

ESC195 Notes

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1 Hyperbolic Functions

- Sometimes, combinations of e^x and e^{-x} are given certain names, for example:

- **Hyperbolic sine:** $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$

- **Hyperbolic cosine:** $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$

- They have the following properties:

$$\frac{d}{dx} \sinh x = \cosh x \quad (1)$$

$$\frac{d}{dx} \cosh x = \sinh x \quad (2)$$

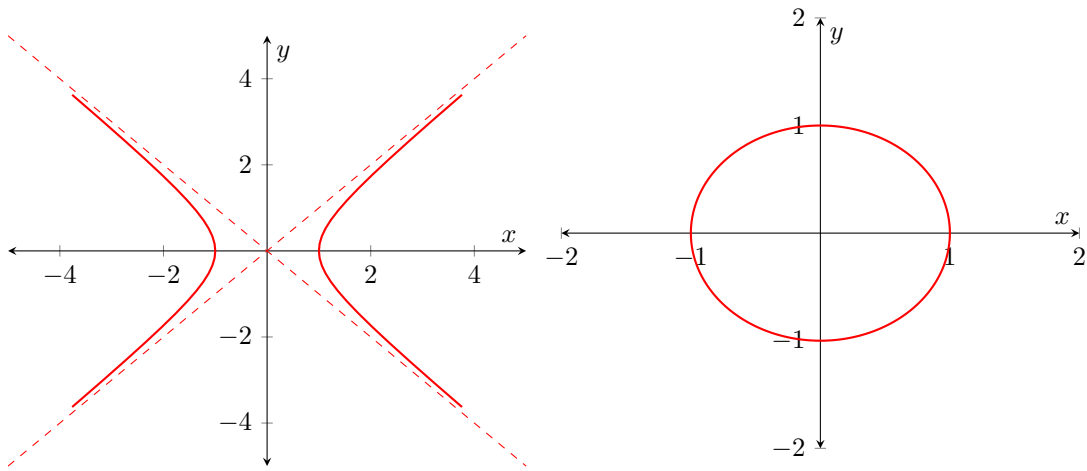
- They are related via:

$$\cosh^2 x - \sinh^2 x = 1 \quad (3)$$

- Both the area of a circular sector and that of a hyperbolic sector is described by:

$$A = \frac{1}{2}t \quad (4)$$

where t is the subtended angle, and the figures are parametrized by $(\cos t, \sin t)$ and $(\cosh t, \sinh t)$.



- The catenary

$$y = a \cosh\left(\frac{x}{a}\right) + C \quad (5)$$

describes the shape of a free hanging rope between two walls separated by a width a .

- The hyperbolic tangent is given by $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. and its derivative is given by:

$$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x \quad (6)$$

- The inverse of $y = \sinh x$ is given by:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \quad (7)$$

Tip: A table of integrals and derivatives revolving around hyperbolic trig functions can be found in the textbook.

2 Indeterminate Forms

- A lot of the times, limits have an indeterminate form, where if we substitute in what x approaches to, we get it in the form of $\frac{0}{0}$, for example:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \quad (8)$$

Theorem: If $f(x) \rightarrow 0$ and $g(x) \rightarrow 0$ as $x \rightarrow \pm, \infty$ or $x \rightarrow c$ or $x \rightarrow c^{+-}$ and if $\frac{f'(x)}{g'(x)} \rightarrow L$, then:

$$\frac{f(x)}{g(x)} \rightarrow L \quad (9)$$

Example 1: Solve: $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

We can set $f(x) = \sin x$, $f'(x) = \cos x$, $g(x) = x$ and $g'(x) = 1$ such that:

$$\lim_{x \rightarrow 0} \frac{f'}{g'} = \lim_{x \rightarrow 0} \cos x = 1 \quad (10)$$

Example 2: Solve $\lim_{x \rightarrow 0^+} \frac{\sin x}{\sqrt{x}}$.

Set $f = \sin x$, $f' = \cos x$, $g = \sqrt{x}$, $g' = \frac{1}{2}x^{-1/2}$ and so:

$$\lim_{x \rightarrow 0^+} 2x^{1/2} \cos x = 0 \implies \lim_{x \rightarrow 0^+} = 0 \quad (11)$$

Example 3: Solve $\lim_{x \rightarrow 0} \frac{e^x - x - 1}{3x^2}$.

If we take the derivative, we get:

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{6x} \quad (12)$$

which is still $\frac{0}{0}$!. We can take derivatives again:

$$\lim_{x \rightarrow 0} \frac{e^x}{6} = \frac{1}{6} \quad (13)$$

so the original limit is $\frac{1}{6}$.

Warning: L'hospital's rule can *only* be used in indeterminate forms. Applying them to limits where

- To prove the L'hospital's rule, we first prove the **Cauchy Mean Value Theorem** as a lemma

Theorem: Cauchy Mean Value Theorem: Given f and g differentiable on (a, b) , continuous on $[a, b]$ and $g' \neq 0$ on (a, b) , there must exist some number r in (a, b) such that:

$$\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad (14)$$

- We then apply **Rolle's Theorem** to prove the Cauchy Mean Value Theorem:

Proof. Set:

$$G(x) = [g(b) - g(a)][f(x) - f(a)] - [g(x) - g(a)][f(b) - f(a)]$$

Note that $G(a) = G(b) = 0$ so it satisfies the conditions of Rolle's Theorem. Taking the derivative, we get:

$$G'(x) = [g(b) - g(a)]f'(x) - g'(x)[f(b) - f(a)] \quad (15)$$

According to Rolle's, there must be some $x = r$ such that $G'(r) = 0$, we can then substitute for this and solve:

$$G'(r) = 0 \implies [g(b) - g(a)]f'(r) = g'(r)[f(b) - f(a)] \quad (16)$$

Which is equivalent to:

$$\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad (17)$$

Furthermore, we have $g'(c) = \frac{g(b) - g(a)}{b - a}$ from the mean value theorem. Since $g' \neq 0$ we have $g(b) - g(a) \neq 0$. \square

- Given $x \rightarrow c^+$ and $f(x), g(x) \rightarrow 0$ where:

$$\lim_{x \rightarrow c^+} \frac{f'(x)}{g'(x)} = L \quad (18)$$

we will now prove that $\lim_{x \rightarrow c^+} \frac{f(x)}{g(x)} = L$.

Proof. Consider the interval $[c, c + h]$ and apply Cauchy MVT. There must be some number c_2 in $[c, c + h]$ such that:

$$\frac{f'(c_2)}{g'(c_2)} = \frac{f(c + h) - f(c)}{g(c + h) - g(c)} = \frac{f(c + h)}{g(c + h)} \quad (19)$$

The last step is a result of the given $f(c) = g(c) = 0$. The LHS can be rewritten as:

$$\lim_{h \rightarrow 0} \frac{f'(c_2)}{g'(c_2)} = \frac{f'(c)}{g'(c)} \quad (20)$$

since c_2 lies in the interval $[c, c + h]$ so if $h \rightarrow 0$, then the interval becomes smaller to contain just c . The RHS can be rewritten as:

$$\lim_{h \rightarrow 0} \frac{f(c + h)}{g(c + h)} = \lim_{x \rightarrow c^+} \frac{f(x)}{g(x)} \quad (21)$$

and therefore:

$$\lim_{x \rightarrow c^+} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} = L \quad (22)$$

□

- To prove the case for $x \rightarrow \pm\infty$, we can let $x = \frac{1}{t}$ and take the limit as $t \rightarrow \infty$.

Example 4: Find $\lim_{x \rightarrow \infty} \frac{\ln x}{x}$.

Taking the derivative of top and bottom, we have:

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = 0 \implies \lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0 \quad (23)$$

Idea: The logarithm function grows very slowly. In fact, any positive power of x will grow faster than $\ln x$.

Example 5: Solve $\lim_{x \rightarrow \infty} \frac{x^3}{e^x}$

This is indeterminate in the form of $\frac{\infty}{\infty}$. We apply L'hospital's rule multiple times:

$$\lim_{x \rightarrow \infty} \frac{x^3}{e^x} \stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{3x^2}{e^x} \left(= \frac{\infty}{\infty} \right) \quad (24)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{6x}{e^x} \left(= \frac{\infty}{\infty} \right) \quad (25)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{6}{e^x} = 0 \quad (26)$$

- Generally, $\lim_{x \rightarrow \infty} \frac{x^m}{e^x} = 0$ where m is any positive integer.
- There are other indeterminate forms, such as 0^0 , for example:

$$\lim_{x \rightarrow 0} x^x \quad (27)$$

The central idea behind this is that $a^b = e^{a \ln b}$. Therefore, this limit is equal to:

$$\lim_{x \rightarrow 0} e^{x \ln x} \quad (28)$$

We can take the limit of the exponent to get:

$$\lim_{x \rightarrow 0} x \ln x = \lim_{x \rightarrow 0} \frac{\ln x}{1/x} \quad (29)$$

Note that the first equation is another indeterminate form with the $0 \cdot \infty$ type, so we had to multiply top and bottom by $\frac{1}{x}$ to get the quotient form. Then we have:

$$\lim_{x \rightarrow 0} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x \rightarrow 0} -x = 0 \quad (30)$$

Therefore:

$$\lim_{x \rightarrow 0} e^{x \ln x} = e^0 = 1 \quad (31)$$

so $\lim_{x \rightarrow 0} x^x = 1$.

Example 6: Solve $\lim_{x \rightarrow \infty} (x+2)^{2/\ln x}$.

This is of the type ∞^0 . The approach is exactly the same as the previous example. We write it in exponential form:

$$= \lim_{x \rightarrow \infty} e^{\frac{2}{\ln x} \ln(x+2)} \quad (32)$$

and looking at the exponent gives:

$$\lim_{x \rightarrow \infty} \frac{2 \ln(x+2)}{\ln x} \quad (33)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{\left(\frac{2}{x+2}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{2x}{x+2} \left(= \frac{\infty}{\infty}\right) \quad (34)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{2}{1} = 2 \quad (35)$$

Therefore:

$$\lim_{x \rightarrow \infty} e^{\frac{2}{\ln x} \ln(x+2)} = e^2 \quad (36)$$

so:

$$\lim_{x \rightarrow \infty} (x+2)^{2/\ln x} = e^2 \quad (37)$$

Example 7: Solve $\lim_{x \rightarrow \infty} \left[\sin \left(\frac{\pi}{x} + \frac{\pi}{2} \right) \right]^x$

This is in the form of 1^∞ . We rewrite it as:

$$\lim_{x \rightarrow \infty} \exp \left(x \ln \left(\sin \left(\frac{\pi}{x} + \frac{\pi}{2} \right) \right) \right) \quad (38)$$

and taking the limit of the exponent:

$$= \lim_{x \rightarrow \infty} x \ln \left(\sin \left(\frac{\pi}{x} + \frac{\pi}{2} \right) \right) \left(= \frac{0}{0}\right) \quad (39)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{\cos \left(\frac{\pi}{x} + \frac{\pi}{2} \right) \cdot \left(-\frac{\pi}{x^2} \right)}{\sin \left(\frac{\pi}{x} + \frac{\pi}{2} \right) \cdot \left(-\frac{1}{x^2} \right)} = \frac{0 \cdot \pi}{1} = 0 \quad (40)$$

Therefore:

$$\lim_{x \rightarrow \infty} \left[\sin \left(\frac{\pi}{x} + \frac{\pi}{2} \right) \right]^x = \lim_{x \rightarrow \infty} \exp \left(x \ln \left(\sin \left(\frac{\pi}{x} + \frac{\pi}{2} \right) \right) \right) = 1 \quad (41)$$

3 Integration

3.1 Recap of Integration

- The definite integral has the geometric interpretation as the area under the curve $f(x)$ between $x = a$ and $x = b$ and the x axis:

$$\int_a^b f(x) dx \quad (42)$$

but can be rigorously defined using a Riemann sum:

$$\int_a^b f(x) dx \equiv \lim_{\|P\|} \sum_{i=1}^n f(x_i^*) \Delta x_i \quad (43)$$

Often, we have a uniform partition, such that $\Delta x_i = \frac{b-a}{n}$ where n is the number of partitions. And if we choose to use the right hand endpoint, then:

$$f(x_i^*) = f(x_i) = f\left(a + \frac{b-a}{n}i\right) \quad (44)$$

Example 8: To solve $\int_0^5 x^2 dx$, we can choose a uniform partition with:

$$\Delta x = \frac{5-0}{n} = \frac{5}{n} \quad (45)$$

and:

$$x_i^* = x_i = i\Delta x \implies f(x_i^*) = (i\Delta x)^2 = \left(i\frac{5}{n}\right)^2 \quad (46)$$

The area approximation is:

$$A \simeq \sum_{i=1}^n \Delta x_i f(x_i^*) = \sum_{i=1}^n \left(\frac{5}{n}\right) \left(i\frac{5}{n}\right)^2 \quad (47)$$

$$= \frac{125}{n^2} \sum_{i=1}^n i^2 = \frac{125}{n^3} \frac{n(n+1)(2n+1)}{6} \quad (48)$$

Taking the limit as $n \rightarrow \infty$, we get:

$$\int_0^5 x^2 dx = \lim_{n \rightarrow \infty} \frac{125}{6} \left(2 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{5^3}{3}. \quad (49)$$

Example 9: To evaluate $\int_1^2 x^{-2} dx$, we can choose

$$x_i^* = \sqrt{x_{i-1}x_i} \quad (50)$$

and a uniform partition of:

$$\Delta x = \frac{2-1}{n} = \frac{1}{n} \quad (51)$$

such that:

$$x_i = 1 + i\Delta x = 1 + \frac{i}{n} = \frac{n+i}{n} \quad (52)$$

and

$$x_{i-1} = \frac{n+i-1}{n} \quad (53)$$

such that the area is:

$$\begin{aligned}
A &\simeq \sum_{i=1}^n \Delta x f(x_i^*) \\
&= \sum_{i=1}^n \frac{1}{n} \left(\frac{1}{x_i^*} \right)^2 \\
&= \sum_{i=1}^n \frac{1}{n} \frac{1}{x_{i-1} x_i} \\
&= \sum_{i=1}^n \frac{1}{n} \frac{n}{n+i-1} \cdot \frac{n}{n+i} \\
&= \sum_{i=1}^n \frac{1}{n+i-1} \cdot \frac{1}{n+i} \\
&= \sum_{i=1}^n \left(\frac{1}{n+i-1} - \frac{1}{n+i} \right) \\
&= n \left[\sum_{i=1}^n \frac{1}{n+i-1} - \sum_{i=1}^n \frac{1}{n+i} \right] \\
&= n \left[\sum_{i=0}^n \frac{1}{n+i} - \sum_{i=1}^n \frac{1}{n+i} \right] \\
&= n \left[\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n} \right] \\
&= n \left(\frac{1}{n} - \frac{1}{2n} \right) \\
&= 1 - \frac{1}{2} = \frac{1}{2}
\end{aligned}$$

The part where we cancel out everything is called a **telescoping series**. Notice how the area doesn't depend on n so we get the exact area, even if we let $n = 1$!

- We need a better way to do integration, so we can define:

$$F(x) \equiv \int_a^x f(t) dt \quad (54)$$

such that $F'(x) = f(x)$. This is the definition of the antiderivative. This leads to the fundamental theorem of calculus:

$$\int_a^b f(t) dt = F(b) - F(a) \quad (55)$$

and the indefinite integral can be written as:

$$\int f(x) dx = G(x) + C \quad (56)$$

The main problem now becomes trying to *find antiderivatives*, which is much easier than Riemann sums, though still more difficult than calculating derivatives.

3.2 Integration by Parts

- **Integration by Parts** attempts to reverse the product rule:

$$(fg)' = fg' + f'g \quad (57)$$

Taking the integral of both sides gives:

$$f(x)g(x) = \int f(x)g'(x) \, dx + \int f'(x)g(x) \, dx \quad (58)$$

$$\int f(x)g'(x) \, dx = \int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx \quad (59)$$

If the second integral is easier than the first, then we have made substantial progress.

Idea: Integration of parts tells us that:

$$\int u \, dv = uv - \int v \, du \quad (60)$$

Example 10: To solve $\int xe^{2x}$, we can let:

$$u = x \quad dv = e^{2x} \, dx \quad (61)$$

$$du = dx \quad v = \frac{1}{2}e^{2x} \quad (62)$$

which gives:

$$\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x} \, dx \quad (63)$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \quad (64)$$

We can check:

$$\frac{d}{dx} \left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \right) \quad (65)$$

$$= xe^{2x} + \frac{1}{2}e^{2x} - \frac{2}{4}e^{2x} \quad (66)$$

$$= xe^{2x} \quad (67)$$

Example 11: To solve $\int x^2 \sin(2x) \, dx$, we let:

$$u = x^2 \quad dv = \sin 2x \, dx \quad (68)$$

$$du = 2x \, dx \quad v = -\frac{1}{2} \cos(2x) \quad (69)$$

which gives:

$$= -\frac{1}{2}x^2 \cos 2x + \int x \cos(2x) \, dx \quad (70)$$

and we can apply integration by parts a second time, if we let:

$$u = x \quad dv = \cos 2x \, dx \quad (71)$$

$$du = dx \quad v = \frac{1}{2} \sin(2x) \quad (72)$$

which gives us:

$$= -\frac{1}{2}x^2 \cos(2x) + \frac{1}{2}x \sin(2x) - \int \frac{1}{2} \sin(2x) \, dx \quad (73)$$

$$= -\frac{1}{2}x^2 \cos(2x) + \frac{1}{2}x \sin(2x) + \frac{1}{4} \cos(2x) + C \quad (74)$$

Example 12: To solve $I = \int e^x \sin x \, dx$, we can let:

$$u = \sin x \qquad dv = e^x \, dx \qquad (75)$$

$$du = \cos x \, dx \qquad v = e^x \qquad (76)$$

to give us:

$$= e^x \sin x - \int e^x \cos x \, dx \qquad (77)$$

We apply integration by parts a second time:

$$u = \cos x \qquad dv = e^x \, dx \qquad (78)$$

$$du = -\sin x \, dx \qquad v = e^x \qquad (79)$$

to get:

$$I = e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x \, dx}_I \qquad (80)$$

$$2I = e^x (\sin x - \cos x) + C' \qquad (81)$$

$$I = \frac{1}{2} e^x (\sin x - \cos x) + C \qquad (82)$$

and we are done.

Example 13: We can also solve integrals that do not appear to have parts, such as $\int \ln x \, dx$. We choose:

$$u = \ln x \qquad dv = dx \qquad (83)$$

$$du = \frac{1}{x} \, dx \qquad v = x \qquad (84)$$

to give us:

$$\ln x - \int dx = x \ln x - x + C \qquad (85)$$

- For a definite integral, we can write IBP as:

$$f(x)g(x) \Big|_a^b - \int_a^b f'(x)g(x) \, dx \qquad (86)$$

Example 14: It is *possible* to apply integration of parts to find the integral of $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$. We can let:

$$u = \frac{1}{\cos x} = \sec x \qquad dv = \sin x \, dx \qquad (87)$$

$$du = \sec x \tan x \qquad v = -\cos x \qquad (88)$$

this gives us:

$$\int \tan x \, dx = -\frac{\cos x}{\cos x} + \int \tan x \, dx \qquad (89)$$

Notice that we could try to subtract the original integral from both sides and get:

$$0 = -1 \qquad (90)$$

which is clearly wrong! However, we forgot the constant of integration, so the correct statement would be:

$$0 + C' = -1 + C \qquad (91)$$

which does not tell us anything interesting. This brings We can see this concretely by repeating the same steps but trying to evaluate the definite integral $\int_a^b \tan x \, dx$ instead, which gives:

$$\int_a^b \tan x \, dx = (-1) \Big|_{x=a}^{x=b} + \int_a^b \tan x \, dx \implies 0 = (-1) - (-1) \implies 0 = 0 \quad (92)$$

which confirms our suspicion that this isn't anything useful, but it's also not an incorrect statement.

Warning: Sometimes it is possible to get more than one answer through various means that differ by a constant factor when solving indefinite integrals. When this happens, nothing is wrong: we simply need to consider the constant of integration.

Idea: But how do we know *which* values of u and dv we should pick? A common strategy is to use **LIATE**:

1. L: Logarithms
2. I: Inverse Trig
3. A: Algebraic
4. T: Trigonometric
5. E: Exponential

If a function consists of two terms, the term that is higher up (closer to L) usually gets differentiated and the term near the bottom (closer to E) usually gets integrated. See [this](#) for how it works, and this [video](#) for a tutorial.

4 Trigonometric Integrals

- The first type of integral we'll deal with is:

$$\int \sin^n x \cos^n x \, dx \quad (93)$$

- In **case 1**, we have either m or n as an odd positive number. We can then use the identity $\sin^2 x + \cos^2 x = 1$ to simplify it.

Example 15: For example, to solve $\int \sin^3 x \cos^2 x \, dx$, we can simplify this to:

$$= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx \quad (94)$$

$$= (\cos^2 x - \cos^4 x) \sin x \, dx \quad (95)$$

and applying a u substitution with $u = \cos x$ and breaking it up into two integrals, we can get:

$$= -\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C \quad (96)$$

- In **case 2**, we have m and n as both even. We then apply the double angle formulas:

$$\sin x \cos x = \frac{1}{2} \sin(2x) \quad (97)$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x \quad (98)$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x \quad (99)$$

Example 16: For example:

$$\int \sin^2 x \cos^4 x \, dx = \int \frac{1}{4} \sin^2(2x) \left(\frac{1}{2} + \frac{1}{2} \cos 2x \right) dx \quad (100)$$

$$= \frac{1}{8} \int \sin^2(2x) \, dx + \frac{1}{8} \int \sin^2 x \cos 2x \, dx \quad (101)$$

$$= \frac{1}{8} \int \left(\frac{1}{2} - \frac{1}{2} \cos 4x \right) dx + \frac{1}{8 \cdot 3 \cdot 2} \sin^3(2x) + C \quad (102)$$

$$= \frac{1}{16} x - \frac{1}{64} \sin(4x) + \frac{1}{48} \sin^3(2x) + C \quad (103)$$

- In **Case 3**, we have:

$$\int \sin^n x \, dx, \int \cos^n x \, dx \quad (104)$$

which we can apply a reduction formula by keep applying integration by parts:

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \quad (105)$$

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx \quad (106)$$

Example 17: To solve the integral $\int \sin^2 x \, dx$, we get:

$$= -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int dx \quad (107)$$

$$= \frac{1}{2} x - \frac{1}{4} \sin 2x + C \quad (108)$$

- In **Case 4**, we have integrals in the following forms:

$$\int \sin(mx) \cos(nx) \, dx \quad (109)$$

$$\int \sin(mx) \sin(nx) \, dx \quad (110)$$

$$\int \cos(mx) \cos(nx) \, dx \quad (111)$$

with $m \neq n$. If $m = n$, then we can apply the double angle formula. To solve these, we apply the following identities:

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)] \quad (112)$$

$$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)] \quad (113)$$

$$\sin A \cos B = \frac{1}{2} [\sin(A - B) + \sin(A + B)] \quad (114)$$

Example 18: For example, we have:

$$\int \sin(3x) \sin(2x) \, dx = \frac{1}{2} \int \cos((3-2)x) \, dx - \frac{1}{2} \int \cos((3+2)x) \, dx \quad (115)$$

$$= \frac{1}{2} \sin x - \frac{1}{10} \sin 5x + C \quad (116)$$

- In **case 5**, we have integrals in the form of either:

$$\int \tan^n x \, dx, \int \cot^n x \, dx \quad (117)$$

To solve these, we apply the following identities:

$$\tan^2 x = \sec^2 x - 1 \quad (118)$$

$$(\tan x)' = \sec^2 x \quad (119)$$

- In **case 6**, we have:

$$\int \sec^n x \, dx, \int \csc^n x \, dx \quad (120)$$

with $n \geq 2$. To solve these, we can make the following substitutions:

$$1 + \tan^2 x = \sec^2 x \quad (121)$$

$$1 + \cot^2 x = \csc^2 x \quad (122)$$

to convert it to a case 5 problem.

- In **case 7**, we have:

$$\int \tan^n x \sec^n x \, dx, \int \cot^n x \csc^n x \, dx \quad (123)$$

Example 19: We have:

$$\tan^3 x \sec^4 x \, dx = \int \tan^3 x \sec^2 x \sec^2 x \, dx \quad (124)$$

$$= \int \tan^3 x (\tan^2 x + 1) \sec^2 x \, dx \quad (125)$$

$$= \int (\tan^5 x + \tan^3 x) \sec^2 x \, dx \quad (126)$$

$$= \frac{1}{6} \tan^6 x + \frac{1}{4} \tan^4 x + C \quad (127)$$

Idea: The basic idea of these types is to apply trigonometric identities to turn the integrals into a form that is easier to deal with. The substitutions are usually very simple but to find them, it requires a lot of practice.

- We can also apply **trigonometric substitutions**, any integrals with any of the three factors below can be solved with this technique:

$$1. \sqrt{a^2 - x^2}: \text{Set } x = a \sin u \implies \sqrt{a^2 - x^2} = a \cos u$$

$$2. \sqrt{a^2 + x^2}: \text{Set } x = a \tan u \implies \sqrt{a^2 + x^2} = a \sec u$$

$$3. \sqrt{x^2 - a^2}: \text{Set } x = a \sec u \implies \sqrt{x^2 - a^2} = a \tan u$$

where the arguments under the square roots are always positive.

Example 20: To solve the integral $\int \frac{x^2}{(4 - x^2)^{3/2}} \, dx$, we can set:

$$x = 2 \sin u \quad (128)$$

$$dx = 2 \cos u \, du \quad (129)$$

$$\sqrt{4 - x^2} = 2 \cos u \quad (130)$$

which gives:

$$= \int \frac{4 \sin^2 u \cdot 2 \cos u \, du}{8 \cos^3 u} \quad (131)$$

$$= \int \tan^2 u \, du \quad (132)$$

$$= \int (\sec^2 u - 1) \, du \quad (133)$$

$$= \tan u - u + C = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C \quad (134)$$

Example 21: The integral $\int \frac{x \, dx}{(2x^2 + 4x - 7)^{1/2}}$ needs a bit more work before we can apply the substitutions. We first apply the square to get:

$$= \int \frac{x \, dx}{\sqrt{2(x+1)^2 - 9}} \quad (135)$$

We can set:

$$\sqrt{2}(x+1) = 3 \sec u \quad (136)$$

$$\sqrt{2} \, dx = 3 \sec u \tan u \, du \quad (137)$$

$$\sqrt{2(x+1)^2 - 9} = 3 \tan u \quad (138)$$

which gives:

$$= \int \frac{\left(\frac{3}{\sqrt{2}} \sec u - 1\right) \left(\frac{3}{\sqrt{2} \sec u \tan u} du\right)}{3 \tan u} \quad (139)$$

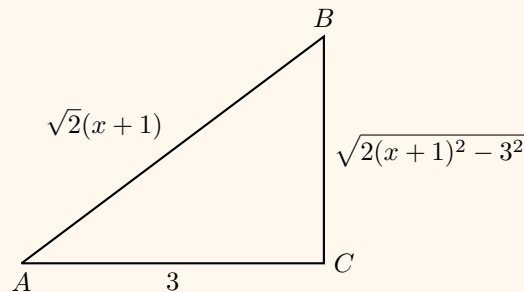
$$= \int \left(\frac{3}{\sqrt{2}} \sec u - 1\right) \left(\frac{1}{\sqrt{2}} \sec u\right) du \quad (140)$$

$$= \frac{3}{2} \int \sec^2 u \, du - \frac{1}{\sqrt{2}} \int \sec u \, du \quad (141)$$

$$= \frac{3}{2} \tan u - \frac{1}{\sqrt{2}} \ln |\sec u + \tan u| + C \quad (142)$$

$$= \frac{1}{2} \sqrt{2x^2 + 4x - 7} - \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2}}{3} (x+1) + \frac{\sqrt{2x^2 + 4x - 7}}{3} \right| + C \quad (143)$$

Idea: We can use triangles to derive the substitution, which comes from the Pythagorean theorem:



and you can clearly see the substitution:

$$3 \sec u = \sqrt{2}(x+1) \implies \cos u = \frac{3}{\sqrt{2}(x+1)} \quad (144)$$

where $u \equiv \angle BAC$.

Example 22: For the integral $\int x \sin^{-1} x \, dx$, we can let:

$$u = \sin^{-1} x \, dv = x \, dx \quad (145)$$

$$du = \frac{dx}{\sqrt{1-x^2}} v = \frac{1}{2} x^2 \quad (146)$$

and applying integration by parts, we get:

$$= \frac{1}{2} x^2 \sin^{-1} x - \int \frac{1}{2} x^2 \frac{dx}{\sqrt{1-x^2}} \quad (147)$$

To solve this secondary integral $\int \frac{x^2 \, dx}{\sqrt{1-x^2}}$, we can let:

$$x = \sin \theta \quad (148)$$

$$dx = \cos \theta \, d\theta \quad (149)$$

$$\sqrt{1-x^2} = \cos \theta \quad (150)$$

which gives:

$$= \frac{\sin^2 \theta \cos \theta \, d\theta}{\cos \theta} \quad (151)$$

$$= \int \sin^2 \theta \, d\theta \quad (152)$$

$$= \frac{1}{2} \theta - \frac{1}{2} \sin \theta \cos \theta + C \quad (153)$$

$$= \frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1-x^2} + C \quad (154)$$

Therefore, we get:

$$\int x \sin^{-1} x \, dx = \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{4} \sin^{-1} x + \frac{1}{4} x \sqrt{1-x^2} + C \quad (155)$$

5 Partial Fractions

- Rational functions are in the form of:

$$R(x) = \frac{P_n(x)}{P_m(x)} \quad (156)$$

where m, n represent the order of the polynomial. If $n \geq m$, it is an **improper** fraction, such as:

$$\frac{x^2 - x}{1 + x} \quad (157)$$

and if $n < m$, we have a proper fraction such as:

$$\frac{x}{x^2 + 3x + 2} \quad (158)$$

- If we have an improper fraction, we use long division to simplify it. For example:

$$\frac{x^3 - 2x^2}{x^2 + 9} = x - 2 + \frac{18 - 9x}{x^2 + 9} \quad (159)$$

which turns the expression into a polynomial (trivial to integrate) as well as a proper fraction.

- There are different types of factors:
 - Linear factors (e.g. $3x + 2$)

- Irreducible quadratic factors (e.g. $x^2 + 1$)

which gives us the different factors:

- **Case 1:** If we have distinct linear factors in the denominator, we can break it into fractions of the form:

$$(x + \alpha) \implies \frac{A}{x + \alpha} \quad (160)$$

Example 23: The partial fraction of $\frac{2x - 17}{x^2 + 3x + 2}$ can be written as the **partial fraction deconvolution**:

$$= \frac{A}{x + 1} + \frac{B}{x + 2} \quad (161)$$

We now need to solve for A and B . We can multiply both sides by $(x + 1)(x + 2)$ to get:

$$2x - 17 = A(x + 2) + B(x + 1) \quad (162)$$

and match up the coefficients. Alternatively, we can pick various values of x (e.g. $x = -2$ and $x = -1$) to solve for the coefficients.

- **Case 2:** If we have repeated linear factors, then the decomposition is in the form of:

$$(x + \alpha)^k \implies \frac{A}{x + \alpha} + \frac{B}{(x + \alpha)^2} + \frac{C}{(x + \alpha)^3} + \cdots + \frac{K}{(x + \alpha)^k} \quad (163)$$

Example 24: To get the decomposition of $\frac{2}{x(x + 1)^2}$, we can get:

$$\frac{2}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2} \quad (164)$$

which gives:

$$2 = A(x + 1)^2 + Bx(x + 1) + Cx \quad (165)$$

matching the coefficients, we get three equations and three unknowns:

$$x^2 : A + B = 0 \quad (166)$$

$$x : 2A + B + C = 0 \quad : A = 2 \quad (167)$$

Solving this system gives $A = 2$, $B = -2$, and $C = -2$. Note that taking the integral of this sum is much easier. We have:

$$\int \frac{d}{x(x + 1)^2} dx = \int \frac{2}{x} dX - \int \frac{2}{x} dx - \int \frac{2}{(x + 1)^2} dx \quad (168)$$

$$= 2 \ln |x| - 2 \ln |x + 1| + \frac{2}{x + 1} + C \quad (169)$$

Idea: As a general rule of thumb, the number of unknown coefficients is equal to the order of the polynomial in the denominator.

- **Case 3:** If we have irreducible quadratic factors, then the partial fraction deconvolution is in the form of:

$$x^2 + px + 8 \implies \frac{Ax + B}{x^2 + px + 8} \quad (170)$$

Example 25: Suppose we have $\frac{2}{(x+1)(x^2+x+1)}$, we can get the partial fraction decomposition as:

$$= \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1} \quad (171)$$

and we work through the deconvolution process in exactly the same way, we remove the denominators on both sides to get (after expanding):

$$2 = Ax^2 + Ax + A + Bx^2 + Bx + Cx + C \quad (172)$$

$$0x^2 + 0x^1 + 2x^0 = (A+B)x^2 + (A+B+C)x^1 + (A+C)x^0 \quad (173)$$

which gives three equations and three unknowns, after we match coefficients:

$$x^2 : A + B = 0 \quad (174)$$

$$x : A + B + C = 0 \quad (175)$$

$$1 : A + C = 2 \quad (176)$$

and solving the system of equations gives $A = 2, B = -2, C = 0$. To get the integral of this second term, we can write the second term as:

$$\int \frac{2x \, dx}{x^2 + 2x + 1} = \underbrace{\int \frac{2x+1}{x^2+x+1} \, dx}_{(1)} - \underbrace{\int \frac{dx}{x^2+x+1}}_{(2)} \quad (177)$$

We “added” 1 and “subtracted” 1 to get these two slightly easier integrals, which we can apply other techniques. The first one can be solved using a u-sub while the second can be solved by completing the square and applying a trigonometric substitution:

$$(1) = \ln |x^2 + x + 1| + C \quad (178)$$

$$(2) = \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}} \tan^{-1} \left[\frac{2}{\sqrt{3}} \left(x + \frac{1}{2}\right) \right] + C \quad (179)$$

allowing us to put everything together.

Example 26: Let’s take an integral we already know the answer of: $\int \frac{2x}{x^2+1} \, dx = \ln(x^2+1) + C$. We can try a partial fraction decomposition:

$$\frac{2x}{x^2+1} = \frac{A}{x+i} + \frac{B}{x-i} = \frac{1}{x+i} + \frac{1}{x-i} \quad (180)$$

which gives:

$$\int \frac{2x}{x^2+1} \, dx = \int \frac{dx}{x+i} + \int \frac{dx}{x-i} \quad (181)$$

In complex analysis, most mathematical functions we are familiar with are still valid, so the integral is:

$$= \ln |x+i| + \ln |x-i| + C \quad (182)$$

and simplifying it gives:

$$\ln(x^2+1) + C \quad (183)$$

Warning: While it is *possible* to use complex numbers to solve irreducible quadratic factors, it isn’t always as easy as the above example. To get the logarithm of a complex number, we can apply the identity (without

proving):

$$\ln(a + ib) = \ln \sqrt{a^2 + b^2} + i \arctan \left(\frac{b}{a} \right) \quad (184)$$

Example 27: Bonus content: Try evaluating the integral $\int \frac{dx}{x^2 + 1}$ with complex analysis. Taking a partial fraction, we get:

$$\frac{1}{x^2 + 1} = \frac{A}{x + i} + \frac{B}{x - i} \quad (185)$$

multiplying both sides, we get:

$$1 = A(x - i) + B(x + i) \quad (186)$$

$$1 = (A + B)x + i(-A + B) \quad (187)$$

we have the systems of two equations:

$$x^1 : A + B = 0 \quad (188)$$

$$x^0 : (B - A)i = 1 \quad (189)$$

which gives $A = \frac{1}{2}i$ and $B = -\frac{1}{2}i$. This gives:

$$= \int \frac{0.5i}{x + i} dx - \int \frac{0.5i}{x - i} dx \quad (190)$$

$$= 0.5i \ln(x + i) - 0.5i \ln(x - i) + C \quad (191)$$

$$= 0.5i \ln \sqrt{x^2 + 1} + (0.5i)i \arctan \left(\frac{b}{x} \right) - (0.5i) \ln \sqrt{x^2 + 1} - (0.5i)i \arctan \left(-\frac{1}{x} \right) \quad (192)$$

$$= -\arctan \left(\frac{1}{x} \right) + C \quad (193)$$

Note that for $x \geq 0$:

$$-\arctan \left(\frac{1}{x} \right) + \frac{\pi}{2} = \arctan x \quad (194)$$

and for $x < 0$:

$$-\arctan \left(\frac{1}{x} \right) - \frac{\pi}{2} = \arctan x \quad (195)$$

- **Case 4:** Repeated irreducible quadratic terms, the decomposition is in the form of:

$$(x^2 + \beta x + 8)^k \implies \frac{A_1 x + B_1}{(x^2 + \beta x + 8)} + \frac{A_2 x + B_2}{(x^2 + \beta x + 8)^2} + \cdots + \frac{A_k x + B_k}{(x^2 + \beta x + 8)^k} \quad (196)$$

These can be extremely messy, but the process is similar to the above examples. For example, we can write:

$$\frac{Ax + B}{(x^2 + \beta x + 8)^2} = \frac{A}{2} \left[\frac{2x + \beta}{(x^2 + \beta x + 8)^2} + \frac{2B/A - \beta}{(x^2 + \beta x + 8)^2} \right] \quad (197)$$

Idea: The general strategy for dealing with a proper fraction integral is to break it up into two terms, one that can be easily be solved via a u-substitution and the second one does not have an x term in the numerator and can be solved using a trigonometric substitution.

- We can also introduce a strategy rationalizing substitutions by turning a function such as:

$$\int \frac{\sqrt{x}}{1 + x} dx \quad (198)$$

into a form that we are familiar with. We can let $u^2 = x \implies 2u \, du = dx$ to give:

$$= \int \frac{u \cdot 2u \, du}{1 + u^2} \quad (199)$$

$$= 2 \int \frac{u^2}{1 + u^2} \, du \quad (200)$$

$$= 2 \int \left(1 - \frac{1}{1 + u^2} \right) \, du \quad (201)$$

$$= 2u - 2 \tan^{-1} u + C \quad (202)$$

$$= 2\sqrt{x} - 2 \tan^{-1} \sqrt{x} + C \quad (203)$$

- Another method is to use a **Weierstrass substitution**, by making the substitution:

$$t = \tan \frac{x}{2} \quad (204)$$

which leads to the following substitutions:

$$\sin x = \frac{2t}{1 + t^2} \quad (205)$$

$$\cos x = \frac{1 - t^2}{1 + t^2} \quad (206)$$

$$dx = \frac{2}{1 + t^2} \, dt \quad (207)$$

This allows us to turn any trigonometric function into a rational function.

Example 28: For example, to solve the integral $\int \frac{dx}{1 + \cos x}$, we make the specified substitution to turn this into:

$$= \int \frac{1}{1 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} \, dt \quad (208)$$

$$= \int \frac{2 \, dt}{(1+t^2) + (1-t^2)} \, dt \quad (209)$$

$$= \int \, dt \quad (210)$$

$$= t + C \quad (211)$$

$$= \tan \left(\frac{x}{2} \right) + C \quad (212)$$