ESC194: Calculus Notes

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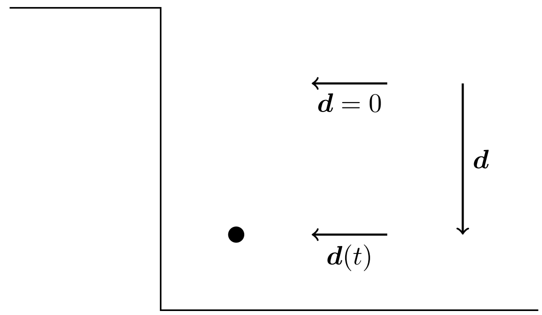
1 Lecture 1

Idea: The goal of the first few lectures is to define the derivative of a function *rigorously* **logically**. There are similar, but not quite the same, problems to the above:

- Defining the *slope* of a tangent to the curve
- Defining *speed* at an *instant*.

However, these concepts are not very well defined.

- Consider a falling object where a ball starts at a height of $d = 0$ and ends at a distance of $d(t)$.



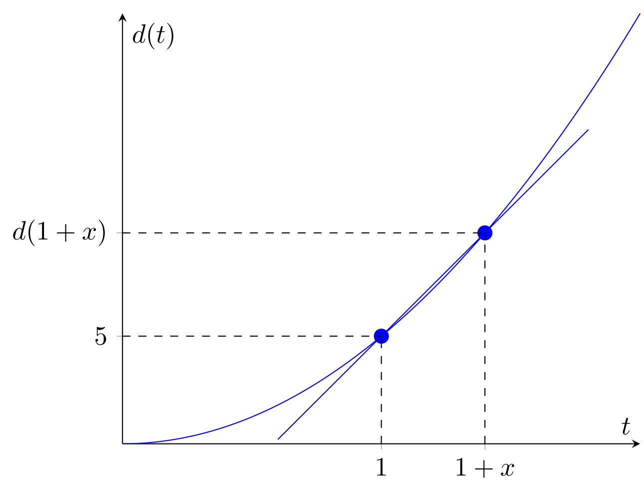
Suppose we are given the function $d(t) = 5t^2$ where d is in meters and t is in seconds. To determine the speed at the *instant* $t = 1$ sec, we can approximate using a secant-line:

$$m_{\text{sec}} = \frac{5(1.1)^2 - 5(1)^2}{1.1 - 1} = 10.5 \text{ m/s} \tag{1}$$

$$m_{\text{sec}} = \frac{5(1.01)^2 - 5(1)^2}{1.01 - 1} = 10.05 \text{ m/s} \tag{2}$$

The instantaneous speed *appears* to approach 10 m/s **exactly** at $t = 1$ sec, but we *cannot* be sure.

- We can do better by introducing a new variable $x[s]$ which represents the interval:



We then define a function $f(x)$ as

$$f(x) \equiv \frac{d(1+x) - d(1)}{x} \tag{3}$$

$$= \frac{5(1+x)^2 - 5(1)^2}{x} \tag{4}$$

$$= \frac{10x + 5x^2}{x} \tag{5}$$

$$= (10 + 5x)\left(\frac{x}{x}\right) \tag{6}$$

Let us *assume* (incorrectly) that $\frac{x}{x} = 1$ for all x. Then:

$$f(x) = 10x + 5 \tag{7}$$

So we could define the "speed at $t = 1$ s" as $f(0)$ which is exactly 10.

Warning: However: $\frac{x}{x} \neq 1$ when $x = 0$ as division by zero is not allowed as a legitimate operation of arithmetic. To explain why, we need to rigorously introduce numbers as defined by **axioms**.

- Example: For every $x \neq 0$ there is another number $\frac{1}{x}$ defined **implicitly** by $x \cdot \frac{1}{x} = 1$.

An implicit definition is when a quantity we are defining does not appear by itself on one side of the equation. Implicit definitions are used for the most basic axioms. If this was not true, then we can contradict ourselves. We would have:

$$0 \cdot \frac{1}{0} = 1 \tag{8}$$

but we can also prove that $0 \cdot a = 0$ for all numbers a, so if $\frac{1}{0}$ is a number, then $0 = 1$.

Furthermore, it is not possible to define $\frac{1}{0}$ if it was a number. A naive approach may be to define it as infinity, but infinity is not a number! If it was, then:

$$1 + \infty = \infty \tag{9}$$

$$2 + \infty = \infty \tag{10}$$

$$1 = 2 \tag{11}$$

Just because you **write** some symbol does not mean it **exists** as a number.

The correct expression for $f(x)$ is instead:

$$f(x) = \begin{cases} 10 + 5x, & x \neq 0 \\ \text{DNE}, & x = 0 \end{cases} \tag{12}$$

While DNE is not a number, $f(x)$ is still a legitimate function. We don't *fix* functions.

- The **limit** can be used to satisfy our intuitive feeling that the answer is exactly 10 m/s. The expression:

$$\lim_{x \to 0} f(x) \tag{13}$$

is a number.

Warning: We can't always trust our intuition on whether a certain relationship will converge or diverge:

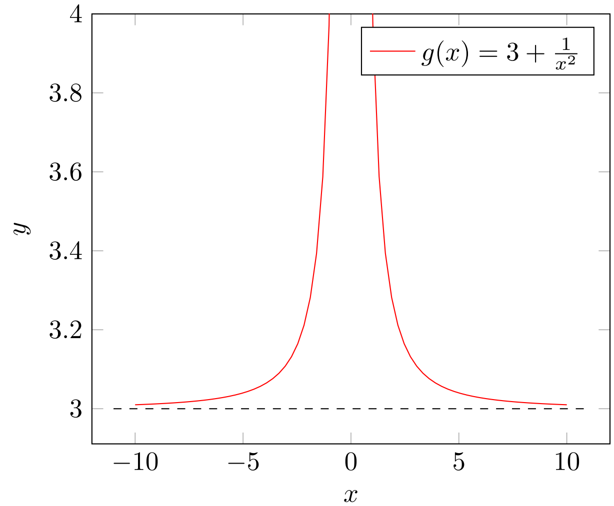
$$A = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{17} + \cdots \tag{14}$$

$$B = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \tag{15}$$

The value of A will converge to 1, but the value of B will diverge[a] and will not exist. This is why we need a rigorous definition of what exactly "approaching" means.

[a] This is known as the harmonic sum.

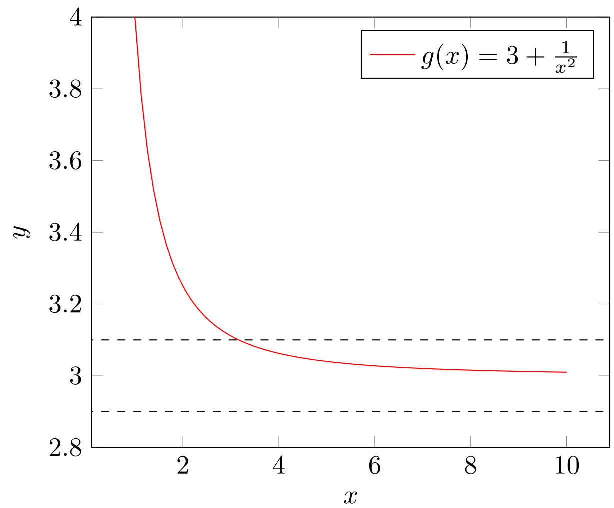
- A simpler type of limit considers what happens at infinity:



It appears that $g(x)$ approaches a value of 3 as x approaches infinity. The challenge is to find a **rigorous definition** of:

$$\lim_{x\to\infty} g(x) \tag{16}$$

so that we can prove it exists as a number. We want to be able to say that we can always find values of x large enough that the values of $g(x)$ will be as close as might be wanted to 3, say within 10^{-10} of 3. Geometrically, this is represented by always being able to pick an x great enough such that $g(x)$ is contained within the dashed lines.



We can solve this via trial and error. Say for a large $x_0 = 10^{100}$, we can show that for all

$$x > x_0 = 10^{100} \tag{17}$$

$$x^2 > 10^{200} \tag{18}$$

$$\frac{1}{x^2} < 10^{-200} \tag{19}$$

$$3 + \frac{1}{x^2} < 3 + 10^{-200} \tag{20}$$

Since $3 + 10^{-200} < 3 + 10^{-10}$, then the weaker case:

$$g(x) < 3 + 10^{-10} \tag{21}$$

must be true for all $x > x_0 = 10^{100}$ However, we need to find the lower bound as well. Next, note that:

$$g(x) = 3 + \frac{1}{x^2} > 3 > 3 - 10^{-10} \tag{22}$$

for all x. Therefore, for all $x > 10^{100}$, then: $3 - 10^{-10} < g(x) < 3 + 10^{-10}$.

- We can generalize this to any arbitrary bound. Some small number $\epsilon > 0$. We want to find some x_0 expressed in terms of ϵ such that for all $x > x_0$, the values of $g(x)$ will be within ϵ of 3.

> **Mnemonic**: The value of ϵ is imposed by the ϵnemy as a challenge.

Again, we use trial and error.

Example 1: Suppose we pick $x_0 = \frac{1}{\epsilon}$. Then for all $x > x_0 = \frac{1}{\epsilon}$, we have:

$$x^2 > \frac{1}{\epsilon^2} \tag{23}$$
$$\frac{1}{x^2} < \epsilon^2 \tag{24}$$
$$3 + \frac{1}{x^2} < 3 + \epsilon^2 \tag{25}$$
$$g(x) < 3 + \epsilon^2 \tag{26}$$
$$\tag{27}$$

which doesn't quite work since in order for the value to be within the bounds, the following must be true:

$$g(x) < 3 + \epsilon^2 \leq 3 + \epsilon \tag{28}$$

which is only true for $\epsilon \leq 1$ and is not true for all values of ϵ.

Example 2: Suppose we pick $x_0 = \frac{1}{\sqrt{\epsilon}}$. Then for all $x > x_0 = \frac{1}{\sqrt{\epsilon}}$, we have:

$$x^2 > \frac{1}{\epsilon} \tag{29}$$
$$\frac{1}{x^2} < \epsilon \tag{30}$$
$$3 + \frac{1}{x^2} < 3 + \epsilon \tag{31}$$
$$g(x) < 3 + \epsilon \tag{32}$$
$$\tag{33}$$

which provides the correct upper bound!

As a result, the value of $y = 3$ passes this challenge test, so we can define a new number:

$$\lim_{x \to \infty} g(x) \tag{34}$$

and assign it to the value of 3.

> **Idea**: There are three steps to find the limit:
> 1. Assume that $\lim\limits_{x \to \infty}$ exists and guess a value for it.
> 2. Show that your guess passes a "challenge" imposed by ϵ
> 3. If you succeed then we can take that $\lim\limits_{x \to \infty} g(x)$ exists as a number since we can assign it to your original guess.

- Similarly, for our original function $f(x)$ we can use a similar way to define the limit as $x \to 0$.but

2 Lecture 2

- Numbers are *bases* elements of mathematics, so they cannot be defined explicitly in terms of anything more basic.
- Instead they are defined *implicitly*, by imposing the rules, or **axioms**, that we require they satisfy.

> **Idea**: The axioms are *inspired* by physical reality, but are not *dictated by it*. They do not exist

- It is important to have as few axioms as possible (to make it philosophically more "pure", and to reduce the risk of contradictions:

1. **Commutative Law**: For each pair $x, y \in \mathrm{Re}$,
$$x + y = y + x \tag{35}$$
and
$$xy = yx \tag{36}$$
2. **Associative Law**: For each triple $x, y, z \in \mathrm{Re}$,
$$x + (y + z) = (x + y) + z \tag{37}$$
and
$$(xy)z = z(yz) \tag{38}$$
3. **Distributive Law** For each triple $x, y, z \in \mathrm{Re}$,
$$x(y + z) = zy + yz \tag{39}$$
and
$$(x + y)z = xz + yz \tag{40}$$
4. **Existence of Identities**: There exists two distinct real numbers, denoted by 0 and 1 for which:
$$x + 0 = 0 + x = x \tag{41}$$
and
$$x \cdot 1 = 1 \cdot x = x \tag{42}$$
for each $x \in \mathrm{Re}$.
5. **Existence of inverses** For each $x \in \mathrm{Re}$, there exists a unique additive inverse which we denote by $-x$ for which
$$x + (-x) = (-x) + x = 0 \tag{43}$$
For each $x \neq 0$ in Re, there exists a unique multiplactive inverse, which we denote by x^{-1} or $1/x$ for which:
$$x \cdot \left(x^{-1}\right) = \left(x^{-1}\right) \cdot x = 1. \tag{44}$$

- It's not important to restrict the number of **definitions**, which are built from axioms, but it gets messy if we make more definitions that are really needed. (e.g. $4 \equiv 3 + 1$)

> **Definition**: **Positive integers** are the "natural numbers": $1, 2, 3, \ldots$ Note that:
> $$2 \equiv 1 + 1$$
> and so forth.

Definition: **Rational numbers** are in the form of:

$$\frac{a}{b} \equiv a \cdot \frac{1}{b}$$

where a, b, are integers are $b \neq 0$. Note that this uses axiom 5 with the definition of fractions to create rational numbers.

- There is no limit to the number of theorems. We can and should prove all *arithmetic* and *algebraic* theorems rigorously logically, starting from the Axioms (e.g. $4 = 2 + 2$).

Example 3: Let us prove $\sqrt{2}$ is irrational by contradiction. Suppose there is a pair of integers: a, b, such that:

$$\left(\frac{a}{b}\right)^2 = 2$$

where all common factors have been removed.Therefore:

$$\therefore a^2 = 2b^2 \tag{45}$$
$$\therefore a^2 \equiv 0 \pmod{2} \tag{46}$$
$$\therefore a \equiv 0 \pmod{2} \tag{47}$$
$$\tag{48}$$

We can write $a = 2q$ where q is some integer such that:

$$\therefore a^2 = 4q^2 \tag{49}$$
$$\therefore 4q^2 = 2b^2 \tag{50}$$
$$\therefore b^2 = 2q^2 \tag{51}$$
$$\therefore b^2 \equiv 0 \pmod{2} \tag{52}$$
$$\therefore b \equiv 0 \pmod{2} \tag{53}$$

However, since a and b are both even, we have contradicted our statement that all common factors have been removed. Thus $\sqrt{2}$ cannot be rational and can only be irrational.

- However, the 5th field axiom only discusses the creation of rational numbers. We could simply add a "root 2 axiom" to create $\sqrt{2}$, just like we did for 0 and 1.

- This is super messy because it would imply we'd need another axiom for every irrational, including every root of every **polynomial function**:

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{54}$$

  where a can be any specified number and n is a positive integer. If we set $P_n(x) = 0$, we get a polynomial equation where we can find the roots.

  Definition: If z is a root of $p_n(x)$ then $p_n(z) = 0$.

  There are n roots[1], most are irrational and are called **algebraic numbers**.

- To prevent creating numerous new axioms, we create a new axiom called **CORA**: Completeness of the Reals Axiom, which tells us that every non-empty set of real numbers that is bounded above has a least upper bound among the real numbers.

[1] Per the fundamental theorem of algebra, which is beyond the scope of this course

Definition: A set of real numbers, $\mathbb{S}$ is bounded above if and only if there exists some number M such that $x \leq M$ for all $x \in \mathbb{S}$. For example:

$$\mathbb{S}_1 = \{1, 3, \frac{17}{5}, 211\} \tag{55}$$

Here, $M = 211$ or 250, etc. We can write the first upper bound as:

$$\text{ub}\mathbb{S}_1 = 211 \tag{56}$$

Definition: The least upper bound is the smallest of all the upper bounds. Here:

$$\text{lub}\mathbb{S}_1 = 211 \tag{57}$$

- Note that we **do not** require that $\text{lub}\mathbb{S} \in \mathbb{S}$ necessarily. For example, if:

  $$\mathbb{S}_2 = \{x : x^2 < 2\} \tag{58}$$

  There are several upper bounds in this set, but is there a least upper bound? We may think intuitively it is $\sqrt{2}$, but we have to be careful! We haven't proved it exists yet. **However,** CORA has *creates* this new number, $\sqrt{2}$, by demanding that it exists.[2]

- Additionally, CORA does the same for all **algebraic irrationals** and **transcendental irrationals**. Without CORA, there would be no irrational numbers!

[2] However, proving that the lower upper bound is $\sqrt{2}$ is rather tricky and was removed from the supplement.

3 Lecture 3

- The absolute value is defined as

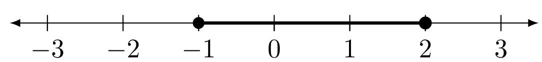
$$|a| = \begin{cases} +a & a \geq 0 \\ -a, & a < 0 \end{cases} \tag{59}$$

  Note that $|a| \geq 0$ always.

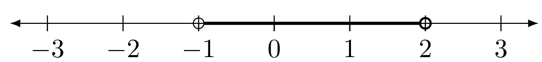
> **Warning**: Note that $|a| = \sqrt{a^2}$. This means that the square root is a function that only has one answer. The square root is defined such that $\sqrt{a} \geq 0$ if it exists and does not exist if $a < 0$. This is **different** from solving the equation:
>
> $$x^2 = 4 \tag{60}$$
>
> We want to perform the inverse of a square, which is *not* the square root! Instead, the inverse of x^2 is $\pm\sqrt{x}$. Just because there are two different values when squared gives the same number, doesn't mean that taking the square root of this number will yield two answers!

- The **real number line** is a geometric analogue of real numbers. It is not necessary (for rigorous proofs), but useful.

- A closed interval is represented by $[a, b]$, and can be written as $a \leq x \leq b$. For example, the interval $[-1, 2]$ can be represented by:

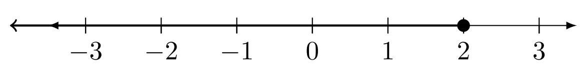


- An open interval does not contain the endpoints. It is represented by (a, b) or $a < x < b$. Similarly, this can be represented on a number line:



- A half-closed or half-open interval is when only one of the ends are closed, and is denoted by $[a, b)$ or $(a, b]$.

- If an interval goes to infinity, such as $(-\infty, b]$ which is equivalent to $x \leq b$. Note that this does not imply infinity is a number (or else the interval could be closed), but instead we can define an expression where infinity is embedded into such that it is rigorously logical.



- There are two sets of numbers involved in **functions**: an x-set and a y-set.

> **Definition**: A function is any **rule** that assigns each x-number to *one* y-number.

  Note that any prescription for the function is acceptable. For example, a table is an example of a function. Neither the x's or y's have to include numbers. There can be holes!

- For each x, only a single value of y can be assigned, i.e. double functions are not allowed.[3]

- Usually we specify functions using an algebraic equation. If for some set of x's, the equation gives a real number y, then it is *assumed* that this *specifies* the set of x's for this function. For example:

$$f(x) = \begin{cases} \frac{10x+5x^2}{x} = 10 + 5x & x \neq 0 \\ \text{DNE} & x = 0 \end{cases} \tag{61}$$

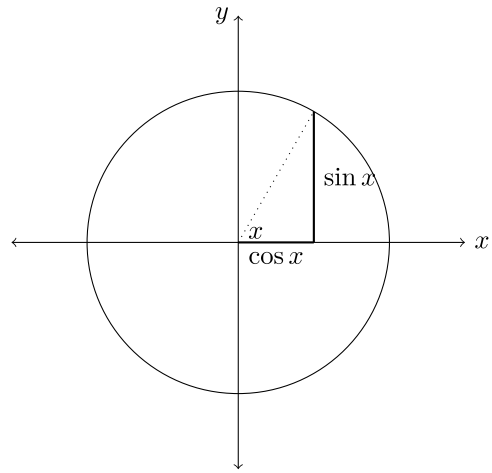
  This is a perfectly good function since it assigns each number in the x-set for this function to some y-value.

- The **domain of a function** is the set of x-values and the **range of a function** as the set of y-values. Here, x is the independent variable and y is the dependent variable.

[3] This is only from how we are defining functions. Other definitions may allow double valued functions.

Note that this doesn't work the other way around, since each x can be used to specify one y but each y can have several corresponding x values! There is some asymmetry involved.

- We want to define trigonometric functions purely algebraically, but for now we have to do it geometrically.



where the angle is in radians. Radians are picked as the unit such that the arc (curved part subtended by angle of θ) is given by

$$s = Rx \tag{62}$$

where R is the radius of the circle.

- Not all algebraic expressions are functions. Suppose we have an ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b} = 1 \tag{63}$$

and solving for y gives:

$$y = \pm b\left(1 - \frac{x^2}{a^2}\right)^{1/2} \tag{64}$$

Since we can't have double valued functions, we can instead break this up into *two separate functions*

- Composite functions can be written in the form of $f(g(x))$.

Example 4: Let $f(x) = x^2 + 2$ and $g(x) = \sin x$. Then:

$$f(g(x)) = \sin^2(x) + 2 \tag{65}$$

$$g(f(x)) = \sin\left(x^2 + 2\right) \tag{66}$$

must be true.

- We also need a rigorous way to define **increasing** and **decreasing functions**.

Definition: $f(x)$ is increasing on an interval I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I.[a]

[a] The motivation behind this definition is to forego the ambiguity when we try to say something like "As x gets bigger, y gets bigger." However a function is a definite relationship between pairs of numbers, none of the values are actually changing!

Similarly, we can define a decreasing function to be the converse, where $f(x_1) > f(x_2)$ for all $x_1 < x_2$ in I.[4]

[4] Note that we can't define an increasing or decreasing function based on its derivative since $y = x^3$ is increasing but it has a derivative of zero at $x = 0$.

Example 5: Prove that $f(x) = x^2 + 3$ is increasing for $x > 0$.

Consider any two numbers x_1, x_2 such that

$$0 < x_1 < x_2. \tag{67}$$

Multiplying by x_1, we get:

$$0 < x_1^2 < \boxed{x_1 x_2}. \tag{68}$$

We can also multiply by x_2, we get:

$$0 < \boxed{x_1 x_2} < x_2^2 \tag{69}$$

Comparing these two inequalities by comparing the boxed compressions, we show that:

$$x_1^2 < x_2^2 \tag{70}$$

$$x_1^2 + 3 < x_2 + 3 \tag{71}$$

$$f(x_1) < f(x_2) \tag{72}$$

Therefore, $f(x)$ is increasing on the interval $x > 0$.

Example 6: Prove that $f(x) = x^2 + 3$ is decreasing for $x < 0$.

Consider any two numbers x_1, x_2 such that

$$x_1 < x_2 < 0. \tag{73}$$

Multiplying by x_1, we get:

$$x_1^2 > \boxed{x_1 x_2} > 0 \tag{74}$$

since we are multiplying by a negative number. We can also multiply by x_2 (which is also negative) to get:

$$\boxed{x_1 x_2} > x_2^2 > 0 \tag{75}$$

Comparing these two inequalities by comparing the boxed expressions, we show that:

$$x_1^2 > x_2^2 \tag{76}$$

$$x_1^2 + 3 > x_2 + 3 \tag{77}$$

$$f(x_1) > f(x_2) \tag{78}$$

Therefore, $f(x)$ is decreasing on the interval $x < 0$.

- We can also define odd and even functions:

Definition: A function $f(x)$ is even if $f(-x) = f(x)$ for all x in the domain of $f(x)$. Similarly, $f(x)$ is odd if $f(-x) = -f(x)$ for all x in the domain of $f(x)$.

- For arithmetic equalities and inequalities, we are given *arithmetic statements*:
  - Equality (e.g. $1 + 2 = 6 - 3$)
  - Inequality (e.g. $3 < 5$)

Theorem: You can add, subtract, multiply, or divide both sides by the same factor (positive or negative) and get another true statement.

However, for *inequalities* with a *negative factor* and for *multiplication and division* only, one has to *reverse the sign* of the inequality.[a]

[a]To convince yourself why this is true, suppose we start with $3 < 5$ and multiply both sides by -7. Is it true that $-21 < -35$?

4 Lecture 4

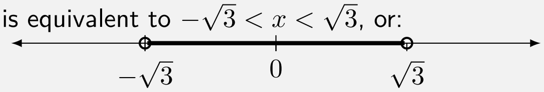
- An arithmetic equation or statement can be true or false. An *algebraic* equation (e.g. $x + 3 = 7x - 1$) is neither true or false. Instead it is used to *specify* a value of x (i.e. a *number*)
- An arithmetic inequality is either true or false. However, an algebraic inequality specifies a set of x. AS a result, we need to make a clear distinction between an arithmetic statement with an algebraic equality or inequality.
- Algebraic inequalities follow the same rules as arithmetic inequalities as summarized in lecture 3.

Example 7: Multiply both sides of $x < -4$ by -3. Do we get the same set of $x's$

$$(-3)(x) > (-3)(4) \tag{79}$$
$$-3x > -12 \tag{80}$$
$$-x < 4 \tag{81}$$

For more practice, refer to the example in Lecture 3 for a discussion on how to prove $x^2 + 3$ decreases for $x < 0$.

Example 8: Show that $x^2 < 3$ is equivalent to $-\sqrt{3} < x < \sqrt{3}$, or:



Note that $x^2 \geq 0$ for all x. $\therefore \sqrt{x^2} < \sqrt{3}$. This is exactly equivalent to defining $|x| = \sqrt{x^2} < \sqrt{3}$. Therefore, there are two possibilities. For $x \geq 0$:

$$x \geq 0 \tag{82}$$
$$\sqrt{x^2} = x \tag{83}$$
$$x < \sqrt{3} \tag{84}$$
$$0 \leq x < \sqrt{3} \tag{85}$$

and similarly for $x < 0$, we have:

$$x < 0 \tag{86}$$
$$\sqrt{x^2} = -x \tag{87}$$
$$-x < \sqrt{3} \tag{88}$$
$$-\sqrt{3} < x < 0 \tag{89}$$

Combining, we get:

$$-\sqrt{3} < x < \sqrt{3} \tag{90}$$

Example 9: What set of x's is represented by $5(x^2 - x - 6) > 0$?

Note that the 5 has no effect. Factoring, we get:

$$(x - 3)(x + 2) > 0 \tag{91}$$

We **break it up** into different cases. First, both factors can be positive. This means that:

$$x - 3 > 0 \tag{92}$$
$$\text{and } x + 2 > 0 \tag{93}$$
$$\therefore x > 3 > -2 \tag{94}$$

which gives $x > 3$. Second, both factors can be negative. This means that:

$$x - 3 < 0 \tag{95}$$
$$\text{and } x + 2 < 0 \tag{96}$$
$$\therefore x < -2 < 3 \tag{97}$$

which gives $xM - 2$. Therefore, the set of x's that satisfy this inequality is:

$$x \in (-\infty, -2) \cup (3, \infty) \tag{98}$$

We should also perform some checks, such as picking numbers in the range $x < -2$, $-2 \leq x \leq 3$, and $x > 3$ to see if they match up with our solution.

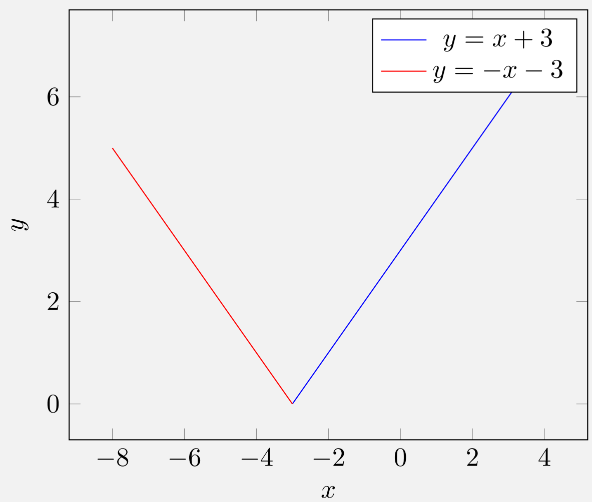
- Similarly, we need a systematic method to approach absolute value functions.

Example 10: What does $f(x) = |x + 3|$ look like?

Intuitively, this should look like the absolute value function $|x|$ but shifted three to the left. We can show this rigorously and algebraically by writing

$$f(x) = \begin{cases} x + 3, & \text{if } (x + 3) \geq 0 \implies x \geq -3 \\ (-x + 3), & \text{if } (x + 3) < 0 \implies x < -3 \end{cases} \tag{99}$$

which we can plot below as:



Example 11: What values of x satisfy $|x + 3| = 5$?

There are two possibilities. First,

$$(x + 3) \geq 0 \implies x \geq -3 \tag{100}$$

Therefore:

$$x + 3 = 5 \implies x = 2 \tag{101}$$

which satisfies the initial restriction posed. The second possibility is when the expression inside the absolute value function is negative:

$$(x + 3) < 0 \implies x < -3 \tag{102}$$

Therefore:

$$-x - 3 = 5 \implies x = -8 \tag{103}$$

which satisfies the $x < -3$ condition. As a result, both $x = 2$ and $x = -8$ satisfy this equality.

Example 12: What values of x satisfy $|x + 3| < 5$?

There are two possibilities. First,

$$(x + 3) \geq 0 \implies x \geq -3 \tag{104}$$

Therefore:

$$x + 3 < 5 \implies x < 2 \tag{105}$$

so we have $-3 \leq x < 2$. The second possibility is when the expression inside the absolute value function is negative:

$$(x + 3) < 0 \implies x < -3 \tag{106}$$

Therefore:

$$-x - 3 < 5 \implies x > -8 \tag{107}$$

so we can also have $-8 < x < -3$. Combining them both together, we get:

$$-8 < x < 2 \tag{108}$$

Idea: Note that the above example represents a *band* of x's centered on -3 of half-width 5. This means that an inequality such as:

$$|x - 7| < 2 \tag{109}$$

represents a band centered on 7 of half-width 2.

This leads to the introduction of $\delta - \epsilon$ proofs, where we want the restriction

$$|x - c| < \delta \tag{110}$$

where c, δ are given numbers. c may be negative or positive but $\delta > 0$ is always positive. This gives the set of x's that satisfy:

$$c - \delta < x < c + \delta \tag{111}$$

We will soon use this in defining the limit where it will be important to exclude the center point $x = c$. We can do this by rewriting the inequality as:

$$0 < |x - c| < \delta \tag{112}$$

forbidding the $x = c$ case. Similarly, we can write:

$$|f(x) - L| < \epsilon \tag{113}$$

to denote the fact that the y value is "sandwiched" in between $c - L$ and $c + L$.

5 Lecture 5

- We want to rigorously create the rigorous definition of a limit. For example, how does one show that for a function such as equation (12), that the limit:
$$\lim_{x \to 0} f(x) \tag{114}$$
does exist.

Idea: We need to build a rigorous "test-definition" for the new number $\lim_{x \to c} f(x)$. We need to be given:

1. c, some particular x value
2. $f(x)$ which may not exist *at* c, but $f(x)$ *is* defined for **all** x *near* c.
3. A guess or candidate for $\lim_{x \to c} f(x)$ which we call L

It is then imposed on you some positive number $\epsilon > 0$, which may be extremely small but never zero. Note that we are not told this exact value for ϵ and will have to allow for *any* $\epsilon > 0$.

The challenge-test is: "Can you find some number $\delta > 0$[a], such that for all x's in the $x - band$, (i.e. in the set $0 < |x - c| < \delta$) the corresponding values of f fall somewhere inside the y-band, i.e. the set $|f(x) - L| < \epsilon$.

Note that only δ is under your control.

[a] Here δoggy δoggy!

6 Lecture 6

- We continue to define the definition of the limit.

> **Definition**: If for any $\epsilon > 0$, a $\delta > 0$ can be found such that for all $0 < |x - c| < \delta$, it can be proved that $|f(x) - L| < \epsilon$, then we can define $\lim_{x\to c} f(x)$ to be a real number and we can assign it to the value L.

- A useful tool is the expression $x = \min\{a, b\}$. For example, for every, for every member you introduce to your club, you get $10, up to a maximum of $50. The expression is then $x = \min\{10N, 50\}$ where N is the number of new numbers.

Example 13: Prove that $\lim_{x\to 5} x^2 = 25$.

1. $\epsilon > 0$ is specified.
2. It is required that $|f(x) - L| < \epsilon$ or
$$|x^2 - 25| < \epsilon \tag{115}$$
3. when $0 < |x - c| < \delta$ or:
$$0 < |x - 5| < \delta \tag{116}$$
4. The left hand side of (2) becomes:
$$\begin{aligned} LHS &= |x^2 - 25| && (117)\\ &= |(x-5)(x+5)| && (118)\\ &= |x-5||x+5| \leq \delta|x+5| && (119)\end{aligned}$$
where we have applied the basic theorem of algebra in the last step. We now need to specify a *second* feature of δ, additional to anything we will specify in ste (5), i.e. in terms of ϵ. Here, we can specify a guess: $\delta \leq 1$. From (3),
$$\begin{aligned} |x-5| &< \delta \leq 1 && (120)\\ 5-1 \leq\ & x \leq 5+1 && (121)\\ 4 \leq\ & x \leq 6 && (122)\\ 9 \leq\ & x+5 \leq 11 && (123)\end{aligned}$$
Note that $x + 5 \geq 9$, then $x + 5 > 0$. As a result, it is positive and:
$$|x+5| = (x+5) \tag{124}$$
Note also that $x + 5 \leq 11$. This is helpful when comparing it to equation (119). Therefore:
$$LHS \leq \delta|x+5| \leq 11\delta \tag{125}$$
5. We now need to pick δ in terms of ϵ. We can try:
$$\delta = \frac{\epsilon}{11} \tag{126}$$
then plugging it into :
$$LHS < 11 \cdot \frac{\epsilon}{11} = \epsilon \tag{127}$$
However, we don't forget our override condition:
$$\delta = \min\{\epsilon/11, 1\} \tag{128}$$

- We can similarly define left and right hand limits. The left hand limit can be written as:
$$\lim_{x\to 0^-} f(x) \tag{129}$$
and similarly for the right hand limit
$$\lim_{x\to 0^+} f(x) \tag{130}$$

- For a right hand limit:

> **Definition**: If for every $\epsilon > 0$, a $\delta > 0$ can be found such that for all $c < x < c + \delta$, one can prove $|f(x) - L| < \epsilon$, then $\lim_{x \to c^+} f(x) = L$

and we can similarly define it for the left hand limit.

- Note that:

$$\lim_{x \to c} f(x) = L \iff \lim_{x \to c^+} = \lim_{x \to c^-} f(x) = L \tag{131}$$

Example 14: Prove that $\lim_{x \to 0^+} x^{1/2} = 0$.

- $\epsilon > 0$ is specified.
- It is required that:

$$|\sqrt{x} - 0| < \epsilon \tag{132}$$

$$|\sqrt{x}| < \epsilon \tag{133}$$

$$\sqrt{x} < \epsilon \tag{134}$$

- when $0 < x < \delta$.
- From (2) and (3), we have:

$$x^{1/2} < \delta^{1/2} \tag{135}$$

under δ control!

- Try $\delta = \epsilon^2$. Then:

$$|\sqrt{x} - 0| < \delta^{1/2} = \epsilon \tag{136}$$

and we are done. We can also write this compactly.

Given $\epsilon > 0$, choose $\delta = \epsilon^2$, then when $0 < x < \delta$, $|\sqrt{x} - 0| < \epsilon$, therefore:

$$\lim_{x \to 0^+} \sqrt{x} = 0. \tag{137}$$

- We can also deal with infinite limits, such as:

$$\lim_{x \to 0} \frac{1}{x^4} \tag{138}$$

We can approach this rigorously: Imagine your ene$M$y imposes some very large number $M > 0$, say $M = 10^6$. The challenge then becomes: "Can you find a $\delta > 0$ such that for all $0 < |x - 0| < \delta$, such that $f(x) > M$?" If yes, we can write:

$$\lim_{x \to 0} \frac{1}{x^4} = \infty \tag{139}$$

> **Warning**: Note that this is **not** an equation as ∞ is **not** a number! All this does is a compact way of saying "$f(x)$ increases without limit as x approaches 0."

7 Lecture 7

- Suppose we wish to rigorously prove an infinite limit.

Example 15: Prove $\dfrac{1}{x^4} = \infty$.

1. Given: $M > 0$
2. It is required that $f(x) > M \implies \frac{1}{x^4} > M$
3. when $0 < |x - c| < \delta \implies 0 < |x| < \delta$
4. LHS of 2 is $|x| < \delta \implies \frac{1}{\delta^4} < \frac{1}{|x|^4}$. Therefore, the LHS of (2) is under δ control.
5. Judgement: Try $\delta = \frac{1}{M^{1/4}}$. Therefore:

$$\text{LHS(2)} > \frac{1}{\delta^4} = M \tag{140}$$

Given $M > 0$, choose $\delta = \frac{1}{M^{1/4}}$ then when $0 < |x - 0| < \delta$, we have $\frac{1}{x^4} > M$.

- Instead of having to prove $\lim_{x\to c} f(x) = L$. What if we were given this statement? This leads to limit theorems:

Theorem: Given $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then:

$$\lim_{x\to c} [f(x) + g(x)] = L + M \tag{141}$$

Proof:

1. Given: $\epsilon > 0$ is specified.
2. It is required that:

$$|f(x) + g(x) - L - M| < \epsilon \tag{142}$$

3. ... when $0 < |x - c| < \delta$
4. The left hand side of (2) is:

$$|(f(x) - L) + (g(x) - M)| \leq |f(x) - L| + |g(x) - M| \tag{143}$$

   where we have applied the triangle inequality. Note that $\lim_{x\to c} f(x) = L$ is given here. As a result, we can now play the role of the ϵnemy and we can specify an number $\epsilon_f > 0$ we want. Suppose we choose $\epsilon_f = \frac{\epsilon}{2}$. It is then guaranteed that some number $\delta_f > 0$ exists for sure such that for all $0 < |x - c| < \delta_f$ it can be proved that $|f(x) - L| < \epsilon_f = \frac{\epsilon}{2}$.

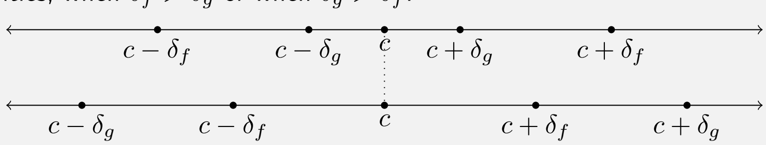
   Similarly for $g(x)$ we can impose any $\epsilon_g > 0$ we want, say $\epsilon_g = \epsilon/2$, and it is guaranteed that some $\delta_g > 0$ exists such that for all $0 < |x - c| < \delta_g$, then $|g(x) - M| < \epsilon_g = \epsilon/2$. Next, note that if x is inside both δ_f and δ_g bands then:

$$\text{LHS (2)} < \epsilon/2 + \epsilon/2 = \epsilon \tag{144}$$

   as requested.

5. Therefore, pick $\delta = \min\{\delta_f, \delta_g\}$.

There are two possibilities, when $\delta_f > \delta_g$ or when $\delta_g > \delta_f$:



so when $0 < |x - c| < \delta = \min\{\delta_g, \delta_f\}$, then x satisfies both $0 < |x - c| < \delta_f$ and $0 < |x - c| < \delta_g$.

Theorem: The product limit theorem says that $_{x\to c} f(x)g(x) = LM$ provided that both limits exist:

$$\lim_{x\to c} f(x) = L, \ \lim_{x\to c} g(x) = M \tag{145}$$

Do not use this theorem unless both limits exist!

Theorem: The polynomial limit theorem says that $\lim_{x\to c} P_n(x) = P_n(c)$ given that $P_n(x)$ is a polynomial.

Theorem: The rational function limit theorem:

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{1}{M},\ M \neq 0 \tag{146}$$

and only applies when both limits exist.

Theorem: The root limit theorem says that:

$$\lim_{x\to c} f(x)^{1/n} = L^{1/n} \tag{147}$$

All of these limit theorems can be proven the same way as the additivity limit theorem, but proofs are not assigned. Therefore, the following is a completely rigorous proof.

Example 16: Determine $\lim_{x\to -2} \frac{x^2-x-6}{x^2-4}$.

We can write it as:

$$\lim_{x\to -2} \frac{x^2-x-6}{x^2-4} = \lim_{x\to -2} \frac{(x-3)\cancel{(x+2)}}{\cancel{(x+2)}(x-2)} \quad \text{(ok since } x+2\neq 0 \text{ here.)} \tag{148}$$

$$= \frac{\lim_{x\to -2}(x-3)}{\lim_{x\to -2}(x-2)} \quad \text{(rational function LT)} \tag{149}$$

$$= \frac{-2-3}{-2-2} \quad \text{(polynomial LT)} \tag{150}$$

$$= \frac{5}{4} \tag{151}$$

- Another useful limit theorem is the sandwich limit theorem.

Theorem: Given:
- $\lim_{x\to c} f(x) = L$
- $\lim_{x\to c} h(x) = L$
- $f(x) \leq g(x) \leq h(x)$ near c, but not necessarily *at* c.

Then:

$$\lim_{x\to c} g(x) = L \tag{152}$$

Example 17: Determine $\lim_{x\to 0} x^2 \cos^2\left(\frac{1}{x^2}\right)$

We might naively try to apply the product LT, but $\lim_{x\to 0} \cos^2\left(\frac{1}{x^2}\right)$ is not defined! Instead, we can apply the sandwich LT. Note that:

$$0 \leq \cos^2\left(\frac{1}{x^2}\right) \leq 1 \tag{153}$$

provided that $x \neq 0$. We can multiply both sides by x^2 since it is a positive quantity, then:

$$0 \leq x^2 \cos^2\left(\frac{1}{x^2}\right) \leq x^2 \tag{154}$$

We can rigorously find the limits of the two extremes of the inequality. We can define:

$$f(x) \equiv 0 \tag{155}$$

$$g(x) \equiv x^2 \cos^2\left(\frac{1}{x^2}\right) \tag{156}$$

$$h(x) \equiv x^2 \tag{157}$$

Note that:

$$\lim_{x\to 0} f(x) = 0 \tag{158}$$

$$\lim_{x\to 0} h(x) = 0 \tag{159}$$

so by the sandwhich limit theorem:

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} x^2 \cos^2\left(\frac{1}{x^2}\right) = 0. \tag{160}$$

8 Lecture 8

- A "continuous function" is intuitively clear, but how do we define it rigorously?

Definition: $f(x)$ is "continuous at c" if
$$\lim_{x\to c} f(x) = f(c) \tag{161}$$

Definition: A function $f(x)$ is discontinuous at c if it is not continuous.

- There are various types of discontinuity:
  - **Jump Discontinuity**- For example:
  $$f(x) = \frac{|x|}{x} \tag{162}$$
  - **Removable Discontinuity**- For example:
  - Discontinuity because either $f(c)$ DNE or $\lim\limits_{x\to c} f(x)$ DNE, or both.
- There are also continuity theorems.

Theorem: If $f(x)$ is continuous at every $x \in [a, b]$, then $f(x)$ is integrable on $[a, b]$ i.e. $\int_a^b f(x)\,\mathrm{d}x$ exists.

Theorem: Given f, g is continuous at a, then $f(x) + g(x)$ is continuous at a.

Proof: Apply the additivity L.T lmao.

- There is also such a thing as a one-sided continuity.

Definition: $f(x)$ is continuous on the right at c if $\lim\limits_{x\to c^+} f(x) = f(c)$.

- amd for an interval.

Definition: $f(x)$ is continuous on (a, b) iff $f(x)$ is continuous at all $x \in (a, b)$.

Definition: $f(x)$ is continuous on $[a, b]$ iff $f(x)$ is continuous on (a, b) and $f(x)$ is continuous from the right of a and from the left of b.

Theorem: If $g(x)$ is continuous at a and $f(x)$ is continuous at $g(a)$, then $f(g(x))$ is continuous at a.

- We can now introduce the **Intermediate Value Theorem**. Recap: We simply want there to be a number q such that $q \cdot q = 2$, i.e. $\sqrt{2}$ exists. We proved such a number does not exist among the rationals. So we imposed a new axiom CORA and then defined:
$$\sqrt{2} = \text{lub}\{x : x^2 < 2\} \tag{163}$$
which CORA guarantees exists as a real number. This however doesn't tell us that $\sqrt{2} \cdot \sqrt{2} = 2$, but we want to use this result. How can we rigorously prove that:
$$[\text{lub}\{x : x^2 < 2\}] \cdot [\text{lub}\{x : x^2 < 2\}] = 2 \tag{164}$$

Theorem:

1. Given that $f(x)$ is continuous on $[a, b]$
2. C is some number such that $f(a) < G(a) < f(b)$.
3. There exists some C in $[a, b]$ such that $f(C) = G$.

- The point of the IVT is that continuous functions don't skip over any y values. Note that this is a property that intuitively we want continuous functions to have.

Example 18: Prove that there is a number c such that $c \cdot c = 2$ using IVT rather than CORA directly.

Consider $f(x) = x^2$ on $[1, 2]$. It is easy to show that $f(x)$ is continuous on $[1, 2]$ per the polynomial continuous theorem. Here $f(1) = 1$ and $f(2) = 4$. Note that $1 < 2 < 4$ so $f(1) < 2 < f(2)$.

The IVT shows that there must be some number c where $1 < c < 2$ in this interval such that $f(c) = c \cdot c = 2$.

Note that if reals consisted of rationals only, then the IVT would not be true!

- Logically we could have started with the IVA (Intermediate Value Axiom) and used it to prove CORT (Completeness of Reals Theorem). But this would be messy: we would need to define functions before we had finished defining numbers.

Example 19: Prove there's at least one root of $x^{17} + 1 = 3x$ in $[0, 1]$.

Define $f(x) = x^{17} + 1 - 3x$. $f(x)$ is continuous by polynomial continuity theorem. $f(0) = 1$ and $f(1) = -1$, so $f(0) > 0 > f(1)$. By IVT there is some c in $(0, 1)$ such that $f(c) = 0$. Therefore c is a root of this equation.

9 Lecture 9

- We want to define a new number $f'(a)$ where it is defined as:

$$f'(a) \equiv \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{165}$$

if it exists where $a \in$ domain of $f(x)$. Note this is known as the derivative of $f(x)$ evaluated at $x = a$.

Example 20: Determine the derivative of $f(x) = x^3$ at $a - 1$. We can evaluate:

$$\begin{aligned}
f'(1) &= \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} && (166)\\
&= \lim_{h \to 0} \frac{(1+h)^3 - 1}{h} && (167)\\
&= \lim_{h \to 0} \frac{1 + 3h + 3h^2 + h^3 - 1}{h} && (168)\\
&= \lim_{h \to 0} \frac{3h + 3h^2 + h^3}{h} && (169)\\
&= \lim_{h \to 0} (3 + 3h + h^2) && \text{provided that } h \neq 0 \quad (170)\\
&= 3 && \text{polynomial limit theorem} \quad (171)
\end{aligned}$$

- h disappears when evaluating the limit, as a result, it is known as a **dummy variable**.
- The tangent line to the curve $f(x)$ at $x = a$ is given by the equation:

$$y_{\text{tangent line}}(x) \equiv f(a) + f'(a)(x - a) \tag{172}$$

- We can define average speed between t_0 and $t_0 + h$ to be:

$$v_{\text{avg}} = \frac{s(t_0 + h) - s(t_0)}{h} \tag{173}$$

where $s(t)$ is the displacement. We can use our rigorous definition of the derivative to make a rigorous definition of "speed at an instant t_0" to be:

$$v(t_0) \equiv \lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h} \tag{174}$$

- We can extend further and define a new *function* $f'(x)$ such that:

$$f'(x) \equiv \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{175}$$

There are two variables here, but it is still rigorously defined since h is still a dummy variable and it will still disappear after we evaluate the limit.

Definition: If $f'(a)$ exists we say $f(x)$ is differentiable at a.

Definition: If $f'(a)$ is differentiable at all $x \in$ domain of $f(x)$, we can say that $f(x)$ is a differentiable function.

Example 21: Let $f(x) = x^2$. Find $f'(x)$:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} && (176)\\
&= \lim_{h\to 0} \frac{(x_2^2 h x_h^2 - x^2}{h} && (177)\\
&= \lim_{h\to 0} \frac{2hx + h^2}{h} && (178)\\
&= \lim_{h\to 0} (2x + h) && (179)
\end{aligned}$$

where we canceled as $h \neq 0$. This is a polynomial function of h as far as $\lim_{h\to 0}$ is concerned! Therefore, by the polynomial limit theorem, we get:

$$f'(x) = 2x \tag{180}$$

Example 22: Let $f(x) = x^n$. Prove that $f'(x) = nx^{n-1}$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+n)^n - x^n}{h} && (181)\\
&= \lim_{h\to 0} \frac{x^n + nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \cdots + h^n - x^n}{h} && (182)\\
&= \lim_{h\to 0} \frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \cdots + h^n}{h} && (183)\\
&= \lim_{h\to 0} nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + \cdots + h^{n-1} && (184)\\
&= nx^{n-1} && (185)
\end{aligned}$$

Later we will show that this is true for any real number, and not just for positive integers.

- Note that there are different notations. For example, the Leibniz notation gives us:

$$f'(x) \equiv \frac{df}{dx} \tag{186}$$

10 Lecture Ten

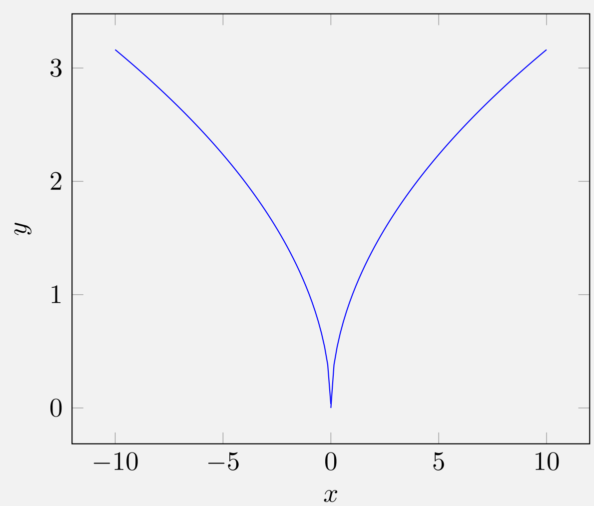
- A few formal definitions:

Definition: $f(x)$ is differentiable on (a, b) if $f(x)$ is differentiable at all $x \in (a, b)$.

Definition: $f(x)$ is differentiable on $[a, b]$ if:
  - $f(x)$ is differentiable on (a, b).
  - The right hand derivative at a exists.
  - The left hand derivative at b exists.

Example 23: There is a cusp in the following example:

Graph with a cusp



Another example would be the absolute value of x, $|x|$.

Warning: From above example, $f(x)$ may be continuous at a point but no differentiable. Differentiability is rarer than continuity!

Theorem: Given $f(x)$ is differentiable at a, then $f(x)$ is continuous at a.

Proof: Consider

$$f(a+h) - f(a) = \left[\frac{f(a+h) - f(a)}{h}\right] h \tag{187}$$

which is acceptable if $h \neq 0$. Then:

$$\begin{aligned} \lim_{h \to 0} \{f(a+h) - f(a)\} &= \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h}\right] \cdot \lim_{h \to 0} h && \text{(Product LT)} \quad (188) \\ &= 0 && \text{(Polynomial LT)} \quad (189) \end{aligned}$$

Note that the use of the product limit theorem requires that both limits exist. We know the first limit exists since we are given that $f'(a)$ exists. As a result:

$$\lim_{h \to 0} \{f(a+h) - f(a)\} = 0 \tag{190}$$

$$\lim_{h \to 0} f(a+h) - \lim_{h \to 0} f(a) = 0 \tag{191}$$

Note that per the polynomial limit theorem, we have $\lim_{h \to 0} f(a) = f(a)$. Making the substitution $x = a + h \implies h = x - a$, we can rewrite our expression as:

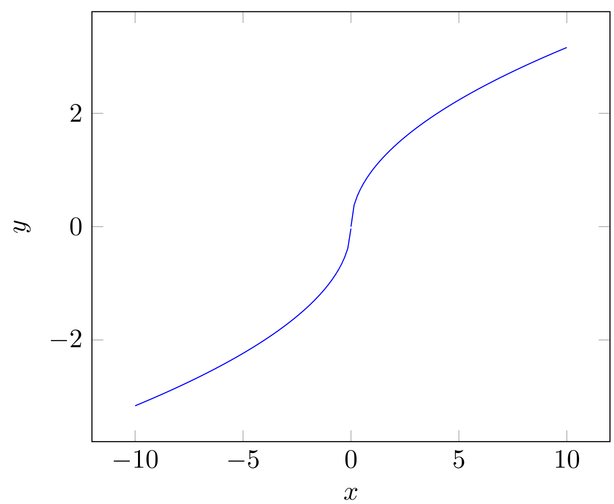
$$\lim_{h \to 0} f(a+h) = \lim_{x \to a} f(x) \tag{192}$$
$$\lim_{x \to a} f(x) = f(a) \tag{193}$$

and therefore $f(x)$ is continuous at $x = a$. Note that the reverse isn't necessarily true!

- **Vertical Tangent Lines** can exist. For example, if $f(x) = x^{1/3}$, then:

$$f'(x) = \frac{1}{3} x^{-2/3} \tag{194}$$

Vertical Tangent Line Example



Definition: A vertical tangent occurs when

$$\lim_{x \to c} |f'(x)| = \infty \tag{195}$$

and $f(x)$ is continuous at c.

- There are a few derivative theorems:

Theorem: The **constant derivative theorem**: For $f(x) = C$, then $f'(x) = 0$.

Theorem: Additivity D.T:

$$(f+g)' = f' + g' \tag{196}$$

which is true if both exist.

Theorem: The product D.T. is:

$$(fg)' = f'g + fg' \tag{197}$$

Theorem: The Power D.T: For $f(x) = Cx^n$, then $f'(x) = nCx^{n-1}$.

Theorem: The polynomial D.T. says that:

$$P_n'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1 \tag{198}$$

Theorem: The reciprocal function D.T. says that:

$$\left(\frac{1}{f}\right)' = \frac{-f'}{f^2} \tag{199}$$

Proof: We can write

$$\left(\frac{1}{f}\right)' = \lim_{h\to 0}\left\{\frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}\right\} \tag{200}$$

$$= \lim_{h\to 0}\left\{\frac{f(x) - f(x+h)}{hf(x)f(x+h)}\right\} \tag{201}$$

$$= \overbrace{\underbrace{\lim_{h\to 0}\left\{\frac{f(x) - f(x+h)}{h}\right\}}_{A} \cdot \underbrace{\lim_{h\to 0}\frac{1}{f(x)}}_{B} \cdot \underbrace{\lim_{h\to 0}\frac{1}{f(x+h)}}_{C}}^{\text{product limit theorem}} \tag{202}$$

Let us now deal with each limit individually. We have:

$$A = -f'(x) \tag{203}$$

per definition. For B, we can apply the constant limit theorem to get:

$$B = \frac{1}{f(x)} \tag{204}$$

To tackle C, because $\frac{1}{f(x)}$ is differentiable, it is continuous, so we can invoke the definition of continuity to get:

$$C = \frac{1}{f(x)} \tag{205}$$

and combining everything together:

$$\left(\frac{1}{f}\right)' = -\frac{f'(x)}{f(x)^2} \tag{206}$$

Example 24: If $f(x) = x^4$, what is $f'(x)$?

Set $g(x) \equiv x^4$, then $f(x) = \frac{1}{g(x)}$. Therefore:

$$f'(x) = -\frac{g'(x)}{g(x)^2} \qquad \text{reciprocal LT} \tag{207}$$

$$= \frac{-4x^3}{(x^4)^2} \tag{208}$$

$$= -4x^{-5} \tag{209}$$

So we have proved $(x^n)' = nx^{n-1}$ for even when n is a negative integer!

Theorem: The quotient derivative theorem says:

$$(f/g)' = \frac{f'g - fg'}{g^2} \tag{210}$$

- We can now tackle **rates of change**. The volume of a sphere is $V = \frac{4}{3}\pi r^3$. Therefore:

$$\frac{dV}{dr} \equiv V' = \frac{4}{3}\pi \underbrace{(3r^2)}_{P.D.T} = \underbrace{4\pi r^2}_{\text{surface area of the sphere!}} \tag{211}$$

Idea: Intuitively, this makes sense! We can interpret the derivative (in Leibniz notation) gives us that $\frac{dV}{dr}$ is a fraction. If we write it as small increments, then:

$$\underbrace{\Delta V}_{\text{small increment of volume}} \simeq 4\pi r^2 \underbrace{\Delta r}_{\text{small increment of radius}} \tag{212}$$

Note that this is only approximate. We can get the actual change in volume as:

$$\Delta V_{\text{actual}} = \frac{4}{3}\pi \left[(r + \Delta r)^3 - r^3\right] \tag{213}$$

$$= \Delta V_{\text{approx}} \left(1 + \frac{\Delta r}{r} + \underbrace{\frac{1}{3}\left(\frac{\Delta r}{r}\right)^2}_{\text{goes to zero}} \right) \tag{214}$$

Therefore As $\Delta r \to 0$, we then have:

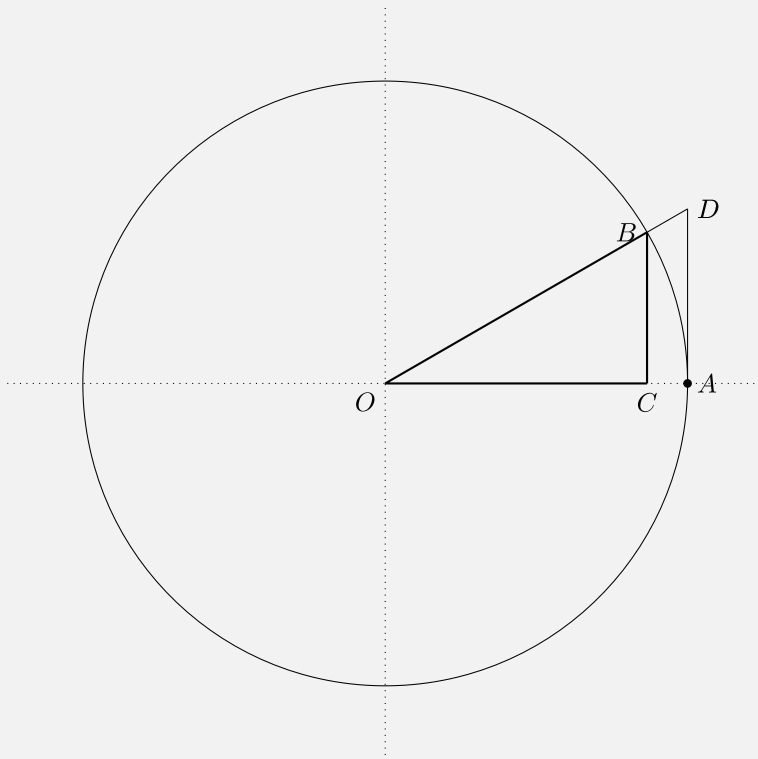
$$\Delta V_{\text{approx}} \to \Delta V_{\text{actual}} \tag{215}$$

11 Lecture 11: Trig Functions / Derivatives

- We can deal with trigonometric functions.

Example 25: Prove $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Note that we cannot use the product limit theorem and both do not exist at zero. We can do this geometrically by drawing a unit circle:



Let $x \equiv \angle BOC$. Then the area of $\triangle OBA$ is:

$$[\triangle OBA] = \frac{1}{2} \sin x \cdot 1 = \frac{1}{2} \sin x \tag{216}$$

The area of sector OBA is then:

$$[OBA] = \frac{1}{2} x \cdot 1^2 = \frac{1}{2} x \tag{217}$$

The area of $\triangle DOA$, using the fact that $DA = \tan x$ is:

$$[\triangle DOA] = \frac{1}{2} \tan x \cdot 1 = \frac{1}{2} \tan x \tag{218}$$

Therefore it is geometrically obvious that:

$$\sin x \leq x \leq \tan x \tag{219}$$

We can divide by two to get:

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x} \tag{220}$$

which is equivalent to:

$$\cos x \leq \frac{\sin x}{x} \leq 1 \tag{221}$$

We can then use the sandwich L.T. to prove that the limit is equal to one.

- We can find the derivative of sine functions:

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} && (222)\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} && (223)\\
&= \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h}+\lim_{h\to 0}\cos x\frac{\sin h}{h} && (224)\\
&= \lim_{h\to 0}\sin x\cdot\lim_{h\to 0}\frac{\cos h-1}{h}+\lim_{h\to 0}\cos x\cdot\lim_{h\to 0}\frac{\sin h}{h} && (225)\\
&= \sin x\cdot 0+\cos x\cdot 1 && (226)\\
&= \cos x && (227)
\end{aligned}$$

Similarly we can show that:

$$\frac{d}{dx}\cos x = -\sin x \tag{228}$$

- For composite functions, we introduce the chain rule:

Theorem: The chain rule is given by:

$$f'(x) = f'(u)u'(x) \tag{229}$$

and is highly suggestive when written in Leibniz notation:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} \tag{230}$$

Example 26: Suppose we have $f(x) = (3x^2+1)^{173}$. What is $f'(x)$?

$$u(x) \equiv 3x^2+1 \tag{231}$$

We can let $f(u) = u^{173} = f(u(x))$. Then:

$$\frac{du}{dx} = 6x \tag{232}$$

$$\frac{df}{du} = 173u^{172} \tag{233}$$

Therefore:

$$\begin{aligned}
\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} &= 6x(173u^{172}) && (234)\\
&= 6x(173)(3x^2+1)^{172} && (235)
\end{aligned}$$

- Sometimes we only have $y(x)$ in the form of an implicit relationship, such as:

$$x^3y^7 - x^2 + y^2 = 0 \tag{236}$$

Note that we can't write $y(x)$ explicitly. While it is less convenient, we can relate y and x with a table and via numerical methods, but we can do it analytically as well. The trick is to apply the $\frac{d}{dx}$ operator to both sides of the implicit equation:

$$\frac{d}{dx}\left(x^3y^7 - x^2 + y\right) = \frac{d}{dx}0 \tag{237}$$

$$\frac{d}{dx}(x^3y^7) - \frac{d}{dx}x^2 + \frac{d}{dx}(y) = 0 \tag{238}$$

(239)

The first term can be evaluated as:

$$\frac{d}{dx}(x^3y^7) = x^3\frac{d}{dx}y^7 + y^7\frac{d}{dx}x^3 \tag{240}$$

$$= 7x^3y^6\frac{dy}{dx} + 3x^2y^7 \tag{241}$$

After doing this for all terms, we get:

$$3x^2y^7 + 7x^3y^6y' - 2x + y' = 0 \tag{242}$$

and solving for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = \frac{2x - 3x^2y^7}{7x^3y^6 + 1} \tag{243}$$

Proof: To prove that $\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{p/q-1}$, we can use the chain rule.

Let $u \equiv x^{p/q}$, and we want to find $u(x)$. Note that:

$$u^q = x^p \tag{244}$$

Define $f(u) \equiv u^q = x^p$. Therefore, $f(u(x))$ is a composite function. We can use the chain rule to get:

$$\frac{df}{dx} = px^{p-1} \tag{245}$$

$$\frac{df}{du} = qu^{q-1} \tag{246}$$

$$\tag{247}$$

We can divide the two to get:

$$u' = \frac{df}{dx}/\frac{df}{du} = \frac{px^{p-1}}{qu^{q-1}} \tag{248}$$

Recall that since $u = x^{p/q}$, we can rewrite:

$$u^{q-1} = x^{p-p/q} \tag{249}$$

therefore simplifying the derivative to:

$$u' = \frac{p}{q}\frac{x^{p-1}}{x^{p-p/q}} \tag{250}$$

$$= \frac{p}{q}x^{p/q-1} \tag{251}$$

Therefore we have proved that:

$$\frac{d}{dx}x^n = nx^{n-1} \tag{252}$$

for any rational number n.

- We can also look at related rates now. For example, suppose we have the volume of a hailstone $V(t)$ and a radius $r(t)$ changing with time t. Suppose we are given that:

$$r(t) = 3t^2 + t \tag{253}$$

in appropriate units. To determine how fast V is changing at $t = 2$ min, then we can use the change rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} \tag{254}$$

$$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)(6t+1) \tag{255}$$

$$= (4\pi r^2)(6t+1) \tag{256}$$

Plugging everything in gives:

$$\left.\frac{dV}{dt}\right|_{t=2} = 20 \tag{257}$$

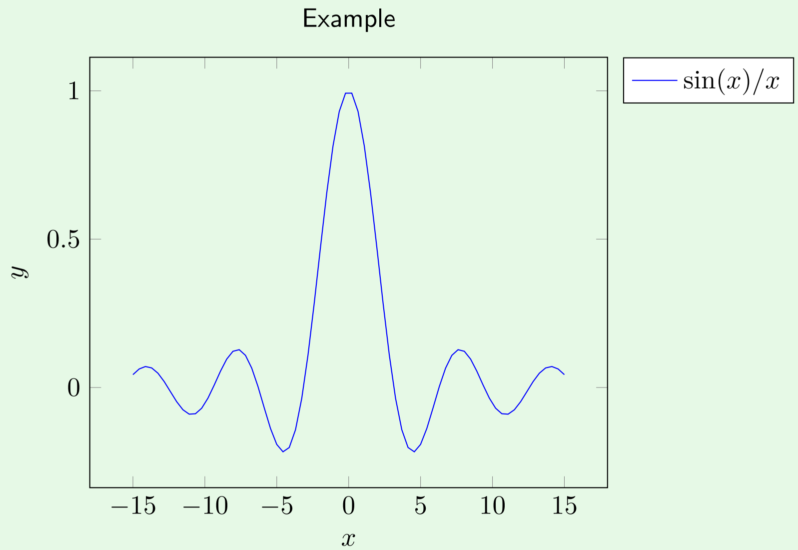
again in appropriate units since I'm too lazy to write them down.

12 Lecture 12: Applications of Derivatives

- Applications of Derivatives:

Definition: $f(x)$ has "an absolute maximum at c" if $f(c) \geq f(x)$ for all $x \in$ domain of $f(x)$. **Note** that $f(c)$ must exist!

For example, if $f(x) = \frac{\sin x}{x}$, it does not have an absolute maximum at $x = 0$!



Definition: $f(x)$ has a "absolute max on $[a, b]$ etc" if $f(c) \geq f(x)$ for all $x \in [a, b]$

Definition: $f(x)$ has a "local max at c" if $f(c) \geq f(x)$ for *some* open interval containing c.

Theorem: The **extreme value theorem** (EVT) says that given $f(x)$ is continuous on $[a, b]$, then $f(x)$ has an absolute maximum $f(c)$ and an absolute minimum $f(x)$ for some $c, d \in [a, b]$.

However, functions do not need to be continuous to have an absolute max.

Proof. The outline of the proof is as follows:

1. Prove all continuous functions on $[a, b]$ are **bounded**
2. Then prove all continuous functions on $[a, b]$ have a max and a min.

Note that this is not the same thing! Remember that $f(x) = \dfrac{\sin x}{x}$ on $[-1, 1]$ is bounded, but does not have an absolute maximum! However, this doesn't violate it since it's not continuous.

We will take (1) to be proven and just prove (2): Consider the set $S = \{f(x) : a \leq x \leq b\}$. Since S is a set of f-values from (1), S is bounded above. By CORA, lub(S) exists as a real number ,call it M. Therefore: $f(x) \leq M$ for all $x \in [a, b]$

We now need to prove that there is some $c \ni [a, b]$ such that $f(c) = M$, i.e. $f(x)$ takes on the value M. We can prove this via contradiction:

Suppose $f(x)$ never equals M. We can then define:

$$g(x) \equiv \frac{1}{M - f(x)} \tag{258}$$

Note that $g(x) > 0$. (It cannot be negative since $M > f(x)$). Therefore, $g(x)$ is also continuous on $[a, b]$ by A.C.T, Q.C.T, and by the fact that $f(x) \neq M$.

Therefore $g(x)$ is also bounded above by part (1). There exists a number K such that $0 < g(x) \leq K$ where

$K > 0$. Taking the inverse, we have:

$$\frac{1}{K} \leq \frac{1}{g(x)} \tag{259}$$
$$\frac{1}{K} \leq M - f(x) \tag{260}$$
$$f(x) \leq M - \frac{1}{K} \tag{261}$$

This makes $M - \frac{1}{K}$ an upper bound of S. However, if M is the least upper bound of S, this gives a contradiction.

Since there is a contradiction, there exists at least one $c \in [a, b]$ such that $f(c) = M$. Therefore, a maximum exists. □

- Fermat's Theorem:

Definition: c is a "critical point" of $f(x)$ if $f'(c) = 0$ or $f'(c)$ DNE.

- We have to be careful however, suppose we look at $f(x) = x^2$ on $x \in [1, 2]$. It is continuous by the polynomial C.T., and $f(2) = 4$ is an absolute maximum of $f(x)$ in $[1, 2]$. However, this doesn't violate Fermat's theorem since it's an absolute max, not a local max!
- Another example is $f(x) = x^3$. We have $f'(0) = 0$ but $f(0)$ is not a local max or min. We **cannot** reverse Fermat's theorem!

Idea: The motivation behind Fermat's theorem is as follows:

1. We often need to find local max, min.
2. But how can we?
3. It's usually easy to calculate $f'(x)$ and then find out where $f'(c) = 0$ or DNE.
4. While these critical points are not necessarily local max, min, only local max, min points will be in this set.[a]

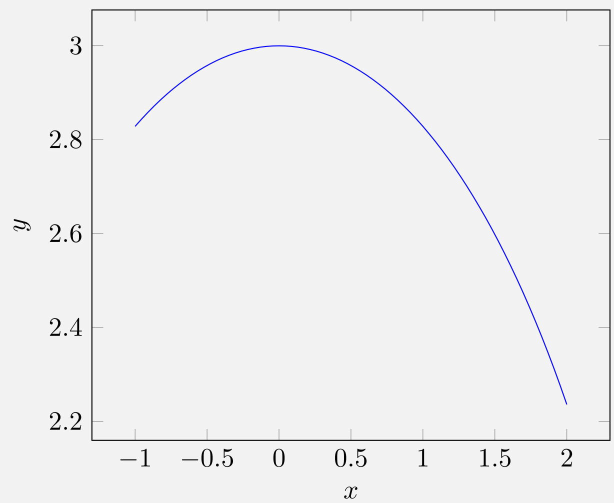
[a] Like using a net to catch fish. Not everything you catch will be the fish you want but the fish you want will be among it.

- This leads to a test for the absolute max/min on $[a, b]$. Given that $f(x)$ is continuous on $x \in [a, b]$. By the EVT there is an absolute max,min on $[a, b]$ for sure. Then, we can:

1. Find all c_{crit} and $f(c_{\text{crit}})$.
2. Find $f(a)$, $f(b)$.
3. The largest of these number is absolute max, and the smallest is the absolute min.

Example 27: Let $f(x) = (9 - x^2)^{1/2}$ on $x \in [-1, 2]$:

Example



We can find the derivative as:

$$f'(x) = \frac{1}{2}(9 - x^2)^{-1/2}(-2x) \tag{262}$$

using the chain rule, polynomial D.T., power D.T.

1. We now look for when $f'(c)0 = 0$, which only happens when $f(0) = 3$. When is it undefined? One might be tempted to say at -3 or 3 but they aren't in the interval.
2. $f(-1) = \sqrt{8}$ and $f(2) = \sqrt{5}$
3. Therefore, the absolute maximum is $f(0) = 3$ and the absolute minimum is $f(2) = \sqrt{5}$.