

# ESC195 Notes

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## Contents

<b>1</b>	<b>Hyperbolic Functions</b>	<b>1</b>
<b>2</b>	<b>Indeterminate Forms</b>	<b>2</b>
<b>3</b>	<b>Integration</b>	<b>6</b>
3.1	Recap of Integration . . . . .	6
3.2	Integration by Parts . . . . .	7
<b>4</b>	<b>Trigonometric Integrals</b>	<b>10</b>
<b>5</b>	<b>Partial Fractions</b>	<b>14</b>
<b>6</b>	<b>Improper Integrals</b>	<b>18</b>
<b>7</b>	<b>Applications of Integrals</b>	<b>22</b>
7.1	Arclength . . . . .	22
7.2	Area of a Surface of Revolution . . . . .	24
<b>8</b>	<b>Applications to Physics and Engineering</b>	<b>25</b>
<b>9</b>	<b>Parametric Equations</b>	<b>28</b>
<b>10</b>	<b>Calculus with Parametric Curves</b>	<b>29</b>
<b>11</b>	<b>Polar Coordinates</b>	<b>32</b>
<b>12</b>	<b>Areas and Lengths in Polar Coordinates</b>	<b>37</b>
<b>13</b>	<b>Infinite Sequences and Series</b>	<b>40</b>

## 1 Hyperbolic Functions

- Sometimes, combinations of  $e^x$  and  $e^{-x}$  are given certain names, for example:

- **Hyperbolic sine:**  $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$

- **Hyperbolic cosine:**  $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$

- They have the following properties:

$$\frac{d}{dx} \sinh x = \cosh x \quad (1)$$

$$\frac{d}{dx} \cosh x = \sinh x \quad (2)$$

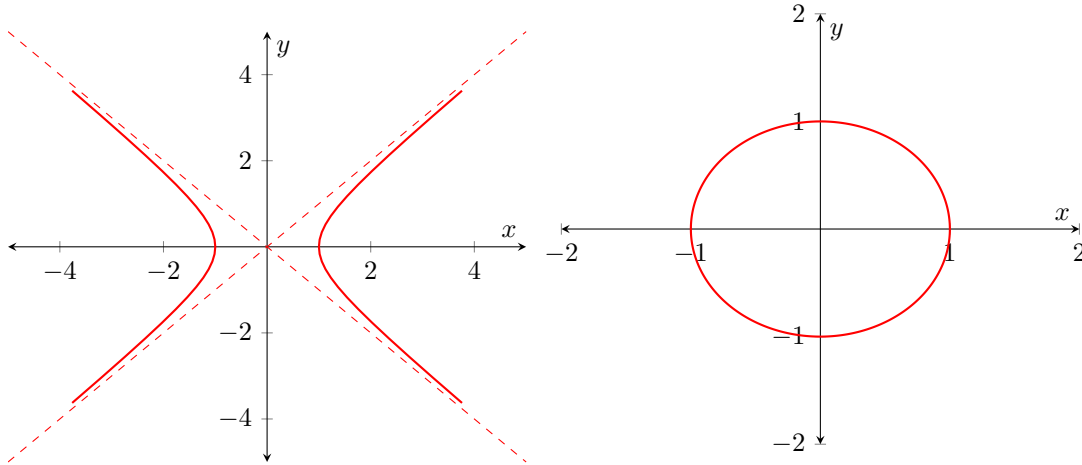
- They are related via:

$$\cosh^2 x - \sinh^2 x = 1 \quad (3)$$

- Both the area of a circular sector and that of a hyperbolic sector is described by:

$$A = \frac{1}{2}t \quad (4)$$

where  $t$  is the subtended angle, and the figures are parametrized by  $(\cos t, \sin t)$  and  $(\cosh t, \sinh t)$ .



- The catenary

$$y = a \cosh\left(\frac{x}{a}\right) + C \quad (5)$$

describes the shape of a free hanging rope between two walls separated by a width  $a$ .

- The hyperbolic tangent is given by  $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ . and its derivative is given by:

$$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x \quad (6)$$

- The inverse of  $y = \sinh x$  is given by:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \quad (7)$$

**Tip:** A table of integrals and derivatives revolving around hyperbolic trig functions can be found in the textbook.

## 2 Indeterminate Forms

- A lot of the times, limits have an indeterminate form, where if we substitute in what  $x$  approaches to, we get it in the form of  $\frac{0}{0}$ , for example:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \quad (8)$$

**Theorem:** If  $f(x) \rightarrow 0$  and  $g(x) \rightarrow 0$  as  $x \rightarrow \pm\infty$  or  $x \rightarrow c$  or  $x \rightarrow c^{+-}$  and if  $\frac{f'(x)}{g'(x)} \rightarrow L$ , then:

$$\frac{f(x)}{g(x)} \rightarrow L \quad (9)$$

**Example 1:** Solve:  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

We can set  $f(x) = \sin x$ ,  $f'(x) = \cos x$ ,  $g(x) = x$  and  $g'(x) = 1$  such that:

$$\lim_{x \rightarrow 0} \frac{f'}{g'} = \lim_{x \rightarrow 0} \cos x = 1 \quad (10)$$

**Example 2:** Solve  $\lim_{x \rightarrow 0^+} \frac{\sin x}{\sqrt{x}}$ .

Set  $f = \sin x$ ,  $f' = \cos x$ ,  $g = \sqrt{x}$ ,  $g' = \frac{1}{2}x^{-1/2}$  and so:

$$\lim_{x \rightarrow 0^+} 2x^{1/2} \cos x = 0 \implies \lim_{x \rightarrow 0^+} = 0 \quad (11)$$

**Example 3:** Solve  $\lim_{x \rightarrow 0} \frac{e^x - x - 1}{3x^2}$ .

If we take the derivative, we get:

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{6x} \quad (12)$$

which is still  $\frac{0}{0}$ !. We can take derivatives again:

$$\lim_{x \rightarrow 0} \frac{e^x}{6} = \frac{1}{6} \quad (13)$$

so the original limit is  $\frac{1}{6}$ .

**Warning:** L'hospital's rule can *only* be used in indeterminate forms. Applying them to limits where

- To prove the L'hospital's rule, we first prove the **Cauchy Mean Value Theorem** as a lemma

**Theorem: Cauchy Mean Value Theorem:** Given  $f$  and  $g$  differentiable on  $(a, b)$ , continuous on  $[a, b]$  and  $g' \neq 0$  on  $(a, b)$ , there must exist some number  $r$  in  $(a, b)$  such that:

$$\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad (14)$$

- We then apply **Rolle's Theorem** to prove the Cauchy Mean Value Theorem:

*Proof.* Set:

$$G(x) = [g(b) - g(a)][f(x) - f(a)] - [g(x) - g(a)][f(b) - f(a)]$$

Note that  $G(a) = G(b) = 0$  so it satisfies the conditions of Rolle's Theorem. Taking the derivative, we get:

$$G'(x) = [g(b) - g(a)]f'(x) - g'(x)[f(b) - f(a)] \quad (15)$$

According to Rolle's, there must be some  $x = r$  such that  $G'(r) = 0$ , we can then substitute for this and solve:

$$G'(r) = 0 \implies [g(b) - g(a)]f'(r) = g'(r)[f(b) - f(a)] \quad (16)$$

Which is equivalent to:

$$\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad (17)$$

Furthermore, we have  $g'(c) = \frac{g(b) - g(a)}{b - a}$  from the mean value theorem. Since  $g' \neq 0$  we have  $g(b) - g(a) \neq 0$ .  $\square$

- Given  $x \rightarrow c^+$  and  $f(x), g(x) \rightarrow 0$  where:

$$\lim_{x \rightarrow c^+} \frac{f'(x)}{g'(x)} = L \quad (18)$$

we will now prove that  $\lim_{x \rightarrow c^+} \frac{f(x)}{g(x)} = L$ .

*Proof.* Consider the interval  $[c, c + h]$  and apply Cauchy MVT. There must be some number  $c_2$  in  $[c, c + h]$  such that:

$$\frac{f'(c_2)}{g'(c_2)} = \frac{f(c + h) - f(c)}{g(c + h) - g(c)} = \frac{f(c + h)}{g(c + h)} \quad (19)$$

The last step is a result of the given  $f(c) = g(c) = 0$ . The LHS can be rewritten as:

$$\lim_{h \rightarrow 0} \frac{f'(c_2)}{g'(c_2)} = \frac{f'(c)}{g'(c)} \quad (20)$$

since  $c_2$  lies in the interval  $[c, c + h]$  so if  $h \rightarrow 0$ , then the interval becomes smaller to contain just  $c$ . The RHS can be rewritten as:

$$\lim_{h \rightarrow 0} \frac{f(c + h)}{g(c + h)} = \lim_{x \rightarrow c^+} \frac{f(x)}{g(x)} \quad (21)$$

and therefore:

$$\lim_{x \rightarrow c^+} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} = L \quad (22)$$

$\square$

- To prove the case for  $x \rightarrow \pm\infty$ , we can let  $x = \frac{1}{t}$  and take the limit as  $t \rightarrow \infty$ .

**Example 4:** Find  $\lim_{x \rightarrow \infty} \frac{\ln x}{x}$ .

Taking the derivative of top and bottom, we have:

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = 0 \implies \lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0 \quad (23)$$

**Idea:** The logarithm function grows very slowly. In fact, any positive power of  $x$  will grow faster than  $\ln x$ .

**Example 5:** Solve  $\lim_{x \rightarrow \infty} \frac{x^3}{e^x}$

This is indeterminate in the form of  $\frac{\infty}{\infty}$ . We apply L'hopital's rule multiple times:

$$\lim_{x \rightarrow \infty} \frac{x^3}{e^x} \stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{3x^2}{e^x} \left( = \frac{\infty}{\infty} \right) \quad (24)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{6x}{e^x} \left( = \frac{\infty}{\infty} \right) \quad (25)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{6}{e^x} = 0 \quad (26)$$

- Generally,  $\lim_{x \rightarrow \infty} \frac{x^m}{e^x} = 0$  where  $m$  is any positive integer.
- There are other indeterminate forms, such as  $0^0$ , for example:

$$\lim_{x \rightarrow 0} x^x \quad (27)$$

The central idea behind this is that  $a^b = e^{a \ln b}$ . Therefore, this limit is equal to:

$$\lim_{x \rightarrow 0} e^{x \ln x} \quad (28)$$

We can take the limit of the exponent to get:

$$\lim_{x \rightarrow 0} x \ln x = \lim_{x \rightarrow 0} \frac{\ln x}{1/x} \quad (29)$$

Note that the first equation is another indeterminate form with the  $0 \cdot \infty$  type, so we had to multiply top and bottom by  $\frac{1}{x}$  to get the quotient form. Then we have:

$$\lim_{x \rightarrow 0} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x \rightarrow 0} -x = 0 \quad (30)$$

Therefore:

$$\lim_{x \rightarrow 0} e^{x \ln x} = e^0 = 1 \quad (31)$$

so  $\lim_{x \rightarrow 0} x^x = 1$ .

**Example 6:** Solve  $\lim_{x \rightarrow \infty} (x+2)^{2/\ln x}$ .

This is of the type  $\infty^0$ . The approach is exactly the same as the previous example. We write it in exponential form:

$$= \lim_{x \rightarrow \infty} e^{\frac{2}{\ln x} \ln(x+2)} \quad (32)$$

and looking at the exponent gives:

$$\lim_{x \rightarrow \infty} \frac{2 \ln(x+2)}{\ln x} \quad (33)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{\left(\frac{2}{x+2}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{2x}{x+2} \left(= \frac{\infty}{\infty}\right) \quad (34)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{2}{1} = 2 \quad (35)$$

Therefore:

$$\lim_{x \rightarrow \infty} e^{\frac{2}{\ln x} \ln(x+2)} = e^2 \quad (36)$$

so:

$$\lim_{x \rightarrow \infty} (x+2)^{2/\ln x} = e^2 \quad (37)$$

**Example 7:** Solve  $\lim_{x \rightarrow \infty} \left[ \sin \left( \frac{\pi}{x} + \frac{\pi}{2} \right) \right]^x$

This is in the form of  $1^\infty$ . We rewrite it as:

$$\lim_{x \rightarrow \infty} \exp \left( x \ln \left( \sin \left( \frac{\pi}{x} + \frac{\pi}{2} \right) \right) \right) \quad (38)$$

and taking the limit of the exponent:

$$= \lim_{x \rightarrow \infty} x \ln \left( \sin \left( \frac{\pi}{x} + \frac{\pi}{2} \right) \right) \left(= \frac{0}{0}\right) \quad (39)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{\cos \left( \frac{\pi}{x} + \frac{\pi}{2} \right) \cdot \left( -\frac{\pi}{x^2} \right)}{\sin \left( \frac{\pi}{x} + \frac{\pi}{2} \right) \cdot \left( -\frac{1}{x^2} \right)} = \frac{0 \cdot \pi}{1} = 0 \quad (40)$$

Therefore:

$$\lim_{x \rightarrow \infty} \left[ \sin \left( \frac{\pi}{x} + \frac{\pi}{2} \right) \right]^x = \lim_{x \rightarrow \infty} \exp \left( x \ln \left( \sin \left( \frac{\pi}{x} + \frac{\pi}{2} \right) \right) \right) = 1 \quad (41)$$

### 3 Integration

#### 3.1 Recap of Integration

- The definite integral has the geometric interpretation as the area under the curve  $f(x)$  between  $x = a$  and  $x = b$  and the  $x$  axis:

$$\int_a^b f(x) dx \quad (42)$$

but can be rigorously defined using a Riemann sum:

$$\int_a^b f(x) dx \equiv \lim_{\|P\|} \sum_{i=1}^n f(x_i^*) \Delta x_i \quad (43)$$

Often, we have a uniform partition, such that  $\Delta x_i = \frac{b-a}{n}$  where  $n$  is the number of partitions. And if we choose to use the right hand endpoint, then:

$$f(x_i^*) = f(x_i) = f\left(a + \frac{b-a}{n}i\right) \quad (44)$$

**Example 8:** To solve  $\int_0^5 x^2 dx$ , we can choose a uniform partition with:

$$\Delta x = \frac{5-0}{n} = \frac{5}{n} \quad (45)$$

and:

$$x_i^* = x_i = i\Delta x \implies f(x_i^*) = (i\Delta x)^2 = \left(i\frac{5}{n}\right)^2 \quad (46)$$

The area approximation is:

$$A \simeq \sum_{i=1}^n \Delta x_i f(x_i^*) = \sum_{i=1}^n \left(\frac{5}{n}\right) \left(i\frac{5}{n}\right)^2 \quad (47)$$

$$= \frac{125}{n^2} \sum_{i=1}^n i^2 = \frac{125}{n^3} \frac{n(n+1)(2n+1)}{6} \quad (48)$$

Taking the limit as  $n \rightarrow \infty$ , we get:

$$\int_0^5 x^2 dx = \lim_{n \rightarrow \infty} \frac{125}{6} \left(2 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{5^3}{3}. \quad (49)$$

**Example 9:** To evaluate  $\int_1^2 x^{-2} dx$ , we can choose

$$x_i^* = \sqrt{x_{i-1}x_i} \quad (50)$$

and a uniform partition of:

$$\Delta x = \frac{2-1}{n} = \frac{1}{n} \quad (51)$$

such that:

$$x_i = 1 + i\Delta x = 1 + \frac{i}{n} = \frac{n+i}{n} \quad (52)$$

and

$$x_{i-1} = \frac{n+i-1}{n} \quad (53)$$

such that the area is:

$$\begin{aligned}
A &\simeq \sum_{i=1}^n \Delta x f(x_i^*) \\
&= \sum_{i=1}^n \frac{1}{n} \left( \frac{1}{x_i^*} \right)^2 \\
&= \sum_{i=1}^n \frac{1}{n} \frac{1}{x_{i-1} x_i} \\
&= \sum_{i=1}^n \frac{1}{n} \frac{n}{n+i-1} \cdot \frac{n}{n+i} \\
&= \sum_{i=1}^n \frac{1}{n+i-1} \cdot \frac{1}{n+i} \\
&= \sum_{i=1}^n \left( \frac{1}{n+i-1} - \frac{1}{n+i} \right) \\
&= n \left[ \sum_{i=1}^n \frac{1}{n+i-1} - \sum_{i=1}^n \frac{1}{n+i} \right] \\
&= n \left[ \sum_{i=0}^n \frac{1}{n+i} - \sum_{i=1}^n \frac{1}{n+i} \right] \\
&= n \left[ \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n} \right] \\
&= n \left( \frac{1}{n} - \frac{1}{2n} \right) \\
&= 1 - \frac{1}{2} = \frac{1}{2}
\end{aligned}$$

The part where we cancel out everything is called a **telescoping series**. Notice how the area doesn't depend on  $n$  so we get the exact area, even if we let  $n = 1$ !

- We need a better way to do integration, so we can define:

$$F(x) \equiv \int_a^x f(t) dt \quad (54)$$

such that  $F'(x) = f(x)$ . This is the definition of the antiderivative. This leads to the fundamental theorem of calculus:

$$\int_a^b f(t) dt = F(b) - F(a) \quad (55)$$

and the indefinite integral can be written as:

$$\int f(x) dx = G(x) + C \quad (56)$$

The main problem now becomes trying to *find antiderivatives*, which is much easier than Riemann sums, though still more difficult than calculating derivatives.

### 3.2 Integration by Parts

- **Integration by Parts** attempts to reverse the product rule:

$$(fg)' = fg' + f'g \quad (57)$$

Taking the integral of both sides gives:

$$f(x)g(x) = \int f(x)g'(x) dx + \int f'(x)g(x) dx \quad (58)$$

$$\int f(x)g'(x) dx = \int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx \quad (59)$$

If the second integral is easier than the first, then we have made substantial progress.

**Idea:** Integration of parts tells us that:

$$\int u dv = uv - \int v du \quad (60)$$

**Example 10:** To solve  $\int xe^{2x}$ , we can let:

$$u = x \quad dv = e^{2x} dx \quad (61)$$

$$du = dx \quad v = \frac{1}{2}e^{2x} \quad (62)$$

which gives:

$$\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x} dx \quad (63)$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \quad (64)$$

We can check:

$$\frac{d}{dx} \left( \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \right) \quad (65)$$

$$= xe^{2x} + \frac{1}{2}e^{2x} - \frac{2}{4}e^{2x} \quad (66)$$

$$= xe^{2x} \quad (67)$$

**Example 11:** To solve  $\int x^2 \sin(2x) dx$ , we let:

$$u = x^2 \quad dv = \sin 2x dx \quad (68)$$

$$du = 2x dx \quad v = -\frac{1}{2} \cos(2x) \quad (69)$$

which gives:

$$= -\frac{1}{2}x^2 \cos 2x + \int x \cos(2x) dx \quad (70)$$

and we can apply integration by parts a second time, if we let:

$$u = x \quad dv = \cos 2x dx \quad (71)$$

$$du = dx \quad v = \frac{1}{2} \sin(2x) \quad (72)$$

which gives us:

$$= -\frac{1}{2}x^2 \cos(2x) + \frac{1}{2}x \sin(2x) - \int \frac{1}{2} \sin(2x) dx \quad (73)$$

$$= -\frac{1}{2}x^2 \cos(2x) + \frac{1}{2}x \sin(2x) + \frac{1}{4} \cos(2x) + C \quad (74)$$



**Example 12:** To solve  $I = \int e^x \sin x \, dx$ , we can let:

$$u = \sin x \qquad dv = e^x \, dx \qquad (75)$$

$$du = \cos x \, dx \qquad v = e^x \qquad (76)$$

to give us:

$$= e^x \sin x - \int e^x \cos x \, dx \qquad (77)$$

We apply integration by parts a second time:

$$u = \cos x \qquad dv = e^x \, dx \qquad (78)$$

$$du = -\sin x \, dx \qquad v = e^x \qquad (79)$$

to get:

$$I = e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x \, dx}_I \qquad (80)$$

$$2I = e^x (\sin x - \cos x) + C' \qquad (81)$$

$$I = \frac{1}{2} e^x (\sin x - \cos x) + C \qquad (82)$$

and we are done.

**Example 13:** We can also solve integrals that do not appear to have parts, such as  $\int \ln x \, dx$ . We choose:

$$u = \ln x \qquad dv = dx \qquad (83)$$

$$du = \frac{1}{x} \, dx \qquad v = x \qquad (84)$$

to give us:

$$\ln x - \int dx = x \ln x - x + C \qquad (85)$$

- For a definite integral, we can write IBP as:

$$f(x)g(x) \Big|_a^b - \int_a^b f'(x)g(x) \, dx \qquad (86)$$

**Example 14:** It is *possible* to apply integration of parts to find the integral of  $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$ . We can let:

$$u = \frac{1}{\cos x} = \sec x \qquad dv = \sin x \, dx \qquad (87)$$

$$du = \sec x \tan x \qquad v = -\cos x \qquad (88)$$

this gives us:

$$\int \tan x \, dx = -\frac{\cos x}{\cos x} + \int \tan x \, dx \qquad (89)$$

Notice that we could try to subtract the original integral from both sides and get:

$$0 = -1 \qquad (90)$$

which is clearly wrong! However, we forgot the constant of integration, so the correct statement would be:

$$0 + C' = -1 + C \qquad (91)$$

which does not tell us anything interesting. This brings We can see this concretely by repeating the same steps but trying to evaluate the definite integral  $\int_a^b \tan x \, dx$  instead, which gives:

$$\int_a^b \tan x \, dx = (-1) \Big|_{x=a}^{x=b} + \int_a^b \tan x \, dx \implies 0 = (-1) - (-1) \implies 0 = 0 \quad (92)$$

which confirms our suspicion that this isn't anything useful, but it's also not an incorrect statement.

**Warning:** Sometimes it is possible to get more than one answer through various means that differ by a constant factor when solving indefinite integrals. When this happens, nothing is wrong: we simply need to consider the constant of integration.

**Idea:** But how do we know *which* values of  $u$  and  $dv$  we should pick? A common strategy is to use **LIATE**:

1. L: Logarithms
2. I: Inverse Trig
3. A: Algebraic
4. T: Trigonometric
5. E: Exponential

If a function consists of two terms, the term that is higher up (closer to L) usually gets differentiated and the term near the bottom (closer to E) usually gets integrated. See [this](#) for how it works, and this [video](#) for a tutorial.

## 4 Trigonometric Integrals

- The first type of integral we'll deal with is:

$$\int \sin^n x \cos^n x \, dx \quad (93)$$

- In **case 1**, we have either  $m$  or  $n$  as an odd positive number. We can then use the identity  $\sin^2 x + \cos^2 x = 1$  to simplify it.

**Example 15:** For example, to solve  $\int \sin^3 x \cos^2 x \, dx$ , we can simplify this to:

$$= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx \quad (94)$$

$$= (\cos^2 x - \cos^4 x) \sin x \, dx \quad (95)$$

and applying a  $u$  substitution with  $u = \cos x$  and breaking it up into two integrals, we can get:

$$= -\frac{1}{3} \cos^3 x + \frac{1}{5} \cos^5 x + C \quad (96)$$

- In **case 2**, we have  $m$  and  $n$  as both even. We then apply the double angle formulas:

$$\sin x \cos x = \frac{1}{2} \sin(2x) \quad (97)$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \cos 2x \quad (98)$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x \quad (99)$$

**Example 16:** For example:

$$\int \sin^2 x \cos^4 x \, dx = \int \frac{1}{4} \sin^2(2x) \left( \frac{1}{2} + \frac{1}{2} \cos 2x \right) dx \quad (100)$$

$$= \frac{1}{8} \int \sin^2(2x) \, dx + \frac{1}{8} \int \sin^2 x \cos 2x \, dx \quad (101)$$

$$= \frac{1}{8} \int \left( \frac{1}{2} - \frac{1}{2} \cos 4x \right) dx + \frac{1}{8 \cdot 3 \cdot 2} \sin^3(2x) + C \quad (102)$$

$$= \frac{1}{16} x - \frac{1}{64} \sin(4x) + \frac{1}{48} \sin^3(2x) + C \quad (103)$$

- In **Case 3**, we have:

$$\int \sin^n x \, dx, \int \cos^n x \, dx \quad (104)$$

which we can apply a reduction formula by keep applying integration by parts:

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \quad (105)$$

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx \quad (106)$$

**Example 17:** To solve the integral  $\int \sin^2 x \, dx$ , we get:

$$= -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int dx \quad (107)$$

$$= \frac{1}{2} x - \frac{1}{4} \sin 2x + C \quad (108)$$

- In **Case 4**, we have integrals in the following forms:

$$\int \sin(mx) \cos(nx) \, dx \quad (109)$$

$$\int \sin(mx) \sin(nx) \, dx \quad (110)$$

$$\int \cos(mx) \cos(nx) \, dx \quad (111)$$

with  $m \neq n$ . If  $m = n$ , then we can apply the double angle formula. To solve these, we apply the following identities:

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)] \quad (112)$$

$$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)] \quad (113)$$

$$\sin A \cos B = \frac{1}{2} [\sin(A - B) + \sin(A + B)] \quad (114)$$

**Example 18:** For example, we have:

$$\int \sin(3x) \sin(2x) \, dx = \frac{1}{2} \int \cos((3-2)x) \, dx - \frac{1}{2} \int \cos((3+2)x) \, dx \quad (115)$$

$$= \frac{1}{2} \sin x - \frac{1}{10} \sin 5x + C \quad (116)$$

- In **case 5**, we have integrals in the form of either:

$$\int \tan^n x \, dx, \int \cot^n x \, dx \quad (117)$$

To solve these, we apply the following identities:

$$\tan^2 x = \sec^2 x - 1 \quad (118)$$

$$(\tan x)' = \sec^2 x \quad (119)$$

- In **case 6**, we have:

$$\int \sec^n x \, dx, \int \csc^n x \, dx \quad (120)$$

with  $n \geq 2$ . To solve these, we can make the following substitutions:

$$1 + \tan^2 x = \sec^2 x \quad (121)$$

$$1 + \cot^2 x = \csc^2 x \quad (122)$$

to convert it to a case 5 problem.

- In **case 7**, we have:

$$\int \tan^n x \sec^n x \, dx, \int \cot^n x \csc^n x \, dx \quad (123)$$

**Example 19:** We have:

$$\tan^3 x \sec^4 x \, dx = \int \tan^3 x \sec^2 x \sec^2 x \, dx \quad (124)$$

$$= \int \tan^3 x (\tan^2 x + 1) \sec^2 x \, dx \quad (125)$$

$$= \int (\tan^5 x + \tan^3 x) \sec^2 x \, dx \quad (126)$$

$$= \frac{1}{6} \tan^6 x + \frac{1}{4} \tan^4 x + C \quad (127)$$

**Idea:** The basic idea of these types is to apply trigonometric identities to turn the integrals into a form that is easier to deal with. The substitutions are usually very simple but to find them, it requires a lot of practice.

- We can also apply **trigonometric substitutions**, any integrals with any of the three factors below can be solved with this technique:

$$1. \sqrt{a^2 - x^2}: \text{Set } x = a \sin u \implies \sqrt{a^2 - x^2} = a \cos u$$

$$2. \sqrt{a^2 + x^2}: \text{Set } x = a \tan u \implies \sqrt{a^2 + x^2} = a \sec u$$

$$3. \sqrt{x^2 - a^2}: \text{Set } x = a \sec u \implies \sqrt{x^2 - a^2} = a \tan u$$

where the arguments under the square roots are always positive.

**Example 20:** To solve the integral  $\int \frac{x^2}{(4 - x^2)^{3/2}} \, dx$ , we can set:

$$x = 2 \sin u \quad (128)$$

$$dx = 2 \cos u \, du \quad (129)$$

$$\sqrt{4 - x^2} = 2 \cos u \quad (130)$$

which gives:

$$= \int \frac{4 \sin^2 u \cdot 2 \cos u \, du}{8 \cos^3 u} \quad (131)$$

$$= \int \tan^2 u \, du \quad (132)$$

$$= \int (\sec^2 u - 1) \, du \quad (133)$$

$$= \tan u - u + C = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C \quad (134)$$

**Example 21:** The integral  $\int \frac{x \, dx}{(2x^2 + 4x - 7)^{1/2}}$  needs a bit more work before we can apply the substitutions. We first apply the square to get:

$$= \int \frac{x \, dx}{\sqrt{2(x+1)^2 - 9}} \quad (135)$$

We can set:

$$\sqrt{2}(x+1) = 3 \sec u \quad (136)$$

$$\sqrt{2} \, dx = 3 \sec u \tan u \, du \quad (137)$$

$$\sqrt{2(x+1)^2 - 9} = 3 \tan u \quad (138)$$

which gives:

$$= \int \frac{\left(\frac{3}{\sqrt{2}} \sec u - 1\right) \left(\frac{3}{\sqrt{2} \sec u \tan u} du\right)}{3 \tan u} \quad (139)$$

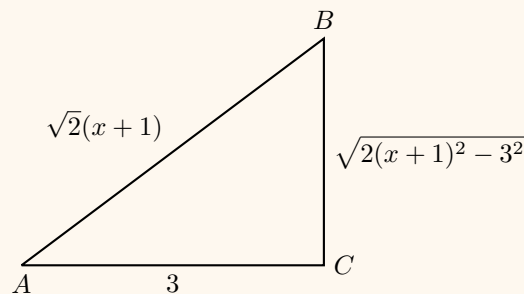
$$= \int \left(\frac{3}{\sqrt{2}} \sec u - 1\right) \left(\frac{1}{\sqrt{2}} \sec u\right) du \quad (140)$$

$$= \frac{3}{2} \int \sec^2 u \, du - \frac{1}{\sqrt{2}} \int \sec u \, du \quad (141)$$

$$= \frac{3}{2} \tan u - \frac{1}{\sqrt{2}} \ln |\sec u + \tan u| + C \quad (142)$$

$$= \frac{1}{2} \sqrt{2x^2 + 4x - 7} - \frac{1}{\sqrt{2}} \ln \left| \frac{\sqrt{2}}{3} (x+1) + \frac{\sqrt{2x^2 + 4x - 7}}{3} \right| + C \quad (143)$$

**Idea:** We can use triangles to derive the substitution, which comes from the Pythagorean theorem:



and you can clearly see the substitution:

$$3 \sec u = \sqrt{2}(x+1) \implies \cos u = \frac{3}{\sqrt{2}(x+1)} \quad (144)$$

where  $u \equiv \angle BAC$ .

**Example 22:** For the integral  $\int x \sin^{-1} x \, dx$ , we can let:

$$u = \sin^{-1} x \, dv = x \, dx \quad (145)$$

$$du = \frac{dx}{\sqrt{1-x^2}} v = \frac{1}{2} x^2 \quad (146)$$

and applying integration by parts, we get:

$$= \frac{1}{2} x^2 \sin^{-1} x - \int \frac{1}{2} x^2 \frac{dx}{\sqrt{1-x^2}} \quad (147)$$

To solve this secondary integral  $\int \frac{x^2 \, dx}{\sqrt{1-x^2}}$ , we can let:

$$x = \sin \theta \quad (148)$$

$$dx = \cos \theta \, d\theta \quad (149)$$

$$\sqrt{1-x^2} = \cos \theta \quad (150)$$

which gives:

$$= \frac{\sin^2 \theta \cos \theta \, d\theta}{\cos \theta} \quad (151)$$

$$= \int \sin^2 \theta \, d\theta \quad (152)$$

$$= \frac{1}{2} \theta - \frac{1}{2} \sin \theta \cos \theta + C \quad (153)$$

$$= \frac{1}{2} \sin^{-1} x - \frac{1}{2} x \sqrt{1-x^2} + C \quad (154)$$

Therefore, we get:

$$\int x \sin^{-1} x \, dx = \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{4} \sin^{-1} x + \frac{1}{4} x \sqrt{1-x^2} + C \quad (155)$$

## 5 Partial Fractions

- Rational functions are in the form of:

$$R(x) = \frac{P_n(x)}{P_m(x)} \quad (156)$$

where  $m, n$  represent the order of the polynomial. If  $n \geq m$ , it is an **improper** fraction, such as:

$$\frac{x^2 - x}{1 + x} \quad (157)$$

and if  $n < m$ , we have a proper fraction such as:

$$\frac{x}{x^2 + 3x + 2} \quad (158)$$

- If we have an improper fraction, we use long division to simplify it. For example:

$$\frac{x^3 - 2x^2}{x^2 + 9} = x - 2 + \frac{18 - 9x}{x^2 + 9} \quad (159)$$

which turns the expression into a polynomial (trivial to integrate) as well as a proper fraction.

- There are different types of factors:
  - Linear factors (e.g.  $3x + 2$ )

- Irreducible quadratic factors (e.g.  $x^2 + 1$ )

which gives us the different factors:

- **Case 1:** If we have distinct linear factors in the denominator, we can break it into fractions of the form:

$$(x + \alpha) \implies \frac{A}{x + \alpha} \quad (160)$$

**Example 23:** The partial fraction of  $\frac{2x - 17}{x^2 + 3x + 2}$  can be written as the **partial fraction deconvolution**:

$$= \frac{A}{x + 1} + \frac{B}{x + 2} \quad (161)$$

We now need to solve for  $A$  and  $B$ . We can multiply both sides by  $(x + 1)(x + 2)$  to get:

$$2x - 17 = A(x + 2) + B(x + 1) \quad (162)$$

and match up the coefficients. Alternatively, we can pick various values of  $x$  (e.g.  $x = -2$  and  $x = -1$ ) to solve for the coefficients.

- **Case 2:** If we have repeated linear factors, then the decomposition is in the form of:

$$(x + \alpha)^k \implies \frac{A}{x + \alpha} + \frac{B}{(x + \alpha)^2} + \frac{C}{(x + \alpha)^3} + \cdots + \frac{K}{(x + \alpha)^k} \quad (163)$$

**Example 24:** To get the decomposition of  $\frac{2}{x(x + 1)^2}$ , we can get:

$$\frac{2}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{x + 1} + \frac{C}{(x + 1)^2} \quad (164)$$

which gives:

$$2 = A(x + 1)^2 + Bx(x + 1) + Cx \quad (165)$$

matching the coefficients, we get three equations and three unknowns:

$$x^2 : A + B = 0 \quad (166)$$

$$x : 2A + B + C = 0 \quad : A = 2 \quad (167)$$

Solving this system gives  $A = 2$ ,  $B = -2$ , and  $C = -2$ . Note that taking the integral of this sum is much easier. We have:

$$\int \frac{d}{x(x + 1)^2} dx = \int \frac{2}{x} dX - \int \frac{2}{x} dx - \int \frac{2}{(x + 1)^2} dx \quad (168)$$

$$= 2 \ln |x| - 2 \ln |x + 1| + \frac{2}{x + 1} + C \quad (169)$$

**Idea:** As a general rule of thumb, the number of unknown coefficients is equal to the order of the polynomial in the denominator.

- **Case 3:** If we have irreducible quadratic factors, then the partial fraction deconvolution is in the form of:

$$x^2 + px + 8 \implies \frac{Ax + B}{x^2 + px + 8} \quad (170)$$

**Example 25:** Suppose we have  $\frac{2}{(x+1)(x^2+x+1)}$ , we can get the partial fraction decomposition as:

$$= \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1} \quad (171)$$

and we work through the deconvolution process in exactly the same way, we remove the denominators on both sides to get (after expanding):

$$2 = Ax^2 + Ax + A + Bx^2 + Bx + Cx + C \quad (172)$$

$$0x^2 + 0x^1 + 2x^0 = (A+B)x^2 + (A+B+C)x^1 + (A+C)x^0 \quad (173)$$

which gives three equations and three unknowns, after we match coefficients:

$$x^2 : A + B = 0 \quad (174)$$

$$x : A + B + C = 0 \quad (175)$$

$$1 : A + C = 2 \quad (176)$$

and solving the system of equations gives  $A = 2, B = -2, C = 0$ . To get the integral of this second term, we can write the second term as:

$$\int \frac{2x \, dx}{x^2 + 2x + 1} = \underbrace{\int \frac{2x+1}{x^2+x+1} \, dx}_{(1)} - \underbrace{\int \frac{dx}{x^2+x+1}}_{(2)} \quad (177)$$

We “added” 1 and “subtracted” 1 to get these two slightly easier integrals, which we can apply other techniques. The first one can be solved using a u-sub while the second can be solved by completing the square and applying a trigonometric substitution:

$$(1) = \ln |x^2 + x + 1| + C \quad (178)$$

$$(2) = \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}} \tan^{-1} \left[ \frac{2}{\sqrt{3}} \left(x + \frac{1}{2}\right) \right] + C \quad (179)$$

allowing us to put everything together.

**Example 26:** Let’s take an integral we already know the answer of:  $\int \frac{2x}{x^2+1} \, dx = \ln(x^2+1) + C$ . We can try a partial fraction decomposition:

$$\frac{2x}{x^2+1} = \frac{A}{x+i} + \frac{B}{x-i} = \frac{1}{x+i} + \frac{1}{x-i} \quad (180)$$

which gives:

$$\int \frac{2x}{x^2+1} \, dx = \int \frac{dx}{x+i} + \int \frac{dx}{x-i} \quad (181)$$

In complex analysis, most mathematical functions we are familiar with are still valid, so the integral is:

$$= \ln |x+i| + \ln |x-i| + C \quad (182)$$

and simplifying it gives:

$$\ln(x^2+1) + C \quad (183)$$

**Warning:** While it is *possible* to use complex numbers to solve irreducible quadratic factors, it isn’t always as easy as the above example. To get the logarithm of a complex number, we can apply the identity (without



proving):

$$\ln(a + ib) = \ln \sqrt{a^2 + b^2} + i \arctan \left( \frac{b}{a} \right) \quad (184)$$

**Example 27:** Bonus content: Try evaluating the integral  $\int \frac{dx}{x^2 + 1}$  with complex analysis. Taking a partial fraction, we get:

$$\frac{1}{x^2 + 1} = \frac{A}{x + i} + \frac{B}{x - i} \quad (185)$$

multiplying both sides, we get:

$$1 = A(x - i) + B(x + i) \quad (186)$$

$$1 = (A + B)x + i(-A + B) \quad (187)$$

we have the systems of two equations:

$$x^1 : A + B = 0 \quad (188)$$

$$x^0 : (B - A)i = 1 \quad (189)$$

which gives  $A = \frac{1}{2}i$  and  $B = -\frac{1}{2}i$ . This gives:

$$= \int \frac{0.5i}{x + i} dx - \int \frac{0.5i}{x - i} dx \quad (190)$$

$$= 0.5i \ln(x + i) - 0.5i \ln(x - i) + C \quad (191)$$

$$= 0.5i \ln \sqrt{x^2 + 1} + (0.5i)i \arctan \left( \frac{b}{x} \right) - (0.5i) \ln \sqrt{x^2 + 1} - (0.5i)i \arctan \left( -\frac{1}{x} \right) \quad (192)$$

$$= -\arctan \left( \frac{1}{x} \right) + C \quad (193)$$

Note that for  $x \geq 0$ :

$$-\arctan \left( \frac{1}{x} \right) + \frac{\pi}{2} = \arctan x \quad (194)$$

and for  $x < 0$ :

$$-\arctan \left( \frac{1}{x} \right) - \frac{\pi}{2} = \arctan x \quad (195)$$

- **Case 4:** Repeated irreducible quadratic terms, the decomposition is in the form of:

$$(x^2 + \beta x + 8)^k \implies \frac{A_1 x + B_1}{(x^2 + \beta x + 8)} + \frac{A_2 x + B_2}{(x^2 + \beta x + 8)^2} + \cdots + \frac{A_k x + B_k}{(x^2 + \beta x + 8)^k} \quad (196)$$

These can be extremely messy, but the process is similar to the above examples. For example, we can write:

$$\frac{Ax + B}{(x^2 + \beta x + 8)^2} = \frac{A}{2} \left[ \frac{2x + \beta}{(x^2 + \beta x + 8)^2} + \frac{2B/A - \beta}{(x^2 + \beta x + 8)^2} \right] \quad (197)$$

**Idea:** The general strategy for dealing with a proper fraction integral is to break it up into two terms, one that can be easily be solved via a u-substitution and the second one does not have an  $x$  term in the numerator and can be solved using a trigonometric substitution.

- We can also introduce a strategy rationalizing substitutions by turning a function such as:

$$\int \frac{\sqrt{x}}{1 + x} dx \quad (198)$$

into a form that we are familiar with. We can let  $u^2 = x \implies 2u \, du = dx$  to give:

$$= \int \frac{u \cdot 2u \, du}{1 + u^2} \quad (199)$$

$$= 2 \int \frac{u^2}{1 + u^2} \, du \quad (200)$$

$$= 2 \int \left(1 - \frac{1}{1 + u^2}\right) \, du \quad (201)$$

$$= 2u - 2 \tan^{-1} u + C \quad (202)$$

$$= 2\sqrt{x} - 2 \tan^{-1} \sqrt{x} + C \quad (203)$$

- Another method is to use a **Weierstrass substitution**, by making the substitution:

$$t = \tan \frac{x}{2} \quad (204)$$

which leads to the following substitutions:

$$\sin x = \frac{2t}{1 + t^2} \quad (205)$$

$$\cos x = \frac{1 - t^2}{1 + t^2} \quad (206)$$

$$dx = \frac{2}{1 + t^2} \, dt \quad (207)$$

This allows us to turn any trigonometric function into a rational function.

**Example 28:** For example, to solve the integral  $\int \frac{dx}{1 + \cos x}$ , we make the specified substitution to turn this into:

$$= \int \frac{1}{1 + \frac{1 - t^2}{1 + t^2}} \cdot \frac{2}{1 + t^2} \, dt \quad (208)$$

$$= \int \frac{2 \, dt}{(1 + t^2) + (1 - t^2)} \, dt \quad (209)$$

$$= \int \, dt \quad (210)$$

$$= t + C \quad (211)$$

$$= \tan\left(\frac{x}{2}\right) + C \quad (212)$$

## 6 Improper Integrals

- Since infinity is not a number, our typical definite integral definition cannot be used for an **improper integral** like:

$$\int_0^\infty f(x) \, dx \quad (213)$$

Instead, we use the following definition:

**Definition:** If  $\lim_{b \rightarrow \infty} \int_a^b f(x) \, dx = L$  exists, then we can define:

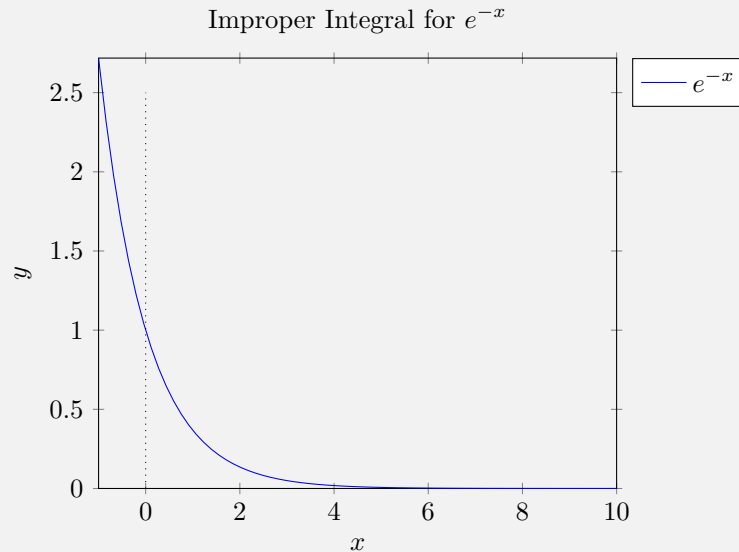
$$\int_a^\infty f(x) \, dx = L \quad (214)$$

**Example 29:** To solve  $\int_0^{\infty} e^{-x} dx$ , we can write it as:

$$= \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx \quad (215)$$

$$= \lim_{b \rightarrow \infty} (1 - e^{-b}) = 1 \quad (216)$$

This is remarkable because even though the area appears infinite (since it is infinitely long), the area is actually finite.



**Example 30:** For the integral  $\int_{-\infty}^{-1} \frac{dx}{x^2}$ , we have:

$$= \lim_{a \rightarrow -\infty} \int_a^{-1} \frac{dx}{x^2} \quad (217)$$

$$= \lim_{a \rightarrow -\infty} \left(1 + \frac{1}{a}\right) = 1 \quad (218)$$

- However, improper integrals can diverge as well.

**Example 31:** For  $\int_3^{\infty} \frac{dx}{x}$ , we get:

$$= \lim_{b \rightarrow \infty} (\ln b - \ln 3) = \infty \quad (219)$$

**Example 32:** For something like  $\int_{-\infty}^{2\pi} \sin x dx$ , the integral does not go to infinity, but since we get:

$$\lim_{a \rightarrow -\infty} (-1 + \cos a) \quad (220)$$

it will diverge, since  $\lim_{a \rightarrow -\infty} \cos a$  does not exist.

- We can generalize this for all reciprocal functions:

**Idea:** For  $\int_a^\infty \frac{dx}{x^p}$  with  $p > 0$ ,  $p \neq 1$ , and  $a > 0$ , we get:

$$= \lim_{b \rightarrow \infty} \int_a^b \frac{dx}{x^p} \quad (221)$$

$$= \lim_{b \rightarrow \infty} \left( \frac{1}{1-p} x^{-p+1} \right) \Big|_a^b \quad (222)$$

$$= \lim_{b \rightarrow \infty} \left( \frac{b^{-p+1}}{1-p} - \frac{a^{-p+1}}{1-p} \right) \quad (223)$$

For  $p > 1$ , we get:

$$= \frac{a^{1-p}}{p-1} \quad (224)$$

and diverges for  $p \leq 1$ .

- There are techniques to check if an improper integral will converge or diverge. This is useful especially if we want to perform a numerical integration but want to verify that it indeed will converge.

**Theorem:** Let  $f, g$  be continuous functions and  $0 \leq f(x) \leq g(x)$  where  $x \in [a, \infty)$ .

- If  $\int_a^\infty g \, dx$  converges, so does  $\int_a^\infty f(x) \, dx$ .
- If  $\int_a^\infty f$  diverges, so does  $\int_a^\infty g(x) \, dx$ .

**Example 33:** The integral  $\int_2^\infty \frac{dx}{\sqrt{1+x^{44/17}}}$  is difficult to evaluate, but we can easily tell that it converges via:

$$\frac{1}{\sqrt{1+x^{44/12}}} < \frac{1}{\sqrt{x^{44/12}}} = \frac{1}{x^{22/12}} \quad (225)$$

Since  $p > 1$ , this converges, so the original integral must also converge.

**Example 34:** For the integral  $\int_3^\infty \frac{dx}{\sqrt{7+x^2}}$ , we can check that it diverges by:

$$(7+x^2)^{1/2} < \sqrt{7} + x \quad (226)$$

We can check this via:  $7+x^2 < 7+2\sqrt{7}+x^2$ . Since:

$$\int_3^\infty \frac{dx}{\sqrt{7}+x} = \ln(\sqrt{7}+x) \Big|_3^\infty \quad (227)$$

which diverges, so the original integral must also diverge.

**Warning:** The notation  $f(x) \Big|_3^\infty$  needs to be defined explicitly since  $\infty$  is not a number. This expression simply implies that we are taking the limit as  $b$  approaches infinity, even though it might look like we're treating  $\infty$  as a number.

- We can look at more interesting examples. Both the lower and upper bounds can be  $\pm\infty$ , such as:

$$\int_{-\infty}^{+\infty} e^{-x^2} \, dx = \sqrt{\pi} \quad (228)$$

**Definition:** We can define an integral from  $-\infty$  to  $+\infty$  as:

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^{\infty} f(x) dx \quad (229)$$

**Warning:** Do *not* evaluate integrals of the above form as:

$$\int_{-\infty}^{\infty} f(x) dx \neq \lim_{b \rightarrow \infty} \int_{-b}^b f(x) dx \quad (230)$$

- For example, take the integral  $\int_{-\infty}^{\infty}$ . If we use the proper definition, then we add two limits that don't exist, so we know this diverges. Note that it might be tempting to write:

$$= \lim_{b \rightarrow \infty} \int_{-b}^b x dx = \lim_{b \rightarrow \infty} \left( \frac{b^2}{2} - \frac{b^2}{2} \right) = 0 \quad (231)$$

but this is only because we are approaching  $-\infty$  and  $+\infty$  at the same rate. If we instead wrote:

$$\lim_{b \rightarrow \infty} \int_{-b}^{2b} x dx = \lim_{b \rightarrow \infty} \left( \frac{4b^2}{2} - \frac{b^2}{2} \right) = \infty \quad (232)$$

If we instead used this approach for our other improper integrals, it wouldn't make a difference since it shouldn't matter the rate at which we approach infinity. Here's another example:

$$\lim_{b \rightarrow \infty} \int_{-b}^{\sqrt{b^2+138}} x dx = \lim_{b \rightarrow \infty} \left( \frac{b^2 + 138}{2} - \frac{b^2}{2} \right) = \lim_{b \rightarrow \infty} \frac{138}{2} = 69 \quad (233)$$

- Improper integrals can also be in the form where there are infinite discontinuities at the bounds of integration. Suppose  $\lim_{x \rightarrow b^-} f(x) = \infty$ . We can treat an integral such as:

$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx \quad (234)$$

**Example 35:** For example, take  $\int_0^1 \frac{dx}{x^{1/3}}$ , and we can evaluate this via:

$$= \lim_{c \rightarrow 0^+} \int_c^1 \frac{dx}{x^{1/3}} \quad (235)$$

$$= \lim_{c \rightarrow 0^+} \frac{3}{2} \left( 1 - c^{2/3} \right) = \frac{3}{2} \quad (236)$$

Again, we have a region that extends to an infinite extend, but it has a finite area. Of course, this won't always be the case.

**Example 36:** Take the example where  $\int_0^1 \frac{dx}{x^2}$ , then we can evaluate this integral via:

$$= \lim_{c \rightarrow 0^+} \int_c^1 \frac{dx}{x^2} \quad (237)$$

$$= \lim_{c \rightarrow 0^+} \left( \frac{1}{c} - 1 \right) = \infty \quad (238)$$

so this integral will diverge.

**Idea:** Notice that we can draw an analogy between:  $\int_0^a \frac{dx}{x^p}$  and  $\int_a^\infty \frac{dx}{x^{1/p}}$ , as they are reflections of one another across the line  $y = x$ . If one diverges, the other will converge, with the exception being  $p = 1$ .

- We can also deal with discontinuities that occur between the given bounds. Similar to before, we break it up into two integrals and *both* integrals must converge for the original integral to converge. For example, take:

$$\int_{-a}^b \frac{1}{|x^{1/2}|} dx \quad (239)$$

with  $a, b > 0$ . For this integral to converge, then both  $\int_{-a}^0 \frac{dx}{|x^{1/2}|}$  and  $\int_0^b \frac{dx}{|x^{1/2}|}$  must converge.

**Warning:** Here is an example of when things go wrong when the integral is not broken up into separate integrals. For example, suppose we wish to evaluate  $\int_{-1}^3 \frac{dx}{x^2}$ . From our previous discussion, we know that  $\int_{-1}^0 \frac{dx}{x^2}$  and  $\int_0^3 \frac{dx}{x^2}$  both diverges. However, one might naively think that:

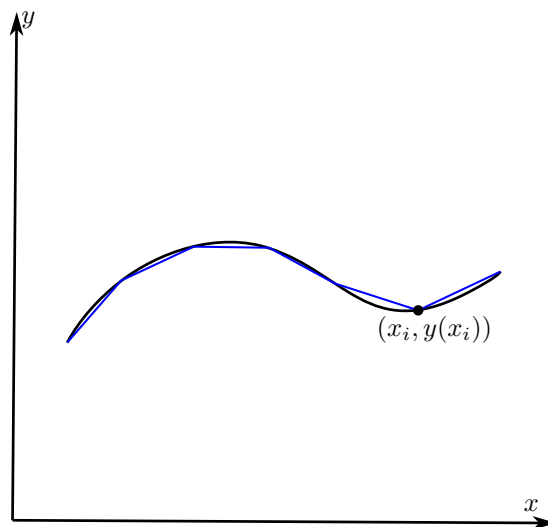
$$\left(-\frac{1}{x}\right) \Big|_{-1}^3 = -\frac{1}{3} - \frac{1}{1} = -\frac{4}{3} \quad (240)$$

which is definitely wrong, since  $\frac{1}{x^2}$  is never negative!

## 7 Applications of Integrals

### 7.1 Arclength

- Suppose we have a curve  $y = f(x)$  where  $x \in [a, b]$  and is differentiable. The problem is to find the length of the curve in this range.



- We can approximate this by partitioning the curve into segments at locations  $x_i$  where:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b \quad (241)$$

such that the arclength is:

$$s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{(x_i - x_{i-1})^2 + (y(x_i) - y(x_{i-1}))^2} \quad (242)$$

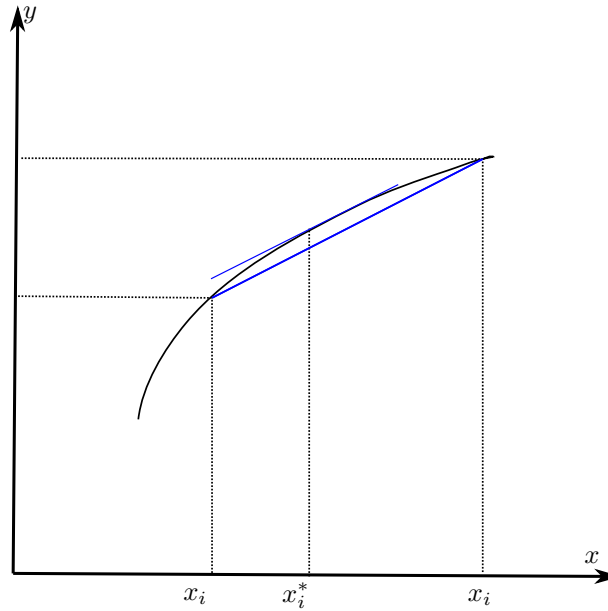
We can use the mean value theorem to write:

$$\frac{\Delta y_i}{\Delta x_i} = \frac{y(x_i) - y(x_{i-1})}{x_i - x_{i-1}} = y'(x_i^*) \quad (243)$$

so we can rewrite:

$$s_i = \sqrt{\Delta x_i^2 + (f'(x_i^*)\Delta x_i)^2} \quad (244)$$

The total length is approximated by the total sum. If we take the limit:



$$s = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n \sqrt{1 + f'(x_i^*)^2} \Delta x_i \quad (245)$$

$$= \int_a^b \sqrt{1 + f'(x)^2} dx \quad (246)$$

**Example 37:** For example, the arclength in  $x \in [0, 44]$  for  $f(x) = x^{3/2}$  can be calculated if we know the derivative:

$$f'(x) = \frac{3}{2}x^{1/2} \quad (247)$$

so:

$$1 + f'(x)^2 = 1 + \frac{9}{4}x \quad (248)$$

which gives:

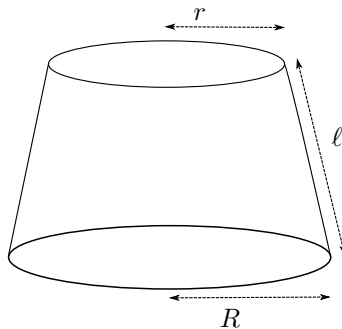
$$s = \int_0^{44} \sqrt{1 + \frac{9}{4}x} dx \quad (249)$$

$$= \left( \frac{4}{9} \cdot \frac{2}{3} \left( 1 + \frac{9}{4}x \right)^{3/2} \right) \Big|_0^{44} \quad (250)$$

$$= 296 \quad (251)$$

## 7.2 Area of a Surface of Revolution

- Consider the new problem of finding the area of a surface of revolution. Similarly, we break it up into smaller segment with width  $\Delta x$ .



- Each small segment is a tapered cone, with an area of:

$$A_i \simeq \pi(f(x_{i-1}) + f(x_i))s_i \quad (252)$$

$$\simeq \pi(f(x_{i-1}) + f(x_i))\sqrt{1 + f'(x_i^*)^2}\Delta x_i \quad (253)$$

From the Intermediate Value Theorem, we have:

$$f(x_{i-1}) + f(x_i) = 2f(x_i^{**}) \quad (254)$$

where  $x_i^{**} \in [x_{i-1}, x_i]$  so the area can be written as:

$$A_i \simeq 2\pi f(x_i^{**})\sqrt{1 + f'(x_i^*)^2}\Delta x_i \quad (255)$$

However, we cannot turn this into an integral just yet since we have both  $x_i^*$  and  $x_i^{**}$ . But in the limit where  $\Delta x_i \rightarrow 0$ , we also have  $x_i^{**} \rightarrow x_i^*$ . We therefore get:

$$A = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2} dx \quad (256)$$

**Example 38:** Suppose we have the function  $y = \sqrt{x}$  rotated across the  $x$  axis and we want the surface area between  $x \in [0, 1]$ . We have  $y' = \frac{1}{2}x^{-1/2}$  and the area becomes:

$$A = \int_0^1 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}} dx \quad (257)$$

$$= \pi \int_0^1 \sqrt{4x + 1} dx \quad (258)$$

Let  $u = 4x + 1$  and  $du = 4dx$ , and we'll get:

$$A = \int_1^5 \pi\sqrt{u}\frac{du}{4} \quad (259)$$

$$= \frac{\pi}{4} \left( \frac{2}{3}u^{3/2} \right) \Big|_1^5 \quad (260)$$

$$= \frac{\pi}{6}(5^{3/2} - 1) \quad (261)$$



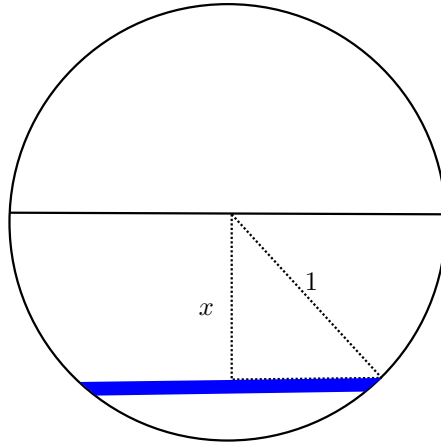
## 8 Applications to Physics and Engineering

- The **hydrostatic pressure** depends on the density  $\rho$ , gravitational constant  $g$  and the depth  $d$ :

$$p = \rho g d \quad (262)$$

and the force of pressure acting on the surface is:

$$F = \rho g d \cdot A = pA \quad (263)$$



**Example 39:** Suppose we have a curved container. The force acting on the entire container can be broken up into segments, each with a force of:

$$F_i = \underbrace{w(x_i^*)\Delta x_i}_{\text{area}} \cdot \underbrace{\rho g x_i^*}_{\text{pressure}} \quad (264)$$

where  $w(x)$  is the width of the container as a function of height. The force exerted on the container is thus:

$$F = \int_a^b \rho g x w(x) dx \quad (265)$$

**Example 40:** Suppose we have a pipe half with a radius of 1m filled with water and we wish to find the force it exerts on the end face of the pipe. We can do this via:

$$F = \int_0^1 \rho g x 2\sqrt{1-x^2} dx \quad (266)$$

$$= 2\rho g \left( -\frac{1}{3}(1-x^2)^{3/2} \right) \Big|_0^1 \quad (267)$$

$$= \frac{2}{3}\rho g = 6533\text{N} \quad (268)$$

- We investigate the **center of mass** of a two dimensional object, which intuitively is the point at which it'll balance, also known as the **centroid**. We can use two principles to help us out:
- Principle 1: Symmetry:** If there is an axis of symmetry, then  $(\bar{x}, \bar{y})$  is on any axis of symmetry. If there are more than one axes of symmetry, then we simply need to find the intersection.
- Principle 2: Additivity:** We can find the centroid of a collection of segments by taking the weighted average of each of the segments it is composed of. For a discrete set, the total area is:

$$A = A_1 + A_2 + \cdots + A_n \quad (269)$$

and the  $x$  location of the centroid is:

$$\bar{x} = \bar{x}_1 \frac{A_1}{A} + \bar{x}_2 \frac{A_2}{A} + \cdots + \bar{x}_n \frac{A_n}{A} \quad (270)$$

and similarly for the  $y$  location:

$$\bar{y} = \bar{y}_1 \frac{A_1}{A} + \bar{y}_2 \frac{A_2}{A} + \cdots + \bar{y}_n \frac{A_n}{A} \quad (271)$$

- Suppose we wish to find the centroid of a curve  $f(x)$  from  $x = a$  to  $x = b$ . We can approximate this region via a series of rectangles such that:

$$A_i = f(x_i^*) \Delta x_i \quad (272)$$

$$\bar{x}_i = \frac{x_{i-1} + x_i}{2} = x_i^* \quad (273)$$

$$\bar{y}_i = \frac{1}{2} f(x_i^*) \quad (274)$$

such that:

$$\bar{x}A = \sum_{i=1}^n \bar{x}_i A_i = \sum_{i=1}^n x_i^* f(x_i^*) \Delta x_i \quad (275)$$

$$\bar{y}A = \frac{1}{2} \sum_{i=1}^n x_i^* f(x_i^*) \Delta x_i \quad (276)$$

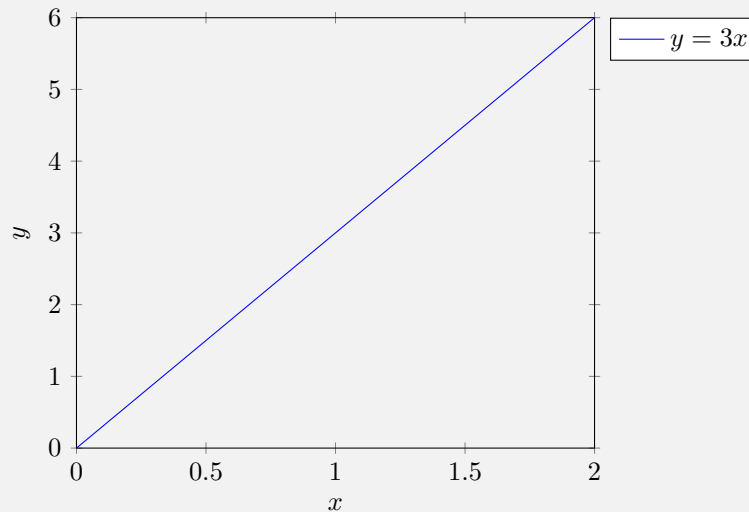
If we take the limit, we get:

$$\bar{x} = \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx} \quad (277)$$

$$\bar{y} = \frac{\int_a^b f(x)^2 dx}{2 \int_a^b f(x) dx} \quad (278)$$

**Example 41:** Suppose we wish to find the area of  $y = 3x$  between  $x = 0$  and  $x = 2$ :

Example



The area is  $A = \int_0^2 3x dx = 6$ . And we have:

$$\bar{x}A = \int_0^2 x(3x) dx = 8 \implies \bar{x} = \frac{4}{3} \quad (279)$$

$$\bar{y}A = \int_0^2 \frac{1}{2} (3x)^2 dx = 12 \implies \bar{y} = 2 \quad (280)$$

which is as we expected from the centroid of a triangle.

- If we want the centroid of the region of intersection between two curves  $f$  and  $g$ , we can use the additivity rule, we can have:

$$\bar{x}A = \bar{x}_gA_g - \bar{x}_fA_f \quad (281)$$

$$\bar{y}A = \bar{y}_gA_g - \bar{y}_fA_f \quad (282)$$

$$A = A_g - A_f \quad (283)$$

or in integral form:

$$\bar{x}A = \int_a^b x [f(x) - g(x)] dx \quad (284)$$

$$\bar{y}A = \frac{1}{2} \int_a^b [f(x)^2 - g(x)^2] dx \quad (285)$$

**Example 42:** Suppose we have two curves  $y = 6$  and  $y = 3$  and we wish to find the centroid of the area between the curves between  $2 < x < 5$ . This gives us:

$$\bar{x}A = \int_2^5 x(6 - 3) dx = \frac{63}{2} \quad (286)$$

$$\bar{y}A = \frac{1}{2} \int_2^5 \frac{1}{2}(36 - 9) dx = \frac{9}{2} \quad (287)$$

which gives us:  $\bar{x} = \frac{7}{2}$  and  $\bar{y} = \frac{9}{2}$ .

- **Pappus's Centroid Theorem** can be used to easily find the volume of revolution. We have:

$$V = 2\pi \bar{R}a \quad (288)$$

where  $\bar{R}$  is the distance from the centroid to the axis of revolution.

**Example 43:** Suppose we have an elliptical torus whose center is The area of the ellipse whose major axis is parallel to the axis of revolution and whose centroid is a distance  $R$  away from the axis. The volume is thus:

$$V = 2\pi R\pi ab = 2\pi^2 abR \quad (289)$$

- We can prove this using the washer method about  $x$ :

$$V_x = \int_a^b \pi(f(x)^2 - g(x)^2) dx \quad (290)$$

$$= 2\pi \int_a^b \frac{1}{2}(f(x)^2 - g(x)^2) dx \quad (291)$$

$$= 2\pi \bar{y}A \quad (292)$$

and using the shell method about  $x$ :

$$V_y = \int_a^b 2\pi x(f(x) - g(x)) dx \quad (293)$$

$$= 2\pi \bar{x}A \quad (294)$$

## 9 Parametric Equations

- Until now, we have described two dimensional curves in the form  $y = f(x)$ . However, we can describe this in parametric equations as well in the form:

$$x = x(t) \quad (295)$$

$$y = y(t) \quad (296)$$

- For example, numerous applications in physics and engineering arise using parametrized coordinates such as projectile motion. The equations of motion are:

$$x(t) = x_0 + v_0 \cos \theta t \quad (297)$$

$$y(t) = y_0 + v_0 \sin \theta t - \frac{1}{2}gt^2 \quad (298)$$

- We can parametrize a straight line using:

$$x(t) = x_c + t(x_1 - x_0) \quad (299)$$

$$y(t) = y_0 + t(y_1 - y_0) \quad (300)$$

with  $t \in (-\infty, \infty)$ . We can check this by letting  $t = 0$ , which gives  $(x_0, y_0)$  and when  $t = 1$  we get  $(x_1, y_1)$ .

- For an ellipse, we can parametrize using:

$$x = a \cos t \quad (301)$$

$$y = b \sin t \quad (302)$$

- Suppose we have two curves parametrized by:

$$C_1 = x_1(t), y_1(t) \quad (303)$$

$$C_2 = x_2(t), y_2(t) \quad (304)$$

An intersection occurs when  $y_1(x) = y_2(x)$  and a collision occurs when  $x_1(t) = x_2(t)$  and  $y_1(t) = y_2(t)$ .

**Example 44:** Suppose we have the following curves:

$$C_1 : x_1(t) = 2t + 6, y_1(t) = 5 - 4t \quad (305)$$

$$C_2 : x_2(t) = 3 - 5 \cos \pi t, y_2 = 1 + 5 \sin \pi t \quad (306)$$

for  $t > 0$ . The first curve is a straight line and the second curve is an offset circle. We can find the intersection via:

$$C_1 : t = \frac{x-6}{2} \implies y_1(x) = 16 - 2x \quad (307)$$

$$C_2 : (x-3)^2 + (y-1)^2 = 25 \quad (308)$$

There will be two points of intersection, and after we solve this, we get  $(6, 5)$  and  $(8, 1)$ . We now need to look at if the two objects will arise at that point at the same time. We do this by looking at each point first

$$(6, 5) : C_1 \implies t = 0 \quad (309)$$

$$C_2|_{t=0} = (-2, 1) \quad (310)$$

$$(8, 1) : C_1 \implies t = 1 \quad (311)$$

$$C_2|_{t=1} = (8, 1) \quad (312)$$

Therefore, there is a collision between the two objects at  $(8, 1)$  at a time  $t = 1$ .

## 10 Calculus with Parametric Curves

- Parametric curves do not need to be functions. As a result, they can have an ordinary tangent, no tangent, or more than one tangent at the same point.
- We can find the secant line of a parametric curve via:

$$m_{\text{secant}} = \frac{y(t_0 + h) - y(t_0)}{x(t_0 + h) - x(t_0)} \quad (313)$$

We can divide both numerator and denominator by  $h$  to get:

$$\frac{\frac{y(t_0+h)-y(t_0)}{h}}{\frac{x(t_0+h)-x(t_0)}{h}} \implies \frac{y'(t_0)}{x'(t_0)} \quad (314)$$

where we took the limit as  $h \rightarrow 0$ .

- We could also have derived this using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \left( \frac{dx}{dt} \right)^{-1} \quad (315)$$

- If  $x'(t_0) = 0$ , then  $x = x_0$  is a vertical tangent and if  $y'(t_0) = 0$ , then we have  $y = y_0$  and have a horizontal tangent. If  $x' = y' = 0$ , then we can't read any information from it.

**Example 45:** Consider the example  $x(t) = \sin(2t)$  and  $y(t) = \sin t$  with  $t \in [0, 2\pi]$ . We have  $x'(t) = 2 \cos 2t$  so we can find the vertical tangents by setting  $2 \cos(2t) = 0$  to get:

$$t \in \left\{ \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \right\} \implies \left( 1, \frac{1}{\sqrt{2}} \right) \left( -1, \frac{1}{\sqrt{2}} \right) \left( 1, -\frac{1}{\sqrt{2}} \right) \left( -1, -\frac{1}{\sqrt{2}} \right) \quad (316)$$

For horizontal tangents, we have  $y'(t) = \cos t \implies \cos t = 0$  which gives:

$$t \in \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\} \implies (0, 1)(0, -1) \quad (317)$$

At  $t = 0$ , we also have  $x(0) = y(0) = 0$ . We can determine the tangent at this point as:

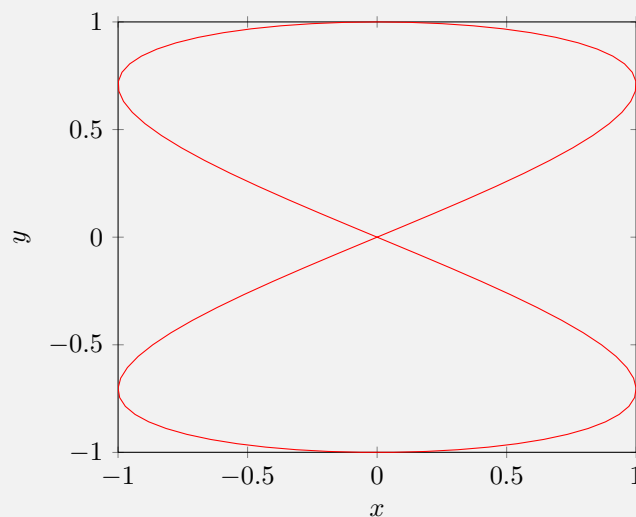
$$m_{\text{tangent}} = \frac{y'(0)}{x'(0)} = \frac{1}{2} \quad (318)$$

However notice that at  $t = \pi$ , we have  $x(\pi) = y(\pi) = 0$  as well so this point has another tangent:

$$m_{\text{tangent}} = \frac{y'(\pi)}{x'(\pi)} = -\frac{1}{2} \quad (319)$$

At this point we can sketch out the conditions we found on the graph and draw out curve:

Parametric Example



- We can determine the **area** of a parametric curve via:

$$A = \int_{t_1}^{t_2} y(t)x'(t) dt \quad (320)$$

Letting  $y = f(x) \implies y(t) = f(x(t))$  gives:

$$A = \int_{t_1}^{t_2} f(x(t))x'(t) dt \quad (321)$$

$$= \int_{a=x(t_1)}^{b=x(t_2)} f(x) dx \quad (322)$$

- We can also determine the area of a closed curve:

**Definition:** A curve is traversed in the positive sense as  $t$  increases, if the enclosed area is on the left (counterclockwise).

The area of a closed loop is then:

$$A = \int_{t_4}^{t_3} y(t)x'(t) dt - \int_{t_4}^{t_5} y(t)x'(t) dt + \int_{t_3}^{t_2} y(t)x'(t) dt - \int_{t_1}^{t_2} y(t)x'(t) dt \quad (323)$$

where  $t_1$  represents the starting point on the bottom,  $t_2$  is the rightmost point,  $t_3$  is an arbitrary point on top,  $t_4$  is the leftmost point, and  $t_5$  is the endpoint that ends off as  $t_1$ . This gives:

$$- \int_{t_3}^{t_4} y(t)x'(t) dt - \int_{t_4}^{t_5} y(t)x'(t) dt - \int_{t_2}^{t_3} y(t)x'(t) dt - \int_{t_1}^{t_2} y(t)x'(t) dt \quad (324)$$

or:

$$A = - \int_{t_1}^{t_5} y(t)x'(t) dt \quad (325)$$

where  $t_1$  and  $t_5$  are the starting and ending points respectively. They can be arbitrarily chosen so long as the points they correspond to are on top of each other.

- Similarly, we can also write the area as:

$$A = \int_{t_1}^{t_5} x(t)y'(t) dt \quad (326)$$

**Example 46:** Let us derive the area of an ellipse, parametrized by:

$$x = a \cos \theta \quad (327)$$

$$y = b \sin \theta \quad (328)$$

for  $\theta \in [0, 2\pi]$ . The area is then:

$$A = - \int_0^{2\pi} b \sin \theta (-a \sin \theta) d\theta \quad (329)$$

$$= ab \int_0^{2\pi} \sin^2 \theta d\theta \quad (330)$$

$$= ab \left[ \frac{1}{2} \theta - \frac{1}{4} \sin 2\theta \right]_0^{2\pi} \quad (331)$$

$$= \pi ab \quad (332)$$

- We can also determine the **arclength** of a parametric curve. We start with the summation:

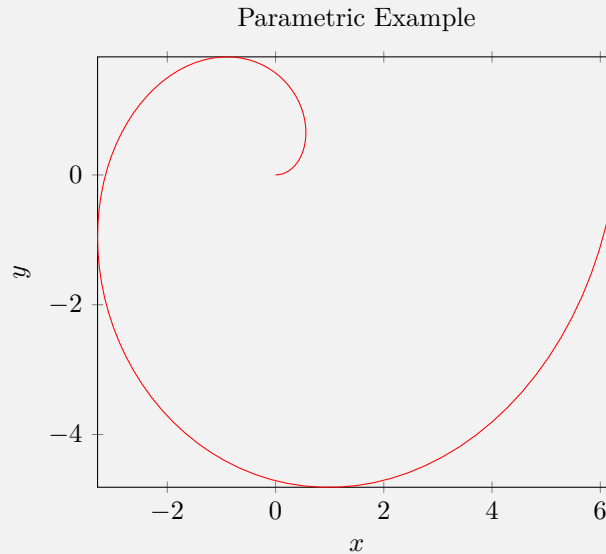
$$s \simeq \sum \sqrt{\Delta x^2 + \Delta y^2} \implies \int_{\alpha}^{\beta} \sqrt{x'(t)^2 + y'(t)^2} dt \quad (333)$$

Note that if we let  $x = t$  and  $y = f(t) = f(x)$ , then we get:

$$s = \int_{\alpha}^{\beta} \sqrt{1 + f'(x)^2} dx \quad (334)$$

which is what we should expect.

**Example 47:** Suppose we have  $x(\theta) = \theta \cos \theta$  and  $y(\theta) = \theta \sin \theta$  for  $\theta \in [0, 2\pi]$ , this gives an Archimedes Spiral:



We can determine the arclength via:

$$s = \int_0^{2\pi} \sqrt{(\cos \theta - \theta \sin \theta)^2 + (\sin \theta + \theta \cos \theta)^2} d\theta \quad (335)$$

$$= \int_0^{2\pi} \sqrt{1 + \theta^2} d\theta \quad (336)$$

$$= \left[ \frac{1}{2} \theta \sqrt{1 + \theta^2} + \frac{1}{2} \ln \left| \theta + \sqrt{1 + \theta^2} \right| \right]_0^{2\pi} \quad (337)$$

$$= \pi \sqrt{1 + 4\pi^2} + \frac{1}{2} \ln \left( 2\pi + \sqrt{1 + 4\pi^2} \right) \quad (338)$$

**Idea:** If we wish to find the speed, we have:

$$v = \frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} \quad (339)$$

which comes from both the fundamental theorem of calculus, as well as from two-dimensional kinematics.

- The surface area can be written as:

$$A = \int_a^b 2\pi y ds = \int_a^b 2\pi y(t) \sqrt{x'(t)^2 + y'(t)^2} dt \quad (340)$$

- What is the *circumference of an ellipse*? If we try to carry out this calculation, we can parametrize the ellipse as before:

$$x = a \sin \theta \quad (341)$$

$$y = b \cos \theta \quad (342)$$

with  $0 \leq \theta \leq 2\pi$  and the arclength as:

$$s = \int_0^{2\pi} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} d\theta \quad (343)$$

$$= \int_0^{2\pi} \sqrt{a^2(1 - \sin 2\theta) + b^2 \sin 2\theta} d\theta \quad (344)$$

$$= \int_0^{2\pi} a \sqrt{1 - \epsilon^2 \sin^2 \theta} d\theta \quad (345)$$

with  $\epsilon \equiv \sqrt{\frac{a^2 - b^2}{a^2}}$ . Unfortunately, this is an elliptic integral of the second kind and has no analytic solution.

## 11 Polar Coordinates

- In polar coordinates, we can represent polar coordinates in terms of the distance  $r$  from the origin and the angle it makes with the positive horizontal axis:

$$(x, y) \iff [r, \theta] \quad (346)$$

- The magnitude of  $r$  gives the distance from the origin, and multiplying  $r$  by  $-1$  rotates the point about the origin by  $\pi$ .
- Polar coordinates are not unique:
  - The pole is  $[0, \theta]$  for all  $\theta$ .
  - $[r, \theta] = [r, \theta + 2n\pi]$  for any integer  $n$ .
  - $[r, \theta] = [-r, \theta + (2n + 1)\pi]$  for any integer  $n$ .
- We can convert between cartesian and polar coordinates using the transformation:

$$x = r \cos \theta \quad (347)$$

$$y = r \sin \theta \quad (348)$$

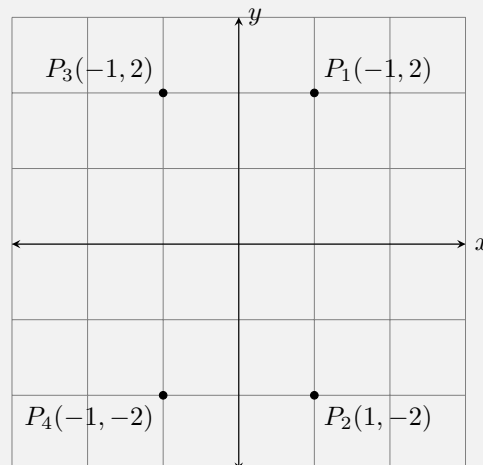
which gives:

$$r = \sqrt{x^2 + y^2} \quad (349)$$

$$\theta = \arctan\left(\frac{y}{x}\right) \quad (350)$$

for  $x \neq 0$ .

**Example 48:** Suppose we have the following four points.





We can represent the four coordinates also as:

$$P_1[\sqrt{5}, 1.107] \quad (351)$$

$$P_2[\sqrt{5}, -1.107] \quad (352)$$

$$P_3[-\sqrt{5}, -1.107] \quad (353)$$

$$P_4[-\sqrt{5}, 1.107] \quad (354)$$

- We can represent straight lines as:

- Straight lines:  $y = mx$ :  $r = \alpha$  with  $\alpha = \arctan(m)$ .
- Vertical lines  $x = a$ : We have  $r \cos \theta = a \implies r = a \sec \theta$ .
- Horizontal lines:  $y = b$ . We have  $r \sin \theta = b \implies r = b \csc \theta$

- We can represent circles in polar coordinates as:

$$x^2 + y^2 = 9 \iff r = 3 \quad (355)$$

- Converting *from* polar coordinates requires a bit of extra work. Suppose we have  $r = 6 \sin \theta$ , then:

$$r^2 = 6r \sin \theta \quad (356)$$

$$x^2 + y^2 = 6y \quad (357)$$

$$x^2 + y^2 - 6y + 9 = 9 \quad (358)$$

$$x^2 + (y - 3)^2 = 9 \quad (359)$$

which represents a circle with radius 3 centered at  $(0, 3)$ .

- Symmetry can also arise in many scenarios. For example:

- Symmetry about  $x$  axis:  $[r_1, \theta]$  and  $[r_1, -\theta]$ .
- Symmetry about  $y$  axis:  $[r, \pi - \theta]$  and  $[r_1, \theta]$ .
- Symmetry about origin:  $[r, \theta]$  and  $[r, \theta + \pi]$ .

which will help when sketching them.

**Example 49:** Suppose we wish to sketch the curve  $r = \frac{1}{2} + \cos \theta$ . Notice that this is periodic so we only need to look at values of  $\theta$  where  $0 \leq \theta < 2\pi$ .

1. Let us first find values of  $\theta$  (if possible) that make  $r = 0$ :

$$0 = \frac{1}{2} + \cos \theta \implies \theta = \frac{2\pi}{3}, \frac{4\pi}{3} \quad (360)$$

2. Find local max and min values of  $r$ :

$$\frac{dr}{d\theta} = -\sin \theta = 0 \implies \theta = 0, \pi \quad (361)$$

At  $\theta = 0$ , we have  $r = \frac{3}{2}$ , at  $\theta = \pi$  we have  $r = -\frac{1}{2}$ . If  $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ , then  $r = \frac{1}{2}$ .

3. We then look at symmetry. Notice that:

$$\frac{1}{2} + \cos(-\theta) + \frac{1}{2} + \cos \theta \quad (362)$$

However:

$$r(\theta - \pi) \neq r(\theta) \quad (363)$$

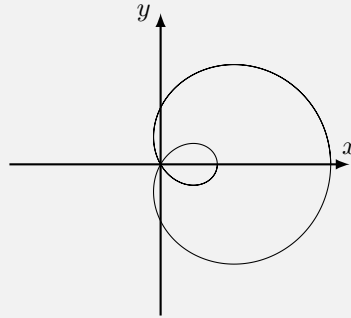
$$r(\pi + \theta) \neq r(\theta) \quad (364)$$

so it is not symmetric about the  $y$  axis or origin.

4. We now look at the relevant intervals. From  $0 \leq \theta < \frac{2\pi}{3}$ , we have  $\frac{dr}{d\theta} < 0$  so the radius is monotonically decreasing.

We can also look at the interval  $\frac{2\pi}{3} \leq \theta < \pi$  and see that  $r$  is negative and  $\frac{dr}{d\theta} < 0$ , so the magnitude of  $r$  *increases*.

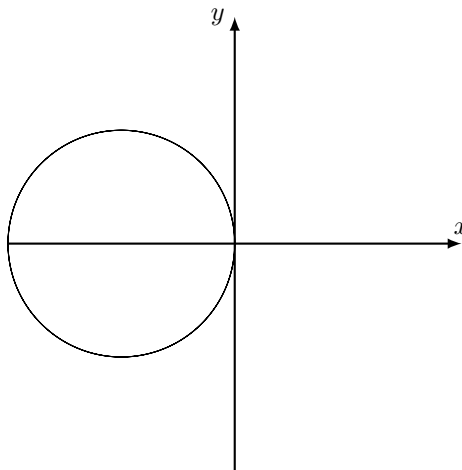
Noting that the function is differentiable at  $\theta = \pi$ , we can reflect the shape about the  $x$  axis to get a Limacon with inner loop.



- There are a few common shapes. Each of these could be flipped or rotated by shifting the argument  $\theta$ , or using negative numbers.

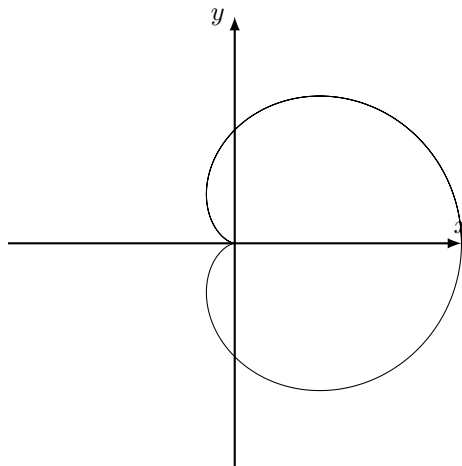
– Circles:

$$r = -2 \cos \theta \quad (365)$$



– Cardioids:

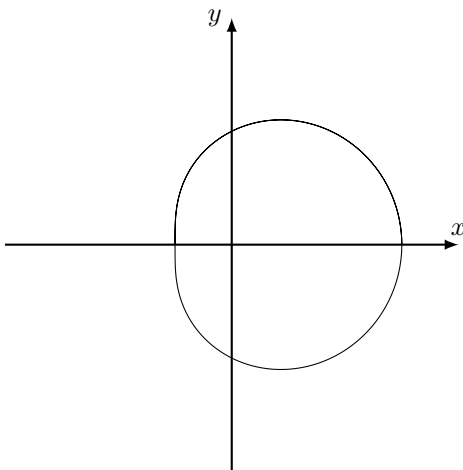
$$r = a + a \cos \theta \quad (366)$$



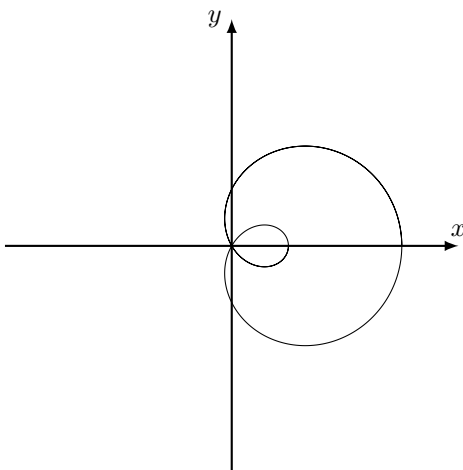
– Limacons:

$$r = a + b \sin \theta \quad (367)$$

There are two types, for  $a > b$ :

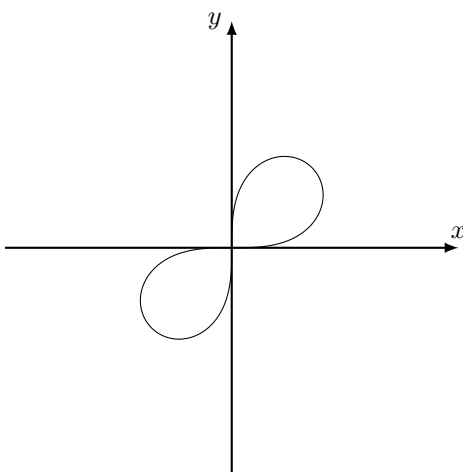


For  $a < b$ :



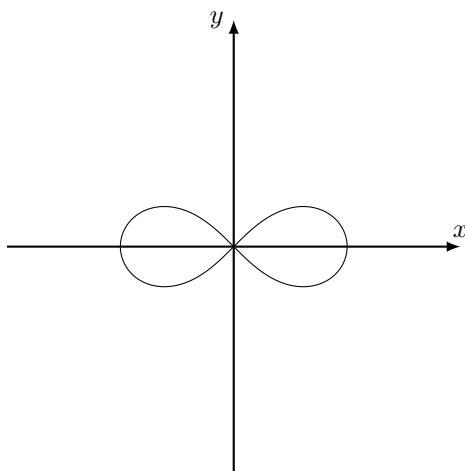
– Lemniscates. Again, there are two types. For:

$$r^2 = a \sin(2\theta) \quad (368)$$



and:

$$r^2 = a \cos(2\theta) \quad (369)$$

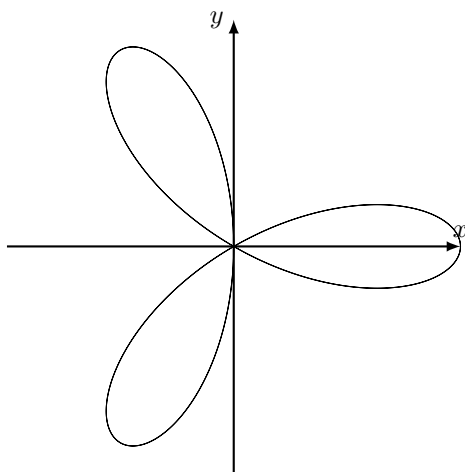


– Petal curves:

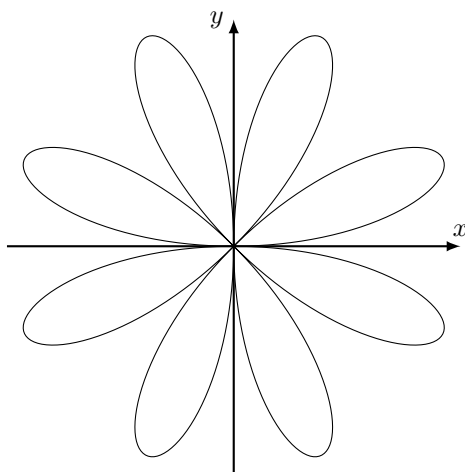
$$r = a \sin(n\theta) \quad (370)$$

$$r = a \cos(n\theta) \quad (371)$$

where  $n$  is an integer. There are  $n$  petals if  $n$  is odd and  $2n$  petals if  $n$  is even. For example, the following is:  
 $r = 2 \cos 3\theta$ :



and for  $r = 2 \sin(4\theta)$ :

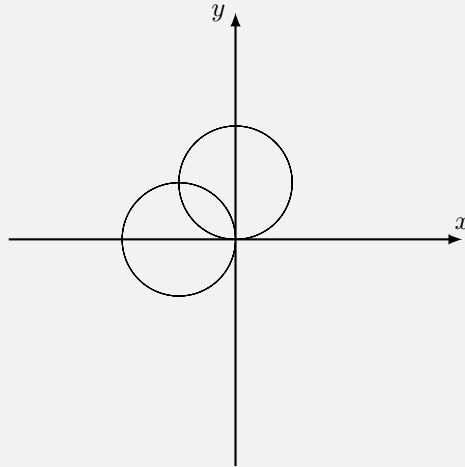


- We can also find the intersection of polar coordinates, but we have to be careful. The following example illustrates why.

**Example 50:** Suppose we have two curves  $r = \sin \theta$  and  $r = -\cos \theta$ . Suppose we try to solve this via:

$$\sin \theta = -\cos \theta \implies \theta = \frac{3\pi}{4}, \frac{7\pi}{4} \quad (372)$$

Plugging this back into  $x = r \cos \theta$  and  $y = r \sin \theta$ , we get  $x = -\frac{1}{2}$  and  $y = -\frac{1}{2}$ . We can also use  $\theta = \frac{7\pi}{4}$  to get:  $x = -\frac{1}{2}$  and  $y = \frac{1}{2}$  which is the same point. However, it represents the curves below:



There is actually two intersection points! The reason for this is that we assumed that the two curves intersect at the same value of  $\theta$ , but this is not necessarily true for the origin, which can be obtained at any angle  $\theta$ .

- As a result, we also have to check the origin.
- It is also possible to find the tangent:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \quad (373)$$

$$= \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} \quad (374)$$

## 12 Areas and Lengths in Polar Coordinates

- Suppose we have a polar curve  $r = \rho(\theta)$  for  $\alpha \leq \theta \leq \beta$ .
- We can determine the area by partitioning the curve into  $\theta_i$  and approximating each subregion as a circular segment. The area of a circular segment is:

$$A = \frac{1}{2} a^2 \Delta \theta \quad (375)$$

We can take the radius to be  $r = \rho(\theta^*)$  where  $\theta_{i-1} \leq \theta^* \leq \theta_i$ . The area of each region is:

$$A_i = \frac{1}{2} \rho(\theta_i^*)^2 \Delta \theta_i \quad (376)$$

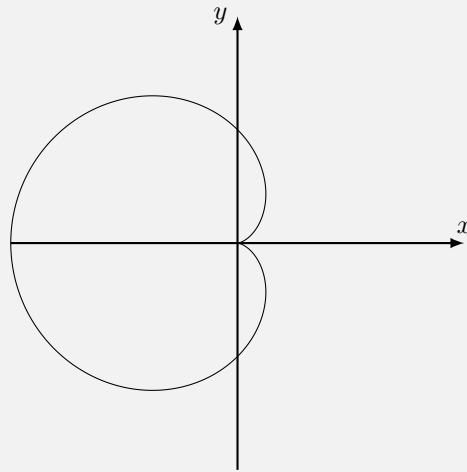
so the total area is:

$$A = \lim_{||P||} \sum_i \frac{1}{2} \rho(\theta_i^*)^2 \Delta \theta_i \quad (377)$$

or

$$A = \frac{1}{2} \int_{\alpha}^{\beta} \rho(\theta)^2 d\theta \quad (378)$$

**Example 51:** Suppose we wish to find the area of  $r = 1 - \cos \theta$ .



The area is then:

$$A = \frac{1}{2} \int_0^{2\pi} (1 - \cos \theta)^2 d\theta \quad (379)$$

$$= \frac{1}{2} \int_0^{2\pi} (1 - 2 \cos \theta + \cos^2 \theta) d\theta \quad (380)$$

$$= \frac{1}{2} \int_0^{2\pi} \left( \frac{3}{2} - 2 \cos \theta + \frac{1}{2} \cos(2\theta) \right) d\theta \quad (381)$$

$$= \frac{1}{2} \left( \frac{3}{2} \theta - 2 \sin \theta + \frac{1}{4} \sin(2\theta) \right) \Big|_0^{2\pi} \quad (382)$$

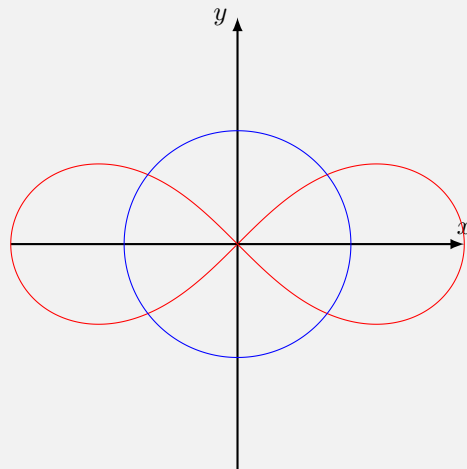
$$= \frac{3}{2} \pi \quad (383)$$

- We can also find the area between two polar curves  $\rho_1$  and  $\rho_2$ . We have:

$$A = \frac{1}{2} \int_{\alpha}^{\beta} \rho_1(\theta)^2 d\theta - \frac{1}{2} \int_{\alpha}^{\beta} \rho_2(\theta)^2 d\theta = \frac{1}{2} \int_{\alpha}^{\beta} (\rho_1^2 - \rho_2^2) d\theta \quad (384)$$

**Warning:** Be careful when applying this formula as it is possible the two functions can overlap between  $\alpha \leq \theta \leq \beta$ . Therefore, we always need a good idea of what's happening.

**Example 52:** Suppose we want to determine the area inside  $r^2 = 4 \cos(2\theta)$  but outside  $r = 1$ . This gives:



We first find the four points of intersection:

$$4 \cos(2\theta) = 1 \implies \cos(2\theta) = \frac{1}{4} \implies \theta = \pm 0.659 \quad (385)$$

or  $\theta = \pi \pm 0.659$ . Due to the symmetry, we only need to find the area of one half of the area we are interested in, which gives:

$$\frac{1}{2}A = \frac{1}{2} \int_{-0.659}^{0.659} (4 \cos 2\theta - 1) d\theta \quad (386)$$

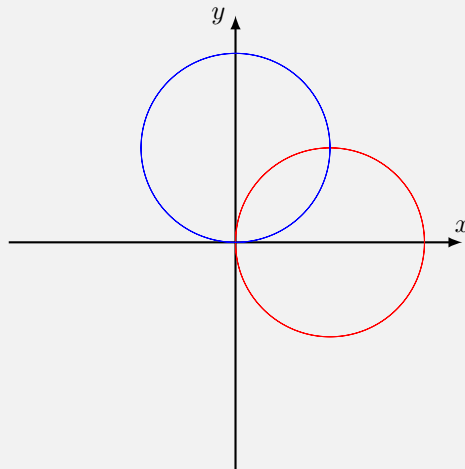
$$= \frac{1}{2} (2 \sin 2\theta - \theta) \Big|_{-0.659}^{0.659} \quad (387)$$

$$= 1.277 \quad (388)$$

so the area is  $A = 2.554$ .

- There are a few challenging examples:

**Example 53:** Suppose we wish to find the area between  $r = \sin \theta$  and  $r = \cos \theta$ :



We know from symmetry that the intersection is at  $\theta = \frac{\pi}{4}$ . We notice that the contribution to the area from each curve  $\rho$  is equal and *independent* from each other. Therefore:

$$A = A_1 + A_2 = \int_0^{\pi/4} \frac{1}{2} \sin^2 \theta d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2} \cos^2 \theta d\theta = \frac{\pi}{8} - \frac{1}{4} \quad (389)$$

- We can determine the arclength by working in parametric form. Let  $x = r(\theta) \cos \theta$  and  $y = r(\theta) \sin \theta$ . Therefore:

$$s = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \quad (390)$$

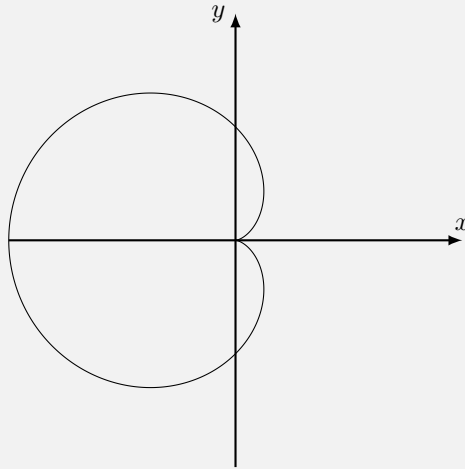
$$= \int_{\alpha}^{\beta} \sqrt{(r' \cos \theta - r \sin \theta)^2 + (r' \sin \theta + r \cos \theta)^2} d\theta \quad (391)$$

$$= \int_{\alpha}^{\beta} \sqrt{(r'^2 \cos^2 \theta + r^2 \sin^2 \theta - 2r'r' \cos \theta \sin \theta) + (r'^2 \sin^2 \theta + r^2 \cos^2 \theta + 2r'r' \cos \theta \sin \theta)} d\theta \quad (392)$$

$$= \int_{\alpha}^{\beta} \sqrt{r^2(\cos^2 \theta + \sin^2 \theta) + r'^2(\cos^2 \theta + \sin^2 \theta)} d\theta \quad (393)$$

$$= \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \quad (394)$$

**Example 54:** Suppose we want to find the arclength of  $r = a(1 - \cos \theta)$  from  $0 \leq \theta < 2\pi$ . This looks like:



We have:

$$s = \int_0^{2\pi} \sqrt{r^2 + (r')^2} d\theta \quad (395)$$

$$= \int_0^{2\pi} \sqrt{a^2(1 - 2\cos \theta + \cos^2 \theta) + a^2 \sin^2 \theta} d\theta \quad (396)$$

$$= a \int_0^{2\pi} \sqrt{2 - 2\cos \theta} d\theta \quad (397)$$

$$= a \int_0^{2\pi} \sqrt{4 \sin^2 \left( \frac{\theta}{2} \right)} d\theta \quad (398)$$

$$= 2a \left[ -2 \cos \left( \frac{\theta}{2} \right) \right]_0^{2\pi} \quad (399)$$

$$= 8a \quad (400)$$

## 13 Infinite Sequences and Series

- We use curly brackets to indicate a sequence, such as:

$$f(n) = \frac{1}{n} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\} \quad (401)$$

Alternatively, we can use  $a_n$  to represent a sequence.

**Definition:** A sequence  $\{a_n\}$  is:

- increasing iff  $a_n < a_{n+1}$
- non-decreasing iff  $a_k \leq a_{n+1}$
- decreasing iff  $a_n > a_{n+1}$
- non-increasing iff  $a_n \geq a_{n+1}$

A function that satisfies any of these are known as **monotonic** functions.

- Bounded functions have an upper or lower bound, while unbounded functions diverge to infinity or negative infinity.

**Example 55:** Suppose we wish to prove that  $2^k$  is unbounded. We wish to find  $k$  such that  $a_k > M$  or  $2^k > M$ . Taking the natural logarithm of both sides, we have:

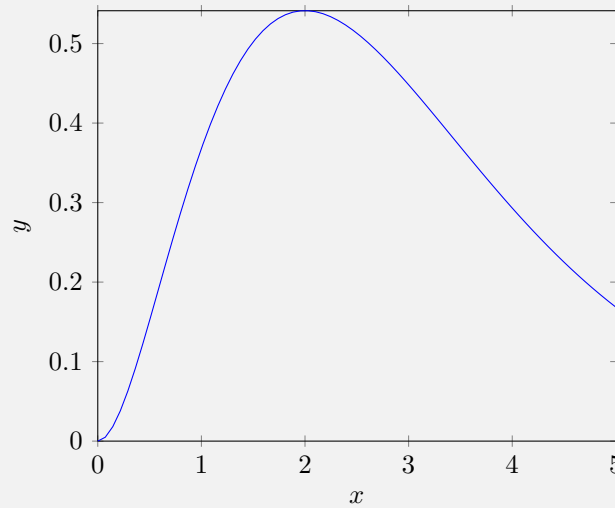
$$k > \frac{\ln M}{\ln 2} \quad (402)$$



which is possible to do and we are done.

**Example 56:** Suppose we wish to find if  $a_n = \frac{n^2}{e^n}$  is bounded or unbounded. This can be approached by working with derivatives through the function  $f(x) = \frac{x^2}{e^x}$ , represented in the following plot:

Example



Taking the derivative  $f'(x) = xe^{-x}(2-x)$ , we see that  $f$  decreases for  $x > 2$  so this means that  $a_n$  decreases for  $n > 2$

**Warning:** Not everything in functions carries over to sequences. For example,  $f(x) = \frac{1}{x - \sqrt{2}}$  is unbounded but  $a_n = \frac{1}{n - \sqrt{2}}$  is bounded since  $n \neq \sqrt{2}$  is impossible.

- We can only take the limit of a sequence as  $n \rightarrow \infty$ .

**Definition:** We can define  $\lim_{n \rightarrow \infty} a_n = L$  iff for every  $\epsilon > 0$ , there exists an integer  $k > 0$  such that if  $n \geq k$ , then  $|a_n - L| < \epsilon$ .

**Example 57:** Let us prove  $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$ . We find  $k$  such that  $\left| \frac{n}{n+1} - 1 \right| < \epsilon$  for  $n \geq k$ . This can be rewritten as:

$$\left| \frac{1}{n+1} \right| < \epsilon \quad (403)$$

or  $|n+1| > \frac{1}{\epsilon}$ . Thus, if we choose  $k = \frac{1}{\epsilon}$  such that if we choose  $n > k = \frac{1}{\epsilon}$ , then:

$$\left| \frac{n}{n+1} - 1 \right| = \left| \frac{1}{n+1} \right| < \left| \frac{1}{n} \right| < \frac{1}{k} = \epsilon \quad (404)$$

Therefore,  $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$ .

**Theorem: Uniqueness of a Limit:** If  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} a_n = M$ , then  $L = M$ .

**Definition:** If a sequence has a limit, it is said to be convergent. Otherwise, it is divergent.

- This leads to the following:
  1. If a sequence is convergent, it is bounded.
  2. If a sequence is unbounded, it is divergent.
  3. A bounded sequence is not necessarily convergent.
- For example,  $a_n = \cos \pi n$  is bounded but not convergent.

**Theorem: Monotonic Sequence Theorem:** A bounded nondecreasing sequence converges to its least upper bound. A bounded non increasing sequence converges to its greatest lower bound.

- The limit has a few properties. Let  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} b_n = M$ . Then:
  1.  $\lim_{n \rightarrow \infty} (a_n + b_n) = L + M$
  2.  $\lim_{n \rightarrow \infty} \alpha a_n = \alpha L$  for  $\alpha \in \mathbb{R}$ .
  3.  $\lim_{n \rightarrow \infty} a_n b_n = L \cdot M$
  4.  $\lim_{n \rightarrow \infty} \frac{1}{b_n} = \frac{1}{M}$  for  $b_n \neq 0, M \neq 0$ .
  5.  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{L}{M}$  for  $b_n \neq 0, M \neq 0$ .

**Theorem: Pinching Theorem for Sequences:** If for large  $n$ ,  $a_n \leq b_n \leq c_n$  and if  $\lim_{n \rightarrow \infty} a_n = L$  and  $\lim_{n \rightarrow \infty} c_n = L$ , then  $\lim_{n \rightarrow \infty} b_n = L$ .

**Example 58:** Suppose we wish to find the limit  $\lim_{n \rightarrow \infty} \frac{\sin(n\pi/6)}{n}$ . We can let:

$$-\frac{1}{n} \leq \frac{\sin(n\pi/6)}{n} \leq \frac{1}{n} \quad (405)$$

Since  $\lim_{n \rightarrow \infty} -\frac{1}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , then the original limit must also be zero.

**Theorem:** Suppose we have a sequence:  $c_n = g(f_n)$ . Given  $\lim_{n \rightarrow \infty} c_n = C$ . If  $f$  is continuous at  $c$ , in the traditional way, then:  $\lim_{n \rightarrow \infty} f(c_n) = f(c)$ .

**Example 59:** Let us look at the function  $\sin\left(\frac{1}{n^2 + 1}\right)$ . We know that  $\lim_{n \rightarrow \infty} \frac{1}{n^2 + 1} = 0$ , so:

$$\lim_{n \rightarrow \infty} \sin\left(\frac{1}{n^2 + 1}\right) = \sin(0) = 0 \quad (406)$$

where we have applied the previous theorem.