity of the approximation to the input parameters.