Medici - Solution Manual

QiLin Xue

1 Determinants

Question 01: The wording is a bit unclear. If it implies that we first multiply row 5 by six, and then add row 7 to it, then the answer is no. Let $\mathbf{A}'$ be the resulting matrix when we multiply row 5 by six. Then by the multilinearity principle, we have:

$$\det \mathbf{A}' = 5 \det \mathbf{A} \tag{1}$$

Adding combinations of other rows do not impact the determinant, since:

$$\det \mathbf{E}(\lambda; i, j) = 1 \tag{2}$$

Question 02: No, because:

$$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1 \tag{3}$$

Question 03: From multilinearity, we have (using the first equality):

$$\frac{2^n}{\det \mathbf{A}} = -4 \implies \det \mathbf{A} = -2^{n-2} \tag{4}$$

From the second set, applying the product rule, we have:

$$-4 = \frac{\det(\mathbf{A})^3}{\det \mathbf{B}} \tag{5}$$

Since we know that $\det \mathbf{A} = -2^{n-2}$ we have:

$$-4 = \frac{-2^{3n-6}}{\det \mathbf{B}} \implies \det \mathbf{B} = 2^{3n-8} \tag{6}$$

Question 04: No, the product rule is only valid for square matrices. As a counterexample, consider $\mathbf{A} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

Question 05: We can rewrite the equation as:

$$\mathbf{A}^2 = -\mathbf{1} \tag{7}$$

Taking the determinant of both sides:

$$\det(\mathbf{A})^2 = (-1)^n \tag{8}$$

and therefore:

$$\det \mathbf{A} = \sqrt{(-1)^n} \tag{9}$$

If n is even, then $\det \mathbf{A} = 1$. Otherwise, $\det \mathbf{A} = i$ where i is the imaginary number.

Question 06: First, we show that if (x, y, z) lies on the plane, then the determinant is zero. The plane can be described with the basis $\mathbf{v}_1 = (x_2 - x_1, y_2 - x_1, z_2 - z_1)$ and $\mathbf{v}_2 = (x_3 - x_1, y_3 - y_1, z_3 - z_1)$. Since the point is on the plane, the point can be described with $\mathbf{v} = (x - x_1, y - y_1, z - z_1)$. Since:

$$\mathbf{v} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \tag{10}$$

, then $\mathbf{v}$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. As a result, we have:

$$\det \begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = \det \begin{bmatrix} x - x_1 & y - y_1 & z - z_1 & 0 \\ x_1 & y_1 & z_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 & 0 \end{bmatrix} \tag{11}$$

Since the first row is a linear combination of the last two rows, the determinant is zero.

Next we show that if the determinant is zero, then the point (x, y, z) lies on the plane. We can write the determinant as:

$$\det \begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{bmatrix} = \det \begin{bmatrix} x - x_1 & y - y_1 & z - z_1 & 0 \\ x_1 & y_1 & z_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 & 0 \end{bmatrix} = 0 \tag{12}$$

Since

$$\det \mathbf{A} = 0 \iff \text{rows are linearly dependent},$$

we know that the rows are linearly dependent. Since none of the points are colinear, the last two rows are linearly independent and therefore the last three rows are also linaerly independent (due to the trailing 1 in the last column of the second row). As a result, including the first row makes the rows linaerly dependent. Since the last entry of the first row is a 0, it means it can be written as a linear combination of the last two rows, which is represented geometrically as being on the defined plane.

Question 07: This is equivalent to solving the equation:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{0} \implies A\mathbf{x} = \mathbf{0} \tag{13}$$

which has a unique solution if and only if $\operatorname{rank}([A|\mathbf{0}]) = 2$, which requires the rows to be linearly dependent, so the rank is zero.

Question 08: Counterexample:

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \tag{14}$$

Note, we found this counterexample by factoring:

$$\det(\mathbf{A}) \det(\mathbf{A} + \mathbf{1}) \tag{15}$$

So we just need to work backwards and find a matrix that isn't invertible, and subtract the identity matrix from it.

Question 09: No, because transformation matrices need to be invertible.

Question 10: Counterexample, let $\mathbf{x} = \mathbf{y} = [1]$.

Question 11: Apply multilinearity lol.

Question 12: Applying $\operatorname{adj}\mathbf{A} = \det \mathbf{A}\mathbf{A}^{-1}$, we have:

$$(\det \mathbf{A})\operatorname{adj}\operatorname{adj}\mathbf{A} = (\det \operatorname{adj}\mathbf{A})\mathbf{A} \tag{16}$$

$$(\det \mathbf{A})\operatorname{adj}\left(\det(\mathbf{A})\mathbf{A}^{-1}\right) = \left(\det\big(\det(\mathbf{A})\mathbf{A}^{-1}\big)\right)\mathbf{A} \tag{17}$$

$$(\det \mathbf{A})\det\big(\det(\mathbf{A})\mathbf{A}^{-1}\big)\cdot\frac{\mathbf{A}}{\det \mathbf{A}} = \left(\det\big(\det(\mathbf{A})\mathbf{A}^{-1}\big)\right)\mathbf{A} \tag{18}$$

If $\mathbf{A}$ is non-invertible, we can use:

$$\mathbf{A}\operatorname{adj}\mathbf{A} = \det \mathbf{A}\mathbf{1} \tag{19}$$

to show that $\det \operatorname{adj}\mathbf{A} = 0$.

Question 13: $\mathbf{A}$ is symmetric if and only if $[a_{ij}] = [a_{ji}]$.