ESC195 Notes

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Contents

1 Hyperbolic Functions

- Sometimes, combinations of e^x and e^{-x} are given certain names, for example:
  - **Hyperbolic sine:** $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$
  - **Hyperbolic cosine:** $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$
- They have the following properties:

$$\frac{d}{dx}\sinh x = \cosh x \tag{1}$$
$$\frac{d}{dx}\cosh x = \sinh x \tag{2}$$

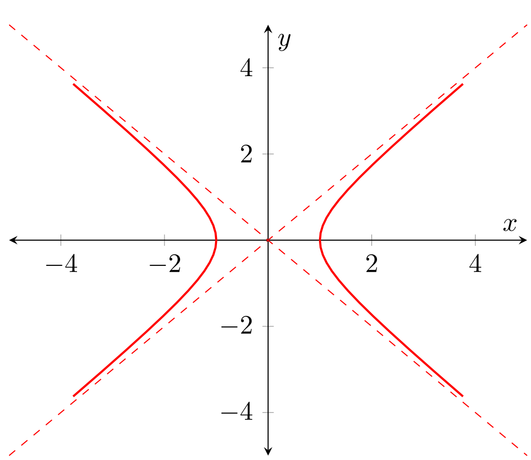
- They are related via:

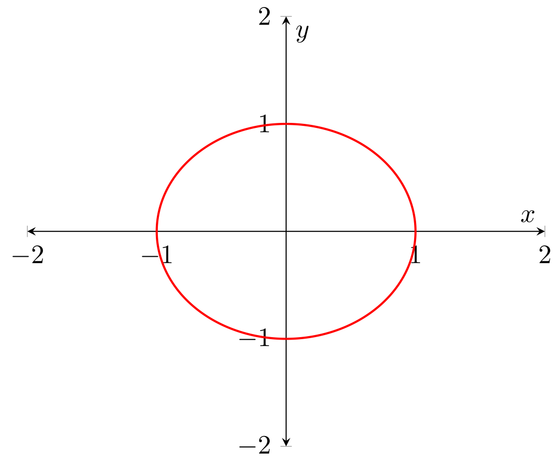
$$\cosh^2 x - \sinh^2 x = 1 \tag{3}$$

- Both the area of a circular sector and that of a hyperbolic sector is described by:

$$A = \frac{1}{2}t \tag{4}$$

  where t is the subtended angle, and the figures are parametized by $(\cos t, \sin t)$ and $(\cosh t, \sinh t)$.





- The catenary

$$y = a\cosh\left(\frac{x}{a}\right) + C \tag{5}$$

describes the shape of a free hanging rope between two walls separated by a width a.

- The hyperbolic tangent is given by $\tanh x = \dfrac{\sinh x}{\cos hx} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. and its derivative is given by:

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \tag{6}$$

- The inverse of $y = \sinh x$ is given by:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \tag{7}$$

Tip: A table of integrals and derivatives revolving around hyperbolic trig functions can be found in the textbook.

2 Indeterminate Forms

- A lot of the times, limits have an indeterminate form, where if we substitute in what x approaches to, we get it in the form of $\dfrac{0}{0}$, for example:

$$\lim_{x\to 0}\frac{\sin x}{x} \tag{8}$$

Theorem: If $f(x) \to 0$ and $g(x) \to 0$ as $x \to \pm, \infty$ or $x \to c$ or $x \to c^{+-}$ and if $\dfrac{f'(x)}{g'(x)} \to L$, then:

$$\frac{f(x)}{g(x)} \to L \tag{9}$$

Example 1: Solve: $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$

We can set $f(x) = \sin x$, $f'(x) = \cos x$, $g(x) = x$ and $g'(x) = 1$ such that:

$$\lim_{x\to 0}\frac{f'}{g'} = \lim_{x\to 0}\cos x = 1 \tag{10}$$

Example 2: Solve $\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}}$.

Set $f = \sin x$, $f' = \cos x$, $g = \sqrt{x}$, $g' = \frac{1}{2}x^{-1/2}$ and so:

$$\lim_{x\to 0^+} 2x^{1/2}\cos x = 0 \implies \lim_{x\to 0^+} = 0 \tag{11}$$

Example 3: Solve $\lim_{x\to 0} \frac{e^x - x - 1}{3x^2}$.

If we take the derivative, we get:

$$\lim_{x\to 0} \frac{e^x - 1}{6x} \tag{12}$$

which is still $\frac{0}{0}$!. We can take derivatives again:

$$\lim_{x\to 0} \frac{e^x}{6} = \frac{1}{6} \tag{13}$$

so the original limit is $\frac{1}{6}$.

Warning: L'hopital's rule can *only* be used in indeterminate forms. Applying them to limits where

- To prove the L'hopital's rule, we first prove the **Cauchy Mean Value Theorem** as a lemma

Theorem: Cauchy Mean Value Theorem: Given f and g differentiable on (a, b), continuous on $[a, b]$ and $g' \neq 0$ on (a, b), there must exist some number r in (a, b) such that:

$$\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)} \tag{14}$$

- We then apply **Rolle's Theorem** to prove the Cauchy Mean Value Theorem:

Proof. Set:

$$\begin{aligned} G(x) =& [g(b) - g(a)][f(x) - f(a)] \\ &- [g(x) - g(a)][f(b) - f(a)] \end{aligned}$$

Note that $G(a) = G(b) = 0$ so it satisfies the conditions of Rolle's Theorem. Taking the derivative, we get:

$$G'(x) = [g(b) - g(a)]f'(x) - g'(x)[f(b) - f(a)] \tag{15}$$

Accoridng to Rolle's, there must be some $x = r$ such that $G'(r) = 0$, we can then substitute for this and solve:

$$G'(r) = 0 \implies [g(b) - g(a)]f'(r) = g'(r)[f(b) - f(a)] \tag{16}$$

Which is equivalent to:

$$\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)} \tag{17}$$

Furthermore, we have $g'(c) = \frac{g(b) - g(a)}{b - a}$ from the mean value theorem. Since $g' \neq 0$ we have $g(b) - g(a) \neq 0$. $\square$

- Given $x \to c^+$ and $f(x), g(x) \to 0$ where:

$$\lim_{x\to c^+} \frac{f'(x)}{g'(x)} = L \tag{18}$$

we will now prove that $\lim_{x\to c^+} \frac{f(x)}{g(x)} = L$.

Proof. Consider the interval $[c, c+h]$ and apply Cauchy MVT. There must be some number c_2 in $[c, c+h]$ such that:

$$\frac{f'(c_2)}{g'(c_2)} = \frac{f(c+h) - f(c)}{g(c+h) - g(c)} = \frac{f(c+h)}{g(c+h)} \tag{19}$$

The last step is a result of the given $f(c) = g(c) = 0$. The LHS can be rewritten as:

$$\lim_{h\to 0} \frac{f'(c_2)}{g'(c_2)} = \frac{f'(c)}{g'(c)} \tag{20}$$

since c_2 lies in the interval $[c, c+h]$ so if $h \to 0$, then the interval becomes smaller to contain just c. The RHS can be rewritten as:

$$\lim_{h\to 0} \frac{f(c+h)}{g(c+h)} = \lim_{x\to c^+} \frac{f(x)}{g(x)} \tag{21}$$

and therefore:

$$\lim_{x\to c^+} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} = L \tag{22}$$

□

- To prove the case for $x \to \pm\infty$, we can let $x = \frac{1}{t}$ and take the limit as $t \to \infty$.

Example 4: Find $\lim\limits_{x\to\infty} \frac{\ln x}{x}$.

Taking the derivative of top and bottom, we have:

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0 \implies \lim_{x\to\infty} \frac{\ln x}{x} = 0 \tag{23}$$

Idea: The logarithm function grows very slowly. In fact, any positive power of x will grow faster than $\ln x$.

Example 5: Solve $\lim\limits_{x\to\infty} \frac{x^3}{e^x}$

This is indeterminate in the form of $\frac{\infty}{\infty}$. We apply L'hopital's rule multiple times:

$$\lim_{x\to\infty} \frac{x^3}{e^x} \overset{*}{=} \lim_{x\to\infty} \frac{3x^2}{e^x} \left(= \frac{\infty}{\infty}\right) \tag{24}$$

$$\overset{*}{=} \lim_{x\to\infty} \frac{6x}{e^x} \left(= \frac{\infty}{\infty}\right) \tag{25}$$

$$\overset{*}{=} \lim_{x\to\infty} \frac{6}{e^x} = 0 \tag{26}$$

- Generally, $\lim\limits_{x\to\infty} \frac{x^m}{e^x} = 0$ where m is any positive integer.
- There are other indeterminate forms, such as 0^0, for example:

$$\lim_{x\to 0} x^x \tag{27}$$

The central idea behind this is that $a^b = e^{a \ln b}$. Therefore, this limit is equal to:

$$\lim_{x\to 0} e^{x\ lnx} \tag{28}$$

We can take the limit of the exponent to get:

$$\lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{1/x} \tag{29}$$

Note that the first equation is another indeterminate form with the $0\cdot\infty$ type, so we had to multiply top and bottom by $\dfrac{1}{x}$ to get the quotient form. Then we have:

$$\lim_{x\to 0}\frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)}=\lim_{x\to 0}-x=0 \tag{30}$$

Therefore:

$$\lim_{x\to 0}e^{x\ln x}=e^0=1 \tag{31}$$

so $\lim\limits_{x\to 0}x^x=1$.

Example 6: Solve $\lim\limits_{x\to\infty}(x+2)^{2/\ln x}$.

This is of the type ∞^0. The approach is exactly the same as the previous example. We write it in exponential form:

$$=\lim_{x\to\infty}e^{\frac{2}{\ln x}\ln(x+2)} \tag{32}$$

and looking at the exponent gives:

$$\lim_{x\to\infty}\frac{2\ln(x+2)}{\ln x} \tag{33}$$

$$\overset{*}{=}\lim_{x\to\infty}\frac{\left(\frac{2}{x+2}\right)}{\left(\frac{1}{x}\right)}=\lim_{x\to\infty}\frac{2x}{x+2}\left(=\frac{\infty}{\infty}\right) \tag{34}$$

$$\overset{*}{=}\lim_{x\to\infty}\frac{2}{1}=2 \tag{35}$$

Therefore:

$$\lim_{x\to\infty}e^{\frac{2}{\ln x}\ln(x+2)}=e^2 \tag{36}$$

so:

$$\lim_{x\to\infty}(x+2)^{2/\ln x}=e^2 \tag{37}$$

Example 7: Solve $\lim\limits_{x\to\infty}\left[\sin\left(\dfrac{\pi}{x}+\dfrac{\pi}{2}\right)\right]^x$

This is in the form of 1^∞. We rewrite it as:

$$\lim_{x\to\infty}\exp\left(x\ln\left(\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\right)\right) \tag{38}$$

and taking the limit of the exponent:

$$=\lim_{x\to\infty}x\ln\left(\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\right)\left(=\frac{0}{0}\right) \tag{39}$$

$$\overset{*}{=}\lim_{x\to\infty}\frac{\cos\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\cdot\left(-\frac{\pi}{x^2}\right)}{\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\cdot\left(-\frac{1}{x^2}\right)}=\frac{0\cdot\pi}{1}=0 \tag{40}$$

Therefore:

$$\lim_{x\to\infty}\left[\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\right]^x=\lim_{x\to\infty}\exp\left(x\ln\left(\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\right)\right)=1 \tag{41}$$

3 Integration

3.1 Recap of Integration

- The definite integral has the geometric interpretation as the area under the curve $f(x)$ between $x = a$ and $x = b$ and the x axis:
$$\int_a^b f(x)\,\mathrm{d}x \tag{42}$$
but can be rigorously defined using a Riemann sum:
$$\int_a^b f(x)\,\mathrm{d}x \equiv \lim_{\|P\|} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \tag{43}$$
Often, we have a uniform partition, such that $\Delta x_i = \dfrac{b-a}{n}$ where n is the number of partitions. And if we choose to use the right hand endpoint, then:
$$f(x_i^*) = f(x_i) = f(x_i) = f\left(a + \frac{b-a}{n}i\right) \tag{44}$$

Example 8: To solve $\displaystyle\int_0^5 x^2\,\mathrm{d}x$, we can choose a uniform partition with:
$$\Delta x = \frac{5-0}{n} = \frac{5}{n} \tag{45}$$
and:
$$x_i^* = x_i = i\Delta x \implies f(x_i^*) = (i\Delta x)^2 = \left(i\frac{5}{n}\right)^2 \tag{46}$$
The area approximation is:
$$A \simeq \sum_{i=1}^{n} \Delta x_i f(x_i^*) = \sum_{i=1}^{n} \left(\frac{5}{n}\right)\left(i\frac{5}{n}\right)^2 \tag{47}$$
$$= \frac{125}{n^2}\sum_{i=1}^{n} i^2 = \frac{125}{n^3}\frac{n(n+1)(2n+1)}{6} \tag{48}$$
Taking the limit as $n \to \infty$, we get:
$$\int_0^5 x^2\,\mathrm{d}x = \lim_{n\to\infty} \frac{125}{6}\left(2 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{5^3}{3}. \tag{49}$$

Example 9: To evaluate $\displaystyle\int_1^2 x^{-2}\,\mathrm{d}x$, we can choose
$$x_i^* = \sqrt{x_{i-1}x_i} \tag{50}$$
and a uniform partition of:
$$\Delta x = \frac{2-1}{n} = \frac{1}{n} \tag{51}$$
such that:
$$x_i = 1 + i\Delta x = 1 + \frac{i}{n} = \frac{n+i}{n} \tag{52}$$
and
$$x_{i-1} = \frac{n+i-1}{n} \tag{53}$$

such that the area is:

$$\begin{aligned}
A &\simeq \sum_{i=1}^{n} \Delta x f(x_i^*) \\
&= \sum_{i=1}^{n} \frac{1}{n}\left(\frac{1}{x_i^*}\right)^2 \\
&= \sum_{i=1}^{n} \frac{1}{n}\frac{1}{x_{i-1}x_i} \\
&= \sum_{i=1}^{n} \frac{1}{n}\frac{n}{n+i-1}\cdot\frac{n}{n+i} \\
&= \sum_{i=1}^{n} n\frac{1}{n+i-1}\cdot\frac{1}{n+i} \\
&= \sum_{i=1}^{n} \left(\frac{1}{n+i-1}-\frac{1}{n+i}\right) \\
&= n\left[\sum_{i=1}^{n}\frac{1}{n+i-1}-\sum_{i=1}^{n}\frac{1}{n+i}\right] \\
&= n\left[\sum_{i=0}^{n}\frac{1}{n+i}-\sum_{i=1}^{n}\frac{1}{n+i}\right] \\
&= n\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}-\frac{1}{n+1}-\frac{1}{n+2}-\cdots-\frac{1}{2n}\right] \\
&= n\left(\frac{1}{n}-\frac{1}{2n}\right) \\
&= 1-\frac{1}{2}=\frac{1}{2}
\end{aligned}$$

The part where we cancel out everything is called a **telescoping series**. Notice how the area doesn't depend on n so we get the exact area, even if we let $n=1!$.

- We need a better way to do integration, so we can define:

$$F(x) \equiv \int_a^x f(t)\,\mathrm{d}t \tag{54}$$

such that $F'(x)=f(x)$. This is the definition of the antiderivative. This leads to the fundamental theorem of calculus:

$$\int_a^b f(t)\,\mathrm{d}t = F(h)-F(a) \tag{55}$$

and the indefinite integral can be written as:

$$\int f(x)\,\mathrm{d}x = G(x)+C \tag{56}$$

The main problem now becomes trying to *find antiderivatives*, which is much easier than Riemann sums, though still more difficult than calculating derivatives.

3.2 Techniques of Integration

- **Integration by Parts** attempts to reverse the product rule:

$$(fg)' = fg' + f'g \tag{57}$$

Taking the integral of both sides gives:

$$f(x)g(x) = \int f(x)g'(x)\,\mathrm{d}x + \int f'(x)g(x)\,\mathrm{d}x \tag{58}$$

$$\int f(x)g'(x)\,\mathrm{d}x = \int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x \tag{59}$$

If the second integral is easier than the first, then we have made substaintial progress.

Idea: Integration of parts tells us that:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u \tag{60}$$

Example 10: To solve $\int xe^{2x}$, we can let:

$$u = x \qquad \mathrm{d}v = e^{2x}\,\mathrm{d}x \tag{61}$$

$$\mathrm{d}u = \mathrm{d}x \qquad v = \frac{1}{2}e^{2x} \tag{62}$$

which gives:

$$\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,\mathrm{d}x \tag{63}$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \tag{64}$$

We can check:

$$\frac{d}{dx}\left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C\right) \tag{65}$$

$$= xe^{2x} + \frac{1}{2}e^{2x} - \frac{2}{4}e^{2x} \tag{66}$$

$$= xe^{2x} \tag{67}$$

Example 11: To solve $\int x^2\sin(2x)\,\mathrm{d}x$, we let:

$$u = x^2 \qquad \mathrm{d}v = \sin 2x\,\mathrm{d}x \tag{68}$$

$$\mathrm{d}u = 2x\,\mathrm{d}x \qquad v = -\frac{1}{2}\cos(2x) \tag{69}$$

which gives:

$$= -\frac{1}{2}x^2\cos 2x + \int x\cos(2x)\,\mathrm{d}x \tag{70}$$

and we can apply integration by parts a second time, if we let:

$$u = x \qquad \mathrm{d}v = \cos 2x\,\mathrm{d}x \tag{71}$$

$$\mathrm{d}u = \mathrm{d}x \qquad v = \frac{1}{2}\sin(2x) \tag{72}$$

which gives us:

$$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) - \int \frac{1}{2}\sin(2x)\,\mathrm{d}x \tag{73}$$

$$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2X) + C \tag{74}$$

Example 12: To solve $I = \int e^x \sin x \, \mathrm{d}x$, we can let:

$$u = \sin x \qquad \mathrm{d}v = e^x \, \mathrm{d}x \tag{75}$$
$$\mathrm{d}u = \cos x \, \mathrm{d}x \qquad v = e^x \tag{76}$$

to give us:

$$= e^x \sin x - \int e^x \cos x \, \mathrm{d}x \tag{77}$$

We apply integration by parts a second time:

$$u = \cos x \qquad \mathrm{d}v = e^x \, \mathrm{d}x \tag{78}$$
$$\mathrm{d}u = -\sin x \, \mathrm{d}x \qquad v = e^x \tag{79}$$

to get:

$$I = e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x \, \mathrm{d}x}_{I} \tag{80}$$
$$2I = e^x (\sin x - \cos x) + C' \tag{81}$$
$$I = \frac{1}{2} e^x (\sin x - \cos x) + C \tag{82}$$

and we are done.

Example 13: We can also solve integrals that do not appear to have parts, such as $\int \ln x \, \mathrm{d}x$. We choose:

$$u = \ln x \qquad \mathrm{d}v = \mathrm{d}x \tag{83}$$
$$\mathrm{d}u = \frac{1}{x} \, \mathrm{d}x \qquad v = x \tag{84}$$

to give us:

$$\ln x - \int \mathrm{d}x = x \ln x - x + C \tag{85}$$

- For a definite integral, we can write IBP as:

$$f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x) \, \mathrm{d}x \tag{86}$$

Example 14: It is *possible* to apply integration of parts to find the integral of $\int \tan x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \, \mathrm{d}x$. We can let:

$$u = \frac{1}{\cos x} = \sec x \qquad \mathrm{d}v = \sin x \, \mathrm{d}x \tag{87}$$
$$\mathrm{d}u = \sec x \tan x \qquad v = -\cos x \tag{88}$$

this gives us:

$$\int \tan x \, \mathrm{d}x = -\frac{\cos x}{\cos x} + \int \tan x \, \mathrm{d}x \tag{89}$$

Notice that we could try to subtract the original integral from both sides and get:

$$0 = -1 \tag{90}$$

which is clearly wrong! However, we forgot the constant of integration, so the correct statement would be:

$$0 + C' = -1 + C \tag{91}$$

which does not tell us anything interesting.