MAT185 Notes

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January 14, 2021

Contents

1 Vector Spaces

- A vector space consistso f three things: a set V together with the two operations of vector addition and scalar multiplication. We use V to describe both the vector space and the set itself. Specifically:

> **Definition**: A **real vector space** is a set V together with two operations called *vector addition* and *scalar multiplication* such that the following *axioms* hold:
>
> 1. **Additive Closure (AC):** For all vectors $\boldsymbol{x}, \boldsymbol{y} \in V$, $\boldsymbol{x} + \boldsymbol{y} \in V$.
> 2. **Scalar Closure (SC):** For all vectors $\boldsymbol{x} \in V$, and scalars $c \in \mathbb{R}$, $c\boldsymbol{x} \in V$.
> 3. **Additive Associativity (AA):** For all vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$, $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$
> 4. **Zero Vector (Z):** There exists a unique vector $\mathbf{0} \in V$ with the property that $\boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$ for all vectors $\boldsymbol{x} \in V$.
> 5. **Additive Inverse (AI):** For each vector $\boldsymbol{x} \in V$, there exists a unique vector $-\boldsymbol{x} \in V$ with the property that $\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}$.
> 6. **Scalar Multiplication Associativity (SMA):** For all vectors $\boldsymbol{x} \in V$, and scalars $c,\ d \in \mathbb{R}$, $(cd)\boldsymbol{x} = c(d\boldsymbol{x})$.
> 7. **Distributivity of Vector Addition (DVA):** For all vectors $\boldsymbol{x}, \boldsymbol{y} \in V$, and scalars $c \in \mathbb{R}$, $c(\boldsymbol{x}+\boldsymbol{y}) = c\boldsymbol{x} + c\boldsymbol{y}$.
> 8. **Distributivity of Scalar Addition (DSA):** For all vectors $\boldsymbol{x} \in V$, and scalars $c, d \in \mathbb{R}$, $(c+d)\boldsymbol{x} = c\boldsymbol{x} + d\boldsymbol{x}$.
> 9. **Identity (I):** For all vectors $\boldsymbol{x} \in V$, $1\boldsymbol{x} = \boldsymbol{x}$.

- Note that *anything* can be a vector space as long as we define operations such that it has properties that satisfies the above properties.
- To define a **complex vector space**, we let the scalars belong in the complex space instead.

> **Theorem**: **The Cancellation Theorem:** Let V be a vector space, and let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$. If:
>
> $$\boldsymbol{x} + \boldsymbol{z} = \boldsymbol{y} + \boldsymbol{z} \tag{1}$$
>
> then:
>
> $$\boldsymbol{x} = \boldsymbol{y} \tag{2}$$

Proof. For $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in V$,

$$\begin{aligned}
\boldsymbol{x} &= \boldsymbol{x} + \mathbf{0} && (Z) \\
&= \boldsymbol{x} + (\boldsymbol{z} + -\boldsymbol{z}) && (AI) \\
&= (\boldsymbol{x} + \boldsymbol{z}) + -\boldsymbol{z} && (AA) \\
&= (\boldsymbol{y} + \boldsymbol{z}) + -\boldsymbol{z} && \text{(given)} \\
&= \boldsymbol{y} + \mathbf{0} && (AI) \\
&= \boldsymbol{y} && (Z)
\end{aligned}$$

Alternatively:

$$\begin{aligned}
\boldsymbol{x} + \boldsymbol{z} &= \boldsymbol{y} + \boldsymbol{z} && \text{(given)} \\
\boldsymbol{x} + \boldsymbol{z} + -\boldsymbol{z} &= \boldsymbol{y} + \boldsymbol{z} + -\boldsymbol{z} \\
\boldsymbol{x} + \mathbf{0} &= \boldsymbol{y} + \mathbf{0} && (AI) \\
\boldsymbol{x} &= \boldsymbol{y} && (Z)
\end{aligned}$$

□

Proposition 1: Let V be a vector space and let $\boldsymbol{x} \in V$. Then $0\boldsymbol{x} = \mathbf{0}$.

Proof. For $\boldsymbol{x} \in V$:

$$\begin{aligned}
0\boldsymbol{x} &= (0+0)\boldsymbol{x} \\
&= 0\boldsymbol{x} + 0\boldsymbol{x} && (DSA) \\
0\boldsymbol{x} + -0\boldsymbol{x} &= 0\boldsymbol{x} + 0\boldsymbol{x} + -0\boldsymbol{x} \\
\mathbf{0} &= 0\boldsymbol{x} + \mathbf{0} && (AI) \\
\mathbf{0} &= 0\boldsymbol{x} && (Z)
\end{aligned}$$

□

Proposition 2: Let V be a vector space, and let $\boldsymbol{x} \in V$. Then $(-1)\boldsymbol{x} = -\boldsymbol{x}$.

Proof. For $\boldsymbol{x} \in V$,

$$\begin{aligned}
-\boldsymbol{x} = -\boldsymbol{x} + \mathbf{0}(Z) \\
&= -\boldsymbol{x} + 0\boldsymbol{x} && (P1) \\
&= -\boldsymbol{x} + (1 + (-1))\boldsymbol{x} \\
&= -\boldsymbol{x} + 1\boldsymbol{x} + (-1)\boldsymbol{x} && (DSA) \\
&= (-\boldsymbol{x} + \boldsymbol{x}) + (-1)\boldsymbol{x} && (AA) \\
&= \mathbf{0} + (-1)\boldsymbol{x} && (P3) \\
&= (-1)\boldsymbol{x} && (P4)
\end{aligned}$$

□

Proposition 3: Let V be a vector space and let $\boldsymbol{x} \in V$. Then $-\boldsymbol{x} + \boldsymbol{x} = \mathbf{0}$.

Proof. For $x \in V$,

$$\begin{aligned}
-\boldsymbol{x}+\boldsymbol{x} &= -\boldsymbol{x}+\boldsymbol{x}+\mathbf{0} && (Z)\\
&= -\boldsymbol{x}+\boldsymbol{x}+((-\boldsymbol{x}+\boldsymbol{x})+-(-\boldsymbol{x}+\boldsymbol{x})) && (AI)\\
&= ((-\boldsymbol{x}+\boldsymbol{x})+(-\boldsymbol{x}+\boldsymbol{x}))+-(-\boldsymbol{x}+\boldsymbol{x}) && (AA)\\
&= (-\boldsymbol{x}+(\boldsymbol{x}+(-\boldsymbol{x}+\boldsymbol{x})))+-(-\boldsymbol{x}+\boldsymbol{x}) && (AA)\\
&= (-\boldsymbol{x}+((\boldsymbol{x}+-\boldsymbol{x})+\boldsymbol{x}))+-(-\boldsymbol{x}+\boldsymbol{x}) && (AA)\\
&= (-\boldsymbol{x}+(\mathbf{0}+\boldsymbol{x}))+-(-\boldsymbol{x}+\boldsymbol{x}) && (Z)\\
&= ((-\boldsymbol{x}+\mathbf{0})+\boldsymbol{x}))+-(-\boldsymbol{x}+\boldsymbol{x}) && (AI)\\
&= (-\boldsymbol{x}+\boldsymbol{x})+-(-\boldsymbol{x}+\boldsymbol{x}) && (Z)\\
&= \mathbf{0} && (AI)
\end{aligned}$$

□

Proposition 4: Let V be a vector space, and let $\boldsymbol{x} \in V$. Then $\mathbf{0}+\boldsymbol{x}=\boldsymbol{x}$.

Proof. For $x \in V$,

$$\begin{aligned}
\mathbf{0}+\boldsymbol{x} &= (\boldsymbol{x}+(-\boldsymbol{x}))+\boldsymbol{x} && (AI)\\
&= \boldsymbol{x}+(-\boldsymbol{x}+\boldsymbol{x}) && (AA)\\
&= \boldsymbol{x}+\mathbf{0} && (P3)\\
&= \boldsymbol{x} && (Z)
\end{aligned}$$

□