MAT185H1S - Linear Algebra
Winter 2021

Notes on Linear Differential Equations with Constant Coefficients (Part II):

Definition: A system of the form

$$\begin{aligned}
x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t), & x_1(0) &= b_1 \\
x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2n}x_n(t), & x_2(0) &= b_2 \\
&\vdots \\
x_n'(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \cdots + a_{nn}x_n(t), & x_n(0) &= b_n
\end{aligned}$$

where each $x_i(t)$ is real-valued function of a real variable, is called a *system of linear differential equations with constant coefficients.*

Writing $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, a vector-valued function of t, and $A = [a_{ij}]$ for the matrix of the coefficients of the system, we may represent the system as

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

Extrapolating our observations and results from the 2×2 case, we give a solution in the case where A is diagonalizable.

Lemma: Let a be an $n \times n$ matrix. If $\mathbf{x}_0$ is an eigenvector of A with eigenvalue λ, then the system $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ has solution $\mathbf{x}(t) = e^{\lambda t}\mathbf{x}_0$.

This lemma yields:

Theorem: Let A be a $n \times n$ diagonalizable matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct). Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for $\mathbb{R}^n$ consisting of eigenvectors of A. If $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$, then the system $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ has solution

$$\mathbf{x}(t) = e^{\lambda_1 t}(c_1\mathbf{v}_1) + e^{\lambda_2 t}(c_2\mathbf{v}_2) + \cdots + e^{\lambda_n t}(c_n\mathbf{v}_n) \tag{1}$$

Proof: As in the 2×2 case, all you need to prove both the Lemma and the Theorem is show $\mathbf{x}' = A\mathbf{x}$.

Example: (cf. Notes on Diagonalization (Part II)). The matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ is diagonalizable and $\alpha = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$ is a basis for ${}^3\mathbb{R}$ consisting of eigenvectors of A corresponding to the eigenvalues 2, 2, and 1 respectively.

For any vector $\mathbf{x}_0 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in {}^3\mathbb{R}$ we can write it uniquely as:

$$\mathbf{x}_0 = (b_1 + 2b_2) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (b_2 + b_3) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - b_2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

as a linear combination of the basis vectors in α.

By the previous theorem the system $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ has the solution

$$\begin{aligned} \mathbf{x}(t) &= e^{2t}(b_1 + 2b_2) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e^{2t}(b_2 + b_3) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - e^{t}(b_2) \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{2t} & 2(e^{2t} - e^{t}) & 0 \\ 0 & e^{t} & 0 \\ 0 & e^{2t} - e^{t} & 0 \end{bmatrix} \mathbf{x}_0 \end{aligned}$$

Compare this with how we computed powers of A.

From a similarity point of view, the matrix $S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ (whose columns are the basis vectors in α) is invertible; and the matrix $D = \mathrm{diag}(2, 2, 1)$ are such that:

$$A = SDS^{-1}$$

and the system $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$ has solution

$$\mathbf{x}(t) = S\Lambda S^{-1}\mathbf{x}_0$$

where $\Lambda = \mathrm{diag}(e^{2t}, e^{2t}, e^{t})$.

Exercise and Discussion: Let A be a diagonalizable matrix and write $A = SDS^{-1}$ where D is diagonal. Show that if $\mathbf{y}(t)$ is a solution to the system

$$\mathbf{y}' = D\mathbf{y}, \quad \mathbf{y}(0) = S^{-1}\mathbf{x}_0$$

then $\mathbf{x}(t) = S\mathbf{y}(t)$ is a solution to the system

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0$$

There are two nice applications of the result from previous exercise. The first is....

Exercise and Discussion: Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

(a) Show that A is diagonalizable and find an invertible matrix S and diagonal matrix D such that $A = SDS^{-1}$

(b) Sketch solutions to $\mathbf{y}' = D\mathbf{y}$ for various $\mathbf{y}(0) = \mathbf{y}_0$.

(c) Applyying the mapping defined by multiplication by S to the path $\mathbf{y}(t)$ and sketch solutions to $\mathbf{x}' = A\mathbf{x}$ for various $\mathbf{x}(0) = \mathbf{x}_0$.

(a) Find the eigenvalues by finding the characteristic polynomial:

$$P_\lambda(A) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{bmatrix} = (\lambda - 2)^2 - 1 = (\lambda - 1)(\lambda - 3)$$

The eigenvalues of A are $\lambda = 1, 3$. We cana then find the eigenspace:

$$E_3 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \text{span} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \text{span} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

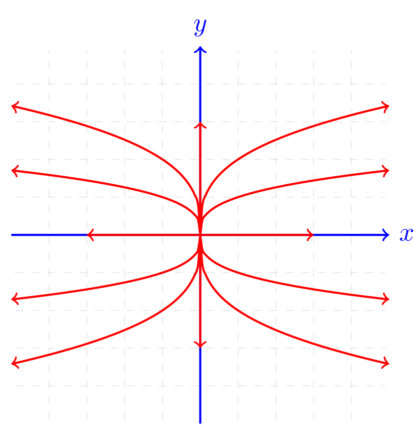
Therefore, $S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, which is invertible and the diagonal matrix is $D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) We wish to find the solution to: $\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad y(0) = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. From the previous theorem, the solution is:

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} b_1 e^{3t} \\ b_2 e^{t} \end{bmatrix} \tag{2}$$

(c) We can plot this out. If we start on one of the axes, we will move on that axis. Otherwise, we have $b_1, b_2 \neq 0$ so $y = b_2 e^t \implies t = \ln \frac{y}{b_2}$. We can then write a function in terms of x:

$$x = b_1 e^{3t} = b_1 \left(\frac{y}{b_2} \right)^3 \tag{3}$$



...and the second is

Exercise and Discussion: Let A be diagonalizable matrix.

(a) Show that the only solution to the system $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{0}$ is the constant solution $\mathbf{x}(t) = \mathbf{0}$.

(b) Using part (a) show that if $\mathbf{x}(t)$ and $\hat{\mathbf{x}}(t)$ are two solutions to the system $\mathbf{x}' = A\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$, then $\mathbf{x}(t) = \hat{\mathbf{x}}(t)$.

(a) A is diagonalizable so we can write it as $A = SDS^{-1}$. If $y = S^{-1}x$, then it solves $y' = Dy$. If $x(0) = 0$), then $y(0) = S^{-1}x(0) = S^{-1}(0) = 0$ as well. However, $y' = Dy$ and since $y(0) = 0$, this means that $y(t) = 0$ for all t (coefficients of general solution will all be zero).

(b) **(Uniqueness of Solutions)**