ESC195 Notes

QiLin Xue

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Contents

1 Areas and Lengths in Polar Coordinates

- Suppose we have a polar curve $r = \rho(\theta)$ for $\alpha \leq \theta \leq \beta$.
- We can determine the area by partitioning the curve into θ_i and approximating each subregion as a circular segment. The area of a circular segment is:

$$A = \frac{1}{2}a^2\Delta\theta \tag{1}$$

We can take the radius to be $r = \rho(\theta^*)$ where $\theta_{i-1} \leq \theta^* \leq \theta_i$. The area of each region is:

$$A_i = \frac{1}{2}\rho(\theta_i^*)^2\Delta\theta_i \tag{2}$$

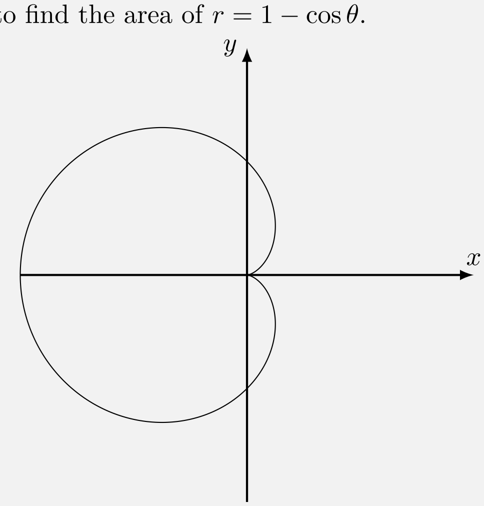
so the total area is:

$$A = \lim_{||P||} \sum_i \frac{1}{2}\rho(\theta_i^*)^2\Delta\theta_i \tag{3}$$

or

$$A = \frac{1}{2}\int_\alpha^\beta \rho(\theta)^2 \, \mathrm{d}\theta \tag{4}$$

Example 1: Suppose we wish to find the area of $r = 1 - \cos\theta$.



The area is then:

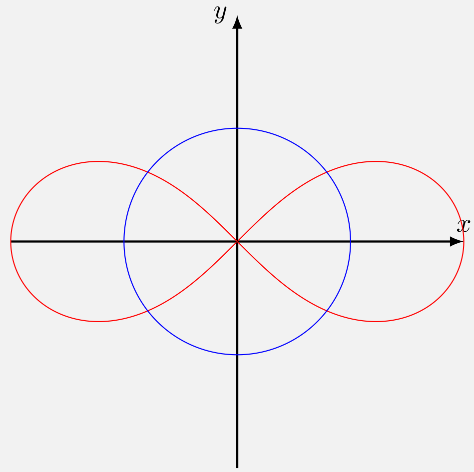
$$\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2 \, \mathrm{d}\theta && (5)\\
&= \frac{1}{2}\int_0^{2\pi} (1-2\cos\theta+\cos^2\theta) \, \mathrm{d}\theta && (6)\\
&= \frac{1}{2}\int_0^{2\pi} \left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos(2\theta)\right) \mathrm{d}\theta && (7)\\
&= \frac{1}{2}\left(\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin(2\theta)\right)\Bigg|_0^{2\pi} && (8)\\
&= \frac{3}{2}\pi && (9)
\end{aligned}$$

- We can also find the area between two polar curves ρ_1 and ρ_2. We have:

$$A = \frac{1}{2}\int_\alpha^\beta \rho_1(\theta)^2 \, \mathrm{d}\theta - \frac{1}{2}\int_\alpha^\beta \rho_2(\theta)^2 \, \mathrm{d}\theta = \frac{1}{2}\int_\alpha^\beta (\rho_1^2-\rho_2^2) \, \mathrm{d}\theta \quad (10)$$

Warning: Be careful when applying this formula as it is possible the two functions can overlap between $\alpha \le \theta \le \beta$. Therefore, we always need a good idea of what's happening.

Example 2: Suppose we want to determine the area inside $r^2 = 4\cos(2\theta)$ but outside $r = 1$. This gives:



We first find the four points of intersection:

$$4\cos(2\theta) = 1 \implies \cos(2\theta) = \frac{1}{4} \implies \theta = \pm 0.659 \quad (11)$$

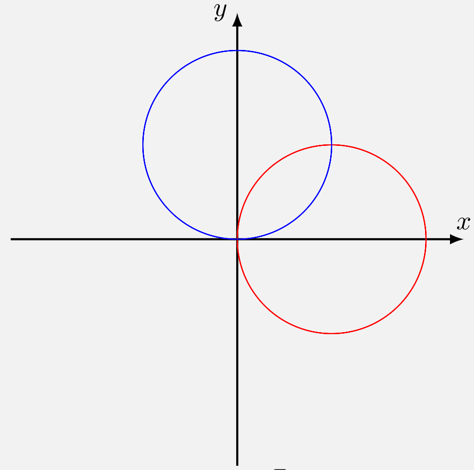
or $\theta = \pi \pm 0.659$. Due to the symmetry, we only need to find the area of one half of the area we are interested in, which gives:

$$\begin{aligned}
\frac{1}{2}A &= \frac{1}{2}\int_{-0.659}^{0.659} (4\cos 2\theta - 1) \, \mathrm{d}\theta && (12)\\
&= \frac{1}{2}(2\sin 2\theta - \theta)\Big|_{-0.659}^{0.659} && (13)\\
&= 1.277 && (14)
\end{aligned}$$

so the area is $A = 2.554$.

- There are a few challenging examples:

Example 3: Suppose we wish to find the area between $r = \sin\theta$ and $r = \cos\theta$:



We know from symmetry that the intersection is at $\theta = \frac{\pi}{4}$. We notice that the contribution to the area from each curve ρ is equal and *independent* from each other. Therefore:

$$A = A_1 + A_2 = \int_0^{\pi/4} \frac{1}{2}\sin^2\theta \,\mathrm{d}\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}\cos^2\theta \,\mathrm{d}\theta = \frac{\pi}{8} - \frac{1}{4} \tag{15}$$

- We can determine the arclength by working in parametric form. Let $x = r(\theta)\cos\theta$ and $y = r(\theta)\sin\theta$. Therefore:

$$s = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,\mathrm{d}\theta \tag{16}$$

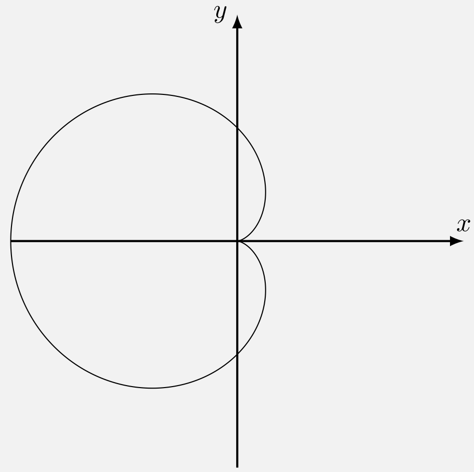
$$= \int_\alpha^\beta \sqrt{(r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2}\,\mathrm{d}\theta \tag{17}$$

$$= \int_\alpha^\beta \sqrt{(r'^2\cos^2\theta + r^2\sin^2\theta - \cancel{2rr'\cos\theta\sin\theta}) + (r'^2\sin^2\theta + r^2\cos^2\theta + \cancel{2r'r\cos\theta\sin\theta})}\,\mathrm{d}\theta \tag{18}$$

$$= \int_\alpha^\beta \sqrt{r^2(\cos^2\theta + \sin^2\theta) + r'^2(\cos^2\theta + \sin^2\theta)} \tag{19}$$

$$= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,\mathrm{d}\theta \tag{20}$$

Example 4: Suppose we want to find the arclength of $r = a(1 - \cos\theta)$ from $0 \le \theta < 2\pi$. This looks like:



We have:

$$s = \int_0^{2\pi} \sqrt{r^2 + (r')^2}\,\mathrm{d}\theta \tag{21}$$
$$= \int_0^{2\pi} \sqrt{a^2(1 - 2\cos\theta + \cos^2\theta) + a^2\sin^2\theta}\,\mathrm{d}\theta \tag{22}$$
$$= a\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\,\mathrm{d}\theta \tag{23}$$
$$= a\int_0^{2\pi} \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\,\mathrm{d}\theta \tag{24}$$
$$= 2a\left[-2\cos\left(\frac{\theta}{2}\right)\right]\Bigg|_0^{2\pi} \tag{25}$$
$$= 8a \tag{26}$$