ESC195 Notes

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Contents

1 Power Series

- We can introduce the power series:

> **Definition:** A power series is a series in the form:
>
> $$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \tag{1}$$

- For example, if we let $c_n = 1$. Then for all n, we get:

$$\sum_{n=0}^{\infty} x_n = 1 + x + x^2 + \cdots = \frac{1}{1-x} \tag{2}$$

and converges if $|x| < 1$.

- A power series about a can be written as:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \tag{3}$$

- Note that for $x = a$, the sum will always converge. However, we are interested for the entire range of values at which it converges..

> **Example 1:** Suppose we have the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$. To test when it converges, we can apply the ratio test:
>
> $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n}\right| \tag{4}$$
>
> $$= |x| \frac{n^2}{n+1}^2 \tag{5}$$
>
> As $n \to \infty$, we get $|x|$. Therefore, the series converges when $|x| < 1$. However, the test says nothing about the endpoints, so we have to test them separately. If $x = 1$, we have:
>
> $$\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{6}$$

We can apply a p-series test to show it converges. For $x = -1$, we have:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \tag{7}$$

and apply the alternating series test to show that it converges. Therefore, the power series converges for:

$$-1 \leq x \leq 1 \tag{8}$$

Example 2: Suppose we have the power series $\displaystyle\sum_{n=0}^{\infty} \frac{(1+5^n)x^n}{n!}$. Using the ratio test, we have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(1+5^{n+1})x^{n+1}}{(n+1)!} \cdot \frac{n!}{(1+5^n)x^n}\right| = \frac{1+5^{n+1}}{1+5^n} \cdot \left|\frac{x}{n+1}\right| \tag{9}$$

which approaches 0 as $n \to \infty$ so it is convergent for all $x \in \mathbb{R}$.

Example 3: Take the power series $\sum n!x^n$. The ratio test then gives:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{n!} \cdot \frac{x^{n+1}}{x^n}\right| = (n+1)|x| \tag{10}$$

This approaches ∞ as $n \to \infty$ so it diverges except for $x = 0$.

Theorem: For a power series $\displaystyle\sum_{n=0}^{\infty} c_n(x-a)^n$, there are three possibilities with respect to convergence:

1. The series converges only when $x = a$
2. The series converges for all x
3. The series converges in some interval $|x-a| < R$ where R is the **radius of convergence**. However, the endpoints must be tested serparately.

Example 4: Take the power series $\displaystyle\sum_{n=0}\sum^{\infty} \frac{(-2)^n(x-1)^n}{n+2}$. The ratio test gives us:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}(x-1)^{n+1}}{n+3} \cdot \frac{n+2}{2^n(x-1)^n}\right| = 2\left(\frac{n+2}{n+3}\right)|x-1| \tag{11}$$

As $n \to \infty$, we get:

$$|x-1| < \frac{1}{2} \therefore R = \frac{1}{2} \tag{12}$$

We now need to check the endpoints. Test $x = \frac{1}{2}$. We get:

$$\sum_{n=0}^{\infty} \frac{(-2)^n\left(-\frac{1}{2}\right)^n}{n+2} = \sum_{n=0}^{\infty} \frac{1}{n+2} = \sum_{i=2}^{\infty} \frac{1}{i} \tag{13}$$

which diverges as it is the harmonic series. We now need to test $x = \frac{3}{2}$. We then get:

$$\sum_{n=0}^{\infty} \frac{(-2)^n\left(\frac{1}{2}\right)^n}{n+2} = \sum \frac{(-1)^n}{n+2} \tag{14}$$

Using the alternating series test, we see that this converges. Therefore, the interval of convergence is $\left(\frac{1}{2}, \frac{3}{2}\right]$.

- It is possible to represent functions as a power series. We saw that for $|x|<1$, the infinite series:

$$\sum_{n=0}^{\infty} = 1 + x + x^2 + \cdots = \frac{1}{1-x} \tag{15}$$

If we let $f(x) = \dfrac{1}{1-x}$, then we can *approximate* it using a truncated power series representation for between $-1<x<1$.

Example 5: Suppose we have the function $\dfrac{x}{x-3}$. If we want to write it as a power series, we can write it as:

$$\begin{aligned}
x \cdot \frac{1}{x-3} &= -x\frac{1}{3-x} && (16)\\
&= -\frac{x}{3}\frac{1}{1-\frac{x}{3}} && (17)\\
&= -\frac{x}{3}\left[1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \cdots\right] && (18)\\
&= -\frac{x}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n && (19)\\
&= -\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^{n+1} && (20)\\
&= -\sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^{n} && (21)
\end{aligned}$$

and it converges for $|x|<3$.

Theorem: Term by Term Differentiation and Integration: Consider the pwoer series $\sum c_n(x-a)^n$ with $R = R_0 > 0$, then

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n \tag{22}$$

is differentiable and continuous on $(a-R_0, a+R_0)$ and:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \tag{23}$$

We can also take the integral:

$$\int f(x)\,\mathrm{d}x = C + c_0(x-a) + \frac{c_1(x-a)^2}{2} + \frac{c_2(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty}\frac{c_n(x-a)^{n+1}}{n+1} \tag{24}$$

Notice that derivatives and infinite sums can be interchanged. Specifically:

$$\frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty}\frac{d}{dx}c_n(x-a)^n \tag{25}$$

$$\int \sum_{n=0}^{\infty} c_n(x-a)^n\,\mathrm{d}x = \sum_{n=0}^{\infty}\int c_n(x-a)^n\,\mathrm{d}x \tag{26}$$

Warning: The radius of convergence between derivatives will always be the same, but the endpoints may change.

Example 6: Suppose we have the function $f(x) = \dfrac{1}{(1+x)^2}$. Note that:

$$\frac{d}{dx}\frac{-1}{1+x} = -\frac{1}{(1+x)^2} \tag{27}$$

so we can write it in terms of its derivative:

$$\frac{d}{dx} - \frac{1}{1+x} = \frac{d}{dx}\left[-\sum_{n=0}^{\infty}(-x)^n\right] = \sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1} = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n \tag{28}$$

Example 7: Let's find the power series representation of $\ln(1-x)$. We notice that it can be written as an integral:

$$\ln(1-x) = -\int \frac{\mathrm{d}x}{1-x} = -\int \sum_{n=0}^{\infty} x^n \,\mathrm{d}x = C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \tag{29}$$

We can determine the constant of integration by setting $x = 0$, which gives $\ln(1) = 0 = C$. Therefore, we can write:

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \tag{30}$$

For $x = 1$, this diverges and for $x = -1$, it conditionally converges.

Example 8: Let us attempt to evaluate $\displaystyle\int_0^{0.1} \frac{\mathrm{d}x}{1+x^4}$ to 6 decimal places without a calculator. We first write it as a power series:

$$\frac{1}{1-(-x)^4} = \sum_{n=0}^{\infty}(-x^4)^n = 1 - x^4 - x^8 - \cdots \tag{31}$$

which converges for $|x| < 1$. Therefore, the integral is:

$$\int \frac{\mathrm{d}x}{1+x^4} = \sum_{n=0}^{\infty}\int(-x^4)^n \,\mathrm{d}x \tag{32}$$

$$= C + \sum_{n=0}^{\infty}(-1)^n \frac{x}{4n+1} \tag{33}$$

$$= C + x - \frac{x^5}{5} + \frac{x^9}{9} - \cdots \tag{34}$$

The integral is then:

$$\int_0^{0.1} \frac{\mathrm{d}x}{1+x^4} = 0.1 - \frac{0.1^5}{5} + \frac{0.1^9}{9} - \cdots = 0.099998 \pm 1.1 \times 10^{-10} \tag{35}$$

Example 9: Let us try to write the power series representaiton of the inverse tangent function $f(x) = \tan^{-1}(x)$. Note that:

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \tag{36}$$

We can write $f(x)$ as the integral:

$$\tan^{-1}(x) = \int \frac{\mathrm{d}x}{1+x^2} = \int (1 - x^2 + x^4 - x^6 + \cdots)\,\mathrm{d}x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \tag{37}$$

We can calculate the constant of integration to be $C = \tan^{-1}(0) = 0$ such that we have:

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \tag{38}$$

with a radius of convergence of $R = 1$.

Remarks: If we substitute in $x = 1$, then we can a special series:

$$\tan^{-1}(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{39}$$

and is known as Leibniz's formula for π.

2 Taylor and Maclaurin Series

- Recall that the power series can be written as:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \tag{40}$$

for $|x-a| < R$, we note that $f(a) = c_0$. However, if we take the derivative:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \tag{41}$$

and we similarly get $f'(a) = c_1$. For the second derivative:

$$f''(x) = 2c_2 + 6c_3(x-a) + \cdots \tag{42}$$

we get $f''(a) = 2c_2$.

- In general:

$$f^(n)(a) = n!c_n \tag{43}$$

Theorem: If $f(x)$ has a power series representation about a:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \tag{44}$$

with $|x-a| < R$. Then the coefficients of the series are $c_n = \frac{f^{(n)(a)}}{n!}$

- For a Taylor series of f about a, we have:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \cdots \tag{45}$$

- For the Maclaurin Series, it is simply a Taylor series taken at $x = a$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots \tag{46}$$

Definition: A definition is called **analytic at a** if it can be represented as a power series about a.

Example 10: Let us attempt to write out the Maclaurin series of $f(x) = e^x$. First note that:

$$f'(x) = e^x = f''(x) = f'''(x) = f^{(n)}(x) \tag{47}$$

Therefore: $f^{(n)}(0) = e^0 = 1$. Therefore, we can write it as the series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{48}$$

We can check that this converges using the ratio test. Let $a_n = \frac{x^n}{n!}$. Then:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \tag{49}$$

which approaches zero as $n \to \infty$. As a result, $R = \infty$

- We ask ourselves the question: When is it true that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Definition: The nth degree Taylor polynomial of f about a can be written as:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{50}$$

Example 11: Let us take a look at e^x about $a = 0$. Then the first, second, third degree series can be written as:

$$T_1(x) = 1 + x \tag{51}$$

$$T_2(x) = 1 + x + \frac{x^2}{2} \tag{52}$$

$$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \tag{53}$$

We can then define the remainder function as:

$$R_n(x) = f(x) - T_n(x) \tag{54}$$

Theorem: If $f(x) = T_n(x) + R_n(x)$ and $\lim_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$. Then f is equal to the sum of its Taylor series.

Given that f has $n+1$ continuous derivatives on an open interval I containing a, tyhen for all $x \in I$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} + R_n(x) \tag{55}$$

where

$$R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n \, \mathrm{d}t \tag{56}$$

Proof. Consider the fundamental theorem of calculus:

$$\int_a^b f'(t)\,\mathrm{d}t = f(b) - f(a) \tag{57}$$

Suppose we evaluate this via integration by parts:

$$\begin{aligned} u &= f'(t) & \mathrm{d}v &= \mathrm{d}t \\ \mathrm{d}u &= f''(t) & v &= t - b \end{aligned}$$

This gives:

$$\int_a^b f'(t)\,\mathrm{d}t = [f'(t)(t-b)]_a^b - \int_a^b f''(t)(t-b)\,\mathrm{d}t \tag{58}$$

$$= (b-a)f'(a) + \int_a^b (b-t)fL''(t)\,\mathrm{d}t \tag{59}$$

We integrate by parts again:

$$u = f''(t) \qquad \mathrm{d}v = (b-t)\,\mathrm{d}t \tag{60}$$

$$\mathrm{d}u = f'''(t)\,\mathrm{d}t \qquad v = -\frac{(b-t)^2}{2} \tag{61}$$

which gives:

$$\int_a^b f^n(t)(b-t)\,\mathrm{d}t = \left[-\frac{(b-t)^2}{2}f''(t)\right]_a^b + \int_a^b \frac{(b-t)^2}{2}f'''(t)\,\mathrm{d}t \tag{62}$$

If we continue this a total of n times, then we eventually get:

$$\int_a^b f'(t)\,\mathrm{d}t = (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \frac{(b-a)^3}{3!}f''(a) + \cdots + \frac{(b-a)^n}{n!}f^{(n)}(a) + \int_a^b \frac{(b-t)^n}{n!}f^{n+1}(t)\,\mathrm{d}t \tag{63}$$

However, remember that this integration is equal to $f(b) - f(a)$. If we let $x = b$, then we get:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x) \tag{64}$$

where from our previous work, we have

$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!}f^{n+1}(t)\,\mathrm{d}t \tag{65}$$

□

- For $|f^{(n+1)}(t)| \leq M$ for $a < t < x$ we can bound the remainder function by:

$$|R_n(x)| \leq \left|\int_0^x \frac{M(x-t)^n}{n!}\,\mathrm{d}t\right| = \left|M\left[\frac{(x-t)^{n+1}}{(n+1)!}\right]_a^x\right| = M\frac{|x-a|^{n+1}}{(n+1)!} \tag{66}$$

- If we instead use the MVT, we can obtain a slightly different expression for the reaminder:

$$R_n(x) = \frac{f^{n+1}(c)(x-a)^{n+1}}{(n+1)!} \tag{67}$$

with $a < c < x$.

Example 12: Suppose we wish to continue the proof that e^x is indeed equal to the sum of its Taylor series, we note again that $f^{(n+1)}(t) = e^t$. For $x > 0$, we can pick an x such that $0 < t < x$ where $e^t < e^x$. The remainder can then be written as:

$$R_n(x) < \frac{e^x x^{n+1}}{(n+1)!} \tag{68}$$

As $n \to \infty$, the remainder approaches zero and as a result:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{69}$$

for all x is a true statement.

Example 13: Let us now find the Maclaurin series for $\cos x$. We have:

$$f(x) = \cos x \qquad f(0) = 1 \tag{70}$$
$$f'(x) = -\sin x \qquad f'(0) = 0 \tag{71}$$
$$f''(x) = -\cos x \qquad f''(0) = -1 \tag{72}$$
$$f'''(x) = \sin x \qquad f'''(0) = 0 \tag{73}$$
$$f^{(4)}(x) = \cos x \qquad f(0) = 1 \tag{74}$$

and it repeats. Therefore, we propose that:

$$\cos x = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \cdots \tag{75}$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{76}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{77}$$

We can use the ratio test to show that the radius of convergence is $R = \infty$. Finally, we need to prove that this sum is $\cos x$. We note that:

$$|f^{n+1}(t)| = \pm\cos t \text{ or } \pm\sin t \leq 1 \tag{78}$$

so we can bound the remainder by:

$$|R_n(x)| \leq \left|\frac{Mx^{n+1}}{(n+1)!}\right| = \left|\frac{x^{n+1}}{(n+1)!}\right| \tag{79}$$