On the simulation of orbital mechanics with the 4th order Runge-Kutta method.

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1. INTRODUCTION

Orbital mechanics is a core discipline within space-mission design and control. By constructing appropriate algorithms, problems pertaining to the motion of rockets and other spacecraft can be solved by simulating ballistics and orbital dynamics of systems under the influence of gravity from a celestial body. In this work, Newton's second law gives the equation of motion of a rocket orbiting the earth:

$$m\ddot{\mathbf{r}} = -\frac{mMG}{|\mathbf{r}|^2}\hat{\mathbf{r}} = -\frac{mMG}{|\mathbf{r}|^3}\mathbf{r}, \tag{1}$$

where G is the gravitational constant and $\mathbf{r}$ the position vector from the center of the earth's mass M to the rocket of mass m which is in orbit. This second order differential equation may be simplified into two first order equations:

$$\frac{d\mathbf{v}}{dt} = -\frac{mMG}{|\mathbf{r}|^3}\mathbf{r}, \tag{2}$$

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \tag{3}$$

where $\mathbf{v}$ is the velocity vector describing the motion of the rocket around stationary earth centered at the origin. Among the large variety of numerical methods used to solve differential equations this work uses the 4th order Runge-Kutta method.

2. RUNGE KUTTA METHOD - ROCKET ORBITING AROUND THE EARTH

The 4th order Runge-Kutta method, being an improvement of Euler's method follows the same principle by evaluating four differentials instead of one at each known point of a function and leading to a series of formulae which ultimately maximize the efficiency of the calculation even for larger increment sizes. This results from the fact that as a fourth-order method its local truncation error is on the order of $O(h^5)$ while its total accumulated error is on the order of $O(h^4)$ [1].

Second order differential equations expressed as a set of first order differential equations just as in (2) and (3) are more intricate to solve since vectors of all dependent parameters must be considered. In general such first order differential equations can be expressed as:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\{\mathbf{x}\}, t), \tag{4}$$

where $\mathbf{x}$ is the vector of all dependent variables and $\mathbf{f}$ the vector function attributing the functional dependence for every variable. Choosing a specific time-increment h the Runge-Kutta method yields:

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \frac{h}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4), \tag{5}$$

$$t_{i+1} = t_i + h, \tag{6}$$

where the $\mathbf{k}$'s are the generic Runge-Kutta coefficients expressed as vectors with each component corresponding to one of the dependent variables. These coefficients are in general expressed as:

$$\mathbf{k}_1 = \mathbf{f}(\{\mathbf{x}\}_i, t_i), \tag{7}$$

$$\mathbf{k}_2 = \mathbf{f}(\left\{\mathbf{x}_i + \frac{h\mathbf{k}_1}{2}\right\}, t_i + \frac{h}{2}), \tag{8}$$

$$\mathbf{k}_3 = \mathbf{f}(\left\{\mathbf{x}_i + \frac{h\mathbf{k}_2}{2}\right\}, t_i + \frac{h}{2}), \tag{9}$$

$$\mathbf{k}_4 = \mathbf{f}(\{\mathbf{x}_i + h\mathbf{k}_3\}, t_i + h), \tag{10}$$

which effectively couple all variables together and perform steps in all of them simultaneously. In the present case of a rocket orbiting stationary earth in two dimensions there is a total of four variables x, y, v_x, v_y, each of which is correlated to four Runge-Kutta coefficients. It is concluded that a total of 16 k's must be calculated to effectively simulate a rocket's orbit around the earth.

3. ROCKET LOOPING AROUND MOON AND BACK TO EARTH

A rocket can be launched from low Earth orbit such that it approaches Moon's surface with the mission to capture images and then return to relay the information via radio signals. It is again convenient to consider the Earth stationary and fixed at the origin and ignore the gravitational effects of the rocket on the Earth and Moon. The dynamics of this system is governed by:

$$\ddot{\mathbf{r}}_M = -\frac{M_E G}{|\mathbf{r}_M|^3}\mathbf{r}_M, \tag{11}$$

$$\ddot{\mathbf{r}}_R = -\frac{M_E G}{|\mathbf{r}_R|^3}\mathbf{r}_R = -\frac{M_M G}{|\mathbf{r}_R - \mathbf{R}_M|^3}(\mathbf{r}_R - \mathbf{R}_M), \qquad (12)$$

where E, M, R relate to the Earth, Moon and Rocket respectively. To maintain the simplicity of this simulation only two dimensional motion is considered as Runga Kutta coefficients are reduced significantly.

4. ECCENTRICITY

Circular or elliptical orbits may be executed by the rocket depending on the initial conditions the computer simulation is given. The orbital eccentricity, is defined as:

$$e = \frac{r_a - r_p}{r_a + r_p}, \qquad (13)$$

where r_a is the apoapsis and r_p the periapsis. The former and latter, being the greatest and least distance between the orbit and main body respectively can be computed by appropriately instructing the program.

5. PROGRAMMING METHODS

At the start of the program three functions which allow the user to chose parameters are defined. In particular, **TimeChoice**, **XYChoice** and **VChoice** enable the investigation of many alternative simulations with different initial conditions and the comparison between trajectories. This feature allows the creation of elliptical, parabolic and circular orbits around the Earth as well as different ”slingshots” around the moon.

Modelling two-dimensional gravitational interactions required the definitions of four functions returning the horizontal and vertical components of both the position and velocity of the rocket. These were **dx/dt**,**dy/dt**,**dVx/dt** and **dVy/dt**. This part of the code was crucial as different simulations require very specific initial conditions to be successfully executed. Such examples are two and three body interactions.

The main function of the program **RungeKutta** contains a loop where expressions similar to (7)-(10) are evaluated for a total of sixteen k’s. The obtained values are used by additional definitions inside the function to create lists of trajectories, velocities and energies of a particular simulation. The automated plotting functions were defined to avoid replicating unnecessary code and visualise the changes in kinetic, potential and total energy for a given orbit. The eccentricity of the rocket’s orbit around the earth was also computed using a simple loop. A simple menu was provided to chose between either a two or three body interaction.

6. RESULTS

The two dimensional simulation of a rocket orbiting the Earth was achieved using the previously discussed Runge-Kutta method. Appropriate initial conditions for the circular orbit around the Earth were obtained from the Horizons JPL Ephemeris system [2]. To ensure a perfect circular orbit was achieved, values close to Ephemeris data were chosen. The trajectory and variations in potential, kinetic and total energy are shown in Figure 1.

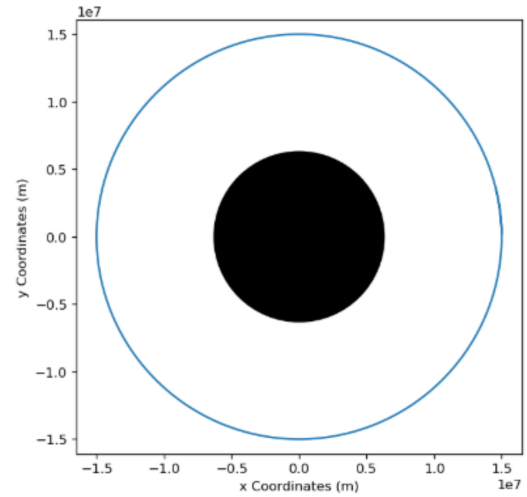

FIG. 1. The circular trajectory of the rocket around the Earth. Initial conditions are:$x_0 = 1.5e7\ m$, $y_0 = 0\ m$, $Vx_0 = 0\ m/s$ and $Vy_0 = 5153\ m/s$.

As expected for a circular orbit there is no change in the potential and kinetic energy. The total energy was found to be conserved leading to the conclusion that the simulation was correct.

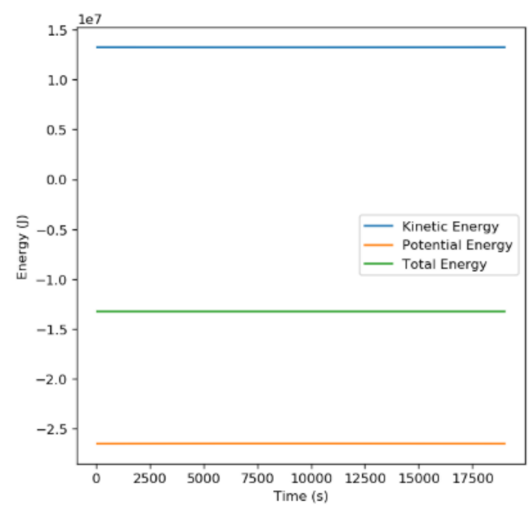

FIG. 2. Potential, kinetic and total energy changes of the circular orbit. As expected for a circular orbit energies are constant.

For every different trajectory the energy plots were used to determine whether the simulations were correct depending on the time-step chosen. More specifically, it was ensured that the total energy was conserved and that the kinetic and potential energy curves showed similar behavior as predicted theoretically. An eccentric and a highly eccentric orbit around the Earth were also achieved by choosing appropriate initial conditions. These conditions were established by the trial and error method and final trajectories along with their energy changes were plotted.

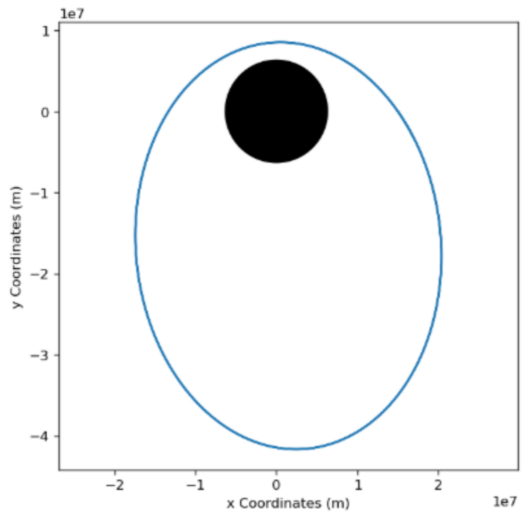

FIG. 3. The eccentric trajectory of the rocket around the Earth. Initial conditions are:$x_0 = 1.5e7\,m$, $y_0 = 0\,m$, $Vx_0 = 3500\,m/s$, $Vy_0 = 5000\,m/s$ and orbit duration 80000 seconds. Eccentricity = 0.66.

As opposed to the energy plot of the circular orbit, the kinetic and potential energies of the elliptical orbit do not remain constant. More specifically, as the kinetic energy increases the potential energy decreases to conserve the total energy.

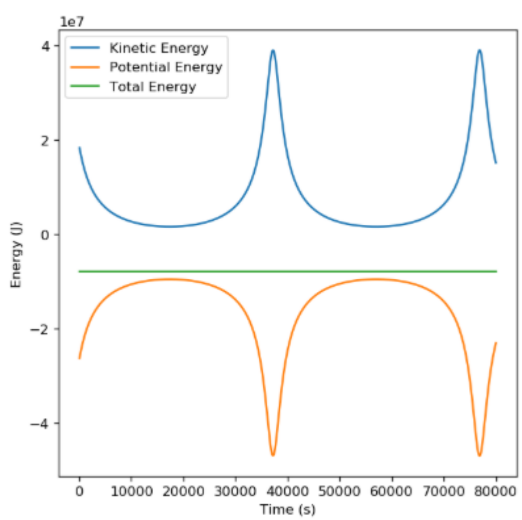

FIG. 4. Potential, kinetic and total energy changes of the eccentric orbit. As expected the total energy remains constant. The kinetic and potential energies seem to undergo opposite changes as expected.

At a further distance from the Earth's centre, the potential energy of the rocket becomes maximum whereas its kinetic energy reaches its minimum value. Keeping the same horizontal component of velocity as in the eccentric orbit, the vertical component was increased to create a highly elliptical orbit as shown in Figure 5.

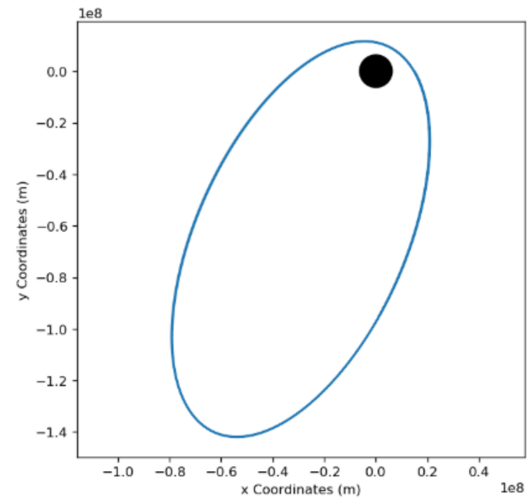

FIG. 5. The highly eccentric trajectory of the rocket around the Earth. Initial conditions are:$x_0 = 1.5e7\,m$, $y_0 = 0\,m$, $Vx_0 = 3500\,m/s$, $Vy_0 = 6000\,m/s$ and orbit duration 500000 seconds. Eccentricity = 0.87.

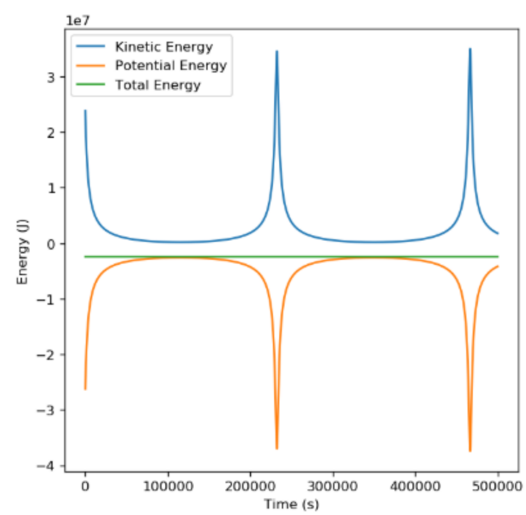

FIG. 6. Potential, kinetic and total energy changes of the highly eccentric orbit. As expected the total energy remains constant. The kinetic and potential energies seem to undergo opposite changes as expected.

Comparing energy plots in Figures 4 and 6 a steeper rate of change in the potential and kinetic energy of the highly eccentric orbit is observed. This is due to the fact that in this case the rocket orbits further from Earth's centre than in the eccentric. To model a three-body interaction, the Runge-Kutta method was applied to solve equations (11) and (12) numerically. To render the determination of the initial results simpler the rocket was set to start at the opposite side of the earth. This ensured that in the trial and error method only a vertical component of velocity was required to allow the rocket to follow the desired trajectory.

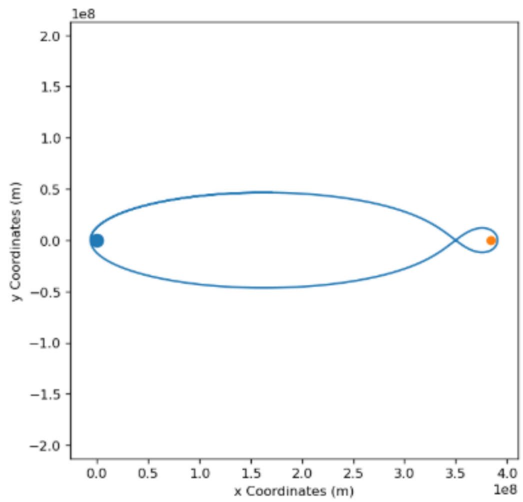

FIG. 7. The slingshot trajectory. Initial conditions are:$x_0 = 0\ m$, $y_0 = 0\ m$, $Vx_0 = 0\ m/s$, $Vy_0 = 10964\ m/s$.

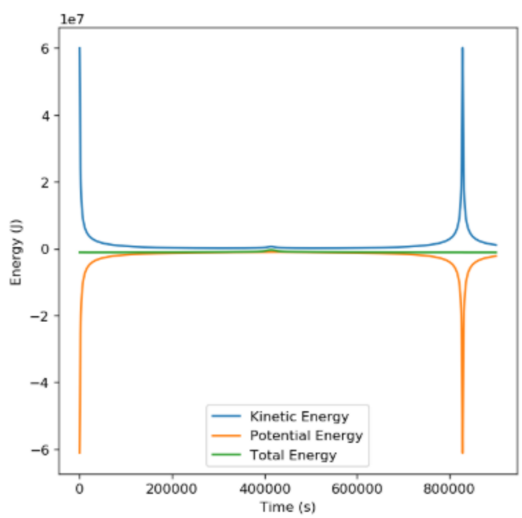

FIG. 8. Potential, kinetic and total energy changes of the 'slingshot'. As expected the total energy remains constant. The kinetic and potential energies seem to undergo opposite changes as expected.

The 'slingshot' trajectory and its energy plot are shown in Figures 7 and 8. More specifically, the rocket is seen to start its orbit at the opposite side of the Earth and as it approaches the gravitational potential of the Moon its direction shifts allowing it to orbit around it. Adequate initial kinetic energy was given to the rocket and escape the Moon's gravitational force and enter Earth's orbit again. The behavior of the kinetic, potential and total energy of this orbit are shown in Figure 8. The rocket starts with maximum kinetic energy which decreases rapidly as it gets further from earth and increases again as it returns. Opposite behaviour is observed for the potential energy while the total energy remains constant.

7. DISCUSSION

The Ephemeris data obtained made the determination of appropriate initial conditions for the rocket's orbit easier. More explicitly, the original initial conditions were divided by a constant number repeatedly to eventually achieve the circular orbit shown in Figure 1. A further indication of the simulation's correctness was the energy plot in in Figure 2. The rocket is observed to execute circular uniform motion around the Earth. As a result the tangential component of its velocity remains constant and no variation in kinetic energy is manifested. The potential energy of the rocket, being dependant on the radius of the circular motion remained constant as a result of the radius remaining unchanged.

Having established the appropriate initial conditions for a perfectly circular orbit the simulation of eccentric orbits was straightforward. Elliptical orbits are non-uniform circular motion, which implies that a drift in the potential and kinetic energy should be observed. The required initial conditions were determined by varying the vertical velocity component and the orbital time period. For the less eccentric orbit shown in Figure 3 the corresponding energies are plotted in Figure 4. Theoretical predictions are validated by observing the changes in the potential and kinetic energy. At larger distances from the Earth's center kinetic energy decreases and potential energy increases as a result of a larger orbital radius. The energy plot in Figure 6 which relates to the highly elliptical trajectory in Figure 5 follows similar trends to those illustrated in Figure 4 for the less eccentric orbit. The difference lies within the steepness of the rate of decrease and increase in kinetic and potential energy as the rocket reaches a larger radius than before.

It is important to note that the Runge-Kutta method should result in approximately identical and accurate simulations for larger time steps. To ensure this was the case, simulations with the same initial conditions were created by increasing the orbital time period, which itself increases the time step of the Runge-Kutta method. The resulting trajectory was identical to that in Figure 3 and its corresponding energy plot is shown in Figure 9. It was found that even at larger steps the total energy was conserved and the kinetic and potential energy curves evolved as previously. This test ensured that the computer program was time-step independent.

Significantly more difficult was the determination of the appropriate initial conditions required to execute a successful 'slingshot' simulation. Although trial and error was used, other methods were incorporated to narrow down these conditions. First of all, by analysing the shape of the orbit it was concluded that the velocity at the far side of the earth must have been solely comprised of a vertical component. Thus, the trial and error method was only necessary to find the adequate vertical component of velocity reducing the overall number of trials.

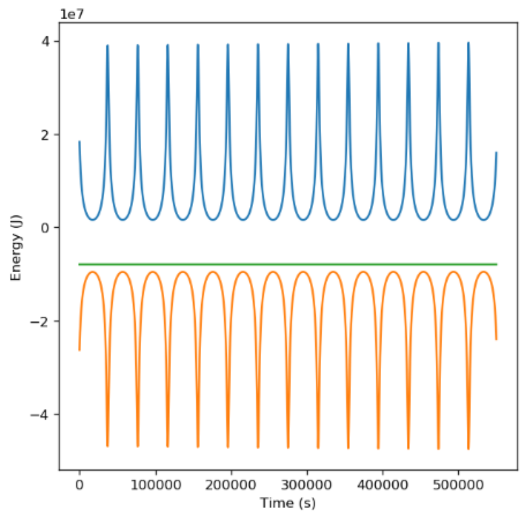

FIG. 9. Potential, kinetic and total energy changes of the less eccentric orbit. As expected the total energy remains constant. The kinetic and potential energies seem to undergo opposite changes as expected. Multiple complete orbits are observed as the kinetic and potential energy values oscillate due to a larger orbital time.

To minimize the number of trials a graphical method to determine the initial parameters of the 'slingshot' simulation was adapted. More explicitly, a simple menu was created, allowing the user to chose multiple values for a given parameter and plot multiple possible trajectories.

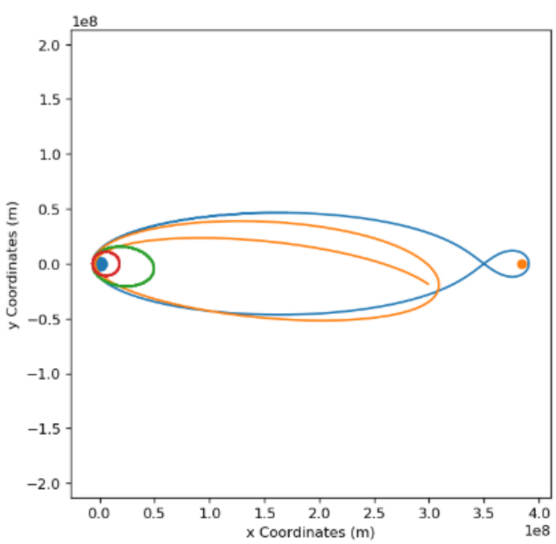

FIG. 10. Four trajectories simulated with different initial velocities, both vertical and horizontal. This method helped narrow down the appropriate initial velocities.

Initially, different vertical and horizontal position coordinates were tried out without success. This led to the conclusion of placing the far side of the Earth at the origin and try out different starting velocities. Four different trajectories are shown in Figure 10, one of which is the successful slingshot simulation. This allowed a point of reference to adjust previous velocity values by observing the shape of the trajectories.

It useful was to consider the sphere of influence of the Moon. The sphere of influence, r_{SOI}, is defined as the point in space where the gravitational influence of the Moon on a body is greater than that of the Earth [3]. This distance is approximated by,

$$r_{SOI} = a(\frac{M_M}{M_E})^{2/5}, \tag{14}$$

where a is the semi major axis of the Moon's orbit around the Earth and M_M, M_E the Moon's and Earth's masses. Using this relation, the distance at which the rocket must travel to be affected by the Moon's gravitational potential was determined. Thus, the rocket's velocity was adjusted to approximate this distance. The final result shown in Figure 7, was achieved by combining all methodologies discussed. The duration of the flight to the Moon and back to Earth can be found by observing the energy plots in Figure 8 and determining the time required for the kinetic or potential energy to reach their initial values. This time period was found to be approximately 9.68 days which is in agreement with results obtained elsewhere [4].

As the rocket enters the Moon's gravitational potential it undergoes slight changes in its kinetic and potential energy which are barely visible in Figure 8. This occurs just at the middle of the figure and corresponds to the small loop around the Moon shown in Figure 7 where the rocket is affected by Moon's gravity.

8. IMPROVEMENTS

There are many possible extensions to the present simulation. A more realistic approach may be attempted by allowing the Moon, Earth and rocket to interact simultaneously. However, this would be computationally more intensive as three second order differential equations would be required to be solved, resulting in a significantly greater number of Runge-Kutta coefficients. Furthermore, these interactions may be modelled in three dimensions, a task which perhaps might be beyond the scope of this work as it gets both computationally and mathematically more complex. Though a different, more sophisticated programming language would render this task more doable and straight forward.

The capabilities of the method used in the present work may be compared to other analytical methods and especially the leapfrog integrator. This is one of the most common methods applied to classical mechanics problems as in this case. Considering classical mechanics, the most complex form this simulation could take is the gravitational interaction of a system of N bodies, a task which requires novel computational methods and substantial processing power.

[1] Dormand, J. R.; Prince, P. J. (October 1978). ”New Runge-Kutta Algorithms for Numerical Simulation in Dynamical Astronomy”. Celestial Mechanics. 18 (3): 223–232.
[2] Pitjeva, Elena V. (August 2006). ”The dynamical model of the planet motions and EPM ephemerides”. Highlights of Astronomy. 14: 470.
[3] Seefelder, Wolfgang (2002). Lunar Transfer Orbits Utilizing Solar Perturbations and Ballistic Capture. Munich: Herbert Utz Verlag. p. 76.
[4] ”How many satellites are orbiting earth in 2018”. Pixalytics. 22 August 2018. Retrieved 27 September 2018.