MAT185 Test 1 Review

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Contents

Note: Axiom propositions and names will be taken from two sources: Prof. Sean Uppal's notes and Prof. GDE's textbook: Medici.

1 Axioms

Medici

A vector space $\mathcal{V}$ over a field Γ of elements $\{\alpha, \beta, \gamma, \ldots\}$, called scalars, is a set of elements $\{\boldsymbol{u}, \boldsymbol{v}, \ bmw, \ldots\}$ called vectors, such that the following axioms are satisfied:

1. There exists an operation of vector addition, denoted $\boldsymbol{u}+\boldsymbol{v}$, such that for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$,
   - **AI.** Closure: $\boldsymbol{u}+\boldsymbol{v} \in \mathcal{V}$.
   - **AII.** Associativity: $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w} = \boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$.
   - **AIII.** Zero: There exists a zero or null vector $\mathbf{0} \in \mathcal{V}$ such that $\boldsymbol{u}+\mathbf{0} = \boldsymbol{u}$.
   - **AIV.** Negative: There exists an negative $-\boldsymbol{u} \in \mathcal{V}$ such that $\boldsymbol{u}+(-\boldsymbol{u}) = \mathbf{0}$.
2. There exists an operation of scalar multiplication, denoted $\alpha\boldsymbol{u}$, such that for all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$ and all $\alpha, \beta \in \Gamma$,
   - **MI.** Closure: $\alpha\boldsymbol{u} \in \mathcal{V}$.
   - **MII.** Associativity: $\alpha(\beta\boldsymbol{u}) = (\alpha\beta)\boldsymbol{u}$.
   - **MIII.** Distributivity:
     (a) $(\alpha+\beta)\boldsymbol{u} = \alpha\boldsymbol{u}+\beta\boldsymbol{u}$
     (b) $\alpha(\boldsymbol{u}+\boldsymbol{v}) = \alpha\boldsymbol{u}+\alpha\boldsymbol{v}$
   - **MIV.** Unitary: For the identity element $1 \in \Gamma$, $1\boldsymbol{u} = \boldsymbol{u}$.

Uppal

A real vector space is a set V together with two operations called vector addition and scalar multiplication such that the following axioms hold. For all vectors $\mathbf{x}, \mathbf{y}, \mathbf{Z} \in V$ and scalars $c, d \in \mathbb{R}$:

1. (AC) Additive Closure: $\mathbf{x}+\mathbf{y} \in V$
2. (SC) Scalar Closure: $c\mathbf{x} \in V$.
3. (AA) Additive Associativity: $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$.
4. (Z) Zero vector: There exists a unique vector $\mathbf{0} \in V$ with the property that $\mathbf{x}+\mathbf{0} = \mathbf{x}$.
5. (AI) Additive Inverse: There exists a unique vector $-\mathbf{x} \in V$ with the property that $\mathbf{x}+(-\mathbf{x}) = \mathbf{0}$.
6. (SMA) Scalar Multipication Associativity: $(cd)\mathbf{x} = c(d\mathbf{x})$.
7. (DVA) Distributivity of Vector Addition: $c(\mathbf{x}+\mathbf{y}) = c\mathbf{x}+c\mathbf{y}$.
8. (DSA) Distributivity of Scalar Addition: $(c+d)\mathbf{x} = c\mathbf{x}+d\mathbf{x}$.
9. (I) Identity: $1\mathbf{x} = \mathbf{x}$.

1.1 Corrolaries

Theorem: **The Cancellation Theorem:** Let $\mathcal{V}$ be a vector space, and let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$. If:

$$\boldsymbol{u} + \boldsymbol{w} = \boldsymbol{v} + \boldsymbol{w} \tag{1}$$

then:

$$\boldsymbol{u} = \boldsymbol{v} \tag{2}$$

Medici

Prop I. For every $\boldsymbol{u}, -\boldsymbol{u} \in \mathcal{V}$, $-\boldsymbol{u} + \boldsymbol{u} = \mathbf{0}$.

Prop II. For every $\boldsymbol{u} \in \mathcal{V}$, $\mathbf{0} + \boldsymbol{u} = \boldsymbol{u}$.

Prop III. Let $\boldsymbol{u} \in V$. Then:

(a) The zero vector $\mathbf{0} \in \mathcal{V}$ is unique.
(b) The negative $-\boldsymbol{u}$ of $\boldsymbol{u}$ is unique.
(c) $-(-\boldsymbol{u}) = \boldsymbol{u}$.

Prop IV. For $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$, $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.

Prop V. For all $\boldsymbol{u} \in \mathcal{V}$ and $\alpha \in \Gamma$:

(a) $0\boldsymbol{v} = \mathbf{0}$
(b) $\alpha\mathbf{0} = \mathbf{0}$
(c) If $\alpha\boldsymbol{v} = \mathbf{0}$, then either $\alpha = 0$ or $\boldsymbol{v} = \mathbf{0}$.

Prop VI. For all $\boldsymbol{u} \in \mathcal{V}$ and $\alpha \in \Gamma$, $(-\alpha)\boldsymbol{v} = -(\alpha\boldsymbol{v}) = \alpha(-\boldsymbol{v})$.

Uppal

Prop I. For every $\mathbf{x} \in V$, then $0\mathbf{x} = \mathbf{0}$.

Prop II. For every $\mathbf{x} \in V$, then $(-1)\mathbf{x} = -\mathbf{x}$.

Prop III. For every $\mathbf{x} \in V$, then $-\mathbf{x} + \mathbf{x} = \mathbf{0}$.

Prop IV. For every $\mathbf{x} \in V$, then $\mathbf{0} + \mathbf{x} = \mathbf{x}$.

This introduces an additional axiom:

10. (C) Commutativity: For all vectors $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.

1.2 Important Facts

You should know and be able to prove the following facts:

- Every vector space is either infinite or contains only the zero vector.

2 Subspaces

Medici

A subspace $\mathcal{U}$ of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$ if and only if $\mathcal{U}$ is itself a vector space over the same fiedl Γ with the same vector addition and scalar multiplication of $\mathcal{V}$.

To show a subset is a subspace:

SI. Zero: There exists a zero vector $\mathbf{0} \in \mathcal{U}$.
SII. Closure under Vector Addition: $\boldsymbol{u} + \boldsymbol{v} \in \mathcal{U}$.
SIII. Closure under Scalar Multiplication: $\alpha\boldsymbol{u} \in \mathcal{U}$.

Uppal

A subspace of a vector V is a subset $W \subseteq V$ that is itself a vector space with the same operations of vector addition and scalar multiplication as in V.

To show a subset is a subspace:

1. (AC & SC): Sums and scalar multiples of vectors from W are in W
2. (Z) W contains the zero vector of V.
3. (AI) The additive inverse of each vector in W is in W.

Alternative Formulation: A non-empty subset W of a vector space V is a subspace of V if and only if $c\mathbf{x}+\mathbf{y} \in W$ whenever $\mathbf{x}, \mathbf{y} \in W$, and $c \in \mathbb{R}$.