Simulating one dimensional free fall using scientific python.

Geri Nicka
School of Physics, University of Bristol.
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INTRODUCTION

The free fall of a projectile through air in one dimension can be modelled by developing an appropriate program using Python. A free falling object will experience a gravitational force in the direction of its motion and a drag force in the opposite direction.

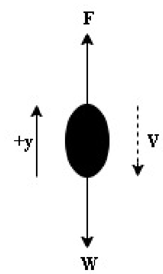

FIG. 1. Forces acting on the free falling projectile.

The drag force, which is to be taken into account in the physical modelling of the free fall is given by :

$$\vec{F} = -kv\hat{v} \tag{1}$$

where k is given by :

$$k = \frac{C_d A \rho_0}{2} \tag{2}$$

in which C_d is the drag coefficient, A is the cross sectional area of the projectile and ρ_0 is the air density. Taking into account that the acceleration of the projectile is varying, and considering Newton's second law for the vertical component of velocity, two first order differential equations are produced,

$$m\frac{dv_y}{dt} = -mg - k|v_y|v_y \tag{3}$$

$$\frac{dy}{dt} = v_y \tag{4}$$

where m is the mass of the free falling object and g the gravitational acceleration ($9.81m/s^2$). In the present modelling, three numerical methods will be applied to solve equations (3) and (4).

EULER'S METHOD

The Euler method is the most straightforward application of numerical integration of first order and first degree ordinary differential equations [1]. Consider a function $f(t, y)$ which produces a specific curve. In order to transfer from point (t_0, y_0) to point (t_1, y_1), the derivative of the curve over the finite distance between the two points must be determined and multiplied by the time elapsed Δt. This can be generalised for two adjacent points (t_n, y_n) and (t_{n+1}, y_{n+1}) for all points on the curve given by $f(t, y)$ and ultimately approximate it.

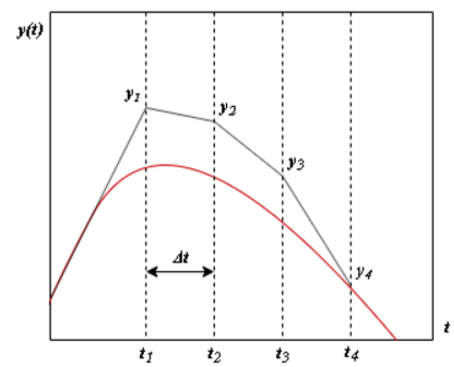

FIG. 2. Approximation of curve given in red using Euler's method with steps Δt.

Applying Euler's method to equations (3) and (4) the height and velocity during a free fall can be approximately predicted from the equations derived below,

$$t_{n+1} = t_n + \Delta t \tag{5}$$

$$y_{n+1} = y_n + \Delta t v_{y,n} \tag{6}$$

$$v_{y,n+1} = v_{y,n} - \Delta t(g + \frac{k}{m}|v_{y,n}|v_{y,n}) \tag{7}$$

Physics challenges in the real world are rarely solved using Euler's method. This is due to the method's weakness in handling specific differential equations; for large increments the solution has the tendency to diverge from the actual results as time evolves. This happens because the error in Euler's method is proportional to the increment squared $(\Delta t)^2$. Clearly, smaller steps compared to larger steps lead to a better approximation of a solution because the error minimizes.

MODIFIED EULER'S METHOD

The Modified Euler's Method is the improved version of the classic Euler method. Its key feature is the reduction in error that the Euler method produces. This is accomplished by taking the value of the middle point between increments, as well as its gradient for each increment [2]. More explicitly, more derivatives at more points are computed in comparison to Euler's method, resulting in a more precise approximation of a known curve. It is frequently used in solving differential equations similar to the form of equations (3) and (4). Thus, every value of velocity can be attributed at the middle of each increment resulting to more accurate position and velocity predictions [3]. The updated equations for this improved method are,

$$v_{mid,n+1} = v_{y,n} - \frac{\Delta t}{2}(g + \frac{k}{m}|v_{y,n}|v_{y,n}) \tag{8}$$

$$v_{n+1} = v_n - \Delta t(g + \frac{k}{m}|v_{mid,n+1}|v_{mid,n+1}) \tag{9}$$

$$y_{n+1} = y_n + \Delta t v_{mid,n+1} \tag{10}$$

The greatest advantage of this regime is that it can be reversed. In other words, it validates energy conservation [4] and prohibits the divergent tendency present when the conventional Euler's method is applied.

ANALYTICAL METHOD

As discussed above, both Euler methods are used to approximate the trajectory of a free falling body under the effect of a drag force in constant or varying air density. The accuracy of these methods can only be determined by comparing them to the results of the analytical method. The position and velocity of the projectile before it reaches the ground are predicted at all times by,

$$y = y_0 - \frac{m}{k}\ln(\cosh\sqrt{\frac{kg}{m}}t) \tag{11}$$

$$v_y = -\sqrt{\frac{mg}{k}}\tanh\sqrt{\frac{kg}{m}}t \tag{12}$$

where y_0 is the initial height of the free fall. Given the fact that Modified Euler is more sophisticated and accurate than Euler itself, it is expected that the approximation curve of the former will be closer to the analytical prediction curve.

SIMULATION METHODS

The functions **Height** and **VerticalVelocity** were defined to be equations (11) and (12) respectively and were responsible for predicting the position and velocity of the projectile at any given time analytically. As it made no sense for position to reach negative values, a "for" loop with specific variables linked to the defined functions was constructed. This allowed the loop to recognise and store consecutive positions into predetermined arrays with zero entries. Finally , an "if" statement was set to break the iterative loop once a stored value of position was recognised as zero. The same method of breaking out of the loop when the projectile reached the ground was implemented for every approximation method.

Equations (3) and (4) were solved by both Euler methods. For each counterpart, iterative loops were used to compute each consecutive position and velocity value between time steps Δt add them until a complete simulation curve for each was produced.

Additionally, a simple user menu was created which allows the user to choose the problem parameters such as the mass, the drag coefficient, cross sectional area, the duration of the free fall and the time step Δt. This enables the investigation of the results given by the two approximation methods for large values of Δt and a varying ratio of k/m

As an attempt to render the simulation more realistic, a height dependant air density was introduced. This created the need to define another two relations for the constant k and use them to make predictions with both Euler methods respectively inside their iterative loops. Air density dependence on height is given by,

$$\rho = \rho_0 \exp(-y/h) \tag{13}$$

with h being the Scale Height at 7.64 km [5].

RESULTS AND DISCUSSION

The curve obtained by the analytical method was used as a reference to check the accuracy of both Euler's and Modified Euler's methods. Height and velocity as functions of time subplots were generated to clearly illustrate their differences.

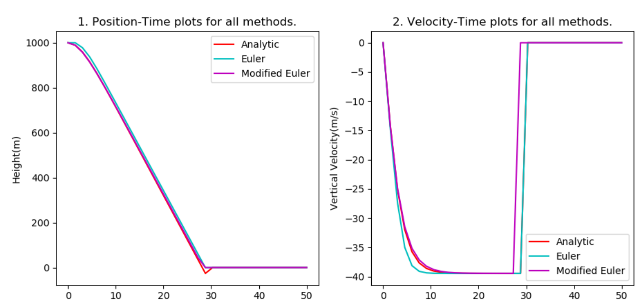

FIG. 3. Height and velocity of projectile accelerating from 1km with: $m = 80kg$, $C_d = 1.2$, $\rho_0 = 1.2kg/m^3$, $A = 0.7m^2$ and $\Delta t = 1.5s$. Results: $u_{y,max} = 39.46m/s$ for both Euler methods.

In Figure 3 it is clearly visible that Modified Euler has approached the analytical curve closer than Euler and is therefore more accurate. It must be noted that since the step size is small, both methods are relatively accurate enough as they predict the same terminal velocity.

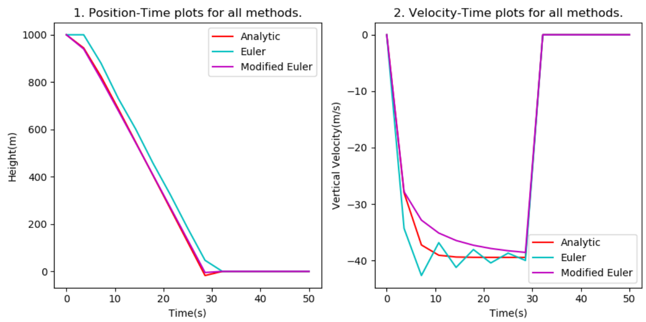

FIG. 4. Height and velocity of projectile accelerating from 1km with: $m = 80kg$, $C_d = 1.2$, $\rho_0 = 1.2kg/m^3$, $A = 0.7m^2$ and $\Delta t = 3.5s$. Results: $u_{y,max} = 42.66m/s$ for Euler method and $v_{y,max} = 38.56m/s$ for Modified Euler method.

The resulting trajectories and velocities for a larger time step in Figure 4 are a solid proof of the fact that the conventional Euler method becomes unstable and diverges when time steps are not small enough. On the other hand, its counterpart tends to remain fairly accurate even for larger steps and retains the shape of the analytical curve but has a larger deviation.

Interesting results in the simulation arise when the ratio k/m decreases by a large factor. For a very large mass, a significantly higher terminal velocity and a sorter duration of the motion are expected. This is due to the increased gravitational force compared to the drag force that causes a much higher acceleration on the object. It is important to note that this occurs only when a free falling object is experiencing a drag force. Otherwise the same acceleration and free fall duration apply regardless of mass as there is only one force acting; gravity.

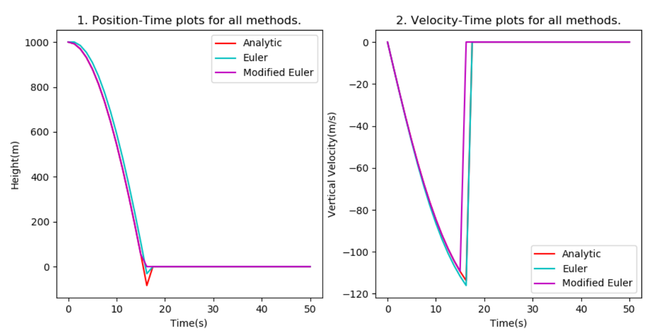

FIG. 5. Height and velocity of projectile accelerating from 1km with: $m = 1000kg$, $C_d = 1.2$, $\rho_0 = 1.2kg/m^3$, $A = 0.7m^2$ and $\Delta t = 1.25s$. Results: $u_{y,max} = 116.06m/s$ for Euler method and $v_{y,max} = 109.28m/s$ for Modified Euler method.

As a final comment for Euler's method it should be clarified that for extremely large steps the approximation not only is inaccurate, but it fails to simulate a real free fall. In the particular case of insensibly large steps the Modified Euler method also yields relatively inaccurate results but still sustains the shape of the analytical curve. Terminal velocities predicted by both methods are not close to the expected ones by any means.

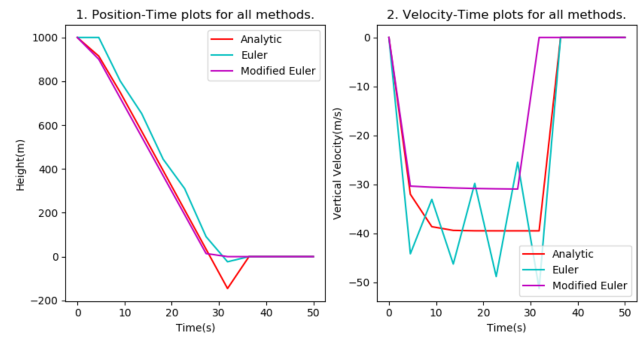

FIG. 6. Height and velocity of projectile accelerating from 1km with: $m = 80kg$, $C_d = 1.2$, $\rho_0 = 1.2kg/m^3$, $A = 0.7m^2$ and $\Delta t = 4.5s$. Results: $u_{y,max} = 51.23m/s$ for Euler method and $v_{y,max} = 30.94m/s$ for modified Euler method.

The results described and discussed above are only valid for the ideal situation of a free fall through air of constant density. If a free fall initiates at a higher altitude, its modelling should revolve around expressing the constant k in terms of height just as in equation (13). Apart from that, the scheme of the simulation is identical as in for a constant air density.

The topic of diving from extremely high altitudes is introduced in this report with the aim to analyse the free fall of Felix Baumgartner at an initial height of 39045 meters.

To achieve the most accurate simulation possible, the parameters present during his fall were taken into account. The cross sectional area in this case was determined to be that of his head -$0.34m^2$- since he jumped facing the earth in order to maximize his terminal velocity [6]. Moreover, the scale height h was set as 7.64km and the drag coefficient was estimated at 1.15. His weight combined with the mass of his protective suit was determined to be 108kg.

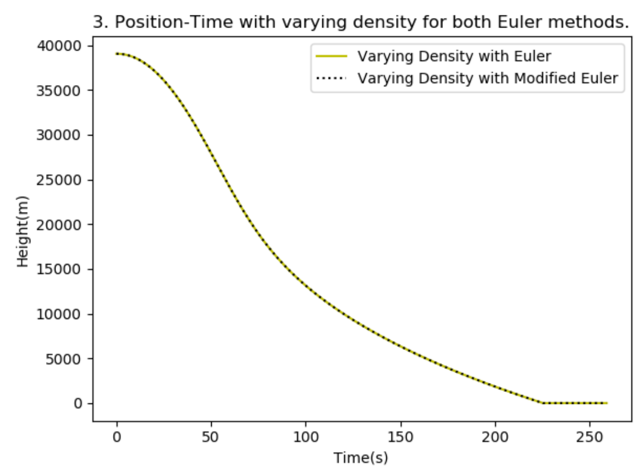

FIG. 7. Height versus time plot for Baumgartner's fall with: $m = 108kg$, $C_d = 1.15$, $A = 0.34m^2$, $y_0 = 39045m$ and $\Delta t = 0.5s$.

Figure (7) shows the trajectories of Baumgartner with varying density for both Euler methods. The two curves coincide almost perfectly. This happens due to the exponential in equation (13) where the initial height is the same for both methods. The difference between the two curves is negligible because it is determined by the different time increments used, which themselves are not large enough to noticeably alter the curves' shape and terminal velocities.

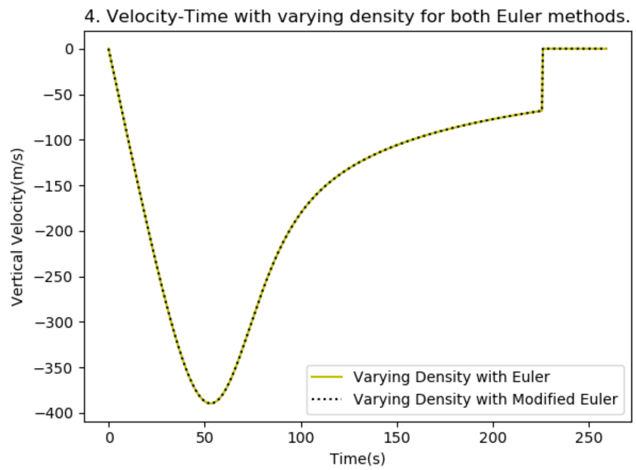

FIG. 8. Velocity versus time plot for Baumgartner's fall with: $m = 108kg$, $C_d = 1.15$, $A = 0.34m^2$, $y_0 = 39045m$ and $\Delta t = 0.5s$. Results: $u_{y,max} = 389.5m/s$ for Euler method and $v_{y,max} = 389.5m/s$ for modified Euler method.

According to the velocity curve in Figure 8, Baumgartner reached a terminal velocity of **389.5** m/s at an approximate time of **53.33** seconds. This maximum velocity was reached at approxiamately **26700** meters above the landing point. It is observed that the terminal velocity reached while propagating through air with varying density is significantly higher than that achieved when air density is taken to be constant. Eventually, the sound barrier at **344** m/s is crossed, which agrees with Baumgartner's achievement.

The lack of powerful drag forces at the start of the downward motion allows the projectile to accelerate rapidly. As the sound barrier is approached, the motion undergoes a much greater oppositely directed force which forces the projectile to reach a terminal velocity. Moments later, velocity starts to decrease exponentially with decreasing height and finally, abruptly reaches zero once Baumgartner lands.

It worth mentioning that for a larger cross sectional area the free fall duration increases significantly. From Figures 9 and 10 it can be clearly seen that for a larger projectile area the landing time increases. Recalling equation (1) and (2) it can be deduced that larger areas result in larger drag forces, thus reducing the overall downward acceleration of the projectile.

In Figure 10 it is seen that velocity has already started to approach zero exponentially but the time needed to reach it is larger due to the increased cross sectional area. The program is designed such that the user himself can chose the cross sectional area and investigate the results.

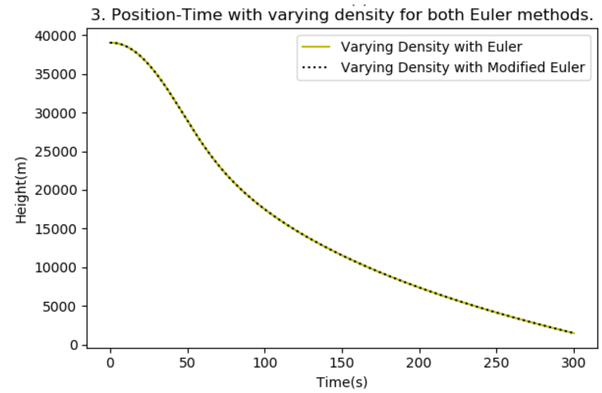

FIG. 9. Height versus time plot for Baumgartner's fall with: $m = 108kg$, $C_d = 1.15$, $A = 0.8m^2$, $y_0 = 39045m$ and $\Delta t = 1.5s$.

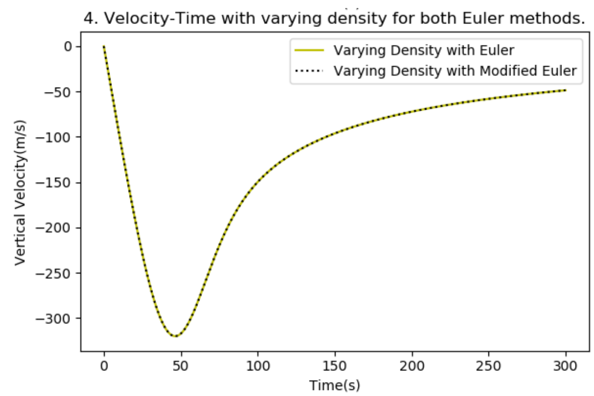

FIG. 10. Velocity versus time plot for Baumgartner's fall with: $m = 108kg$, $C_d = 1.15$, $A = 0.8m^2$, $y_0 = 39045m$ and $\Delta t = 1.5s$.

IMPROVEMENTS

For the two Euler methods, instead of using two loops, which essentially operated in the same way, a function with the same mechanism could have been defined. That would make the program more compact and allow the function to be called at any time instead of initiating the same loops twice or more times. However, there was no practical need of using the same loops more than once. Generally, user defined functions are easier to work with and render the program structure less complicated.

Furthermore, the code could have been adapted to check the user input for every parameter using simple "if" statements. In the case of undesired values such as negative height or time, a "while" loop could return the user back to choosing a correct parameter by also printing why the previous value chosen was invalid.

The approximation methods could be controlled by the program to accept user input values for Δt in a certain range which rules out the possibility of failure in approximation as examined earlier for large steps.

Moreover, graphs of comparison between the errors produced by Euler and Modified Euler could have been included in the code to quantitatively express their deviations and indicate at which point they start to fail in the approximation.

It was assumed that as Baumgartner was falling he did not change orientation though space. In real world conditions this cannot be possible as torque at different points on the sky diver causes rotation and subsequently leads to a change in cross sectional area. The program could be modified to account for a varying cross section.

Last but not least, the drag coefficient was assumed to be a constant. This is true for low velocities but as the sound barrier is approached the drag force heavily depends on velocity [7]. The extension of adapting the program to use the drag coefficient as a function of time would yield results for terminal velocity even closer to the actual one achieved.

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