

Medici - Solution Manual

QiLin Xue

Contents

1	The ABCs of Matrices	1
2	Elementary Matrices	3
3	Linear Equations	3
4	Vector Space	3

1 The ABCs of Matrices

Question 01:

(a) Let us write \mathbf{A} as:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad (1)$$

Then its transpose is:

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad (2)$$

Their sum is:

$$\mathbf{A} + \mathbf{A}^T = \begin{bmatrix} 2a_{11} & a_{21} + a_{12} & \cdots & a_{n1} + a_{1n} \\ a_{12} + a_{21} & 2a_{22} & \cdots & a_{n2} + a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} + a_{n1} & a_{n2} + a_{2n} & \cdots & 2a_{nn} \end{bmatrix} \quad (3)$$

which from inspection is symmetric. More rigorously, during the addition, each value a_{ij} in matrix \mathbf{A} , with $1 \leq i, j \leq n$, was added to a_{ji} such that in the sum $\mathbf{A} + \mathbf{A}^T$, the value indexed by ij is: $a_{ij} + a_{ji}$. For the sum to be symmetric, we must demand that:

$$(\mathbf{A} + \mathbf{A}^T)^T = \mathbf{A} + \mathbf{A}^T \quad (4)$$

To take the transpose, we swap the indices i and j such that if the value at a certain index was initially $a_{ij} + a_{ji}$, then after taking the transpose, it would be $a_{ji} + a_{ij}$, which is the same value as before.

(b) Similarly, if the value at a certain index in \mathbf{A} was initially a_{ij} , then the value at that same index in $\mathbf{A} - \mathbf{A}^T$ would be: $a_{ij} - a_{ji}$. For the sum to be skew-symmetric, the following property needs to be satisfied:

$$(\mathbf{A} - \mathbf{A}^T)^T = -(\mathbf{A} - \mathbf{A}^T) \quad (5)$$

After taking the transpose, the value at the same index as before would be $a_{ji} - a_{ij}$, which is equivalent to $-(a_{ij} - a_{ji})$, the negative of the previous value.

Question 02: We define the invertible matrix $P = ABC$ such that:

$$ABCP^{-1} = I \quad (6)$$

$$A \underbrace{(BCP^{-1})}_{A^{-1}} = I \quad (7)$$

$$(8)$$

$$(BCP^{-1}) = A^{-1} \quad (9)$$

$$(BCP^{-1})A = I \quad (10)$$

$$B \underbrace{(CP^{-1}A)}_{B^{-1}} = I \quad (11)$$

$$(CP^{-1}A) = B^{-1} \quad (12)$$

$$C \underbrace{(P^{-1}AB)}_{C^{-1}} = I \quad (13)$$

and we are done.

Question 03: We want to show that $AB = BA \iff (A + B)^2 = A^2 + 2AB + B^2$. First we show that this holds backwards:

$$(A + B)^2 = A^2 + AB + AB + B^2 \quad (\text{given}) \quad (14)$$

$$A^2 + AB + BA + B^2 = A^2 + AB + AB + B^2 \quad (15)$$

$$BA = AB \quad (\text{cancellation}) \quad (16)$$

Notice that the steps are reversible. We can explicitly complete the proof by working forwards. Assume $BA = AB$ is a given, then:

$$BA = AB \quad (\text{given}) \quad (17)$$

$$A^2 + AB + BA + B^2 = A^2 + AB + AB + B^2 \quad (18)$$

$$(A + B)^2 = A^2 + 2AB + B^2 \quad (\text{factor}) \quad (19)$$

Question 04: The case for $n = 1$ is trivial so we prove the base case for $n = 2$. Let $A_1 = [a_{ij}]$ be a $m \times n$ matrix and let $A_2 = [b_{ij}]$ be a $n \times p$ matrix. We then have:

$$(A_1 A_2)^T = \left(\left[\sum_{k=1}^n a_{ik} b_{kj} \right] \right)^T \quad (20)$$

$$= \left[\sum_{k=1}^n a_{ki} b_{jk} \right] \quad (21)$$

We also know that:

$$A_2^T A_1^T = [b_{ji}] [a_{ji}] \quad (22)$$

$$= \left[\sum_{k=1}^n b_{jk} a_{ki} \right] \quad (23)$$

and thus, the two are equivalent: $(A_1 A_2)^T = A_2^T A_1^T$. We now assume this holds true for $n = m$ and show that

it's true for $n = m + 1$, such that:

$$\left(\underbrace{\mathbf{A}_1 \mathbf{A}_2 \cdots \mathbf{A}_m}_{\mathbf{P}} \mathbf{A}_{m+1} \right)^T = (\mathbf{P} \mathbf{A}_{m+1})^T \quad (24)$$

$$= \mathbf{A}_{m+1}^T \mathbf{P}^T \quad (\text{base case}) \quad (25)$$

$$= \mathbf{A}_{m+1}^T (\mathbf{A}_m^T \cdots \mathbf{A}_2^T \mathbf{A}_1^T) \quad (\text{assumption}) \quad (26)$$

$$= \mathbf{A}_{m+1}^T \mathbf{A}_m^T \cdots \mathbf{A}_2^T \mathbf{A}_1^T \quad (27)$$

which is what we want. Since this is true for $n = 2$, then it must hold true for all $n \in \mathbb{Z}^+$.

Question 05:

(a) Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ so that the trace of the sum is:

$$\text{Tr}(\mathbf{A} + \mathbf{B}) = \sum_{i=1}^n (a_{ii} + b_{ii}) = \sum_{i=1}^n a_{ii} + \sum_{i=1}^n b_{ii} = \text{Tr} \mathbf{A} + \text{Tr} \mathbf{B} \quad (28)$$

where we have applied the linearity of the summation.

(b) Similarly, we have:

$$\text{Tr}(\mathbf{AB}) = \sum_{i=1}^n (a_{ii} b_{ii}) = \sum_{i=1}^n (b_{ii} a_{ii}) = \text{Tr} \mathbf{B} \text{Tr} \mathbf{A} \quad (29)$$

since scalar multiplication is commutative.

(c) We take the trace of both sides, using the properties we derived above:

$$\text{Tr}(\mathbf{AB} - \mathbf{BA}) = \text{Tr} \mathbf{0} \quad (30)$$

$$\text{Tr} \mathbf{AB} - \text{Tr} \mathbf{BA} = n \quad (31)$$

$$0 = n \quad (32)$$

Since $n > 0$, this is impossible.

2 Elementary Matrices

3 Linear Equations

4 Vector Space

Question 01: For the sake of practice, we will test all nine axioms:

1. (AC) Addition gives a vector in \mathbb{R}^2
2. (SC) Scalar multiplication gives a vector in \mathbb{R}^2
3. (AA) We can test for additive associativity via:

$$((x_1, x_2) \oplus (y_1, y_2)) \oplus (z_1, z_2) = (x_1 + y_1, 0) \oplus (z_1, 0) \quad (33)$$

$$= (x_1 + y_1 + z_1, 0) \quad (34)$$

and similarly:

$$(x_1, x_2) \oplus ((y_1, y_2) \oplus (z_1, z_2)) = (x_1, x_2) \oplus (y_1 + z_1, 0) \quad (35)$$

$$= (x_1 + y_1 + z_1, 0) \quad (36)$$

so this property also holds.

4. (Z) Let the zero vector be $\mathbf{0} = (z_1, z_2)$. Then we want:

$$(x_1, x_2) \oplus (z_1, z_2) = (x_1, x_2) \quad (37)$$

$$(x_1 + z_1, 0) = (x_1, x_2) \quad (38)$$

Since it's possible for $x_2 \neq 0$, this property will not hold and the set V is not a vector space.

5. (AI) Let the additive inverse of \mathbf{x} be $\mathbf{z} = (z_1, z_2)$. We want:

$$(x_1, x_2) \oplus (z_1, z_2) = (0, 0) \quad (39)$$

$$(x_1 + z_1, 0) = (0, 0) \quad (40)$$

If $z_1 = -x_1$, then $(-x_1, z_2)$ is an additive inverse of \mathbf{x} where $z_2 \in \mathbb{R}$ has no restrictions. However, there are infinitely many such inverses so this axiom is still violated as no unique inverse can be found.

6. (SMA) We want to show that:

$$c \odot (d \odot (x_1, y_1)) = (c \cdot d) \odot (x_1, y_1) \quad (41)$$

$$c \odot (dx_1, 0) = cd \odot (x_1, y_1)(cdx_1, 0) = (cdx_1, 0) \quad (42)$$

which is always true, so this axiom is satisfied.

7. (DVA) We want to show that:

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = (c \odot (x_1, y_1)) \oplus (c \odot (x_2, y_2)) \quad (43)$$

$$c \odot (x_1 + x_2, 0) = (cx_1, 0) \oplus (cx_2, 0) \quad (44)$$

$$(cx_1 + cx_2, 0) = (cx_1 + cx_2, 0) \quad (45)$$

This is always true, so this axiom is satisfied.

8. (DSA) We want to show that:

$$(c + d) \odot (x_1, y_1) = c \odot (x_1, y_1) \oplus d \odot (x_1, y_1) \quad (46)$$

$$((c + d)x_1, 0) = (cx_1 + dx_1, 0) \quad (47)$$

which is also always true.

9. (I) We want to show that:

$$1 \odot (x_1, y_1) = (x_1, y_1) \quad (48)$$

$$(x_1, 0) = (x_1, y_1) \quad (49)$$

Since it's possible to choose a $y_1 \neq 0$, this axiom is not satisfied.

Since this set V doesn't satisfy (Z), (AI), and (I), it is not a vector space.

Question 02:

(a) Let $\mathbf{0} = (z_1, z_2)$. We want:

$$(z_1, z_2) + (x_1, y_1) = (x_1, y_1) \quad (50)$$

$$(x_1 + z_1 + 1, y_1 + z_2 + 1) = (x_1, y_1) \quad (51)$$

The unique solution that accomplishes this is: $\mathbf{0} = (z_1, z_2) = (-1, -1)$.

(b) Let the additive inverse by $\mathbf{z} = (z_1, z_2)$. We want:

$$(2, 3) + (z_1, z_2) = (-1, -1) \quad (52)$$

$$(2 + z_1 + 1, 3 + z_2 + 1) = (-1, -1) \quad (53)$$

We get the additive inverse to be:

$$-(2, 3) = (-4, -5) \quad (54)$$

Question 03: Let $\mathbf{0} = (z_1, z_2)$. We want:

$$(z_1, z_2) + (x_1, y_1) = (x_1, y_1) \quad (55)$$

$$(x_1 + z_1 + k, y_1 + z_2 + k) = (x_1, y_1) \quad (56)$$

The unique solution that accomplishes this is: $\mathbf{0} = (z_1, z_2) = (-k, -k)$. Let the additive inverse of (x, y) $\mathbf{a} = (a_1, a_2)$. We want:

$$(x, y) + (a_1, a_2) = (-k, -k) \quad (57)$$

$$(x + a_1 + k, y + a_2 + k) = (-k, -k) \quad (58)$$

We get the additive inverse to be:

$$-(x, y) = (-x - 2k, -y - 2k) \quad (59)$$

Question 04: *Note:* I will be using Uppal's notation and proposition order (if applicable). See my notes for details.

$$-\mathbf{0} = -\mathbf{0} + (\mathbf{0} + -\mathbf{0}) \quad (Z) \quad (60)$$

$$= (-\mathbf{0} + \mathbf{0}) - \mathbf{0} \quad (AA) \quad (61)$$

$$= \mathbf{0} + -\mathbf{0} \quad (P3) \quad (62)$$

$$= \mathbf{0} \quad (Z) \quad (63)$$