ESC103: Midterm 1 Review

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1 Basic Vectors

The **linear combination** of vectors $\vec{v}$ and $\vec{w}$ is given by:

$$c\vec{v} + d\vec{v} \tag{1}$$

where c and d are scalars. Note that **vector addition** is both **associative** and **commutative.**

The length of a vector in $\vec{v}$ in $\mathbb{R}^N$ is given by:

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + V_N^2} \tag{2}$$

The **dot product** (also known as scalar product) is defined as:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 + \cdots + v_N w_N \tag{3}$$

for two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^N$. Using these, we can derive the following ideas and theorems.

Idea: The dot product of a vector with itself gives the square of its length:

$$\|\vec{v}\|^2 = \vec{v} \cdot \vec{v} \tag{4}$$

Idea: The **angle between two vectors** is given by:

$$\cos\theta = \frac{\vec{w} \cdot \vec{v}}{\|\vec{w}\|\|\vec{v}\|} \tag{5}$$

Idea: The **triangle inequality** is:

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{v}\| \tag{6}$$

Idea: The **Cauchy-Schwarz-Bunakowsky Inequality** is:

$$|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|\|\vec{w}\| \tag{7}$$

Idea: The **projection** of $\vec{w}$ on $\vec{v}$ can be written as:

$$\vec{u} = \mathsf{proj}_{\vec{v}}\vec{w} = \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|}\frac{\vec{v}}{\|\vec{v}\|} \tag{8}$$

You can view the last part $\frac{\vec{v}}{\|\vec{v}\|}$ as a unit vector pointing in the direction of $\vec{v}$ such that the **scalar projection** can be defined as:

$$|\vec{u}| = |\mathsf{proj}_{\vec{v}}\vec{w}| = \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|} \tag{9}$$

2 Plane Geometry

The **cross product** is defined as:

$$\vec{u} = \vec{v} \times \vec{w} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix} \tag{10}$$

which has a magnitude of $\|\vec{u} \times \vec{w}\| = \|\vec{u}\|\|\vec{w}\| \sin\theta$. The direction can be determined using the **right hand rule**. Cross products have the following properties:

Properties: The properties of a cross product is as follows.

- Consider 3 vectors, $\vec{v}$, $\vec{w}$, and $\vec{z}$. Then:

$$\vec{v} \times (\vec{w} + \vec{z}) = \vec{v} \times \vec{w} + \vec{v} \times \vec{z} \tag{11}$$

- The cross product is not commutative, but they are anti-commutative:

$$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v} \tag{12}$$

- When crossed with the zero vector, we have:

$$\vec{v} \times \vec{0} = \vec{0} \times \vec{v} = \vec{0} \tag{13}$$

- When multiplied by a scalar,

$$c(\vec{v} \times \vec{w}) = (c\vec{v}) \times \vec{w} = \vec{v} \times (c\vec{w}) \tag{14}$$

We can write out any point on a plane given a linear combination of two vectors $\vec{d_1}$ and $\vec{d_2}$ which span the plane:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} c\vec{d_1} + d\vec{d_2} \tag{15}$$

where $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ is any known point on the plane. Additionally, we can define a plane by its normal vector:

$$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{16}$$

by taking the cross product between any two vectors $\overrightarrow{P_0P}$ that lie on the plane such that:

$$\overrightarrow{P_0P} \cdot \vec{n} = 0 \tag{17}$$

TODO: Will revisit this part with more plane related things (e.g. projections onto a plane, distance from line to plane, alternate representations of planes.)

3 Matrix Multiplication and Linear Transformations

A matrix can be used to represent systems of linear equations. For example, the following set:

$$x - 2y = 1 \tag{18}$$
$$3x + 2y = 11 \tag{19}$$

can be represented by a **row picture**, which represents the classical "find the intersection" visual approach. It can also be represented via a **column picture**:

$$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} \tag{20}$$

It can also be represented via matrices:

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}}_{\text{Matrix A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{vector } \vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 11 \end{bmatrix}}_{\text{vector } \vec{b}} \tag{21}$$

We can perform **matrix multiplication** if A has n columns and B has n rows.

Idea: hen multiplying two matrices, the entry in row i and column j of AB is:

$$\text{(Row i of A)} \cdot \text{(column j of B)} \tag{22}$$

Recall that A and B can only be multiplied of A is $m \times n$ and B is $n \times p$. The size of the resulting matrix is therefore $m \times p$.

Properties:

1. $A + B = B + A$ (commutative)
2. $c(A + B) = cA + cB$ (where c is scalar)
3. $A + (B + C) = (A + B) + C$ (associative)
4. $C(A + B) = CA + CB$ (distributive from left)
5. $(A + B)C = AC + BC$ (distributive from right)
6. $A(BC) = (AB)C$ (associative)

Matrices can also be viewed as **linear transformations**. L represents a linear operator iff:

1. $L(c\vec{v}) = cL(\vec{v})$
2. $L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$

Idea: All linear transformations can be summarized by matrices and represented by matrix multiplication of a vector. We can determine the matrix associated with the transformation by analyzing what happens to the unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$, under the transformation.

Using the above information, we can show the following ideas:

Idea: The **projection** of vector $\vec{w}$ on $\vec{v}$ can be written using the linear transformation T_2 such that:

$$\vec{u} = T_2(\vec{w}) = \frac{1}{v_1^2 + v_2^2 + v_3^2} \begin{bmatrix} v_1^2 & v_1 v_2 & v_1 v_3 \\ v_2 v_1 & v_2^2 & v_2 v_3 \\ v_3 v_1 & v_3 v_2 & v_3^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \tag{23}$$

Idea: The **identity** matrix:

$$I(\vec{w}) = \vec{w} \tag{24}$$

where:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{25}$$

Idea: Suppose we have a transformation T_1 and T_2, we can define a **composition** of these transformations as:

$$T_3(\vec{v}) = T_2(T_1(\vec{v})) = M_{T_2} M_{T_1} \vec{v} = M_{T_3} \tag{26}$$

where M represents the matrix associated with the linear transformations.

Idea: We want to derive the **double angle** formulas with matrix multiplication. Suppose we wish to determine the matrix associated with the transformation of rotating a vector by an angle θ counterclockwise. We think about where the $\vec{i}$ and $\vec{j}$ vectors go to, which are $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ respectively, so the matrix associated with it is thus:

$$M_{T_\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{27}$$

which is known as the **rotation** matrix. So rotating an angle by 2θ is equivalent to applying the transformation $T_\theta(T_\theta(\vec{v}))$:

$$M_{T_\theta}^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{28}$$

or:

$$\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \tag{29}$$

4 Eigenvalues, Inverse, and Determinants

The motivation behind this section is that most vectors change direction when they are multiplied by a matrix, except a few certain ones which have very special properties.

Definition: **Eigenvectors** are special vectors associated with a certain transformation T such that they don't change directions under a linear transformation. We can denote these vectors $\vec{w}$ as solutions to the linear equation (where $\vec{w} \neq \vec{0}$):

$$M\vec{w} = \lambda\vec{w} \tag{30}$$

where the scalar λ is the **eigenvalue** of matrix M.

Many times, we can determine the eigenvectors and eigenvalues by drawing a picture and using intuition. In the lecture, the following matrices were used as examples:

- The projection matrix. (Answer: $\vec{w}$: any vector parallel or perpendicular to the projection. λ: 1 and 0 respectively.)
- A reflection matrix. (Answer: $\vec{w}$: any vector parallel or perpendicular to the line of reflection. λ: 1 and -1 respectively.)
- A rotation matrix. (Answer: None, at least in $\mathbb{R}^2$.)

Idea: However, sometimes intuition fails. We can solve the eigenvalue eigenvector equation as follows (this uses information from later on in this section):

$$M\vec{w} = \lambda\vec{w} \tag{31}$$

$$(M - I\lambda)\vec{w} = \vec{0} \tag{32}$$

Since $M - I\lambda$ is not invertible, then the determinant of $M - I\lambda$ is zero, or:

$$\det(M - \lambda I) = 0 \tag{33}$$

This is the equation we need to solve to find the eigenvalue λ.

Another problem in linear algebra is finding **inverses**. Suppose $\vec{u}$ and $\vec{T}$ were given and we wish to find $\vec{w}$ in the following equation:

$$\vec{u} = T(\vec{w}) \tag{34}$$

If we can find the inverse T^{-1} where $T^{-1}T = TT^{-1} = I$, then:

$$\vec{w} = T^{-1}(\vec{u}) \tag{35}$$

Idea: Calculating T^{-1} is equivalent to expanding $T^{-1}T$ and demanding each entry corresponds to the corresponding entry in the identity matrix. For example, if $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $T^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, then:

$$\begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{36}$$

which gives the system of four equations and four unknowns when expanded. It turns out that the inverse T^{-1} is:

$$T^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{37}$$

The quantity that we factor out $ad - bc$ is the **determinant** of the matrix, and in order for the inverse to exist, it cannot equal zero. If the inverse exists, we call it **invertible.**

Definition: The determinant of $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be written as:

$$\det(M) = ad - bc \tag{38}$$

A few other ideas that follow:

Idea: The inverse of the projection matrix does not exist. This can be interpreted both with determinants (rigorously) and geometrically (it's a irreversible process).

Idea: In the eigenvector eigenvalue equation:

$$M\vec{v} = \lambda\vec{v} \tag{39}$$

the quantity $M - \lambda I$ is not invertible.