

ESC195 Notes

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1 Hyperbolic Functions

- Sometimes, combinations of e^x and e^{-x} are given certain names, for example:

- **Hyperbolic sine:** $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$

- **Hyperbolic cosine:** $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$

- They have the following properties:

$$\frac{d}{dx} \sinh x = \cosh x \quad (1)$$

$$\frac{d}{dx} \cosh x = \sinh x \quad (2)$$

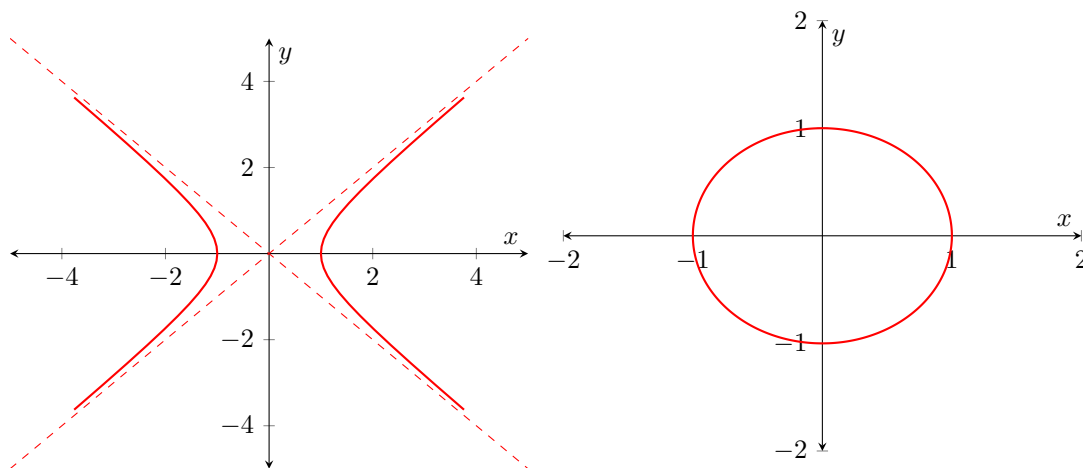
- They are related via:

$$\cosh^2 x - \sinh^2 x = 1 \quad (3)$$

- Both the area of a circular sector and that of a hyperbolic sector is described by:

$$A = \frac{1}{2}t \quad (4)$$

where t is the subtended angle, and the figures are parametrized by $(\cos t, \sin t)$ and $(\cosh t, \sinh t)$.



- The catenary

$$y = a \cosh\left(\frac{x}{a}\right) + C \quad (5)$$

describes the shape of a free hanging rope between two walls separated by a width a .

- The hyperbolic tangent is given by $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. and its derivative is given by:

$$\frac{d}{dx} \tanh x = \text{sech}^2 x \quad (6)$$

- The inverse of $y = \sinh x$ is given by:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \quad (7)$$

Tip: A table of integrals and derivatives revolving around hyperbolic trig functions can be found in the textbook.

2 Indeterminate Forms

- A lot of the times, limits have an indeterminate form, where if we substitute in what x approaches to, we get it in the form of $\frac{0}{0}$, for example:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \quad (8)$$

Theorem: If $f(x) \rightarrow 0$ and $g(x) \rightarrow 0$ as $x \rightarrow \pm, \infty$ or $x \rightarrow c$ or $x \rightarrow c^{+-}$ and if $\frac{f'(x)}{g'(x)} \rightarrow L$, then:

$$\frac{f(x)}{g(x)} \rightarrow L \quad (9)$$

Example 1: Solve: $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

We can set $f(x) = \sin x$, $f'(x) = \cos x$, $g(x) = x$ and $g'(x) = 1$ such that:

$$\lim_{x \rightarrow 0} \frac{f'}{g'} = \lim_{x \rightarrow 0} \cos x = 1 \quad (10)$$

Example 2: Solve $\lim_{x \rightarrow 0^+} \frac{\sin x}{\sqrt{x}}$.

Set $f = \sin x$, $f' = \cos x$, $g = \sqrt{x}$, $g' = \frac{1}{2}x^{-1/2}$ and so:

$$\lim_{x \rightarrow 0^+} 2x^{1/2} \cos x = 0 \implies \lim_{x \rightarrow 0^+} = 0 \quad (11)$$

Example 3: Solve $\lim_{x \rightarrow 0} \frac{e^x - x - 1}{3x^2}$.

If we take the derivative, we get:

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{6x} \quad (12)$$

which is still $\frac{0}{0}$!. We can take derivatives again:

$$\lim_{x \rightarrow 0} \frac{e^x}{6} = \frac{1}{6} \quad (13)$$

so the original limit is $\frac{1}{6}$.

Warning: L'hospital's rule can *only* be used in indeterminate forms. Applying them to limits where

- To prove the L'hospital's rule, we first prove the **Cauchy Mean Value Theorem** as a lemma

Theorem: Cauchy Mean Value Theorem: Given f and g differentiable on (a, b) , continuous on $[a, b]$ and $g' \neq 0$ on (a, b) , there must exist some number r in (a, b) such that:

$$\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad (14)$$

- We then apply **Rolle's Theorem** to prove the Cauchy Mean Value Theorem:

Proof. Set:

$$\begin{aligned} G(x) &= [g(b) - g(a)][f(x) - f(a)] \\ &\quad - [g(x) - g(a)][f(b) - f(a)] \end{aligned}$$

Note that $G(a) = G(b) = 0$ so it satisfies the conditions of Rolle's Theorem. Taking the derivative, we get:

$$G'(x) = [g(b) - g(a)]f'(x) - g'(x)[f(b) - f(a)] \quad (15)$$

According to Rolle's, there must be some $x = r$ such that $G'(r) = 0$, we can then substitute for this and solve:

$$G'(r) = 0 \implies [g(b) - g(a)]f'(r) = g'(r)[f(b) - f(a)] \quad (16)$$

Which is equivalent to:

$$\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad (17)$$

Furthermore, we have $g'(c) = \frac{g(b) - g(a)}{b - a}$ from the mean value theorem. Since $g' \neq 0$ we have $g(b) - g(a) \neq 0$. \square

- Given $x \rightarrow c^+$ and $f(x), g(x) \rightarrow 0$ where:

$$\lim_{x \rightarrow c^+} \frac{f'(x)}{g'(x)} = L \quad (18)$$

we will now prove that $\lim_{x \rightarrow c^+} \frac{f(x)}{g(x)} = L$.

Proof. Consider the interval $[c, c + h]$ and apply Cauchy MVT. There must be some number c_2 in $[c, c + h]$ such that:

$$\frac{f'(c_2)}{g'(c_2)} = \frac{f(c + h) - f(c)}{g(c + h) - g(c)} = \frac{f(c + h)}{g(c + h)} \quad (19)$$

The last step is a result of the given $f(c) = g(c) = 0$. The LHS can be rewritten as:

$$\lim_{h \rightarrow 0} \frac{f'(c_2)}{g'(c_2)} = \frac{f'(c)}{g'(c)} \quad (20)$$

since c_2 lies in the interval $[c, c + h]$ so if $h \rightarrow 0$, then the interval becomes smaller to contain just c . The RHS can be rewritten as:

$$\lim_{h \rightarrow 0} \frac{f(c + h)}{g(c + h)} = \lim_{x \rightarrow c^+} \frac{f(x)}{g(x)} \quad (21)$$

and therefore:

$$\lim_{x \rightarrow c^+} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} = L \quad (22)$$

\square

- To prove the case for $x \rightarrow \pm\infty$, we can let $x = \frac{1}{t}$ and take the limit as $t \rightarrow \infty$.

Example 4: Find $\lim_{x \rightarrow \infty} \frac{\ln x}{x}$.

Taking the derivative of top and bottom, we have:

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = 0 \implies \lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0 \quad (23)$$

Idea: The logarithm function grows very slowly. In fact, any positive power of x will grow faster than $\ln x$.

Example 5: Solve $\lim_{x \rightarrow \infty} \frac{x^3}{e^x}$

This is indeterminate in the form of $\frac{\infty}{\infty}$. We apply L'hopital's rule multiple times:

$$\lim_{x \rightarrow \infty} \frac{x^3}{e^x} \stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{3x^2}{e^x} \left(= \frac{\infty}{\infty} \right) \quad (24)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{6x}{e^x} \left(= \frac{\infty}{\infty} \right) \quad (25)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{6}{e^x} = 0 \quad (26)$$

- Generally, $\lim_{x \rightarrow \infty} \frac{x^m}{e^x} = 0$ where m is any positive integer.
- There are other indeterminate forms, such as 0^0 , for example:

$$\lim_{x \rightarrow 0} x^x \quad (27)$$

The central idea behind this is that $a^b = e^{a \ln b}$. Therefore, this limit is equal to:

$$\lim_{x \rightarrow 0} e^{x \ln x} \quad (28)$$

We can take the limit of the exponent to get:

$$\lim_{x \rightarrow 0} x \ln x = \lim_{x \rightarrow 0} \frac{\ln x}{1/x} \quad (29)$$

Note that the first equation is another indeterminate form with the $0 \cdot \infty$ type, so we had to multiply top and bottom by $\frac{1}{x}$ to get the quotient form. Then we have:

$$\lim_{x \rightarrow 0} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x \rightarrow 0} -x = 0 \quad (30)$$

Therefore:

$$\lim_{x \rightarrow 0} e^{x \ln x} = e^0 = 1 \quad (31)$$

so $\lim_{x \rightarrow 0} x^x = 1$.

Example 6: Solve $\lim_{x \rightarrow \infty} (x+2)^{2/\ln x}$.

This is of the type ∞^0 . The approach is exactly the same as the previous example. We write it in exponential form:

$$= \lim_{x \rightarrow \infty} e^{\frac{2}{\ln x} \ln(x+2)} \quad (32)$$

and looking at the exponent gives:

$$\lim_{x \rightarrow \infty} \frac{2 \ln(x+2)}{\ln x} \quad (33)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{\left(\frac{2}{x+2}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{2x}{x+2} \left(= \frac{\infty}{\infty}\right) \quad (34)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{2}{1} = 2 \quad (35)$$

Therefore:

$$\lim_{x \rightarrow \infty} e^{\frac{2}{\ln x} \ln(x+2)} = e^2 \quad (36)$$

so:

$$\lim_{x \rightarrow \infty} (x+2)^{2/\ln x} = e^2 \quad (37)$$

Example 7: Solve $\lim_{x \rightarrow \infty} \left[\sin \left(\frac{\pi}{x} + \frac{\pi}{2} \right) \right]^x$

This is in the form of 1^∞ . We rewrite it as:

$$\lim_{x \rightarrow \infty} \exp \left(x \ln \left(\sin \left(\frac{\pi}{x} + \frac{\pi}{2} \right) \right) \right) \quad (38)$$

and taking the limit of the exponent:

$$= \lim_{x \rightarrow \infty} x \ln \left(\sin \left(\frac{\pi}{x} + \frac{\pi}{2} \right) \right) \left(= \frac{0}{0} \right) \quad (39)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{\cos \left(\frac{\pi}{x} + \frac{\pi}{2} \right) \cdot \left(-\frac{\pi}{x^2} \right)}{\sin \left(\frac{\pi}{x} + \frac{\pi}{2} \right) \cdot \left(-\frac{1}{x^2} \right)} = \frac{0 \cdot \pi}{1} = 0 \quad (40)$$

Therefore:

$$\lim_{x \rightarrow \infty} \left[\sin \left(\frac{\pi}{x} + \frac{\pi}{2} \right) \right]^x = \lim_{x \rightarrow \infty} \exp \left(x \ln \left(\sin \left(\frac{\pi}{x} + \frac{\pi}{2} \right) \right) \right) = 1 \quad (41)$$

3 Integration

3.1 Recap of Integration

- The definite integral has the geometric interpretation as the area under the curve $f(x)$ between $x = a$ and $x = b$ and the x axis:

$$\int_a^b f(x) \, dx \quad (42)$$

but can be rigorously defined using a Riemann sum:

$$\int_a^b f(x) \, dx \equiv \lim_{\|P\|} \sum_{i=1}^n f(x_i^*) \Delta x_i \quad (43)$$

Often, we have a uniform partition, such that $\Delta x_i = \frac{b-a}{n}$ where n is the number of partitions. And if we choose to use the right hand endpoint, then:

$$f(x_i^*) = f(x_i) = f(x_i) = f \left(a + \frac{b-a}{n} i \right) \quad (44)$$

Example 8: To solve $\int_0^5 x^2 \, dx$, we can choose a uniform partition with:

$$\Delta x = \frac{5-0}{n} = \frac{5}{n} \quad (45)$$

and:

$$x_i^* = x_i = i\Delta x \implies f(x_i^*) = (i\Delta x)^2 = \left(i \frac{5}{n} \right)^2 \quad (46)$$

The area approximation is:

$$A \simeq \sum_{i=1}^n \Delta x_i f(x_i^*) = \sum_{i=1}^n \left(\frac{5}{n} \right) \left(i \frac{5}{n} \right)^2 \quad (47)$$

$$= \frac{125}{n^2} \sum_{i=1}^n i^2 = \frac{125}{n^3} \frac{n(n+1)(2n+1)}{6} \quad (48)$$

Taking the limit as $n \rightarrow \infty$, we get:

$$\int_0^5 x^2 \, dx = \lim_{n \rightarrow \infty} \frac{125}{6} \left(2 + \frac{2}{n} + \frac{1}{n^2} \right) = \frac{5^3}{3}. \quad (49)$$

Example 9: To evaluate $\int_1^2 x^{-2} dx$, we can choose

$$x_i^* = \sqrt{x_{i-1}x_i} \quad (50)$$

and a uniform partition of:

$$\Delta x = \frac{2-1}{n} = \frac{1}{n} \quad (51)$$

such that:

$$x_i = 1 + i\Delta x = 1 + \frac{i}{n} = \frac{n+i}{n} \quad (52)$$

and

$$x_{i-1} = \frac{n+i-1}{n} \quad (53)$$

such that the area is:

$$\begin{aligned} A &\simeq \sum_{i=1}^n \Delta x f(x_i^*) \\ &= \sum_{i=1}^n \frac{1}{n} \left(\frac{1}{x_i^*} \right)^2 \\ &= \sum_{i=1}^n \frac{1}{n} \frac{1}{x_{i-1}x_i} \\ &= \sum_{i=1}^n \frac{1}{n} \frac{n}{n+i-1} \cdot \frac{n}{n+i} \\ &= \sum_{i=1}^n n \frac{1}{n+i-1} \cdot \frac{1}{n+i} \\ &= \sum_{i=1}^n \left(\frac{1}{n+i-1} - \frac{1}{n+i} \right) \\ &= n \left[\sum_{i=1}^n \frac{1}{n+i-1} - \sum_{i=1}^n \frac{1}{n+i} \right] \\ &= n \left[\sum_{i=0}^n \frac{1}{n+i} - \sum_{i=1}^n \frac{1}{n+i} \right] \\ &= n \left[\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n} \right] \\ &= n \left(\frac{1}{n} - \frac{1}{2n} \right) \\ &= 1 - \frac{1}{2} = \frac{1}{2} \end{aligned}$$

The part where we cancel out everything is called a **telescoping series**. Notice how the area doesn't depend on n so we get the exact area, even if we let $n = 1!$.

- We need a better way to do integration, so we can define:

$$F(x) \equiv \int_a^x f(t) dt \quad (54)$$

such that $F'(x) = f(x)$. This is the definition of the antiderivative. This leads to the fundamental theorem of calculus:

$$\int_a^b f(t) dt = F(b) - F(a) \quad (55)$$

and the indefinite integral can be written as:

$$\int f(x) \, dx = G(x) + C \quad (56)$$

The main problem now becomes trying to *find antiderivatives*, which is much easier than Riemann sums, though still more difficult than calculating derivatives.

3.2 Integration by Parts

- **Integration by Parts** attempts to reverse the product rule:

$$(fg)' = fg' + f'g \quad (57)$$

Taking the integral of both sides gives:

$$f(x)g(x) = \int f(x)g'(x) \, dx + \int f'(x)g(x) \, dx \quad (58)$$

$$\int f(x)g'(x) \, dx = \int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx \quad (59)$$

If the second integral is easier than the first, then we have made substantial progress.

Idea: Integration of parts tells us that:

$$\int u \, dv = uv - \int v \, du \quad (60)$$

Example 10: To solve $\int xe^{2x}$, we can let:

$$u = x \quad dv = e^{2x} \, dx \quad (61)$$

$$du = dx \quad v = \frac{1}{2}e^{2x} \quad (62)$$

which gives:

$$\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x} \, dx \quad (63)$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \quad (64)$$

We can check:

$$\frac{d}{dx} \left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \right) \quad (65)$$

$$= xe^{2x} + \frac{1}{2}e^{2x} - \frac{2}{4}e^{2x} \quad (66)$$

$$= xe^{2x} \quad (67)$$

Example 11: To solve $\int x^2 \sin(2x) \, dx$, we let:

$$u = x^2 \quad dv = \sin 2x \, dx \quad (68)$$

$$du = 2x \, dx \quad v = -\frac{1}{2} \cos(2x) \quad (69)$$

which gives:

$$= -\frac{1}{2}x^2 \cos 2x + \int x \cos(2x) dx \quad (70)$$

and we can apply integration by parts a second time, if we let:

$$u = x \quad dv = \cos 2x dx \quad (71)$$

$$du = dx \quad v = \frac{1}{2} \sin(2x) \quad (72)$$

which gives us:

$$= -\frac{1}{2}x^2 \cos(2x) + \frac{1}{2}x \sin(2x) - \int \frac{1}{2} \sin(2x) dx \quad (73)$$

$$= -\frac{1}{2}x^2 \cos(2x) + \frac{1}{2}x \sin(2x) + \frac{1}{4} \cos(2x) + C \quad (74)$$

Example 12: To solve $I = \int e^x \sin x dx$, we can let:

$$u = \sin x \quad dv = e^x dx \quad (75)$$

$$du = \cos x dx \quad v = e^x \quad (76)$$

to give us:

$$= e^x \sin x - \int e^x \cos x dx \quad (77)$$

We apply integration by parts a second time:

$$u = \cos x \quad dv = e^x dx \quad (78)$$

$$du = -\sin x dx \quad v = e^x \quad (79)$$

to get:

$$I = e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x dx}_I \quad (80)$$

$$2I = e^x (\sin x - \cos x) + C' \quad (81)$$

$$I = \frac{1}{2} e^x (\sin x - \cos x) + C \quad (82)$$

and we are done.

Example 13: We can also solve integrals that do not appear to have parts, such as $\int \ln x dx$. We choose:

$$u = \ln x \quad dv = dx \quad (83)$$

$$du = \frac{1}{x} dx \quad v = x \quad (84)$$

to give us:

$$\ln x - \int dx = x \ln x - x + C \quad (85)$$

- For a definite integral, we can write IBP as:

$$f(x)g(x) \Big|_a^b - \int_a^b f'(x)g(x) dx \quad (86)$$

Example 14: It is *possible* to apply integration of parts to find the integral of $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$. We can let:

$$u = \frac{1}{\cos x} = \sec x \qquad dv = \sin x \, dx \qquad (87)$$

$$du = \sec x \tan x \qquad v = -\cos x \qquad (88)$$

this gives us:

$$\int \tan x \, dx = -\frac{\cos x}{\cos x} + \int \tan x \, dx \qquad (89)$$

Notice that we could try to subtract the original integral from both sides and get:

$$0 = -1 \qquad (90)$$

which is clearly wrong! However, we forgot the constant of integration, so the correct statement would be:

$$0 + C' = -1 + C \qquad (91)$$

which does not tell us anything interesting. This brings We can see this concretely by repeating the same steps but trying to evaluate the definite integral $\int_a^b \tan x \, dx$ instead, which gives:

$$\int_a^b \tan x \, dx = (-1) \Big|_{x=a}^{x=b} + \int_a^b \tan x \, dx \implies 0 = (-1) - (-1) \implies 0 = 0 \qquad (92)$$

which confirms our suspicion that this isn't anything useful, but it's also not an incorrect statement.

Warning: Sometimes it is possible to get more than one answer through various means that differ by a constant factor when solving indefinite integrals. When this happens, nothing is wrong: we simply need to consider the constant of integration.

Idea: But how do we know *which* values of u and dv we should pick? A common strategy is to use **LIATE**:

1. L: Logarithms
2. I: Inverse Trig
3. A: Algebraic
4. T: Trigonometric
5. E: Exponential

If a function consists of two terms, the term that is higher up (closer to L) usually gets differentiated and the term near the bottom (closer to E) usually gets integrated. See [this](#) for how it works, and [this video](#) for a tutorial.

4 Trigonometric Integrals

- The first type of integral we'll deal with is:

$$\int \sin^n x \cos^n x \, dx \qquad (93)$$

- In **case 1**, we have either m or n as an odd positive number. We can then use the identity $\sin^2 x + \cos^2 x = 1$ to simplify it.

Example 15: For example, to solve $\int \sin^3 x \cos^2 x \, dx$, we can simplify this to:

$$= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx \quad (94)$$

$$= (\cos^2 x - \cos^4 x) \sin x \, dx \quad (95)$$

and applying a u substitution with $u = \cos x$ and breaking it up into two integrals, we can get:

$$= -\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C \quad (96)$$

- In **case 2**, we have m and n as both even. We then apply the double angle formulas:

$$\sin x \cos x = \frac{1}{2} \sin(2x) \quad (97)$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x \quad (98)$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x \quad (99)$$

Example 16: For example:

$$\int \sin^2 x \cos^4 x \, dx = \int \frac{1}{4} \sin^2(2x) \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) dx \quad (100)$$

$$= \frac{1}{8} \int \sin^2(2x) \, dx + \frac{1}{8} \int \sin^2 x \cos 2x \, dx \quad (101)$$

$$= \frac{1}{8} \int \left(\frac{1}{2} - \frac{1}{2} \cos 4x \right) dx + \frac{1}{8 \cdot 3 \cdot 2} \sin^3(2x) + C \quad (102)$$

$$= \frac{1}{16} x - \frac{1}{64} \sin(4x) + \frac{1}{48} \sin^3(2x) + C \quad (103)$$

- In **Case 3**, we have:

$$\int \sin^n x \, dx, \int \cos^n x \, dx \quad (104)$$

which we can apply a reduction formula by keep applying integration by parts:

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \quad (105)$$

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx \quad (106)$$

Example 17: To solve the integral $\int \sin^2 x \, dx$, we get:

$$= -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int dx \quad (107)$$

$$= \frac{1}{2} x - \frac{1}{4} \sin 2x + C \quad (108)$$

- In **Case 4**, we have integrals in the following forms:

$$\int \sin(mx) \cos(nx) \, dx \quad (109)$$

$$\int \sin(mx) \sin(nx) \, dx \quad (110)$$

$$\int \cos(mx) \cos(nx) \, dx \quad (111)$$

with $m \neq n$. If $m = n$, then we can apply the double angle formula. To solve these, we apply the following identities:

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)] \quad (112)$$

$$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)] \quad (113)$$

$$\sin A \cos B = \frac{1}{2} [\sin(A - B) + \sin(A + B)] \quad (114)$$

Example 18: For example, we have:

$$\int \sin(3x) \sin(2x) dx = \frac{1}{2} \int \cos((3-2)x) dx - \frac{1}{2} \int \cos((3+2)x) dx \quad (115)$$

$$= \frac{1}{2} \sin x - \frac{1}{10} \sin 5x + C \quad (116)$$

- In **case 5**, we have integrals in the form of either:

$$\int \tan^n x dx, \int \cot^n x dx \quad (117)$$

To solve these, we apply the following identities:

$$\tan^2 x = \sec^2 x - 1 \quad (118)$$

$$(\tan x)' = \sec^2 x \quad (119)$$

- In **case 6**, we have:

$$\int \sec^n x dx, \int \csc^n x dx \quad (120)$$

with $n \geq 2$. To solve these, we can make the following substitutions:

$$1 + \tan^2 x = \sec^2 x \quad (121)$$

$$1 + \cot^2 x = \csc^2 x \quad (122)$$

to convert it to a case 5 problem.

- In **case 7**, we have:

$$\int \tan^n x \sec^n x dx, \int \cot^n x \csc^n x dx \quad (123)$$

Example 19: We have:

$$\tan^3 x \sec^4 x dx = \int \tan^3 x \sec^2 x \sec^2 x dx \quad (124)$$

$$= \int \tan^3 x (\tan^2 x + 1) \sec^2 x dx \quad (125)$$

$$= \int (\tan^5 x + \tan^3 x) \sec^2 x dx \quad (126)$$

$$= \frac{1}{6} \tan^6 x + \frac{1}{4} \tan^4 x + C \quad (127)$$

Idea: The basic idea of these types is to apply trigonometric identities to turn the integrals into a form that is easier to deal with. The substitutions are usually very simple but to find them, it requires a lot of practice.

- We can also apply **trigonometric substitutions**, any integrals with any of the three factors below can be solved with this technique:

$$1. \sqrt{a^2 - x^2}: \text{Set } x = a \sin u \implies \sqrt{a^2 - x^2} = a \cos u$$

2. $\sqrt{a^2 + x^2}$: Set $x = a \tan u \implies \sqrt{a^2 + x^2} = a \sec u$

3. $\sqrt{x^2 - a^2}$: Set $x = a \sec u \implies \sqrt{x^2 - a^2} = a \tan u$

where the arguments under the square roots are always positive.

Example 20: To solve the integral $\int \frac{x^2}{(4 - x^2)^{3/2}} dx$, we can set:

$$x = 2 \sin u \quad (128)$$

$$dx = 2 \cos u \, du \quad (129)$$

$$\sqrt{4 - x^2} = 2 \cos u \quad (130)$$

which gives:

$$= \int \frac{4 \sin^2 u \cdot 2 \cos u \, du}{8 \cos^3 u} \quad (131)$$

$$= \int \tan^2 u \, du \quad (132)$$

$$= \int (\sec^2 u - 1) \, du \quad (133)$$

$$= \tan u - u + C = \frac{x}{\sqrt{4 - x^2}} - \sin^{-1} \left(\frac{x}{2} \right) + C \quad (134)$$

Example 21: The integral $\int \frac{x \, dx}{(2x^2 + 4x - 7)^{1/2}}$ needs a bit more work before we can apply the substitutions. We first apply the square to get:

$$= \int \frac{x \, dx}{\sqrt{2(x+1)^2 - 9}} \quad (135)$$

We can set:

$$\sqrt{2}(x+1) = 3 \sec u \quad (136)$$

$$\sqrt{2} \, dx = 3 \sec u \tan u \, du \quad (137)$$

$$\sqrt{2(x+1)^2 - 9} = 3 \tan u \quad (138)$$

which gives:

$$= \int \frac{\left(\frac{3}{\sqrt{2}} \sec u - 1 \right) \left(\frac{3}{\sqrt{2} \sec u \tan u} du \right)}{3 \tan u} \quad (139)$$

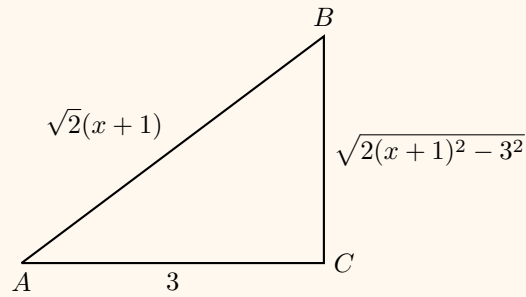
$$= \int \left(\frac{3}{\sqrt{2}} \sec u - 1 \right) \left(\frac{1}{\sqrt{2}} \sec u \right) du \quad (140)$$

$$= \frac{3}{2} \int \sec^2 u \, du - \frac{1}{\sqrt{2}} \int \sec u \, du \quad (141)$$

$$= \frac{3}{2} \tan u - \frac{1}{\sqrt{2}} \ln |\sec u + \tan u| + C \quad (142)$$

$$= \frac{1}{2} \sqrt{2x^2 + 4x - 7} - \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2}}{3}(x+1) + \frac{\sqrt{2x^2 + 4x - 7}}{3} \right| + C \quad (143)$$

Idea: We can use triangles to derive the substitution, which comes from the Pythagorean theorem:



and you can clearly see the substitution:

$$3 \sec u = \sqrt{2}(x+1) \implies \cos u = \frac{3}{\sqrt{2}(x+1)} \quad (144)$$

where $u \equiv \angle BAC$.

Example 22: For the integral $\int x \sin^{-1} x \, dx$, we can let:

$$u = \sin^{-1} x \, dv = x \, dx \quad (145)$$

$$du = \frac{dx}{\sqrt{1-x^2}} v = \frac{1}{2} x^2 \quad (146)$$

and applying integration by parts, we get:

$$= \frac{1}{2} x^2 \sin^{-1} x - \int \frac{1}{2} x^2 \frac{dx}{\sqrt{1-x^2}} \quad (147)$$

To solve this secondary integral $\int \frac{x^2 \, dx}{\sqrt{1-x^2}}$, we can let:

$$x = \sin \theta \quad (148)$$

$$dx = \cos \theta \, d\theta \quad (149)$$

$$\sqrt{1-x^2} = \cos \theta \quad (150)$$

which gives:

$$= \frac{\sin^2 \theta \cos \theta \, d\theta}{\cos \theta} \quad (151)$$

$$= \int \sin^2 \theta \, d\theta \quad (152)$$

$$= \frac{1}{2} \theta - \frac{1}{2} \sin \theta \cos \theta + C \quad (153)$$

$$= \frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1-x^2} + C \quad (154)$$

Therefore, we get:

$$\int x \sin^{-1} x \, dx = \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{4} \sin^{-1} x + \frac{1}{4} x \sqrt{1-x^2} + C \quad (155)$$

5 Partial Fractions

- Rational functions are in the form of:

$$R(x) = \frac{P_n(x)}{P_m(x)} \quad (156)$$

where m, n represent the order of the polynomial. If $n \geq m$, it is an **improper** fraction, such as:

$$\frac{x^2 - x}{1 + x} \quad (157)$$

and if $n < m$, we have a proper fraction such as:

$$\frac{x}{x^2 + 3x + 2} \quad (158)$$

- If we have an improper fraction, we use long division to simplify it. For example:

$$\frac{x^3 - 2x^2}{x^2 + 9} = x - 2 + \frac{18 - 9x}{x^2 + 9} \quad (159)$$

which turns the expression into a polynomial (trivial to integrate) as well as a proper fraction.

- There are different types of factors:
 - Linear factors (e.g. $3x + 2$)
 - Irreducible quadratic factors (e.g. $x^2 + 1$)

which gives us the different factors:

- **Case 1:** If we have distinct linear factors in the denominator, we can break it into fractions of the form:

$$(x + \alpha) \implies \frac{A}{x + \alpha} \quad (160)$$

Example 23: The partial fraction of $\frac{2x - 17}{x^2 + 3x + 2}$ can be written as the **partial fraction deconvolution**:

$$= \frac{A}{x + 1} + \frac{B}{x + 2} \quad (161)$$

We now need to solve for A and B . We can multiply both sides by $(x + 1)(x + 2)$ to get:

$$2x - 17 = A(x + 2) + B(x + 1) \quad (162)$$

and match up the coefficients. Alternatively, we can pick various values of x (e.g. $x = -2$ and $x = -1$) to solve for the coefficients.

- **Case 2:** If we have repeated linear factors, then the decomposition is in the form of:

$$(x + \alpha)^k \implies \frac{A}{x + \alpha} + \frac{B}{(x + \alpha)^2} + \frac{C}{(x + \alpha)^3} + \cdots + \frac{K}{(x + \alpha)^k} \quad (163)$$

Example 24: To get the decomposition of $\frac{2}{x(x + 1)^2}$, we can get:

$$\frac{2}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2} \quad (164)$$

which gives:

$$2 = A(x + 1)^2 + Bx(x + 1) + Cx \quad (165)$$

matching the coefficients, we get three equations and three unknowns:

$$x^2 : A + B = 0 \quad (166)$$

$$x : 2A + B + C = 0 \quad : A = 2 \quad (167)$$

Solving this system gives $A = 2$, $B = -2$, and $C = -2$. Note that taking the integral of this sum is much

easier. We have:

$$\int \frac{d}{x(x+1)^2} dx = \int \frac{2}{x} dX - \int \frac{2}{x} dx - \int \frac{2}{(x+1)^2} dx \quad (168)$$

$$= 2 \ln |x| - 2 \ln |x+1| + \frac{2}{x+1} + C \quad (169)$$

Idea: As a general rule of thumb, the number of unknown coefficients is equal to the order of the polynomial in the denominator.

- **Case 3:** If we have irreducible quadratic factors, then the partial fraction deconvolution is in the form of:

$$x^2 + px + 8 \implies \frac{Ax + B}{x^2 + px + 8} \quad (170)$$

Example 25: Suppose we have $\frac{2}{(x+1)(x^2+x+1)}$, we can get the partial fraction decomposition as:

$$= \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1} \quad (171)$$

and we work through the deconvolution process in exactly the same way, we remove the denominators on both sides to get (after expanding):

$$2 = Ax^2 + Ax + A + Bx^2 + Bx + Cx + C \quad (172)$$

$$0x^2 + 0x^1 + 2x^0 = (A+B)x^2 + (A+B+C)x^1 + (A+C)x^0 \quad (173)$$

which gives three equations and three unknowns, after we match coefficients:

$$x^2 : A + B = 0 \quad (174)$$

$$x : A + B + C = 0 \quad (175)$$

$$1 : A + C = 2 \quad (176)$$

and solving the system of equations gives $A = 2, B = -2, C = 0$. To get the integral of this second term, we can write the second term as:

$$\int \frac{2x dx}{x^2 + 2x + 1} = \underbrace{\int \frac{2x+1}{x^2+x+1} dx}_{(1)} - \underbrace{\int \frac{dx}{x^2+x+1}}_{(2)} \quad (177)$$

We “added” 1 and “subtracted” 1 to get these two slightly easier integrals, which we can apply other techniques. The first one can be solved using a u-sub while the second can be solved by completing the square and applying a trigonometric substitution:

$$(1) = \ln |x^2 + x + 1| + C \quad (178)$$

$$(2) = \int \frac{dx}{(x + \frac{1}{2})^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}} \tan^{-1} \left[\frac{2}{\sqrt{3}} \left(x + \frac{1}{2} \right) \right] + C \quad (179)$$

allowing us to put everything together.

Example 26: Let's take an integral we already know the answer of: $\int \frac{2x}{x^2+1} dx = \ln(x^2+1) + C$. We can try a partial fraction decomposition:

$$\frac{2x}{x^2+1} = \frac{A}{x+i} + \frac{B}{x-i} = \frac{1}{x+i} + \frac{1}{x-i} \quad (180)$$

which gives:

$$\int \frac{2x}{x^2 + 1} dx = \int \frac{dx}{x+i} + \int \frac{dx}{x-i} \quad (181)$$

In complex analysis, most mathematical functions we are familiar with are still valid, so the integral is:

$$= \ln |x+i| + \ln |x-i| + C \quad (182)$$

and simplifying it gives:

$$\ln(x^2 + 1) + C \quad (183)$$

Warning: While it is *possible* to use complex numbers to solve irreducible quadratic factors, it isn't always as easy as the above example. To get the logarithm of a complex number, we can apply the identity (without proving):

$$\ln(a+ib) = \ln \sqrt{a^2 + b^2} + i \arctan\left(\frac{b}{a}\right) \quad (184)$$

Example 27: Bonus content: Try evaluating the integral $\int \frac{dx}{x^2 + 1}$ with complex analysis. Taking a partial fraction, we get:

$$\frac{1}{x^2 + 1} = \frac{A}{x+i} + \frac{B}{x-i} \quad (185)$$

multiplying both sides, we get:

$$1 = A(x-i) + B(x+i) \quad (186)$$

$$1 = (A+B)x + i(-A+B) \quad (187)$$

we have the systems of two equations:

$$x^1 : A + B = 0 \quad (188)$$

$$x^0 : (B - A)i = 1 \quad (189)$$

which gives $A = \frac{1}{2}i$ and $B = -\frac{1}{2}i$. This gives:

$$= \int \frac{0.5i}{x+i} dx - \int \frac{0.5i}{x-i} dx \quad (190)$$

$$= 0.5i \ln(x+i) - 0.5i \ln(x-i) + C \quad (191)$$

$$= 0.5i \ln \sqrt{x^2 + 1} + (0.5i)i \arctan\left(\frac{b}{x}\right) - (0.5i) \ln \sqrt{x^2 + 1} - (0.5i)i \arctan\left(-\frac{1}{x}\right) \quad (192)$$

$$= -\arctan\left(\frac{1}{x}\right) + C \quad (193)$$

Note that for $x \geq 0$:

$$-\arctan\left(\frac{1}{x}\right) + \frac{\pi}{2} = \arctan x \quad (194)$$

and for $x < 0$:

$$-\arctan\left(\frac{1}{x}\right) - \frac{\pi}{2} = \arctan x \quad (195)$$

- **Case 4:** Repeated irreducible quadratic terms, the decomposition is in the form of:

$$(x^2 + \beta x + 8)^k \implies \frac{A_1 x + B_1}{(x^2 + \beta x + 8)} + \frac{A_2 x + B_2}{(x^2 + \beta x + 8)^2} + \cdots + \frac{A_k x + B_k}{(x^2 + \beta x + 8)^k} \quad (196)$$

These can be extremely messy, but the process is similar to the above examples. For example, we can write:

$$\frac{Ax + B}{(x^2 + \beta x + 8)^2} = \frac{A}{2} \left[\frac{2x + \beta}{(x^2 + \beta x + 8)^2} + \frac{2B/A - \beta}{(x^2 + \beta x + 8)^2} \right] \quad (197)$$

Idea: The general strategy for dealing with a proper fraction integral is to break it up into two terms, one that can be easily be solved via a u-substitution and the second one does not have an x term in the numerator and can be solved using a trigonometric substitution.

- We can also introduce a strategy rationalizing substitutions by turning a function such as:

$$\int \frac{\sqrt{x}}{1+x} dx \quad (198)$$

into a form that we are familiar with. We can let $u^2 = x \implies 2u du = dx$ to give:

$$= \int \frac{u \cdot 2u du}{1+u^2} \quad (199)$$

$$= 2 \int \frac{u^2}{1+u^2} du \quad (200)$$

$$= 2 \int \left(1 - \frac{1}{1+u^2} \right) du \quad (201)$$

$$= 2u - 2 \tan^{-1} u + C \quad (202)$$

$$= 2\sqrt{x} - 2 \tan^{-1} \sqrt{x} + C \quad (203)$$

- Another method is to use a **Weierstrass substitution**, by making the substitution:

$$t = \tan \frac{x}{2} \quad (204)$$

which leads to the following substitutions:

$$\sin x = \frac{2t}{1+t^2} \quad (205)$$

$$\cos x = \frac{1-t^2}{1+t^2} \quad (206)$$

$$dx = \frac{2}{1+t^2} dt \quad (207)$$

This allows us to turn any trigonometric function into a rational function.

Example 28: For example, to solve the integral $\int \frac{dx}{1+\cos x}$, we make the specified substitution to turn this into:

$$= \int \frac{1}{1 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt \quad (208)$$

$$= \int \frac{2 dt}{(1+t^2) + (1-t^2)} dt \quad (209)$$

$$= \int dt \quad (210)$$

$$= t + C \quad (211)$$

$$= \tan \left(\frac{x}{2} \right) + C \quad (212)$$

6 Improper Integrals

- Since infinity is not a number, our typical definite integral definition cannot be used for an **improper integral** like:

$$\int_0^{\infty} f(x) dx \quad (213)$$

Instead, we use the following definition:

Definition: If $\lim_{b \rightarrow \infty} \int_a^b f(x) dx = L$ exists, then we can define:

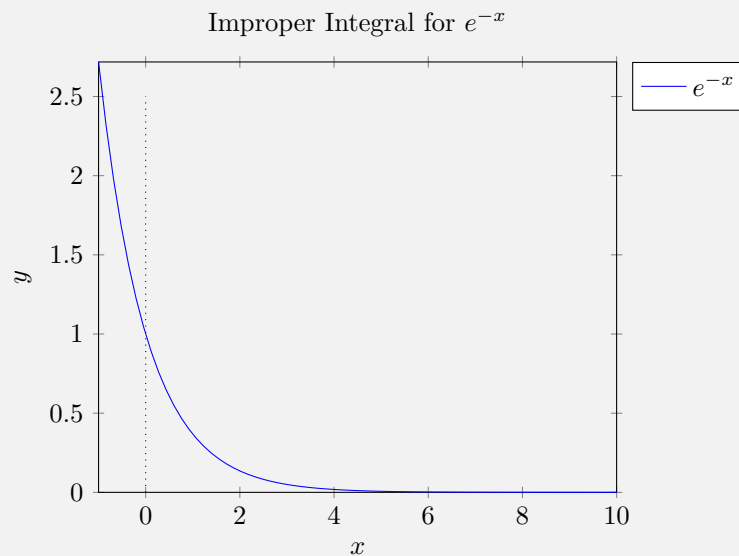
$$\int_a^{\infty} f(x) dx = L \quad (214)$$

Example 29: To solve $\int_0^{\infty} e^{-x} dx$, we can write it as:

$$= \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx \quad (215)$$

$$= \lim_{b \rightarrow \infty} (1 - e^{-b}) = 1 \quad (216)$$

This is remarkable because even though the area appears infinite (since it is infinitely long), the area is actually finite.



Example 30: For the integral $\int_{-\infty}^{-1}$, we have:

$$= \lim_{a \rightarrow -\infty} \int_a^{-1} \frac{dx}{x^2} \quad (217)$$

$$= \lim_{a \rightarrow -\infty} \left(1 + \frac{1}{a} \right) = 1 \quad (218)$$

- However, improper integrals can diverge as well.

Example 31: For $\int_3^\infty \frac{dx}{x}$, we get:

$$= \lim_{b \rightarrow \infty} (\ln b - \ln 3) = \infty \quad (219)$$

Example 32: For something like $\int_{-\infty}^{2\pi} \sin x \, dx$, the integral does not go to infinity, but since we get:

$$\lim_{a \rightarrow -\infty} (-1 + \cos a) \quad (220)$$

it will diverge, since $\lim_{a \rightarrow -\infty} \cos a$ does not exist.

- We can generalize this for all reciprocal functions:

Idea: For $\int_a^\infty \frac{dx}{x^p}$ with $p > 0$, $p \neq 1$, and $a > 0$, we get:

$$= \lim_{b \rightarrow \infty} \int_a^b \frac{dx}{x^p} \quad (221)$$

$$= \lim_{b \rightarrow \infty} \left(\frac{1}{1-p} x^{-p+1} \right) \Big|_a^b \quad (222)$$

$$= \lim_{b \rightarrow \infty} \left(\frac{b^{-p+1}}{1-p} - \frac{a^{-p+1}}{1-p} \right) \quad (223)$$

For $p > 1$, we get:

$$= \frac{a^{1-p}}{p-1} \quad (224)$$

and diverges for $p \leq 1$.

- There are techniques to check if an improper integral will converge or diverge. This is useful especially if we want to perform a numerical integration but want to verify that it indeed will converge.

Theorem: Let f, g be continuous functions and $0 \leq f(x) \leq g(x)$ where $x \in [a, \infty)$.

- If $\int_a^\infty g \, dx$ converges, so does $\int_a^\infty f(x) \, dx$.
- If $\int_a^\infty f$ diverges, so does $\int_a^\infty g(x) \, dx$.

Example 33: The integral $\int_2^\infty \frac{dx}{\sqrt{1+x^{44/17}}}$ is difficult to evaluate, but we can easily tell that it converges via:

$$\frac{1}{\sqrt{1+x^{44/12}}} < \frac{1}{\sqrt{x^{44/12}}} = \frac{1}{x^{22/12}} \quad (225)$$

Since $p > 1$, this converges, so the original integral must also converge.

Example 34: For the integral $\int_3^\infty \frac{dx}{\sqrt{7+x^2}}$, we can check that it diverges by:

$$(7+x^2)^{1/2} < \sqrt{7} + x \quad (226)$$

We can check this via: $7 + x^2 < 7 + 2\sqrt{7} + x^2$. Since:

$$\int_3^\infty \frac{dx}{\sqrt{7} + x} = \ln(\sqrt{7} + x) \Big|_3^\infty \quad (227)$$

which diverges, so the original integral must also diverge.

Warning: The notation $f(x) \Big|_3^\infty$ needs to be defined explicitly since ∞ is not a number. This expression simply implies that we are taking the limit as b approaches infinity, even though it might look like we're treating ∞ as a number.

- We can look at more interesting examples. Both the lower and upper bounds can be $\pm\infty$, such as:

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi} \quad (228)$$

Definition: We can define an integral from $-\infty$ to $+\infty$ as:

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^{\infty} f(x) dx \quad (229)$$

Warning: Do *not* evaluate integrals of the above form as:

$$\int_{-\infty}^{\infty} f(x) dx \neq \lim_{b \rightarrow \infty} \int_{-b}^b f(x) dx \quad (230)$$

- For example, take the integral $\int_{-\infty}^{\infty}$. If we use the proper definition, then we add two limits that don't exist, so we know this diverges. Note that it might be tempting to write:

$$= \lim_{b \rightarrow \infty} \int_{-b}^b x dx = \lim_{b \rightarrow \infty} \left(\frac{b^2}{2} - \frac{b^2}{2} \right) = 0 \quad (231)$$

but this is only because we are approaching $-\infty$ and $+\infty$ at the same rate. If we instead wrote:

$$\lim_{b \rightarrow \infty} \int_{-b}^{2b} x dx = \lim_{b \rightarrow \infty} \left(\frac{4b^2}{2} - \frac{b^2}{2} \right) = \infty \quad (232)$$

If we instead used this approach for our other improper integrals, it wouldn't make a difference since it shouldn't matter the rate at which we approach infinity. Here's another example:

$$\lim_{b \rightarrow \infty} \int_{-b}^{\sqrt{b^2+138}} x dx = \lim_{b \rightarrow \infty} \left(\frac{b^2 + 138}{2} - \frac{b^2}{2} \right) = \lim_{b \rightarrow \infty} \frac{138}{2} = 69 \quad (233)$$

- Improper integrals can also be in the form where there are infinite discontinuities at the bounds of integration. Suppose $\lim_{x \rightarrow b^-} f(x) = \infty$. We can treat an integral such as:

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx \quad (234)$$

Example 35: For example, take $\int_0^1 \frac{dx}{x^{1/3}}$, and we can evaluate this via:

$$= \lim_{c \rightarrow 0^+} \int_c^1 \frac{dx}{x^{1/3}} \quad (235)$$

$$= \lim_{c \rightarrow 0^+} \frac{3}{2} \left(1 - c^{2/3}\right) = \frac{3}{2} \quad (236)$$

Again, we have a region that extends to an infinite extend, but it has a finite area. Of course, this won't always be the case.

Example 36: Take the example where $\int_0^1 \frac{dx}{x^2}$, then we can evaluate this integral via:

$$= \lim_{c \rightarrow 0^+} \int_c^1 \frac{dx}{x^2} \quad (237)$$

$$= \lim_{c \rightarrow 0^+} \left(\frac{1}{c} - 1\right) = \infty \quad (238)$$

so this integral will diverge.

Idea: Notice that we can draw an analogy between: $\int_0^a \frac{dx}{x^p}$ and $\int_a^\infty \frac{dx}{x^{1/p}}$, as they are reflections of one another across the line $y = x$. If one diverges, the other will converge, with the exception being $p = 1$.

- We can also deal with discontinuities that occur between the given bounds. Similar to before, we break it up into two integrals and *both* integrals must converge for the original integral to converge. For example, take:

$$\int_{-a}^b \frac{1}{|x^{1/2}|} dx \quad (239)$$

with $a, b > 0$. For this integral to converge, then both $\int_{-a}^0 \frac{dx}{|x^{1/2}|}$ and $\int_0^b \frac{dx}{|x^{1/2}|}$ must converge.

Warning: Here is an example of when things go wrong when the integral is not broken up into separate integrals. For example, suppose we wish to evaluate $\int_{-1}^3 \frac{dx}{x^2}$. From our previous discussion, we know that $\int_{-1}^0 \frac{dx}{x^2}$ and $\int_0^3 \frac{dx}{x^2}$ both diverges. However, one might naively think that:

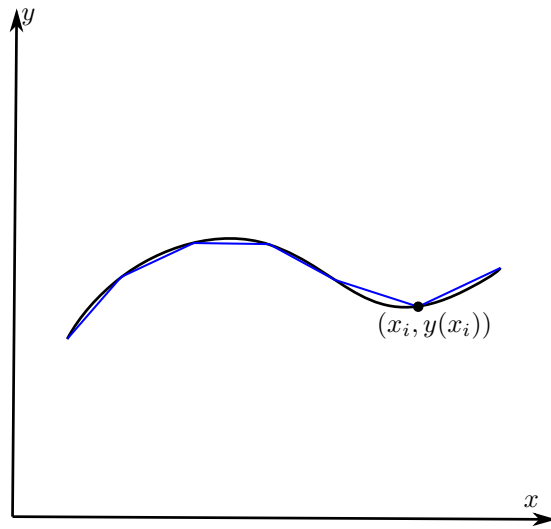
$$\left(-\frac{1}{x}\right) \Big|_{-1}^3 = -\frac{1}{3} - \frac{1}{1} = -\frac{4}{3} \quad (240)$$

which is definitely wrong, since $\frac{1}{x^2}$ is never negative!

7 Applications of Integrals

7.1 Arclength

- Suppose we have a curve $y = f(x)$ where $x \in [a, b]$ and is differentiable. The problem is to find the length of the curve in this range.



- We can approximate this by partitioning the curve into segments at locations x_i where:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b \quad (241)$$

such that the arclength is:

$$s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{(x_i - x_{i-1})^2 + (y(x_i) - y(x_{i-1}))^2} \quad (242)$$

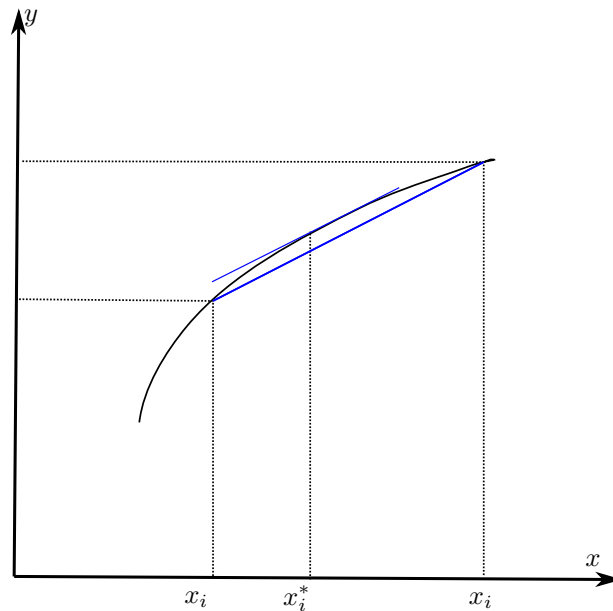
We can use the mean value theorem to write:

$$\frac{\Delta y_i}{\Delta x_i} = \frac{y(x_i) - y(x_{i-1})}{x_i - x_{i-1}} = y'(x_i^*) \quad (243)$$

so we can rewrite:

$$s_i = \sqrt{\Delta x_i^2 + (f'(x_i^*)\Delta x_i)^2} \quad (244)$$

The total length is approximated by the total sum. If we take the limit:



$$s = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n \text{sqrt} 1 + f'(x_i^*)^2 \Delta x_i \quad (245)$$

$$= \int_a^b \sqrt{1 + f'(x)^2} dx \quad (246)$$

Example 37: For example, the arclength in $x \in [0, 44]$ for $f(x) = x^{3/2}$ can be calculated if we know the derivative:

$$f'(x) = \frac{3}{2}x^{1/2} \quad (247)$$

so:

$$1 + f'(x)^2 = 1 + \frac{9}{4}x \quad (248)$$

which gives:

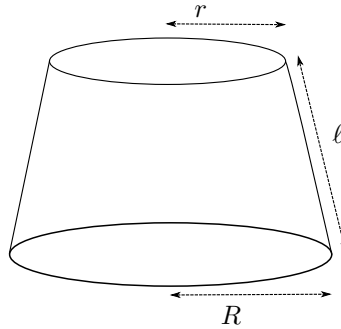
$$s = \int_0^{44} \sqrt{1 + \frac{9}{4}x} dx \quad (249)$$

$$= \left(\frac{4}{9} \cdot \frac{2}{3} \left(1 + \frac{9}{4}x \right)^{3/2} \right) \Big|_0^{44} \quad (250)$$

$$= 296 \quad (251)$$

7.2 Area of a Surface of Revolution

- Consider the new problem of finding the area of a surface of revolution. Similarly, we break it up into smaller segment with width Δx .



- Each small segment is a tapered cone, with an area of:

$$A_i \simeq \pi(f(x_{i-1}) + f(x_i))s_i \quad (252)$$

$$\simeq \pi(f(x_{i-1}) + f(x_i))\sqrt{1 + f'(x_i^*)^2}\Delta x_i \quad (253)$$

From the Intermediate Value Theorem, we have:

$$f(x_{i-1}) + f(x_i) = 2f(x_i^{**}) \quad (254)$$

where $x_i^{**} \in [x_{i-1}, x_i]$ so the area can be written as:

$$A_i \simeq 2\pi f(x_i^{**})\sqrt{1 + f'(x_i^*)^2}\Delta x_i \quad (255)$$

However, we cannot turn this into an integral just yet since we have both x_i^* and x_i^{**} . But in the limit where $\Delta x_i \rightarrow 0$, we also have $x_i^{**} \rightarrow x_i^*$. We therefore get:

$$A = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2} dx \quad (256)$$

Example 38: Suppose we have the function $y = \sqrt{x}$ rotated across the x axis and we want the surface area between $x \in [0, 1]$. We have $y' = \frac{1}{2}x^{-1/2}$ and the area becomes:

$$A = \int_0^1 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}} dx \quad (257)$$

$$= \pi \int_0^1 \sqrt{4x + 1} dx \quad (258)$$

Let $u = 4x + 1$ and $du = 4 dx$, and we'll get:

$$A = \int_1^5 \pi\sqrt{u}\frac{du}{4} \quad (259)$$

$$= \frac{\pi}{4} \left(\frac{2}{3} u^{3/2} \right) \Big|_1^5 \quad (260)$$

$$= \frac{\pi}{6} (5^{3/2} - 1) \quad (261)$$

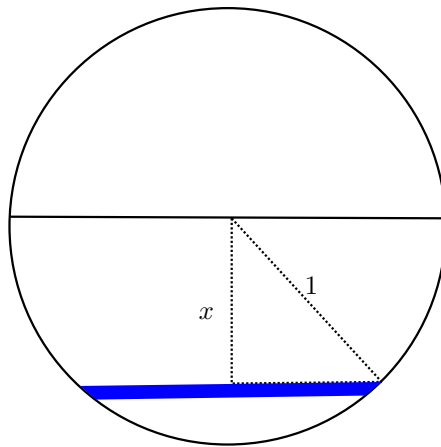
8 Applications to Physics and Engineering

- The **hydrostatic pressure** depends on the density ρ , gravitational constant g and the depth d :

$$p = \rho g d \quad (262)$$

and the force of pressure acting on the surface is:

$$F = \rho g d \cdot A = pA \quad (263)$$



Example 39: Suppose we have a curved container. The force acting on the entire container can be broken up into segments, each with a force of:

$$F_i = \underbrace{w(x_i^*)\Delta x_i}_{\text{area}} \cdot \underbrace{\rho g x_i^*}_{\text{pressure}} \quad (264)$$

where $w(x)$ is the width of the container as a function of height. The force exerted on the container is thus:

$$F = \int_a^b \rho g x w(x) dx \quad (265)$$

Example 40: Suppose we have a pipe half with a radius of 1m filled with water and we wish to find the force it exerts on the end face of the pipe. We can do this via:

$$F = \int_0^1 \rho g x 2\sqrt{1-x^2} dx \quad (266)$$

$$= 2\rho g \left(-\frac{1}{3}(1-x^2)^{3/2} \right) \Big|_0^1 \quad (267)$$

$$= \frac{2}{3}\rho g = 6533\text{N} \quad (268)$$

- We investigate the **center of mass** of a two dimensional object, which intuitively is the point at which it'll balance, also known as the **centroid**. We can use two principles to help us out:
- **Principle 1: Symmetry:** If there is an axis of symmetry, then (\bar{x}, \bar{y}) is on any axis of symmetry. If there are more than one axes of symmetry, then we simply need to find the intersection.
- **Principle 2: Additivity:** We can find the centroid of a collection of segments by taking the weighted average of each of the segments it is composed of. For a discrete set, the total area is:

$$A = A_1 + A_2 + \cdots + A_n \quad (269)$$

and the x location of the centroid is:

$$\bar{x} = \bar{x}_1 \frac{A_1}{A} + \bar{x}_2 \frac{A_2}{A} + \cdots + \bar{x}_n \frac{A_n}{A} \quad (270)$$

and similarly for the y location:

$$\bar{y} = \bar{y}_1 \frac{A_1}{A} + \bar{y}_2 \frac{A_2}{A} + \cdots + \bar{y}_n \frac{A_n}{A} \quad (271)$$

- Suppose we wish to find the centroid of a curve $f(x)$ from $x = a$ to $x = b$. We can approximate this region via a series of rectangles such that:

$$A_i = f(x_i^*) \Delta x_i \quad (272)$$

$$\bar{x}_i = \frac{x_{i-1} + x_i}{2} = x_i^* \quad (273)$$

$$\bar{y}_i = \frac{1}{2} f(x_i^*) \quad (274)$$

such that:

$$\bar{x}A = \sum_{i=1}^n \bar{x}_i A_i = \sum_{i=1}^n x_i^* f(x_i^*) \Delta x_i \quad (275)$$

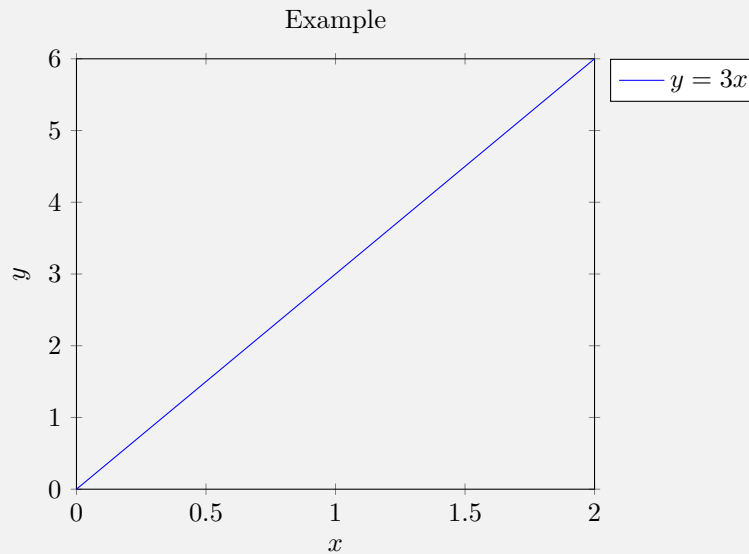
$$\bar{y}A = \frac{1}{2} \sum_{i=1}^n x_i^* f(x_i^*) \Delta x_i \quad (276)$$

If we take the limit, we get:

$$\bar{x} = \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx} \quad (277)$$

$$\bar{y} = \frac{\int_a^b f(x)^2 dx}{2 \int_a^b f(x) dx} \quad (278)$$

Example 41: Suppose we wish to find the area of $y = 3x$ between $x = 0$ and $x = 2$:



The area is $A = \int_0^2 3x \, dx = 6$. And we have:

$$\bar{x}A = \int_0^2 x(3x) \, dx = 8 \implies \bar{x} = \frac{4}{3} \quad (279)$$

$$\bar{y}A = \int_0^2 \frac{1}{2}(3x)^2 \, dx = 12 \implies \bar{y} = 2 \quad (280)$$

which is as we expected from the centroid of a triangle.

- If we want the centroid of the region of intersection between two curves f and g , we can use the additivity rule, we can have:

$$\bar{x}A = \bar{x}_g A_g - \bar{x}_f A_f \quad (281)$$

$$\bar{y}A = \bar{y}_g A_g - \bar{y}_f A_f \quad (282)$$

$$A = A_g - A_f \quad (283)$$

or in integral form:

$$\bar{x}A = \int_a^b x[f(x) - g(x)] \, dx \quad (284)$$

$$\bar{y}A = \frac{1}{2} \int_a^b [f(x)^2 - g(x)^2] \, dx \quad (285)$$

Example 42: Suppose we have two curves $y = 6$ and $y = 3$ and we wish to find the centroid of the area between the curves between $2 < x < 5$. This gives us:

$$\bar{x}A = \int_2^5 x(6 - 3) \, dx = \frac{63}{2} \quad (286)$$

$$\bar{y}A = \frac{1}{2} \int_2^5 \frac{1}{2}(36 - 9) \, dx = \frac{9}{2} \quad (287)$$

which gives us: $\bar{x} = \frac{7}{2}$ and $\bar{y} = \frac{9}{2}$.

- **Pappus's Centroid Theorem** can be used to easily find the volume of revolution. We have:

$$V = 2\pi \bar{R}a \quad (288)$$

where \bar{R} is the distance from the centroid to the axis of revolution.

Example 43: Suppose we have an elliptical torus whose center is The area of the ellipse whose major axis is parallel to the axis of revolution and whose centroid is a distance R away from the axis. The volume is thus:

$$V = 2\pi R\pi ab = 2\pi^2 abR \quad (289)$$

- We can prove this using the washer method about x :

$$V_x = \int_a^b \pi(f(x)^2 - g(x)^2) dx \quad (290)$$

$$= 2\pi \int_a^b \frac{1}{2}(f(x)^2 - g(x)^2) dx \quad (291)$$

$$= 2\pi \bar{y}A \quad (292)$$

and using the shell method about x :

$$V_y = \int_a^b 2\pi x(f(x) - g(x)) dx \quad (293)$$

$$= 2\pi \bar{x}A \quad (294)$$

9 Parametric Equations

- Until now, we have described two dimensional curves in the form $y = f(x)$. However, we can describe this in parametric equations as well in the form:

$$x = x(t) \quad (295)$$

$$y = y(t) \quad (296)$$

- For example, numerous applications in physics and engineering arise using parametrized coordinates such as projectile motion. The equations of motion are:

$$x(t) = x_0 + v_0 \cos \theta t \quad (297)$$

$$y(t) = y_0 + v_0 \sin \theta t - \frac{1}{2}gt^2 \quad (298)$$

- We can parametrize a straight line using:

$$x(t) = x_c + t(x_1 - x_0) \quad (299)$$

$$y(t) = y_0 + t(y_1 - y_0) \quad (300)$$

with $t \in (-\infty, \infty)$. We can check this by letting $t = 0$, which gives (x_0, y_0) and when $t = 1$ we get (x_1, y_1) .

- For an ellipse, we can parametrize using:

$$x = a \cos t \quad (301)$$

$$y = b \sin t \quad (302)$$

- Suppose we have two curves parametrized by:

$$C_1 = x_1(t), y_1(t) \quad (303)$$

$$C_2 = x_2(t), y_2(t) \quad (304)$$

An intersection occurs when $y_1(x) = y_2(x)$ and a collision occurs when $x_1(t) = x_2(t)$ and $y_1(t) = y_2(t)$.

Example 44: Suppose we have the following curves:

$$C_1 : x_1(t) = 2t + 6, y_1(t) = 5 - 4t \quad (305)$$

$$C_2 : x_2(t) = 3 - 5 \cos \pi t, y_2 = 1 + 5 \sin \pi t \quad (306)$$

for $t > 0$. The first curve is a straight line and the second curve is an offset circle. We can find the intersection via:

$$C_1 : t = \frac{x-6}{2} \implies y_1(x) = 16 - 2x \quad (307)$$

$$C_2 : (x-3)^2 + (y-1)^2 = 25 \quad (308)$$

There will be two points of intersection, and after we solve this, we get $(6, 5)$ and $(8, 1)$. We now need to look at if the two objects will arise at that point at the same time. We do this by looking at each point first

$$(6, 5) : C_1 \implies t = 0 \quad (309)$$

$$C_2|_{t=0} = (-2, 1) \quad (310)$$

$$(8, 1) : C_1 \implies t = 1 \quad (311)$$

$$C_2|_{t=1} = (8, 1) \quad (312)$$

Therefore, there is a collision between the two objects at $(8, 1)$ at a time $t = 1$.

10 Calculus with Parametric Curves

- Parametric curves do not need to be functions. As a result, they can have an ordinary tangent, no tangent, or more than one tangent at the same point.
- We can find the secant line of a parametric curve via:

$$m_{\text{secant}} = \frac{y(t_0 + h) - y(t_0)}{x(t_0 + h) - x(t_0)} \quad (313)$$

We can divide both numerator and denominator by h to get:

$$\frac{\frac{y(t_0+h)-y(t_0)}{h}}{\frac{x(t_0+h)-x(t_0)}{h}} \implies \frac{y'(t_0)}{x'(t_0)} \quad (314)$$

where we took the limit as $h \rightarrow 0$.

- We could also have derived this using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \left(\frac{dx}{dt} \right)^{-1} \quad (315)$$

- If $x'(t_0) = 0$, then $x = x_0$ is a vertical tangent and if $y'(t_0) = 0$, then we have $y = y_0$ and have a horizontal tangent. If $x' = y' = 0$, then we can't read any information from it.

Example 45: Consider the example $x(t) = \sin(2t)$ and $y(t) = \sin t$ with $t \in [0, 2\pi]$. We have $x'(t) = 2 \cos 2t$ so we can find the vertical tangents by setting $2 \cos(2t) = 0$ to get:

$$t \in \left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\} \implies \left(1, \frac{1}{\sqrt{2}} \right) \left(-1, \frac{1}{\sqrt{2}} \right) \left(1, -\frac{1}{\sqrt{2}} \right) \left(-1, -\frac{1}{\sqrt{2}} \right) \quad (316)$$

For horizontal tangents, we have $y'(t) = \cos t \implies \cos t = 0$ which gives:

$$t \in \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\} \implies (0, 1)(0, -1) \quad (317)$$

At $t = 0$, we also have $x(0) = y(0) = 0$. We can determine the tangent at this point as:

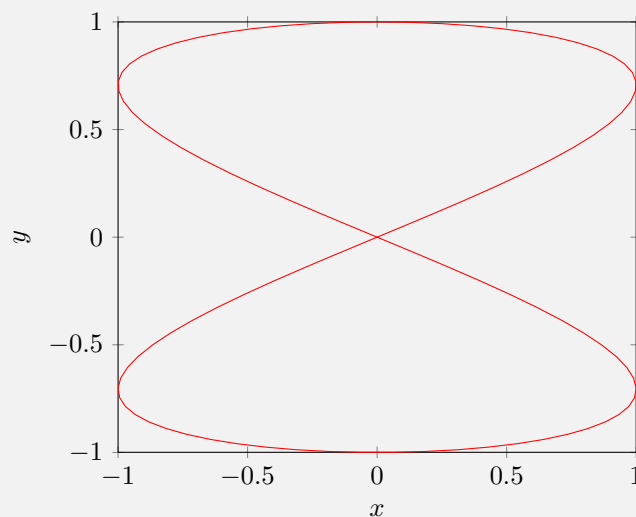
$$m_{\text{tangent}} = \frac{y'(0)}{x'(0)} = \frac{1}{2} \quad (318)$$

However notice that at $t = \pi$, we have $x(\pi) = y(\pi) = 0$ as well so this point has another tangent:

$$m_{\text{tangent}} = \frac{y'(\pi)}{x'(\pi)} = -\frac{1}{2} \quad (319)$$

At this point we can sketch out the conditions we found on the graph and draw out curve:

Parametric Example



- We can determine the **area** of a parametric curve via:

$$A = \int_{t_1}^{t_2} y(t)x'(t) dt \quad (320)$$

Letting $y = f(x) \implies y(t) = f(x(t))$ gives:

$$A = \int_{t_1}^{t_2} f(x(t))x'(t) dt \quad (321)$$

$$= \int_{a=x(t_1)}^{b=x(t_2)} f(x) dx \quad (322)$$

- We can also determine the area of a closed curve:

Definition: A curve is traversed in the positive sense as t increases, if the enclosed area is on the left (counterclockwise).

The area of a closed loop is then:

$$A = \int_{t_4}^{t_3} y(t)x'(t) dt - \int_{t_4}^{t_5} y(t)x'(t) dt + \int_{t_3}^{t_2} y(t)x'(t) dt - \int_{t_1}^{t_2} y(t)x'(t) dt \quad (323)$$

where t_1 represents the starting point on the bottom, t_2 is the rightmost point, t_3 is an arbitrary point on top, t_4 is the leftmost point, and t_5 is the endpoint that ends off as t_1 . This gives:

$$- \int_{t_3}^{t_4} y(t)x'(t) dt - \int_{t_4}^{t_5} y(t)x'(t) dt - \int_{t_2}^{t_3} y(t)x'(t) dt - \int_{t_1}^{t_2} y(t)x'(t) dt \quad (324)$$

or:

$$A = - \int_{t_1}^{t_5} y(t)x'(t) dt \quad (325)$$

where t_1 and t_5 are the starting and ending points respectively. They can be arbitrarily chosen so long as the points they correspond to are on top of each other.

- Similarly, we can also write the area as:

$$A = \int_{t_1}^{t_5} x(t)y'(t) dt \quad (326)$$

Example 46: Let us derive the area of an ellipse, parametrized by:

$$x = a \cos \theta \quad (327)$$

$$y = b \sin \theta \quad (328)$$

for $\theta \in [0, 2\pi]$. The area is then:

$$A = - \int_0^{2\pi} b \sin \theta (-a \sin \theta) d\theta \quad (329)$$

$$= ab \int_0^{2\pi} \sin^2 \theta d\theta \quad (330)$$

$$= ab \left[\frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right]_0^{2\pi} \quad (331)$$

$$= \pi ab \quad (332)$$

- We can also determine the **arclength** of a parametric curve. We start with the summation:

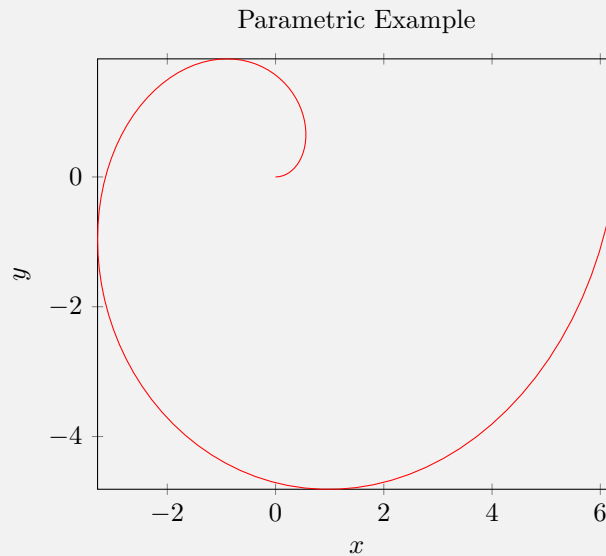
$$s \simeq \sum \sqrt{\Delta x^2 + \Delta y^2} \implies \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2} dt \quad (333)$$

Note that if we let $x = t$ and $y = f(t) = f(x)$, then we get:

$$s = \int_{\alpha}^{\beta} \sqrt{1 + f'(x)^2} dx \quad (334)$$

which is what we should expect.

Example 47: Suppose we have $x(\theta) = \theta \cos \theta$ and $y(\theta) = \theta \sin \theta$ for $\theta \in [0, 2\pi]$, this gives an Archimedes Spiral:



We can determine the arclength via:

$$s = \int_0^{2\pi} \sqrt{(\cos \theta - \theta \sin \theta)^2 + (\sin \theta + \theta \cos \theta)^2} d\theta \quad (335)$$

$$= \int_0^{2\pi} \sqrt{1 + \theta^2} d\theta \quad (336)$$

$$= \left[\frac{1}{2} \theta \sqrt{1 + \theta^2} + \frac{1}{2} \ln \left| \theta + \sqrt{1 + \theta^2} \right| \right]_0^{2\pi} \quad (337)$$

$$= \pi \sqrt{1 + 4\pi^2} + \frac{1}{2} \ln \left(2\pi + \sqrt{1 + 4\pi^2} \right) \quad (338)$$

Idea: If we wish to find the speed, we have:

$$v = \frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} \quad (339)$$

which comes from both the fundamental theorem of calculus, as well as from two-dimensional kinematics.

- The surface area can be written as:

$$A = \int_a^b 2\pi y ds = \int_a^b 2\pi y(t) \sqrt{x'(t)^2 + y'(t)^2} dt \quad (340)$$

- What is the *circumference of an ellipse*? If we try to carry out this calculation, we can parametrize the ellipse as before:

$$x = a \sin \theta \quad (341)$$

$$y = b \cos \theta \quad (342)$$

with $0 \leq \theta \leq 2\pi$ and the arclength as:

$$s = \int_0^{2\pi} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta \quad (343)$$

$$= \int_0^{2\pi} \sqrt{a^2(1 - \sin 2\theta) + b^2 \sin 2\theta} d\theta \quad (344)$$

$$= \int_0^{2\pi} a \sqrt{1 - \epsilon^2 \sin^2 \theta} d\theta \quad (345)$$

with $\epsilon \equiv \sqrt{\frac{a^2 - b^2}{a^2}}$. Unfortunately, this is an elliptic integral of the second kind and has no analytic solution.

11 Polar Coordinates

- In polar coordinates, we can represent polar coordinates in terms of the distance r from the origin and the angle it makes with the positive horizontal axis:

$$(x, y) \iff [r, \theta] \quad (346)$$

- The magnitude of r gives the distance from the origin, and multiplying r by -1 rotates the point about the origin by π .
- Polar coordinates are not unique:
 - The pole is $[0, \theta]$ for all θ .
 - $[r, \theta] = [r, \theta + 2n\pi]$ for any integer n .
 - $[r, \theta] = [-r, \theta + (2n + 1)\pi]$ for any integer n .
- We can convert between cartesian and polar coordinates using the transformation:

$$x = r \cos \theta \quad (347)$$

$$y = r \sin \theta \quad (348)$$

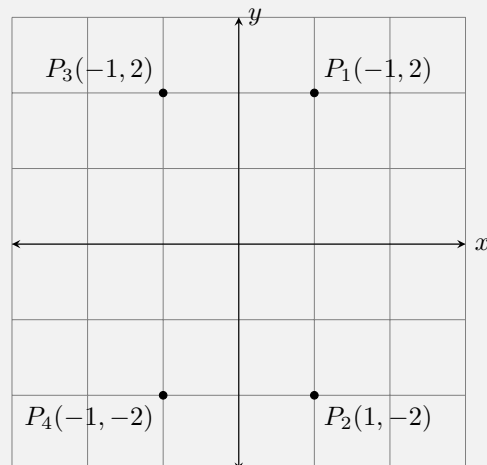
which gives:

$$r = \sqrt{x^2 + y^2} \quad (349)$$

$$\theta = \arctan\left(\frac{y}{x}\right) \quad (350)$$

for $x \neq 0$.

Example 48: Suppose we have the following four points.



We can represent the four coordinates also as:

$$P_1[\sqrt{5}, 1.107] \quad (351)$$

$$P_2[\sqrt{5}, -1.107] \quad (352)$$

$$P_3[-\sqrt{5}, -1.107] \quad (353)$$

$$P_4[-\sqrt{5}, 1.107] \quad (354)$$

- We can represent straight lines as:

- Straight lines: $y = mx$: $r = \alpha$ with $\alpha = \arctan(m)$.
- Vertical lines $x = a$: We have $r \cos \theta = a \implies r = a \sec \theta$.
- Horizontal lines: $y = b$. We have $r \sin \theta = b \implies r = b \csc \theta$

- We can represent circles in polar coordinates as:

$$x^2 + y^2 = 9 \iff r = 3 \quad (355)$$

- Converting *from* polar coordinates requires a bit of extra work. Suppose we have $r = 6 \sin \theta$, then:

$$r^2 = 6r \sin \theta \quad (356)$$

$$x^2 + y^2 = 6y \quad (357)$$

$$x^2 + y^2 - 6y + 9 = 9 \quad (358)$$

$$x^2 + (y - 3)^2 = 9 \quad (359)$$

which represents a circle with radius 3 centered at $(0, 3)$.

- Symmetry can also arise in many scenarios. For example:

- Symmetry about x axis: $[r_1, \theta]$ and $[r_1, -\theta]$.
- Symmetry about y axis: $[r, \pi - \theta]$ and $[r_1, \theta]$.
- Symmetry about origin: $[r, \theta]$ and $[r, \theta + \pi]$.

which will help when sketching them.

Example 49: Suppose we wish to sketch the curve $r = \frac{1}{2} + \cos \theta$. Notice that this is periodic so we only need to look at values of θ where $0 \leq \theta < 2\pi$.

1. Let us first find values of θ (if possible) that make $r = 0$:

$$0 = \frac{1}{2} + \cos \theta \implies \theta = \frac{2\pi}{3}, \frac{4\pi}{3} \quad (360)$$

2. Find local max and min values of r :

$$\frac{dr}{d\theta} = -\sin \theta = 0 \implies \theta = 0, \pi \quad (361)$$

At $\theta = 0$, we have $r = \frac{3}{2}$, at $\theta = \pi$ we have $r = -\frac{1}{2}$. If $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, then $r = \frac{1}{2}$.

3. We then look at symmetry. Notice that:

$$\frac{1}{2} + \cos(-\theta) + \frac{1}{2} + \cos \theta \quad (362)$$

However:

$$r(\theta - \pi) \neq r(\theta) \quad (363)$$

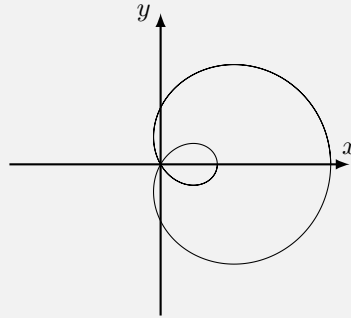
$$r(\pi + \theta) \neq r(\theta) \quad (364)$$

so it is not symmetric about the y axis or origin.

4. We now look at the relevant intervals. From $0 \leq \theta < \frac{2\pi}{3}$, we have $\frac{dr}{d\theta} < 0$ so the radius is monotonically decreasing.

We can also look at the interval $\frac{2\pi}{3} \leq \theta < \pi$ and see that r is negative and $\frac{dr}{d\theta} < 0$, so the magnitude of r *increases*.

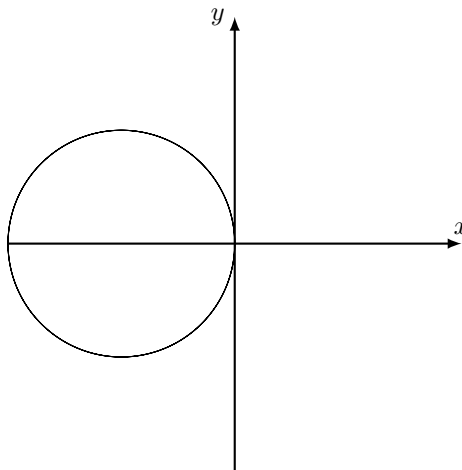
Noting that the function is differentiable at $\theta = \pi$, we can reflect the shape about the x axis to get a Limacon with inner loop.



- There are a few common shapes. Each of these could be flipped or rotated by shifting the argument θ , or using negative numbers.

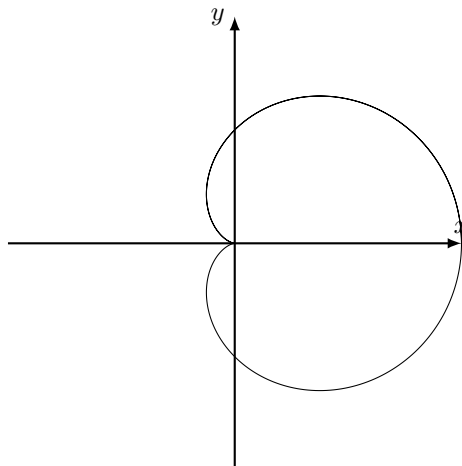
– Circles:

$$r = -2 \cos \theta \quad (365)$$



– Cardioids:

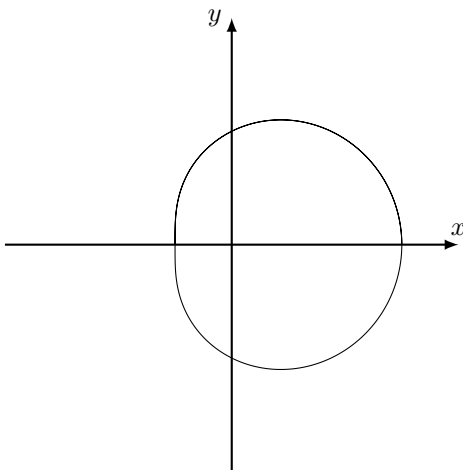
$$r = a + a \cos \theta \quad (366)$$



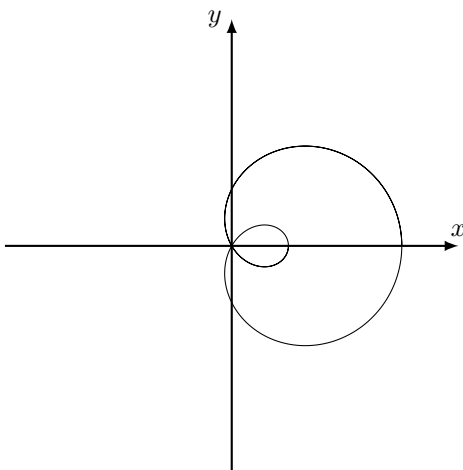
– Limacons:

$$r = a + b \sin \theta \quad (367)$$

There are two types, for $a > b$:

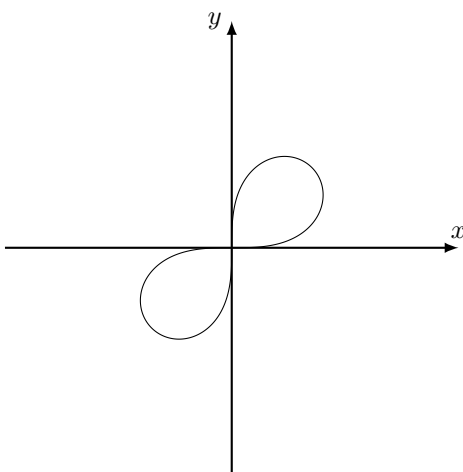


For $a < b$:



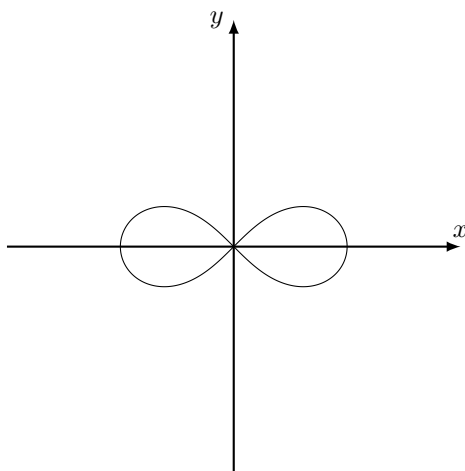
– Lemniscates. Again, there are two types. For:

$$r^2 = a \sin(2\theta) \quad (368)$$



and:

$$r^2 = a \cos(2\theta) \quad (369)$$

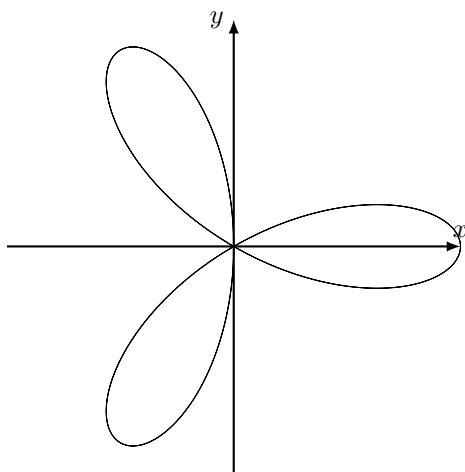


– Petal curves:

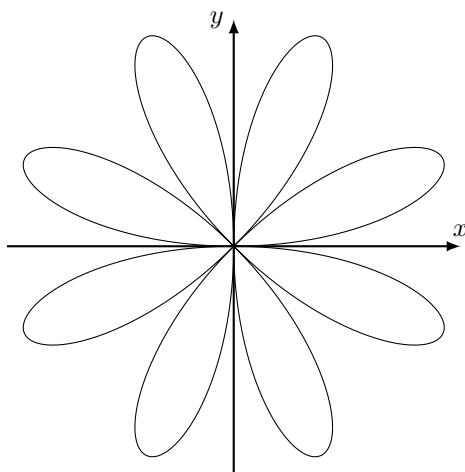
$$r = a \sin(n\theta) \quad (370)$$

$$r = a \cos(n\theta) \quad (371)$$

where n is an integer. There are n petals if n is odd and $2n$ petals if n is even. For example, the following is:
 $r = 2 \cos 3\theta$:



and for $r = 2 \sin(4\theta)$:

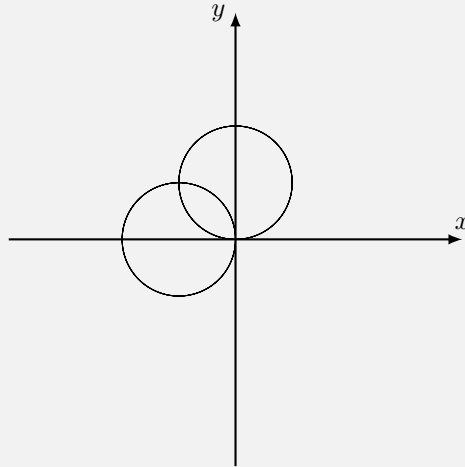


- We can also find the intersection of polar coordinates, but we have to be careful. The following example illustrates why.

Example 50: Suppose we have two curves $r = \sin \theta$ and $r = -\cos \theta$. Suppose we try to solve this via:

$$\sin \theta = -\cos \theta \implies \theta = \frac{3\pi}{4}, \frac{7\pi}{4} \quad (372)$$

Plugging this back into $x = r \cos \theta$ and $y = r \sin \theta$, we get $x = -\frac{1}{2}$ and $y = -\frac{1}{2}$. We can also use $\theta = \frac{7\pi}{4}$ to get: $x = -\frac{1}{2}$ and $y = \frac{1}{2}$ which is the same point. However, it represents the curves below:



There is actually two intersection points! The reason for this is that we assumed that the two curves intersect at the same value of θ , but this is not necessarily true for the origin, which can be obtained at any angle θ .

- As a result, we also have to check the origin.
- It is also possible to find the tangent:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \quad (373)$$

$$= \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} \quad (374)$$

12 Areas and Lengths in Polar Coordinates

- Suppose we have a polar curve $r = \rho(\theta)$ for $\alpha \leq \theta \leq \beta$.
- We can determine the area by partitioning the curve into θ_i and approximating each subregion as a circular segment. The area of a circular segment is:

$$A = \frac{1}{2} a^2 \Delta \theta \quad (375)$$

We can take the radius to be $r = \rho(\theta^*)$ where $\theta_{i-1} \leq \theta^* \leq \theta_i$. The area of each region is:

$$A_i = \frac{1}{2} \rho(\theta_i^*)^2 \Delta \theta_i \quad (376)$$

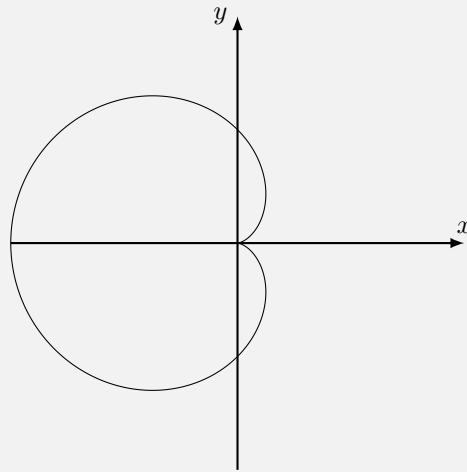
so the total area is:

$$A = \lim_{||P||} \sum_i \frac{1}{2} \rho(\theta_i^*)^2 \Delta \theta_i \quad (377)$$

or

$$A = \frac{1}{2} \int_{\alpha}^{\beta} \rho(\theta)^2 d\theta \quad (378)$$

Example 51: Suppose we wish to find the area of $r = 1 - \cos \theta$.



The area is then:

$$A = \frac{1}{2} \int_0^{2\pi} (1 - \cos \theta)^2 d\theta \quad (379)$$

$$= \frac{1}{2} \int_0^{2\pi} (1 - 2\cos \theta + \cos^2 \theta) d\theta \quad (380)$$

$$= \frac{1}{2} \int_0^{2\pi} \left(\frac{3}{2} - 2\cos \theta + \frac{1}{2} \cos(2\theta) \right) d\theta \quad (381)$$

$$= \frac{1}{2} \left(\frac{3}{2}\theta - 2\sin \theta + \frac{1}{4} \sin(2\theta) \right) \Big|_0^{2\pi} \quad (382)$$

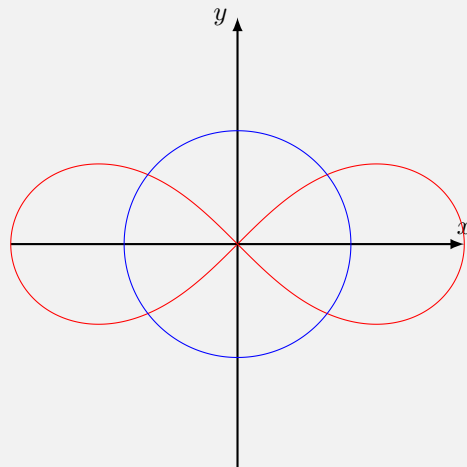
$$= \frac{3}{2}\pi \quad (383)$$

- We can also find the area between two polar curves ρ_1 and ρ_2 . We have:

$$A = \frac{1}{2} \int_{\alpha}^{\beta} \rho_1(\theta)^2 d\theta - \frac{1}{2} \int_{\alpha}^{\beta} \rho_2(\theta)^2 d\theta = \frac{1}{2} \int_{\alpha}^{\beta} (\rho_1^2 - \rho_2^2) d\theta \quad (384)$$

Warning: Be careful when applying this formula as it is possible the two functions can overlap between $\alpha \leq \theta \leq \beta$. Therefore, we always need a good idea of what's happening.

Example 52: Suppose we want to determine the area inside $r^2 = 4\cos(2\theta)$ but outside $r = 1$. This gives:



We first find the four points of intersection:

$$4 \cos(2\theta) = 1 \implies \cos(2\theta) = \frac{1}{4} \implies \theta = \pm 0.659 \quad (385)$$

or $\theta = \pi \pm 0.659$. Due to the symmetry, we only need to find the area of one half of the area we are interested in, which gives:

$$\frac{1}{2}A = \frac{1}{2} \int_{-0.659}^{0.659} (4 \cos 2\theta - 1) d\theta \quad (386)$$

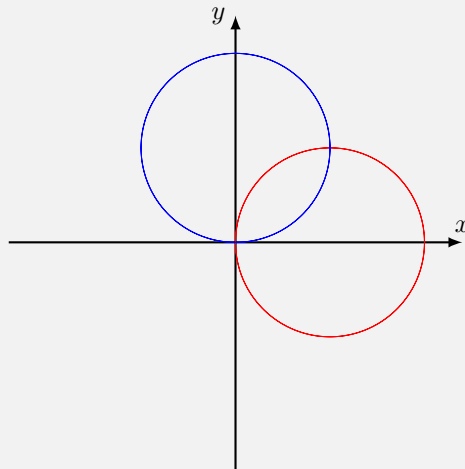
$$= \frac{1}{2} (2 \sin 2\theta - \theta) \Big|_{-0.659}^{0.659} \quad (387)$$

$$= 1.277 \quad (388)$$

so the area is $A = 2.554$.

- There are a few challenging examples:

Example 53: Suppose we wish to find the area between $r = \sin \theta$ and $r = \cos \theta$:



We know from symmetry that the intersection is at $\theta = \frac{\pi}{4}$. We notice that the contribution to the area from each curve ρ is equal and *independent* from each other. Therefore:

$$A = A_1 + A_2 = \int_0^{\pi/4} \frac{1}{2} \sin^2 \theta d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2} \cos^2 \theta d\theta = \frac{\pi}{8} - \frac{1}{4} \quad (389)$$

- We can determine the arclength by working in parametric form. Let $x = r(\theta) \cos \theta$ and $y = r(\theta) \sin \theta$. Therefore:

$$s = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \quad (390)$$

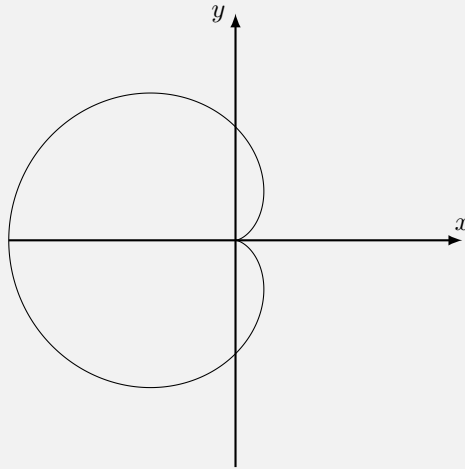
$$= \int_{\alpha}^{\beta} \sqrt{(r' \cos \theta - r \sin \theta)^2 + (r' \sin \theta + r \cos \theta)^2} d\theta \quad (391)$$

$$= \int_{\alpha}^{\beta} \sqrt{(r'^2 \cos^2 \theta + r^2 \sin^2 \theta - 2r'r' \cos \theta \sin \theta) + (r'^2 \sin^2 \theta + r^2 \cos^2 \theta + 2r'r' \cos \theta \sin \theta)} d\theta \quad (392)$$

$$= \int_{\alpha}^{\beta} \sqrt{r^2(\cos^2 \theta + \sin^2 \theta) + r'^2(\cos^2 \theta + \sin^2 \theta)} d\theta \quad (393)$$

$$= \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \quad (394)$$

Example 54: Suppose we want to find the arclength of $r = a(1 - \cos \theta)$ from $0 \leq \theta < 2\pi$. This looks like:



We have:

$$s = \int_0^{2\pi} \sqrt{r^2 + (r')^2} d\theta \quad (395)$$

$$= \int_0^{2\pi} \sqrt{a^2(1 - 2\cos \theta + \cos^2 \theta) + a^2 \sin^2 \theta} d\theta \quad (396)$$

$$= a \int_0^{2\pi} \sqrt{2 - 2\cos \theta} d\theta \quad (397)$$

$$= a \int_0^{2\pi} \sqrt{4 \sin^2 \left(\frac{\theta}{2} \right)} d\theta \quad (398)$$

$$= 2a \left[-2 \cos \left(\frac{\theta}{2} \right) \right] \Big|_0^{2\pi} \quad (399)$$

$$= 8a \quad (400)$$

13 Infinite Sequences and Series

- We use curly brackets to indicate a sequence, such as:

$$f(n) = \frac{1}{n} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\} \quad (401)$$

Alternatively, we can use a_n to represent a sequence.

Definition: A sequence $\{a_n\}$ is:

- increasing iff $a_n < a_{n+1}$
- non-decreasing iff $a_k \leq a_{n+1}$
- decreasing iff $a_n > a_{n+1}$
- non-increasing iff $a_n \geq a_{n+1}$

A function that satisfies any of these are known as **monotonic** functions.

- Bounded functions have an upper or lower bound, while unbounded functions diverge to infinity or negative infinity.

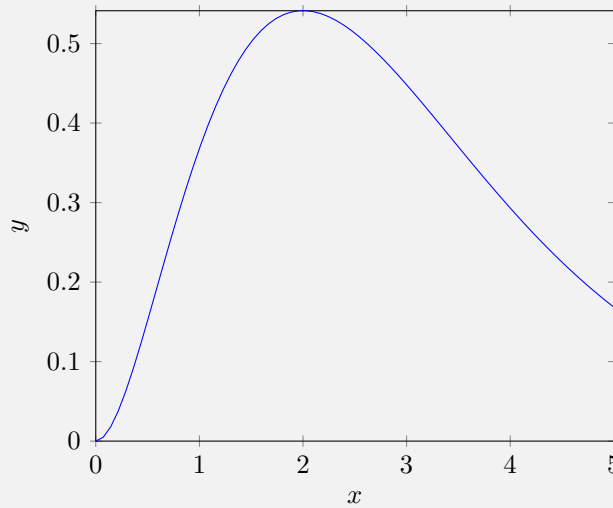
Example 55: Suppose we wish to prove that 2^k is unbounded. We wish to find k such that $a_k > M$ or $2^k > M$. Taking the natural logarithm of both sides, we have:

$$k > \frac{\ln M}{\ln 2} \quad (402)$$

which is possible to do and we are done.

Example 56: Suppose we wish to find if $a_n = \frac{n^2}{e^n}$ is bounded or unbounded. This can be approached by working with derivatives through the function $f(x) = \frac{x^2}{e^x}$, represented in the following plot:

Example



Taking the derivative $f'(x) = xe^{-x}(2-x)$, we see that f decreases for $x > 2$ so this means that a_n decreases for $n > 2$

Warning: Not everything in functions carries over to sequences. For example, $f(x) = \frac{1}{x - \sqrt{2}}$ is unbounded but $a_n = \frac{1}{n - \sqrt{2}}$ is bounded since $n \neq \sqrt{2}$ is impossible.

- We can only take the limit of a sequence as $n \rightarrow \infty$.

Definition: We can define $\lim_{n \rightarrow \infty} a_n = L$ iff for every $\epsilon > 0$, there exists an integer $k > 0$ such that if $n \geq k$, then $|a_n - L| < \epsilon$.

Example 57: Let us prove $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$. We find k such that $\left| \frac{n}{n+1} - 1 \right| < \epsilon$ for $n \geq k$. This can be rewritten as:

$$\left| \frac{1}{n+1} \right| < \epsilon \quad (403)$$

or $|n+1| > \frac{1}{\epsilon}$. Thus, if we choose $k = \frac{1}{\epsilon}$ such that if we choose $n > k = \frac{1}{\epsilon}$, then:

$$\left| \frac{n}{n+1} - 1 \right| = \left| \frac{1}{n+1} \right| < \left| \frac{1}{n} \right| < \frac{1}{k} = \epsilon \quad (404)$$

Therefore, $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$.

Theorem: Uniqueness of a Limit: If $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} a_n = M$, then $L = M$.

Definition: If a sequence has a limit, it is said to be convergent. Otherwise, it is divergent.

- This leads to the following:
 1. If a sequence is convergent, it is bounded.
 2. If a sequence is unbounded, it is divergent.
 3. A bounded sequence is not necessarily convergent.
- For example, $a_n = \cos \pi n$ is bounded but not convergent.

Theorem: Monotonic Sequence Theorem: A bounded nondecreasing sequence converges to its least upper bound. A bounded non increasing sequence converges to its greatest lower bound.

- The limit has a few properties. Let $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} b_n = M$. Then:

1. $\lim_{n \rightarrow \infty} (a_n + b_n) = L + M$
2. $\lim_{n \rightarrow \infty} \alpha a_n = \alpha L$ for $\alpha \in \mathbb{R}$.
3. $\lim_{n \rightarrow \infty} a_n b_n = L \cdot M$
4. $\lim_{n \rightarrow \infty} \frac{1}{b_n} = \frac{1}{M}$ for $b_n \neq 0, M \neq 0$.
5. $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L}{M}$ for $b_n \neq 0, M \neq 0$.

Theorem: Pinching Theorem for Sequences: If for large n , $a_n \leq b_n \leq c_n$ and if $\lim_{n \rightarrow \infty} a_n = L$ and $\lim_{n \rightarrow \infty} c_n = L$, then $\lim_{n \rightarrow \infty} b_n = L$.

Example 58: Suppose we wish to find the limit $\lim_{n \rightarrow \infty} \frac{\sin(n\pi/6)}{n}$. We can let:

$$-\frac{1}{n} \leq \frac{\sin(n\pi/6)}{n} \leq \frac{1}{n} \quad (405)$$

Since $\lim_{n \rightarrow \infty} -\frac{1}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$, then the original limit must also be zero.

Theorem: Suppose we have a sequence: $c_n = g(f_n)$. Given $\lim_{n \rightarrow \infty} c_n = C$. If f is continuous at c , in the traditional way, then: $\lim_{n \rightarrow \infty} f(c_n) = f(c)$.

Example 59: Let us look at the function $\sin\left(\frac{1}{n^2 + 1}\right)$. We know that $\lim_{n \rightarrow \infty} \frac{1}{n^2 + 1} = 0$, so:

$$\lim_{n \rightarrow \infty} \sin\left(\frac{1}{n^2 + 1}\right) = \sin(0) = 0 \quad (406)$$

where we have applied the previous theorem.

14 Sequences

- We begin with some **important limits**:

- For $x > 0$, $\lim_{n \rightarrow \infty} x^{1/n} = 1$.
- If $|x| < 1$, then $\lim_{n \rightarrow \infty} x^n = 0$.

Proof. We know the function is decreasing: $|x|^{n+1} = |x||x^n| < |x|^n$. Alternatively, we want to show that $|x^k - 0| < \epsilon$ for all $n > k$. We want to find k such that:

$$|x^k - 0| = |x^n| = |x|^k < \epsilon \quad (407)$$

or: $|x| < \epsilon^{1/n}$. We know that:

$$\lim_{n \rightarrow \infty} \epsilon^{1/n} = 1 \quad (408)$$

and since $|x| < \epsilon^{1/k}$, we must have $|x^n| < \epsilon$ for all $n > k$. \square

- For $\alpha > 0$, $\lim_{n \rightarrow \infty} \frac{1}{n^\alpha} = 0$.

Proof. Note that:

$$0 < \frac{1}{n^\alpha} = \left(\frac{1}{n}\right)^\alpha \quad (409)$$

We can pick an odd positive integer p such that $1/p < \alpha$ such that:

$$\left(\frac{1}{n}\right)^\alpha \leq \left(\frac{1}{n}\right)^{1/p} \implies \lim_{n \rightarrow \infty} \left(\frac{1}{n}\right)^{1/p} = \left(\lim_{n \rightarrow \infty} \frac{1}{n}\right)^{1/p} = 0 \quad (410)$$

\square

- $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$ for $x \in \mathbb{R}$.
- $\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$
- $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$.
- $\lim_{n \rightarrow \infty} n^{1/n} = 1$.
- $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$

Proof. First let's deal with the $x = 0$ case, which is trivial. Now:

$$\ln \left(1 + \frac{x}{n}\right)^n = n \ln \left(1 + \frac{x}{n}\right) \quad (411)$$

$$= \frac{x \ln(1 + x/n)}{x/n} \quad (412)$$

$$= x \left(\frac{\ln(1 + x/n) - \ln(1)}{x/n} \right) \quad (413)$$

Taking the limit, we have:

$$\lim_{n \rightarrow \infty} \frac{\ln(1 + x/h) - \ln 1}{x/h} = \lim_{h \rightarrow 0} \frac{\ln(1 + h) - \ln 1}{h} \quad (414)$$

which is the first principles definition of the derivative of $\ln(x)$ at $x = 1$, which gives:

$$\lim_{h \rightarrow 0} \ln(1 + x/n)^n = x \cdot 1 = x \implies \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x \quad (415)$$

\square

- Sequences can also be defined recursively. We need a base term, e.g. $a_1 = 1$ and also a general relationship, such as:

$$a_n = \sqrt{6 + a_{n-1}} \quad (416)$$

this gives the sequence $\{1, \sqrt{7}, \sqrt{6 + \sqrt{7}}, \dots\}$

- How do we find the **limit** of such a recursively defined function? To do so, we first need to show that the limit actually exists. To do so, we must have both:

$$\lim_{n \rightarrow \infty} a_n = L \quad (417)$$

$$\lim_{n \rightarrow \infty} a_{n-1} = L \quad (418)$$

Therefore, we get:

$$L = \sqrt{6 + L} \implies L = 3, -2 \quad (419)$$

Since it is increasing, we must have $L = 3$.

15 Series

- Suppose we wish to add the infinite series:

$$I = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \quad (420)$$

- We can define the partial sum to be:

$$s_0 = a_0 = \sum_{k=0}^0 a_n \quad (421)$$

$$s_1 = a_0 + a_1 = \sum_{k=0}^1 a_n \quad (422)$$

$$s_2 = a_0 + a_1 + a_2 = \sum_{k=0}^2 a_n \quad (423)$$

$$\vdots \quad (424)$$

$$s_n = a_0 + a_1 + \cdots + a_n = \sum_{k=0}^n a_k \quad (425)$$

- We can then consider the sequence $\{s_n\} = \{a_0, a_0 + a_1, a_0 + a_1 + a_2, \cdots\}$. This sequence converges if the sum converges. Specifically, if $\lim_{n \rightarrow \infty} \{s_n\} = L$, then $\sum_{k=0}^{\infty} a_n = L$.

Example 60: Suppose we wish to evaluate:

$$\sum_{k=0}^{\infty} \frac{1}{(k+2)(k+3)} = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} \quad (426)$$

We can use partial fractions to write:

$$\frac{1}{(k+2)(k+3)} = \frac{1}{k+2} - \frac{1}{k+3} \quad (427)$$

so the sum becomes:

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots - \frac{1}{n+3} = \frac{1}{2} - \frac{1}{n+3} \quad (428)$$

which is known as a telescoping sequence. Taking the limit as $n \rightarrow \infty$, we get that the sum converges to $\frac{1}{2}$.

- The sum of a geometric series is:

$$x^0 + x^1 + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \quad (429)$$

which converges when $|x| < 1$.

Proof. Let $S_n = 1 + x + x^2 + \cdots + x^n$ and $xS_n = x + x^2 + x^3 + \cdots + x^{n+1}$. Then subtracting the two, we get:

$$S_n - xS_n = 1 - x^{n+1} \implies S_n = \frac{1 - x^{n+1}}{1 - x} \quad (430)$$

and for $|x| < 1$, the limit gives us $\frac{1}{1-x}$ and if $|x| > 1$, the limit diverges. \square

- Suppose we wish to write the repeating fraction as a decimal: $0.\overline{285714}$. This is equal to:

$$= \frac{28574}{10^6} + \frac{285714}{10^{12}} + \cdots \quad (431)$$

$$= \frac{28574}{10^6} \left(1 + \frac{1}{10^6} + \frac{1}{10^{12}} + \cdots \right) \quad (432)$$

Evaluating this infinite series, we get:

$$\frac{2}{7} \quad (433)$$

Example 61: Suppose we wish to write out $\frac{x}{4-x^2}$ as a sum for $|x| < 2$. We have:

$$\frac{x}{4-x^2} = \frac{x}{4} \left(\frac{1}{1-x^2/4} \right) \quad (434)$$

$$= \frac{x}{4} \sum_{k=0}^{\infty} \left(\frac{x^2}{4} \right)^k \quad (435)$$

$$= \frac{x}{4} \sum_{k=0}^{\infty} \left(\frac{x}{2} \right)^{2k} \quad (436)$$

$$= \frac{1}{2} \left[\frac{x}{2} + \left(\frac{x}{2} \right)^3 + \left(\frac{x}{2} \right)^5 + \cdots \right] \quad (437)$$

Theorem: Here are a few important properties that arise when applying limit laws:

- If $\sum_{k=0}^{\infty} a_k = n$ and $\sum_{k=0}^{\infty} b_k = M$, then $\sum_{k=0}^{\infty} (a_k + b_k) = n + M$.
- If $\sum_{k=0}^{\infty} a_k = L$, then $\sum_{k=0}^{\infty} \alpha a_k = \alpha L$ for $\alpha \in \mathbb{R}$.

Theorem: If $\sum_{k=0}^{\infty} a_k$ converges iff $\sum_{k=j}^{\infty} a_k$ converges where j is a positive integer.

Example 62: Suppose we are given that $\sum_{k=4}^{\infty} \frac{3^{k-1}}{3^{3k+1}}$ converges, then $\sum_{k=0}^{\infty} \frac{3^{k-1}}{3^{3k+1}}$ converges.

Theorem: If $\sum_{k=0}^{\infty} a_k$ converges, then $a_k \rightarrow 0$ as $k \rightarrow \infty$.

Theorem: (Test for Divergence:) This is the contrapositive of the previous theorem. If $a_k \not\rightarrow 0$ as $k \rightarrow \infty$, then $\sum_{k=0}^{\infty} a_k$ diverges.

16 Convergence Tests

- We start with the integral test:

Theorem: If f is continuous, decreasing, and positive on $[1, \infty)$, then: $\sum_{k=1}^{\infty} f(k)$ converges if and only if $\int_1^{\infty} f(x) \, dx$ converges.

Example 63: Suppose we take the harmonic sum:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \quad (438)$$

However, the integral $\int_1^{\infty} \frac{dx}{x} = \lim_{b \rightarrow \infty} \ln b$ diverges.