ESC103: Mathematics and Computation Notes

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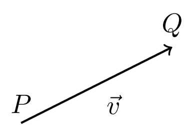
1 Lecture 1

- A **vector** is a quantity with a magnitude, a direction, and units.
- A **scalar** is a quantity with a magnitude, a sign, and units.

> **Idea**: At the heart of linear algebra is two operations and they both deal with vectors. We *add* vectors and we multiply vectors by scalars.

- We can draw a vector from point P to point Q by drawing an arrow:



where P is the **tail** and Q is the **head** of the vector. The vector can be written algebraically as:

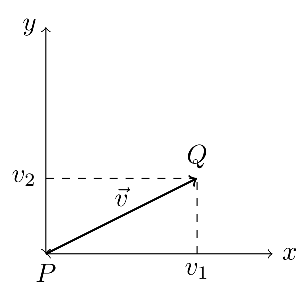
$$\vec{v} = \vec{PQ} \tag{1}$$

Note that this is not equal to:

$$\vec{PQ} \neq \overrightarrow{QP} \tag{2}$$

because even though their magnitudes are the same, their direction is not.

- We can draw a two-dimensional vector in $\mathbb{R}^2$ by translating the vector $\vec{v}$ to the origin (since we are not changing direction or magnitude).



By breaking it up into components, we can write $\vec{v}$ as a column vector:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{3}$$

Note that this is not the same as a row vector:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq \begin{bmatrix} v_1 & v_2 \end{bmatrix} \tag{4}$$

- To add two vectors, we can add them components wise:

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} \tag{5}$$

  and similarly for subtraction:

$$\vec{v} - \vec{w} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 - w_1 \\ v_2 - w_2 \end{bmatrix} \tag{6}$$

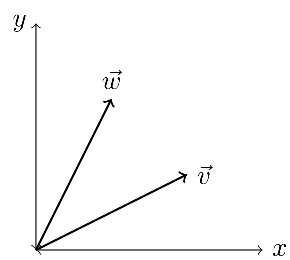
- To multiply a vector by a scalar, we multiply each component by that scalar:

$$c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix} \tag{7}$$

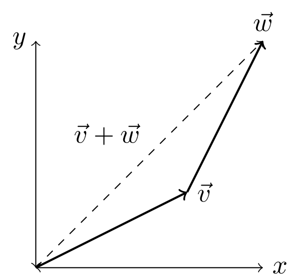
- Note that subtraction is just a combination of addition and scalar multiplication:

$$\vec{v} - \vec{w} = \vec{v} + (-1) \cdot \vec{w} \tag{8}$$

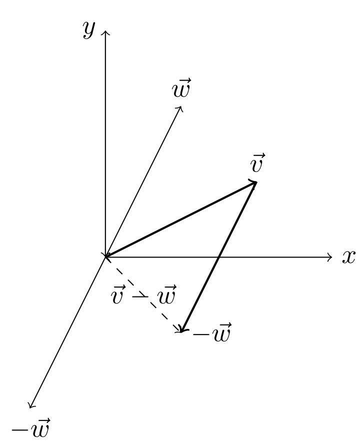
- Geometrically, we can add two vectors $\vec{v}$ and $\vec{w}$:



  We can do this by translating one of the vectors and add them tip to tail:

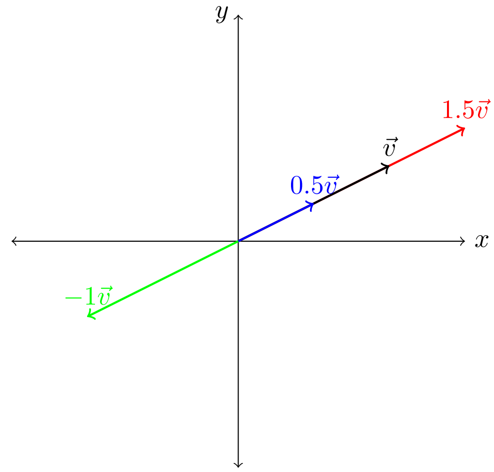


- To subtract two vectors, we first flip the second vector and then add. Suppose we wish to represent $\vec{v} - \vec{w}$ geometrically, then:



  Note that this is equivalent from adding tail to tail.

- To multiply a vector by a scalar, we do not change the direction but instead we change the magnitude or length. Geometrically:



where all vectors originate from the origin.

- The **zero vector** is defined as:

$$\vec{0} \equiv \vec{v} - \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9}$$

> **Idea**: Scalar addition can be represented as the addition of one dimensional vectors in $\mathbb{R}^1$. For example, the number line is essentially just a coordinate axis and we can represent numbers in a similar way:
>
> $$3 = 2 + 1 \tag{10}$$
>
> and geometrically:
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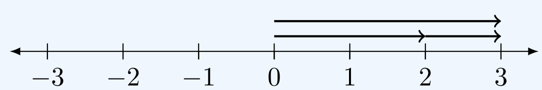
> 
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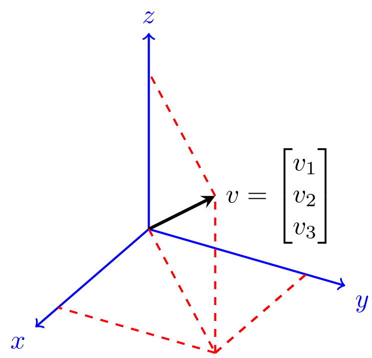
2 Lecture 2

- The heart of linear algebra are **linear combinations**. Let c and d be scalars. Then:

$$c\vec{v} + d\vec{w} \tag{11}$$

  is a linear combination (LC) of $\vec{v}$ and $\vec{w}$.

- For vectors in three-dimensional ($\mathbb{R}^3$)



- The **transpose** of a matrix changes swaps the ig index with the ji index such that:

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{12}$$

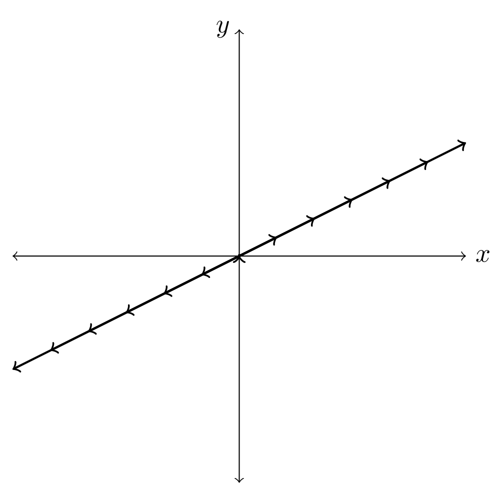
- Addition, subtraction, and multiplication with scalars behave in the same ways.
- There are a few properties that vectors behave:
  - The **commutative property** says that:

$$\vec{v} + \vec{w} + w \vec{+} \vec{v} \tag{13}$$

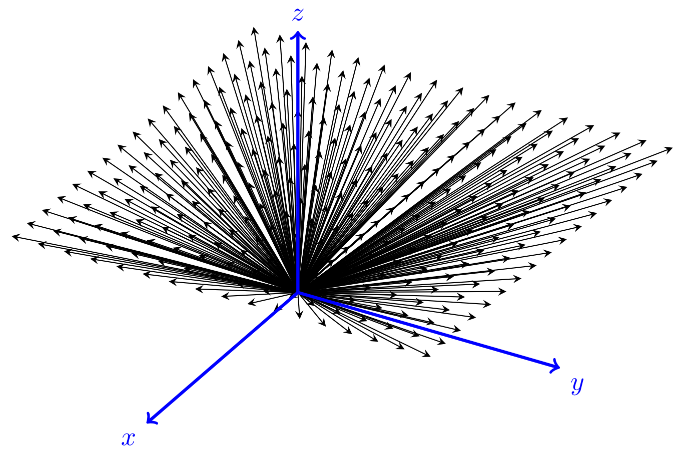
  - The **associative property** says that;

$$\vec{v} + \vec{w} + \vec{z} = (\vec{v} + \vec{w}) + \vec{z} = \vec{v} + (\vec{w} + \vec{z}) \tag{14}$$

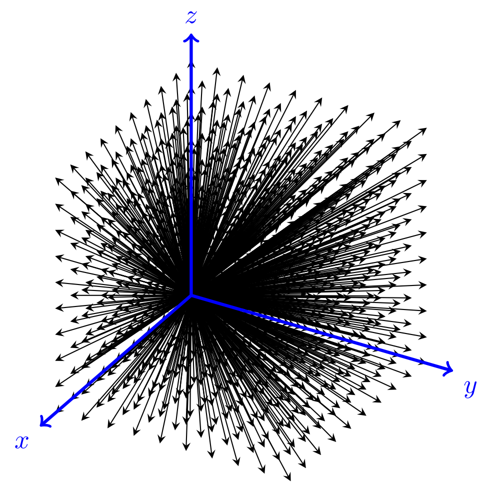
- To introduce **vector spaces**, suppose we have three vectors $\vec{v}$, $\vec{w}$, $\vec{z}$ and three scalars c, d, e.
  - Linear combinations of just $c\vec{v}$ gives a one-dimensional line given that $c \neq 0$.



  - Linear combination of $c\vec{v} + d\vec{w}$ given that $\vec{v}$ and $vecw$ are not parallel (not colinear) give a plane:



- The linear combinations of $c\vec{v}+d\vec{w}+e\vec{z}$ where the vectors are not coplanar (lie on the same plane) give all of $\mathbb{R}^3$.



- The length of a vector of $\vec{v}$ in $\mathbb{R}^N$ is given by:

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_N^2} \tag{15}$$

Proof: We prove via induction. Suppose:

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_N^2} \tag{16}$$

is true. We now prove that this is true for $\|w\|$ as well where w is in $\mathbb{R}^{N+1}$. We can write:

$$\vec{v'} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \ddots \\ v_N \\ 0 \end{bmatrix}, \vec{v''} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \ddots \\ v_{N-1} \\ v_N \end{bmatrix}, \vec{v'''} = \begin{bmatrix} 0 \\ 0\ 0 \\ \ddots \\ 0 \\ v_N \end{bmatrix} \tag{17}$$

Since these two are orthogonal, we can use Pythagorean theorem:

$$\|\vec{v''}\| = \sqrt{(\|\vec{v'}\|)^2 + (\|\vec{v'''}\|)^2} = \sqrt{v_1^2 + v_2^2 + \cdots + v_{N-1}^2} \tag{18}$$

Since this is true for $N = 2$, this must be true for all N.

- A few properties of the absolute magnitude:
  - When multiplied by a scalar:
$$\|c\vec{v}\| = c|\vec{v}| \tag{19}$$
  - When the magnitude is zero:
$$\|\vec{v}\| = 0 \iff \vec{v} = \vec{0} \tag{20}$$
- A **unit vector** is any vector with length equal to one. The most famous ones:
$$\vec{i} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \vec{j} = \begin{bmatrix}0\\1\\0\end{bmatrix}, \vec{k} = \begin{bmatrix}0\\0\\1\end{bmatrix} \tag{21}$$
- To turn any vector $\vec{v}$ into a unit vector, we can multiply it by the scalar $\frac{1}{\|\vec{v}\|}$.
- Suppose we wish to find the distance between P_1 and P_2, we can use the following steps:
  1. Define a coordinate system and orient the vectors in the given system. Draw a diagram.
  2. We can write the linear combination algebraically:
$$\overrightarrow{OP_1} + \overrightarrow{P_1P_2} = \overrightarrow{OP_2} \implies \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} \tag{22}$$
  3. We can then solve for $\|\overrightarrow{P_1P_2}\|$ and we are done.

3 Lecture 3: Dot Product

- Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$, then the dot product of two vectors in $\mathbb{R}^3$ is:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 \tag{23}$$

  also sometimes known as the **scalar product**, which comes from the fact that the dot product gives a scalar quantity. This is a *definition*.

- There are a few properties of dot products:

  – The distributive property: $\vec{v} \cdot (\vec{w} + \vec{z}) = \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{z}$

  Proof: Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$. Then:

$$\begin{aligned} \vec{v} \cdot (\vec{w} + \vec{z}) &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \cdot \left(\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \right) && (24) \\ &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \cdot \begin{bmatrix} w_1 + z_1 \\ w_2 + z_2 \\ w_3 + z_3 \end{bmatrix} && (25) \\ &= \begin{bmatrix} v_1(w_1 + z_1) \\ v_2(w_2 + z_2) \\ v_3(w_3 + z_2) \end{bmatrix} && (26) \\ &= \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{z} && (27) \end{aligned}$$

  – The commutative property: $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.

  – Associative: $c(\vec{v} \cdot \vec{w}) = (c\vec{v}) \cdot \vec{w} = \vec{v} \cdot (c\vec{w})$

- There is an important connection between the length of a vector and the dot product. Recall that:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \tag{28}$$

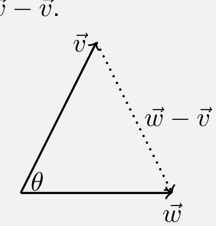
  We can square this to notice that the RHS is just the dot product of a vector with itself:

$$\|\vec{v}\|^2 = v_1^2 + v_2^2 + v_3^2 = \vec{v} \cdot \vec{v} \tag{29}$$

  Thus:

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} \tag{30}$$

Example 1: Suppose we wish to find the angle between two vectors $\vec{v}$ and $\vec{w}$. Traditionally, we may want to complete the triangle by drawing the vector $\vec{w} - \vec{v}$.



We can then write:

$$\begin{aligned}
\|\vec{w}-\vec{v}\|^2 &= (\vec{w}-\vec{v})\cdot(\vec{w}-\vec{v}) && (31)\\
&= \vec{w}\cdot\vec{w} - 2\vec{v}\cdot\vec{w} + \vec{v}\cdot\vec{v} && (32)\\
&= \|\vec{w}\|^2 + \|\vec{v}\|^2 - 2\vec{w}\cdot\vec{v} && (33)
\end{aligned}$$

This resembles the cosine law:

$$\|\vec{w}-\vec{v}\|^2 = \|\vec{w}\|^2 + \|\vec{v}\|^2 - 2\|\vec{w}\|\|\vec{v}\|\cos\theta \tag{34}$$

By comparison, we can conclude that:

$$\vec{w}\cdot\vec{v} = \|\vec{v}\|\|\vec{w}\|\cos\theta \tag{35}$$

This is *not* the definition of the dot product, but only a geometric interpretation of it. We can solve for $\cos\theta$ to be in this case:

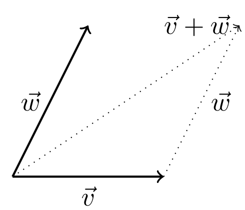
$$\cos\theta = \frac{\vec{w}\cdot\vec{v}}{\|\vec{w}\|\|\vec{v}\|} \tag{36}$$

- Note that the sign of the dot product tells us some important information about the angle θ. For example, if we know that $\vec{v}\cdot\vec{w} > 0$, then this is true if and only if $0 \leq \theta < \pi/2$ (or in other words, acute). Similarly, $\vec{v}\cdot\vec{w} = 0 \iff \theta = \pi/2$ and $\vec{v}\cdot\vec{w} > 0 \iff \theta > \pi/2$

Theorem: The triangle inequality tells us:

$$\|\vec{v}+\vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\| \tag{37}$$

- We can visualize the triangle inequality intuitively by drawing a diagram:



and we can intuitively see that $\vec{v}+\vec{w}$ has to be have a smaller length than the sum of the two lengths of $\vec{v}$ and $\vec{w}$. However, we need to do this algebraically:

Proof: We write:

$$\begin{aligned}
\|\vec{v}+\vec{w}\|^2 &= (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w}) && (38)\\
&= \vec{v}\cdot\vec{v} + 2\vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w} && (39)\\
&= \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\|\vec{w}\|\cos\theta && (40)
\end{aligned}$$

To get the inequality, we can replace $\cos\theta$ by its least upper bound. Then:

$$\begin{aligned}
\|\vec{v}+\vec{w}\|^2 &\leq \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\|\vec{w}\|(1) && (41)\\
&\leq (\|\vec{v}\| + \|\vec{w}\|)^2 && (42)
\end{aligned}$$

Taking the root of both sides (since none of the magnitudes can be negative), we are then able to prove the triangle inequality.

Theorem: The **Cauchy-Schwarz-Bunakowsky Inequality** relates the absolute magnitude of the dot product:

$$|\vec{v}\cdot\vec{w} \leq \|\vec{v}\|\|\vec{w}\| \tag{43}$$

4 Lecture 4: Projections

- For two nonzero vectors $\vec{v}$ and $\vec{w}$ such that:

$$\vec{v} \cdot \vec{w} = 0 \implies \cos\theta = 0 \implies \theta = \frac{\pi}{2} \tag{44}$$

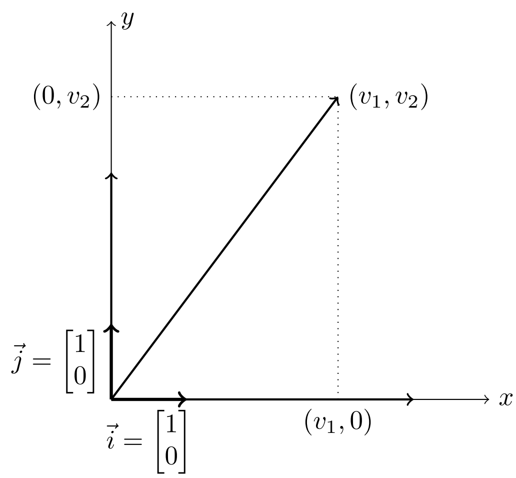
If $\vec{v}$ and/or $\vec{w}$ is the zero vector,

$$\vec{v} \cdot \vec{w} = 0 \tag{45}$$

is also true.

> **Definition**: $\vec{v}$ and $\vec{w}$ are orthogonal if and only if $\vec{v} \cdot \vec{w} = 0$.

- Projections are what we use to define points in 2-d and 3-d. For example:



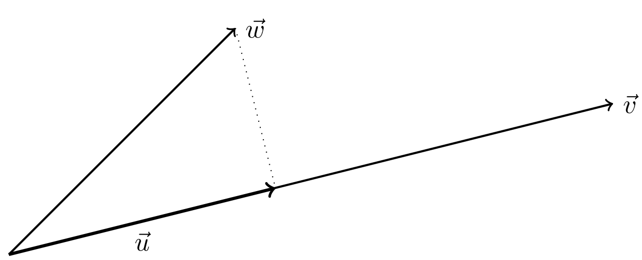
We can write $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ as:

$$\vec{v} = v_1\vec{i} + v_2\vec{j} \tag{46}$$

$$= v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{47}$$

$$= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{48}$$

- Let's generalize this concept to project one vector $\vec{w}$ on another vector $\vec{v}$



We can say that: $\vec{u}$ is the projection of $\vec{w}$ on $\vec{v}$. Based on the way we have constructed $\vec{u}$ we know it has certain properties:

  – $\vec{u}$ is parallel to $\vec{v}$, so it can be expressed as a multiple of $\vec{v}$ such that:

$$\vec{u} = c\vec{v} \tag{49}$$

  where c is a scalar.

- We can say that $\vec{w} - \vec{u}$ (and by extension $\vec{u} - \vec{w}$) is orthogonal to $\vec{v}$:

$$(\vec{w} - \vec{u}) \cdot \vec{v} = 0 \tag{50}$$

Using these two properties, we can solve for the unknown c:

$$(\vec{w} - \vec{u}) \cdot \vec{v} = 0 \tag{51}$$
$$\vec{w} \cdot \vec{v} - \vec{u} \cdot \vec{v} = 0 \tag{52}$$
$$\vec{w} \cdot \vec{v} - (c\vec{v}) \cdot \vec{v} = 0 \tag{53}$$
$$\vec{w} \cdot \vec{v} - c\,(\vec{v} \cdot \vec{v}) = 0 \tag{54}$$

Solving for c gives:

$$c = \frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \tag{55}$$

so we have:

$$\boxed{\vec{u} = \frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v}} \tag{56}$$

Definition: The projection of $\vec{w}$ on $\vec{v}$ can be written as:

$$\vec{u} = \text{proj}_{\vec{v}}\vec{w} = \frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v} = \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|}\frac{1}{\|\vec{v}\|}\vec{v} \tag{57}$$

where the last part $\frac{1}{\|\vec{v}\|}\vec{v}$ is a unit vector pointing in the direction of $\vec{v}$.

- Suppose we wish to project a vector $\vec{v}$ onto another vector (e.g. z-axis) in three dimensions. *The same formula applies.*
- Suppose instead we wish to project $\vec{v}$ on a plane, such as the xy plane? If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Then the projection would be $\begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}$.

5 Lecture 5

- Given v and w, the definition of cross product gives:

$$\vec{u} = \vec{v} \times \vec{w} = \begin{bmatrix} v_2w_3 = v_3w + 2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix} \tag{58}$$

  which is perpendicular to both $\vec{v}$ and $\vec{w}$.

- This is however, not the only orthogonal vector since any scalar multiple of $\vec{u} = \vec{v} \times \vec{w}$ will be orthogonal to both $\vec{v}$ and $\vec{w}$.

- We can prove this orthogonal property by taking the dot product with both $\vec{v}$ and $\vec{w}$.

- There are a few properties:

  – Consider 3 vectors, $\vec{v}$, $\vec{w}$, and $\vec{z}$. Then:

$$\vec{v} \times (\vec{w} + \vec{z}) = \vec{v} \times \vec{w} + \vec{v} \times \vec{z} \tag{59}$$

  – The cross product is not commutative, but they are anti-commutative:

$$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v} \tag{60}$$

  – When crossed with the zero vector, we have:

$$\vec{v} \times \vec{0} = \vec{0} \times \vec{v} = \vec{0} \tag{61}$$

  – When multiplied by a scalar,

$$c(\vec{v} \times \vec{w}) = (c\vec{v}) \times \vec{w} = \vec{v} \times (c\vec{w}) \tag{62}$$

Warning: The cross product its **not** associative. In general:

$$\vec{v} \times (\vec{w} \times \vec{z}) \neq (\vec{v} \times \vec{w}) \times \vec{z} \tag{63}$$

- The direction of $\vec{u} = \vec{v} \times \vec{w}$ can be easily determined using the right hand rule.

- We can determine the magnitude to be $\|\vec{v}\|\|\vec{w}\| \sin\theta$ where $\sin\theta$ is the angle in between. The Lagrange identity shows that:

$$\|\vec{v} \times \vec{w}\|^2 = |\vec{v}\|^2|\vec{w}\|^2 - (\vec{v} \cdot \vec{w})^2 \tag{64}$$

  We can introduce the cosine formula to get:

$$\|\vec{v} \times \vec{w}\|^2 = |\vec{v}\|^2|\vec{w}\|^2 - |\vec{v}\|^2|\vec{w}\|^2 \cos^2\theta = |\vec{v}\|^2|\vec{w}\|^2(1 - \cos^2\theta) \tag{65}$$

  Using the Pythagorean identity, we let $\sin^2\theta = 1 - \cos^2\theta$.

- The cross product $\vec{v} \times \vec{w}$ has a magnitude equal to the area of the parallelogram defined by $\vec{v}$ and $\vec{w}$.

6 Lecture Six: Equations of Points, Lines, Planes

- For a line in three dimensions through the origin, we can write a point on this line as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\vec{d} \tag{66}$$

  where c is a scalar and $\vec{d}$ is any nonzero vector that is parallel to the line.

- For a line not through the origin, a point on this line can be expressed as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + c\vec{d} \tag{67}$$

  where $P_0(x_0, y_0, z_0)$ is a known point on the line.

- We can take the 2D line equation $y = mx + b$ and write it as a two dimensional vector equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} + c \begin{bmatrix} 1 \\ m \end{bmatrix} \tag{68}$$

- For a point on a plane that passes through the origin, we can use linear combinations of $\vec{d_1}$ and $\vec{d_2}$ where both are parallel to the plane:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\vec{d_1} + d\vec{d_2}. \tag{69}$$

  If the plane was not in the origin, we can write it as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + c\vec{d_1} + d\vec{d_2}. \tag{70}$$

- A generic normal vector $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ parallel to a line $\overrightarrow{P_0P}$ can be written as:

$$\overrightarrow{P_0P} \cdot \vec{n} = 0 \tag{71}$$

  which can be represented as:

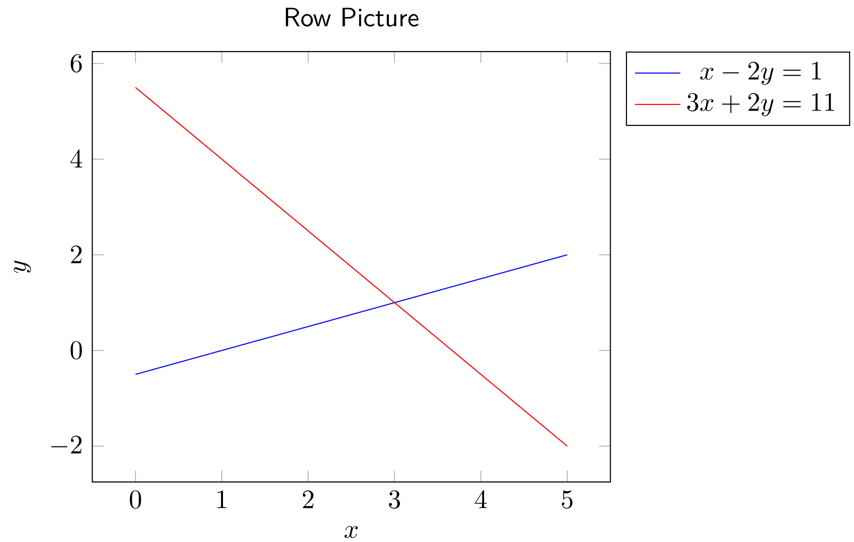
$$\begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{72}$$

7 Lecture Seven: Systems of equation

- Suppose we wish to solve the system of equation:

$$x - 2y = 1 \tag{73}$$
$$3x + 2y = 11 \tag{74}$$

We can do this numerous ways. If we look at the row picture:

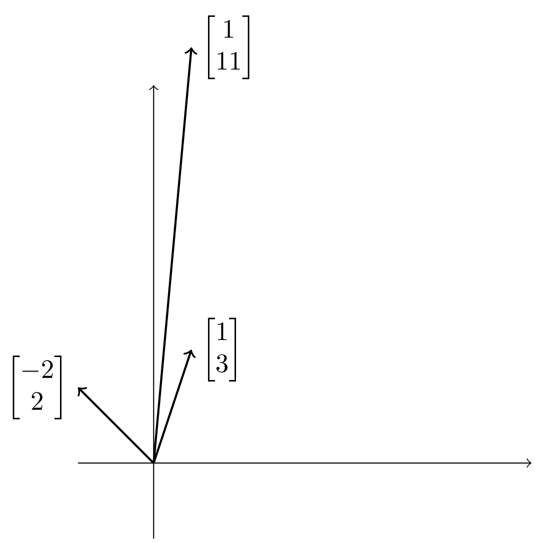


where the intersection point is at $(3, 1)$.

- Now let's look at the column picture. Instead of seeing two equations, we are going to express these two equations as a single vector equation:

$$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} \tag{75}$$

The solution requires us to find linear combinations of the vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ that equal $\begin{bmatrix} 1 \\ 11 \end{bmatrix}$



and the linear combination that gives the solution is:

$$3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} \tag{76}$$

- A little later in the course, we will use matrices to represent these systems, for example:

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}}_{\text{Matrix A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{vector } \vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 11 \end{bmatrix}}_{\text{vector } \vec{b}} \tag{77}$$

  where the matrix-vector product $A\vec{x}$ on the LHS is defined to be the equivalent of the column picture, e.g.:

$$A\vec{x} = x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} \tag{78}$$

- This leads to the dot product rule of calculating $A\vec{x}$:

$$\underbrace{\begin{bmatrix} & & & & & \\ a_i & b_i & c_i & d_i & e_i & f_i \\ & & & & & \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_j \\ b_j \\ c_j \\ d_j \\ e_j \\ f_j \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} \\ a_i a_j + b_i b_j + c_i c_j + d_i d_j + e_i e_j + f_i f_j \\ \\ \end{bmatrix}}_{A\vec{x}} \tag{79}$$

> **Idea**: What is a matrix? In the simplest of terms, it is a rectangular array of numbers, such as:
>
> $$A = \begin{bmatrix} 4 & 8 & 3 \\ 2 & 1 & -9 \end{bmatrix} \tag{80}$$
>
> which has two rows and three columns. $\therefore$ this isa 2×3 matrix. The general way to denote a matrix is via:
>
> $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \tag{81}$$
>
> where a_{ij} is the entry in row i and column j. Addition and scalar multiplication works in the same way.

- We can also multiply two matrices A and B if and only if A has n columns and B has n rows.

8 Lecture Eight: Matrix Multiplication

- When multiplying two matrices, the entry in row i and column j of AB is:

$$\text{(Row i of A)} \cdot \text{(column j of B)} \tag{82}$$

- Recall that A and B can only be multiplied of A is $m \times n$ and B is $n \times p$. The size of the resulting matrix is therefore $m \times p$.

Example 2: Suppose we wish to multiply $A = \begin{bmatrix} 2 & 4 & 8 \\ 1 & 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 9 \\ 4 & 2 \\ 3 & 7 \end{bmatrix}$. To determine AB, we get:

$$AB = \begin{bmatrix} 2 \cdot 1 + 4 \cdot 4 + 8 \cdot 3 & 2 \cdot 9 + 4 \cdot 2 + 8 \cdot 7 \\ 1 \cdot 1 + 3 \cdot 4 + 3 \cdot 6 & 1 \cdot 9 + 3 \cdot 2 + 6 \cdot 7 \end{bmatrix} = \begin{bmatrix} 42 & 82 \\ 31 & 57 \end{bmatrix} \tag{83}$$

- This leads properties of matrices:

1. $A + B = B + A$ (commutative)
2. $c(A + B) = cA + cB$ (where c is scalar)
3. $A + (B + C) = (A + B) + C$ (associative)
4. $C(A + B) = CA + CB$ (distributive from left)
5. $(A + B)C = AC + BC$ (distributive from right)
6. $A(BC) = (AB)C$ (associative)

- We can take exponents:

$$AA = A^2 \tag{84}$$
$$A^p A^q = A^{p+q} \tag{85}$$
$$(A^p)^q = A^{pq} \tag{86}$$

and later on we will see the inverse:

$$A^{-1} \tag{87}$$

- We can also view matrices as **transformations**. A linear transformation L is a function that maps that maps a vector in $\mathbb{R}^n$ to a vector in $\mathbb{R}^n$ with the following properties:

$$L : \mathbb{R}^n \to \mathbb{R}^n \tag{88}$$

This is analogous to the mapping:

$$y = f(x) \tag{89}$$

or

$$f : x \to y \tag{90}$$

- If $\vec{v}$ and $\vec{w} \in \mathbb{R}^n$, then $L(\vec{v})$ and $L(\vec{w}) \in \mathbb{R}^n$. It has the following properties:

1. $L(c\vec{v}) = cL(\vec{v})$
2. $L(\vec{v} + \vec{w}) = L(\vec{v}) + L(\vec{w})$

Example 3: Suppose we define a transformation T_1 that adds a constant vector $\vec{u}_0$ to every vector where:

$$T_1 : \mathbb{R}^n \to \mathbb{R}^n \tag{91}$$

Is this a linear transformation? If so, then the following must be true:

$$T_1(\vec{v}) + T_2(\vec{w}) = T_1(\vec{v} + \vec{w}) \tag{92}$$

$$\vec{v} + \vec{w} + 2\vec{u}_0 = \vec{v} + \vec{w} + \vec{u}_0 \tag{93}$$

which is only true when $\vec{u}$ is the zero vector.