ECE159 Notes

QiLin Xue

March 13, 2021

Contents

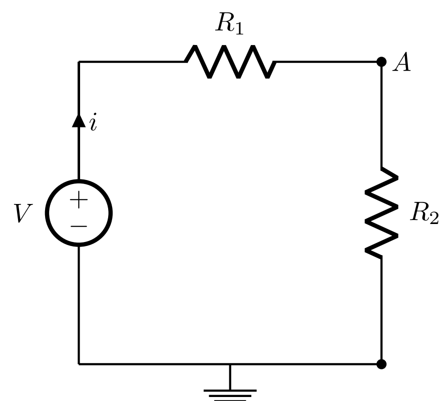
1 DC circuit Analysis **1**
- 1.1 Voltage and Current Division . . . 1
- 1.2 Mesh Analysis . . . 2
- 1.3 Nodal Analysis . . . 3
- 1.4 Superposition . . . 3
- 1.5 Source Transformation . . . 4
- 1.6 Thevenin Equivalent Circuit . . . 5

2 Operational Amplifiers **6**
- 2.1 Ideal OP Amp . . . 7

1 DC circuit Analysis

1.1 Voltage and Current Division

For a series circuit, we can calculate the equivalent resistance by adding them.



For this circuit, we have:

$$R_{\text{eq}} = R_1 + R_2 \tag{1}$$

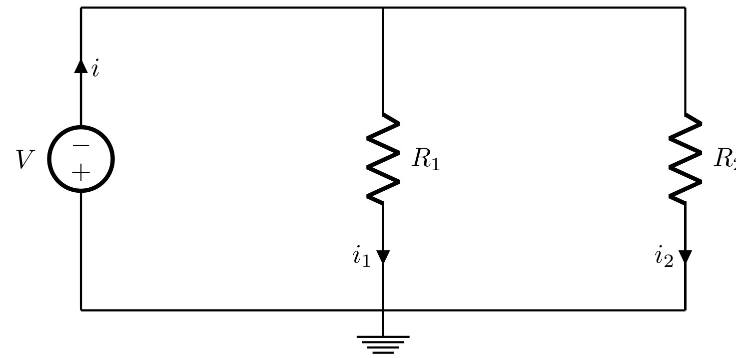
and the voltage:

$$V_A = \frac{R_2}{R_1 + R_2} V \tag{2}$$

Proof. The current is given by $i = \dfrac{V}{R_1 + R_2}$ and the voltage at A is given by:

$$V_A = V - iR_1 = V\left(1 - \frac{R_1}{R_1 + R_2}\right) = \frac{R_2}{R_1 + R_2} V \tag{3}$$

For a parallel circuit, the effective resistance is the harmonic sum:



In this circuit, we have:

$$R_{\text{eq}} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} = \frac{R_1 R_2}{R_1 + R_2} \tag{4}$$

and the current in the branches is given by:

$$i_1 = \frac{R_2}{R_1 + R_2} i \tag{5}$$

Proof. The voltage drop across R_1 and R_2 must be the same, so:

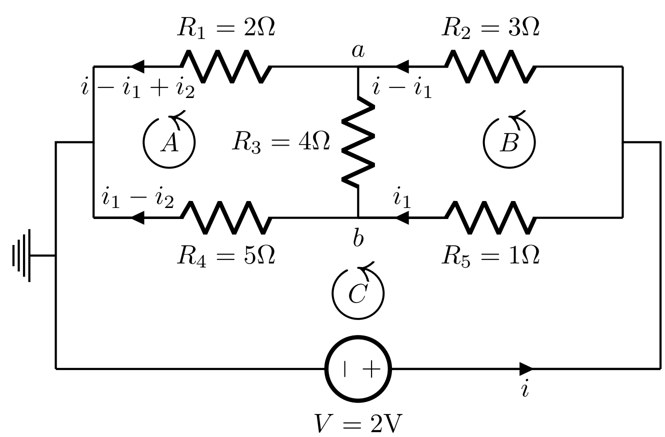
$$i_1 R_1 = i_2 R_2 \tag{6}$$

We also have $i_2 = i - i_1$, which gives:

$$i_1 R_1 = (i - i_1) R_2 \implies i_1 = \frac{R_2}{R_1 + R_2} i \tag{7}$$

1.2 Mesh Analysis

In mesh analysis, Kirchoff's Voltage law $\sum V = 0$ is written for each independent loop, and a system of equation is solved. The number of independent loops can be determined by finding the minimum number of wire cuts needed such that there are no loops.



We have three independent loops, labelled as A, B, C, which gives the system of three equations:

$$0 = -i_2 R_3 - (i - i_1 + i_2) R_1 + (i_1 - i_2) R_4 \tag{8}$$
$$0 = -(i - i_1) R_2 + i_2 R_3 + i_1 R_5 \tag{9}$$
$$0 = V - i_1 R_5 - (i_1 - i_2) R_4 \tag{10}$$

Solving this system gives:

$$(i, i_1, i_2) = \left(-\frac{144}{181}\text{A}, \frac{82}{181}\text{A}, \frac{26}{181}\text{A}\right) \tag{11}$$

which can be used to determine the currents in all resistors and the associated voltages.

1.3 Nodal Analysis

In nodal analysis, we deal with the potentials at each node and write out Kirchoff's current law $\sum I = 0$ for the current *leaving* (or alternatively, entering) each node for each node where the potential is unknown. Let the nodes labelled a and b have potentials V_a and V_b. Then we have:

$$0 = \frac{V_a - V}{R_2} + \frac{V_a - V_b}{R_3} + \frac{V_a - 0}{R_1} \tag{12}$$

$$0 = \frac{V_b - V}{R_5} + \frac{V_b - V_a}{R_3} + \frac{V_b - 0}{R_4} \tag{13}$$

which after solving, gives:

$$(V_a, V_b) = \left(\frac{176}{181}\text{V}, \frac{280}{181}\text{V}\right) \tag{14}$$

which can be easily confirmed via the mesh analysis done above.

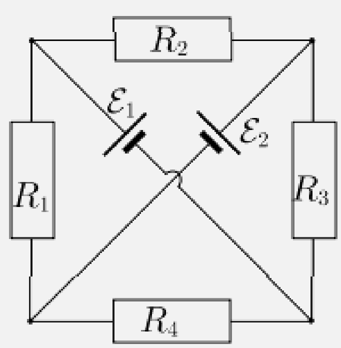
1.4 Superposition

Kirchoff's equations are linear. Each term includes only a first power of a current or a voltage, hence we can apply superposition. Let there be n independent voltage sources and m independent current sources. The current in the j^{th} wire can be found as:

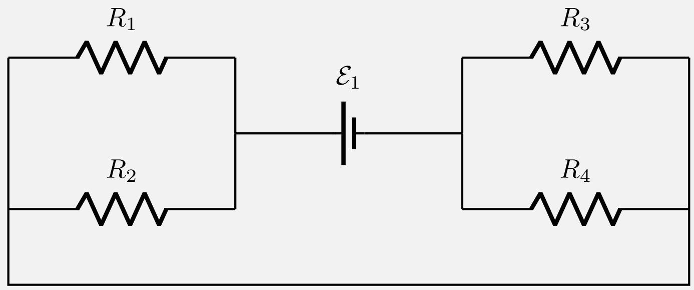
$$I_j = \sum_{k=1}^{n+m} I_j(k) \tag{15}$$

where $I_j(k)$ is the current in that wire when only the k^{th} battery (or current source) is included into the circuit. All other batteries are short circuited and all other current sources are removed by cutting off a connection wire.

Example 1: In the following circuit, we have $\mathcal{E}_1 = \mathcal{E}_2$ and all the resistors are equal: $R_1 = R_2 = R_3 = R_4 = R$. Let us attempt to find the current through each resistor.



We make use of the superposition principle and start by short-circuiting $\mathcal{E}_2$. This leads to the following equivalent circuit:



Since all resistors have a resistance of R, this leads to the following currents in each resistor (going back to the original diagram):

$$R_1 \to \frac{\mathcal{E}}{2R}\,\text{(downwards)}$$
$$R_2 \to \frac{\mathcal{E}}{2R}\,\text{(rightwards)}$$
$$R_3 \to \frac{\mathcal{E}}{2R}\,\text{(downwards)}$$
$$R_4 \to \frac{\mathcal{E}}{2R}\,\text{(leftwards)}$$

and similarly if we short $\mathcal{E}_1$, we get the following currents:

$$R_1 \to \frac{\mathcal{E}}{2R}\,\text{(downwards)}$$
$$R_2 \to \frac{\mathcal{E}}{2R}\,\text{(leftwards)}$$
$$R_3 \to \frac{\mathcal{E}}{2R}\,\text{(downwards)}$$
$$R_4 \to \frac{\mathcal{E}}{2R}\,\text{(rightwards)}$$

By considering the superposition of these currents, we get after adding them together:

$$R_1 \to \frac{\mathcal{E}}{R}\,\text{(downwards)}$$
$$R_2 \to 0$$
$$R_3 \to \frac{\mathcal{E}}{R}\,\text{(downwards)}$$
$$R_4 \to 0$$

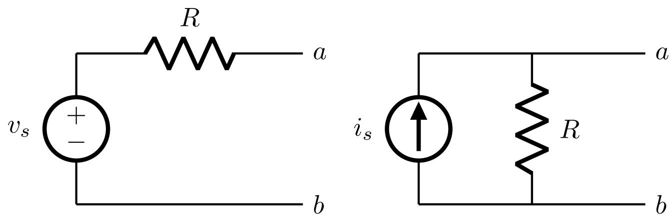
Alternatively we can solve this problem via symmetry. Note that there is reflection symmetry across the vertical line. This means that the current does not have a preferred direction of going either right or left, so that the current in R_2 and R_4 will be zero. As a result, we can simply replace these two resistors with an open gap, which results in the other four circuit elements in series:

$$I = \frac{2\mathcal{E}}{2R} = \frac{\mathcal{E}}{R}$$

which travels in a "zigzag" pattern.

1.5 Source Transformation

A source transformation is the process of replacing a voltage source v_s in series with a resistor R by a current source i_s in parallel with a resistor R, or vice versa. For example, these two circuits are equivalent:



where $v_s = i_s R$.

Proof. We can show that this is valid by letting $V_a = V$ and $V_b = 0$. and calculating the current at a. For the first circuit, we have:

$$i = \frac{v_s - V}{R} \tag{16}$$

For the second circuit, the current through the resistor is $i_R = \dfrac{V}{R}$ (downwards) and letting $i_s = \dfrac{v_s}{R}$ we have the current at a as:

$$i = \frac{v_s}{R} - \frac{V}{R} \tag{17}$$

which we see is the same for both cases.

Example 2: Suppose we have n batteries with a voltage of $\mathcal{E}_i$ and internal resistances of r_i with $i = 1, 2, \ldots, n$, all connected in parallel. What is the effective electromotive force and the internal resistance of such a system of batteries?

We treat the batteries as n current sources providing a current of $I_i = \dfrac{\mathcal{E}_i}{r_i}$. When determining an equivalent circuit, all the current sources add up to a total current of:

$$I_{\text{total}} = \sum_{i=1}^{n} \frac{\mathcal{E}_i}{r_i}$$

and applying the idea again, this is equivalent to an effective voltage source of $\mathcal{E}_{\text{eff}} = I_{\text{total}} R_{\text{eff}}$ with the effective resistance being:

$$R_{\text{eff}} = \left(\sum_{i=1}^{n} r_i^{-1} \right)^{-1}$$

since we are adding resistors in parallel. Putting everything together gives:

$$\mathcal{E}_{\text{eff}} = \left(\sum_{i=1}^{n} r_i^{-1} \right)^{-1} \sum_{i=1}^{n} \frac{\mathcal{E}_i}{r_i}$$

1.6 Thevenin Equivalent Circuit

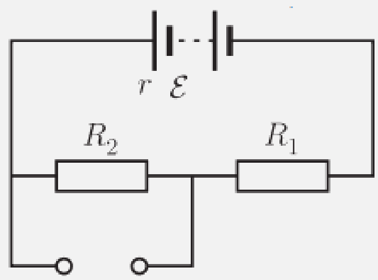
Thevenin's Theorem tells us that any two-terminal circuit can be replaced by an equivalent voltage source V_{th} in series with a resistor R_{th}. These are related via:

$$R_{\text{th}} = \frac{V_{\text{th}}}{I_{\text{sc}}} \tag{18}$$

To calculate V_{th}, we find the *open circuit* voltage between the two terminals. To calculate R_{th}, there are two options:

1. Remove all independent voltage and current sources and calculate the equivalent resistance across the terminals.
2. Connect the two terminals via a wire with negligible resistance. Calculate the current I_{sc} through this wire and we can calculate R_{th} using equation 18.

Example 3: Take the following circuit. What is the maximal power which can be dissipated on a load connected to the leads of the circuit?



Recall that any circuit which consists of only resistors and batteries and has two ports A and B is equivalent to a series connection of a battery and a resistance. In other words, we consider the Thevenin equivalent of the circuit. The equivalent internal resistance can be found easily replacing the battery by a wire and using series and parallel connections.

$$R_{\text{eq}} = \frac{R_2(R_1+r)}{R_2+(R_1+r)}$$

The equivalent emf $\mathcal{E}_{\text{eq}}$ is the potential drop across R_2. The net resistance about the battery is $R_2+(R_1+r)$, so the current through the battery is ,

$$I = \mathcal{E}/r \implies I = \frac{\mathcal{E}}{R_1+R_2+r}.$$

The potential drop about R_2 is then

$$V = IR_2 = \mathcal{E}\frac{R_2}{R_1+R_2+r}.$$

This means that entire circuit in the figure can be substituted with an equivalent battery with

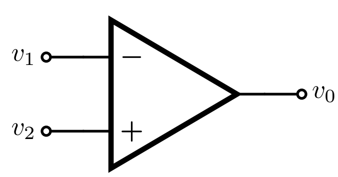
$$\mathcal{E}_{\text{eq}} = \mathcal{E}\frac{R_2}{R_1+R_2+r}.$$

By the maximal power transfer theorem, this means that the resistance of the load attached needs to be $R = R_{\text{eq}}$. The potential drop across it is $\frac{\mathcal{E}}{2}$, so we have:

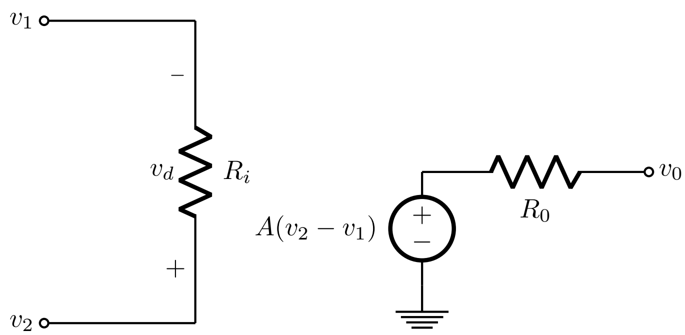
$$\begin{aligned} P_{\max} = \frac{\mathcal{E}_{\text{eq}}^2}{4R_{\text{eq}}} &= \frac{1}{4}\mathcal{E}^2\left(\frac{R_2}{R_1+R_2+r}\right)^2 \cdot \frac{R_2+(R_1+r)}{R_2\times(R_1+r)} \\ &= \frac{1}{4}\frac{R_2}{(r+R_1+R_2)(R_1+r)}\mathcal{E}^2 \end{aligned}$$

2 Operational Amplifiers

An operational amplifier (op amp) is an active circuit element designed to perform mathematical operations of addition, subtraction, multiplication, division, differentiation, and integration.



It is equivalent to the following circuit:



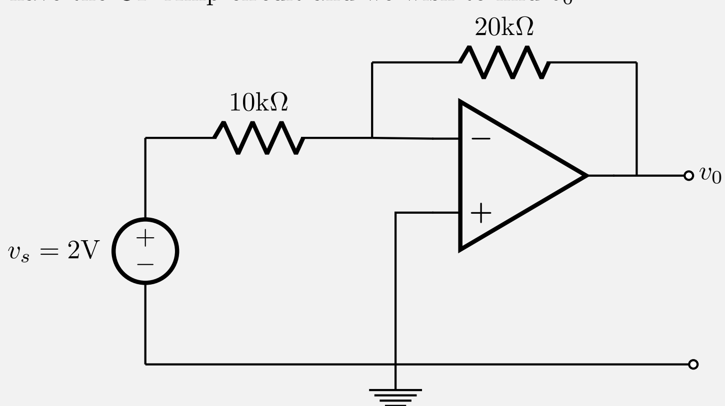
where $A \gg 1$ is known as the open-loop voltage gain. The input resistance R_i is typically very big.

2.1 Ideal OP Amp

For an ideal OP Amp, we have an infinite open-loop gain, infinite input resistance, and zero output resistance. This gives the following results:

- The currents into both input terminals are zero.
- Voltage across input terminals are zero.

Example 4: Suppose we have the OP Amp circuit and we wish to find v_0



Since the terminals ahve equal voltage, and the positive end is connected to the ground, then we must have:

$$2V - i(10\text{k}\Omega) = 0 \implies i = 0.2\text{mA}$$

and thus:

$$v_0 = -i \cdot 20\text{k}\Omega = -4\text{V}$$

This is known as an **inverting amplifier**.