# CIV102: Structures and Materials Notes

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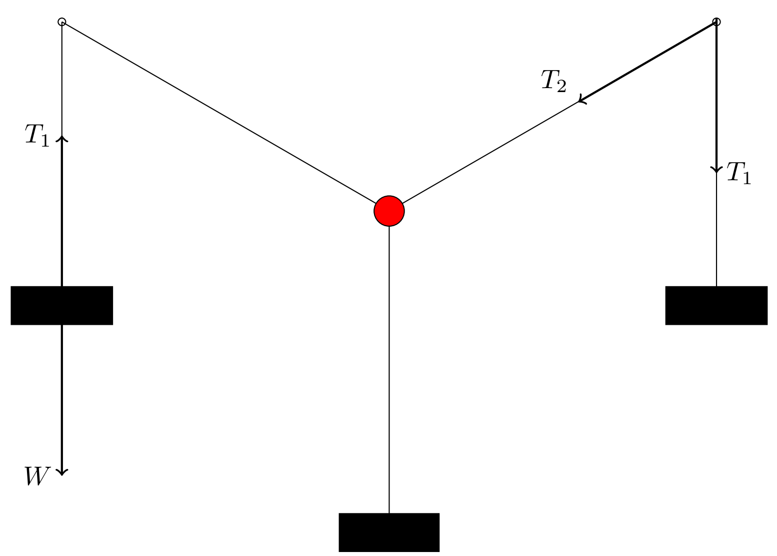
## 1 Lecture 1

**Idea**: The three basic principles of engineering are:

- $F = ma$
- You can't push on a rope.
- A necessary condition for solving any given engineering problem is to know the answer before starting.

## 2 Lecture 2: Basic Concepts

- Suppose we have a system in equilibrium with three forces acting on it. We have three weights $W$ pulling down on massless wires, two of which loop through a pulley to hold the center mass in equilibrium.




- To determine the forces in the wires, we draw a free body diagram on any weight, to get:

$$T_1 = W \tag{1}$$

- The moment (or torque) is defined as the cross product between a position vector and the force vector[1]:

$$\vec{M} = \vec{r} \times \vec{F}$$

- Using this we can balance moments on each pulley of radius $R$:

$$T_1 R = T_2 R \implies T_1 = T_2 \tag{2}$$

Note that the force of tension at the bottom is equal to the tension force at the top since the wire is assumed to be massless.

[1] In a flat plane, $M$ is instead a pseudovector (behaves like a scalar) and it is not necessary to write the vector symbol.

- Therefore, the three tension forces acting on the center mass are equal. We can balance forces in the $x$ and $y$ directions:

$$\sum F_y = 0 = T\cos\alpha - T\cos\beta \tag{3}$$

$$\sum F_y = 0 = T\sin\alpha + T\sin\beta - T \tag{4}$$

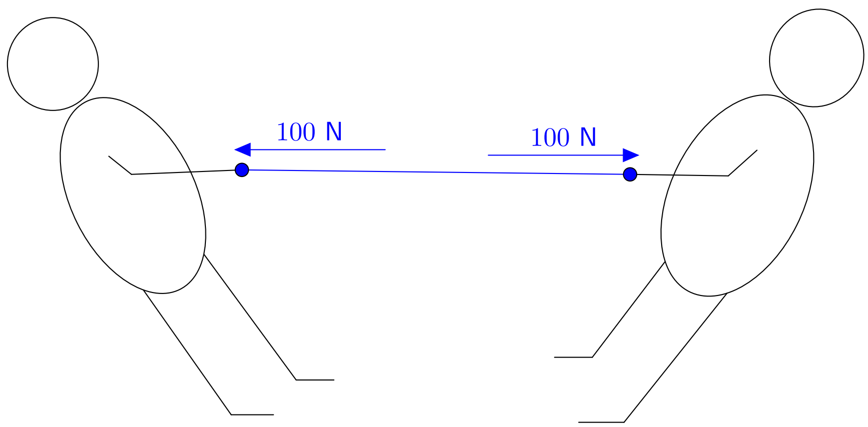
  where $\alpha$ and $\beta$ are the angles the top two wires make with the horizontal. solving this system gives us:

$$\alpha = \beta = 30^\circ \tag{5}$$

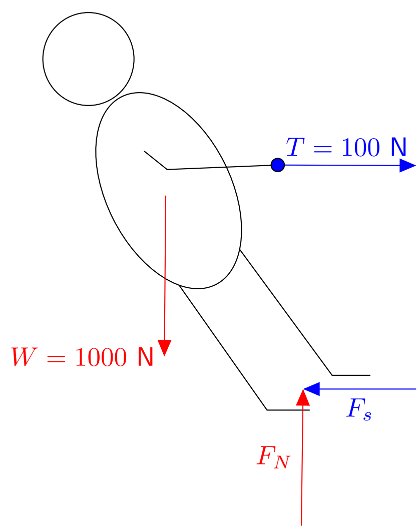
  In other words, the force vectors form an equilateral triangle when arrange tip to tail.

- However in reality, $\alpha = \beta$ are slightly larger than $60^\circ$, caused by the extra downwards force caused by the mass of the washer and strings (which we did not include in our analysis)

- Suppose two people play a game of tug of war as shown in the image below where the tensile force is $T = 100$ N




  Since the rope is in equilibrium, the two forces the two people exert must be equal to each other. Now let us look at the equilibrium of the person to the left (Bill):




- From balancing forces in orthogonal directions, we can see that $F_s = T = 100$ N and $F_N = W = 1000$ N.

- There is also a balance of torque. A **couple** is a pair of forces with equal magnitude and opposite direction that act on a different line of action (causing a net moment).

- The moment of the $F_s$ and $T$ couple is (assuming the perpendicular distance between them is $1.5$ m)

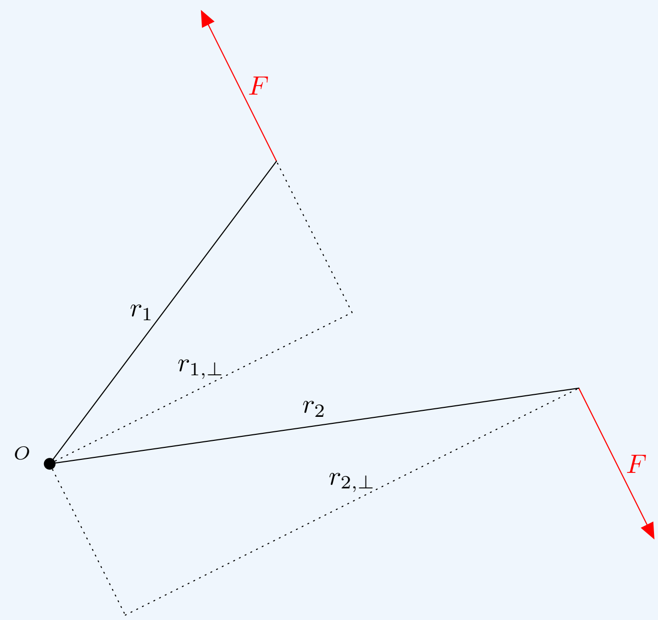
$$M = 100 \times 1.5 = 150\,\mathrm{N\cdot m}\ \text{[CW]} \tag{6}$$

- The moment of the $W$ and $F_N$ couple must also be $M$. If the horizontal between the center of gravity and the feet is $L$, then we have:

$$1000 \times L = 150 \implies L = 0.15 \text{ m} \tag{7}$$

**Theorem**: In a system with $N$ couples, then the sum of the moments of all the couples have to sum up to zero. In other words, for any reference point, the net torque due to a couple is invariant.

To prove this, consider the torque due to an arbitrary couple about a coordinate $O$ as shown in the diagram below:




The net torque about $O$ is:

$$\sum \tau = Fr_{2,\perp} - Fr_{1,\perp} = F(r_{2,\perp} - r_{1,\perp}) \tag{8}$$

Notice that $r_{2,\perp} - r_{1,\perp}$ gives the perpendicular distance between the two forces in the couple $L$, and thus:

$$\sum \tau = FL \tag{9}$$

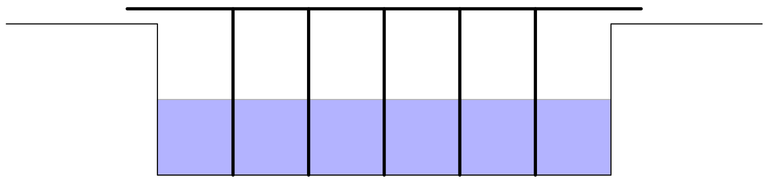
does not depend on the axis of rotation. Thus, for any combination of couples, we just need to sum up their moments to get the net torque. Additionally, we can prove this algebraically[a].

[a]See Wikipedia

- The gravitational acceleration $g$ varies globally, and engineers have to account for it. In Toronto $g = 9.805$ m/s$^2$ but for the purposes of this course, we will take $g = 9.81$ m/s.

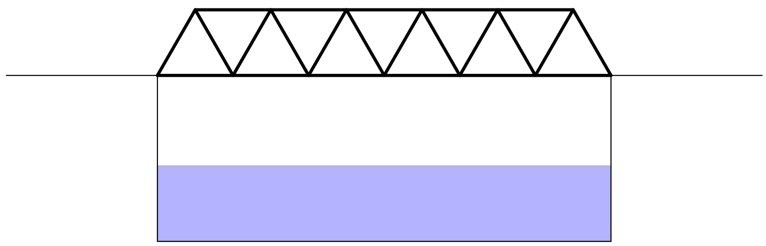
# 3 Lecture 3 - Building Bridges

- Suppose we have a simple bridge over water:



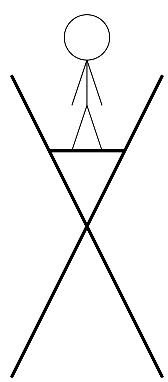
where the horizontal platform is known as the **beam** and the vertical supports are known as **posts** (which comes from the german word for tree)

- Disadvantages of this bridge is that it can easily break and become unusable in the event of a flood. This can be modified to become a **truss** bridge:

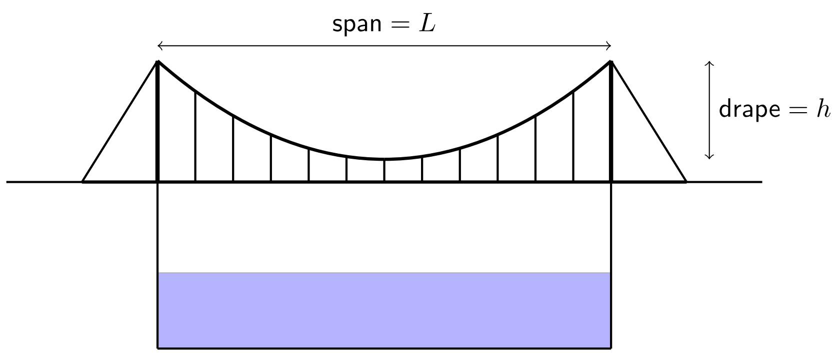


However this type of bridge needs constant maintenance and none of the truss bridges the Romans have built lasted to today.

- One of the oldest bridges (est. 5000 BC) was built across a large pond, known as the "Sweet Track." The bridge was built in an X-shape and people could walk on top:



- Another form of bridge is suspended by hanging chains, known as a **suspension bridge**. A strong *main cable* is pulled over two towers and fixed into concrete supports. Secondary *hanging cables* run vertically and add support to the bridge.

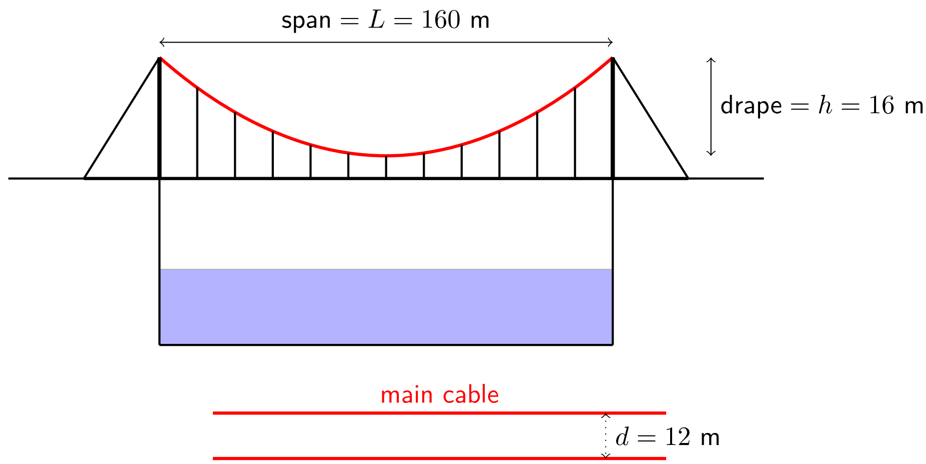



The ratio of the drape to the span is very important in determining how well the bridge is able to support the load. A typical ratio would be $L : h = 10 : 1$.

- The **dead load** refers to the weight of the bridge as is, and the **live load** refers to the actual cars and people that need to cross the bridge.
- Under its own weight or under a constant force, a piece of rope or flexible material will form the shape of a **catenary**, which is a shape that minimizes the potential energy of the system.
- The **catenary** shape very closely resembles that of a parabola, at least for small enough values, such that for most purposes it can be approximated as such.

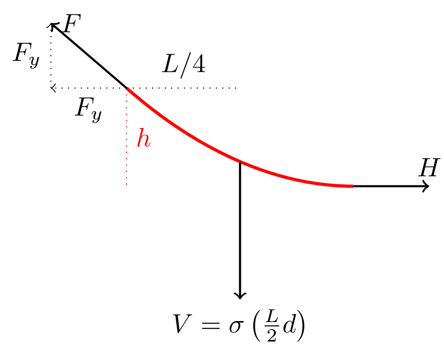
# 4 Lecture 4: Statics of Suspension Bridge

- The top-down view of a suspension bridge is seen below:




- We assume that both the live and the dead load is $5\text{kN}\,\text{m}^{-2}$ each, which is approximately equivalent to the force exerted by a crowd of people standing close together. We can say that the total weight is $\sigma = 10\text{kN}\,\text{m}^{-2}$.

- We can balance forces on half of the suspension cable:




And we see that $F_y = V$ and $F_x = H$. We can determine

$$V = 9600\text{kN} \tag{10}$$

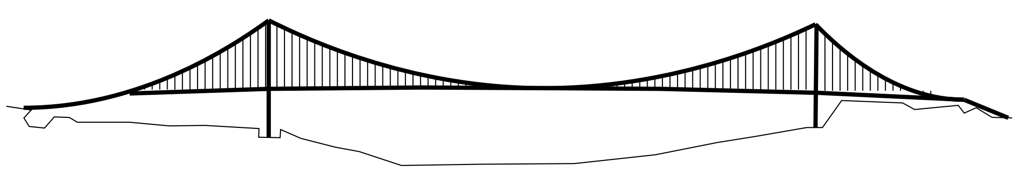
These pairs of forces form two couples, so we can determine the magnitude of the forces by summing the moments to zero:

$$V\frac{L}{4} = Hh \implies H = \frac{\sigma L^2 d}{8h} = 24{,}000\text{kN} \tag{11}$$

Therefore, the total force of tension at the top is given by

$$F = \sqrt{F_x^2 + F_y^2} = 12{,}900\text{kN} \tag{12}$$

- The Golden Gate bridge in San Francisco was an engineering triumph:



It also revolutionized safety procedures and hugely influenced life in the city. It is sometimes known as the "Mona Lisa" of bridges.

- The Quebec Bridge, which had a span half of that of the Golden Gate bridge, was not a feat of engineering triumph.

# 5 Lecture 5

- There is a linear relationship between the force of a wire and the deformation. The slope was called the stiffness:

$$k = \frac{F}{\Delta L} \tag{13}$$

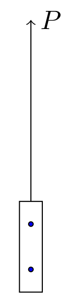
  The problem with this is that the slope would be different for different sizes of the same object, so it didn't reflect a particular object's characteristics.

- Young modified this relationship to instead measure the tensile strength $\sigma = F/A$ in terms of the strain $\epsilon = \frac{\Delta L}{L}$, such that:

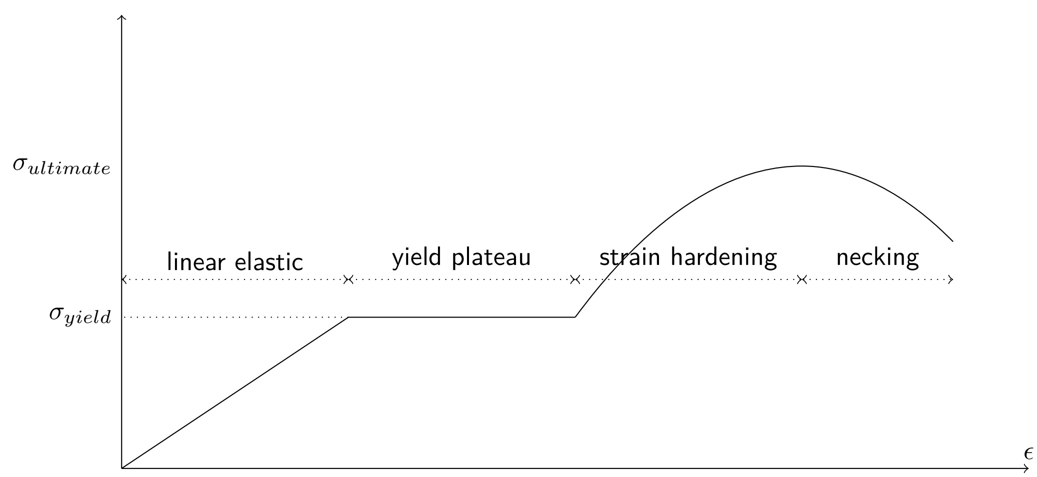
$$E = \frac{\sigma}{\epsilon} \tag{14}$$

  This provides a method to experimentally determine the value of $E$, by loading different weights and measuring the strain for each.

- A **Coupon Test** is the testing of a small representative piece of a material to determine the properties. We do this via the setup below, where we place two indicators a length $L_0$ apart. As we load weight onto the rectangle, the distance between the indicators will change to $L_0 + \Delta L$.




- We can increase the strain by adding weights below and measuring the force $P$. Dividing it through by the area gives the tensile strength.




- The **yield plateau** occurs when the atoms in the metal are no longer vibrating back and forth between one another, but sliding across one another. As a result, we experience **permanent plastic deformations** during this period.

- At a certain point, the imperfections build up such that yielding becomes more difficult, so in order to overcome them, the stress increases. This process is known as **strain hardening**.

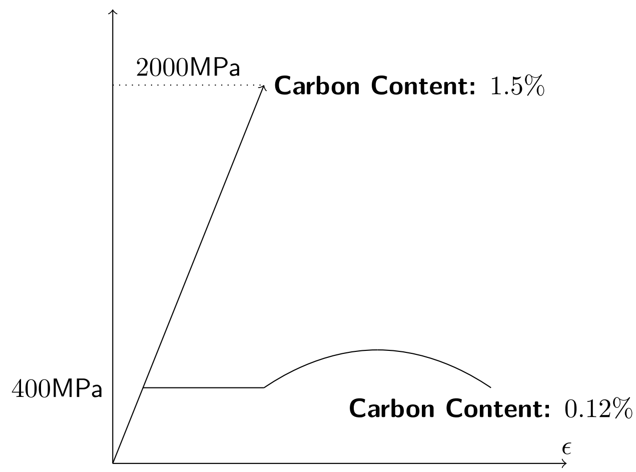
- **Necking** occurs after the ultimate stress point where the cable will form a neck shape, reducing the tensile strength.

- **Rupture** occurs after the cable cannot support the strain at all and then break.

- If the applied load is taken off, the cable will contract back to $\sigma = 0$ following the same slope. After the initial yield plateau, the cable will not be able to naturally go back to its original rest length.

# 6 Lecture Six: Stress-Strain Response

- For a **brittle** material, the slope of the stress-strain graph will be very high and it will have almost no warning before breaking.
  - Applications: Piano wires
- Other wires, such as **flossing wires**, which are used to reinforce concrete, there is a different behaviour, where yielding occurs very early.




- **Work** is defined as:

$$\text{Work} = \text{Force} \cdot \text{Distance} \tag{15}$$

  and is measured in Joules ([J] = [N m]) and **energy** is defined as the capacity to do work
- Body fat can store around $40\text{MJ kg}^{-1}$ in body fat, where $7.5\%$ of it can be transferred. One rule of thumb is that one pound of body fat carries you $35$ miles.
- Power is the rate at which work is being done. Originally defined as:

$$1 \text{ HP} = 746\text{W} \tag{16}$$

- The work, or energy stored in deforming a wire is the area under the curve of the force vs displacement curve. The **elastic strain energy** gives the maximum recoverable energy, or:

$$E = \frac{1}{2}F_{\text{max}}\Delta_{\text{max}} = \frac{1}{2}\sigma A \times \epsilon L \tag{17}$$

  The energy density is:

$$E/V = \frac{1}{2}\sigma\epsilon \tag{18}$$

- The resilience of a material is directly related to its ability to store elastic strain energy.
- Some vocabulary terms:
  - Weight [$\text{kN m}^{-3}$] - weight density of the material
  - Stiffness $E$[MPa] - The Young's Modulus: How difficult it is to stretch
  - Tensile strength [MPa]- Critical points at which the material starts to yield, or break (ultimate).
  - Compressive strength [MPa] - yield strength equivalent of tensile strength, except for compressions.
  - Resilience [$\text{MJ m}^{-3}$] - maximum energy which a material can absorb per unit volume before experiencing permanent deformations.

– Toughness $[\mathrm{MJ\,m^{-3}}]$ - maximum energy which a material can absorb per unit volume before experiencing permanent deformations.

– Ductility is the maximum elongation before it experiences failure.

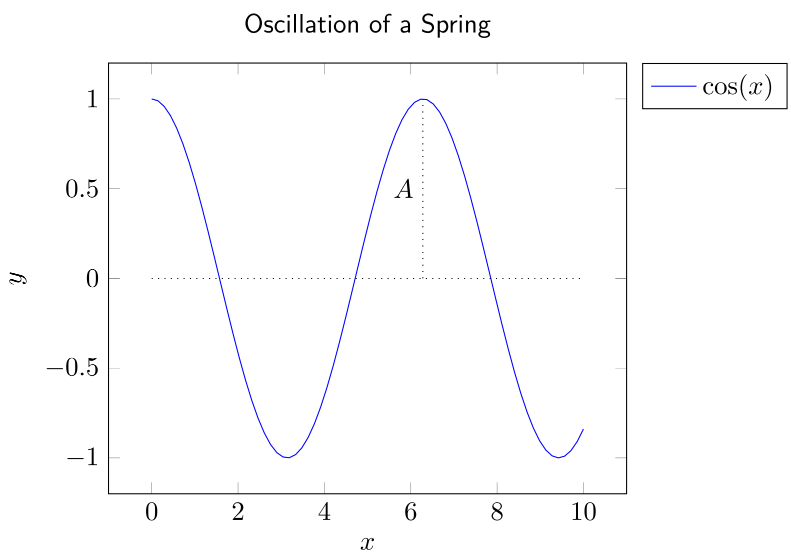
– $\alpha[10^{-6}{}^{\circ}\mathrm{C}^{-1}]$ - thermal expansion coefficient

# 7 Lecture 7: Power of Springing Bodies

- The force from an elastic spring is linear with respect to the disturbance:

$$|F| = k|\Delta x| \tag{19}$$

- When a spring is perturbed from its equilibrium position, it will oscillate back and forth:




and the period is given by:

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{20}$$

- The differential equation is given by:

$$\frac{d^2y}{dt^2} = -\frac{k}{m}y - g \tag{21}$$

and comes from Newton's second law. The solution to this equation gives:

$$y = A\cos\left(\sqrt{\frac{k}{m}}t\right) \tag{22}$$

- The natural frequency is given by:

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{23}$$

# 8 Lecture 8: Factors of Safety

- In lecture 4, we derived that the maximum tension is:

$$F_{\text{max}} = 12900\text{kN} \tag{24}$$

We wish to actually magnify this by a factor of safety. The failure stress is:

$$\sigma_{\text{ultimate}} = 1860\text{MPa} \tag{25}$$

and the area of one wire ($\varnothing = 5$mm) s:

$$19.63\text{mm}^2 \tag{26}$$

Thus each wire will have a maximum force of

$$F = 36.52\text{kN} \tag{27}$$

so we require that:

$$N > 12924/36.52 = 354 \text{ wires} \tag{28}$$

- For perfect materials and workmanship, Rankine recommended the safety factor for dead loads to be $2$ and for live loads to be $4$.
- For good ordinary materials and workmanship, in the case of metals, the factors of safety should be $3$ and $6$.
- For working with timber, the factors of safety should be $4 \to 5$ and $8 \to 10$
- For masonry, the factors of safety should be $4$ and $8$.

# 9 Lecture 9: Weight a Moment! What is I?

- For a system to be in equilibrium, the forces need to sum up to zero.
- The **moment of inertia** describes how much an object resists a change in its angular velocity.
- Recall the period of a mass on a spring under a linear force is:

$$T = 2\pi\sqrt{\frac{m}{k}} \implies T \propto \sqrt{m} \tag{29}$$

  The rotational analog of mass is rotational inertia, so the period of an object under the influence of a linear restoring torque is propotoinal to:

$$T \propto \sqrt{I} \tag{30}$$

- The strong axis refers to the orientation of the body which has greater rotational inertia and the **weak axis** refers to the orientation with the less rotational inertia.
- The moment of inertia of a point mass $m$ measured from a pivot a radius $r$ away is given by:

$$I = mr^2 \tag{31}$$

  If we have a mass on both ends of a rod, and apply a force of $F = \frac{M}{r}$ on a mass perpendicular to the rod, then the effective force on both masses is:

$$F_{\text{effective}} = \frac{M}{2r} \tag{32}$$

- Some angular kinematics results:

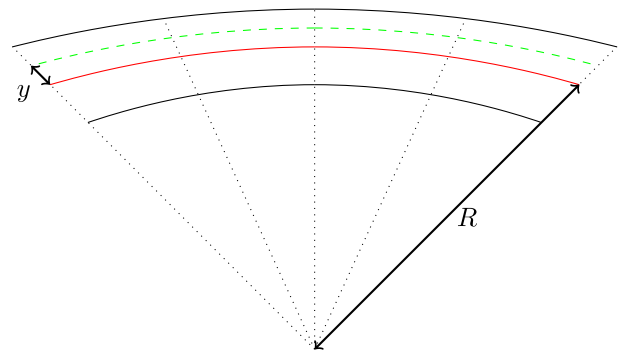
$$s = \theta r \tag{33}$$

$$\frac{d^2 s}{dt^2} = a \tag{34}$$

$$\alpha = \frac{d^2\theta}{dt^2} = \frac{1}{r}\frac{d^2 s}{dt^2} \tag{35}$$

# 10 Lecture Ten: Bending of Beams

- Plane sections refer to vertical lines which are perpendicular to the length of a member. After the member is bent, these vertical lines rotate but otherwise remain straight.




We can divide up the plane sections such that they subtend an angle of $\phi$ and the length of the red line (neutral section) they subtend is of unit length. As a result:

$$\phi = \frac{1}{R} \tag{36}$$

where $R$ is the radius. Lengths above the red mark get stretched while lengths below the red mark gets shrunk. If we take a section $y$ above the neutral length, it gets deformed a length:

$$L_{\text{def}} = \phi(R + y) = \phi R + \phi y \tag{37}$$

and the strain would be given by:

$$\epsilon = \frac{L_{\text{def}} - L_0}{L_0} = \frac{\phi R + \phi y - 1}{1} = \phi y \tag{38}$$

As a result, the stress increases linearly:

$$\sigma_y = E\phi y \tag{39}$$

- If we integrate the stresses in the $y$ direction, then we get:

$$N = \int_A \sigma_y dA = E\phi \int_A y dA \tag{40}$$

We can calculate the curvature as:

$$\phi = \frac{M}{EI} \tag{41}$$

so

$$\sigma_y = \frac{My}{I}. \tag{42}$$

which is referred as **Navier's equation.**

- The moment can be calculated by:

$$M = \int_A \sigma_y y dA = E\phi \int_A y^2 dA = E\phi I \tag{43}$$

where $I$ here is the second moment of inertia: which is the moment of inertia divided by density.

- The moment of inertia of a beam is:

$$I = \frac{1}{12} bh^3 \tag{44}$$