ESC195 Notes

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Contents

1 Hyperbolic Functions

- Sometimes, combinations of e^x and e^{-x} are given certain names, for example:
  - **Hyperbolic sine:** $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$
  - **Hyperbolic cosine:** $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$
- They have the following properties:

$$\frac{d}{dx}\sinh x = \cosh x \tag{1}$$

$$\frac{d}{dx}\cosh x = \sinh x \tag{2}$$

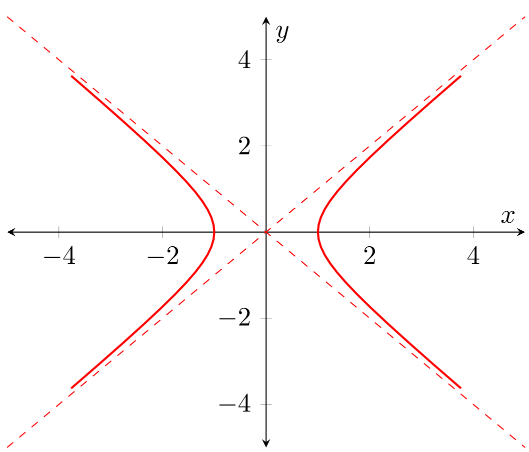
- They are related via:

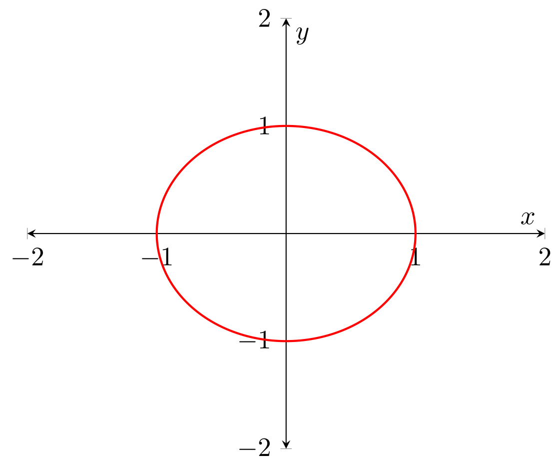
$$\cosh^2 x - \sinh^2 x = 1 \tag{3}$$

- Both the area of a circular sector and that of a hyperbolic sector is described by:

$$A = \frac{1}{2}t \tag{4}$$

  where t is the subtended angle, and the figures are parametized by $(\cos t, \sin t)$ and $(\cosh t, \sinh t)$.





- The catenary

$$y = a\cosh\left(\frac{x}{a}\right) + C \tag{5}$$

  describes the shape of a free hanging rope between two walls separated by a width a.

- The hyperbolic tangent is given by $\tanh x = \frac{\sinh x}{\cos hx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. and its derivative is given by:

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \tag{6}$$

- The inverse of $y = \sinh x$ is given by:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \tag{7}$$

> **Tip:** A table of integrals and derivatives revolving around hyperbolic trig functions can be found in the textbook.

2 Indeterminate Forms

- A lot of the times, limits have an indeterminate form, where if we substitute in what x approaches to, we get it in the form of $\frac{0}{0}$, for example:

$$\lim_{x\to 0}\frac{\sin x}{x} \tag{8}$$

Theorem: If $f(x)\to 0$ and $g(x)\to 0$ as $x\to\pm,\infty$ or $x\to c$ or $x\to c^{+-}$ and if $\frac{f'(x)}{g'(x)}\to L$, then:

$$\frac{f(x)}{g(x)}\to L \tag{9}$$

Example 1: Solve: $\lim_{x\to 0}\frac{\sin x}{x}$

We can set $f(x)=\sin x$, $f'(x)=\cos x$, $g(x)=x$ and $g'(x)=1$ such that:

$$\lim_{x\to 0}\frac{f'}{g'}=\lim_{x\to 0}\cos x=1 \tag{10}$$

Example 2: Solve $\lim_{x\to 0^+}\frac{\sin x}{\sqrt{x}}$.

Set $f=\sin x$, $f'=\cos x$, $g=\sqrt{x}$, $g'=\frac{1}{2}x^{-1/2}$ and so:

$$\lim_{x\to 0^+}2x^{1/2}\cos x=0 \implies \lim_{x\to 0^+}=0 \tag{11}$$

Example 3: Solve $\lim_{x\to 0}\frac{e^x-x-1}{3x^2}$.

If we take the derivative, we get:

$$\lim_{x\to 0}\frac{e^x-1}{6x} \tag{12}$$

which is still $\frac{0}{0}$!. We can take derivatives again:

$$\lim_{x\to 0}\frac{e^x}{6}=\frac{1}{6} \tag{13}$$

so the original limit is $\frac{1}{6}$.

Warning: L'hopital's rule can *only* be used in indeterminate forms. Applying them to limits where

- To prove the L'hopital's rule, we first prove the **Cauchy Mean Value Theorem** as a lemma

Theorem: Cauchy Mean Value Theorem: Given f and g differentiable on (a,b), continuous on $[a,b]$ and $g'\neq 0$ on (a,b), there must exist some number r in (a,b) such that:

$$\frac{f'(r)}{g'(r)}=\frac{f(b)-f(a)}{g(b)-g(a)} \tag{14}$$

- We then apply **Rolle's Theorem** to prove the Cauchy Mean Value Theorem:

Proof. Set:

$$\begin{aligned}G(x) =&[g(b)-g(a)][f(x)-f(a)]\\&-[g(x)-g(a)][f(b)-f(a)]\end{aligned}$$

Note that $G(a) = G(b) = 0$ so it satisfies the conditions of Rolle's Theorem. Taking the derivative, we get:

$$G'(x) = [g(b)-g(a)]f'(x) - g'(x)[f(b)-f(a)] \tag{15}$$

Accoridng to Rolle's, there must be some $x = r$ such that $G'(r) = 0$, we can then substitute for this and solve:

$$G'(r) = 0 \implies [g(b)-g(a)]f'(r) = g'(r)[f(b)-f(a)] \tag{16}$$

Which is equivalent to:

$$\frac{f'(r)}{g'(r)} = \frac{f(b)-f(a)}{g(b)-g(a)} \tag{17}$$

Furthermore, we have $g'(c) = \dfrac{g(b)-g(a)}{b-a}$ from the mean value theorem. Since $g' \neq 0$ we have $g(b)-g(a) \neq 0$. □

- Given $x \to c^+$ and $f(x), g(x) \to 0$ where:

$$\lim_{x\to c^+} \frac{f'(x)}{g'(x)} = L \tag{18}$$

we will now prove that $\displaystyle\lim_{x\to c^+} \frac{f(x)}{g(x)} = L$.

Proof. Consider the interval $[c, c+h]$ and apply Cauchy MVT. There must be some number c_2 in $[c, c+h]$ such that:

$$\frac{f'(c_2)}{g'(c_2)} = \frac{f(c+h)-f(c)}{g(c+h)-g(c)} = \frac{f(c+h)}{g(c+h)} \tag{19}$$

The last step is a result of the given $f(c) = g(c) = 0$. The LHS can be rewritten as:

$$\lim_{h\to 0} \frac{f'(c_2)}{g'(c_2)} = \frac{f'(c)}{g'(c)} \tag{20}$$

since c_2 lies in the interval $[c, c+h]$ so if $h \to 0$, then the interval becomes smaller to contain just c. The RHS can be rewritten as:

$$\lim_{h\to 0} \frac{f(c+h)}{g(c+h)} = \lim_{x\to c^+} \frac{f(x)}{g(x)} \tag{21}$$

and therefore:

$$\lim_{x\to c^+} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} = L \tag{22}$$

□

- To prove the case for $x \to \pm\infty$, we can let $x = \dfrac{1}{t}$ and take the limit as $t \to \infty$.

Example 4: Find $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x}$.

Taking the derivative of top and bottom, we have:

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0 \implies \lim_{x\to\infty} \frac{\ln x}{x} = 0 \tag{23}$$

Idea: The logarithm function grows very slowly. In fact, any positive power of x will grow faster than $\ln x$.

Example 5: Solve $\displaystyle\lim_{x\to\infty} \frac{x^3}{e^x}$

This is indeterminate in the form of $\frac{\infty}{\infty}$. We apply L'hopital's rule multiple times:

$$\lim_{x\to\infty}\frac{x^3}{e^x}\overset{*}{=}\lim_{x\to\infty}\frac{3x^2}{e^x}\left(=\frac{\infty}{\infty}\right) \tag{24}$$

$$\overset{*}{=}\lim_{x\to\infty}\frac{6x}{e^x}\left(=\frac{\infty}{\infty}\right) \tag{25}$$

$$\overset{*}{=}\lim_{x\to\infty}\frac{6}{e^x}=0 \tag{26}$$

- Generally, $\lim_{x\to\infty}\frac{x^m}{e^x}=0$ where m is any positive integer.
- There are other indeterminate forms, such as 0^0, for example:

$$\lim_{x\to 0}x^x \tag{27}$$

The central idea behind this is that $a^b=e^{a\ln b}$. Therefore, this limit is equal to:

$$\lim_{x\to 0}e^{x\ lnx} \tag{28}$$

We can take the limit of the exponent to get:

$$\lim_{x\to 0}x\ln x=\lim_{x\to 0}\frac{\ln x}{1/x} \tag{29}$$

Note that the first equation is another indeterminate form with the $0\cdot\infty$ type, so we had to multiply top and bottom by $\frac{1}{x}$ to get the quotient form. Then we have:

$$\lim_{x\to 0}\frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)}=\lim_{x\to 0}-x=0 \tag{30}$$

Therefore:

$$\lim_{x\to 0}e^{x\ln x}=e^0=1 \tag{31}$$

so $\lim_{x\to 0}x^x=1$.

Example 6: Solve $\lim_{x\to\infty}(x+2)^{2/\ln x}$.

This is of the type ∞^0. The approach is exactly the same as the previous example. We write it in exponential form:

$$=\lim_{x\to\infty}e^{\frac{2}{\ln x}\ln(x+2)} \tag{32}$$

and looking at the exponent gives:

$$\lim_{x\to\infty}\frac{2\ln(x+2)}{\ln x} \tag{33}$$

$$\overset{*}{=}\lim_{x\to\infty}\frac{\left(\frac{2}{x+2}\right)}{\left(\frac{1}{x}\right)}=\lim_{x\to\infty}\frac{2x}{x+2}\left(=\frac{\infty}{\infty}\right) \tag{34}$$

$$\overset{*}{=}\lim_{x\to\infty}\frac{2}{1}=2 \tag{35}$$

Therefore:

$$\lim_{x\to\infty}e^{\frac{2}{\ln x}\ln(x+2)}=e^2 \tag{36}$$

so:

$$\lim_{x\to\infty}(x+2)^{2/\ln x}=e^2 \tag{37}$$

Example 7: Solve $\lim_{x\to\infty}\left[\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\right]^x$

This is in the form of 1^∞. We rewrite it as:

$$\lim_{x\to\infty}\exp\left(x\ln\left(\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\right)\right) \tag{38}$$

and taking the limit of the exponent:

$$= \lim_{x\to\infty} x\ln\left(\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\right)\left(=\frac{0}{0}\right) \tag{39}$$

$$\stackrel{*}{=} \lim_{x\to\infty}\frac{\cos\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\cdot\left(-\frac{\pi}{x^2}\right)}{\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\cdot\left(-\frac{1}{x^2}\right)} = \frac{0\cdot\pi}{1} = 0 \tag{40}$$

Therefore:

$$\lim_{x\to\infty}\left[\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\right]^x = \lim_{x\to\infty}\exp\left(x\ln\left(\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\right)\right) = 1 \tag{41}$$

3 Integration

3.1 Recap of Integration

- The definite integral has the geometric interpretation as the area under the curve $f(x)$ between $x = a$ and $x = b$ and the x axis:

$$\int_a^b f(x)\,\mathrm{d}x \tag{42}$$

but can be rigorously defined using a Riemann sum:

$$\int_a^b f(x)\,\mathrm{d}x \equiv \lim_{\|P\|}\sum_{i=1}^n f(x_i^*)\Delta x_i \tag{43}$$

Often, we have a uniform partition, such that $\Delta x_i = \dfrac{b-a}{n}$ where n is the number of partitions. And if we choose to use the right hand endpoint, then:

$$f(x_i^*) = f(x_i) = f(x_i) = f\left(a + \frac{b-a}{n}i\right) \tag{44}$$

Example 8: To solve $\displaystyle\int_0^5 x^2\,\mathrm{d}x$, we can choose a uniform partition with:

$$\Delta x = \frac{5-0}{n} = \frac{5}{n} \tag{45}$$

and:

$$x_i^* = x_i = i\Delta x \implies f(x_i^*) = (i\Delta x)^2 = \left(i\frac{5}{n}\right)^2 \tag{46}$$

The area approximation is:

$$A \simeq \sum_{i=1}^n \Delta x_i f(x_i^*) = \sum_{i=1}^n \left(\frac{5}{n}\right)\left(i\frac{5}{n}\right)^2 \tag{47}$$

$$= \frac{125}{n^2}\sum_{i=1}^n i^2 = \frac{125}{n^3}\frac{n(n+1)(2n+1)}{6} \tag{48}$$

Taking the limit as $n\to\infty$, we get:

$$\int_0^5 x^2\,\mathrm{d}x = \lim_{n\to\infty}\frac{125}{6}\left(2+\frac{2}{n}+\frac{1}{n^2}\right) = \frac{5^3}{3}. \tag{49}$$

Example 9: To evaluate $\int_1^2 x^{-2}\,\mathrm{d}x$, we can choose

$$x_i^* = \sqrt{x_{i-1}x_i} \tag{50}$$

and a uniform partition of:

$$\Delta x = \frac{2-1}{n} = \frac{1}{n} \tag{51}$$

such that:

$$x_i = 1 + i\Delta x = 1 + \frac{i}{n} = \frac{n+i}{n} \tag{52}$$

and

$$x_{i-1} = \frac{n+i-1}{n} \tag{53}$$

such that the area is:

$$\begin{aligned}
A &\simeq \sum_{i=1}^{n} \Delta x f(x_i^*) \\
&= \sum_{i=1}^{n} \frac{1}{n}\left(\frac{1}{x_i^*}\right)^2 \\
&= \sum_{i=1}^{n} \frac{1}{n}\frac{1}{x_{i-1}x_i} \\
&= \sum_{i=1}^{n} \frac{1}{n}\frac{n}{n+i-1}\cdot\frac{n}{n+i} \\
&= \sum_{i=1}^{n} n\frac{1}{n+i-1}\cdot\frac{1}{n+i} \\
&= \sum_{i=1}^{n} \left(\frac{1}{n+i-1} - \frac{1}{n+i}\right) \\
&= n\left[\sum_{i=1}^{n}\frac{1}{n+i-1} - \sum_{i=1}^{n}\frac{1}{n+i}\right] \\
&= n\left[\sum_{i=0}^{n}\frac{1}{n+i} - \sum_{i=1}^{n}\frac{1}{n+i}\right] \\
&= n\left[\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n}\right] \\
&= n\left(\frac{1}{n} - \frac{1}{2n}\right) \\
&= 1 - \frac{1}{2} = \frac{1}{2}
\end{aligned}$$

The part where we cancel out everything is called a **telescoping series**. Notice how the area doesn't depend on n so we get the exact area, even if we let $n = 1$!.

- We need a better way to do integration, so we can define:

$$F(x) \equiv \int_a^x f(t)\,\mathrm{d}t \tag{54}$$

such that $F'(x) = f(x)$. This is the definition of the antiderivative. This leads to the fundamental theorem of calculus:

$$\int_a^b f(t)\,\mathrm{d}t = F(h) - F(a) \tag{55}$$

and the indefinite integral can be written as:

$$\int f(x)\,\mathrm{d}x = G(x) + C \tag{56}$$

The main problem now becomes trying to *find antiderivatives*, which is much easier than Riemann sums, though still more difficult than calculating derivatives.

3.2 Integration by Parts

- **Integration by Parts** attempts to reverse the product rule:

$$(fg)' = fg' + f'g \tag{57}$$

Taking the integral of both sides gives:

$$f(x)g(x) = \int f(x)g'(x)\,\mathrm{d}x + \int f'(x)g(x)\,\mathrm{d}x \tag{58}$$

$$\int f(x)g'(x)\,\mathrm{d}x = \int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x \tag{59}$$

If the second integral is easier than the first, then we have made substaintial progress.

Idea: Integration of parts tells us that:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u \tag{60}$$

Example 10: To solve $\int xe^{2x}$, we can let:

$$u = x \qquad \mathrm{d}v = e^{2x}\,\mathrm{d}x \tag{61}$$

$$\mathrm{d}u = \mathrm{d}x \qquad v = \frac{1}{2}e^{2x} \tag{62}$$

which gives:

$$\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,\mathrm{d}x \tag{63}$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \tag{64}$$

We can check:

$$\frac{d}{dx}\left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C\right) \tag{65}$$

$$= xe^{2x} + \frac{1}{2}e^{2x} - \frac{2}{4}e^{2x} \tag{66}$$

$$= xe^{2x} \tag{67}$$

Example 11: To solve $\int x^2\sin(2x)\,\mathrm{d}x$, we let:

$$u = x^2 \qquad \mathrm{d}v = \sin 2x\,\mathrm{d}x \tag{68}$$

$$\mathrm{d}u = 2x\,\mathrm{d}x \qquad v = -\frac{1}{2}\cos(2x) \tag{69}$$

which gives:

$$= -\frac{1}{2}x^2\cos 2x + \int x\cos(2x)\,\mathrm{d}x \tag{70}$$

and we can apply integration by parts a second time, if we let:

$$u = x \qquad \mathrm{d}v = \cos 2x\,\mathrm{d}x \tag{71}$$

$$\mathrm{d}u = \mathrm{d}x \qquad v = \frac{1}{2}\sin(2x) \tag{72}$$

which gives us:

$$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) - \int \frac{1}{2}\sin(2x)\,\mathrm{d}x \tag{73}$$

$$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2X) + C \tag{74}$$

Example 12: To solve $I = \displaystyle\int e^x \sin x\,\mathrm{d}x$, we can let:

$$u = \sin x \qquad \mathrm{d}v = e^x\,\mathrm{d}x \tag{75}$$

$$\mathrm{d}u = \cos x\,\mathrm{d}x \qquad v = e^x \tag{76}$$

to give us:

$$= e^x\sin x - \int e^x\cos x\,\mathrm{d}x \tag{77}$$

We apply integration by parts a second time:

$$u = \cos x \qquad \mathrm{d}v = e^x\,\mathrm{d}x \tag{78}$$

$$\mathrm{d}u = -\sin x\,\mathrm{d}x \qquad v = e^x \tag{79}$$

to get:

$$I = e^x\sin x - e^x\cos x - \underbrace{\int e^x\sin x\,\mathrm{d}x}_{I} \tag{80}$$

$$2I = e^x(\sin x - \cos x) + C' \tag{81}$$

$$I = \frac{1}{2}e^x(\sin x - \cos x) + C \tag{82}$$

and we are done.

Example 13: We can also solve integrals that do not appear to have parts, such as $\displaystyle\int \ln x\,\mathrm{d}x$. We choose:

$$u = \ln x \qquad \mathrm{d}v = \mathrm{d}x \tag{83}$$

$$\mathrm{d}u = \frac{1}{x}\,\mathrm{d}x \qquad v = x \tag{84}$$

to give us:

$$\ln x - \int \mathrm{d}x = x\ln x - x + C \tag{85}$$

- For a definite integral, we can write IBP as:

$$f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,\mathrm{d}x \tag{86}$$

Example 14: It is *possible* to apply integration of parts to find the integral of $\int \tan x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \, \mathrm{d}x$. We can let:

$$u = \frac{1}{\cos x} = \sec x \qquad \mathrm{d}v = \sin x \, \mathrm{d}x \tag{87}$$

$$\mathrm{d}u = \sec x \tan x \qquad v = -\cos x \tag{88}$$

this gives us:

$$\int \tan x \, \mathrm{d}x = -\frac{\cos x}{\cos x} + \int \tan x \, \mathrm{d}x \tag{89}$$

Notice that we could try to subtract the original integral from both sides and get:

$$0 = -1 \tag{90}$$

which is clearly wrong! However, we forgot the constant of integration, so the correct statement would be:

$$0 + C' = -1 + C \tag{91}$$

which does not tell us anything interesting. This brings We can see this concretely by repeating the same steps but trying to evaluate the definite integral $\int_a^b \tan x \, \mathrm{d}x$ instead, which gives:

$$\int_a^b \tan \mathrm{d}x = (-1)\Big|_{x=a}^{x=b} + \int_a^b \tan x \, \mathrm{d}x \implies 0 = (-1) - (-1) \implies 0 = 0 \tag{92}$$

which confirms our suspcision that this isn't anything useful, but it's also not an incorrect statement.

Warning: Sometimes it is possible to get more than one answer through various means that differ by a constant factor when solving indefinite integrals. When this happens, nothing is wrong: we simply need to consider the constant of integration.

Idea: But how do we know *which* values of u and $\mathrm{d}v$ we should pick? A common strategy is to use **LIATE**:

1. L: Logarithms
2. I: Inverse Trig
3. A: Algebraic
4. T: Trigonometric
5. E: Exponential

If a function consists of two terms, the term that is higher up (closer to L) usually gets differentiated and the term near the bottom (closer to E) usually gets integrated. See this for how it works, and this video for a tutorial.

4 Trigonometric Integrals

- The first type of integral we'll deal with is:

$$\int \sin^n x \cos^n x \, \mathrm{d}x \tag{93}$$

- In **case 1**, we have either m or n as an odd positive number. We can then use the identity $\sin^2 x + \cos^2 x = 1$ to simplify it.

Example 15: For example, to solve $\int \sin^3 x \cos^2 x \, \mathrm{d}x$, we can simplify this to:

$$= \int (1-\cos^2 x)\cos^2 x \sin x \, \mathrm{d}X \tag{94}$$
$$= (\cos^2 x - \cos^4 x)\sin x \, \mathrm{d}x \tag{95}$$

and applying a u substitution with $u = \cos x$ and breaking it up into two integrals, we can get:

$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C \tag{96}$$

- In **case 2**, we have m and n as both even. We then apply the double angle formulas:

$$\sin x \cos x = \frac{1}{2}\sin(2x) \tag{97}$$
$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x \tag{98}$$
$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x \tag{99}$$

Example 16: For example:

$$\int \sin^2 x \cos^4 \mathrm{d}x = \int \frac{1}{4}\sin^2(2x)\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)\mathrm{d}x \tag{100}$$
$$= \frac{1}{8}\int \sin^2(2x)\,\mathrm{d}x + \frac{1}{8}\int \sin^2 x \cos 2x \, \mathrm{d}x \tag{101}$$
$$= \frac{1}{8}\int \left(\frac{1}{2}-\frac{1}{2}\cos 4x\right)\mathrm{d}x + \frac{1}{8\cdot 3\cdot 2}\sin^3(2x) + C \tag{102}$$
$$= \frac{1}{16}x - \frac{1}{64}\sin(4x) + \frac{1}{48}\sin^3(2x) + C \tag{103}$$

- In **Case 3**, we have:

$$\int \sin^n \mathrm{d}x \,, \int \cos^n \mathrm{d}x \tag{104}$$

which we can apply a reduction formula by keep applying integration by parts:

$$\int \sin^n \mathrm{d}x = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, \mathrm{d}x \tag{105}$$
$$\int \cos^n \mathrm{d}x = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, \mathrm{d}x \tag{106}$$

Example 17: To solve the integral $\int \sin^2 x \, \mathrm{d}x$, we get:

$$= -\frac{1}{2}\sin x \cos x + \frac{1}{2}\int \mathrm{d}x \tag{107}$$
$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C \tag{108}$$

- In **Case 4**, we have integrals in the following forms:

$$\int \sin(mx)\cos(nx)\,\mathrm{d}x \tag{109}$$
$$\int \sin(mx)\sin(nx)\,\mathrm{d}x \tag{110}$$
$$\int \cos(mx)\cos(nx)\,\mathrm{d}x \tag{111}$$

with $m \neq n$. If $m = n$, then we can apply the double angle formula. To solve these, we apply the following identities:

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \tag{112}$$
$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right] \tag{113}$$
$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right] \tag{114}$$

Example 18: For example, we have:

$$\int \sin(3x)\sin(2x)\,\mathrm{d}x = \frac{1}{2}\int \cos((3-2)x)\,\mathrm{d}x - \frac{1}{2}\cos((3+2)x)\,\mathrm{d}x \tag{115}$$
$$= \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C \tag{116}$$

- In **case 5**, we have integrals in the form of either:

$$\int \tan^n x\,\mathrm{d}x\,,\ \int \cot^n x\,\mathrm{d}x \tag{117}$$

To solve these, we apply the following identities:

$$\tan^2 x = \sec^2 x - 1 \tag{118}$$
$$(\tan x)' = \sec^2 x \tag{119}$$

- In **case 6**, we have:

$$\int \sec^n x\,\mathrm{d}x\,,\ \int \csc^n x\,\mathrm{d}x \tag{120}$$

with $n \geq 2$. To solve these, we can make the following substitutions:

$$1 + \tan^2 x = \sec^2 x \tag{121}$$
$$1 + \cot^2 x = \csc^2 x \tag{122}$$

to convert it to a case 5 problem.

- In **case 7**, we have:

$$\int \tan^n x \sec^n x\,\mathrm{d}x\,,\ \int \cot^n x \csc^n x\,\mathrm{d}x \tag{123}$$

Example 19: We have:

$$\tan^3 x \sec^4 x\,\mathrm{d}x = \int \tan^3 x \sec^2 x \sec^2 x\,\mathrm{d}x \tag{124}$$
$$= \int \tan^3 x \left(\tan^2 x + 1\right)\sec^2 x\,\mathrm{d}X \tag{125}$$
$$= \int (\tan^5 x + \tan^3 x)\sec^2 x\,\mathrm{d}x \tag{126}$$
$$= \frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + C \tag{127}$$

Idea: The basic idea of these types is to apply trigonometric identities to turn the integrals into a form that is easier to deal with. The substitutions are usually very simple but to find them, it requires a lot of practice.

- We can also apply **trigonometric substitutions**, any integrals with any of the three factors below can be solved with this technique:

  1. $\sqrt{a^2 - x^2}$: Set $x = a\sin u \implies \sqrt{a^2 - x^2} = a\cos u$

2. $\sqrt{a^2+x^2}$: Set $x = a\tan u \implies \sqrt{a^2+x^2} = a\sec u$
3. $\sqrt{x^2-a^2}$: Set $x = a\sec u \implies \sqrt{x^2-a^2} = a\tan u$

where the arguments under the square roots are always positive.

Example 20: To solve the integral $\int \frac{x^2}{(4-x^2)^{3/2}}\,\mathrm{d}x$, we can set:

$$x = 2\sin u \tag{128}$$
$$\mathrm{d}x = 2\cos u\,\mathrm{d}u \tag{129}$$
$$\sqrt{4-x^2} = 2\cos u \tag{130}$$

which gives:

$$= \int \frac{4\sin^2 u \cdot 2\cos u\,\mathrm{d}u}{8\cos^3 u} \tag{131}$$
$$= \int \tan^2 u\,\mathrm{d}u \tag{132}$$
$$= \int (\sec^2 u - 1)\,\mathrm{d}u \tag{133}$$
$$= \tan u - u + C \qquad = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C \tag{134}$$

Example 21: The integral $\int \frac{x\,\mathrm{d}x}{(2x^2+4x-7)^{1/2}}$ needs a bit more work before we can apply the sibstutitions. We first apply the square to get:

$$= \int \frac{x\,\mathrm{d}x}{\sqrt{2(x+1)^2-9}} \tag{135}$$

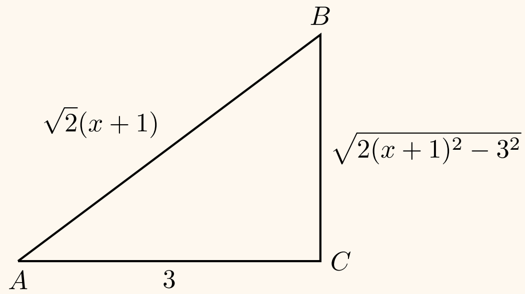
We can set:

$$\sqrt{2}(x+1) = 3\sec u \tag{136}$$
$$\sqrt{2}\,\mathrm{d}x = 3\sec u\tan u\,\mathrm{d}u \tag{137}$$
$$\sqrt{2(x+1)^2-9} = 3\tan u \tag{138}$$

which gives:

$$= \int \frac{\left(\frac{3}{\sqrt{2}}\sec u - 1\right)\left(\frac{3}{\sqrt{2}\sec u\tan u\mathrm{d}u}\right)}{3\tan u} \tag{139}$$
$$= \int \left(\frac{3}{\sqrt{2}}\sec u - 1\right)\left(\frac{1}{\sqrt{2}}\sec u\right)\mathrm{d}u \tag{140}$$
$$= \frac{3}{2}\int \sec^2 u\,\mathrm{d}u - \frac{1}{\sqrt{2}}\int \sec u\,\mathrm{d}u \tag{141}$$
$$= \frac{3}{2}\tan u - \frac{1}{\sqrt{2}}\ln|\sec u + \tan u| + C \tag{142}$$
$$= \frac{1}{2}\sqrt{2x^2+4x-7} - \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}}{3}(x+1) + \frac{\sqrt{2x^2+4x-7}}{3}\right| + C \tag{143}$$

Idea: We can use triangles to derive the substitution, which comes from the Pythagorean theorem:



and you can clearly see the substitution:

$$3\sec u = \sqrt{2}(x+1) \implies \cos u = \frac{3}{\sqrt{2}(x+1)} \tag{144}$$

where $u \equiv \angle BAC$.

Example 22: For the integral $\int x\sin^{-1} x\,\mathrm{d}x$, we can let:

$$u = \sin^{-1} x\,\mathrm{d}v = x\,\mathrm{d}x \tag{145}$$

$$\mathrm{d}u = \frac{\mathrm{d}x}{\sqrt{1-x^2}} v = \frac{1}{2}x^2 \tag{146}$$

and applying integration by parts, we get:

$$= \frac{1}{2}x^2\sin^{-1} x - \int \frac{1}{2}x^2 \frac{\mathrm{d}x}{\sqrt{1-x^2}} \tag{147}$$

To solve this secondary integral $\int \frac{x^2\,\mathrm{d}x}{\sqrt{1-x^2}}$, we can let:

$$x = \sin\theta \tag{148}$$

$$\mathrm{d}x = \cos\theta\,\mathrm{d}\theta \tag{149}$$

$$\sqrt{1-x^2} = \cos\theta \tag{150}$$

which gives:

$$= \frac{\sin^2\theta\cos\theta\,\mathrm{d}\theta}{\cos\theta} \tag{151}$$

$$= \int \sin^2\theta\,\mathrm{d}\theta \tag{152}$$

$$= \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta + C \tag{153}$$

$$= \frac{1}{2}\sin^{-1} - \frac{1}{2}x\sqrt{1-x^2} + C \tag{154}$$

Therefore, we get:

$$\int x\sin^{-1} x\,\mathrm{d}x = \frac{1}{2}x^2\sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{1}{4}x\sqrt{1-x^2} + C \tag{155}$$

5 Partial Fractions

- Rational functions are in the form of:

$$R(x) = \frac{P_n(x)}{P_m(x)} \tag{156}$$

where m, n represent the order of the polynomial. If $n \geq m$, it is an **improper** fraction, such as:

$$\frac{x^2 - x}{1 + x} \tag{157}$$

and if $n < m$, we have a proper fraction such as:

$$\frac{x}{x^2 + 3x + 2} \tag{158}$$

- If we have an improper fraction, we use long division to simplify it. For example:

$$\frac{x^3 - 2x^2}{x^2 + 9} = x - 2 + \frac{18 - 9x}{x^2 + 9} \tag{159}$$

which turns the expression into a polynomial (trivial to integrate) as well as a proper fraction.

- There are different types of factors:
  - Linear factors (e.g. $3x + 2$)
  - Irreducible quadratic factors (e.g. $x^2 + 1$)

which gives us the different factors:

- **Case 1:** If we have distinct linear factors in the denominator, we can break it into fractions of the form:

$$(x + \alpha) \implies \frac{A}{x + \alpha} \tag{160}$$

Example 23: The partial fraction of $\frac{2x - 17}{x^2 + 3x + 2}$ can be written as the **partial fraction deconvolution**:

$$= \frac{A}{x + 1} + \frac{B}{x + 2} \tag{161}$$

We now need to solve for A and B. We can multiply both sides by $(x + 1)(x + 2)$ to get:

$$2x - 17 = A(x + 2) + B(x + 1) \tag{162}$$

and match up the coefficients. Alternatively, we can pick various values of x (e.g. $x = -2$ and $x = -1$) to solve for the coefficients.

- **Case 2:** If we have repeated linear factors, then the decomposition is in the form of:

$$(x + \alpha)^k \implies \frac{A}{x + \alpha} + \frac{B}{(x + \alpha)^2} + \frac{C}{(x + \alpha)^3} + \cdots + \frac{K}{(x + \alpha)^k} \tag{163}$$

Example 24: To get the decomposition of $\frac{2}{x(x + 1)^2}$, we can get:

$$\frac{2}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2} \tag{164}$$

which gives:

$$2 = A(x + 1)^2 + Bx(x + 1) + Cx \tag{165}$$

matching the coefficients, we get three equations and three unknowns:

$$x^2 : A + B = 0 \tag{166}$$

$$x : 2A + B + C = 01 \qquad : A = 2 \tag{167}$$

Solving this system gives $A = 2$, $B = -2$, and $C = -2$. Note that taking the integral of this sum is much easier. We have:

$$\int \frac{d}{x(x+1)^2}\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}X - \int \frac{2}{x}\,\mathrm{d}x - \int \frac{2}{(x+1)^2}\,\mathrm{d}x \tag{168}$$

$$= 2\ln|x| - 2\ln|x+1| + \frac{2}{x+1} + C \tag{169}$$

Idea: As a general rule of thumb, the number of unknown coefficients is equal to the order of the polynomial in the denominator.

- **Case 3:** If we have irreducible quadratic factors, then the partial fraction deconvolution is in the form of:

$$x^2 + px + 8 \implies \frac{Ax+B}{x^2+px+8} \tag{170}$$

Example 25: Suppose we have $\frac{2}{(x+1)(x^2+x+1)}$, we can get the partial fraction decomposition as:

$$= \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1} \tag{171}$$

and we work through the deconvolution process in exactly the same way, we remove the denominators on both sides to get (after expanding):

$$2 = Ax^2 + Ax + A + Bx^2 + Bx + Cx + C \tag{172}$$

$$0x^2 + 0x^1 + 2x^0 = (A+B)x^2 + (A+B+C)x^1 + (A+C)x^0 \tag{173}$$

which gives three equations and three unknowns, after we match coefficients:

$$x^2 : A + B = 0 \tag{174}$$

$$x : A + B + C = 0 \tag{175}$$

$$1 : A + C = 2 \tag{176}$$

and solving the system of equations gives $A = 2, B = -2, C = 0$. To get the integral of this second term, we can write the second term as:

$$\int \frac{2x\,\mathrm{d}x}{x^2+2x+1} = \underbrace{\int \frac{2x+1}{x^2+x+1}\,\mathrm{d}x}_{(1)} - \underbrace{\int \frac{\mathrm{d}x}{x^2+x+1}}_{(2)} \tag{177}$$

We "added" 1 and "subtracted" 1 to get these two slightly easier integrals, which we can apply other techniques. The first one can be solved using a u-sub while the second can be solved by completing the square and applying a trigonometric substitution:

$$(1) = \ln|x^2+x+1| + C \tag{178}$$

$$(2) = \int \frac{\mathrm{d}x}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right] + C \tag{179}$$

allowing us to put everything together.

Example 26: Let's take an integral we already know the answer of: $\int \frac{2x}{x^2+1}\,\mathrm{d}x = \ln\left(x^2+1\right) + C$. We can try a partial fraction decomposition:

$$\frac{2x}{x^2+1} = \frac{A}{x+i} + \frac{B}{x-i} = \frac{1}{x+i} + \frac{1}{x-i} \tag{180}$$

which gives:

$$\int \frac{2x}{x^2+1}\,\mathrm{d}x = \int \frac{\mathrm{d}x}{x+i} + \int \frac{\mathrm{d}x}{x-i} \tag{181}$$

In complex analysis, most mathematical functions we are familiar with are sitll valid, so the integral is:

$$= \ln|x+i| + \ln|x-i| + C \tag{182}$$

and simplifying it gives:

$$\ln\big(x^2+1\big) + C \tag{183}$$

Warning: While it is *possible* to use complex numbers to solve irreducible quadratic factors, it isn't always as easy as the above example. To get the logarithm of a complex number, we can apply the identity (without proving):

$$\ln(a+ib) = \ln\sqrt{a^2+b^2} + i\arctan\left(\frac{b}{a}\right) \tag{184}$$

Example 27: Bonus content: Try evaluating the integral $\int \frac{\mathrm{d}x}{x^2+1}$ with complex analysis. Taking a partial fraction, we get:

$$\frac{1}{x^2+1} = \frac{A}{x+i} + \frac{B}{x-i} \tag{185}$$

multiplying both sides, we get:

$$1 = A(x-i) + B(x+i) \tag{186}$$
$$1 = (A+B)x + i(-A+B) \tag{187}$$

we have the systems of two equations:

$$x^1 : A+B = 0 \tag{188}$$
$$x^0 : (B-A)i = 1 \tag{189}$$

which gives $A = \frac{1}{2}i$ and $B = -\frac{1}{2}i$. This gives:

$$= \int \frac{0.5i}{x+i}\,\mathrm{d}x - \int \frac{0.5i}{x-i}\,\mathrm{d}x \tag{190}$$
$$= 0.5i\ln(x+i) - 0.5i\ln(x-i) + C \tag{191}$$
$$= 0.5i\ln\sqrt{x^2+1} + (0.5i)i\arctan\left(\frac{b}{x}\right) - (0.5i)\ln\sqrt{x^2+1} - (0.5i)i\arctan\left(-\frac{1}{x}\right) \tag{192}$$
$$= -\arctan\left(\frac{1}{x}\right) + C \tag{193}$$

Note that for $x \geq 0$:

$$-\arctan\left(\frac{1}{x}\right) + \frac{\pi}{2} = \arctan x \tag{194}$$

and for $x < 0$:

$$-\arctan\left(\frac{1}{x}\right) - \frac{\pi}{2} = \arctan x \tag{195}$$

- **Case 4:** Repeated irreducible quadratic terms, the decomposition is in the form of:

$$(x^2+\beta x+8)^k \implies \frac{A_1x+B_1}{(x^2+\beta x+8)} + \frac{A_2x+B_2}{(x^2+\beta x+8)^2} + \cdots + \frac{A_kx+B_k}{(x^2+\beta x+x)^k} \tag{196}$$

These can be extremely messy, but the process is similar to the above examples. For example, we can write:

$$\frac{Ax+B}{(x^2+\beta x+8)^2}=\frac{A}{2}\left[\frac{2x+\beta}{(x^2+\beta x+8)^2}+\frac{2B/A-\beta}{(x^2+\beta x+8)^2}\right] \tag{197}$$

Idea: The general strategy for dealing with a proper fraction integral is to break it up into two terms, one that can be easily be solved via a u-substitution and the second one does not have an x term in the numerator and can be solved using a trigonometric substitution.

- We can also introduce a strategy rationalizing substitutions by turning a function such as:

$$\int \frac{\sqrt{x}}{1+x}\,\mathrm{d}x \tag{198}$$

into a form that we are familiar with. We can let $u^2 = x \implies 2u\,\mathrm{d}u = \mathrm{d}x$ to give:

$$= \int \frac{u\cdot 2u\,\mathrm{d}u}{1+u^2} \tag{199}$$

$$= 2\int \frac{u^2}{1+u^2}\,\mathrm{d}u \tag{200}$$

$$= 2\int \left(1-\frac{1}{1+u^2}\right)\mathrm{d}u \tag{201}$$

$$= 2u - 2\tan^{-1} u + C \tag{202}$$

$$= 2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C \tag{203}$$

- Another method is to use a **Weierstrass substitution**, by making the substitution:

$$t = \tan\frac{x}{2} \tag{204}$$

which leads to the following substitutions:

$$\sin x = \frac{2t}{1+t^2} \tag{205}$$

$$\cos x = \frac{1-t^2}{1+t^2} \tag{206}$$

$$\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t \tag{207}$$

This allows us to turn any trigonometric function into a rational function.

Example 28: For example, to solve the integral $\displaystyle\int \frac{\mathrm{d}x}{1+\cos x}$, we make the specified substitution to turn this into:

$$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,\mathrm{d}t \tag{208}$$

$$= \int \frac{2\,\mathrm{d}t}{(1+t^2)+(1-t^2)}\,\mathrm{d}t \tag{209}$$

$$= \int \mathrm{d}t \tag{210}$$

$$= t + C \tag{211}$$

$$= \tan\left(\frac{x}{2}\right) + C \tag{212}$$

6 Improper Integrals

- Since infinity is not a number, our typical definite integral definition cannot be used for an **improper integral** like:

$$\int_0^\infty f(x)\,\mathrm{d}x \tag{213}$$

Instead, we use the following definition:

Definition: If $\lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x = L$ exists, then we can define:

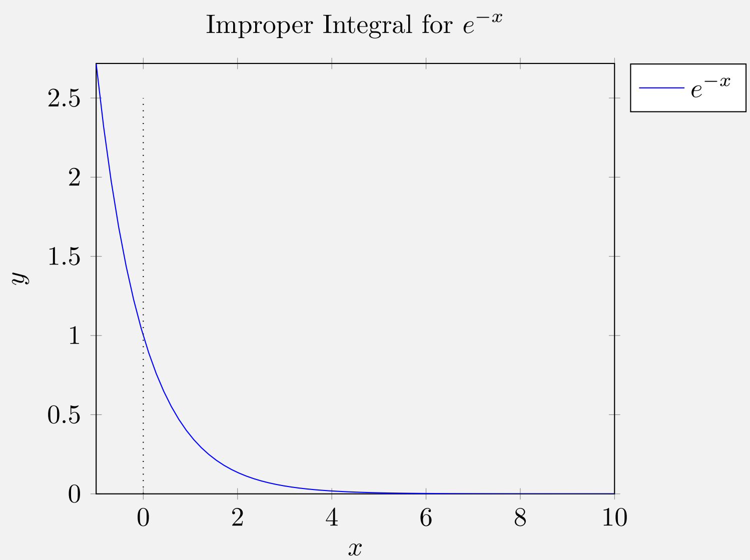
$$\int_a^\infty f(x)\,\mathrm{d}x = L \tag{214}$$

Example 29: To solve $\int_0^\infty e^{-x}\,\mathrm{d}x$, we can write it as:

$$= \lim_{b\to\infty}\int_0^b e^{-x}\,\mathrm{d}x \tag{215}$$

$$= \lim_{b\to\infty}\left(1-e^{-b}\right) = 1 \tag{216}$$

This is remarkable because even though the area appears infinite (since it is infinitely long), the area is actually finite.



Example 30: For the integral $\int_{-\infty}^{-1}$, we have:

$$= \lim_{a\to-\infty}\int_a^{-1}\frac{\mathrm{d}x}{x^2} \tag{217}$$

$$= \lim_{a\to-\infty}\left(1+\frac{1}{a}\right) = 1 \tag{218}$$

- However, improper integrals can diverge as well.

Example 31: For $\int_3^\infty \frac{\mathrm{d}x}{x}$, we get:

$$= \lim_{b\to\infty} (\ln b - \ln 3) = \infty \tag{219}$$

Example 32: For something like $\int_{-\infty}^{2\pi} \sin x \, \mathrm{d}x$, the integral does not go to infinity, but since we get:

$$\lim_{a\to-\infty} (-1 + \cos a) \tag{220}$$

it will diverge, since $\lim_{a\to-\infty} \cos a$ does not exist.

- We can generalize this for all reciprocal functions:

Idea: For $\int_a^\infty \frac{\mathrm{d}x}{x^p}$ with $p > 0$, $p \neq 1$, and $a > 0$, we get:

$$= \lim_{b\to\infty} \int_a^b \frac{\mathrm{d}x}{x^p} \tag{221}$$

$$= \lim_{b\to\infty} \left(\frac{1}{1-p} x^{-p+1}\right)\Bigg|_a^b \tag{222}$$

$$= \lim_{b\to\infty} \left(\frac{b^{-p+1}}{1-p} - \frac{a^{-p+1}}{1-p}\right) \tag{223}$$

For $p > 1$, we get:

$$= \frac{a^{1-p}}{p-1} \tag{224}$$

and diverges for $p \leq 1$.

- There are techniques to check if an improper integral will converge or diverge. This is useful especially if we want to perform a numerical integration but want to verify that it indeed will converge.

Theorem: Let f, g be continuous functions and $0 \leq f(x) \leq g(x)$ where $x \in [a, \infty)$,.
  - If $\int_a^\infty g \, \mathrm{d}x$ converges, so does $\int_a^\infty f(x) \, \mathrm{d}x$.
  - If $\int_a^\infty f$ diverges, so does $\int_a^\infty g(x) \, \mathrm{d}x$.

Example 33: The integral $\int_2^\infty \frac{\mathrm{d}x}{\sqrt{1+x^{44/17}}}$ is difficult to evaluate, but we can easily tell that it converges via:

$$\frac{1}{\sqrt{1+x^{44/12}}} < \frac{1}{\sqrt{x^{44/12}}} = \frac{1}{x^{22/12}} \tag{225}$$

Since $p > 1$, this converges, so the original integral must also converge.

Example 34: For the integral $\int_3^\infty \frac{\mathrm{d}x}{\sqrt{7+x^2}}$, we can check that it diverges by:

$$(7 + x^2)^{1/2} < \sqrt{7} + x \tag{226}$$

We can check this via: $7 + x^2 < 7 + 2\sqrt{7} + x^2$. Since:

$$\int_3^\infty \frac{\mathrm{d}x}{\sqrt{7}+x} = \ln\left(\sqrt{7}+x\right)\Big|_3^\infty \tag{227}$$

which diverges, so the original integral must also diverge.

Warning: The notation $f(x)\Big|_3^\infty$ needs to be defined explicitly since ∞ is not a number. This expression simply implies that we are taking the limit as b approaches infinity, even though it might look like we're treating ∞ as a number.

- We can look at more interesting examples. Both the lower and upper bounds can be $\pm\infty$, such as:

$$\int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi} \tag{228}$$

Definition: We can define an integral from $-\infty$ to $+\infty$ as:

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x \tag{229}$$

Warning: Do *not* evaluate integrals of the above form as:

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x \neq \lim_{b\to\infty}\int_{-b}^b f(x)\,\mathrm{d}x \tag{230}$$

- For example, take the integral $\displaystyle\int_{-\infty}^\infty$. If we use the proper definition, then we add two limits that don't exist, so we know this diverges. Note that it might be tempting to write:

$$= \lim_{b\to\infty}\int_{-b}^b x\,\mathrm{d}x = \lim_{b\to\infty}\left(\frac{b^2}{2}-\frac{b^2}{2}\right) = 0 \tag{231}$$

but this is only because we are approaching $-\infty$ and $+\infty$ at the same rate. If we instead wrote:

$$\lim_{b\to\infty}\int_{-b}^{2b} x\,\mathrm{d}x = \lim_{b\to\infty}\left(\frac{4b^2}{2}-\frac{b^2}{2}\right) = \infty \tag{232}$$

If we instead used this approach for our other improper integrals, it wouldn't make a difference since it shouldn't matter the rate at which we approach infinity. Here's another example:

$$\lim_{b\to\infty}\int_{-b}^{\sqrt{b^2+138}} x\,\mathrm{d}x = \lim_{b\to\infty}\left(\frac{b^2+138}{2}-\frac{b^2}{2}\right) = \lim_{b\to\infty}\frac{138}{2} = 69 \tag{233}$$

- Improper integrals can also be in the form where there are infinite dicontinuities at the bounds of integration. Suppose $\displaystyle\lim_{x\to b^-} f(x) = \infty$. We can treat an integral such as:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{c\to b^-}\int_a^c f(x)\,\mathrm{d}x \tag{234}$$

Example 35: For example, take $\int_0^1 \frac{\mathrm{d}x}{x^{1/3}}$, and we can evaluate this via:

$$= \lim_{c\to 0^+}\int_c^1 \frac{\mathrm{d}x}{x^{1/3}} \tag{235}$$

$$= \lim_{c\to 0^+}\frac{3}{2}\left(1-c^{2/3}\right) = \frac{3}{2} \tag{236}$$

Again, we have a region that extends to an infinite extend, but it has a finite area. Of course, this won't always be the case.

Example 36: Take the example where $\int_0^1 \frac{\mathrm{d}x}{x^2}$, then we can evaluate this integral via:

$$= \lim_{c\to 0^+}\int_c^1 \frac{\mathrm{d}x}{x^2} \tag{237}$$

$$= \lim_{c\to 0^+}\left(\frac{1}{c}-1\right) = \infty \tag{238}$$

so this integral will diverge.

Idea: Notice that we can draw an anaology between: $\int_0^a \frac{\mathrm{d}x}{x^p}$ and $\int_a^\infty \frac{\mathrm{d}x}{x^{1/p}}$, as they are reflections of one another across the line $y=x$. If one diverges, the other will converge, with the exception being $p=1$.

- We can also deal with discontinuities that occur between the given bounds. Similar to before, we break it up into two integrals and *both* integrals must converge for the original integral to converge. For example, take:

$$\int_{-a}^b \frac{1}{|x^{1/2}|}\,\mathrm{d}x \tag{239}$$

with $a,b>0$. For this integral to converge, then both $\int_{-a}^0 \frac{\mathrm{d}x}{|x^{1/2}|}$ and $\int_0^b \frac{\mathrm{d}x}{|x^{1/2}|}$ must converge.

Warning: Here is an example of when things go wrong when the integral is not broken up into separate integrals. For example, suppose we wish to evaluate $\int_{-1}^3 \frac{\mathrm{d}x}{x^2}$. From our previous discussion, we know that $\int_{-1}^0 \frac{\mathrm{d}x}{x^2}$ and $\int_0^3 \frac{\mathrm{d}x}{x^2}$ both diverges. However, one might naively think that:

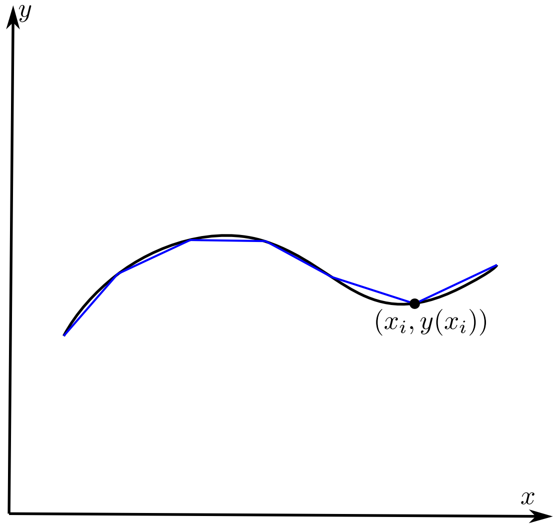
$$\left(-\frac{1}{x}\right)\Bigg|_{-1}^{3} = -\frac{1}{3}-\frac{1}{1} = -\frac{4}{3} \tag{240}$$

which is definitely wrong, since $\frac{1}{x^2}$ is never negative!

7 Applications of Integrals

7.1 Arclength

- Suppose w ehave a curve $y=f(x)$ where $x\in[a,b]$ and is differentiable. The problem is to find the length of the curve in this range.



- We can approximate this by partitioning the curve into segments at locations x_i where:

$$a = x_0 < x_1 < x_@ < \cdots < x_{n-1} < x_n = b \tag{241}$$

such that the arclength is:

$$s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{(x_i - x_{i-1})^2 + (y(x_i) - y(x_{i-1}))^2} \tag{242}$$

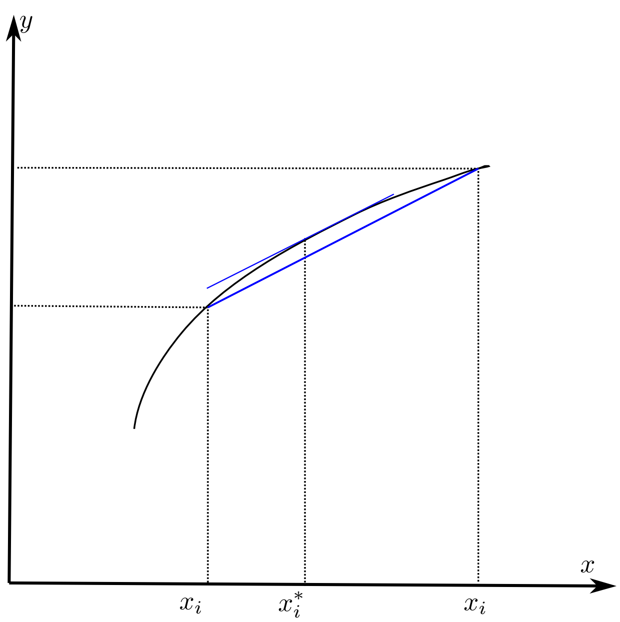
We can use the mean value theorem to write:

$$\frac{\Delta y_i}{\Delta x_i} = \frac{y(x_i) - y(x_{i-1})}{x_i - x_{i-1}} = y'(x_i^*) \tag{243}$$

so we can rewrite:

$$s_i = \sqrt{\Delta x_i^2 + \left(f'(x_i^*)\Delta x_i\right)^2} \tag{244}$$

The total length is approximated by the total sum. If we take the limit:



$$s = \lim_{\|P\|\to 0} \sum_{i=1}^{n} sqrt1 + f'(x_i^*)^2 \Delta x_i \tag{245}$$
$$= \int_a^b \sqrt{1+f'(x)^2}\,\mathrm{d}x \tag{246}$$

Example 37: For example, the arclength in $x \in [0, 44]$ for $f(x) = x^{3/2}$ can be calculated if we know the derivative:

$$f'(x) = \frac{3}{2}x^{1/2} \tag{247}$$

so:

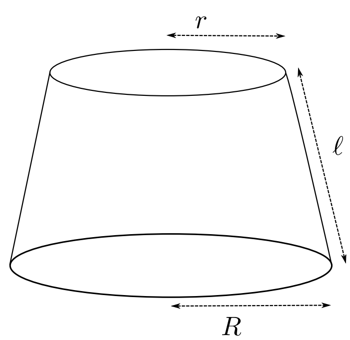
$$1 + f'(x)^2 = 1 + \frac{9}{4}x \tag{248}$$

which gives:

$$s = \int_0^{44} \sqrt{1+\frac{9}{4}x}\,\mathrm{d}x \tag{249}$$
$$= \left(\frac{4}{9}\cdot\frac{2}{3}\left(1+\frac{9}{4}x\right)^{3/2}\right)\Bigg|_0^{44} \tag{250}$$
$$= 296 \tag{251}$$

7.2 Area of a Surface of Revolution

- Consider the new problem of finding the area of a surface of revolution. Similarly, we break it up into smaller segment with width Δx.



- Each small segment is a tapered cone, with an area of:

$$A_i \simeq \pi(f(x_{i-1}) + f(x_i))s_i \tag{252}$$
$$\simeq \pi(f(x_{i-1}) + f(x_i))\sqrt{1+f'(x_i^*)^2}\Delta x_i \tag{253}$$

From the Intermediate Value Theorem, we have:

$$f(x_{i-1}) + f(x_i) = 2f(x_i^{**}) \tag{254}$$

where $x_i^{**} \in [x_{i-1}, x_i]$ so the area can be written as:

$$A_i \simeq 2\pi f(x_i^{**})\sqrt{1+f'(x_i^*)^2}\Delta x_i \tag{255}$$

However, we cannot turn this into an integral just yet since we have both x_i^* and x_i^{**}. But in the limit where $\Delta x_i \to 0$, we also have $x_i^{**} \to x_i^*$. We therefore get:

$$A = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,\mathrm{d}x \tag{256}$$

Example 38: Suppose we have the function $y = \sqrt{x}$ rotated across the x axis and we want the surface area between $x \in [0,1]$. We have $y' = \frac{1}{2}x^{-1/2}$ and the area becomes:

$$A = \int_0^1 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\,\mathrm{d}x \tag{257}$$

$$= \pi\int_0^1 \sqrt{4x+1}\,\mathrm{d}x \tag{258}$$

Let $u = 4x+1$ and $\mathrm{d}u = 4\,\mathrm{d}x$, and we'll get:

$$A = \int_1^5 \pi\sqrt{u}\frac{\mathrm{d}u}{4} \tag{259}$$

$$= \frac{\pi}{4}\left(\frac{2}{3}u^{3/2}\right)\bigg|5_1 \tag{260}$$

$$= \frac{\pi}{6}(5^{3/2}-1) \tag{261}$$

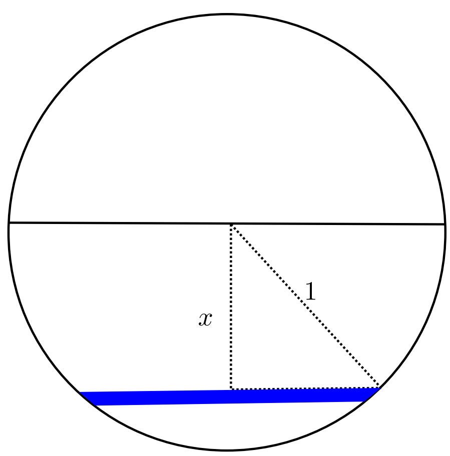
8 Applications to Physics and Engineering

- The **hydrostatic pressure** depends on the density ρ, gravitational constant g and the depth d:

$$p = \rho g d \tag{262}$$

and the force of pressure acting on the surface is:

$$F = \rho g d \cdot A = pA \tag{263}$$



Example 39: Suppose we have a curved container. The force acting on the entire container can be broken up into segments, each with a force of:

$$F_i = \underbrace{w(x_i^*)\Delta x_i}_{\text{area}} \cdot \underbrace{\rho g x_i^*}_{\text{pressure}} \tag{264}$$

where $w(x)$ is the width of the container as a function of height. The force exerted on the container is thus:

$$F = \int_a^b \rho g x w(x)\,\mathrm{d}x \tag{265}$$

Example 40: Suppose we have a pipe half with a radius of 1m filled with water and we wish to find the force it exerts on the end face of the pipe. We can do this via:

$$F = \int_0^1 \rho g x 2\sqrt{1-x^2}\,\mathrm{d}x \tag{266}$$

$$= 2\rho g\left(-\frac{1}{3}(1-x^2)^{3/2}\right)\Big|_0^1 \tag{267}$$

$$= \frac{2}{3}\rho g = 6533\text{N} \tag{268}$$

- We investigate the **center of mass** of a two dimensional object, which intuitively is the point at which it'll balance, also known as the **centroid**. We can use two principles to help us out:
- **Principle 1: Symmetry:** If there is an axis of symmetry, then $(\bar{x}, \bar{y})$ is on any axis of symmetry. If there are more than one axes of symmetry, then we simply need to find the intersection.
- **Principle 2: Additivity:** We can find the centroid of a collection of segments by taking the weighted average of each of the segments it is composed of. For a discrete set, the total area is:

$$A = A_1 + A_2 + \cdots + A_n \tag{269}$$

and the x location of the centroid is:

$$\bar{x} = \bar{x_1}\frac{A_1}{A} + \bar{x_2}\frac{A_2}{A} + \cdots + \bar{x}_n\frac{A_n}{A} \tag{270}$$

and similarly for the y location:

$$\bar{y} = \bar{y_1}\frac{A_1}{A} + \bar{y_2}\frac{A_2}{A} + \cdots + \bar{y}_n\frac{A_n}{A} \tag{271}$$

- Suppose we wish to find the centroid of a curve $f(x)$ from $x = a$ to $x = b$. We can approximate this region via a series of recxtangles such that:

$$A_i = f(x_i^*)\Delta x_i \tag{272}$$

$$\bar{x}_i = \frac{x_{i-1} + x_i}{2} = x_i^* \tag{273}$$

$$\bar{y}_i = \frac{1}{2}f(x_i^*) \tag{274}$$

such that:

$$\bar{x}A = \sum_{i=1}^n \bar{x}_i A_i = \sum_{i=1}^n x_i^* f(x_i^*)\Delta x_i \tag{275}$$

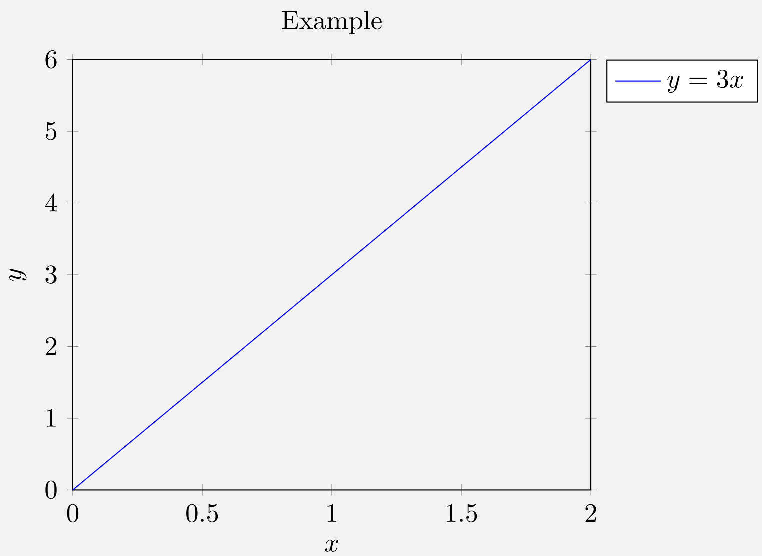
$$\bar{y}A = \frac{1}{2}\sum_{i=1}^n x_i^* f(x_i^*)\Delta x_i \tag{276}$$

If we take the limit, we get:

$$\bar{x} = \frac{\int_a^b x f(x)\,\mathrm{d}x}{\int_a^b f(x)\,\mathrm{d}x} \tag{277}$$

$$\bar{y} = \frac{\int_a^b f(x)^2\,\mathrm{d}x}{2\int_a^b f(x)\,\mathrm{d}x} \tag{278}$$

Example 41: Suppose we wish to find the area of $y = 3x$ between $x = 0$ and $x = 2$:



The area is $A = \int_0^2 3x \, \mathrm{d}x = 6$. And we have:

$$\bar{x}A = \int_0^2 x(3x) \, \mathrm{d}x = 8 \implies \bar{x} = \frac{4}{3} \tag{279}$$

$$\bar{y}A = \int_0^2 \frac{1}{2}(3x)^2 \, \mathrm{d}x = 12 \implies \bar{y} = 2 \tag{280}$$

which is as we expected from the centroid of a triangle.

- If we want the centroid of the region of intersection between two curves f and g, we can use the additivity rule, we can have:

$$\bar{x}A = \bar{x}_g A_g - \bar{x}_f A_f \tag{281}$$

$$\bar{y}A = \bar{y}_g A_g - \bar{y}_f A_f \tag{282}$$

$$A = A_g - A_f \tag{283}$$

or in integral form:

$$\bar{x}A = \int_a^b x \left[f(x) - g(x)\right] \mathrm{d}x \tag{284}$$

$$\bar{y}A = \frac{1}{2}\int_a^b \left[f(x)^2 - g(x)^2\right] \mathrm{d}x \tag{285}$$

Example 42: Suppose we have two curves $y = 6$ and $y = 3$ and we wish to find the centroid of the area between the curves between $2 < x < 5$. This gives us:

$$\bar{x}A = \int_2^5 x(6-3) \, \mathrm{d}x = \frac{63}{2} \tag{286}$$

$$\bar{y}A = \frac{1}{2}\int_2^5 \frac{1}{2}(36-9) \, \mathrm{d}x = \frac{9}{2} \tag{287}$$

which gives us: $\bar{x} = \frac{7}{2}$ and $\bar{y} = \frac{9}{2}$.

- **Pappu's Centroid Theorem** can be used to easily find the volume of revolution. We have:

$$V = 2\pi \bar{R} a \tag{288}$$

where $\bar{R}$ is the distance from the centroid to the axis of revolution.

Example 43: Suppose we have an elliptical torus whose center is The area of the ellipse whose major axis is parallel to the axis of revolution and whose centroid is a distance R away from the axis. The volume is thus:

$$V = 2\pi R\pi ab = 2\pi^2 abR \tag{289}$$

- We can prove this using the washer method about x:

$$\begin{aligned} V_x &= \int_a^b \pi(f(x)^2 - g(x)^2)\,\mathrm{d}x && (290)\\ &= 2\pi\int_a^b \frac{1}{2}(f(x)^2 - g(x)^2)\,\mathrm{d}x && (291)\\ &= 2\pi\bar{y}A && (292)\end{aligned}$$

and using the shell method about x:

$$\begin{aligned} V_y &= \int_a^b 2\pi x(f(x) - g(x))\,\mathrm{d}x && (293)\\ &= 2\pi\bar{x}A && (294)\end{aligned}$$

9 Parametric Equations

- Until now, we have described two dimensional curves in the form $y = f(x)$. However, we can describe this in parametric equations as well in the form:

$$x = x(t) \tag{295}$$
$$y = y(t) \tag{296}$$

- For example, numerous applications in physics and engineering arise using parametized coordinates such as projectile motion. The equations of motion are:

$$x(t) = x_0 + v_0 \cos\theta t \tag{297}$$
$$y(t) = y_0 + v_0 \sin\theta t - \frac{1}{2}gt^2 \tag{298}$$

- We can parametize a straight line using:

$$x(t) = x_c + t(x_1 - x_0) \tag{299}$$
$$y(t) = y_0 + t(y_1 - y_0) \tag{300}$$

with $t \in (-\infty, \infty)$. We can check this by letting $t = 0$, which gives (x_0, y_0) and when $t = 1$ we get (x_1, y_1).

- For an ellipse, we can parametize using:

$$x = a\cos t \tag{301}$$
$$y = b\sin t \tag{302}$$

- Suppose we have two curves parametized by:

$$C_1 = x_1(t),\, y_1(t) \tag{303}$$
$$C_2 = x_2(t),\, y_2(t) \tag{304}$$

An intersection occurs when $y_1(x) = y_2(x)$ and a collision occurs when $x_1(t) = x_2(t)$ and $y_1(t) = y_2(t)$.

Example 44: Suppose we have the following curves:

$$C_1 : x_1(t) = 2t + 6,\, y_1(t) = 5 - 4t \tag{305}$$
$$C_2 : x_2(t) = 3 - 5\cos\pi t,\, y_2 = 1 + 5\sin\pi t \tag{306}$$

for $t > 0$. The first curve is a straight line and the second curve is an offset circle. We can find the intersection via:

$$C_1 : t = \frac{x-6}{2} \implies y_1(x) = 16 - 2x \tag{307}$$
$$C_2 =: (x-3)^2 + (y-1)^2 = 25 \tag{308}$$

There will be two points of intersection, and after we solve this, we get $(6, 5)$ and $(8, 1)$. We now need to look at if the two objects will arise at that point at the same time. We do this by looking at each point first

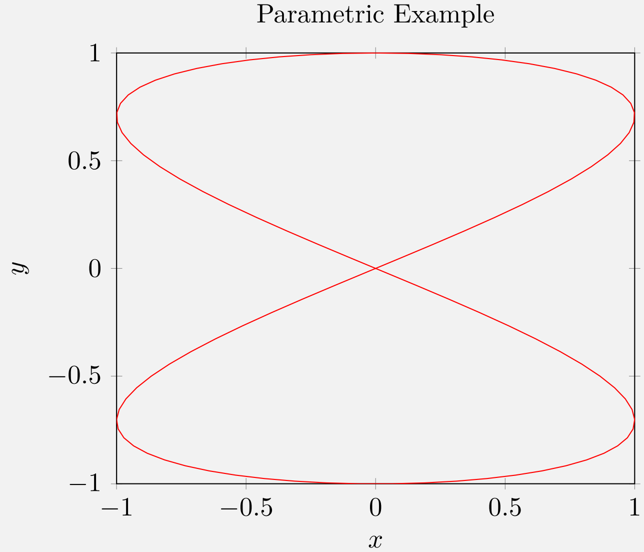
$$(6, 5) : C_1 \implies t = 0 \tag{309}$$
$$C_2|_{t=0} = (-2, 1) \tag{310}$$
$$(8, 1) : C_1 \implies t = 1 \tag{311}$$
$$C_2|_{t=1} = (8, 1) \tag{312}$$

Therefore, there is a collision between the two objects at $(8, 1)$ at a time $t = 1$.

10 Calculus with Parametric Curves

- Parametric curves do not need to be functions. As a result, they can have an ordinary tangent, no tangent, or more than one tangent at the same point.
- We can find the secant line of a parametric curve via:
$$m_{\text{secant}} = \frac{y(t_0+h)-y(t_0)}{x(t_0+h)-x(t_0)} \tag{313}$$
We can divide both numerator and denominator by h to get:
$$\frac{\frac{y(t_0+h)-y(t_0)}{h}}{\frac{x(t_0+h)-x(t_0)}{h}} \implies \frac{y'(t_0)}{x'(t_0)} \tag{314}$$
where we took the limit as $h \to 0$.
- We could also have derived this using the chain rule:
$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\left(\frac{dx}{dt}\right)^{-1} \tag{315}$$
- If $x'(t_0)=0$, then $x=x_0$ is a vertical tangent and if $y'(t_0)=0$, then we have $y=y_0$ and have a horizontal tangent. If $x'=y'=0$, then we can't read any information from it.

Example 45: Consider the example $x(t)=\sin(2t)$ and $y(t)=\sin t$ with $t\in[0,2\pi]$. We have $x'(t)=2\cos 2t$ so we can find the vertical tangents by setting $2\cos(2t)=0$ to get:
$$t \in \{\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}\} \implies \left(1,\frac{1}{\sqrt{2}}\right)\left(-1,\frac{1}{\sqrt{2}}\right)\left(1,-\frac{1}{\sqrt{2}}\right)\left(-1,-\frac{1}{\sqrt{2}}\right) \tag{316}$$
For horizontal tangents, we have $y'(t)=\cos t \implies \cos t = 0$ which gives:
$$t\in\{\frac{\pi}{2},\frac{3\pi}{2}\} \implies (0,1)(0,-1) \tag{317}$$
At $t=0$, we also have $x(0)=y(0)=0$. We can determine the tangent at this point as:
$$m_{\text{tangent}} = \frac{y'(0)}{x'(0)} = \frac{1}{2} \tag{318}$$
However notice that at $t=\pi$, we have $x(\pi)=y(\pi)=0$ as well so this point has another tangent:
$$m_{\text{tangent}} = \frac{y'(\pi)}{x'(\pi)} = -\frac{1}{2} \tag{319}$$
At this point we can sketch out the conditions we found on the graph and draw out curve:



- We can determine the **area** of a parametric curve via:

$$A = \int_{t_1}^{t_2} y(t)x'(t)\,\mathrm{d}t \tag{320}$$

Letting $y = f(x) \implies y(t) = f(x(t))$ gives:

$$\begin{aligned} A &= \int_{t_1}^{t_2} f(x(t))x'(t)\,\mathrm{d}t && (321) \\ &= \int_{a=x(t_1)}^{b=x(t_2)} f(x)\,\mathrm{d}x && (322) \end{aligned}$$

- We can also determine the area of a closed curve:

> **Definition:** A curve is traversed in the positive sense as t increases, if the enclosed area is on the left (counterclockwise).

The area of a closed loop is then:

$$A = \int_{t_4}^{t_3} y(t)x'(t)\,\mathrm{d}t - \int_{t_4}^{t_5} y(t)x'(t)\,\mathrm{d}t + \int_{t_3}^{t_2} y(t)x'(t)\,\mathrm{d}t - \int_{t_1}^{t_2} y(t)x'(t)\,\mathrm{d}t \tag{323}$$

where t_1 represents the starting point on the bottom, t_2 is the rightmost point, t_3 is an arbitrary point on top, t_4 is the leftmost point, and t_5 is the endpoint that ends off as t_1. This gives:

$$-\int_{t_3}^{t_4} y(t)x'(t)\,\mathrm{d}t - \int_{t_4}^{t_5} y(t)x'(t)\,\mathrm{d}t - \int_{t_2}^{t_3} y(t)x'(t)\,\mathrm{d}t - \int_{t_1}^{t_2} y(t)x'(t)\,\mathrm{d}t \tag{324}$$

or:

$$A = -\int_{t_1}^{t_5} y(t)x'(t)\,\mathrm{d}t \tag{325}$$

where t_1 and t_5 are the starting and ending points respectively. They can be arbitarrily chosen so long as the points they coorespond to are on top of each other.

- Similarly, we can also write the area as:

$$A = \int_{t_1}^{t_5} x(t)y'(t)\,\mathrm{d}t \tag{326}$$

Example 46: Let us derive the area of an ellipse, parametized by:

$$\begin{aligned} x &= a\cos\theta && (327) \\ y &= b\sin\theta && (328) \end{aligned}$$

for $\theta \in [0, 2\pi]$. The area is then:

$$\begin{aligned} A &= -\int_0^{2\pi} b\sin\theta(-a\sin\theta)d\theta && (329) \\ &= ab\int_0^{2\pi} \sin^2\theta\,\mathrm{d}\theta && (330) \\ &= ab\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} && (331) \\ &= \pi ab && (332) \end{aligned}$$

- We can also determine the **arclength** of a parametric curve. We start with the summation:

$$s \simeq \sum \sqrt{\Delta x^2 + \Delta y^2} \implies \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t \tag{333}$$

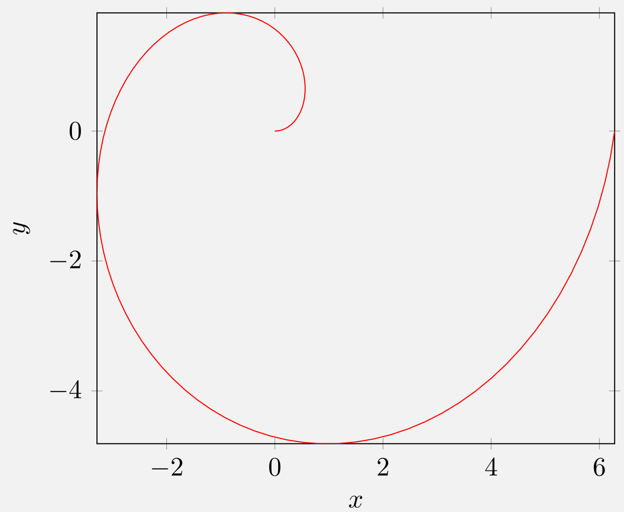
Note that if we let $x = t$ and $y = f(t) = f(x)$, then we get:

$$s = \int_\alpha^\beta \sqrt{1 + f'(x)^2}\,\mathrm{d}x \tag{334}$$

which is what we should expect.

Example 47: Suppose we have $x(\theta) = \theta\cos\theta$ and $y(\theta) = \theta\sin\theta$ for $\theta \in [0, 2\pi]$, this gives an Archimedes Spiral:

Parametric Example



We can determine the arclength via:

$$s = \int_0^{2\pi} \sqrt{(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2}\,\mathrm{d}\theta \tag{335}$$

$$= \int_0^{2\pi} \sqrt{1 + \theta^2}\,\mathrm{d}\theta \tag{336}$$

$$= \left[\frac{1}{2}\theta\sqrt{1+\theta^2} + \frac{1}{2}\ln\left|\theta + \sqrt{1+\theta^2}\right|\right]\Bigg|_0^{2\pi} \tag{337}$$

$$= \pi\sqrt{1+4\pi^2} + \frac{1}{2}\ln\left(2\pi + \sqrt{1+4\pi^2}\right) \tag{338}$$

Idea: If we wish to find the speed, we have:

$$v = \frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} \tag{339}$$

which comes from both the fundamental theorem of calculus, as well as from two-dimensional kinematics.

- The surface area can be written as:

$$A = \int_a^b 2\pi y\,\mathrm{d}s = \int_a^b 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t \tag{340}$$

- What is the *circumference of an ellipse?* If we try to carry out this calculation, we can parametize the ellipse as before:

$$x = a\sin\theta \tag{341}$$
$$y = b\cos\theta \tag{342}$$

with $0 \leq \theta \leq 2\pi$ and the arclength as:

$$s = \int_0^{2\pi} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,\mathrm{d}\theta \tag{343}$$

$$= \int_0^{2\pi} \sqrt{a^2(1-\sin 2\theta) + b^2\sin 2\theta}\,\mathrm{d}\theta \tag{344}$$

$$= \int_0^{2\pi} a\sqrt{1-\epsilon^2\sin^2\theta} \tag{345}$$

with $\epsilon \equiv \sqrt{\dfrac{a^2-b^2}{a^2}}$. Unfortunately, this is an elliptic integral of the second kind and has no analytic solution.

11 Polar Coordinates

- In polar coordinates, we can represent polar coordinates in terms of the distance r from the origin and the angle it makes with the positive horizontal axis:
$$(x, y) \iff [r, \theta] \tag{346}$$
- The magnitude of r gives the distance from the origin, and multiplying r by -1 rotates the point about the origin by π.
- Polar coordinates are not unique:
  - The pole is $[0, \theta]$ for all θ.
  - $[r, \theta] = [r, \theta + 2n\pi]$ for any integer n.
  - $[r, \theta] = [-r, \theta + (2n+1)\pi]$ for any integer n.
- We can convert between cartesian and polar coordinates using the transformation:

$$x = r\cos\theta \tag{347}$$

$$y = r\sin\theta \tag{348}$$

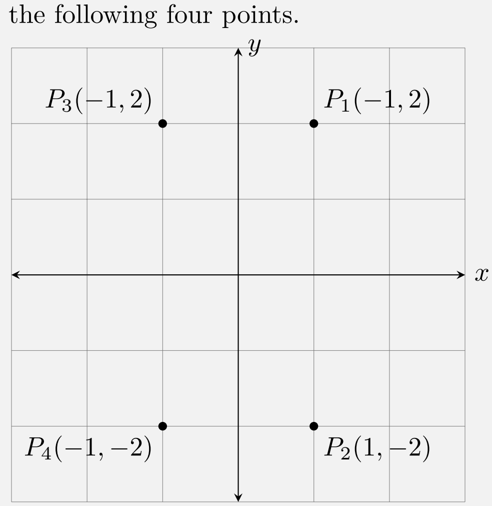
which gives:

$$r = \sqrt{x^2+y^2} \tag{349}$$

$$\theta = \arctan\left(\frac{y}{x}\right) \tag{350}$$

for $x \neq 0$.

Example 48: Suppose we have the following four points.



We can represent the four coordinates also as:

$$P_1[\sqrt{5}, 1.107] \tag{351}$$
$$P_2[\sqrt{5}, -1.107] \tag{352}$$
$$P_3[-\sqrt{5}, -1.107] \tag{353}$$
$$P_4[-\sqrt{5}, 1.107] \tag{354}$$

- We can represent straight lines as:
  - Straight lines: $y = mx$: $r = \alpha$ with $\alpha = \arctan(m)$.
  - Vertical lines $x = a$: We have $r\cos\theta = a \implies r = a\sec\theta$.
  - Horizontal lines: $y = b$. We have $r\sin\theta = b \implies r = b\csc\theta$
- We can represent circles in polar coordinates as:

$$x^2 + y^2 = 9 \iff r = 3 \tag{355}$$

- Converting *from* polar coordinates requires a bit of extra work. Suppose we have $r = 6\sin\theta$, then:

$$r^2 = 6r\sin\theta \tag{356}$$
$$x^2 + y^2 = 6y \tag{357}$$
$$x^2 + y^2 - 6y + 9 = 9 \tag{358}$$
$$x^2 + (y-3)^2 = 9 \tag{359}$$

which represents a circle with radius 3 centered at $(0, 3)$.

- Symmetry can also arise in many scenarios. For example:
  - Symmetry about x axis: $[r_1, \theta]$ and $[r_1, -\theta]$.
  - Symmetry about y axis: $[r, \pi - \theta]$ and $[r_1, \theta]$.
  - Symmetry about origin: $[r, \theta]$ and $[r, \theta + \pi]$.

which will help when sketching them.

Example 49: Suppose we wish to sketch the curve $r = \frac{1}{2} + \cos\theta$. Notice that this is periodic so we only need to look at values of θ where $0 \le \theta < 2\pi$.

1. Let us first find values of θ (if possible) that make $r = 0$:

$$0 = \frac{1}{2} + \cos\theta \implies \theta = \frac{2\pi}{3}, \frac{4\pi}{3} \tag{360}$$

2. Find local max and min values of r:

$$\frac{dr}{d\theta} = -\sin\theta = 0 \implies \theta = 0, \pi \tag{361}$$

At $\theta = 0$, we have $r = \frac{3}{2}$, at $\theta = \pi$ we have $r = -\frac{1}{2}$. If $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, then $r = \frac{1}{2}$.

3. We then loko at symmetry. Notice that:

$$\frac{1}{2} + \cos(-\theta) + \frac{1}{2} + \cos\theta \tag{362}$$

However:

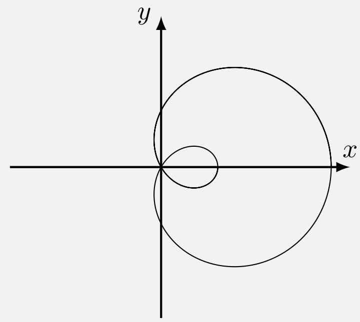
$$r(\theta - \pi) \neq r(\theta) \tag{363}$$
$$r(\pi + \theta) \neq r(\theta) \tag{364}$$

so it is not symmetryic about the y axis or origin.

4. We now look at the relevant intervals. From $0 \le \theta < \frac{2\pi}{3}$, we have $\frac{dr}{d\theta} < 0$ so the radius is monotonically decreasing.

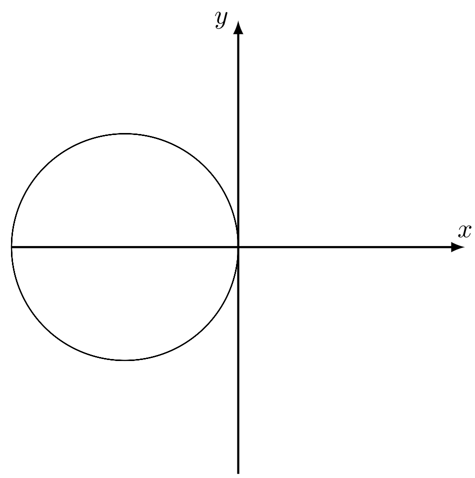
   We can also look at the interval $\frac{2\pi}{3} \le \theta < \pi$ and see that r is negative and $\frac{dr}{d\theta} < 0$, so the manigutde of r *increases.*

   Noting that the function is differentiable at $\theta = \pi$, we can reflect the shape about the x axis to get a Limacon with inner loop.



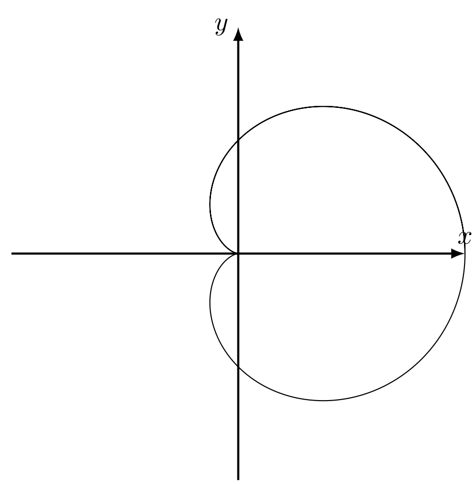
- There are a few common shapes. Each of these could be flipped or rotated by shifting the argument θ, or using negative numbers.
  - Circles:

$$r = -2\cos\theta \tag{365}$$



  - Cardioids:

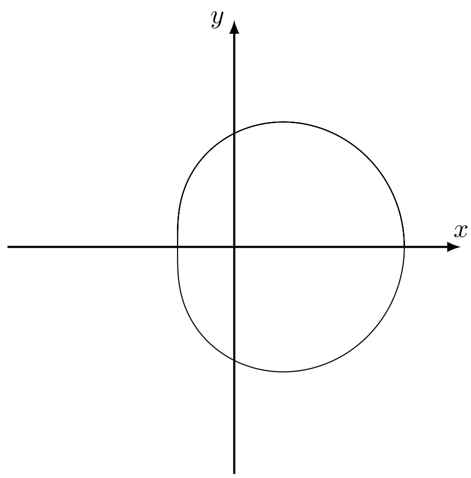
$$r = a + a\cos\theta \tag{366}$$



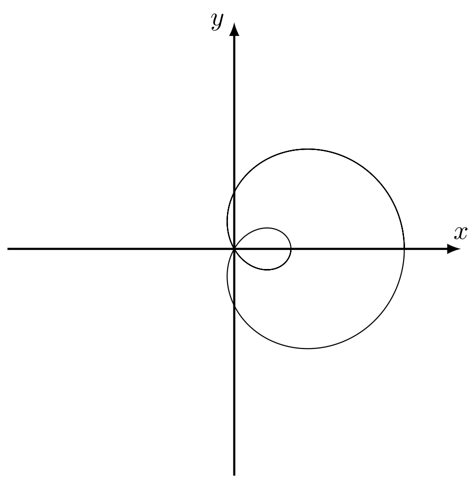
- Limacons:

$$r = a + b\sin\theta \tag{367}$$

  There are two types, for $a > b$:

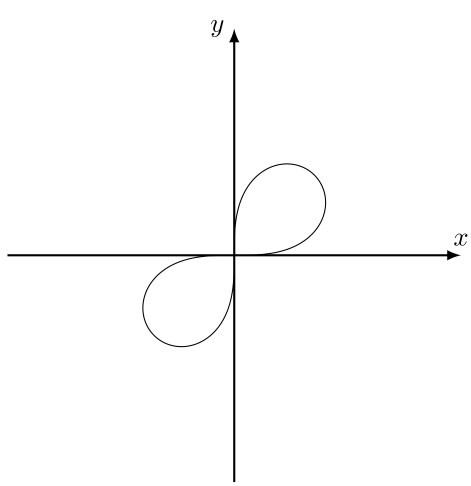


  For $a < b$:



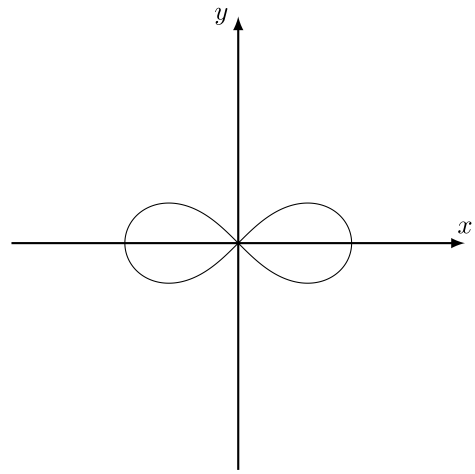
- Leminiscates. Again, there are two types. For:

$$r^2 = a\sin(2\theta) \tag{368}$$



  and:

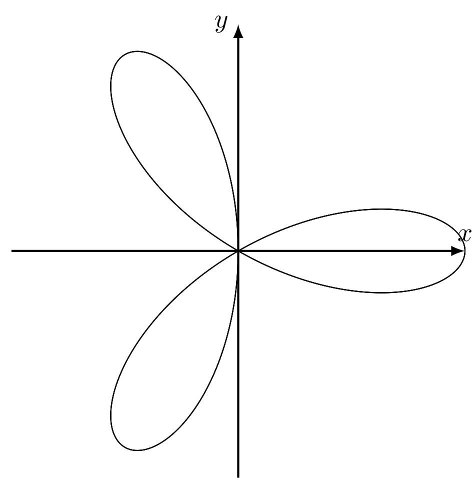
$$r^2 = a\cos(2\theta) \tag{369}$$



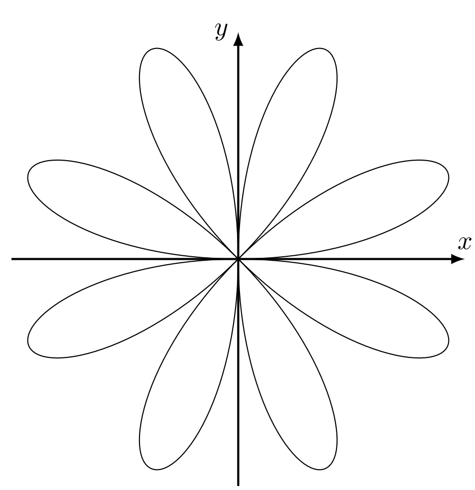
- Petal curves:

$$r = a\sin(n\theta) \tag{370}$$
$$r = a\cos(n\theta) \tag{371}$$

 where n is an integer. There are n petals if n is odd and $2n$ petals if n is even. For example, the following is: $r = 2\cos 3\theta$:



 and for $r = 2\sin(4\theta)$:

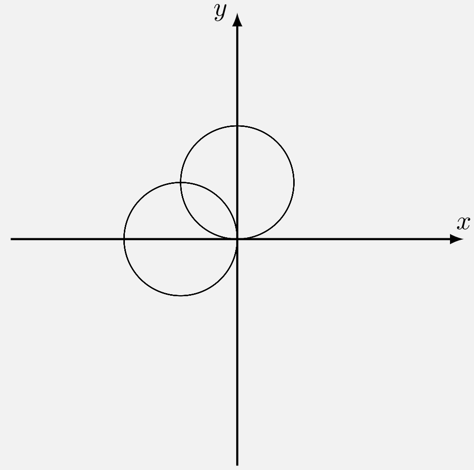


- We can also find the intersection of polar coordinates, but we have to be careful. The following example illustrates why.

Example 50: Suppose we have two curves $r = \sin\theta$ and $r = -\cos\theta$. Suppose we try to solve this via:

$$\sin\theta = -\cos\theta \implies \theta = \frac{3\pi}{4}, \frac{7\pi}{4} \tag{372}$$

Plugging this back into $x = r\cos\theta$ and $y = r\sin\theta$, we get $x = -\frac{1}{2}$ and $y = -\frac{1}{2}$. We can also use $\theta = \frac{7\pi}{4}$ to get: $x = -\frac{1}{2}$ and $y = \frac{1}{2}$ which is the same point. However, it represents the curves below:



There is actually two intersection points! The reason for this is that we assumed that the two curves intersect at the same value of θ, but this is not necessarily true for the origin, which can be obtained at any angle θ.

- As a result, we also have to check the origin.
- It is also possible to find the tangent:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \tag{373}$$

$$= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \tag{374}$$

12 Areas and Lengths in Polar Coordinates

- Suppose we have a polar curve $r = \rho(\theta)$ for $\alpha \leq \theta \leq \beta$.
- We can determine the area by partitioning the curve into θ_i and approximating each subregion as a circular segment. The area of a circular segment is:

$$A = \frac{1}{2}a^2\Delta\theta \tag{375}$$

We can take the radius to be $r = \rho(\theta^*)$ where $\theta_{i-1} \leq \theta^* \leq \theta_i$. The area of each region is:

$$A_i = \frac{1}{2}\rho(\theta_i^*)^2\Delta\theta_i \tag{376}$$

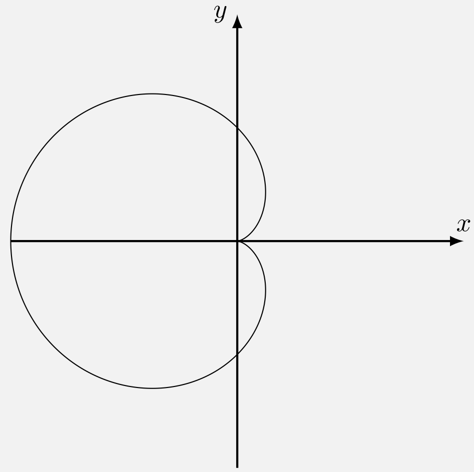
so the total area is:

$$A = \lim_{||P||}\sum_i \frac{1}{2}\rho(\theta_i^*)^2\Delta\theta_i \tag{377}$$

or

$$A = \frac{1}{2}\int_\alpha^\beta \rho(\theta)^2 \, \mathrm{d}\theta \tag{378}$$

Example 51: Suppose we wish to find the area of $r = 1 - \cos\theta$.



The area is then:

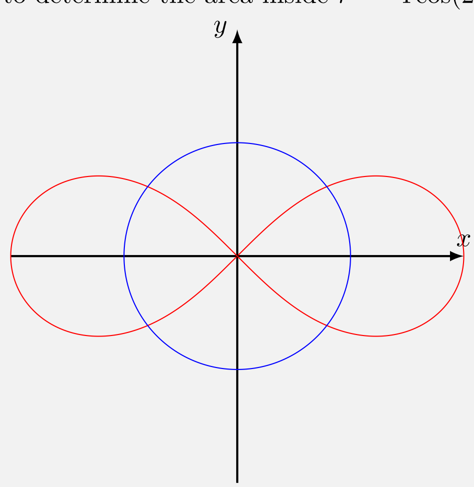
$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} (1-\cos\theta)^2 \,\mathrm{d}\theta && (379) \\ &= \frac{1}{2}\int_0^{2\pi} (1-2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta && (380) \\ &= \frac{1}{2}\int_0^{2\pi} \left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos(2\theta)\right)\mathrm{d}\theta && (381) \\ &= \frac{1}{2}\left(\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin(2\theta)\right)\Bigg|_0^{2\pi} && (382) \\ &= \frac{3}{2}\pi && (383) \end{aligned}$$

- We can also find the area between two polar curves ρ_1 and ρ_2. We have:

$$A = \frac{1}{2}\int_\alpha^\beta \rho_1(\theta)^2\,\mathrm{d}\theta - \frac{1}{2}\int_\alpha^\beta \rho_2(\theta)^2\,\mathrm{d}\theta = \frac{1}{2}\int_\alpha^\beta (\rho_1^2-\rho_2^2)\,\mathrm{d}\theta \qquad (384)$$

Warning: Be careful when applying this formula as it is possible the two functions can overlap between $\alpha \le \theta \le \beta$. Therefore, we always need a good idea of what's happening.

Example 52: Suppose we want to determine the area inside $r^2 = 4\cos(2\theta)$ but outside $r = 1$. This gives:



We first find the four points of intersection:

$$4\cos(2\theta) = 1 \implies \cos(2\theta) = \frac{1}{4} \implies \theta = \pm 0.659 \tag{385}$$

or $\theta = \pi \pm 0.659$. Due to the symmetry, we only need to find the area of one half of the area we are interested in, which gives:

$$\frac{1}{2}A = \frac{1}{2}\int_{-0.659}^{0.659} (4\cos 2\theta - 1)\,\mathrm{d}\theta \tag{386}$$

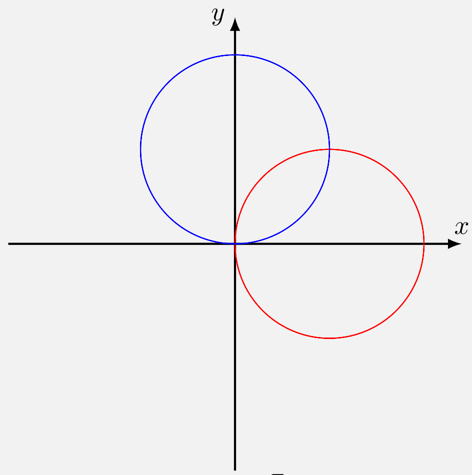
$$= \frac{1}{2}(2\sin 2\theta - \theta)\Big|_{-0.659}^{0.659} \tag{387}$$

$$= 1.277 \tag{388}$$

so the area is $A = 2.554$.

- There are a few challenging examples:

Example 53: Suppose we wish to find the area between $r = \sin\theta$ and $r = \cos\theta$:



We know from symmetry that the intersection is at $\theta = \dfrac{\pi}{4}$. We notice that the contribution to the area from each curve ρ is equal and *independent* from each other. Therefore:

$$A = A_1 + A_2 = \int_0^{\pi/4} \frac{1}{2}\sin^2\theta\,\mathrm{d}\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}\cos^2\theta\,\mathrm{d}\theta = \frac{\pi}{8} - \frac{1}{4} \tag{389}$$

- We can determine the arclength by working in parametric form. Let $x = r(\theta)\cos\theta$ and $y = r(\theta)\sin\theta$. Therefore:

$$s = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,\mathrm{d}\theta \tag{390}$$

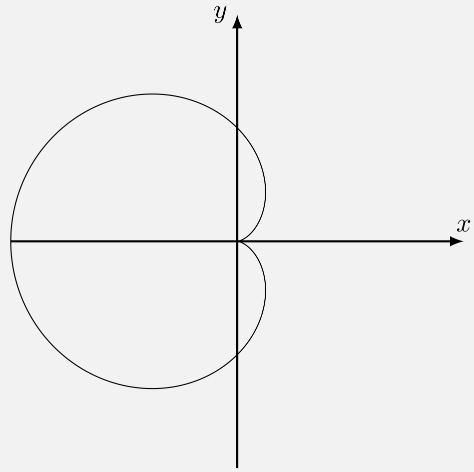
$$= \int_\alpha^\beta \sqrt{(r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2}\,\mathrm{d}\theta \tag{391}$$

$$= \int_\alpha^\beta \sqrt{(r'^2\cos^2\theta + r^2\sin^2\theta - \cancel{2rr'\cos\theta\sin\theta}) + (r'^2\sin^2\theta + r^2\cos^2\theta + \cancel{2r'r\cos\theta\sin\theta})}\,\mathrm{d}\theta \tag{392}$$

$$= \int_\alpha^\beta \sqrt{r^2(\cos^2\theta + \sin^2\theta) + r'^2(\cos^2\theta + \sin^2\theta)} \tag{393}$$

$$= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,\mathrm{d}\theta \tag{394}$$

Example 54: Suppose we want to find the arclength of $r = a(1-\cos\theta)$ from $0 \le \theta < 2\pi$. This looks like:



We have:

$$s = \int_0^{2\pi} \sqrt{r^2 + (r')^2}\,\mathrm{d}\theta \tag{395}$$
$$= \int_0^{2\pi} \sqrt{a^2(1-2\cos\theta+\cos^2\theta) + a^2\sin^2\theta}\,\mathrm{d}\theta \tag{396}$$
$$= a\int_0^{2\pi} \sqrt{2-2\cos\theta}\,\mathrm{d}\theta \tag{397}$$
$$= a\int_0^{2\pi} \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\,\mathrm{d}\theta \tag{398}$$
$$= 2a\left[-2\cos\left(\frac{\theta}{2}\right)\right]\Bigg|_0^{2\pi} \tag{399}$$
$$= 8a \tag{400}$$

13 Infinite Sequences and Series

- We use curly brackets to indicate a sequence, such as:

$$f(n) = \frac{1}{n} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\} \tag{401}$$

Alternatively, we can use a_n to represent a sequence.

Definition: A sequence $\{a_n\}$ is:
- increasing iff $a_n < a_{n+1}$
- non-decreasing iff $a_k \le a_{n+1}$
- decreasing iff $a_n > a_{n+1}$
- non-increasing iff $a_n \ge a_{n+1}$

A function that satisfies any of these are known as **monotonic** functions.

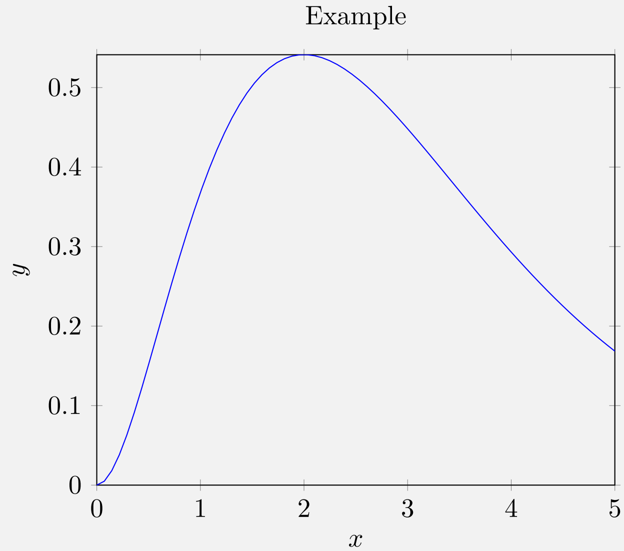
- Bounded functions have an upper or lower bound, while unbounded functions diverge to infinity or negative infinity.

Example 55: Suppose we wish to prove that 2^k is unbounded. We wish to find k such that $a_k > M$ or $2^k > M$. Taking the natural logarithm of both sides, we have:

$$k > \frac{\ln M}{\ln 2} \tag{402}$$

which is possible to do and we are done.

Example 56: Suppose we wish to find if $a_n = \frac{n^2}{e^n}$ is bounded or unbounded. This can be approached by working with derivatives through the function $f(x) = \frac{x^2}{e^x}$, represented in the following plot:



Taking the derivative $f'(x) = xe^{-x}(2-x)$, we see that f decreases for $x > 2$ so this means that a_n decreases for $n > 2$

Warning: Not everything in functions carries over to sequences. For example, $f(x) = \frac{1}{x-\sqrt{2}}$ is unbounded but $a_n = \frac{1}{n-\sqrt{2}}$ is bounded since $n \neq \sqrt{2}$ is impossible.

- We can only take the limit of a sequence as $n \to \infty$.

Definition: We can define $\lim_{n\to\infty} a_n = L$ iff for every $\epsilon > 0$, there exists an integer $k > 0$ such that if $n \geq k$, then $|a_n - L| < \epsilon$.

Example 57: Let us prove $\lim_{n\to\infty} \frac{n}{n+1} = 1$. We find k such that $\left|\frac{n}{n+1} - 1\right| < \epsilon$ for $n \geq k$. This can be rewritten as:

$$\left|\frac{1}{n+1}\right| < \epsilon \tag{403}$$

or $|n+1| > \frac{1}{\epsilon}$. Thus, if we choose $k = \frac{1}{\epsilon}$ such that if we choose $n > k = \frac{1}{\epsilon}$, then:

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{1}{n+1}\right| < \left|\frac{1}{n}\right| < \frac{1}{k} = \epsilon \tag{404}$$

Therefore, $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Theorem: Uniqueness of a Limit: If $\lim_{n\to\infty} = L$ and $\lim_{n\to\infty} a_n = M$, then $L = M$.

Definition: If a sequence has a limit, it is said to be convergent. Otherwise, it is divergent.

- This leads to the following:
  1. If a sequence is convergent, it is bounded.
  2. If a sequence is unbounded, it is divergent.
  3. A bounded sequence is not necessarily convergent.
- For example, $a_n = \cos \pi n$ is bounded but not convergent.

Theorem: **Monotonic Sequence Theorem:** A bounded nondecreasing sequence converges to its least upper bound. A bounded non increasing sequence converges to its greatest lower bound.

- The limit has a few properties. Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Then:
  1. $\lim_{n\to\infty}(a_n + b_n) = L + M$
  2. $\lim_{n\to\infty} \alpha a_n = \alpha L$ for $\alpha \in \mathbb{R}$.
  3. $\lim_{n\to\infty} a_n b_n = L \cdot M$
  4. $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{M}$ for $b_n \neq 0, M \neq 0$.
  5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ for $b_n \neq 0, M \neq 0$.

Theorem: **Pinching Theorem for Sequences:** If for large n, $a_n \leq b_n \leq c_n$ and if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} = L$, then $\lim_{n\to\infty} b_n = L$.

Example 58: Suppose we wish to find the limit $\lim_{n\to\infty} \frac{\sin(n\pi/6)}{n}$. We can let:

$$-\frac{1}{n} \leq \frac{\sin(n\pi/6)}{n} \leq \frac{1}{n} \tag{405}$$

Since $\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$, then the original limit must also be zero.

Theorem: Supose we have a sequence: $c_n = g(f_n)$. Given $\lim_{n\to\infty} c_n = C$. If f is continuous at c, in the traditional way, then: $\lim_{n\to\infty} f(c_n) = f(c)$.

Example 59: Let us look at the function $\sin\left(\frac{1}{n^2+1}\right)$. We know that $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$, so:

$$\lim_{n\to\infty} \sin\left(\frac{1}{n^2+1}\right) = \sin(0) = 0 \tag{406}$$

where we have applied the previous theorem.

14 Sequences

- We begin with some **important limits**:

- For $x > 0$, $\lim_{n\to\infty} x^{1/n} = 1$.
- If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

  Proof. We know the function is decreasing: $|x|^{n+1} = |x||x^n| < |x|^n$. Alternatively, we want to show taht $|x^k - 0| < \epsilon$ for all $n > k$. We want to find k such that:

  $$|x^k = 0| = |x^n| = |x|^k < \epsilon \tag{407}$$

  or: $|x| < \epsilon^{1/n}$. We know that:

  $$\lim_{n\to\infty} \epsilon^{1/k} = 1 \tag{408}$$

  and since $|x| < \epsilon^{1/k}$, we must have $|x^n| < \epsilon$ for all $n > k$. □

- For $\alpha > 0$, $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$.

  Proof. Note that:

  $$0 < \frac{1}{n^\alpha} = \left(\frac{1}{n}\right)^\alpha \tag{409}$$

  We can pick an odd positive integer p such that $1/p < \alpha$ such that:

  $$\left(\frac{1}{n}\right)^\alpha \le \left(\frac{1}{n}\right)^{1/p} \implies \lim_{n\to\infty}\left(\frac{1}{n}\right)^{1/p} = \left(\lim_{n\to\infty}\frac{1}{n}\right)^{1/p} = 0 \tag{410}$$

  □

- $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for $x \in \mathbb{R}$.
- $\lim_{n\to\infty} \frac{n!}{n^n} = 0$
- $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.
- $\lim_{n\to\infty} n^{1/n} = 1$.
- $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

  Proof. First let's deal with the $x = 0$ case, which is trivial. Now:

  $$\ln\left(1 + \frac{x}{n}\right)^n = n\ln\left(1 + \frac{x}{n}\right) \tag{411}$$
  $$= \frac{x\ln(1 + x/n)}{x/n} \tag{412}$$
  $$= x\left(\frac{\ln(1 + x/n) - \ln(1)}{x/n}\right) \tag{413}$$

  Taking the limit, we have:

  $$\lim_{n\to\infty} \frac{\ln(1 + x/h) - \ln 1}{x/h} = \lim_{h\to 0} \frac{\ln(1 + h) - \ln 1}{h} \tag{414}$$

  which is the first principles definition of the derivative of $\ln(x)$ at $x = 1$, which gives:

  $$\lim_{h\to\infty} \ln(1 + x/n)^n = x \cdot 1 = x \implies \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x \tag{415}$$

  □

- Sequences can also be defined recursively. We need a base term, e.g. $a_1 = 1$ and also a general relationship, such as:

  $$a_n = \sqrt{6 + a_{n-1}} \tag{416}$$

  this gives the sequence $\{1, \sqrt{7}, \sqrt{6 + \sqrt{7}}, \dots\}$

- How do we find the **limit** of such a recursively defined function? To do so, we first need to show that the limit actually exists. To do so, we must have both:

$$\lim_{n\to\infty} a_n = L \tag{417}$$
$$\lim_{n\to\infty} a_{n-1} = L \tag{418}$$

Therefore, we get:

$$L = \sqrt{6+L} \implies L = 3, -2 \tag{419}$$

Since it is increasing, we must have $L = 3$.

15 Series

- Suppose we wish to add the infinite series:

$$I = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \tag{420}$$

- We can define the partial sum to be:

$$s_0 = a_0 = \sum_{k=0}^{0} a_n \tag{421}$$
$$s_1 = a_0 + a_1 = \sum_{k=0}^{1} a_n \tag{422}$$
$$s_2 = a_0 + a_1 + a_2 = \sum_{k=0}^{2} a_n \tag{423}$$
$$\vdots \tag{424}$$
$$s_n = a_0 + a_1 + \cdots + a_n = \sum_{k=0}^{n} a_k \tag{425}$$

- We can then consider the sequence $\{s_n\} = \{a_0, a_0 + a_2, a_0 + a_1 + a_2, \cdots\}$. This sequence converges if the sum converges. Specifically, if $\lim\limits_{n\to\infty} \{s_n\} = L$, then $\sum\limits_{k=0}^{\infty} a_n = L$.

Example 60: Suppose we wish to evaluate:

$$\sum_{k=0}^{\infty} \frac{1}{(k+2)(k+3)} = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} \tag{426}$$

We can use partial fractions to write:

$$\frac{1}{(k+2)(k+3)} = \frac{1}{k+2} - \frac{1}{k+3} \tag{427}$$

so the sum becomes:

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} + \cdots - \frac{1}{n+3} = \frac{1}{2} - \frac{1}{n+3} \tag{428}$$

which is known as a telescoping sequence. Taking the limit as $n \to \infty$, we get that the sum converges to $\frac{1}{2}$.

- The sum of a geometric series is:

$$x^0 + x^1 + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \tag{429}$$

which converges when $|x| < 1$.

Proof. Let $S_n = 1 + x + x^2 + \cdots + x^n$ and $xS_n = x + x^2 + x^3 + \cdots + x^{n+1}$. Then subtracting the two, we get:

$$S_n - xS_n = 1 - x^{n+1} \implies S_n = \frac{1 - x^{n+1}}{1 - x} \tag{430}$$

and for $|x| < 1$, the limit gives us $\dfrac{1}{1-x}$ and if $|x| > 1$, the limit diverges. $\square$

- Suppose we wish to write the repeating fraction as a decimal: $0.\overline{285714}$. This is equal to:

$$= \frac{28574}{10^6} + \frac{285714}{10^12} + \cdots \tag{431}$$

$$= \frac{28574}{10^6}\left(1 + \frac{1}{10^6} + \frac{1}{10^{12}} + \cdots\right) \tag{432}$$

Evaluating this infinite series, we get:

$$\frac{2}{7} \tag{433}$$

Example 61: Suppose we wish to write out $\dfrac{x}{4-x^2}$ as a sum for $|x| < 2$. We have:

$$\frac{x}{4-x^2} = \frac{x}{4}\left(\frac{1}{1-x^2/4}\right) \tag{434}$$

$$= \frac{x}{4}\sum_{k=0}^{\infty}\left(\frac{x^2}{4}\right)^k \tag{435}$$

$$= \frac{x}{4}\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k} \tag{436}$$

$$= \frac{1}{2}\left[\frac{x}{2} + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^5 + \cdots\right] \tag{437}$$

Theorem: Here are a few important properties that arise when applying limit laws:

– If $\sum_{k=0}^{\infty} a_k = n$ and $\sum_{k=0}^{\infty} b_k = M$, then $\sum_{k=0}^{\infty}(a_k + b_k) = L + M$.

– If $\sum_{k=0}^{\infty} a_k = L$, then $\sum_{k=0}^{\infty} \alpha a_k = \alpha L$ for $\alpha \in \mathbb{R}$.

Theorem: If $\sum_{k=0}^{\infty} a_k$ converges iff $\sum_{k=j}^{\infty} a_k$ converges where j is a positive integer.

Example 62: Suppose we are given that $\sum_{k=4}^{\infty} \frac{3^{k-1}}{3^{3k+1}}$ converges, then $\sum_{k=0}^{\infty} \frac{3^{k-1}}{3^{3k+1}}$ converges.

Theorem: If $\sum_{k=0}^{\infty} a_k$ converges, then $a_k \to 0$ as $k \to \infty 0$.

Theorem: **(Test for Divergence:)** This is the contrapositive of the previous theorem. If $a_k \not\to 0$ as $k \to \infty$, then $\sum_{k=0}^{\infty} a_k$ diverges.

16 Convergence Tests

- We start with the integral test:

Theorem: If f is continuous, decreasing, and positive on $[1, \infty)$, then: $\sum_{k=1}^{\infty} f(k)$ converges if and only if $\int_1^{\infty} f(x)\,\mathrm{d}x$ converges.

Example 63: Suppose we take the harmonic sum:

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \tag{438}$$

However, the integral $\int_1^{\infty} \frac{\mathrm{d}x}{x} = \lim_{b\to\infty} \ln b$ diverges.

- The p-series is:

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \tag{439}$$

which will converge if $p > 1$ since $\int_1^{\infty} \frac{\mathrm{d}x}{x^p}$ converges iff $p > 1$.

Example 64: Suppose we wish to look at $\sum_{n=5}^{\infty} \frac{1}{n^2+9}$. First, we notice that:

$$\lim_{t\to\infty} \int_5^t \frac{\mathrm{d}x}{x^2+9} = \lim_{t\to\infty} \left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]\Bigg|_5^t \tag{440}$$

which converges, and after checking the relevant conditions, this means that the sum converges too.

Definition: The **remainder** for a sequence $\{f_n\}$ is given as:

$$R_n = f(n) - f_n \tag{441}$$

where $f(n)$ denotes a continuous functoin while f_n is discrete.

- For a decreasing function, $R_n \leq \int_n^{\infty} f(x)\,\mathrm{d}x$ and $R_n \geq \int_{n+1}^{\infty} f(x)\,\mathrm{d}x$. This means that:

$$S_n + \int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq S \leq S_n + \int_n^{\infty} f(x)\,\mathrm{d}x \tag{442}$$

- We can also use the **comparison test**

Theorem: Given $\sum a_k$ and $\sum b_k$ with $a_k > 0$ and $b_k > 0$:

1. If $\sum b_k$ is convergent, and if $a_k \leq b_k$ for all sufficiently large k, then $\sum a_k$ converges.
2. If $\sum b_k$ is diverge and $a_k > b_k$ for all k sufficiently large, then $\sum a_k$ diverges.

Proof. Assume $a_k \leq b_k$ for all k we can define:

$$S_n = \sum_{k=1}^{n} a_k \tag{443}$$

as the sequence of partial sums where:

$$b_k = \sum_{k=1}^{n} b_k \tag{444}$$

where $t = \sum_{k=1}^{\infty} b_k$ exists. This implies that $\{S_n\}$ is increasing since $a_k > 0$ and so:

$$S_n \leq t_n < t \tag{445}$$

where $\{S_n\}$ is a bounded sequence. By the monotonic sequence theorem, $\{S_n\}$ has a limit and $\sum_{k=1}^{\infty} a_k$ is defined to be equal to that limit. Therefore, $\sum a_k$ converges. □

Example 65: Suppose we wish to determine if $\sum_{n=1}^{\infty} \frac{7}{17n^2 + 3\sqrt{n} + 5}$ converges. Notice that for $n \geq 1$, we have:

$$17n^2 + 3\sqrt{n} + 5 > 17n^2 \tag{446}$$

and so:

$$\frac{7}{17n^2 + 3\sqrt{n} + 5} < \frac{7}{17n^2} \tag{447}$$

Since $\frac{7}{17} \sum \frac{1}{n^2}$ converges, then the original sum must also converge.

Example 66: Suppose we wish to determine if $\sum_{k=1}^{\infty} \frac{\ln(n/1000)}{n}$ converges. We want to find a k such that:

$$\frac{\ln(k/1000)}{k} > \frac{1}{k} \tag{448}$$

wihch means that we want to pick $k > 1000e > 2718$. Therefore, since $\sum_{k=2719}^{\infty} \frac{1}{k}$ is divergent, then the original sum is also divergent.

- Suppose we wish to determine if $\sum n = 2^{\infty}$ converges. This looks like $\frac{1}{n^3}$, but we notice that:

$$\frac{1}{n^3 - n} > \frac{1}{n^3} \tag{449}$$

and we run into trouble. This means we have to turn to the **limit comparison test**

Theorem: The limit comparison test: Given $\sum a_k$, $\sum b_k$ where $a_k > 0$ and $b_k > 0$:

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then both series converge or diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and if $\sum b_n$ converges, then $\sum_{a_n}$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and if $\sum b_n$ diverges, then $\sum a_n$ diverges.

Proof. We are given that:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \implies \left|\frac{a_n}{b_n} - c\right| < \epsilon \tag{450}$$

for $n < N$. We are working *backwards* here, so we are free to choose *any* value of ϵ and this will hold true. We can choose $\epsilon = \frac{c}{2}$ such that:

$$\frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2} \tag{451}$$

which gives:

$$\frac{c}{2}b_n < a_n < \frac{3c}{2}b_n \tag{452}$$

with $n > N$. If $\sum b_n$ converges, so does $\frac{3c}{2}\sum b_n$ since c is just a number. Thereforfe, $\sum a_n$ converges by the comparison test. If $\sum b_n$ diverges, so does $\frac{c}{2}\sum b_n$ and again by the comparison test, $\sum a_n$ diverges. □

Example 67: We continue our previous discussion of $\sum_{n=2}^{\infty}\frac{1}{n^3-n}$. We let $a_n = \frac{1}{n^3-n}$ and $b_n = \frac{1}{n^3}$. Both a_n and b_n are convergent and:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{1-\frac{1}{n^2}} = 1 > 0 \tag{453}$$

so the original sequence is convergent.

Example 68: We can also revisit $\sum \frac{\ln(n/1000)}{n}$. We consider $a_n = \frac{\ln(n/1000)}{n}$ and $b_n = \frac{1}{n}$. Since $\sum b_n$ diverges and $\lim_{n\to\infty}\frac{a_n}{b_n}$ diverges too, then $\sum a_n$ diverges as well.

17 Alternating Series

- Some series have both positive and negative terms, such as:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \tag{454}$$

Theorem: The **Alternating Series Test**: Let $\{a_k\}$ be a sequence of positive numbers. If and only if $a_{k+1} < a_k$ and $a_k \to 0$ as $k \to \infty$, then:

$$\sum_{k=1}^{\infty}(-1)^{k-1}a_k \tag{455}$$

converges.

Proof. Let $S_2 = a_1 - a_2 > 0$ and $s_4 = s_2 + (a_3 - a_4)$. We can generalize this to:

$$S_{2n} = S_{2n-2} + (a_{2n-1} - a_{2n}) > S_{2n-2} \tag{456}$$

such that $\{S_{2n}\}$ is monotonically increasing. However, we also have:

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \tag{457}$$

Since $S_{2n} < a_1$ for all n, we can apply the monotonic limit theorem to show that the limit L exists. We then have:

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1} = L \tag{458}$$

□

Example 69: Take the sum $1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{9} + \frac{1}{3} - \cdots$. Although $a_n \to 0$, the terms are not decreasing in mangitude, so it is divergent.

- For an alternating sequence, the limit will be between S_n and S_{n+1} so we can estimate the error as:

$$|L - S_n| \le a_{n+1} \tag{459}$$

- For example, the series expansion for e^{-1} is:

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots \tag{460}$$

If we continue to the $\frac{1}{5!}$ term, then we get:

$$e^{-1} \simeq 0.3666 \pm \frac{1}{6!} \tag{461}$$

- We introduce the absolute convergence and the ratio and root tests.

Definition: If $\sum |a_k|$ converges, we say that $\sum a_k$ is absolutely convergent. If $\sum a_k$ converges, but $\sum |a_k|$ does not, we say $\sum a_k$ is conditionally convergent.

Theorem: If $\sum |a_k|$ converges, then $\sum a_k$ converges.

Proof. Let:

$$-|a_n| \leq a_n \leq |a_n| \tag{462}$$

$$0 \leq a_n + |a_n| \leq 2|a_n| \tag{463}$$

$$0 \leq b_n \leq 2|a_n| \tag{464}$$

Note: Let $\sum a_n = \sum b_n - \sum |a_n|$. Since both $\sum b_n$ and $\sum |a_n|$ is convergent, then the original sum must be convergent as well. $\square$

- For example, $\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}$ is conditionally convergent.

Theorem: The **Root Test:** Given $\sum a_k$, $a_k \geq 0$. If $(a_k)^{1/k} \to p$ as $k \to \infty$, then:

1. If $p < 1$, then $\sum a_k$ converges.
2. If $p > 1$, then $\sum a_k$ diverges.
3. If $p = 1$ the test is inconclusive.

Proof. Given $p < 1$, choose μ such that $p < \mu < 1$. Since $(a_k)^{1/k} \to p$, we have:

$$(a_k)^{1/k} < \mu \tag{465}$$

or

$$a_k < \mu^k \tag{466}$$

for k sufficiently large. But $\sum \mu^k$ converges (geometric series, $x < 1$), so $\sum a_k$ converges as well. $\square$

Example 70: Take the series $\sum \left(\frac{n^2+1}{2n^1+1}\right)^n$. Note that $a_n^{1/k} = \frac{2k}{k+1} \to \frac{1}{2}$ so the series is convergent.

Theorem: The **ratio test:** GTiven $\sum a_k$, with $a_k > 0$. If $\frac{a_{k+1}}{a_k} \to \lambda$ as $k \to \infty$, then:

1. If $\lambda < 1$, $\sum a_k$ converges.
2. If $\lambda > 1$, $\sum a_k$ diverges.
3. If $\lambda = 1$, the test is inconclusive.

Proof. Given $\lambda < 1$, we can choose μ such that $\lambda < \mu < 1$. Thus:

$$\frac{a_{k+1}}{a_k} < \mu \tag{467}$$

for k sufficiently large, say $k > K$. We have:

$$a_{K+1} < \mu a_K \tag{468}$$
$$a_{K+2} < \mu a_{K+1} < \mu^2 a_K \tag{469}$$
$$\vdots \tag{470}$$
$$a_{K+j} < \mu^j a_K \tag{471}$$

for $j = 1, 2, 3, \ldots$. Let $n = K + j$. Then we can rewrite the last line as:

$$a_n < \mu^{n-k} a_K = \frac{a_K}{\mu^K}\mu^n \tag{472}$$

Since the factor $\dfrac{a_K}{\mu^K}$ is some constant and μ^n converges, then the original sum is convergent. □

Tip: The ratio test is usually the most straightforward and the most useful test to employ.

Example 71: Suppose we take the sum $\sum \dfrac{k^2}{e^k}$. We have:

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2} = \frac{(k+1)^2}{k^2} \cdot \frac{1}{e} \tag{473}$$

As $k \to \infty$, we get $\dfrac{1}{e} < 1$ so the sum is convergent.

18 Power Series

- We can introduce the power series:

Definition: A power series is a series in the form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \tag{474}$$

- For example, if we let $c_n = 1$. Then for all n, we get:

$$\sum_{n=0}^{\infty} x_n = 1 + x + x^2 + \cdots = \frac{1}{1-x} \tag{475}$$

and converges if $|x| < 1$.

- A power series about a can be written as:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \tag{476}$$

- Note that for $x = a$, the sum will always converge. However, we are interested for the entire range of values at which it converges..

Example 72: Suppose we have the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$. To test when it converges, we can apply the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n}\right| \tag{477}$$

$$= |x| \frac{n^2}{n+1}^2 \tag{478}$$

As $n \to \infty$, we get $|x|$. Therefore, the series converges when $|x| < 1$. However, the test says nothing about the endpoints, so we have to test them separately. If $x = 1$, we have:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{479}$$

We can apply a p-series test to show it converges. For $x = -1$, we have:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \tag{480}$$

and apply the alternating series test to show that it converges. Therefore, the power series converges for:

$$-1 \leq x \leq 1 \tag{481}$$

Example 73: Suppose we have the power series $\sum_{n=0}^{\infty} \frac{(1+5^n)x^n}{n!}$. Using the ratio test, we have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(1+5^{n+1})x^{n+1}}{(n+1)!} \cdot \frac{n!}{(1+5^n)x^n}\right| = \frac{1+5^{n+1}}{1+5^n} \cdot \left|\frac{x}{n+1}\right| \tag{482}$$

which approaches 0 as $n \to \infty$ so it is convergent for all $x \in \mathbb{R}$.

Example 74: Take the power series $\sum n!x^n$. The ratio test then gives:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{n!} \cdot \frac{x^{n+1}}{x^n}\right| = (n+1)|x| \tag{483}$$

This approaches ∞ as $n \to \infty$ so it diverges except for $x = 0$.

Theorem: For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are three possibilities with respect to convergence:

1. The series converges only when $x = a$
2. The series converges for all x
3. The series converges in some interval $|x - a| < R$ where R is the **radius of convergence**. However, the endpoints must be tested serparately.

Example 75: Take the power series $\sum_{n=0} \sum^{\infty} \frac{(-2)^n(x-1)^n}{n+2}$. The ratio test gives us:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}(x-1)^{n+1}}{n+3} \cdot \frac{n+2}{2^n(x-1)^n}\right| = 2\left(\frac{n+2}{n+3}\right)|x-1| \tag{484}$$

As $n \to \infty$, we get:

$$|x-1| < \frac{1}{2} \therefore R = \frac{1}{2} \tag{485}$$

We now need to check the endpoints. Test $x = \frac{1}{2}$. We get:

$$\sum_{n=0}^{\infty} \frac{(-2)^n \left(-\frac{1}{2}\right)^n}{n+2} = \sum_{n=0}^{\infty} \frac{1}{n+2} = \sum_{i=2}^{\infty} \frac{1}{i} \tag{486}$$

which diverges as it is the harmonic series. We now need to test $x = \frac{3}{2}$. We then get:

$$\sum_{n=0}^{\infty} \frac{(-2)^n \left(\frac{1}{2}\right)^n}{n+2} = \sum \frac{(-1)^n}{n+2} \tag{487}$$

Using the alternating series test, we see that this converges. Therefore, the interval of convergence is $\left(\frac{1}{2}, \frac{3}{2}\right]$.

- It is possible to represent functions as a power series. We saw that for $|x| < 1$, the infinite series:

$$\sum_{n=0}^{\infty} = 1 + x + x^2 + \cdots = \frac{1}{1-x} \tag{488}$$

If we let $f(x) = \frac{1}{1-x}$, then we can *approximate* it using a truncated power series representation for between $-1 < x < 1$.

Example 76: Suppose we have the function $\frac{x}{x-3}$. If we want to write it as a power series, we can write it as:

$$\begin{aligned}
x \cdot \frac{1}{x-3} &= -x\frac{1}{3-x} && (489)\\
&= -\frac{x}{3}\frac{1}{1-\frac{x}{3}} && (490)\\
&= -\frac{x}{3}\left[1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \cdots\right] && (491)\\
&= -\frac{x}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n && (492)\\
&= -\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^{n+1} && (493)\\
&= -\sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^n && (494)
\end{aligned}$$

and it converges for $|x| < 3$.

Theorem: **Term by Term Differentiation and Integration:** Consider the pwoer series $\sum c_n(x-a)^n$ with $R = R_0 > 0$, then

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n \tag{495}$$

is differentiable and continuous on $(a - R_0, a + R_0)$ and:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \tag{496}$$

We can also take the integral:

$$\int f(x)\,\mathrm{d}x = C + c_0(x-a) + \frac{c_1(x-a)^2}{2} + \frac{c_2(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1} \tag{497}$$

Notice that derivatives and infinite sums can be interchanged. Specifically:

$$\frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty}\frac{d}{dx}c_n(x-a)^n \tag{498}$$

$$\int \sum_{n=0}^{\infty} c_n(x-a)^n\,\mathrm{d}x = \sum_{n=0}^{\infty}\int c_n(x-a)^n\,\mathrm{d}x \tag{499}$$

Warning: The radius of convergence between derivatives will always be the same, but the endpoints may change.

Example 77: Suppose we have the function $f(x) = \frac{1}{(1+x)^2}$. Note that:

$$\frac{d}{dx}\frac{-1}{1+x} = -\frac{1}{(1+x)^2} \tag{500}$$

so we can write it in terms of its derivative:

$$\frac{d}{dx} - \frac{1}{1+x} = \frac{d}{dx}\left[-\sum_{n=0}^{\infty}(-x)^n\right] = \sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1} = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n \tag{501}$$

Example 78: Let's find the power series representation of $\ln(1-x)$. We notice that it can be written as an integral:

$$\ln(1-x) = -\int \frac{\mathrm{d}x}{1-x} = -\int \sum_{n=0}^{\infty} x^n\,\mathrm{d}x = C - \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1} \tag{502}$$

We can determine the constant of integration by setting $x = 0$, which gives $\ln(1) = 0 = C$. Therefore, we can write:

$$\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n} \tag{503}$$

For $x = 1$, this diverges and for $x = -1$, it conditionally converges.

Example 79: Let us attempt to evaluate $\int_0^{0.1}\frac{\mathrm{d}x}{1+x^4}$ to 6 decimal places without a calculator. We first write it as a power series:

$$\frac{1}{1-(-x)^4} = \sum_{n=0}^{\infty}(-x^4)^n = 1 - x^4 - x^8 - \cdots \tag{504}$$

which converges for $|x| < 1$. Therefore, the integral is:

$$\int \frac{\mathrm{d}x}{1+x^4} = \sum_{n=0}^{\infty}\int(-x^4)^n\,\mathrm{d}x \tag{505}$$

$$= C + \sum_{n=0}^{\infty}(-1)^n\frac{x}{4n+1} \tag{506}$$

$$= C + x - \frac{x^5}{5} + \frac{x^9}{9} - \cdots \tag{507}$$

The integral is then:

$$\int_0^{0.1} \frac{\mathrm{d}x}{1+x^4} = 0.1 - \frac{0.1^5}{5} + \frac{0.1^9}{9} - \cdots = 0.099998 \pm 1.1 \times 10^{-10} \tag{508}$$

Example 80: Let us try to write the power series representaiton of the inverse tangent function $f(x) = \tan^{-1}(x)$. Note that:

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \tag{509}$$

We can write $f(x)$ as the integral:

$$\tan^{-1}(x) = \int \frac{\mathrm{d}x}{1+x^2} = \int (1 - x^2 + x^4 - x^6 + \cdots)\,\mathrm{d}x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \tag{510}$$

We can calculate the constant of integration to be $C = \tan^{-1}(0) = 0$ such that we have:

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \tag{511}$$

with a radius of convergence of $R = 1$.

Remarks: If we substitute in $x = 1$, then we can a special series:

$$\tan^{-1}(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{512}$$

and is known as Leibniz's formula for π.

19 Taylor and Maclaurin Series

- Recall that the power series can be written as:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \tag{513}$$

for $|x-a| < R$, we note that $f(a) = c_0$. However, if we take the derivative:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \tag{514}$$

and we similarly get $f'(a) = c_1$. For the second derivative:

$$f''(x) = 2c_2 + 6c_3(x-a) + \cdots \tag{515}$$

we get $f''(a) = 2c_2$.

- In general:

$$f^(n)(a) = n!c_n \tag{516}$$

Theorem: If $f(x)$ has a power series representation about a:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \tag{517}$$

with $|x-a| < R$. Then the coefficients of the series are $c_n = \dfrac{f^{(n)(a)}}{n!}$

- For a Taylor series of f about a, we have:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \cdots \tag{518}$$

- For the Maclaurin Series, it is simply a Taylor series taken at $x = a$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \cdots \tag{519}$$

Definition: A definition is called **analytic at a** if it can be represented as a power series about a.

Example 81: Let us attempt to write out the Maclaurin series of $f(x) = e^x$. First note that:

$$f'(x) = e^x = f''(x) = f'''(x) = f^{(n)}(x) \tag{520}$$

Therefore: $f^{(n)}(0) = e^0 = 1$. Therefore, we can write it as the series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{521}$$

We can check that this converges using the ratio test. Let $a_n = \frac{x^n}{n!}$. Then:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \tag{522}$$

which approaches zero as $n \to \infty$. As a result, $R = \infty$

- We ask ourselves the question: When is it true that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Definition: The nth degree Taylor polynomial of f about a can be written as:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{523}$$

Example 82: Let us take a look at e^x about $a = 0$. Then the first, second, third degree series can be written as:

$$T_1(x) = 1 + x \tag{524}$$

$$T_2(x) = 1 + x + \frac{x^2}{2} \tag{525}$$

$$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \tag{526}$$

We can then define the remainder function as:

$$R_n(x) = f(x) - T_n(x) \tag{527}$$

Theorem: If $f(x) = T_n(x) + R_n(x)$ and $\lim_{n\to\infty} R_n(x) = 0$ for $|x - a| < R$. Then f is equal to the sum of its Taylor series.

Given that f has $n+1$ continuous derivatives on an open interval I containing a, tyhen for all $x \in I$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} + R_n(x) \tag{528}$$

where

$$R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n \, \mathrm{d}t \tag{529}$$

Proof. Consider the fundamental theorem of calculus:

$$\int_a^b f'(t)\,\mathrm{d}t = f(b) - f(a) \tag{530}$$

Suppose we evaluate this via integration by parts:

$$\begin{aligned} u &= f'(t) & \mathrm{d}v &= \mathrm{d}t \\ \mathrm{d}u &= f''(t) & v &= t - b \end{aligned}$$

This gives:

$$\int_a^b f'(t)\,\mathrm{d}t = [f'(t)(t-b)]_a^b - \int_a^b f''(t)(t-b)\,\mathrm{d}t \tag{531}$$

$$= (b-a)f'(a) + \int_a^b (b-t)fL''(t)\,\mathrm{d}t \tag{532}$$

We integrate by parts again:

$$u = f''(t) \qquad \mathrm{d}v = (b-t)\,\mathrm{d}t \tag{533}$$

$$\mathrm{d}u = f'''(t)\,\mathrm{d}t \qquad v = -\frac{(b-t)^2}{2} \tag{534}$$

which gives:

$$\int_a^b f^n(t)(b-t)\,\mathrm{d}t = \left[-\frac{(b-t)^2}{2}f''(t)\right]_a^b + \int_a^b \frac{(b-t)^2}{2}f'''(t)\,\mathrm{d}t \tag{535}$$

If we continue this a total of n times, then we eventually get:

$$\int_a^b f'(t)\,\mathrm{d}t = (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \frac{(b-a)^3}{3!}f''(a) + \cdots + \frac{(b-a)^n}{n!}f^{(n)}(a) + \int_a^b \frac{(b-t)^n}{n!}f^{n+1}(t)\,\mathrm{d}t \tag{536}$$

However, remember that this integration is equal to $f(b) - f(a)$. If we let $x = b$, then we get:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x) \tag{537}$$

where from our previous work, we have

$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!}f^{n+1}(t)\,\mathrm{d}t \tag{538}$$

□

- For $|f^{(n+1)}(t)| \leq M$ for $a < t < x$ we can bound the remainder function by:

$$|R_n(x)| \leq \left|\int_0^x \frac{M(x-t)^n}{n!}\,\mathrm{d}t\right| = \left|M\left[\frac{(x-t)^{n+1}}{(n+1)!}\right]_a^x\right| = M\frac{|x-a|^{n+1}}{(n+1)!} \tag{539}$$

- If we instead use the MVT, we can obtain a slightly different expression for the reaminder:

$$R_n(x) = \frac{f^{n+1}(c)(x-a)^{n+1}}{(n+1)!} \tag{540}$$

with $a < c < x$.

Example 83: Suppose we wish to continue the proof that e^x is indeed equal to the sum of its Taylor series, we note again that $f^{(n+1)}(t) = e^t$. For $x > 0$, we can pick an x such that $0 < t < x$ where $e^t < e^x$. The remainder can then be written as:

$$R_n(x) < \frac{e^x x^{n+1}}{(n+1)!} \tag{541}$$

As $n \to \infty$, the remainder approaches zero and as a result:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{542}$$

for all x is a true statement.

Example 84: Let us now find the Maclaurin series for $\cos x$. We have:

$$f(x) = \cos x \qquad f(0) = 1 \tag{543}$$
$$f'(x) = -\sin x \qquad f'(0) = 0 \tag{544}$$
$$f''(x) = -\cos x \qquad f''(0) = -1 \tag{545}$$
$$f'''(x) = \sin x \qquad f'''(0) = 0 \tag{546}$$
$$f^{(4)}(x) = \cos x \qquad f(0) = 1 \tag{547}$$

and it repeats. Therefore, we propose that:

$$\cos x = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \cdots \tag{548}$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{549}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{550}$$

We can use the ratio test to show that the radius of convergence is $R = \infty$. Finally, we need to prove that this sum is $\cos x$. We note that:

$$|f^{n+1}(t)| = \pm \cos t \text{ or } \pm \sin t \leq 1 \tag{551}$$

so we can bound the remainder by:

$$|R_n(x)| \leq \left|\frac{Mx^{n+1}}{(n+1)!}\right| = \left|\frac{x^{n+1}}{(n+1)!}\right| \tag{552}$$

- An important idea is that it does not matter where the coefficients in the power series expansion comes from. There is only one unique set of coefficients so if one possible set is found, then it is the only set.

Example 85: Let us find the Maclaurin series for $\sin x$:

$$\sin x = -\frac{d}{dx} \cos x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{553}$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} 2n \frac{x^{2n-1}}{2n!} \tag{554}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{555}$$

Example 86: Consider the function $x \sin x$. The power expansion is thus:

$$x \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)!} \tag{556}$$

Example 87: Let us try to find the Taylor series of $\cos x$ about $\dfrac{17\pi}{4}$. This gives:

$$f(x) = \cos x \qquad f\left(\frac{17\pi}{4}\right) = \frac{1}{\sqrt{2}} \tag{557}$$
$$f'(x) = -\sin x \qquad f'\left(\frac{17\pi}{4}\right) = -\frac{1}{\sqrt{2}} \tag{558}$$
$$f''(x) = -\cos x \qquad f''\left(\frac{17\pi}{4}\right) = -\frac{1}{\sqrt{2}} \tag{559}$$
$$f'''(x) = \sin x \qquad f'''\left(\frac{17\pi}{4}\right) = \frac{1}{\sqrt{2}} \tag{560}$$
$$f''''(x) = \cos x \qquad f''''\left(\frac{17\pi}{4}\right) = \frac{1}{\sqrt{2}} \tag{561}$$

Example 88: We have already seen that:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \tag{562}$$

fpr $-1 < x \leq 1$. We can verify this by verifying coefficients:

$$f(x) = \ln(1+x) \tag{563}$$
$$f'(x) = \frac{1}{1+x} \tag{564}$$
$$f''(x) = -\frac{1}{(1+x)^2} \tag{565}$$
$$f'''(x) = \frac{2}{(1+x)^3} \tag{566}$$
$$f''''(x) = \frac{-3!}{(1+x)^4} \tag{567}$$
$$f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{(1+x)^n} \tag{568}$$

Suppose we wish to bound the n^{th} derivative using the remainder function:

$$R_n = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n \, \mathrm{d}t \tag{569}$$
$$= \frac{1}{n!}\int_0^x (-1)^{n+2}\frac{n!}{(1+t)^{n+1}}(x-t)^n \, \mathrm{d}t \tag{570}$$
$$= (-1)^n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}} \, \mathrm{d}t \tag{571}$$

Let us work with nonzero values of x: $0 \leq x \leq 1$. Then:

$$|R_n(x)| = \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}} \, \mathrm{d}t \tag{572}$$
$$\leq \int_0^x (x-t)^n \, \mathrm{d}t \qquad (1+t) > 1 \tag{573}$$
$$= \frac{x^{n+1}}{n+1} \tag{574}$$

As $n \to \infty$, this approaches zero. Now let $-1 < x < 0$. Then:

$$|R_n(x)| = \left|\int_0^x \frac{(x-t)^n}{(1+t)^{n+1}} \, \mathrm{d}t\right| \tag{575}$$
$$= \int_x^0 \left(\frac{t-x}{1+t}\right)^n \frac{\mathrm{d}t}{1+t} \tag{576}$$

Note that from the mean value theorem, a number z exists, where $x < z < 0$ such that[a]:

$$\int_x^0 \left(\frac{t-x}{1+t}\right)^n \frac{\mathrm{d}t}{1+t} = \left(\frac{z-x}{1+z}\right)^n \left(\frac{-x}{1+z}\right) \tag{577}$$

Now $|x| < 1$ so $|x| - |z| < 1 - |z|$. This implies:

$$\frac{|x|-|z|}{1-|z|} < 1 \tag{578}$$

$$\frac{-x+z}{1+z} < 1 \tag{579}$$

We then have the limit:

$$\lim_{n\to 0} \left(\frac{z-x}{1+z}\right)^n = 0 \tag{580}$$

and therefore:

$$R_n(x) \to 0 \tag{581}$$

as $n \to \infty$.

[a] To see it explicitly, we can interpret the function as the area under the curve from x to 0 of: $\frac{(t-x)^n}{(1+t)^{n+1}}$. The mean value theorem tells us that the average height of this function has to occur at a value of $t = z$ where $x < z < 0$ and the total area can be represented as the average height (at $t = z$) multiplied by the width, which is $0 - x = -x$ (note that since x is negative, we can interpret this as multiplying it by -1 to get a positive area).

- It is possible to multiply and divide different power series

Example 89: Suppose we have the function $\frac{e^x}{1-x}$. Then the power series is given as:

$$= \left(1 + x + \frac{x^2}{2} + \cdots\right)(1 + x + x^2 + \cdots) \tag{582}$$

$$= 1 + 2x + \frac{5}{2}x^2 + \frac{16}{6}x^3 + \cdots \tag{583}$$

Example 90: We can determine the power expansion for $\tan x$ by using long division. We have:

$$\tan x = \frac{\sin x}{\cos x} \tag{584}$$

$$= \frac{x - x^3/3! + x^5/5! + \cdots}{1 - x^2/2 + x^4/4! + \cdots} \tag{585}$$

$$= x + \frac{x^3}{2} + \frac{2}{15}x^5 + \cdots \tag{586}$$

which for obvious reasons, I have omitted the long division steps. The radius of convergence is $|x| < R = \frac{\pi}{2}$ since the function diverges at $\tan\frac{\pi}{2}$.

20 Applications of Taylor Polynomials

- Recall that the n^{th} degree polynomial is:

$$T_n = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \tag{587}$$

The first degree polynomial is a linear approximation and the second degree is a quadratic approximation (at least, near $x = a$).

- We can use Taylor series to estimate erros:
  1. Alternating series: $|R_n(x)| < |a_{n+1}|$
  2. Taylor's formula: $|R_n| < \dfrac{M(x-a)^{n+1}}{(n+1)!}$

Example 91: Suppose we want to use the Taylor expansion of $f(x) = \sqrt{x}$ at $a = 1$ and use it to evaluate $\sqrt{1.25}$. The first four derivatives are:

$$f(x) = x^{1/2} \qquad f(1) = 1 \tag{588}$$

$$f'(x) = \frac{1}{2}x^{-1/2} \qquad f'(1) = \frac{1}{2} \tag{589}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \qquad f''(1) = -\frac{1}{4} \tag{590}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} \qquad f'''(1) = \frac{3}{8} \tag{591}$$

$$f''''(x) = -\frac{15}{16}x^{-7/2} \qquad f''''(1) = -\frac{15}{16} \tag{592}$$

which gives:

$$\sqrt{x} \simeq T_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{4}\frac{(x-1)^2}{2!} + \frac{3}{8}\frac{(x-1)^3}{3!} \tag{593}$$

and the error is:

$$|R_3(x)| < |a_4| = \frac{15}{16}\frac{(x-1)^4}{4!} \tag{594}$$

so:

$$\sqrt{1.125} \simeq 1 + \frac{0.25}{2} - \frac{0.25^2}{8} + \frac{0.25^3}{16} \pm \frac{5}{128}0.25^4 \approx 1.11816 \pm 0.00015 \tag{595}$$

Example 92: Consider the maclaurin series of $\cos x$ about $a = 0$. We want to find the error for $-\frac{\pi}{4} < x < \frac{\pi}{4}$, we have:

$$\cos x \simeq T_3(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \tag{596}$$

and the error would be:

$$|R_3(x)| < \left|\frac{x^8}{8!}\right| < \frac{(\pi/4)^8}{8!} \approx 3.6 \times 10^{-6} \tag{597}$$

Note that we can also use our alternating series result to get the same error.