Medici - Solution Manual

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Contents

1 The ABCs of Matrices

Question 01:

(a) Let us write $\boldsymbol{A}$ as:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{1}$$

Then its transpose is:

$$\boldsymbol{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{2}$$

Their sum is:

$$\boldsymbol{A} + \boldsymbol{A}^T = \begin{bmatrix} 2a_{11} & a_{21}+a_{12} & \cdots & a_{n1}+a_{1n} \\ a_{12}+a_{21} & 2a_{22} & \cdots & a_{n2}+a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}+a_{n1} & a_{n2}+a_{2n} & \cdots & 2a_{nn} \end{bmatrix} \tag{3}$$

which from inspection is symmetric. More rigorously, during the addition, each value a_{ij} in matrix $\boldsymbol{A}$, with $1 \leq i, j \leq n$, was added to a_{ji} such that in the sum $\boldsymbol{A} + \boldsymbol{A}^T$, the value indexed by ij is: $a_{ij} + a_{ji}$. For the sum to be symmetric, we must demand that:

$$(\boldsymbol{A} + \boldsymbol{A}^T)^T = \boldsymbol{A} + \boldsymbol{A}^T \tag{4}$$

To take the transpose, we swap the indices i and j such that if the value at a certain index was initially $a_{ij} + a_{ji}$, then after taking the transpose, it would be $a_{ji} + a_{ij}$, which is the same value as before.

(b) Similarly, if the value at a certain index in $\boldsymbol{A}$ was initially a_{ij}, then the value at that same index in $\boldsymbol{A} - \boldsymbol{A}^T$ would be: $a_{ij} - a_{ji}$. For the sum to be skew-symmetric, the following property needs to be satisfied:

$$(\boldsymbol{A} - \boldsymbol{A}^T)^T = -(\boldsymbol{A} - \boldsymbol{A}^T) \tag{5}$$

After taking the transpose, the value at the same index as before would be $a_{ji} - a_{ij}$, which is equivalent to $-(a_{ij} - a_{ji})$, the negative of the previous value.

Question 02: We define the invertible matrix $\boldsymbol{P} = \boldsymbol{ABC}$ such that:

$$\boldsymbol{ABCP}^{-1} = \boldsymbol{I} \tag{6}$$

$$\boldsymbol{A}\underbrace{\left(\boldsymbol{BCP}^{-1}\right)}_{\boldsymbol{A}^{-1}} = \boldsymbol{I} \tag{7}$$

$$\tag{8}$$

$$\left(\boldsymbol{BCP}^{-1}\right) = \boldsymbol{A}^{-1} \tag{9}$$

$$\left(\boldsymbol{BCP}^{-1}\right)\boldsymbol{A} = \boldsymbol{I} \tag{10}$$

$$\boldsymbol{B}\underbrace{\left(\boldsymbol{CP}^{-1}\boldsymbol{A}\right)}_{\boldsymbol{B}^{-1}} = \boldsymbol{I} \tag{11}$$

$$\left(\boldsymbol{CP}^{-1}\boldsymbol{A}\right) = \boldsymbol{B}^{-1} \tag{12}$$

$$\boldsymbol{C}\underbrace{\left(\boldsymbol{P}^{-1}\boldsymbol{AB}\right)}_{\boldsymbol{C}^{-1}} = \boldsymbol{I} \tag{13}$$

and we are done.

Question 03: We want to show that $\boldsymbol{AB} = \boldsymbol{BA} \iff (\boldsymbol{A}+\boldsymbol{B})^2 = \boldsymbol{A}^2 + 2\boldsymbol{AB} + \boldsymbol{B^2}$. First we show that this holds backwards:

$$(\boldsymbol{A}+\boldsymbol{B})^2 = \boldsymbol{A}^2 + \boldsymbol{AB} + \boldsymbol{AB} + \boldsymbol{B^2} \qquad \text{(given)} \tag{14}$$

$$\boldsymbol{A}^2 + \boldsymbol{AB} + \boldsymbol{BA} + \boldsymbol{B^2} = \boldsymbol{A}^2 + \boldsymbol{AB} + \boldsymbol{AB} + \boldsymbol{B^2} \tag{15}$$

$$\boldsymbol{BA} = \boldsymbol{AB} \qquad \text{(cancellation)} \tag{16}$$

Notice that the steps are reversible. We can explicitly complete the proof by working forwards. Assume $\boldsymbol{BA} = \boldsymbol{AB}$ is a given, then:

$$\boldsymbol{BA} = \boldsymbol{AB} \qquad \text{(given)} \tag{17}$$

$$\boldsymbol{A}^2 + \boldsymbol{AB} + \boldsymbol{BA} + \boldsymbol{B^2} = \boldsymbol{A}^2 + \boldsymbol{AB} + \boldsymbol{AB} + \boldsymbol{B^2} \tag{18}$$

$$(\boldsymbol{A}+\boldsymbol{B})^2 = \boldsymbol{A}^2 + 2\boldsymbol{AB} + \boldsymbol{B^2} \qquad \text{(factor)} \tag{19}$$

Question 04: The case for $n = 1$ is trivial so we prove the base case for $n = 2$. Let $\boldsymbol{A}_1 = [a_{ij}]$ be a $m \times n$ matrix and let $\boldsymbol{A}_2 = [b_{ij}]$ be a $n \times p$ matrix. We then have:

$$(\boldsymbol{A}_1\boldsymbol{A}_2)^T = \left(\left[\sum_{k=1}^{n} a_{ik}b_{kj}\right]\right)^T \tag{20}$$

$$= \left[\sum_{k=1}^{n} a_{ki}b_{jk}\right] \tag{21}$$

We also know that:

$$\boldsymbol{A}_2^T\boldsymbol{A}_1^T = [b_{ji}][a_{ji}] \tag{22}$$

$$= \left[\sum_{k=1}^{n} b_{jk}a_{ki}\right] \tag{23}$$

and thus, the two are equivalent: $(\boldsymbol{A}_1\boldsymbol{A}_2)^T = \boldsymbol{A}_2^T\boldsymbol{A}_1^T$. We now assume this holds true for $n = m$ and show that it's true for $n = m + 1$, such that:

$$\left(\underbrace{\boldsymbol{A}_1\boldsymbol{A}_2\cdots\boldsymbol{A}_m}_{\boldsymbol{P}}\boldsymbol{A}_{m+1}\right)^T = \left(\boldsymbol{P}\boldsymbol{A}_{m+1}\right)^T \tag{24}$$
$$= \boldsymbol{A}_{m+1}^T\boldsymbol{P}^T \qquad \text{(base case)} \tag{25}$$
$$= \boldsymbol{A}_{m+1}^T\left(\boldsymbol{A}_m^T\cdots\boldsymbol{A}_2^T\boldsymbol{A}_1^T\right) \qquad \text{(assumption)} \tag{26}$$
$$= \boldsymbol{A}_{m+1}^T\boldsymbol{A}_m^T\cdots\boldsymbol{A}_2^T\boldsymbol{A}_1^T \tag{27}$$

which is what we want. Since this is true for $n = 2$, then it must hold true for all $n \in \mathbb{Z}^+$.

Question 05:

(a) Let $\boldsymbol{A} = [a_{ij}]$ and $\boldsymbol{B} = [b_{ij}]$ so that the trace of the sum is:

$$\operatorname{Tr}(\boldsymbol{A}+\boldsymbol{B}) = \sum_{i=1}^{n}(a_{ii}+b_{ii}) = \sum_{i=1}^{n}a_{ii} + \sum_{i=1}^{n}b_{ii} = \operatorname{Tr}\boldsymbol{A} + \operatorname{Tr}\boldsymbol{B} \tag{28}$$

where we have applied the linearity of the summation.

(b) Similarly, we have:

$$\operatorname{Tr}(\boldsymbol{A}\boldsymbol{B}) = \sum_{i=1}^{n}(a_{ii}b_{ii}) = \sum_{i=1}^{n}(b_{ii}a_{ii}) = \operatorname{Tr}\boldsymbol{B}\operatorname{Tr}\boldsymbol{A} \tag{29}$$

since scalar multiplication is commutative.

(c) We take the trace of both sides, using the properties we derived above:

$$\operatorname{Tr}(\boldsymbol{A}\boldsymbol{B}-\boldsymbol{B}\boldsymbol{A}) = \operatorname{Tr}\boldsymbol{1} \tag{30}$$
$$\operatorname{Tr}\boldsymbol{A}\boldsymbol{B} - \operatorname{Tr}\boldsymbol{A}\boldsymbol{B} = n \tag{31}$$
$$0 = n \tag{32}$$

Since $n > 0$, this is impossible.

2 Elementary Matrices

3 Linear Equations

4 Vector Space

Question 01: For the sake of practice, we will test all nine axioms:

1. (AC) Addition gives a vector in $\mathbb{R}^2$
2. (SC) Scalar multiplication gives a vector in $\mathbb{R}^2$
3. (AA) We can test for additive associativity via:

$$((x_1,x_2)\oplus(y_1,y_2))\oplus(z_1,z_2) = (x_1+y_1,0)\oplus(z_1,0) \tag{33}$$
$$= (x_1+y_1+z_1,0) \tag{34}$$

and similarly:

$$(x_1,x_2)\oplus((y_1,y_2)\oplus(z_1,z_2)) = (x_1,x_2)\oplus(y_1+z_1,0) \tag{35}$$
$$= (x_1+y_1+z_1,0) \tag{36}$$

so this property also holds.

4. (Z) Let the zero vector be $\mathbf{0} = (z_1, z_2)$. Then we want:

$$(x_1, x_2) \oplus (z_1, z_2) = (x_1, x_2) \tag{37}$$
$$(x_1 + z_1, 0) = (x_1, x_2) \tag{38}$$

Since it's possible for $x_2 \neq 0$, this property will not hold and the set V is not a vector space.

5. (AI) Let the additive inverse of $\boldsymbol{x}$ be $\boldsymbol{z} = (z_1, z_2)$. We want:

$$(x_1, x_2) \oplus (z_1, z_2) = (0, 0) \tag{39}$$
$$(x_1 + z_1, 0) = (0, 0) \tag{40}$$

If $z_1 = -x_1$, then $(-x_1, z_2)$ is an additive inverse of $\boldsymbol{x}$ where $z_2 \in \mathbb{R}$ has nor restrictions. However, there are infinitely many such inverses so this axiom is still violated as no unique inverse can be found.

6. (SMA) We want to show that:

$$c \odot (d \odot (x_1, y_1)) = (c \cdot d) \odot (x_1, y_1) \tag{41}$$
$$c \odot (dx_1, 0) = cd \odot (x_1, y_1)(cdx_1, 0) \qquad = (cdx_1, 0) \tag{42}$$

which is always true, so this axiom is satisfied.

7. (DVA) We want to show that:

$$c \odot ((x_1, y_1) \oplus (x_2, y_2)) = (c \odot (x_1, y_1)) \oplus (c \odot (x_2, y_2)) \tag{43}$$
$$c \odot (x_1 + x_2, 0) = (cx_1, 0) \oplus (cx_2, 0) \tag{44}$$
$$(cx_1 + cx_2, 0) = (cx_1 + cx_2, 0) \tag{45}$$

This is always true, so this axiom is satisfied.

8. (DSA) We want to show that:

$$(c + d) \odot (x_1, y_1) = c \odot (x_1, y_1) \oplus d \odot (x_1, y_1) \tag{46}$$
$$((c + d)x_1, 0) = (cx_1 + dx_1, 0) \tag{47}$$

which is also always true.

9. (I) We want to show that:

$$1 \odot (x_1, y_1) = (x_1, y_1) \tag{48}$$
$$(x_1, 0) = (x_1, y_1) \tag{49}$$

Since it's possible to choose a $y_1 \neq 0$, this axiom is not satisfied.

Since this set V doesn't satisfy (Z), (AI), and (I), it is not a vector space.

Question 02:

(a) Let $\mathbf{0} = (z_1, z_2)$. We want:

$$(z_1, z_2) + (x_1, y_1) = (x_1, y_1) \tag{50}$$
$$(x_1 + z_1 + 1, y_1 + z_2 + 1) = (x_1, y_1) \tag{51}$$

The unique solution that accomplishes this is: $\mathbf{0} = (z_1, z_2) = (-1, -1)$.

(b) Let the additive inverse by $\boldsymbol{z} = (z_1, z_2)$. We want:

$$(2, 3) + (z_1, z_2) = (-1, -1) \tag{52}$$
$$(2 + z_1 + 1, 3 + z_2 + 1) = (-1, -1) \tag{53}$$

We get the additive inverse to be:

$$-(2, 3) = (-4, -5) \tag{54}$$

Question 03: Let $\mathbf{0} = (z_1, z_2)$. We want:

$$(z_1, z_2) + (x_1, y_1) = (x_1, y_1) \tag{55}$$
$$(x_1 + z_1 + k, y_1 + z_2 + k) = (x_1, y_1) \tag{56}$$

The unique solution that accomplishes this is: $\mathbf{0} = (z_1, z_2) = (-k, -k)$. Let the additive inverse of (x, y) $\boldsymbol{a} = (a_1, a_2)$. We want:

$$(x, y) + (a_1, a_2) = (-k, -k) \tag{57}$$
$$(x + a_1 + k, y + a_2 + k) = (-k, -k) \tag{58}$$

We get the additive inverse to be:

$$-(x, y) = (-x - 2k, -y - 2k) \tag{59}$$

Question 04: *Note:* I will be using Uppal's notation and proposition order (if applicable). See my notes for details.

$$-\mathbf{0} = -\mathbf{0} + (\mathbf{0} + -\mathbf{0}) \qquad (Z) \tag{60}$$
$$= (-\mathbf{0} + \mathbf{0}) - \mathbf{0} \qquad (AA) \tag{61}$$
$$= \mathbf{0} + -\mathbf{0} \qquad (P3) \tag{62}$$
$$= \mathbf{0} \qquad (Z) \tag{63}$$