

ESC103: Mathematics and Computation Notes

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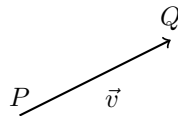
Fall 2020

1 Lecture 1

- A **vector** is a quantity with a magnitude, a direction, and units.
- A **scalar** is a quantity with a magnitude, a sign, and units.

Idea: At the heart of linear algebra is two operations and they both deal with vectors. We *add* vectors and we multiply vectors by scalars.

- We can draw a vector from point P to point Q by drawing an arrow:



where P is the **tail** and Q is the **head** of the vector. The vector can be written algebraically as:

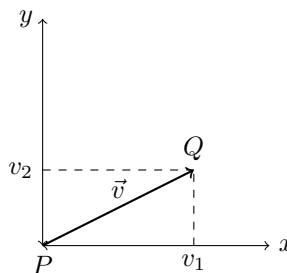
$$\vec{v} = \vec{PQ} \quad (1)$$

Note that this is not equal to:

$$\vec{PQ} \neq \vec{QP} \quad (2)$$

because even though their magnitudes are the same, their direction is not.

- We can draw a two-dimensional vector in \mathbb{R}^2 by translating the vector \vec{v} to the origin (since we are not changing direction or magnitude).



By breaking it up into components, we can write \vec{v} as a column vector:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (3)$$

Note that this is not the same as a row vector:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \neq [v_1 \quad v_2] \quad (4)$$

- To add two vectors, we can add them components wise:

$$\vec{v} + \vec{w} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} \quad (5)$$

and similarly for subtraction:

$$\vec{v} - \vec{w} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 - w_1 \\ v_2 - w_2 \end{bmatrix} \quad (6)$$

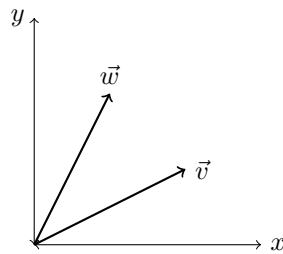
- To multiply a vector by a scalar, we multiply each component by that scalar:

$$c\vec{v} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix} \quad (7)$$

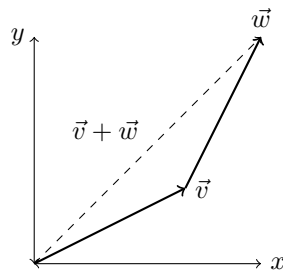
- Note that subtraction is just a combination of addition and scalar multiplication:

$$\vec{v} - \vec{w} = \vec{v} + (-1) \cdot \vec{w} \quad (8)$$

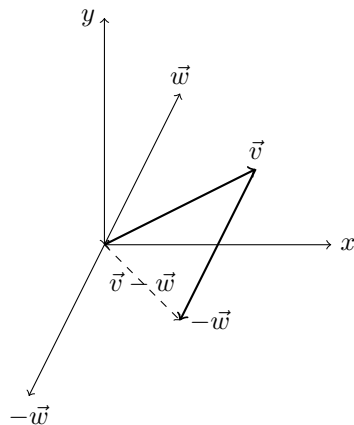
- Geometrically, we can add two vectors \vec{v} and \vec{w} :



We can do this by translating one of the vectors and add them tip to tail:

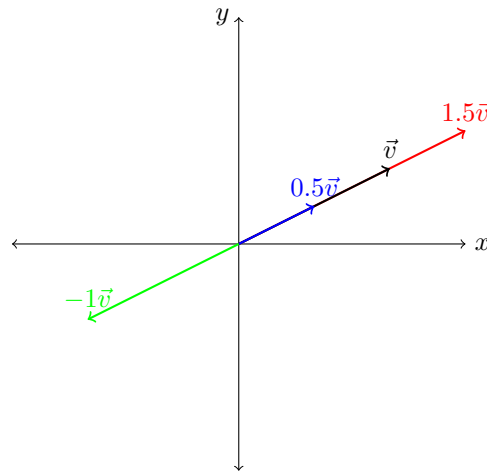


- To subtract two vectors, we first flip the second vector and then add. Suppose we wish to represent $\vec{v} - \vec{w}$ geometrically, then:



Note that this is equivalent from adding tail to tail.

- To multiply a vector by a scalar, we do not change the direction but instead we change the magnitude or length. Geometrically:



where all vectors originate from the origin.

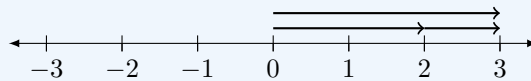
- The **zero vector** is defined as:

$$\vec{0} \equiv \vec{v} - \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (9)$$

Idea: Scalar addition can be represented as the addition of one dimensional vectors in \mathbb{R}^1 . For example, the number line is essentially just a coordinate axis and we can represent numbers in a similar way:

$$3 = 2 + 1 \quad (10)$$

and geometrically:



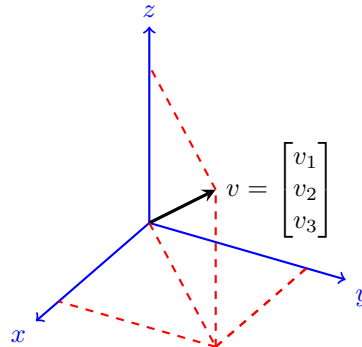
2 Lecture 2

- The heart of linear algebra are **linear combinations**. Let c and d be scalars. Then:

$$c\vec{v} + d\vec{w} \quad (11)$$

is a linear combination (LC) of \vec{v} and \vec{w} .

- For vectors in three-dimensional (\mathbb{R}^3)



- The **transpose** of a matrix changes swaps the ig index with the ji index such that:

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^T = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad (12)$$

- Addition, subtraction, and multiplication with scalars behave in the same ways.
- There are a few properties that vectors behave:

- The **commutative property** says that:

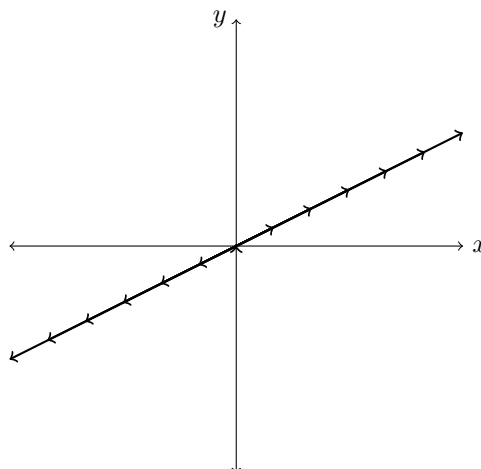
$$\vec{v} + \vec{w} = \vec{w} + \vec{v} \quad (13)$$

- The **associative property** says that;

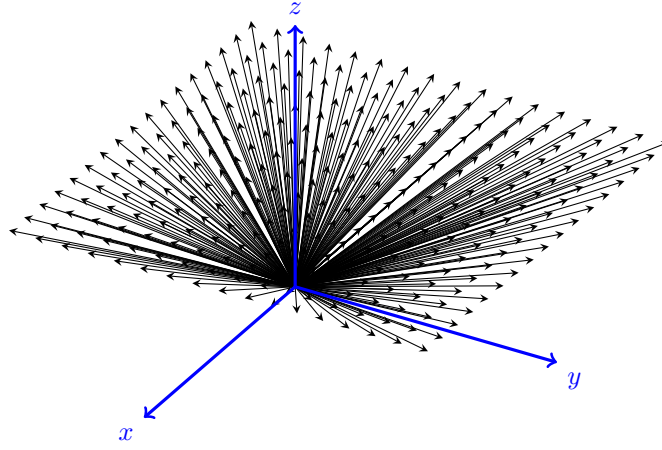
$$\vec{v} + \vec{w} + \vec{z} = (\vec{v} + \vec{w}) + \vec{z} = \vec{v} + (\vec{w} + \vec{z}) \quad (14)$$

- To introduce **vector spaces**, suppose we have three vectors \vec{v} , \vec{w} , \vec{z} and three scalars c , d , e .

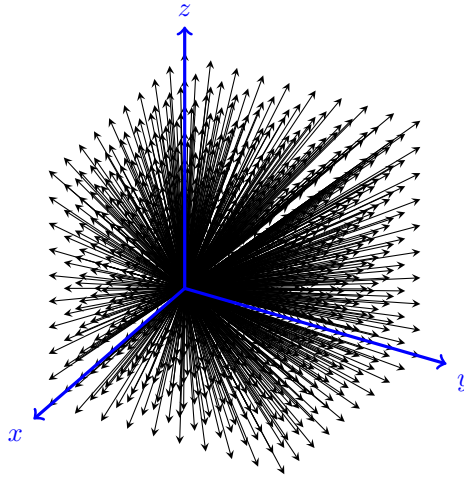
- Linear combinations of just $c\vec{v}$ gives a one-dimensional line given that $c \neq 0$.



- Linear combination of $c\vec{v} + d\vec{w}$ given that \vec{v} and \vec{w} are not parallel (not colinear) give a plane:



- The linear combinations of $c\vec{v}+d\vec{w}+e\vec{z}$ where the vectors are not coplanar (lie on the same plane) give all of \mathbb{R}^3 .



- The length of a vector of \vec{v} in \mathbb{R}^N is given by:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_N^2} \quad (15)$$

Proof: We prove via induction. Suppose:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_N^2} \quad (16)$$

is true. We now prove that this is true for $\|\vec{w}\|$ as well where w is in \mathbb{R}^{N+1} . We can write:

$$\vec{v}' = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_N \\ 0 \end{bmatrix}, \vec{v}'' = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{N-1} \\ v_N \end{bmatrix}, \vec{v}''' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ v_N \end{bmatrix} \quad (17)$$

Since these two are orthogonal, we can use Pythagorean theorem:

$$\|\vec{v}'''\| = \sqrt{(\|\vec{v}'\|)^2 + (\|\vec{v}''\|)^2} = \sqrt{v_1^2 + v_2^2 + \cdots + v_{N-1}^2} \quad (18)$$

Since this is true for $N = 2$, this must be true for all N .

- A few properties of the absolute magnitude:

- When multiplied by a scalar:

$$\|c\vec{v}\| = c\|\vec{v}\| \quad (19)$$

- When the magnitude is zero:

$$\|\vec{v}\| = 0 \iff \vec{v} = \vec{0} \quad (20)$$

- A **unit vector** is any vector with length equal to one. The most famous ones:

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (21)$$

- To turn any vector \vec{v} into a unit vector, we can multiply it by the scalar $\frac{1}{\|\vec{v}\|}$.
- Suppose we wish to find the distance between P_1 and P_2 , we can use the following steps:
 1. Define a coordinate system and orient the vectors in the given system. Draw a diagram.
 2. We can write the linear combination algebraically:

$$\overrightarrow{OP_1} + \overrightarrow{P_1P_2} = \overrightarrow{OP_2} \implies \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} \quad (22)$$

3. We can then solve for $\|\overrightarrow{P_1P_2}\|$ and we are done.

3 Lecture 3: Dot Product

- Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$, then the dot product of two vectors in \mathbb{R}^3 is:

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 \quad (23)$$

also sometimes known as the **scalar product**, which comes from the fact that the dot product gives a scalar quantity. This is a *definition*.

- There are a few properties of dot products:
 - The distributive property: $\vec{v} \cdot (\vec{w} + \vec{z}) = \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{z}$

Proof: Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$. Then:

$$\vec{v} \cdot (\vec{w} + \vec{z}) = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \cdot \left(\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \right) \quad (24)$$

$$= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \cdot \begin{bmatrix} w_1 + z_1 \\ w_2 + z_2 \\ w_3 + z_3 \end{bmatrix} \quad (25)$$

$$= \begin{bmatrix} v_1(w_1 + z_1) \\ v_2(w_2 + z_2) \\ v_3(w_3 + z_3) \end{bmatrix} \quad (26)$$

$$= \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{z} \quad (27)$$

- The commutative property: $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
- Associative: $c(\vec{v} \cdot \vec{w}) = (c\vec{v}) \cdot \vec{w} = \vec{v} \cdot (c\vec{w})$

- There is an important connection between the length of a vector and the dot product. Recall that:

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \quad (28)$$

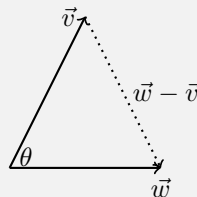
We can square this to notice that the RHS is just the dot product of a vector with itself:

$$\|\vec{v}\|^2 = v_1^2 + v_2^2 + v_3^2 = \vec{v} \cdot \vec{v} \quad (29)$$

Thus:

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} \quad (30)$$

Example 1: Suppose we wish to find the angle between two vectors \vec{v} and \vec{w} . Traditionally, we may want to complete the triangle by drawing the vector $\vec{w} - \vec{v}$.



We can then write:

$$\|\vec{w} - \vec{v}\|^2 = (\vec{w} - \vec{v}) \cdot (\vec{w} - \vec{v}) \quad (31)$$

$$= \vec{w} \cdot \vec{w} - 2\vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{v} \quad (32)$$

$$= \|\vec{w}\|^2 + \|\vec{v}\|^2 - 2\vec{w} \cdot \vec{v} \quad (33)$$

This resembles the cosine law:

$$\|\vec{w} - \vec{v}\|^2 = \|\vec{w}\|^2 + \|\vec{v}\|^2 - 2\|\vec{w}\|\|\vec{v}\|\cos\theta \quad (34)$$

By comparison, we can conclude that:

$$\vec{w} \cdot \vec{v} = \|\vec{v}\|\|\vec{w}\|\cos\theta \quad (35)$$

This is *not* the definition of the dot product, but only a geometric interpretation of it. We can solve for $\cos\theta$ to be in this case:

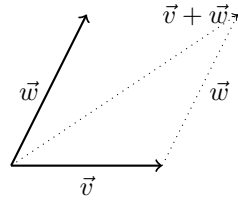
$$\cos\theta = \frac{\vec{w} \cdot \vec{v}}{\|\vec{w}\|\|\vec{v}\|} \quad (36)$$

- Note that the sign of the dot product tells us some important information about the angle θ . For example, if we know that $\vec{v} \cdot \vec{w} > 0$, then this is true if and only if $0 \leq \theta < \pi/2$ (or in other words, acute). Similarly, $\vec{v} \cdot \vec{w} = 0 \iff \theta = \pi/2$ and $\vec{v} \cdot \vec{w} < 0 \iff \theta > \pi/2$

Theorem: The triangle inequality tells us:

$$\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\| \quad (37)$$

- We can visualize the triangle inequality intuitively by drawing a diagram:



and we can intuitively see that $\vec{v} + \vec{w}$ has to be have a smaller length than the sum of the two lengths of \vec{v} and \vec{w} . However, we need to do this algebraically:

Proof: We write:

$$\|\vec{v} + \vec{w}\|^2 = (\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) \quad (38)$$

$$= \vec{v} \cdot \vec{v} + 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w} \quad (39)$$

$$= \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\|\vec{w}\|\cos\theta \quad (40)$$

To get the inequality, we can replace $\cos\theta$ by its least upper bound. Then:

$$\|\vec{v} + \vec{w}\|^2 \leq \|\vec{v}\|^2 + \|\vec{w}\|^2 + 2\|\vec{v}\|\|\vec{w}\|(1) \quad (41)$$

$$\leq (\|\vec{v}\| + \|\vec{w}\|)^2 \quad (42)$$

Taking the root of both sides (since none of the magnitudes can be negative), we are then able to prove the triangle inequality.

Theorem: The **Cauchy-Schwarz-Bunakowsky Inequality** relates the absolute magnitude of the dot product:

$$|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\|\|\vec{w}\| \quad (43)$$

4 Lecture 4: Projections

- For two nonzero vectors \vec{v} and \vec{w} such that:

$$\vec{v} \cdot \vec{w} = 0 \implies \cos \theta = 0 \implies \theta = \frac{\pi}{2} \quad (44)$$

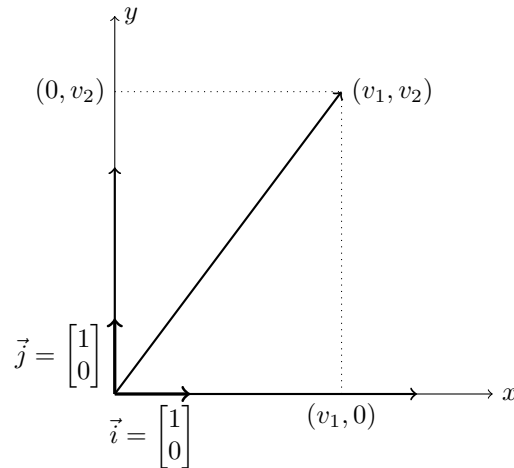
If \vec{v} and/or \vec{w} is the zero vector,

$$\vec{v} \cdot \vec{w} = 0 \quad (45)$$

is also true.

Definition: \vec{v} and \vec{w} are orthogonal if and only if $\vec{v} \cdot \vec{w} = 0$.

- Projections are what we use to define points in 2-d and 3-d. For example:



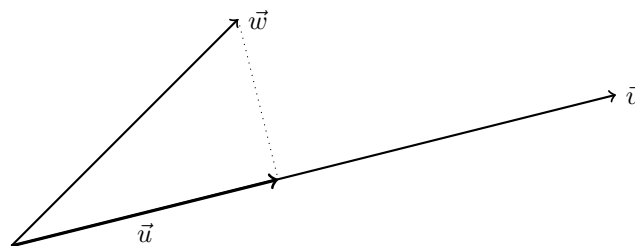
We can write $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ as:

$$\vec{v} = v_1 \vec{i} + v_2 \vec{j} \quad (46)$$

$$= v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (47)$$

$$= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (48)$$

- Let's generalize this concept to project one vector \vec{w} on another vector \vec{v}



We can say that: \vec{u} is the projection of \vec{w} on \vec{v} . Based on the way we have constructed \vec{u} we know it has certain properties:

- \vec{u} is parallel to \vec{v} , so it can be expressed as a multiple of \vec{v} such that:

$$\vec{u} = c\vec{v} \quad (49)$$

where c is a scalar.

– We can say that $\vec{w} - \vec{u}$ (and by extension $\vec{u} - \vec{w}$) is orthogonal to \vec{v} :

$$(\vec{w} - \vec{u}) \cdot \vec{v} = 0 \quad (50)$$

Using these two properties, we can solve for the unknown c :

$$(\vec{w} - \vec{u}) \cdot \vec{v} = 0 \quad (51)$$

$$\vec{w} \cdot \vec{v} - \vec{u} \cdot \vec{v} = 0 \quad (52)$$

$$\vec{w} \cdot \vec{v} - (c\vec{v}) \cdot \vec{v} = 0 \quad (53)$$

$$\vec{w} \cdot \vec{v} - c(\vec{v} \cdot \vec{v}) = 0 \quad (54)$$

Solving for c gives:

$$c = \frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \quad (55)$$

so we have:

$$\vec{u} = \frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} \quad (56)$$

Definition: The projection of \vec{w} on \vec{v} can be written as:

$$\vec{u} = \text{proj}_{\vec{v}} \vec{w} = \frac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} = \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|} \frac{1}{\|\vec{v}\|} \vec{v} \quad (57)$$

where the last part $\frac{1}{\|\vec{v}\|} \vec{v}$ is a unit vector pointing in the direction of \vec{v} .

- Suppose we wish to project a vector \vec{v} onto another vector (e.g. z-axis) in three dimensions. *The same formula applies.*
- Suppose instead we wish to project \vec{v} on a plane, such as the xy plane? If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Then the projection would be $\begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}$.

5 Lecture 5

- Given v and w , the definition of cross product gives:

$$\vec{u} = \vec{v} \times \vec{w} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix} \quad (58)$$

which is perpendicular to both \vec{v} and \vec{w} .

- This is however, not the only orthogonal vector since any scalar multiple of $\vec{u} = \vec{v} \times \vec{w}$ will be orthogonal to both \vec{v} and \vec{w} .
- We can prove this orthogonal property by taking the dot product with both \vec{v} and \vec{w} .
- There are a few properties:
 - Consider 3 vectors, \vec{v} , \vec{w} , and \vec{z} . Then:

$$\vec{v} \times (\vec{w} + \vec{z}) = \vec{v} \times \vec{w} + \vec{v} \times \vec{z} \quad (59)$$

- The cross product is not commutative, but they are anti-commutative:

$$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v} \quad (60)$$

- When crossed with the zero vector, we have:

$$\vec{v} \times \vec{0} = \vec{0} \times \vec{v} = \vec{0} \quad (61)$$

- When multiplied by a scalar,

$$c(\vec{v} \times \vec{w}) = (c\vec{v}) \times \vec{w} = \vec{v} \times (c\vec{w}) \quad (62)$$

Warning: The cross product is **not** associative. In general:

$$\vec{v} \times (\vec{w} \times \vec{z}) \neq (\vec{v} \times \vec{w}) \times \vec{z} \quad (63)$$

- The direction of $\vec{u} = \vec{v} \times \vec{w}$ can be easily determined using the right hand rule.
- We can determine the magnitude to be $\|\vec{v}\| \|\vec{w}\| \sin \theta$ where $\sin \theta$ is the angle in between. The Lagrange identity shows that:

$$\|\vec{v} \times \vec{w}\|^2 = \|\vec{v}\|^2 \|\vec{w}\|^2 - (\vec{v} \cdot \vec{w})^2 \quad (64)$$

We can introduce the cosine formula to get:

$$\|\vec{v} \times \vec{w}\|^2 = \|\vec{v}\|^2 \|\vec{w}\|^2 - \|\vec{v}\|^2 \|\vec{w}\|^2 \cos^2 \theta = \|\vec{v}\|^2 \|\vec{w}\|^2 (1 - \cos^2 \theta) \quad (65)$$

Using the Pythagorean identity, we let $\sin^2 \theta = 1 - \cos^2 \theta$.

- The cross product $\vec{v} \times \vec{w}$ has a magnitude equal to the area of the parallelogram defined by \vec{v} and \vec{w} .

6 Lecture Six: Equations of Points, Lines, Planes

- For a line in three dimensions through the origin, we can write a point on this line as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\vec{d} \quad (66)$$

where c is a scalar and \vec{d} is any nonzero vector that is parallel to the line.

- For a line not through the origin, a point on this line can be expressed as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + c\vec{d} \quad (67)$$

where $P_0(x_0, y_0, z_0)$ is a known point on the line.

- We can take the 2D line equation $y = mx + b$ and write it as a two dimensional vector equation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix} + c \begin{bmatrix} 1 \\ m \end{bmatrix} \quad (68)$$

- For a point on a plane that passes through the origin, we can use linear combinations of \vec{d}_1 and \vec{d}_2 where both are parallel to the plane:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = c\vec{d}_1 + d\vec{d}_2. \quad (69)$$

If the plane was not in the origin, we can write it as:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + c\vec{d}_1 + d\vec{d}_2. \quad (70)$$

- A generic normal vector $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ parallel to a line $\overrightarrow{P_0P}$ can be written as:

$$\overrightarrow{P_0P} \cdot \vec{n} = 0 \quad (71)$$

which can be represented as:

$$\begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad (72)$$

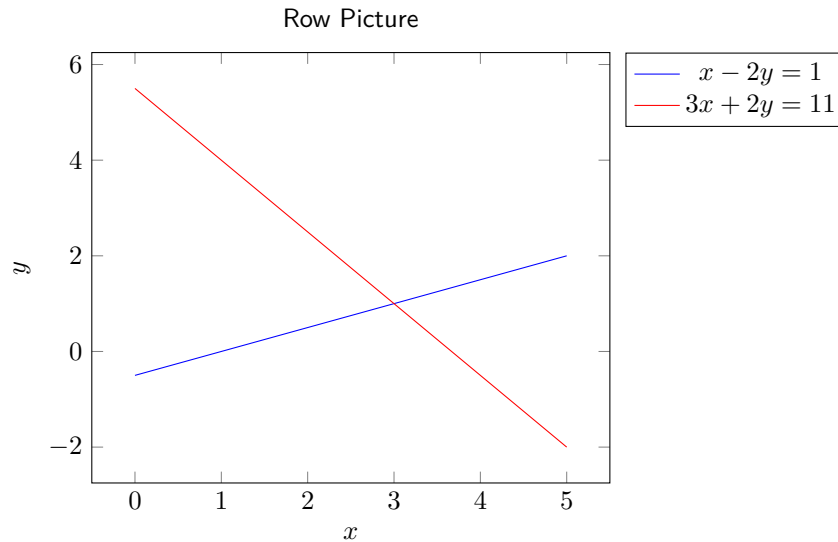
7 Lecture Seven: Systems of equation

- Suppose we wish to solve the system of equation:

$$x - 2y = 1 \quad (73)$$

$$3x + 2y = 11 \quad (74)$$

We can do this numerous ways. If we look at the row picture:

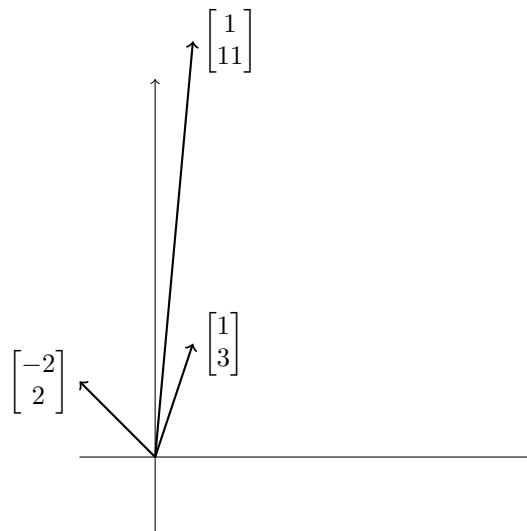


where the intersection point is at $(3, 1)$.

- Now let's look at the column picture. Instead of seeing two equations, we are going to express these two equations as a single vector equation:

$$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} \quad (75)$$

The solution requires us to find linear combinations of the vectors $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ that equal $\begin{bmatrix} 1 \\ 11 \end{bmatrix}$



and the linear combination that gives the solution is:

$$3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} \quad (76)$$

- A little later in the course, we will use matrices to represent these systems, for example:

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}}_{\text{Matrix } A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{vector } \vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 11 \end{bmatrix}}_{\text{vector } \vec{b}} \quad (77)$$

where the matrix-vector product $A\vec{x}$ on the LHS is defined to be the equivalent of the column picture, e.g.:

$$A\vec{x} = x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} \quad (78)$$

- This leads to the dot product rule of calculating $A\vec{x}$:

$$\underbrace{\begin{bmatrix} a_i & b_i & c_i & d_i & e_i & f_i \end{bmatrix}}_A \underbrace{\begin{bmatrix} a_j \\ b_j \\ c_j \\ d_j \\ e_j \\ f_j \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} a_i a_j + b_i b_j + c_i c_j + d_i d_j + e_i e_j + f_i f_j \end{bmatrix}}_{A\vec{x}} \quad (79)$$

Idea: What is a matrix? In the simplest of terms, it is a rectangular array of numbers, such as:

$$A = \begin{bmatrix} 4 & 8 & 3 \\ 2 & 1 & -9 \end{bmatrix} \quad (80)$$

which has two rows and three columns. \therefore this is a 2×3 matrix. The general way to denote a matrix is via:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad (81)$$

where a_{ij} is the entry in row i and column j . Addition and scalar multiplication works in the same way.

- We can also multiply two matrices A and B if and only if A has n columns and B has n rows.