ESC195 Notes

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Contents

1 Maximum and Minimum Values

- We can extend the idea of maximum and minimum values to multiple dimensions.

Definition: f is said to have a local maximum at $\vec{x}_0$ iff $f(\vec{x}_0) \geq f(\vec{x})$ fir $\vec{x}$ in some neighbourhood of $\vec{x}_0$. f is said to have a local minimum at $\vec{x}_0$ iff $f(\vec{x}_0) \leq f(\vec{x})$ for $\vec{x}$ in some neighbourhood of $\vec{x}_0$.

Theorem: If f has a local extreme values at $\vec{x}_0$, then either $\nabla f(\vec{x}_0) = \vec{0}$ or $\nabla f(\vec{x}_0)$ DNE.

Proof. Let $g(x) = f(x, y)$. Then:

$$\frac{dg}{dx}(x_0) = 0 = \frac{\partial f}{\partial x}(x_0, y_0) \tag{1}$$

Then if $z = f(x, y)$, we have:

$$g(x, y, z) = z \cdot f(x, y) = 0 \tag{2}$$

which implies $\nabla g = \hat{k}$ for $f_x = f_y = 0$. □

Definition: Points where $\nabla f = \vec{0}$ or DNE called critical points.

Definition: Points where $\nabla f = \vec{0}$ are called stationary points.

Definition: Stationary points which are not local extremes are called saddle points.

Definition: Let $f(x, y) = 20 - x^2 - y^2$. The gradient is:

$$\nabla f = (-2x, -2y) \tag{3}$$

which exists everywhere, but is zero at $(0, 0)$ which is a stationary point. To see what type of stationary point it is, set $x = h$ and $y = k$ where h and k are very small. Then:

$$f(0, 0) = 20 \tag{4}$$

$$f(h, k) = 20 - k^2 - h^2 \leq 20 \tag{5}$$

for all h, k. As a result, $f(0, 0)$ is a local maximum.

Example 1: Suppose we have $f(x,y) = xy$. The gradient is:

$$\nabla f = (y,x) \tag{6}$$

If we set the gradient to $\vec{0}$, we find that it is equal to zero at $(0,0)$. Again, set $x = h$ and $y = k$. If both have the same sign, then it is positive. If they have opposite signs, then f is negative. This results in a saddle point.

Example 2: Suppose we have a function $f(x,y) = 2x^2 + y^2 - xy - 7y$. The gradient is:

$$\nabla f = (4x - y, 2y - x - 7) \tag{7}$$

which is zero at $(1,-4)$. To test if it's a maximum/minimum/saddle point, we can test points. However, this would require four test cases and I don't feel like writing it out. But if you do have the patience, you'll find that it's a local minimum.

Example 3: Suppose we have a cone given by $f(x,y) = -\sqrt{x^2+y^2}$. The gradient is:

$$\nabla f = (-(x^2+y^2)^{-1/2} \cdot 2x, -(x^2+y^2)^{-1/2} \cdot 2y) \tag{8}$$

which does not exist at $(0,0)$.

Theorem: **Second Derivatives Test:** For $f(x,y)$ with continuous second order partial derivatives, and $\nabla f(x_0,y_0) = \vec{0}$, set $A = \frac{\partial^2 f}{\partial x^2}(x_0,y_0)$, $B = \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)$, $C = \frac{\partial^2 f}{\partial y^2}(x_0,y_0)$. They form the discriminant[a]:

$$D = AC - B^2 \tag{9}$$

1. If $D < 0$, then (x_0,y_0) is a saddle point.
2. If $D > 0$, and $A, C > 0$, then (x_0,y_0) is a local minimum.
3. If $D > 0$, and $A, C < 0$ then (x_0,y_0) is a local maximum.

[a] This comes from more advanced calculus, but here's the intuition from where it'll come from (credit to Nathan): You can take linear approximations and they're planes, you can also take "quadric approximations" of surfaces at a stationary point and you would only get either a quadratic paraboloid (which is a max or min) or a hyperbolic paraboloid (has a saddle) - and this test basically looks at the quadric approximation and tells you which case it is.

Example 4: For $f(x,y) = xy$, at $(0,0)$ we have: $f_{xx} = A = 0$, $f_{yy} = C = 0$, and $f_{xy} = 1 = B$. Then:

$$D = AC - B^2 = -1 < 0 \tag{10}$$

so $(0,0)$ is a saddle point.

Example 5: For $f(X,y) = 2x^2 + y^2 - xy - 7y$, we have $\nabla f = \vec{0}$ at $(1,4)$. Then:

$$A = 4 \tag{11}$$
$$B = -1 \tag{12}$$
$$C = 2 \tag{13}$$

so $D = 8 - 1 = 7 > 0$ and we have a local minimum.

Theorem: If f is continuous on a bounded, closed set, then f takes on both an absolute minimum and an absolute maximum on that set.

Example 6: Let $f(x,y)=(x-4)^2+y^2$ on the set $\{(x,y): 0\leq x\leq 2, x^3\leq y\leq 4x\}$. The gradient is given by:

$$\nabla f = (2(x-4), 2y) \tag{14}$$

and the gradient is zero at $(4,0)$. To find the extrema, we have to look at the boundaries. The first boundary is $y=x^3$ from $0\leq x\leq 2$. We can parametrize this with $x=t$ such that $y=t^3$ from $0\leq t\leq 2$, and we have the vector function:

$$\vec{r}_1(t)=(t,t^3) \tag{15}$$

We then need to find when $f(\vec{r}_1(t))=f_1(t)$ has an extrema. Using the chain rule:

$$\begin{aligned} f_1'(t) &= \nabla f\cdot \vec{r}'(t) & (16)\\ &= (2(t-4), 2t^3)\cdot(1,3t^2) & (17)\\ &= 2t-8+6t^5 & (18)\end{aligned}$$

Setting $f_1'(t)=0$, we get $t=1$ or $(1,1)$ where $f(1,1)=10$. For this boundary, we can test the second derivative and get:

$$f_1''(t)=2+30t^4=32>0 \tag{19}$$

so that *in this boundary*, we have a local minimum.