ESC195 Notes

QiLin Xue

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Contents

1 Hyperbolic Functions

- Sometimes, combinations of e^x and e^{-x} are given certain names, for example:
  - **Hyperbolic sine:** $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$
  - **Hyperbolic cosine:** $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$
- They have the following properties:

$$\frac{d}{dx}\sinh x = \cosh x \tag{1}$$
$$\frac{d}{dx}\cosh x = \sinh x \tag{2}$$

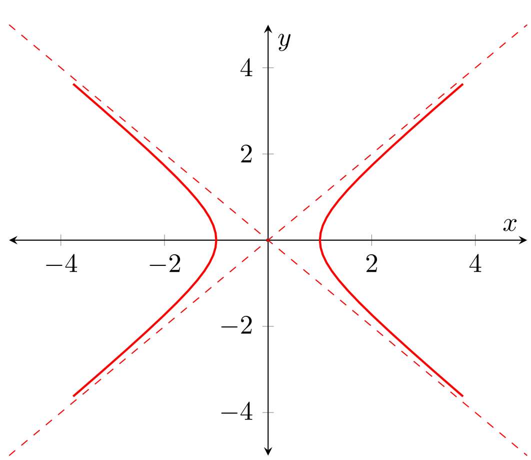
- They are related via:

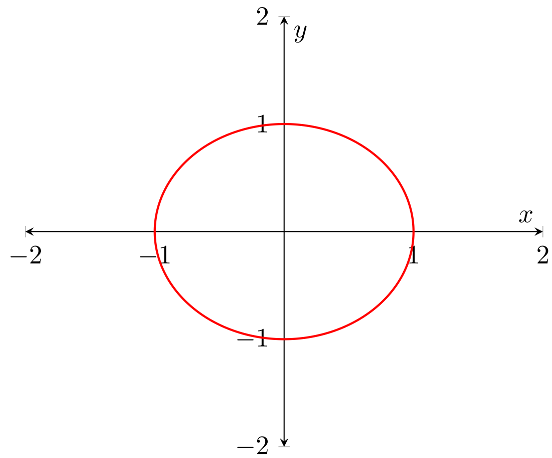
$$\cosh^2 x - \sinh^2 x = 1 \tag{3}$$

- Both the area of a circular sector and that of a hyperbolic sector is described by:

$$A = \frac{1}{2}t \tag{4}$$

  where t is the subtended angle, and the figures are parametized by $(\cos t, \sin t)$ and $(\cosh t, \sinh t)$.





- The catenary
$$y = a\cosh\left(\frac{x}{a}\right) + C \tag{5}$$
describes the shape of a free hanging rope between two walls separated by a width a.
- The hyperbolic tangent is given by $\tanh x = \dfrac{\sinh x}{\cos hx} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. and its derivative is given by:
$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \tag{6}$$
- The inverse of $y = \sinh x$ is given by:
$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \tag{7}$$

> **Tip:** A table of integrals and derivatives revolving around hyperbolic trig functions can be found in the textbook.

2 Indeterminate Forms

- A lot of the times, limits have an indeterminate form, where if we substitute in what x approaches to, we get it in the form of $\dfrac{0}{0}$, for example:
$$\lim_{x\to 0}\frac{\sin x}{x} \tag{8}$$

> **Theorem:** If $f(x) \to 0$ and $g(x) \to 0$ as $x \to \pm, \infty$ or $x \to c$ or $x \to c^{+-}$ and if $\dfrac{f'(x)}{g'(x)} \to L$, then:
> $$\frac{f(x)}{g(x)} \to L \tag{9}$$

Example 1: Solve: $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$

We can set $f(x) = \sin x$, $f'(x) = \cos x$, $g(x) = x$ and $g'(x) = 1$ such that:
$$\lim_{x\to 0}\frac{f'}{g'} = \lim_{x\to 0}\cos x = 1 \tag{10}$$

Example 2: Solve $\lim_{x\to 0^+} \frac{\sin x}{\sqrt{x}}$.

Set $f = \sin x$, $f' = \cos x$, $g = \sqrt{x}$, $g' = \frac{1}{2}x^{-1/2}$ and so:

$$\lim_{x\to 0^+} 2x^{1/2}\cos x = 0 \implies \lim_{x\to 0^+} = 0 \tag{11}$$

Example 3: Solve $\lim_{x\to 0} \frac{e^x - x - 1}{3x^2}$.

If we take the derivative, we get:

$$\lim_{x\to 0} \frac{e^x - 1}{6x} \tag{12}$$

which is still $\frac{0}{0}$!. We can take derivatives again:

$$\lim_{x\to 0} \frac{e^x}{6} = \frac{1}{6} \tag{13}$$

so the original limit is $\frac{1}{6}$.

Warning: L'hopital's rule can *only* be used in indeterminate forms. Applying them to limits where

- To prove the L'hopital's rule, we first prove the **Cauchy Mean Value Theorem** as a lemma

Theorem: **Cauchy Mean Value Theorem**: Given f and g differentiable on (a,b), continuous on $[a,b]$ and $g' \neq 0$ on (a,b), there must exist some number r in (a,b) such that:

$$\frac{f'(r)}{g'(r)} = \frac{f(b)-f(a)}{g(b)-g(a)} \tag{14}$$

- We then apply **Rolle's Theorem** to prove the Cauchy Mean Value Theorem:

Proof. Set:

$$\begin{aligned} G(x) =& [g(b)-g(a)][f(x)-f(a)] \\ &-[g(x)-g(a)][f(b)-f(a)] \end{aligned}$$

Note that $G(a) = G(b) = 0$ so it satisfies the conditions of Rolle's Theorem. Taking the derivative, we get:

$$G'(x) = [g(b)-g(a)]f'(x) - g'(x)[f(b)-f(a)] \tag{15}$$

Accoridng to Rolle's, there must be some $x = r$ such that $G'(r) = 0$, we can then substitute for this and solve:

$$G'(r) = 0 \implies [g(b)-g(a)]f'(r) = g'(r)[f(b)-f(a)] \tag{16}$$

Which is equivalent to:

$$\frac{f'(r)}{g'(r)} = \frac{f(b)-f(a)}{g(b)-g(a)} \tag{17}$$

Furthermore, we have $g'(c) = \frac{g(b)-g(a)}{b-a}$ from the mean value theorem. Since $g' \neq 0$ we have $g(b)-g(a) \neq 0$. ☐

- Given $x \to c^+$ and $f(x), g(x) \to 0$ where:

$$\lim_{x\to c^+} \frac{f'(x)}{g'(x)} = L \tag{18}$$

we will now prove that $\lim_{x\to c^+} \frac{f(x)}{g(x)} = L$.

Proof. Consider the interval $[c, c+h]$ and apply Cauchy MVT. There must be some number c_2 in $[c, c+h]$ such that:

$$\frac{f'(c_2)}{g'(c_2)} = \frac{f(c+h)-f(c)}{g(c+h)-g(c)} = \frac{f(c+h)}{g(c+h)} \tag{19}$$

The last step is a result of the given $f(c) = g(c) = 0$. The LHS can be rewritten as:

$$\lim_{h\to 0} \frac{f'(c_2)}{g'(c_2)} = \frac{f'(c)}{g'(c)} \tag{20}$$

since c_2 lies in the interval $[c, c+h]$ so if $h \to 0$, then the interval becomes smaller to contain just c. The RHS can be rewritten as:

$$\lim_{h\to 0} \frac{f(c+h)}{g(c+h)} = \lim_{x\to c^+} \frac{f(x)}{g(x)} \tag{21}$$

and therefore:

$$\lim_{x\to c^+} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} = L \tag{22}$$

□

- To prove the case for $x \to \pm\infty$, we can let $x = \frac{1}{t}$ and take the limit as $t \to \infty$.

Example 4: Find $\lim_{x\to\infty} \frac{\ln x}{x}$.

Taking the derivative of top and bottom, we have:

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0 \implies \lim_{x\to\infty} \frac{\ln x}{x} = 0 \tag{23}$$

Idea: The logarithm function grows very slowly. In fact, any positive power of x will grow faster than $\ln x$.

Example 5: Solve $\lim_{x\to\infty} \frac{x^3}{e^x}$

This is indeterminate in the form of $\frac{\infty}{\infty}$. We apply L'hopital's rule multiple times:

$$\lim_{x\to\infty} \frac{x^3}{e^x} \overset{*}{=} \lim_{x\to\infty} \frac{3x^2}{e^x} \left(= \frac{\infty}{\infty}\right) \tag{24}$$

$$\overset{*}{=} \lim_{x\to\infty} \frac{6x}{e^x} \left(= \frac{\infty}{\infty}\right) \tag{25}$$

$$\overset{*}{=} \lim_{x\to\infty} \frac{6}{e^x} = 0 \tag{26}$$

- Generally, $\lim_{x\to\infty} \frac{x^m}{e^x} = 0$ where m is any positive integer.
- There are other indeterminate forms, such as 0^0, for example:

$$\lim_{x\to 0} x^x \tag{27}$$

The central idea behind this is that $a^b = e^{a \ln b}$. Therefore, this limit is equal to:

$$\lim_{x\to 0} e^{x\ lnx} \tag{28}$$

We can take the limit of the exponent to get:

$$\lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{1/x} \tag{29}$$

Note that the first equation is another indeterminate form with the $0 \cdot \infty$ type, so we had to multiply top and bottom by $\dfrac{1}{x}$ to get the quotient form. Then we have:

$$\lim_{x\to 0} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0} -x = 0 \tag{30}$$

Therefore:

$$\lim_{x\to 0} e^{x\ln x} = e^0 = 1 \tag{31}$$

so $\lim\limits_{x\to 0} x^x = 1$.

Example 6: Solve $\lim\limits_{x\to\infty} (x+2)^{2/\ln x}$.

This is of the type ∞^0. The approach is exactly the same as the previous example. We write it in exponential form:

$$= \lim_{x\to\infty} e^{\frac{2}{\ln x}\ln(x+2)} \tag{32}$$

and looking at the exponent gives:

$$\lim_{x\to\infty} \frac{2\ln(x+2)}{\ln x} \tag{33}$$

$$\stackrel{*}{=} \lim_{x\to\infty} \frac{\left(\frac{2}{x+2}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{2x}{x+2} \left(= \frac{\infty}{\infty}\right) \tag{34}$$

$$\stackrel{*}{=} \lim_{x\to\infty} \frac{2}{1} = 2 \tag{35}$$

Therefore:

$$\lim_{x\to\infty} e^{\frac{2}{\ln x}\ln(x+2)} = e^2 \tag{36}$$

so:

$$\lim_{x\to\infty} (x+2)^{2/\ln x} = e^2 \tag{37}$$

Example 7: Solve $\lim\limits_{x\to\infty} \left[\sin\left(\dfrac{\pi}{x} + \dfrac{\pi}{2}\right)\right]^x$

This is in the form of 1^∞. We rewrite it as:

$$\lim_{x\to\infty} \exp\left(x\ln\left(\sin\left(\frac{\pi}{x} + \frac{\pi}{2}\right)\right)\right) \tag{38}$$

and taking the limit of the exponent:

$$= \lim_{x\to\infty} x\ln\left(\sin\left(\frac{\pi}{x} + \frac{\pi}{2}\right)\right) \left(= \frac{0}{0}\right) \tag{39}$$

$$\stackrel{*}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{\pi}{x} + \frac{\pi}{2}\right) \cdot \left(-\frac{\pi}{x^2}\right)}{\sin\left(\frac{\pi}{x} + \frac{\pi}{2}\right) \cdot \left(-\frac{1}{x^2}\right)} = \frac{0 \cdot \pi}{1} = 0 \tag{40}$$

Therefore:

$$\lim_{x\to\infty} \left[\sin\left(\frac{\pi}{x} + \frac{\pi}{2}\right)\right]^x = \lim_{x\to\infty} \exp\left(x\ln\left(\sin\left(\frac{\pi}{x} + \frac{\pi}{2}\right)\right)\right) = 1 \tag{41}$$

3 Integration

3.1 Recap of Integration

- The definite integral has the geometric interpretation as the area under the curve $f(x)$ between $x=a$ and $x=b$ and the x axis:

$$\int_a^b f(x)\,\mathrm{d}x \tag{42}$$

but can be rigorously defined using a Riemann sum:

$$\int_a^b f(x)\,\mathrm{d}x \equiv \lim_{\|P\|}\sum_{i=1}^n f(x_i^*)\Delta x_i \tag{43}$$

Often, we have a uniform partition, such that $\Delta x_i = \dfrac{b-a}{n}$ where n is the number of partitions. And if we choose to use the right hand endpoint, then:

$$f(x_i^*) = f(x_i) = f(x_i) = f\left(a + \frac{b-a}{n}i\right) \tag{44}$$

Example 8: To solve $\displaystyle\int_0^5 x^2\,\mathrm{d}x$, we can choose a uniform partition with:

$$\Delta x = \frac{5-0}{n} = \frac{5}{n} \tag{45}$$

and:

$$x_i^* = x_i = i\Delta x \implies f(x_i^*) = (i\Delta x)^2 = \left(i\frac{5}{n}\right)^2 \tag{46}$$

The area approximation is:

$$A \simeq \sum_{i=1}^n \Delta x_i f(x_i^*) = \sum_{i=1}^n \left(\frac{5}{n}\right)\left(i\frac{5}{n}\right)^2 \tag{47}$$

$$= \frac{125}{n^2}\sum_{i=1}^n i^2 = \frac{125}{n^3}\frac{n(n+1)(2n+1)}{6} \tag{48}$$

Taking the limit as $n\to\infty$, we get:

$$\int_0^5 x^2\,\mathrm{d}x = \lim_{n\to\infty}\frac{125}{6}\left(2+\frac{2}{n}+\frac{1}{n^2}\right) = \frac{5^3}{3}. \tag{49}$$

Example 9: To evaluate $\displaystyle\int_1^2 x^{-2}\,\mathrm{d}x$, we can choose

$$x_i^* = \sqrt{x_{i-1}x_i} \tag{50}$$

and a uniform partition of:

$$\Delta x = \frac{2-1}{n} = \frac{1}{n} \tag{51}$$

such that:

$$x_i = 1 + i\Delta x = 1 + \frac{i}{n} = \frac{n+i}{n} \tag{52}$$

and

$$x_{i-1} = \frac{n+i-1}{n} \tag{53}$$

such that the area is:

$$\begin{aligned}
A &\simeq \sum_{i=1}^{n} \Delta x f(x_i^*) \\
&= \sum_{i=1}^{n} \frac{1}{n}\left(\frac{1}{x_i^*}\right)^2 \\
&= \sum_{i=1}^{n} \frac{1}{n}\frac{1}{x_{i-1}x_i} \\
&= \sum_{i=1}^{n} \frac{1}{n}\frac{n}{n+i-1}\cdot\frac{n}{n+i} \\
&= \sum_{i=1}^{n} n\frac{1}{n+i-1}\cdot\frac{1}{n+i} \\
&= \sum_{i=1}^{n} \left(\frac{1}{n+i-1}-\frac{1}{n+i}\right) \\
&= n\left[\sum_{i=1}^{n}\frac{1}{n+i-1}-\sum_{i=1}^{n}\frac{1}{n+i}\right] \\
&= n\left[\sum_{i=0}^{n}\frac{1}{n+i}-\sum_{i=1}^{n}\frac{1}{n+i}\right] \\
&= n\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}-\frac{1}{n+1}-\frac{1}{n+2}-\cdots-\frac{1}{2n}\right] \\
&= n\left(\frac{1}{n}-\frac{1}{2n}\right) \\
&= 1-\frac{1}{2}=\frac{1}{2}
\end{aligned}$$

The part where we cancel out everything is called a **telescoping series**. Notice how the area doesn't depend on n so we get the exact area, even if we let $n = 1$!.

- We need a better way to do integration, so we can define:

$$F(x) \equiv \int_a^x f(t)\,\mathrm{d}t \tag{54}$$

such that $F'(x) = f(x)$. This is the definition of the antiderivative. This leads to the fundamental theorem of calculus:

$$\int_a^b f(t)\,\mathrm{d}t = F(h) - F(a) \tag{55}$$

and the indefinite integral can be written as:

$$\int f(x)\,\mathrm{d}x = G(x) + C \tag{56}$$

The main problem now becomes trying to *find antiderivatives*, which is much easier than Riemann sums, though still more difficult than calculating derivatives.

3.2 Integration by Parts

- **Integration by Parts** attempts to reverse the product rule:

$$(fg)' = fg' + f'g \tag{57}$$

Taking the integral of both sides gives:

$$f(x)g(x) = \int f(x)g'(x)\,\mathrm{d}x + \int f'(x)g(x)\,\mathrm{d}x \tag{58}$$

$$\int f(x)g'(x)\,\mathrm{d}x = \int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x \tag{59}$$

If the second integral is easier than the first, then we have made substaintial progress.

Idea: Integration of parts tells us that:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u \tag{60}$$

Example 10: To solve $\int xe^{2x}$, we can let:

$$u = x \qquad \mathrm{d}v = e^{2x}\,\mathrm{d}x \tag{61}$$

$$\mathrm{d}u = \mathrm{d}x \qquad v = \frac{1}{2}e^{2x} \tag{62}$$

which gives:

$$\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,\mathrm{d}x \tag{63}$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \tag{64}$$

We can check:

$$\frac{d}{dx}\left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C\right) \tag{65}$$

$$= xe^{2x} + \frac{1}{2}e^{2x} - \frac{2}{4}e^{2x} \tag{66}$$

$$= xe^{2x} \tag{67}$$

Example 11: To solve $\int x^2\sin(2x)\,\mathrm{d}x$, we let:

$$u = x^2 \qquad \mathrm{d}v = \sin 2x\,\mathrm{d}x \tag{68}$$

$$\mathrm{d}u = 2x\,\mathrm{d}x \qquad v = -\frac{1}{2}\cos(2x) \tag{69}$$

which gives:

$$= -\frac{1}{2}x^2\cos 2x + \int x\cos(2x)\,\mathrm{d}x \tag{70}$$

and we can apply integration by parts a second time, if we let:

$$u = x \qquad \mathrm{d}v = \cos 2x\,\mathrm{d}x \tag{71}$$

$$\mathrm{d}u = \mathrm{d}x \qquad v = \frac{1}{2}\sin(2x) \tag{72}$$

which gives us:

$$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) - \int \frac{1}{2}\sin(2x)\,\mathrm{d}x \tag{73}$$

$$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2X) + C \tag{74}$$

Example 12: To solve $I = \int e^x \sin x \, \mathrm{d}x$, we can let:

$$u = \sin x \qquad \mathrm{d}v = e^x \, \mathrm{d}x \tag{75}$$
$$\mathrm{d}u = \cos x \, \mathrm{d}x \qquad v = e^x \tag{76}$$

to give us:

$$= e^x \sin x - \int e^x \cos x \, \mathrm{d}x \tag{77}$$

We apply integration by parts a second time:

$$u = \cos x \qquad \mathrm{d}v = e^x \, \mathrm{d}x \tag{78}$$
$$\mathrm{d}u = -\sin x \, \mathrm{d}x \qquad v = e^x \tag{79}$$

to get:

$$I = e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x \, \mathrm{d}x}_{I} \tag{80}$$
$$2I = e^x (\sin x - \cos x) + C' \tag{81}$$
$$I = \frac{1}{2} e^x (\sin x - \cos x) + C \tag{82}$$

and we are done.

Example 13: We can also solve integrals that do not appear to have parts, such as $\int \ln x \, \mathrm{d}x$. We choose:

$$u = \ln x \qquad \mathrm{d}v = \mathrm{d}x \tag{83}$$
$$\mathrm{d}u = \frac{1}{x} \, \mathrm{d}x \qquad v = x \tag{84}$$

to give us:

$$\ln x - \int \mathrm{d}x = x \ln x - x + C \tag{85}$$

- For a definite integral, we can write IBP as:

$$f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x) \, \mathrm{d}x \tag{86}$$

Example 14: It is *possible* to apply integration of parts to find the integral of $\int \tan x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \, \mathrm{d}x$. We can let:

$$u = \frac{1}{\cos x} = \sec x \qquad \mathrm{d}v = \sin x \, \mathrm{d}x \tag{87}$$
$$\mathrm{d}u = \sec x \tan x \qquad v = -\cos x \tag{88}$$

this gives us:

$$\int \tan x \, \mathrm{d}x = -\frac{\cos x}{\cos x} + \int \tan x \, \mathrm{d}x \tag{89}$$

Notice that we could try to subtract the original integral from both sides and get:

$$0 = -1 \tag{90}$$

which is clearly wrong! However, we forgot the constant of integration, so the correct statement would be:

$$0 + C' = -1 + C \tag{91}$$

which does not tell us anything interesting. This brings We can see this concretely by repeating the same steps but trying to evaluate the definite integral $\int_a^b \tan x \, \mathrm{d}x$ instead, which gives:

$$\int_a^b \tan \mathrm{d}x = (-1)\Big|_{x=a}^{x=b} + \int_a^b \tan x \, \mathrm{d}x \implies 0 = (-1) - (-1) \implies 0 = 0 \tag{92}$$

which confirms our suspcision that this isn't anything useful, but it's also not an incorrect statement.

Warning: Sometimes it is possible to get more than one answer through various means that differ by a constant factor when solving indefinite integrals. When this happens, nothing is wrong: we simply need to consider the constant of integration.

Idea: But how do we know *which* values of u and $\mathrm{d}v$ we should pick? A common strategy is to use **LIATE**:

1. L: Logarithms
2. I: Inverse Trig
3. A: Algebraic
4. T: Trigonometric
5. E: Exponential

If a function consists of two terms, the term that is higher up (closer to L) usually gets differentiated and the term near the bottom (closer to E) usually gets integrated. See this for how it works, and this video for a tutorial.

4 Trigonometric Integrals

- The first type of integral we'll deal with is:

$$\int \sin^n x \cos^n x \, \mathrm{d}x \tag{93}$$

- In **case 1**, we have either m or n as an odd positive number. We can then use the identity $\sin^2 x + \cos^2 x = 1$ to simplify it.

Example 15: For example, to solve $\int \sin^3 x \cos^2 x \, \mathrm{d}x$, we can simplify this to:

$$= \int (1 - \cos^2 x) \cos^2 x \sin x \, \mathrm{d}X \tag{94}$$

$$= (\cos^2 x - \cos^4 x) \sin x \, \mathrm{d}x \tag{95}$$

and applying a u substitution with $u = \cos x$ and breaking it up into two integrals, we can get:

$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C \tag{96}$$

- In **case 2**, we have m and n as both even. We then apply the double angle formulas:

$$\sin x \cos x = \frac{1}{2}\sin(2x) \tag{97}$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x \tag{98}$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x \tag{99}$$

Example 16: For example:

$$\begin{aligned}
\int \sin^2 x \cos^4 \mathrm{d}x &= \int \frac{1}{4}\sin^2(2x)\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)\mathrm{d}x && (100)\\
&= \frac{1}{8}\int \sin^2(2x)\,\mathrm{d}x + \frac{1}{8}\int \sin^2 x \cos 2x\,\mathrm{d}x && (101)\\
&= \frac{1}{8}\int\left(\frac{1}{2}-\frac{1}{2}\cos 4x\right)\mathrm{d}x + \frac{1}{8\cdot 3\cdot 2}\sin^3(2x) + C && (102)\\
&= \frac{1}{16}x - \frac{1}{64}\sin(4x) + \frac{1}{48}\sin^3(2x) + C && (103)
\end{aligned}$$

- In **Case 3**, we have:

$$\int \sin^n \mathrm{d}x\,,\ \int \cos^n \mathrm{d}x \tag{104}$$

which we can apply a reduction formula by keep applying integration by parts:

$$\int \sin^n \mathrm{d}x = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,\mathrm{d}x \tag{105}$$

$$\int \cos^n \mathrm{d}x = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,\mathrm{d}x \tag{106}$$

Example 17: To solve the integral $\int \sin^2 x\,\mathrm{d}x$, we get:

$$\begin{aligned}
&= -\frac{1}{2}\sin x\cos x + \frac{1}{2}\int \mathrm{d}x && (107)\\
&= \frac{1}{2}x - \frac{1}{4}\sin 2x + C && (108)
\end{aligned}$$

- In **Case 4**, we have integrals in the following forms:

$$\int \sin(mx)\cos(nx)\,\mathrm{d}x \tag{109}$$

$$\int \sin(mx)\sin(nx)\,\mathrm{d}x \tag{110}$$

$$\int \cos(mx)\cos(nx)\,\mathrm{d}x \tag{111}$$

with $m \neq n$. If $m = n$, then we can apply the double angle formula. To solve these, we apply the following identities:

$$\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \tag{112}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right] \tag{113}$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right] \tag{114}$$

Example 18: For example, we have:

$$\begin{aligned}
\int \sin(3x)\sin(2x)\,\mathrm{d}x &= \frac{1}{2}\int \cos((3-2)x)\,\mathrm{d}x - \frac{1}{2}\cos((3+2)x)\,\mathrm{d}x && (115)\\
&= \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C && (116)
\end{aligned}$$

- In **case 5**, we have integrals in the form of either:

$$\int \tan^n x\,\mathrm{d}x\,,\ \int \cot^n x\,\mathrm{d}x \tag{117}$$

To solve these, we apply the following identities:

$$\tan^2 x = \sec^2 x - 1 \tag{118}$$
$$(\tan x)' = \sec^2 x \tag{119}$$

- In **case 6**, we have:

$$\int \sec^n x \, \mathrm{d}x \, , \int \csc^n x \, \mathrm{d}x \tag{120}$$

with $n \geq 2$. To solve these, we can make the following substitutions:

$$1 + \tan^2 x = \sec^2 x \tag{121}$$
$$1 + \cot^2 x = \csc^2 x \tag{122}$$

to convert it to a case 5 problem.

- In **cae 7**, we have:

$$\int \tan^n x \sec^n x \, \mathrm{d}x \, , \int \cot^n x \csc^n x \, \mathrm{d}x \tag{123}$$

Example 19: We have:

$$\tan^3 x \sec^4 x \, \mathrm{d}x = \int \tan^3 x \sec^2 x \sec^2 x \, \mathrm{d}x \tag{124}$$
$$= \int \tan^3 x \left(\tan^2 x + 1\right) \sec^2 x \, \mathrm{d}X \tag{125}$$
$$= \int (\tan^5 x + \tan^3 x) \sec^2 x \, \mathrm{d}x \tag{126}$$
$$= \frac{1}{6} \tan^6 x + \frac{1}{4} \tan^4 x + C \tag{127}$$

Idea: The basic idea of these types is to apply trigonometric identities to turn the integrals into a form that is easier to deal with. The substitutions are usually very simple but to find them, it requires a lot of practice.

- We can also apply **trigonometric substitutions**, any integrals with any of the three factors below can be solved with this technique:

  1. $\sqrt{a^2 - x^2}$: Set $x = a \sin u \implies \sqrt{a^2 - x^2} = a \cos u$
  2. $\sqrt{a^2 + x^2}$: Set $x = a \tan u \implies \sqrt{a^2 + x^2} = a \sec u$
  3. $\sqrt{x^2 - a^2}$: Set $x = a \sec u \implies \sqrt{x^2 - a^2} = a \tan u$

  where the arguments under the square roots are always positive.

Example 20: To solve the integral $\displaystyle\int \frac{x^2}{(4 - x^2)^{3/2}} \mathrm{d}x$, we can set:

$$x = 2 \sin u \tag{128}$$
$$\mathrm{d}x = 2 \cos u \, \mathrm{d}u \tag{129}$$
$$\sqrt{4 - x^2} = 2 \cos u \tag{130}$$

which gives:

$$= \int \frac{4\sin^2 u \cdot 2\cos u \,\mathrm{d}u}{8\cos^3 u} \tag{131}$$

$$= \int \tan^2 u \,\mathrm{d}u \tag{132}$$

$$= \int (\sec^2 u - 1)\,\mathrm{d}u \tag{133}$$

$$= \tan u - u + C \qquad = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C \tag{134}$$

Example 21: The integral $\int \frac{x\,\mathrm{d}x}{(2x^2+4x-7)^{1/2}}$ needs a bit more work before we can apply the sibstutitions. We first apply the square to get:

$$= \int \frac{x\,\mathrm{d}x}{\sqrt{2(x+1)^2-9}} \tag{135}$$

We can set:

$$\sqrt{2}(x+1) = 3\sec u \tag{136}$$

$$\sqrt{2}\,\mathrm{d}x = 3\sec u \tan u \,\mathrm{d}u \tag{137}$$

$$\sqrt{2(x+1)^2-9} = 3\tan u \tag{138}$$

which gives:

$$= \int \frac{\left(\frac{3}{\sqrt{2}}\sec u - 1\right)\left(\frac{3}{\sqrt{2}\sec u \tan u \mathrm{d}u}\right)}{3\tan u} \tag{139}$$

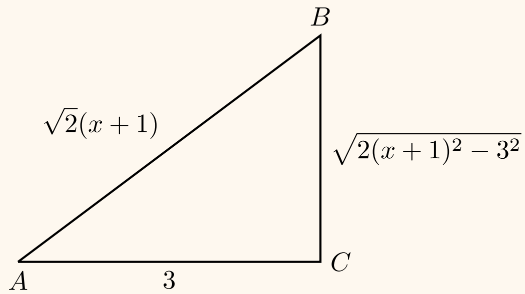
$$= \int \left(\frac{3}{\sqrt{2}}\sec u - 1\right)\left(\frac{1}{\sqrt{2}}\sec u\right)\mathrm{d}u \tag{140}$$

$$= \frac{3}{2}\int \sec^2 u \,\mathrm{d}u - \frac{1}{\sqrt{2}}\int \sec u \,\mathrm{d}u \tag{141}$$

$$= \frac{3}{2}\tan u - \frac{1}{\sqrt{2}}\ln|\sec u + \tan u| + C \tag{142}$$

$$= \frac{1}{2}\sqrt{2x^2+4x-7} - \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}}{3}(x+1) + \frac{\sqrt{2x^2+4x-7}}{3}\right| + C \tag{143}$$

Idea: We can use triangles to derive the substitution, which comes from the Pythagorean theorem:



and you can clearly see the substitution:

$$3\sec u = \sqrt{2}(x+1) \implies \cos u = \frac{3}{\sqrt{2}(x+1)} \tag{144}$$

where $u \equiv \angle BAC$.

Example 22: For the integral $\int x \sin^{-1} x \, \mathrm{d}x$, we can let:

$$u = \sin^{-1} x \, \mathrm{d}v = x \, \mathrm{d}x \tag{145}$$

$$\mathrm{d}u = \frac{\mathrm{d}x}{\sqrt{1-x^2}} v = \frac{1}{2}x^2 \tag{146}$$

and applying integration by parts, we get:

$$= \frac{1}{2}x^3 \sin^{-1} x - \int \frac{1}{2}x^2 \frac{\mathrm{d}x}{\sqrt{1-x^2}} \tag{147}$$

To solve this secondary integral $\int \frac{d^2 \, \mathrm{d}x}{\sqrt{1-x^2}}$, we can let:

$$x = \sin\theta \tag{148}$$

$$\mathrm{d}x = \cos\theta \, \mathrm{d}\theta \tag{149}$$

$$\sqrt{1-x^2} = \cos\theta \tag{150}$$

which gives:

$$= \frac{\sin^2\theta \cos\theta \, \mathrm{d}\theta}{\cos\theta} \tag{151}$$

$$= \int \sin^2\theta \, \mathrm{d}\theta \tag{152}$$

$$= \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta + C \tag{153}$$

$$= \frac{1}{2}\sin^{-1} - \frac{1}{2}x\sqrt{1-x^2} + C \tag{154}$$

Therefore, we get:

$$\int x \sin^{-1} x \, \mathrm{d}x = \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{1}{4}x\sqrt{1-x^2} + C \tag{155}$$