

# ESC194: Midterm 1 Review

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Fall 2020

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# 1 Curve sketching

## 1.1 Formally Defining Horizontal Asymptotes

Horizontal asymptotes are formally defined as:

**Definition:** A horizontal asymptote occurs when  $\lim_{x \rightarrow \infty} f(x) = L$ . We can say that  $f(x)$  goes to  $L$  as  $x$  goes to infinity if for any  $\epsilon > 0$ , a number  $A$  can be found s.t. for all  $x > A$ ,  $|f(x) - L| < \epsilon$ .

The idea behind this revolves around finding  $f$  values as close to  $L$  as might be wanted by going to large enough  $x$  values.

An important theorem to determine horizontal asymptotes of reciprocal functions:

**Theorem:** The reciprocal horizontal asymptote limit:

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^r} = 0 \quad (1)$$

## 1.2 Prelims

We can use Fermat's theorem to determine critical points:

**Definition:**  $c$  is a critical point of  $f(x)$  if  $f'(c) = 0$  or  $f'(c)$  DNE.

Here are some key features that might be seen on a graph:

- **Concavity:** If the graph of  $y = f(x)$  lies above all its tangents in  $I$ , then  $f(x)$  is concave up in  $I$ . If it lies below, then it is concave down.
- **Cusp:** A point  $c$  is a cusp if  $f(x)$  is continuous at  $x = c$  but  $\lim_{x \rightarrow c^-} f(x) = \pm\infty$  and  $\lim_{x \rightarrow c^+} f(x) = \mp\infty$ .
- **Vertical Tangent:** A vertical tangent occurs when  $\lim_{x \rightarrow c} |f'(x)| = \infty$  and  $f(x)$  is continuous at  $c$ .
- **Slant Asymptote:** If  $\lim_{x \rightarrow \infty} [f(x) - (mx + b)] = 0$ , then  $y = mx + b$  is a slant asymptote to  $f(x)$  at  $+\infty$ .
- **Inflection point:** A point of inflection is at  $c$  if  $f(x)$  is continuous at  $c$  and the sign of concavity changes at  $c$ .

A function is increasing on an interval  $I$  if  $f(x_1) < f(x_2)$  for all  $x_1 < x_2$  in  $I$ . Although we can use this definition to find local max/mins, there are a few cutie (QT/quick test) ways to do so:

- **QT1: Increasing/Decreasing Test.** If  $f$  is differentiable on the interval  $I$ , we show that if  $f' > 0$ ,  $f$  is increasing. If  $f' < 0$ ,  $f$  is decreasing. If  $f' = 0$ ,  $f$  is constant.
- **QT2: First Derivative Test** Given that  $I$  contains a critical point and  $f$  is continuous at  $c_{\text{crit}}$ , and  $f$  is differentiable in  $I$  but not necessarily at  $c_{\text{crit}}$ . Then, if  $f' > 0$  to the left of  $c_{\text{crit}}$  and  $f' < 0$  to the right, then  $c_{\text{crit}}$  is a local max. If it's the opposite, we get the local minimum.
- **QT3: Concavity** Given that  $f(x)$  is twice differentiable on  $I$ , then  $f''(x)$  exists on  $I$ . As a result if  $f''(x) > 0$ ,  $f$  is concave up. If  $f'' < 0$ ,  $f$  is concave down.
- **QT4: Second Derivative Test** Given that  $f''(x)$  is continuous near  $c$  and  $f'(c) = 0$ , then if  $f''(c) > 0$ ,  $f(c)$  is a local minimum. If  $f''(c) < 0$ ,  $f(c)$  is a local maximum. If  $f''(c) = 0$ , there is no verdict.

In general, the recipe to test for local max and min is to:

- Find all  $c_{\text{crit}}$ .

- If QT4 applies, use it.
- If it doesn't, and if QT2 applies, use it.
- If QT2 doesn't apply, use the basic definition of increasing/decreasing.

### 1.3 Curve Sketching Steps

1. Determine general behaviour:
  - Find Domain / Range / Limits at  $\infty$ .
  - Determine endpoints if they exist.
  - Find vertical, horizontal, slant asymptotes if they exist:
2. Determine  $x$  and  $y$  intercepts.
3. Establish if  $f(x)$  is symmetrical, even, odd, and/or periodic.
4. Find  $f'(x)$  and use this to:
  - Find all critical points and  $f(c_{\text{crit}})$ .
  - Find when  $f(x)$  is increasing/decreasing.
  - Apply QT2.
  - Find vertical tangents / cusps if they exist.
5. Find  $f''(x)$  and use it to:
  - Find when  $f(x)$  is concave up/down.
  - Find points of inflection if they exist.
  - Optional: Use QT4 to confirm local max/min
6. Determine the absolute maximum and min by choosing the largest and smallest values of  $f$ , if they exist.

## 2 Optimization

Here is a checklist for solving optimization problems. If we want to optimize  $f$ :

- Check critical points.
- Check for endpoints.
- Check for local max, min.
- Check  $\lim_{x \rightarrow \infty}$  and  $\lim_{x \rightarrow -\infty}$ .
- Make a decision.

### 2.1 Numerical Methods

**Theorem:** The **method of successive bisections** can be performed if  $f$  is a continuous function and we can find values  $a$  and  $b$  such that  $f(b) < 0 < f(a)$ . These two values can be determined by trial and error. By IVT, the root must exist in between  $a$  and  $b$ . To use this method, we calculate the halfway point  $x_{h1}$ . If  $f(x_{h1})$  is positive, we replace  $a$  with  $x_{h1}$ . If it's negative, we replace  $b$  with  $x_{h1}$ .

**Theorem:** Using **Newton's Method** is much faster computationally. However, there is the added restriction that  $f(x)$  must be differentiable. It works in the following steps:

1. Make a first guess for the root,  $x_1$
2. Find the equation for the tangent line at  $(x_1, f(x_1))$
3. Find the  $x$  intercept of the tangent line, and let

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad (2)$$

and continue with  $x_2$ . Note however, that this doesn't always work such as when it diverges away from the root such as  $x^{1/3}$ .

Here are the overall steps that are recommended:

1. Try Newton's Method first if function is differentiable.
2. If the  $x_n$  values converge, great!
3. If they do not, try another value.
4. If they still diverge, use the method of successive bisections.

### 3 Formal Definition of an Integral

The summation notation is denoted below:

**Definition:** If  $a_m, a_{m+1}, a_{m+2}, \dots, a_n$  are real numbers and  $m$  and  $n$  are integers such that  $m \leq n$ , then:

$$\sum_{i=m}^n a_i = a_m + a_{m+1} + \dots + a_{n-1} + a_n \quad (3)$$

There are a few theorems:

- For a constant  $\alpha$ :

$$\sum_{i=m}^n \alpha a_i = \alpha \sum_{i=m}^n a_i \quad (4)$$

- It is also linear:

$$\sum_{i=m}^n (a_i + b_i) = \sum_{i=m}^n a_i + \sum_{i=m}^n b_i \quad (5)$$

- $\sum_{i=1}^n \alpha = \alpha n$

- $\sum_{i=1}^n i = \frac{n(n+1)}{2}$

- $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

- $\sum_{i=1}^n i^3 = \left( \frac{n(n+1)}{2} \right)^2$

- $\sum_{i=1}^n i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

One way of defining an integral is thinking of the area under the curve. This introduces the concept of a **Riemann Sum**:

$$\sum_{i=1}^n f(x_i^*) \Delta x_i \quad (6)$$

where  $x_i$  represents points in the **partition** of the domain in which we want to approximate the area. The approximation gets more and more precise as the size  $\Delta x_i$  decreases. A few technical definitions to help:

**Definition:** A **partition** is a finite subset of the closed interval  $[a, b]$ , which contains the points  $a$  and  $b$ . Denoted by  $P$ .

**Definition:** The **norm** of  $P = \|P\|$  which is the length of the longest subinterval:

$$\|P\| = \max(\Delta x_1, \Delta x_2, \dots, \Delta x_n) \quad (7)$$

Which can all be tied together to formally define the definite integral.

**Definition:** If  $f$  is a function defined on a closed interval  $[a, b]$ , let  $P$  be a partition of  $[a, b]$  with partition  $x_0, x_1, x_2, \dots, x_n$  where:

$$a = x_0 < x_1 < x_2 < \dots < x_n = b \quad (8)$$

Choose points  $x_i^*$  within each subinterval  $[x_{i-1}, x_i]$  and let  $\Delta x_i = x_i - x_{i-1}$ , and  $\|P\| = \max\{\Delta x_i\}$ . Then the **definite integral** of  $f$  from  $a$  to  $b$  is:

$$\int_a^b f(x) dx \equiv \lim_{\|P\|} \sum_{i=1}^n f(x_i^*) \Delta x_i \quad (9)$$

if the limit exists. If the limit does exist, then  $f$  is called integrable on the interval  $[a, b]$ . The sign  $\int$  is called the integral sign.  $f(x)$  is known as the **integrand**, and  $a, b$  are the limits of integration. The output is a single number that does not depend on  $x$ .

We can formally show that the definite integral can take on a specific value  $I$  with a  $\delta - \epsilon$  statement:

**Idea:** If we have:

$$\int_a^b f(x) dx = I \quad (10)$$

then for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that:

$$\left| I - \sum_{i=1}^n f(x_i^*) \Delta x_i \right| < \epsilon \quad (11)$$

for all partitions  $P$  of  $[a, b]$  with  $\|P\| < \delta$  and all possible choices of  $x_i^*$  in  $[x_{i-1}, x_i]$ .

However, going through this proof would be a nightmare. Instead, we can show **integrability** via the following theorem:

**Theorem:** Continuous and/or piecewise continuous on  $[a, b]$  guarantees integrability on  $[a, b]$ ,

**Definition:** A function is **piecewise continuous** if it only has a finite number of jump discontinuities.

Now that we know when the integral exists, we can find ways of calculating it from scratch:

**Idea:** Going through with a full Riemann sum calculation is also tedious. As a result, here are a few conventions to make it easier:

- We usually select regular partitions:

$$\Delta x = \Delta x_1 = \Delta x_2 = \cdots = \Delta x_n = \frac{b-a}{n} \quad (12)$$

- And we select  $x_i^*$  to be the RH end point such that:

$$x_i^* = x_i = a + i\Delta x = a + i\frac{b-a}{n} \quad (13)$$

Therefore, the integral can be written as:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(a + i\frac{b-a}{n}\right) \frac{b-a}{n} \quad (14)$$

## 4 Properties of Integration

### 4.1 Definite Integral Properties

There are a few properties:

- Constant:

$$\int_a^b c dx = c(b-a) \quad (15)$$

- Additivity:

$$\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx \quad (16)$$

- Constant Multiple:

$$\int_a^b c(f)x dx = c \int_a^b f(x) dx \quad (17)$$

- Changing Limits:

$$\int_a^b f(x) dx = \int_a^z f(x) dx + \int_z^b f(x) dx \quad (18)$$

There are also **order properties** of integrals. If  $a < b$ , then:

- If  $f(x) \geq 0$  for  $a \leq x \leq b$ , then

$$\int_a^b f(x) dx \geq 0 \quad (19)$$

- If  $f(x) \geq g(x)$  for  $a \leq x \leq b$ , then:

$$\int_a^b f dx \geq \int_a^b g(x) dx \quad (20)$$

- If  $m \leq f(x) \leq M$  for  $a \leq x \leq b$ , then:

$$m(b-a) \leq \int_a^b f dx \leq M(b-a) \quad (21)$$

- Absolute values:

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx \quad (22)$$

## 4.2 Fundamental Theorem of Calculus

The **first fundamental theorem of calculus** states that:

**Theorem:** Let  $f$  be continuous on  $[a, b]$ . The function  $F$  is defined on  $[a, b]$  by:

$$F(x) = \int_a^x f(t) dt \quad (23)$$

is continuous on  $[a, b]$ , differentiable on  $(a, b)$ , and has derivative:

$$F'(x) = f(x) \quad (24)$$

for all  $x \in (a, b)$ .

Rarely (never) will you get a simple question like this. Sometimes, the upper bound is a function  $g(x)$  instead. If this is the case, then:

**Idea:** Assuming that  $f$  is continuous in  $[a, b]$ , then the function  $F$  is defined on  $[a, b]$  by:

$$F(x) = \int_a^{g(x)} f(t) dt \quad (25)$$

has a derivative:

$$F'(x) = g'(x)f(g(x)) \quad (26)$$

for  $x \in (a, b)$ . To see why this is true, we can apply the chain rule:

$$F'(x) = \frac{d}{dx} \int_a^{g(x)} f(t) dt = g'(x)f(g(x)) \quad (27)$$

The **second fundamental theorem of calculus** states that:

**Theorem:** Let  $f$  be continuous on  $[a, b]$ . If  $G$  is any antiderivative for  $f$  on  $[a, b]$ , then:

$$\int_a^b f(t) dt = G(b) - G(a) \quad (28)$$

This can alternatively be written as:

$$\int_a^b F'(x) dx = F(b) - F(a) \quad (29)$$

which can be interpreted as the net change of  $F(x)$ . For example:

$$\Delta x = \int_a^b v(t) dt \quad (30)$$

gives the displacement from  $t = a$  to  $t = b$ . The proofs for these two theorems are provided below:

**Proof: (1st theorem)** For  $x$  and  $x + h$  in  $(a, b)$ ,

$$F(x + h) - F(x) = \int_a^{x+h} f(t) dt - \int_a^x f(t) dt \quad (31)$$

$$= \int_a^x f(t) dt + \int_x^{x+h} f(t) dt - \int_a^x f(t) dt \quad (32)$$

$$= \int_x^{x+h} f(t) dt \quad (33)$$



For  $h \neq 0$ , we have:

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) dt \quad (34)$$

We can separate it into cases. If  $h > 0$ , then we can write per the extreme value theorem the minimum value of  $f$  as  $f(u) = m$  and the maximum value as  $f(v) = M$  for  $u, v \in [x, x+h]$  such that:

$$mh \leq \int_x^{x+h} f(t) dt \leq Mh \quad (35)$$

or:

$$f(u)h \leq \int_x^{x+h} f(t) dt \leq f(v)h \quad (36)$$

which we can rewrite, after dividing through by  $h$ :

$$f(u) \leq \frac{F(x+h) - F(x)}{h} \leq f(v) \quad (37)$$

As  $h \rightarrow 0$ , we have  $u \rightarrow x$  and  $v \rightarrow x$ . Therefore:

$$\lim_{h \rightarrow 0} f(u) = \lim_{u \rightarrow x} f(u) = f(x) \quad (38)$$

$$\lim_{h \rightarrow 0} f(v) = \lim_{v \rightarrow x} f(v) = f(x) \quad (39)$$

which gives us:

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = f(x) \quad (40)$$

or:

$$\frac{d}{dx} \int_a^x f(t) dt = f(x) \quad (41)$$

**Proof: (2nd theorem)** Given that  $F(x) = \int_a^x f(t) dt$  is an antiderivative of  $f$  and given that  $G$  is an antiderivative, then:

$$F'(x) = G'(x) \implies F(x) = G(x) + C \quad (42)$$

We know that  $F(a) = 0$ , so  $G(a) + C = 0$  or  $C = -G(a)$ , which gives:

$$\int_a^b f(t) dt = F(b) = G(b) - G(a) \quad (43)$$

### 4.3 Integration Tricks

The **u-substitution** essentially reverses the chain rule.

**Idea:** Suppose we have an integral in the form:

$$\int f(g(x))g'(x) dx \quad (44)$$

If we let  $u = g(x)$ , then  $du = g'(x)dx$ . So we can simplify the integral to:

$$\int f(u) du = F(u) + C = F(g(x)) + C \quad (45)$$

Once we have the indefinite integral, we can use back substitution to find the definite integral. We can avoid this step using a change of variables.

**Theorem:**

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du \quad (46)$$

In general, here are a few tips, in no particular order:

- Refer to the table of integrals at the back of the book. You are allowed to use them.
- Look for symmetry and periodicity.
- Draw a picture. Sometimes, you can avoid a complicated integral and use plain old geometry this way!
- For u-substitution, look for derivative-function pairs.
- If there are not too many terms, you can sometimes expand functions into a polynomial.
- Check if the integral even exists!
- Apply the first theorem of calculus, if applicable.
- See if the integral (or a similar one) is in the book.

## 5 Areas and Volumes

### 5.1 Areas Between Curves

Suppose we wish to find the **area between two curves**  $f(x)$  and  $g(x)$ . We can do this by partitioning the area into infinitesimally small rectangles:

$$\Delta A_i = [f(x_i^*) - g(x_i^*)] \Delta x_i \quad (47)$$

so that the area is given by:

$$A = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n [f(x_i^*) - g(x_i^*)] \Delta x_i \quad (48)$$

$$= \int_a^b [f(x) - g(x)] dx \quad (49)$$

If we let  $f(x) \geq g(x)$  when  $x \in [a, b]$ , then this gives the difference in their areas  $A_1 - A_2$ . If the condition  $f(x) \geq g(x)$  is not satisfied, then we must break up the integral into multiple parts (if we interpret the area as having a positive area only). We can modify the area formula to be:

$$A = \int_a^b |f(x) - g(x)| dx \quad (50)$$

Suppose we have a curve  $x = f(y)$  and  $x = g(y)$  instead. The area between  $y = a$  and  $y = b$  works in the same way:

$$A = \int_a^b |f(y) - g(y)| dy \quad (51)$$

### 5.2 Volumes

We can determine the **volume** of a solid by partitioning it into thin cylinders whose axes are parallel to the  $x$  axis. Then we can break up the volume into thin sections:

$$V_i \simeq A_i \Delta x_i \quad (52)$$

so the volume is:

$$V = \int_a^b A(x) dx \quad (53)$$

which is the general formula for the volume of any shape. If we can figure out  $A(x)$  and the necessary bounds, we can find the volume for any change.

**Idea:** For **solids of revolution**, we rotate a curve  $f(x)$  about the  $x$  axis. The volume of this solid using the **disk method** is:

$$V = \int_a^b \pi f(x)^2 dx \quad (54)$$

Similarly around the  $y$  axis:

$$V = \int_c^d \pi g(y)^2 dy \quad (55)$$

For the volume by rotating the region between two curves  $f(x)$  and  $g(x)$ , we get:

$$V = \int_a^b \pi (f(x)^2 - g(x)^2) dx \quad (56)$$

which is known as the **washer method**.

Sometimes, the disk and washer method is too difficult to apply.

**Idea:** We can use the **shell method about the y-axis** to find the volume when a curve is rotated about the  $y$  axis. Suppose we wish to rotate a curve  $f(x)$  from  $x = a$  to  $x = b$  around the  $y$  axis. Then the volume is:

$$V = \int_a^b 2\pi x f(x) dx \quad (57)$$

Similarly, if a curve is rotated about the  $x$  axis, we use the **shell method about the x-axis**:

$$V = \int_a^b 2\pi y f(y) dy \quad (58)$$

## 6 Misc

I honestly don't know where this section belongs, so I'm just copying and pasting from my notes (which I actually spent a decent amount of effort on):

- The average of a discrete set  $\{a_1, a_2, \dots, a_N\}$  is given by:

$$a_{\text{avg}} = \frac{1}{N} \sum_i^N a_i \quad (59)$$

- For a continuous distribution, we can extend this to:

$$f_{\text{avg}} = \frac{1}{N} \sum_i^N f(x_i^*) \quad (60)$$

Taking the limit as  $N \rightarrow \infty$ , we get:

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx \quad (61)$$

**Theorem: Mean Value Theorem for Integrals:** If  $f$  is continuous on  $[a, b]$ , then there exists a number  $c$  in  $[a, b]$  such that:

$$f(c) = f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx \quad (62)$$

**Proof:** Define  $F(x) = \int_a^x f(t) dt$ . If we apply the mean value theorem to  $F$ , then:

$$F'(c) = \frac{F(b) - F(a)}{b - a} \quad (63)$$

for some  $c \in [a, b]$ . Now since:

$$F'(x) = f(x) \quad (64)$$

it becomes apparent that:

$$f(c) = \frac{\int_a^b f(t) dt - \int_a^a f(t) dt}{b - a} = \frac{1}{b - a} \int_a^b f(t) dt \quad (65)$$

- We can also introduce **inverse functions**.

**Definition:** A function  $f(x)$  is said to be one-to-one if  $f(x_1) = f(x_2)$  implies  $x_1 = x_2$ . Alternatively, we can say that  $f(x_1) \neq f(x_2)$  whenever  $x_1 \neq x_2$ .

- We can use the **horizontal line test**. If any horizontal line crosses the function more than one time, then it is not one-to-one.

**Definition:** Let  $f$  be a 1-1 function with domain  $A$  and range  $B$ . Then its inverse function,  $f^{-1}$  has domain  $B$  and range  $A$ , and is defined by:

$$f^{-1}(x) = y \iff f(y) = x \quad (66)$$

Therefore:

$$f^{-1}(f(x)) = f(f^{-1}(x)) = x \quad (67)$$

**Warning:** To prevent confusion, not that:

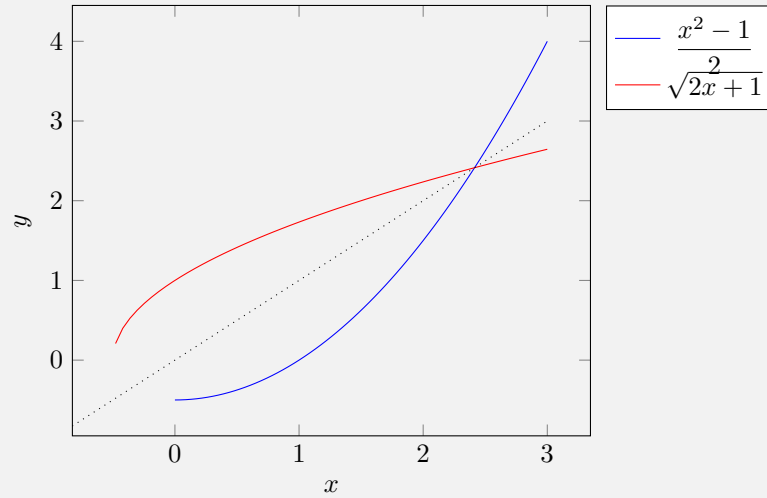
$$\frac{1}{f(x)} = [f(x)]^{-1} \neq f^{-1}(x) \quad (68)$$

- Geometrically, the inverse of a function represents a reflection of each point across the line  $y = x$ .

**Example 1:** If  $g(x) = \sqrt{2x + 1}$ , it is implied that  $x \geq -1/2$ , so it is a one-to-one function. Therefore, the inverse function is:

$$g^{-1}(x) = \frac{x^2 - 1}{2} \quad (69)$$

### Inverse Function Example



**Theorem:** If  $f$  is either an increasing or decreasing function, then  $f$  is 1-1, and hence, has an inverse.

*Proof.* Say  $f(x)$  is decreasing, then  $x_1 < x_2 \implies f(x_1) > f(x_2)$  and if  $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$ .  $\square$

**Theorem:** Let  $f$  be a 1-1 function defined on an interval  $I$ . If  $f$  is continuous, then  $f^{-1}$  is also continuous. (Proof provided in Appendix F)

- Let  $g(x) = f^{-1}(x)$ . Then:

$$f(g(x)) = x \quad (70)$$

$$\frac{d}{dx} f(g(x)) = 1 \quad (71)$$

$$f'(g(x))g'(x) = 1 \quad (72)$$

$$g'(x) = \frac{1}{f'(g(x))} \quad (73)$$

or:

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \quad (74)$$

which is equivalent to:

$$\frac{dy}{dx} = \frac{1}{\frac{dy}{dx}} \quad (75)$$

**Theorem:** The inverse of composite functions is given by:

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1} \quad (76)$$

*Proof.* Let  $y = (f \circ g)^{-1}(x)$ . Then:

$$x = (f \circ g)(y) = f(g(y)) \quad (77)$$

so we have:

$$g(y) = f^{-1}(x) \quad (78)$$

$$y = g^{-1}(f^{-1}(x)) \quad (79)$$

$$= (g^{-1} \circ f^{-1})(x) \quad (80)$$

$\square$

## 7 Logarithms and Exponentials

**Warning:** Note that in this section, I make the assumption you are already familiar with general logarithm and exponential properties, so I won't spend time writing those down.

**Definition:** A **logarithm function** is a nonconstant differentiable function  $f$  defined for  $x \in \{\mathbb{R}, (0, \infty)\}$  such that for all  $a > 0$  and  $b > 0$ :

$$f(a \cdot b) = f(a) + f(b) \quad (81)$$

It has the following properties:

- $f(1) = 0$
- $f(1/x) = -f(x)$
- $f(x/y) = f(x) - f(y)$
- $f'(x) = \frac{1}{x} f'(1)$ .

This leads to the definition of the **natural logarithm**:

**Definition:** The natural logarithm is defined as:

$$\ln(x) = \int_1^x \frac{dt}{t} \quad (82)$$

Note that  $\ln(x)$  is not the antiderivative of  $\frac{1}{t}$ . We can instead write:

$$\int \frac{dt}{t} = \ln|x| + C \quad (83)$$

since  $x$  can be negative as well.

**Theorem: Feynman's trick of Differentiation<sup>a</sup>** (otherwise known as logarithmic differentiation): The following was popularized by Richard Feynman during the first of his Feynman Lectures. If we have a function:

$$g(x) = g_1(x)g_2(x)g_3(x) \cdots g_n(x) \quad (84)$$

Then taking the natural logarithm of both sides, applying the chain rule, and simplifying gives:

$$g'(x) = g(x) \left( \frac{g'_1}{g_1} + \frac{g'_2}{g_2} + \cdots + \frac{g'_n}{g_n} \right) \quad (85)$$

<sup>a</sup>Note that this is not a formal name. I just chose it to name it after Feynman because I'm a huge Feynman stan and I first heard about it in the preface to the Feynman Lectures where he was talking about mathematical tricks.

Exponential functions can be introduced:

**Definition:** If  $z$  is a real number, then  $e^z$  is the number such that:

$$\ln(e^z) = z \quad (86)$$

More formally, we can write the exponential function as  $\exp(x) = e^x$ . The most useful property of  $e^x$  is that:

$$\frac{d}{dx} e^x = e^x \quad (87)$$

We can extend this to general logarithmic and exponential functions. If  $x > 0$ , then we can define:

**Definition:** The **general exponential** function is defined as

$$x^z = e^{z \ln x} \quad (88)$$

if  $x > 0$ .

Similarly:

**Definition:** The **general logarithm** can be defined as:

$$\log_p(x) = \frac{\ln x}{\ln p} \quad (89)$$

such that:

$$\frac{d}{dx} a^x = \ln(a) a^x \quad (90)$$

and

$$\frac{d}{dx} \log_p(x) = \frac{1}{x \ln p} \quad (91)$$

## 7.1 Bounding $e$

**Idea:** We can first bound  $e^x$  by setting a lower limit (which happens to be the Taylor series!). Notice that via integration:

$$e^x = 1 + \int_0^x e^t dt \quad (92)$$

Since  $e^x$  is always increasing, we can claim that  $e^x > 1$  for  $x > 0$  such that:

$$e^x = 1 + \int_0^x e^t dt > 1 + \int_0^x 1 dt = 1 + x \quad (93)$$

We can then repeat the previous step to show that

$$e^x = 1 + \int_0^x e^t dt > 1 + \int_0^x (1 + t) dt = 1 + x + \frac{x^2}{2} \quad (94)$$

Repeating the process, we eventually get:

$$e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \quad (95)$$

Instead of choosing to bound  $e^x$ , we can also choose to bound  $e$ . We have that:

$$\ln x = \int_1^x \frac{dt}{t} \quad (96)$$

such that:

$$\ln \left( 1 + \frac{1}{n} \right) = \int_1^{1+1/n} \frac{dt}{t} < \int_1^{1+1/n} 1 dt \quad (97)$$

Since  $\frac{1}{t} < 1$  for  $t > 0$ . The upper bound then becomes:

$$1 + \frac{1}{n} - 1 = \frac{1}{n} \implies \ln \left( 1 + \frac{1}{n} \right) < \frac{1}{n} \quad (98)$$

We can similarly repeat this process:

$$1 + \frac{1}{n} < e^{1/n} \implies \left( 1 + \frac{1}{n} \right)^n < e \quad (99)$$

Note that if we take the limit as  $n \rightarrow \infty$ , *intuitively* we would expect the upper bound to become closer and closer to the true value. We shall explore this further, and we can write the lower bound as:

$$\ln \left( 1 + \frac{1}{n} \right) = \int_1^{1+1/n} \frac{dt}{t} > \int_1^{1+1/n} \frac{dt}{1+1/n} \quad (100)$$

since  $\frac{1}{t} > \frac{1}{1+1/n}$ . We can write this in logarithm form to get:

$$\ln \left( 1 + \frac{1}{n} \right) > \left( \frac{1}{1+1/n} \right) \left( 1 + \frac{1}{n} - 1 \right) = \frac{1}{n+1} \implies \left( 1 + \frac{1}{n} \right)^{n+1} > e \quad (101)$$

Putting it altogether, we have the following statement:

**Idea:**  $e$  can be estimated with its lower and upper bound with the following:

$$\left( 1 + \frac{1}{n} \right)^n < e < \left( 1 + \frac{1}{n} \right)^{n+1} \quad (102)$$

## 8 Inverse Trigonometric Functions

We can define the inverse function of trigonometric functions by restricting their domain, such as from  $-\pi/2$  to  $\pi/2$  for  $\sin(x)$ .

**Definition:** The inverse function for  $\sin(x)$  is given by :

$$\sin^{-1}(x) = \arcsin(x) \quad (103)$$

**Warning:** You need to be very careful with the domain and range. Sometimes, if  $x$  falls out of the domain, it can lead to a different answer altogether, or it could be undefined.

There's a lot of formulas for this one, but to derive formula such as  $\sin(\tan^{-1}(x))$ , you just need to draw a picture of a right angled triangle with one of the legs as  $x$  and either the hypotenuse or the other leg as 1. If proofs are not needed, there is a formula sheet with all properties at the end of the book.