ESC195 Notes

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Contents

1 Hyperbolic Functions

- Sometimes, combinations of e^x and e^{-x} are given certain names, for example:
  - **Hyperbolic sine:** $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$
  - **Hyperbolic cosine:** $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$
- They have the following properties:

$$\frac{d}{dx}\sinh x = \cosh x \tag{1}$$
$$\frac{d}{dx}\cosh x = \sinh x \tag{2}$$

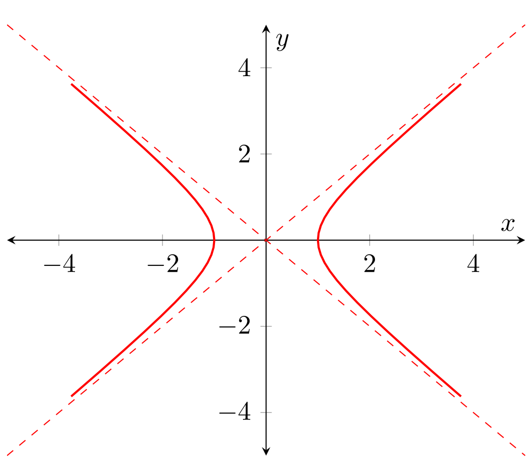
- They are related via:

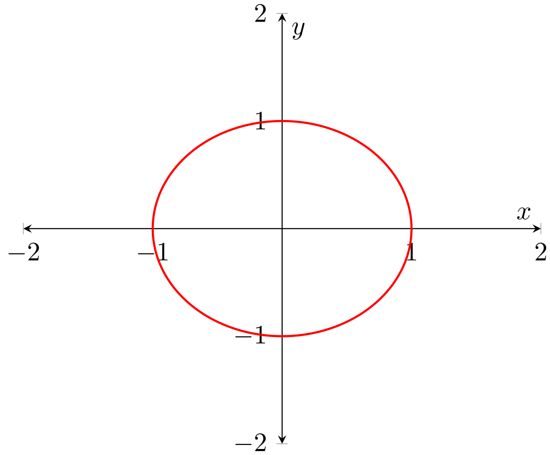
$$\cosh^2 x - \sinh^2 x = 1 \tag{3}$$

- Both the area of a circular sector and that of a hyperbolic sector is described by:

$$A = \frac{1}{2}t \tag{4}$$

  where t is the subtended angle, and the figures are parametized by $(\cos t, \sin t)$ and $(\cosh t, \sinh t)$.





- The catenary

$$y = a\cosh\left(\frac{x}{a}\right) + C \tag{5}$$

  describes the shape of a free hanging rope between two walls separated by a width a.

- The hyperbolic tangent is given by $\tanh x = \dfrac{\sinh x}{\cos hx} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. and its derivative is given by:

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \tag{6}$$

- The inverse of $y = \sinh x$ is given by:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \tag{7}$$

> **Tip:** A table of integrals and derivatives revolving around hyperbolic trig functions can be found in the textbook.

2 Indeterminate Forms

- A lot of the times, limits have an indeterminate form, where if we substitute in what x approaches to, we get it in the form of $\dfrac{0}{0}$, for example:

$$\lim_{x\to 0}\frac{\sin x}{x} \tag{8}$$

> **Theorem:** If $f(x) \to 0$ and $g(x) \to 0$ as $x \to \pm, \infty$ or $x \to c$ or $x \to c^{+-}$ and if $\dfrac{f'(x)}{g'(x)} \to L$, then:
>
> $$\frac{f(x)}{g(x)} \to L \tag{9}$$

Example 1: Solve: $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$

We can set $f(x) = \sin x$, $f'(x) = \cos x$, $g(x) = x$ and $g'(x) = 1$ such that:

$$\lim_{x\to 0}\frac{f'}{g'} = \lim_{x\to 0}\cos x = 1 \tag{10}$$

Example 2: Solve $\displaystyle\lim_{x\to 0+}\frac{\sin x}{\sqrt{x}}$.

Set $f = \sin x$, $f' = \cos x$, $g = \sqrt{x}$, $g' = \frac{1}{2}x^{-1/2}$ and so:

$$\lim_{x\to 0^+} 2x^{1/2}\cos x = 0 \implies \lim_{x\to 0^+} = 0 \tag{11}$$

Example 3: Solve $\lim_{x\to 0} \frac{e^x - x - 1}{3x^2}$.

If we take the derivative, we get:

$$\lim_{x\to 0} \frac{e^x - 1}{6x} \tag{12}$$

which is still $\frac{0}{0}$!. We can take derivatives again:

$$\lim_{x\to 0} \frac{e^x}{6} = \frac{1}{6} \tag{13}$$

so the original limit is $\frac{1}{6}$.

Warning: L'hopital's rule can *only* be used in indeterminate forms. Applying them to limits where

- To prove the L'hopital's rule, we first prove the **Cauchy Mean Value Theorem** as a lemma

Theorem: Cauchy Mean Value Theorem: Given f and g differentiable on (a, b), continuous on $[a, b]$ and $g' \neq 0$ on (a, b), there must exist some number r in (a, b) such that:

$$\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)} \tag{14}$$

- We then apply **Rolle's Theorem** to prove the Cauchy Mean Value Theorem:

Proof. Set:

$$\begin{aligned} G(x) =& [g(b) - g(a)][f(x) - f(a)] \\ &- [g(x) - g(a)][f(b) - f(a)] \end{aligned}$$

Note that $G(a) = G(b) = 0$ so it satisfies the conditions of Rolle's Theorem. Taking the derivative, we get:

$$G'(x) = [g(b) - g(a)]f'(x) - g'(x)[f(b) - f(a)] \tag{15}$$

Accoridng to Rolle's, there must be some $x = r$ such that $G'(r) = 0$, we can then substitute for this and solve:

$$G'(r) = 0 \implies [g(b) - g(a)]f'(r) = g'(r)[f(b) - f(a)] \tag{16}$$

Which is equivalent to:

$$\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)} \tag{17}$$

Furthermore, we have $g'(c) = \frac{g(b) - g(a)}{b - a}$ from the mean value theorem. Since $g' \neq 0$ we have $g(b) - g(a) \neq 0$. □

- Given $x \to c^+$ and $f(x), g(x) \to 0$ where:

$$\lim_{x\to c^+} \frac{f'(x)}{g'(x)} = L \tag{18}$$

we will now prove that $\lim_{x\to c^+} \frac{f(x)}{g(x)} = L$.

Proof. Consider the interval $[c, c+h]$ and apply Cauchy MVT. There must be some number c_2 in $[c, c+h]$ such that:

$$\frac{f'(c_2)}{g'(c_2)} = \frac{f(c+h)-f(c)}{g(c+h)-g(c)} = \frac{f(c+h)}{g(c+h)} \tag{19}$$

The last step is a result of the given $f(c) = g(c) = 0$. The LHS can be rewritten as:

$$\lim_{h\to 0}\frac{f'(c_2)}{g'(c_2)} = \frac{f'(c)}{g'(c)} \tag{20}$$

since c_2 lies in the interval $[c, c+h]$ so if $h \to 0$, then the interval becomes smaller to contain just c. The RHS can be rewritten as:

$$\lim_{h\to 0}\frac{f(c+h)}{g(c+h)} = \lim_{x\to c^+}\frac{f(x)}{g(x)} \tag{21}$$

and therefore:

$$\lim_{x\to c^+}\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} = L \tag{22}$$

$\square$

- To prove the case for $x \to \pm\infty$, we can let $x = \frac{1}{t}$ and take the limit as $t \to \infty$.

Example 4: Find $\lim_{x\to\infty}\frac{\ln x}{x}$.

Taking the derivative of top and bottom, we have:

$$\lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0 \implies \lim_{x\to\infty}\frac{\ln x}{x} = 0 \tag{23}$$

Idea: The logarithm function grows very slowly. In fact, any positive power of x will grow faster than $\ln x$.

Example 5: Solve $\lim_{x\to\infty}\frac{x^3}{e^x}$

This is indeterminate in the form of $\frac{\infty}{\infty}$. We apply L'hopital's rule multiple times:

$$\lim_{x\to\infty}\frac{x^3}{e^x} \overset{*}{=} \lim_{x\to\infty}\frac{3x^2}{e^x}\left(=\frac{\infty}{\infty}\right) \tag{24}$$

$$\overset{*}{=} \lim_{x\to\infty}\frac{6x}{e^x}\left(=\frac{\infty}{\infty}\right) \tag{25}$$

$$\overset{*}{=} \lim_{x\to\infty}\frac{6}{e^x} = 0 \tag{26}$$

- Generally, $\lim_{x\to\infty}\frac{x^m}{e^x} = 0$ where m is any positive integer.
- There are other indeterminate forms, such as 0^0, for example:

$$\lim_{x\to 0} x^x \tag{27}$$

The central idea behind this is that $a^b = e^{a\ln b}$. Therefore, this limit is equal to:

$$\lim_{x\to 0} e^{x\ lnx} \tag{28}$$

We can take the limit of the exponent to get:

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0}\frac{\ln x}{1/x} \tag{29}$$

Note that the first equation is another indeterminate form with the $0 \cdot \infty$ type, so we had to multiply top and bottom by $\frac{1}{x}$ to get the quotient form. Then we have:

$$\lim_{x \to 0} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x \to 0} -x = 0 \tag{30}$$

Therefore:

$$\lim_{x \to 0} e^{x \ln x} = e^0 = 1 \tag{31}$$

so $\lim_{x \to 0} x^x = 1$.

Example 6: Solve $\lim_{x \to \infty} (x+2)^{2/\ln x}$.

This is of the type ∞^0. The approach is exactly the same as the previous example. We write it in exponential form:

$$= \lim_{x \to \infty} e^{\frac{2}{\ln x} \ln(x+2)} \tag{32}$$

and looking at the exponent gives:

$$\lim_{x \to \infty} \frac{2 \ln(x+2)}{\ln x} \tag{33}$$

$$\stackrel{*}{=} \lim_{x \to \infty} \frac{\left(\frac{2}{x+2}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \to \infty} \frac{2x}{x+2} \left(= \frac{\infty}{\infty}\right) \tag{34}$$

$$\stackrel{*}{=} \lim_{x \to \infty} \frac{2}{1} = 2 \tag{35}$$

Therefore:

$$\lim_{x \to \infty} e^{\frac{2}{\ln x} \ln(x+2)} = e^2 \tag{36}$$

so:

$$\lim_{x \to \infty} (x+2)^{2/\ln x} = e^2 \tag{37}$$

Example 7: Solve $\lim_{x \to \infty} \left[\sin\left(\frac{\pi}{x} + \frac{\pi}{2}\right)\right]^x$

This is in the form of 1^∞. We rewrite it as:

$$\lim_{x \to \infty} \exp\left(x \ln\left(\sin\left(\frac{\pi}{x} + \frac{\pi}{2}\right)\right)\right) \tag{38}$$

and taking the limit of the exponent:

$$= \lim_{x \to \infty} x \ln\left(\sin\left(\frac{\pi}{x} + \frac{\pi}{2}\right)\right) \left(= \frac{0}{0}\right) \tag{39}$$

$$\stackrel{*}{=} \lim_{x \to \infty} \frac{\cos\left(\frac{\pi}{x} + \frac{\pi}{2}\right) \cdot \left(-\frac{\pi}{x^2}\right)}{\sin\left(\frac{\pi}{x} + \frac{\pi}{2}\right) \cdot \left(-\frac{1}{x^2}\right)} = \frac{0 \cdot \pi}{1} = 0 \tag{40}$$

Therefore:

$$\lim_{x \to \infty} \left[\sin\left(\frac{\pi}{x} + \frac{\pi}{2}\right)\right]^x = \lim_{x \to \infty} \exp\left(x \ln\left(\sin\left(\frac{\pi}{x} + \frac{\pi}{2}\right)\right)\right) = 1 \tag{41}$$

3 Integration

3.1 Recap of Integration

- The definite integral has the geometric interpretation as the area under the curve $f(x)$ between $x = a$ and $x = b$ and the x axis:

$$\int_a^b f(x) \, dx \tag{42}$$

but can be rigorously defined using a Riemann sum:

$$\int_a^b f(x)\,\mathrm{d}x \equiv \lim_{\|P\|} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \tag{43}$$

Often, we have a uniform partition, such that $\Delta x_i = \dfrac{b-a}{n}$ where n is the number of partitions. And if we choose to use the right hand endpoint, then:

$$f(x_i^*) = f(x_i) = f(x_i) = f\left(a + \frac{b-a}{n} i\right) \tag{44}$$

Example 8: To solve $\displaystyle\int_0^5 x^2\,\mathrm{d}x$, we can choose a uniform partition with:

$$\Delta x = \frac{5-0}{n} = \frac{5}{n} \tag{45}$$

and:

$$x_i^* = x_i = i\Delta x \implies f(x_i^*) = (i\Delta x)^2 = \left(i\frac{5}{n}\right)^2 \tag{46}$$

The area approximation is:

$$A \simeq \sum_{i=1}^{n} \Delta x_i f(x_i^*) = \sum_{i=1}^{n} \left(\frac{5}{n}\right)\left(i\frac{5}{n}\right)^2 \tag{47}$$

$$= \frac{125}{n^2}\sum_{i=1}^{n} i^2 = \frac{125}{n^3}\frac{n(n+1)(2n+1)}{6} \tag{48}$$

Taking the limit as $n \to \infty$, we get:

$$\int_0^5 x^2\,\mathrm{d}x = \lim_{n\to\infty} \frac{125}{6}\left(2 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{5^3}{3}. \tag{49}$$

Example 9: To evaluate $\displaystyle\int_1^2 x^{-2}\,\mathrm{d}x$, we can choose

$$x_i^* = \sqrt{x_{i-1}x_i} \tag{50}$$

and a uniform partition of:

$$\Delta x = \frac{2-1}{n} = \frac{1}{n} \tag{51}$$

such that:

$$x_i = 1 + i\Delta x = 1 + \frac{i}{n} = \frac{n+i}{n} \tag{52}$$

and

$$x_{i-1} = \frac{n+i-1}{n} \tag{53}$$

such that the area is:

$$\begin{aligned}
A &\simeq \sum_{i=1}^{n} \Delta x f(x_i^*) \\
&= \sum_{i=1}^{n} \frac{1}{n}\left(\frac{1}{x_i^*}\right)^2 \\
&= \sum_{i=1}^{n} \frac{1}{n}\frac{1}{x_{i-1}x_i} \\
&= \sum_{i=1}^{n} \frac{1}{n}\frac{n}{n+i-1}\cdot\frac{n}{n+i} \\
&= \sum_{i=1}^{n} n\frac{1}{n+i-1}\cdot\frac{1}{n+i} \\
&= \sum_{i=1}^{n} \left(\frac{1}{n+i-1}-\frac{1}{n+i}\right) \\
&= n\left[\sum_{i=1}^{n}\frac{1}{n+i-1}-\sum_{i=1}^{n}\frac{1}{n+i}\right] \\
&= n\left[\sum_{i=0}^{n}\frac{1}{n+i}-\sum_{i=1}^{n}\frac{1}{n+i}\right] \\
&= n\left[\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n-1}-\frac{1}{n+1}-\frac{1}{n+2}-\cdots-\frac{1}{2n}\right] \\
&= n\left(\frac{1}{n}-\frac{1}{2n}\right) \\
&= 1-\frac{1}{2}=\frac{1}{2}
\end{aligned}$$

The part where we cancel out everything is called a **telescoping series**. Notice how the area doesn't depend on n so we get the exact area, even if we let $n = 1!$.

- We need a better way to do integration, so we can define:

$$F(x) \equiv \int_a^x f(t)\,\mathrm{d}t \tag{54}$$

such that $F'(x) = f(x)$. This is the definition of the antiderivative. This leads to the fundamental theorem of calculus:

$$\int_a^b f(t)\,\mathrm{d}t = F(h) - F(a) \tag{55}$$

and the indefinite integral can be written as:

$$\int f(x)\,\mathrm{d}x = G(x) + C \tag{56}$$

The main problem now becomes trying to *find antiderivatives*, which is much easier than Riemann sums, though still more difficult than calculating derivatives.

3.2 Integration by Parts

- **Integration by Parts** attempts to reverse the product rule:

$$(fg)' = fg' + f'g \tag{57}$$

Taking the integral of both sides gives:

$$f(x)g(x) = \int f(x)g'(x)\,\mathrm{d}x + \int f'(x)g(x)\,\mathrm{d}x \tag{58}$$

$$\int f(x)g'(x)\,\mathrm{d}x = \int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x \tag{59}$$

If the second integral is easier than the first, then we have made substaintial progress.

Idea: Integration of parts tells us that:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u \tag{60}$$

Example 10: To solve $\int xe^{2x}$, we can let:

$$u = x \qquad \mathrm{d}v = e^{2x}\,\mathrm{d}x \tag{61}$$

$$\mathrm{d}u = \mathrm{d}x \qquad v = \frac{1}{2}e^{2x} \tag{62}$$

which gives:

$$\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,\mathrm{d}x \tag{63}$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \tag{64}$$

We can check:

$$\frac{d}{dx}\left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C\right) \tag{65}$$

$$= xe^{2x} + \frac{1}{2}e^{2x} - \frac{2}{4}e^{2x} \tag{66}$$

$$= xe^{2x} \tag{67}$$

Example 11: To solve $\int x^2 \sin(2x)\,\mathrm{d}x$, we let:

$$u = x^2 \qquad \mathrm{d}v = \sin 2x\,\mathrm{d}x \tag{68}$$

$$\mathrm{d}u = 2x\,\mathrm{d}x \qquad v = -\frac{1}{2}\cos(2x) \tag{69}$$

which gives:

$$= -\frac{1}{2}x^2\cos 2x + \int x\cos(2x)\,\mathrm{d}x \tag{70}$$

and we can apply integration by parts a second time, if we let:

$$u = x \qquad \mathrm{d}v = \cos 2x\,\mathrm{d}x \tag{71}$$

$$\mathrm{d}u = \mathrm{d}x \qquad v = \frac{1}{2}\sin(2x) \tag{72}$$

which gives us:

$$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) - \int \frac{1}{2}\sin(2x)\,\mathrm{d}x \tag{73}$$

$$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2X) + C \tag{74}$$

Example 12: To solve $I = \int e^x \sin x \, \mathrm{d}x$, we can let:

$$u = \sin x \qquad \mathrm{d}v = e^x \, \mathrm{d}x \tag{75}$$
$$\mathrm{d}u = \cos x \, \mathrm{d}x \qquad v = e^x \tag{76}$$

to give us:

$$= e^x \sin x - \int e^x \cos x \, \mathrm{d}x \tag{77}$$

We apply integration by parts a second time:

$$u = \cos x \qquad \mathrm{d}v = e^x \, \mathrm{d}x \tag{78}$$
$$\mathrm{d}u = -\sin x \, \mathrm{d}x \qquad v = e^x \tag{79}$$

to get:

$$I = e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x \, \mathrm{d}x}_{I} \tag{80}$$
$$2I = e^x (\sin x - \cos x) + C' \tag{81}$$
$$I = \frac{1}{2} e^x (\sin x - \cos x) + C \tag{82}$$

and we are done.

Example 13: We can also solve integrals that do not appear to have parts, such as $\int \ln x \, \mathrm{d}x$. We choose:

$$u = \ln x \qquad \mathrm{d}v = \mathrm{d}x \tag{83}$$
$$\mathrm{d}u = \frac{1}{x} \, \mathrm{d}x \qquad v = x \tag{84}$$

to give us:

$$\ln x - \int \mathrm{d}x = x \ln x - x + C \tag{85}$$

- For a definite integral, we can write IBP as:

$$f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x) \, \mathrm{d}x \tag{86}$$

Example 14: It is *possible* to apply integration of parts to find the integral of $\int \tan x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \, \mathrm{d}x$. We can let:

$$u = \frac{1}{\cos x} = \sec x \qquad \mathrm{d}v = \sin x \, \mathrm{d}x \tag{87}$$
$$\mathrm{d}u = \sec x \tan x \qquad v = -\cos x \tag{88}$$

this gives us:

$$\int \tan x \, \mathrm{d}x = -\frac{\cos x}{\cos x} + \int \tan x \, \mathrm{d}x \tag{89}$$

Notice that we could try to subtract the original integral from both sides and get:

$$0 = -1 \tag{90}$$

which is clearly wrong! However, we forgot the constant of integration, so the correct statement would be:

$$0 + C' = -1 + C \tag{91}$$

which does not tell us anything interesting. This brings We can see this concretely by repeating the same steps but trying to evaluate the definite integral $\int_a^b \tan x \, \mathrm{d}x$ instead, which gives:

$$\int_a^b \tan \mathrm{d}x = (-1)\Big|_{x=a}^{x=b} + \int_a^b \tan x \, \mathrm{d}x \implies 0 = (-1) - (-1) \implies 0 = 0 \tag{92}$$

which confirms our suspcision that this isn't anything useful, but it's also not an incorrect statement.

Warning: Sometimes it is possible to get more than one answer through various means that differ by a constant factor when solving indefinite integrals. When this happens, nothing is wrong: we simply need to consider the constant of integration.

Idea: But how do we know *which* values of u and $\mathrm{d}v$ we should pick? A common strategy is to use **LIATE**:

1. L: Logarithms
2. I: Inverse Trig
3. A: Algebraic
4. T: Trigonometric
5. E: Exponential

If a function consists of two terms, the term that is higher up (closer to L) usually gets differentiated and the term near the bottom (closer to E) usually gets integrated. See this for how it works, and this video for a tutorial.

4 Trigonometric Integrals

- The first type of integral we'll deal with is:

$$\int \sin^n x \cos^n x \, \mathrm{d}x \tag{93}$$

- In **case 1**, we have either m or n as an odd positive number. We can then use the identity $\sin^2 x + \cos^2 x = 1$ to simplify it.

Example 15: For example, to solve $\int \sin^3 x \cos^2 x \, \mathrm{d}x$, we can simplify this to:

$$= \int (1 - \cos^2 x) \cos^2 x \sin x \, \mathrm{d}X \tag{94}$$

$$= (\cos^2 x - \cos^4 x) \sin x \, \mathrm{d}x \tag{95}$$

and applying a u substitution with $u = \cos x$ and breaking it up into two integrals, we can get:

$$= -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C \tag{96}$$

- In **case 2**, we have m and n as both even. We then apply the double angle formulas:

$$\sin x \cos x = \frac{1}{2}\sin(2x) \tag{97}$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x \tag{98}$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x \tag{99}$$

Example 16: For example:

$$\int \sin^2 x \cos^4 \mathrm{d}x = \int \frac{1}{4}\sin^2(2x)\left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)\mathrm{d}x \tag{100}$$
$$= \frac{1}{8}\int \sin^2(2x)\,\mathrm{d}x + \frac{1}{8}\int \sin^2 x\cos 2x\,\mathrm{d}x \tag{101}$$
$$= \frac{1}{8}\int\left(\frac{1}{2}-\frac{1}{2}\cos 4x\right)\mathrm{d}x + \frac{1}{8\cdot 3\cdot 2}\sin^3(2x) + C \tag{102}$$
$$= \frac{1}{16}x - \frac{1}{64}\sin(4x) + \frac{1}{48}\sin^3(2x) + C \tag{103}$$

- In **Case 3**, we have:

$$\int \sin^n \mathrm{d}x\,,\ \int \cos^n \mathrm{d}x \tag{104}$$

which we can apply a reduction formula by keep applying integration by parts:

$$\int \sin^n \mathrm{d}x = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,\mathrm{d}x \tag{105}$$
$$\int \cos^n \mathrm{d}x = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,\mathrm{d}x \tag{106}$$

Example 17: To solve the integral $\int \sin^2 x\,\mathrm{d}x$, we get:

$$= -\frac{1}{2}\sin x\cos x + \frac{1}{2}\int \mathrm{d}x \tag{107}$$
$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C \tag{108}$$

- In **Case 4**, we have integrals in the following forms:

$$\int \sin(mx)\cos(nx)\,\mathrm{d}x \tag{109}$$
$$\int \sin(mx)\sin(nx)\,\mathrm{d}x \tag{110}$$
$$\int \cos(mx)\cos(nx)\,\mathrm{d}x \tag{111}$$

with $m \neq n$. If $m = n$, then we can apply the double angle formula. To solve these, we apply the following identities:

$$\sin A\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \tag{112}$$
$$\cos A\cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right] \tag{113}$$
$$\sin A\cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right] \tag{114}$$

Example 18: For example, we have:

$$\int \sin(3x)\sin(2x)\,\mathrm{d}x = \frac{1}{2}\int \cos((3-2)x)\,\mathrm{d}x - \frac{1}{2}\cos((3+2)x)\,\mathrm{d}x \tag{115}$$
$$= \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C \tag{116}$$

- In **case 5**, we have integrals in the form of either:

$$\int \tan^n x\,\mathrm{d}x\,,\ \int \cot^n x\,\mathrm{d}x \tag{117}$$

To solve these, we apply the following identities:

$$\tan^2 x = \sec^2 x - 1 \tag{118}$$
$$(\tan x)' = \sec^2 x \tag{119}$$

- In **case 6**, we have:

$$\int \sec^n x \, \mathrm{d}x \, , \int \csc^n x \, \mathrm{d}x \tag{120}$$

with $n \geq 2$. To solve these, we can make the following substitutions:

$$1 + \tan^2 x = \sec^2 x \tag{121}$$
$$1 + \cot^2 x = \csc^2 x \tag{122}$$

to convert it to a case 5 problem.

- In **case 7**, we have:

$$\int \tan^n x \sec^n x \, \mathrm{d}x \, , \int \cot^n x \csc^n x \, \mathrm{d}x \tag{123}$$

Example 19: We have:

$$\tan^3 x \sec^4 x \, \mathrm{d}x = \int \tan^3 x \sec^2 x \sec^2 x \, \mathrm{d}x \tag{124}$$
$$= \int \tan^3 x \left(\tan^2 x + 1\right) \sec^2 x \, \mathrm{d}X \tag{125}$$
$$= \int (\tan^5 x + \tan^3 x) \sec^2 x \, \mathrm{d}x \tag{126}$$
$$= \frac{1}{6} \tan^6 x + \frac{1}{4} \tan^4 x + C \tag{127}$$

Idea: The basic idea of these types is to apply trigonometric identities to turn the integrals into a form that is easier to deal with. The substitutions are usually very simple but to find them, it requires a lot of practice.

- We can also apply **trigonometric substitutions**, any integrals with any of the three factors below can be solved with this technique:

  1. $\sqrt{a^2 - x^2}$: Set $x = a \sin u \implies \sqrt{a^2 - x^2} = a \cos u$
  2. $\sqrt{a^2 + x^2}$: Set $x = a \tan u \implies \sqrt{a^2 + x^2} = a \sec u$
  3. $\sqrt{x^2 - a^2}$: Set $x = a \sec u \implies \sqrt{x^2 - a^2} = a \tan u$

  where the arguments under the square roots are always positive.

Example 20: To solve the integral $\int \frac{x^2}{(4 - x^2)^{3/2}} \, \mathrm{d}x$, we can set:

$$x = 2 \sin u \tag{128}$$
$$\mathrm{d}x = 2 \cos u \, \mathrm{d}u \tag{129}$$
$$\sqrt{4 - x^2} = 2 \cos u \tag{130}$$

which gives:

$$= \int \frac{4\sin^2 u \cdot 2\cos u \,\mathrm{d}u}{8\cos^3 u} \tag{131}$$

$$= \int \tan^2 u \,\mathrm{d}u \tag{132}$$

$$= \int (\sec^2 u - 1)\,\mathrm{d}u \tag{133}$$

$$= \tan u - u + C \qquad = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C \tag{134}$$

Example 21: The integral $\int \frac{x\,\mathrm{d}x}{(2x^2+4x-7)^{1/2}}$ needs a bit more work before we can apply the sibstutitions. We first apply the square to get:

$$= \int \frac{x\,\mathrm{d}x}{\sqrt{2(x+1)^2-9}} \tag{135}$$

We can set:

$$\sqrt{2}(x+1) = 3\sec u \tag{136}$$

$$\sqrt{2}\,\mathrm{d}x = 3\sec u \tan u \,\mathrm{d}u \tag{137}$$

$$\sqrt{2(x+1)^2-9} = 3\tan u \tag{138}$$

which gives:

$$= \int \frac{\left(\frac{3}{\sqrt{2}}\sec u - 1\right)\left(\frac{3}{\sqrt{2}\sec u \tan u \mathrm{d}u}\right)}{3\tan u} \tag{139}$$

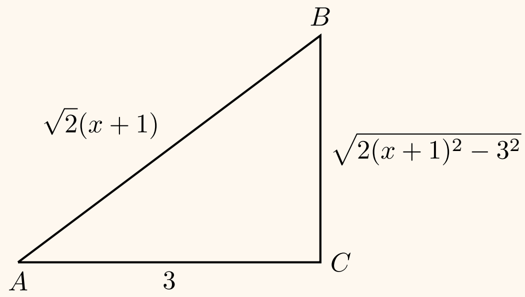
$$= \int \left(\frac{3}{\sqrt{2}}\sec u - 1\right)\left(\frac{1}{\sqrt{2}}\sec u\right)\mathrm{d}u \tag{140}$$

$$= \frac{3}{2}\int \sec^2 u\,\mathrm{d}u - \frac{1}{\sqrt{2}}\int \sec u\,\mathrm{d}u \tag{141}$$

$$= \frac{3}{2}\tan u - \frac{1}{\sqrt{2}}\ln|\sec u + \tan u| + C \tag{142}$$

$$= \frac{1}{2}\sqrt{2x^2+4x-7} - \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}}{3}(x+1) + \frac{\sqrt{2x^2+4x-7}}{3}\right| + C \tag{143}$$

Idea: We can use triangles to derive the substitution, which comes from the Pythagorean theorem:



and you can clearly see the substitution:

$$3\sec u = \sqrt{2}(x+1) \implies \cos u = \frac{3}{\sqrt{2}(x+1)} \tag{144}$$

where $u \equiv \angle BAC$.

Example 22: For the integral $\int x \sin^{-1} x \, \mathrm{d}x$, we can let:

$$u = \sin^{-1} x \, \mathrm{d}v = x \, \mathrm{d}x \tag{145}$$

$$\mathrm{d}u = \frac{\mathrm{d}x}{\sqrt{1-x^2}} v = \frac{1}{2}x^2 \tag{146}$$

and applying integration by parts, we get:

$$= \frac{1}{2}x^2 \sin^{-1} x - \int \frac{1}{2}x^2 \frac{\mathrm{d}x}{\sqrt{1-x^2}} \tag{147}$$

To solve this secondary integral $\int \frac{x^2 \, \mathrm{d}x}{\sqrt{1-x^2}}$, we can let:

$$x = \sin\theta \tag{148}$$

$$\mathrm{d}x = \cos\theta \, \mathrm{d}\theta \tag{149}$$

$$\sqrt{1-x^2} = \cos\theta \tag{150}$$

which gives:

$$= \frac{\sin^2\theta \cos\theta \, \mathrm{d}\theta}{\cos\theta} \tag{151}$$

$$= \int \sin^2\theta \, \mathrm{d}\theta \tag{152}$$

$$= \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta + C \tag{153}$$

$$= \frac{1}{2}\sin^{-1} - \frac{1}{2}x\sqrt{1-x^2} + C \tag{154}$$

Therefore, we get:

$$\int x \sin^{-1} x \, \mathrm{d}x = \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{1}{4}x\sqrt{1-x^2} + C \tag{155}$$

5 Partial Fractions

- Rational functions are in the form of:

$$R(x) = \frac{P_n(x)}{P_m(x)} \tag{156}$$

  where m, n represent the order of the polynomial. If $n \geq m$, it is an **improper** fraction, such as:

$$\frac{x^2 - x}{1+x} \tag{157}$$

  and if $n < m$, we have a proper fraction such as:

$$\frac{x}{x^2+3x+2} \tag{158}$$

- If we have an improper fraction, we use long division to simplify it. For example:

$$\frac{x^3 - 2x^2}{x^2+9} = x - 2 + \frac{18-9x}{x^2+9} \tag{159}$$

  which turns the expression into a polynomial (trivial to integrate) as well as a proper fraction.
- There are different types of factors:
  - Linear factors (e.g. $3x+2$)

   – Irreducible quadratic factors (e.g. x^2+1)

which gives us the different factors:

- **Case 1:** If we have distinct linear factors in the denominator, we can break it into fractions of the form:

$$(x+\alpha) \implies \frac{A}{x+\alpha} \tag{160}$$

Example 23: The partial fraction of $\dfrac{2x-17}{x^2+3x+2}$ can be written as the **partial fraction deconvolution**:

$$= \frac{A}{x+1} + \frac{B}{x+2} \tag{161}$$

We now need to solve for A and B. We can multiply both sides by $(x+1)(x+2)$ to get:

$$2x-17 = A(x+2) + B(x+1) \tag{162}$$

and match up the coefficients. Alternatively, we can pick various values of x (e.g. $x=-2$ and $x=-1$) to solve for the coefficients.

- **Case 2:** If we have repeated linear factors, then the decomposition is in the form of:

$$(x+\alpha)^k \implies \frac{A}{x+\alpha} + \frac{B}{(x+\alpha)^2} + \frac{C}{(x+\alpha)^3} + \cdots + \frac{K}{(x+\alpha)^k} \tag{163}$$

Example 24: To get the decomposition of $\dfrac{2}{x(x+1)^2}$, we can get:

$$\frac{2}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \tag{164}$$

which gives:

$$2 = A(x+1)^2 + Bx(x+1) + Cx \tag{165}$$

matching the coefficients, we get three equations and three unknowns:

$$x^2 : A+B = 0 \tag{166}$$

$$x : 2A+B+C = 01 \qquad : A = 2 \tag{167}$$

Solving this system gives $A=2$, $B=-2$, and $C=-2$. Note that taking the integral of this sum is much easier. We have:

$$\int \frac{d}{x(x+1)^2}\,\mathrm{d}x = \int \frac{2}{x}\,\mathrm{d}X - \int \frac{2}{x}\,\mathrm{d}x - \int \frac{2}{(x+1)^2}\,\mathrm{d}x \tag{168}$$

$$= 2\ln|x| - 2\ln|x+1| + \frac{2}{x+1} + C \tag{169}$$

Idea: As a general rule of thumb, the number of unknown coefficients is equal to the order of the polynomial in the denominator.

- **Case 3:** If we have irreducible quadratic factors, then the partial fraction deconvolution is in the form of:

$$x^2+px+8 \implies \frac{Ax+B}{x^2+px+8} \tag{170}$$

Example 25: Suppose we have $\frac{2}{(x+1)(x^2+x+1)}$, we can get the partial fraction decomposition as:

$$= \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1} \tag{171}$$

and we work through the deconvolution process in exactly the same way, we remove the denominators on both sides to get (after expanding):

$$2 = Ax^2 + Ax + A + Bx^2 + Bx + Cx + C \tag{172}$$
$$0x^2 + 0x^1 + 2x^0 = (A+B)x^2 + (A+B+C)x^1 + (A+C)x^0 \tag{173}$$

which gives three equations and three unknowns, after we match coefficients:

$$x^2 : A + B = 0 \tag{174}$$
$$x : A + B + C = 0 \tag{175}$$
$$1 : A + C = 2 \tag{176}$$

and solving the system of equations gives $A = 2, B = -2, C = 0$. To get the integral of this second term, we can write the second term as:

$$\int \frac{2x\,\mathrm{d}x}{x^2+2x+1} = \underbrace{\int \frac{2x+1}{x^2+x+1}\,\mathrm{d}x}_{(1)} - \underbrace{\int \frac{\mathrm{d}x}{x^2+x+1}}_{(2)} \tag{177}$$

We “added” 1 and “subtracted” 1 to get these two slightly easier integrals, which we can apply other techniques. The first one can be solved using a u-sub while the second can be solved by completing the square and applying a trigonometric substitution:

$$(1) = \ln|x^2+x+1| + C \tag{178}$$
$$(2) = \int \frac{\mathrm{d}x}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right] + C \tag{179}$$

allowing us to put everything together.

Example 26: Let’s take an integral we already know the answer of: $\int \frac{2x}{x^2+1}\,\mathrm{d}x = \ln\left(x^2+1\right) + C$. We can try a partial fraction decomposition:

$$\frac{2x}{x^2+1} = \frac{A}{x+i} + \frac{B}{x-i} = \frac{1}{x+i} + \frac{1}{x-i} \tag{180}$$

which gives:

$$\int \frac{2x}{x^2+1}\,\mathrm{d}x = \int \frac{\mathrm{d}x}{x+i} + \int \frac{\mathrm{d}x}{x-i} \tag{181}$$

In complex analysis, most mathematical functions we are familiar with are sitll valid, so the integral is:

$$= \ln|x+i| + \ln|x-i| + C \tag{182}$$

and simplifying it gives:

$$\ln\left(x^2+1\right) + C \tag{183}$$

Warning: While it is *possible* to use complex numbers to solve irreducible quadratic factors, it isn’t always as easy as the above example. To get the logarithm of a complex number, we can apply the identity (without proving):

$$\ln(a+ib) = \ln\sqrt{a^2+b^2} + i\arctan\left(\frac{b}{a}\right) \tag{184}$$

Example 27: Bonus content: Try evaluating the integral $\int \frac{\mathrm{d}x}{x^2+1}$ with complex analysis. Taking a partial fraction, we get:

$$\frac{1}{x^2+1} = \frac{A}{x+i} + \frac{B}{x-i} \tag{185}$$

multiplying both sides, we get:

$$1 = A(x-i) + B(x+i) \tag{186}$$
$$1 = (A+B)x + i(-A+B) \tag{187}$$

we have the systems of two equations:

$$x^1 : A+B = 0 \tag{188}$$
$$x^0 : (B-A)i = 1 \tag{189}$$

which gives $A = \frac{1}{2}i$ and $B = -\frac{1}{2}i$. This gives:

$$= \int \frac{0.5i}{x+i}\,\mathrm{d}x - \int \frac{0.5i}{x-i}\,\mathrm{d}x \tag{190}$$
$$= 0.5i\ln(x+i) - 0.5i\ln(x-i) + C \tag{191}$$
$$= 0.5i\ln\sqrt{x^2+1} + (0.5i)i\arctan\left(\frac{b}{x}\right) - (0.5i)\ln\sqrt{x^2+1} - (0.5i)i\arctan\left(-\frac{1}{x}\right) \tag{192}$$
$$= -\arctan\left(\frac{1}{x}\right) + C \tag{193}$$

Note that for $x \geq 0$:

$$-\arctan\left(\frac{1}{x}\right) + \frac{\pi}{2} = \arctan x \tag{194}$$

and for $x < 0$:

$$-\arctan\left(\frac{1}{x}\right) - \frac{\pi}{2} = \arctan x \tag{195}$$

- **Case 4:** Repeated irreducible quadratic terms, the decomposition is in the form of:

$$(x^2+\beta x+8)^k \implies \frac{A_1x+B_1}{(x^2+\beta x+8)} + \frac{A_2x+B_2}{(x^2+\beta x+8)^2} + \cdots + \frac{A_kx+B_k}{(x^2+\beta x+x)^k} \tag{196}$$

These can be extremely messy, but the process is similar to the above examples. For example, we can write:

$$\frac{Ax+B}{(x^2+\beta x+8)^2} = \frac{A}{2}\left[\frac{2x+\beta}{(x^2+\beta x+8)^2} + \frac{2B/A-\beta}{(x^2+\beta x+8)^2}\right] \tag{197}$$

Idea: The general strategy for dealing with a proper fraction integral is to break it up into two terms, one that can be easily be solved via a u-substitution and the second one does not have an x term in the numerator and can be solved using a trigonometric substitution.

- We can also introduce a strategy rationalizing substitutions by turning a function such as:

$$\int \frac{\sqrt{x}}{1+x}\,\mathrm{d}x \tag{198}$$

into a form that we are familiar with. We can let $u^2 = x \implies 2u\,\mathrm{d}u = \mathrm{d}x$ to give:

$$= \int \frac{u \cdot 2u\,\mathrm{d}u}{1+u^2} \tag{199}$$
$$= 2\int \frac{u^2}{1+u^2}\,\mathrm{d}u \tag{200}$$
$$= 2\int \left(1 - \frac{1}{1+u^2}\right)\mathrm{d}u \tag{201}$$
$$= 2u - 2\tan^{-1} u + C \tag{202}$$
$$= 2\sqrt{x} - 2\tan^{-1}\sqrt{x} + C \tag{203}$$

- Another method is to use a **Weierstrass substitution**, by making the substitution:

$$t = \tan\frac{x}{2} \tag{204}$$

which leads to the following substitutions:

$$\sin x = \frac{2t}{1+t^2} \tag{205}$$
$$\cos x = \frac{1-t^2}{1+t^2} \tag{206}$$
$$\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t \tag{207}$$

This allows us to turn any trigonometric function into a rational function.

Example 28: For example, to solve the integral $\int \frac{\mathrm{d}x}{1+\cos x}$, we make the specified substitution to turn this into:

$$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,\mathrm{d}t \tag{208}$$
$$= \int \frac{2\,\mathrm{d}t}{(1+t^2)+(1-t^2)}\,\mathrm{d}t \tag{209}$$
$$= \int \mathrm{d}t \tag{210}$$
$$= t + C \tag{211}$$
$$= \tan\left(\frac{x}{2}\right) + C \tag{212}$$

6 Improper Integrals

- Since infinity is not a number, our typical definite integral definition cannot be used for an **improper integral** like:

$$\int_0^\infty f(x)\,\mathrm{d}x \tag{213}$$

Instead, we use the following definition:

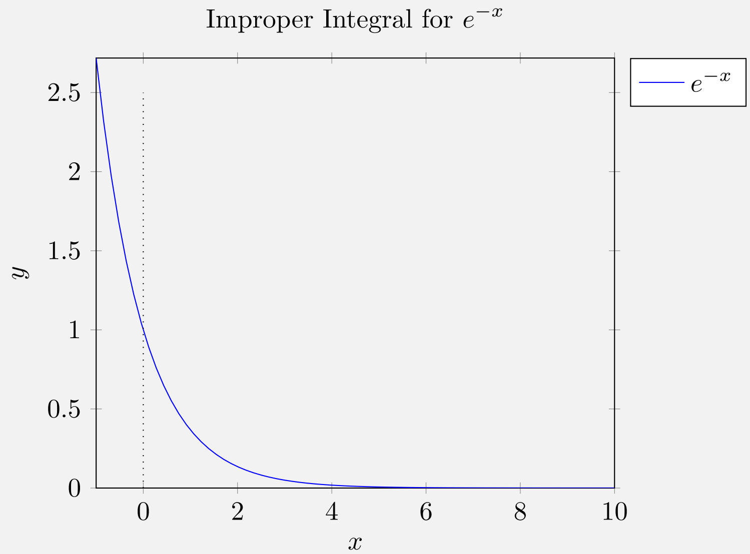
Definition: If $\lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x = L$ exists, then we can define:

$$\int_a^\infty f(x)\,\mathrm{d}x = L \tag{214}$$

Example 29: To solve $\int_0^\infty e^{-x}\,\mathrm{d}x$, we can write it as:

$$= \lim_{b\to\infty} \int_0^b e^{-x}\,\mathrm{d}x \tag{215}$$
$$= \lim_{b\to\infty} \left(1 - e^{-b}\right) = 1 \tag{216}$$

This is remarkable because even though the area appears infinite (since it is infinitely long), the area is actually finite.



Example 30: For the integral $\int_{-\infty}^{-1}$, we have:

$$= \lim_{a\to-\infty} \int_a^{-1} \frac{\mathrm{d}x}{x^2} \tag{217}$$
$$= \lim_{a\to-\infty} \left(1 + \frac{1}{a}\right) = 1 \tag{218}$$

- However, improper integrals can diverge as well.

Example 31: For $\int_3^\infty \frac{\mathrm{d}x}{x}$, we get:

$$= \lim_{b\to\infty} (\ln b - \ln 3) = \infty \tag{219}$$

Example 32: For something like $\int_{-\infty}^{2\pi} \sin x\,\mathrm{d}x$, the integral does not go to infinity, but since we get:

$$\lim_{a\to-\infty} (-1 + \cos a) \tag{220}$$

it will diverge, since $\lim_{a\to-\infty} \cos a$ does not exist.

- We can generalize this for all reciprocal functions:

Idea: For $\int_a^\infty \frac{\mathrm{d}x}{x^p}$ with $p > 0$, $p \neq 1$, and $a > 0$, we get:

$$= \lim_{b\to\infty} \int_a^b \frac{\mathrm{d}x}{x^p} \tag{221}$$

$$= \lim_{b\to\infty} \left(\frac{1}{1-p}x^{-p+1}\right)\Bigg|_a^b \tag{222}$$

$$= \lim_{b\to\infty} \left(\frac{b^{-p+1}}{1-p} - \frac{a^{-p+1}}{1-p}\right) \tag{223}$$

For $p > 1$, we get:

$$= \frac{a^{1-p}}{p-1} \tag{224}$$

and diverges for $p \leq 1$.

- There are techniques to check if an improper integral will converge or diverge. This is useful especially if we want to perform a numerical integration but want to verify that it indeed will converge.

Theorem: Let f, g be continuous functions and $0 \leq f(x) \leq g(x)$ where $x \in [a, \infty)$,.
  - If $\int_a^\infty g\,\mathrm{d}x$ converges, so does $\int_a^\infty f(x)\,\mathrm{d}x$.
  - If $\int_a^\infty f$ diverges, so does $\int_a^\infty g(x)\,\mathrm{d}x$.

Example 33: The integral $\int_2^\infty \frac{\mathrm{d}x}{\sqrt{1+x^{44/17}}}$ is difficult to evaluate, but we can easily tell that it converges via:

$$\frac{1}{\sqrt{1+x^{44/12}}} < \frac{1}{\sqrt{x^{44/12}}} = \frac{1}{x^{22/12}} \tag{225}$$

Since $p > 1$, this converges, so the original integral must also converge.

Example 34: For the integral $\int_3^\infty \frac{\mathrm{d}x}{\sqrt{7+x^2}}$, we can check that it diverges by:

$$(7+x^2)^{1/2} < \sqrt{7} + x \tag{226}$$

We can check this via: $7 + x^2 < 7 + 2\sqrt{7} + x^2$. Since:

$$\int_3^\infty \frac{\mathrm{d}x}{\sqrt{7}+x} = \ln\left(\sqrt{7}+x\right)\Big|_3^\infty \tag{227}$$

which diverges, so the original integral must also diverge.

Warning: The notation $f(x)\Big|_3^\infty$ needs to be defined explicitly since ∞ is not a number. This expression simply implies that we are taking the limit as b approaches infinity, even though it might look like we're treating ∞ as a number.

- We can look at more interesting examples. Both the lower and upper bounds can be $\pm\infty$, such as:

$$\int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi} \tag{228}$$

Definition: We can define an integral from $-\infty$ to $+\infty$ as:

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{a} f(x)\,\mathrm{d}x + \int_{a}^{\infty} f(x)\,\mathrm{d}x \tag{229}$$

Warning: Do *not* evaluate integrals of the above form as:

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \neq \lim_{b\to\infty}\int_{-b}^{b} f(x)\,\mathrm{d}x \tag{230}$$

- For example, take the integral $\int_{-\infty}^{\infty}$. If we use the proper definition, then we add two limits that don't exist, so we know this diverges. Note that it might be tempting to write:

$$= \lim_{b\to\infty}\int_{-b}^{b} x\,\mathrm{d}x = \lim_{b\to\infty}\left(\frac{b^2}{2}-\frac{b^2}{2}\right) = 0 \tag{231}$$

but this is only because we are approaching $-\infty$ and $+\infty$ at the same rate. If we instead wrote:

$$\lim_{b\to\infty}\int_{-b}^{2b} x\,\mathrm{d}x = \lim_{b\to\infty}\left(\frac{4b^2}{2}-\frac{b^2}{2}\right) = \infty \tag{232}$$

If we instead used this approach for our other improper integrals, it wouldn't make a difference since it shouldn't matter the rate at which we approach infinity. Here's another example:

$$\lim_{b\to\infty}\int_{-b}^{\sqrt{b^2+138}} x\,\mathrm{d}x = \lim_{b\to\infty}\left(\frac{b^2+138}{2}-\frac{b^2}{2}\right) = \lim_{b\to\infty}\frac{138}{2} = 69 \tag{233}$$

- Improper integrals can also be in the form where there are infinite dicontinuities at the bounds of integration. Suppose $\lim_{x\to b^-} f(x) = \infty$. We can treat an integral such as:

$$\int_{a}^{b} f(x)\,\mathrm{d}x = \lim_{c\to b^-}\int_{a}^{c} f(x)\,\mathrm{d}x \tag{234}$$

Example 35: For example, take $\int_0^1 \frac{\mathrm{d}x}{x^{1/3}}$, and we can evaluate this via:

$$\begin{aligned} &= \lim_{c\to 0^+}\int_c^1 \frac{\mathrm{d}x}{x^{1/3}} && (235)\\ &= \lim_{c\to 0^+}\frac{3}{2}\left(1-c^{2/3}\right) = \frac{3}{2} && (236)\end{aligned}$$

Again, we have a region that extends to an infinite extend, but it has a finite area. Of course, this won't always be the case.

Example 36: Take the example where $\int_0^1 \frac{\mathrm{d}x}{x^2}$, then we can evaluate this integral via:

$$\begin{aligned} &= \lim_{c\to 0^+}\int_c^1 \frac{\mathrm{d}x}{x^2} && (237)\\ &= \lim_{c\to 0^+}\left(\frac{1}{c}-1\right) = \infty && (238)\end{aligned}$$

so this integral will diverge.

Idea: Notice that we can draw an anaology between: $\int_0^a \frac{\mathrm{d}x}{x^p}$ and $\int_a^\infty \frac{\mathrm{d}x}{x^{1/p}}$, as they are reflections of one another across the line $y = x$. If one diverges, the other will converge, with the exception being $p = 1$.

- We can also deal with discontinuities that occur between the given bounds. Similar to before, we break it up into two integrals and *both* integrals must converge for the original integral to converge. For example, take:

$$\int_{-a}^{b} \frac{1}{|x^{1/2}|}\,\mathrm{d}x \tag{239}$$

with $a, b > 0$. For this integral to converge, then both $\int_{-a}^0 \frac{\mathrm{d}x}{|x^{1/2}|}$ and $\int_0^b \frac{\mathrm{d}x}{|x^{1/2}|}$ must converge.

Warning: Here is an example of when things go wrong when the integral is not broken up into separate integrals. For example, suppose we wish to evaluate $\int_{-1}^3 \frac{\mathrm{d}x}{x^2}$. From our previous discussion, we know that $\int_{-1}^0 \frac{\mathrm{d}x}{x^2}$ and $\int_0^3 \frac{\mathrm{d}x}{x^2}$ both diverges. However, one might naively think that:

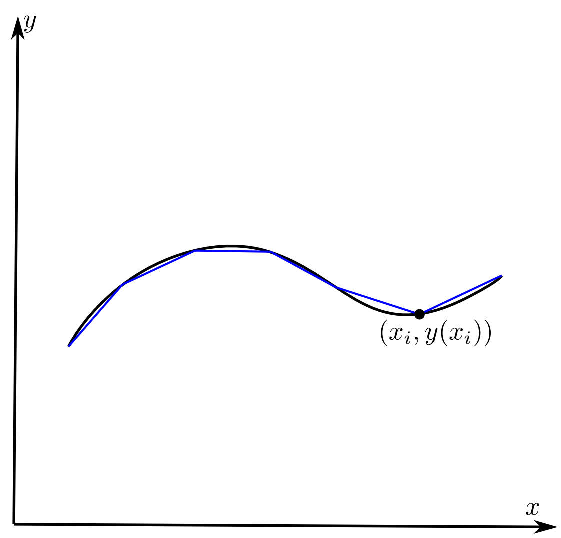
$$\left(-\frac{1}{x}\right)\Bigg|_{-1}^{3} = -\frac{1}{3} - \frac{1}{1} = -\frac{4}{3} \tag{240}$$

which is definitely wrong, since $\frac{1}{x^2}$ is never negative!

7 Applications of Integrals

7.1 Arclength

- Suppose w ehave a curve $y = f(x)$ where $x \in [a, b]$ and is differentiable. The problem is to find the length of the curve in this range.



- We can approximate this by partitioning the curve into segments at locations x_i where:

$$a = x_0 < x_1 < x_@ < \cdots < x_{n-1} < x_n = b \tag{241}$$

such that the arclength is:

$$s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{(x_i - x_{i-1})^2 + (y(x_i) - y(x_{i-1}))^2} \tag{242}$$

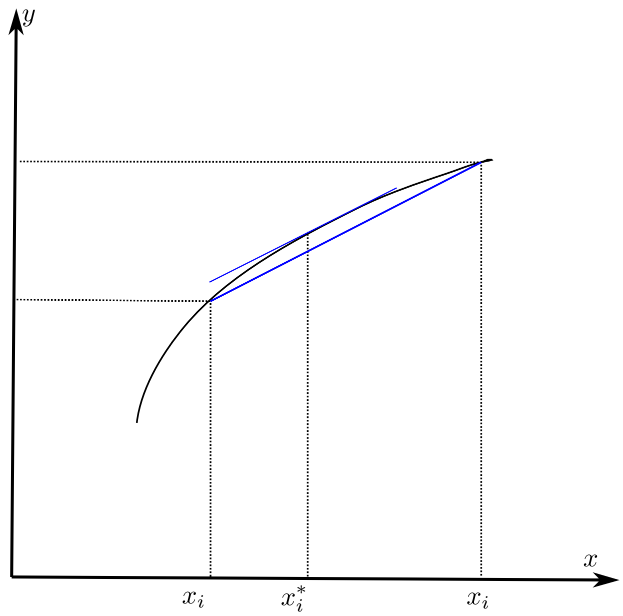
We can use the mean value theorem to write:

$$\frac{\Delta y_i}{\Delta x_i} = \frac{y(x_i) - y(x_{i-1})}{x_i - x_{i-1}} = y'(x_i^*) \tag{243}$$

so we can rewrite:

$$s_i = \sqrt{\Delta x_i^2 + (f'(x_i^*)\Delta x_i)^2} \tag{244}$$

The total length is approximated by the total sum. If we take the limit:



$$s = \lim_{\|P\|\to 0} \sum_{i=1}^{n} sqrt1 + f'(x_i^*)^2 \Delta x_i \tag{245}$$

$$= \int_a^b \sqrt{1 + f'(x)^2}\,\mathrm{d}x \tag{246}$$

Example 37: For example, the arclength in $x \in [0, 44]$ for $f(x) = x^{3/2}$ can be calculated if we know the derivative:

$$f'(x) = \frac{3}{2}x^{1/2} \tag{247}$$

so:

$$1 + f'(x)^2 = 1 + \frac{9}{4}x \tag{248}$$

which gives:

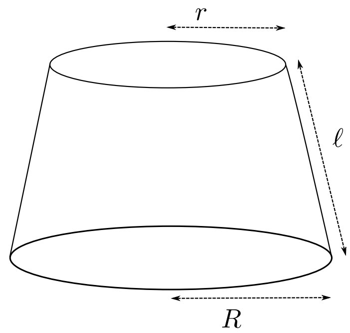
$$s = \int_0^{44} \sqrt{1 + \frac{9}{4}x}\,\mathrm{d}x \tag{249}$$

$$= \left(\frac{4}{9} \cdot \frac{2}{3}\left(1 + \frac{9}{4}x\right)^{3/2}\right)\Bigg|_0^{44} \tag{250}$$

$$= 296 \tag{251}$$

7.2 Area of a Surface of Revolution

- Consider the new problem of finding the area of a surface of revolution. Similarly, we break it up into smaller segment with width Δx.



- Each small segment is a tapered cone, with an area of:

$$A_i \simeq \pi(f(x_{i-1}) + f(x_i))s_i \tag{252}$$
$$\simeq \pi(f(x_{i-1}) + f(x_i))\sqrt{1 + f'(x_i^*)^2}\Delta x_i \tag{253}$$

From the Intermediate Value Theorem, we have:

$$f(x_{i-1}) + f(x_i) = 2f(x_i^{**}) \tag{254}$$

where $x_i^{**} \in [x_{i-1}, x_i]$ so the area can be written as:

$$A_i \simeq 2\pi f(x_i^{**})\sqrt{1 + f'(x_i^*)^2}\Delta x_i \tag{255}$$

However, we cannot turn this into an integral just yet since we have both x_i^* and x_i^{**}. But in the limit where $\Delta x_i \to 0$, we also have $x_i^{**} \to x_i^*$. We therefore get:

$$A = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\,\mathrm{d}x \tag{256}$$

Example 38: Suppose we have the function $y = \sqrt{x}$ rotated across the x axis and we want the surface area between $x \in [0, 1]$. We have $y' = \dfrac{1}{2}x^{-1/2}$ and the area becomes:

$$A = \int_0^1 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\,\mathrm{d}x \tag{257}$$
$$= \pi\int_0^1 \sqrt{4x+1}\,\mathrm{d}x \tag{258}$$

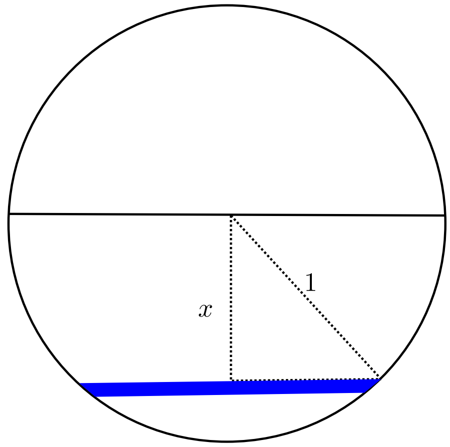
Let $u = 4x + 1$ and $\mathrm{d}u = 4\,\mathrm{d}x$, and we'll get:

$$A = \int_1^5 \pi\sqrt{u}\frac{\mathrm{d}u}{4} \tag{259}$$
$$= \frac{\pi}{4}\left(\frac{2}{3}u^{3/2}\right)\Big| 5_1 \tag{260}$$
$$= \frac{\pi}{6}(5^{3/2} - 1) \tag{261}$$

8 Applications to Physics and Engineering

- The **hydrostatic pressure** depends on the density ρ, gravitational constant g and the depth d:

$$p = \rho g d \tag{262}$$



and the force of pressure acting on the surface is:

$$F = \rho g d \cdot A = pA \tag{263}$$

Example 39: Suppose we have a curved container. The force acting on the entire container can be broken up into segments, each with a force of:

$$F_i = \underbrace{w(x_i^*)\Delta x_i}_{\text{area}} \cdot \underbrace{\rho g x_i^*}_{\text{pressure}} \tag{264}$$

where $w(x)$ is the width of the container as a function of height. The force exerted on the container is thus:

$$F = \int_a^b \rho g x w(x)\,\mathrm{d}x \tag{265}$$

Example 40: Suppose we have a pipe half with a radius of 1m filled with water and we wish to find the force it exerts on the end face of the pipe. We can do this via:

$$F = \int_0^1 \rho g x 2\sqrt{1-x^2}\,\mathrm{d}x \tag{266}$$

$$= 2\rho g \left(-\frac{1}{3}(1-x^2)^{3/2}\right)\Bigg|_0^1 \tag{267}$$

$$= \frac{2}{3}\rho g = 6533\text{N} \tag{268}$$

- We investigate the **center of mass** of a two dimensional object, which intuitively is the point at which it'll balance, also known as the **centroid**. We can use two principles to help us out:
- **Principle 1: Symmetry:** If there is an axis of symmetry, then $(\bar{x}, \bar{y})$ is on any axis of symmetry. If there are more than one axes of symmetry, then we simply need to find the intersection.
- **Principle 2: Additivity:** We can find the centroid of a collection of segments by taking the weighted average of each of the segments it is composed of. For a discrete set, the total area is:

$$A = A_1 + A_2 + \cdots + A_n \tag{269}$$

and the x location of the centroid is:

$$\bar{x} = \bar{x_1}\frac{A_1}{A} + \bar{x_2}\frac{A_2}{A} + \cdots + \bar{x}_n\frac{A_n}{A} \tag{270}$$

and similarly for the y location:

$$\bar{y} = \bar{y}_1\frac{A_1}{A} + \bar{y}_2\frac{A_2}{A} + \cdots + \bar{y}_n\frac{A_n}{A} \tag{271}$$

- Suppose we wish to find the centroid of a curve $f(x)$ from $x = a$ to $x = b$. We can approximate this region via a series of recxtangles such that:

$$A_i = f(x_i^*)\Delta x_i \tag{272}$$

$$\bar{x}_i = \frac{x_{i-1} + x_i}{2} = x_i^* \tag{273}$$

$$\bar{y}_i = \frac{1}{2}f(x_i^*) \tag{274}$$

such that:

$$\bar{x}A = \sum_{i=1}^{n}\bar{x}_i A_i = \sum_{i=1}^{n} x_i^* f(x_i^*)\Delta x_i \tag{275}$$

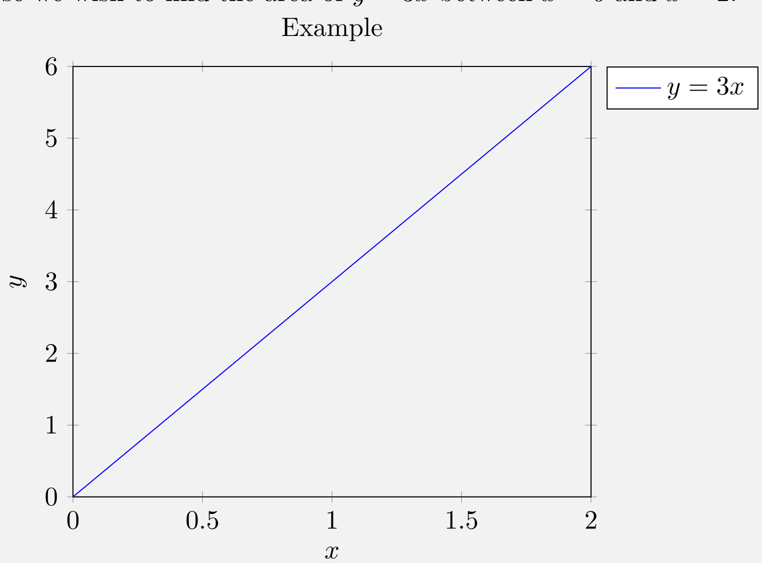
$$\bar{y}A = \frac{1}{2}\sum_{i=1}^{n} x_i^* f(x_i^*)\Delta x_i \tag{276}$$

If we take the limit, we get:

$$\bar{x} = \frac{\int_a^b x f(x)\,\mathrm{d}x}{\int_a^b f(x)\,\mathrm{d}x} \tag{277}$$

$$\bar{y} = \frac{\int_a^b f(x)^2\,\mathrm{d}x}{2\int_a^b f(x)\,\mathrm{d}x} \tag{278}$$

Example 41: Suppose we wish to find the area of $y = 3x$ between $x = 0$ and $x = 2$:



The area is $A = \int_0^2 3x\,\mathrm{d}x = 6$. And we have:

$$\bar{x}A = \int_0^2 x(3x)\,\mathrm{d}x = 8 \implies \bar{x} = \frac{4}{3} \tag{279}$$

$$\bar{y}A = \int_0^2 \frac{1}{2}(3x)^2\,\mathrm{d}x = 12 \implies \bar{y} = 2 \tag{280}$$

which is as we expected from the centroid of a triangle.

- If we want the centroid of the region of intersection between two curves f and g, we can use the additivity rule, we can have:

$$\bar{x}A = \bar{x}_g A_g - \bar{x}_f A_f \tag{281}$$
$$\bar{y}A = \bar{y}_g A_g - \bar{y}_f A_f \tag{282}$$
$$A = A_g - A_f \tag{283}$$

or in integral form:

$$\bar{x}A = \int_a^b x\left[f(x) - g(x)\right] \mathrm{d}x \tag{284}$$
$$\bar{y}A = \frac{1}{2}\int_a^b \left[f(x)^2 - g(x)^2\right] \mathrm{d}x \tag{285}$$

Example 42: Suppose we have two curves $y = 6$ and $y = 3$ and we wish to find the centroid of the area between the curves between $2 < x < 5$. This gives us:

$$\bar{x}A = \int_2^5 x(6-3)\,\mathrm{d}x = \frac{63}{2} \tag{286}$$
$$\bar{y}A = \frac{1}{2}\int_2^5 \frac{1}{2}(36-9)\,\mathrm{d}x = \frac{9}{2} \tag{287}$$

which gives us: $\bar{x} = \dfrac{7}{2}$ and $\bar{y} = \dfrac{9}{2}$.

- **Pappu's Centroid Theorem** can be used to easily find the volume of revolution. We have:

$$V = 2\pi\bar{R}a \tag{288}$$

where $\bar{R}$ is the distance from the centroid to the axis of revolution.

Example 43: Suppose we have an elliptical torus whose center is The area of the ellipse whose major axis is parallel to the axis of revolution and whose centroid is a distance R away from the axis. The volume is thus:

$$V = 2\pi R\pi ab = 2\pi^2 abR \tag{289}$$

- We can prove this using the washer method about x:

$$V_x = \int_a^b \pi(f(x)^2 - g(x)^2)\,\mathrm{d}x \tag{290}$$
$$= 2\pi\int_a^b \frac{1}{2}(f(x)^2 - g(x)^2)\,\mathrm{d}x \tag{291}$$
$$= 2\pi\bar{y}A \tag{292}$$

and using the shell method about x:

$$V_y = \int_a^b 2\pi x(f(x) - g(x))\,\mathrm{d}x \tag{293}$$
$$= 2\pi\bar{x}A \tag{294}$$

9 Parametric Equations

- Until now, we have described two dimensional curves in the form $y = f(x)$. However, we can describe this in parametric equations as well in the form:

$$x = x(t) \tag{295}$$
$$y = y(t) \tag{296}$$

- For example, numerous applications in physics and engineering arise using parametized coordinates such as projectile motion. The equations of motion are:

$$x(t) = x_0 + v_0 \cos\theta t \tag{297}$$
$$y(t) = y_0 + v_0 \sin\theta t - \frac{1}{2}gt^2 \tag{298}$$

- We can parametize a straight line using:

$$x(t) = x_c + t(x_1 - x_0) \tag{299}$$
$$y(t) = y_0 + t(y_1 - y_0) \tag{300}$$

  with $t \in (-\infty, \infty)$. We can check this by letting $t = 0$, which gives (x_0, y_0) and when $t = 1$ we get (x_1, y_1).

- For an ellipse, we can parametize using:

$$x = a\cos t \tag{301}$$
$$y = b\sin t \tag{302}$$

- Suppose we have two curves parametized by:

$$C_1 = x_1(t),\, y_1(t) \tag{303}$$
$$C_2 = x_2(t),\, y_2(t) \tag{304}$$

  An intersection occurs when $y_1(x) = y_2(x)$ and a collision occurs when $x_1(t) = x_2(t)$ and $y_1(t) = y_2(t)$.

Example 44: Suppose we have the following curves:

$$C_1 : x_1(t) = 2t + 6,\, y_1(t) = 5 - 4t \tag{305}$$
$$C_2 : x_2(t) = 3 - 5\cos\pi t,\, y_2 = 1 + 5\sin\pi t \tag{306}$$

for $t > 0$. The first curve is a straight line and the second curve is an offset circle. We can find the intersection via:

$$C_1 : t = \frac{x-6}{2} \implies y_1(x) = 16 - 2x \tag{307}$$
$$C_2 =: (x-3)^2 + (y-1)^2 = 25 \tag{308}$$

There will be two points of intersection, and after we solve this, we get $(6, 5)$ and $(8, 1)$. We now need to look at if the two objects will arise at that point at the same time. We do this by looking at each point first

$$(6,5) : C_1 \implies t = 0 \tag{309}$$
$$C_2|_{t=0} = (-2, 1) \tag{310}$$
$$(8,1) : C_1 \implies t = 1 \tag{311}$$
$$C_2|_{t=1} = (8, 1) \tag{312}$$

Therefore, there is a collision between the two objects at $(8, 1)$ at a time $t = 1$.

10 Calculus with Parametric Curves

- Parametric curves do not need to be functions. As a result, they can have an ordinary tangent, no tangent, or more than one tangent at the same point.
- We can find the secant line of a parametric curve via:

$$m_{\text{secant}} = \frac{y(t_0+h)-y(t_0)}{x(t_0+h)-x(t_0)} \tag{313}$$

  We can divide both numerator and denominator by h to get:

$$\frac{\frac{y(t_0+h)-y(t_0)}{h}}{\frac{x(t_0+h)-x(t_0)}{h}} \implies \frac{y'(t_0)}{x'(t_0)} \tag{314}$$

  where we took the limit as $h \to 0$.
- We could also have derived this using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\left(\frac{dx}{dt}\right)^{-1} \tag{315}$$

- If $x'(t_0) = 0$, then $x = x_0$ is a vertical tangent and if $y'(t_0) = 0$, then we have $y = y_0$ and have a horizontal tangent. If $x' = y' = 0$, then we can't read any information from it.

Example 45: Consider the example $x(t) = \sin(2t)$ and $y(t) = \sin t$ with $t \in [0, 2\pi]$. We have $x'(t) = 2\cos 2t$ so we can find the vertical tangents by setting $2\cos(2t) = 0$ to get:

$$t \in \{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\} \implies \left(1, \frac{1}{\sqrt{2}}\right)\left(-1, \frac{1}{\sqrt{2}}\right)\left(1, -\frac{1}{\sqrt{2}}\right)\left(-1, -\frac{1}{\sqrt{2}}\right) \tag{316}$$

For horizontal tangents, we have $y'(t) = \cos t \implies \cos t = 0$ which gives:

$$t \in \{\frac{\pi}{2}, \frac{3\pi}{2}\} \implies (0,1)(0,-1) \tag{317}$$

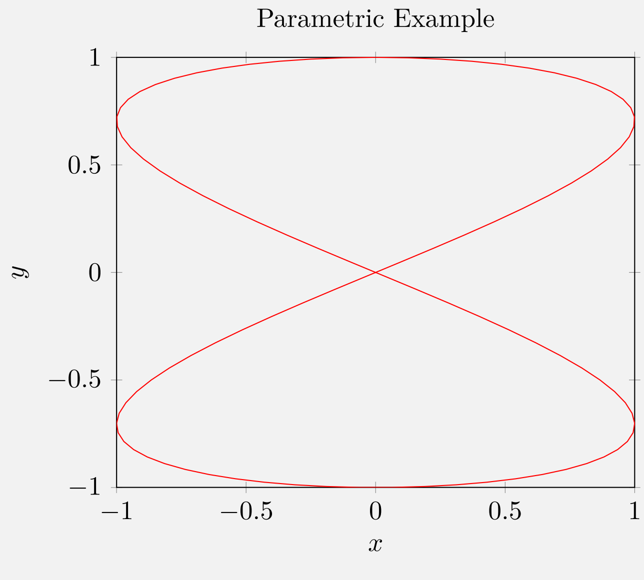
At $t = 0$, we also have $x(0) = y(0) = 0$. We can determine the tangent at this point as:

$$m_{\text{tangent}} = \frac{y'(0)}{x'(0)} = \frac{1}{2} \tag{318}$$

However notice that at $t = \pi$, we have $x(\pi) = y(\pi) = 0$ as well so this point has another tangent:

$$m_{\text{tangent}} = \frac{y'(\pi)}{x'(\pi)} = -\frac{1}{2} \tag{319}$$

At this point we can sketch out the conditions we found on the graph and draw out curve:



- We can determine the **area** of a parametric curve via:

$$A = \int_{t_1}^{t_2} y(t)x'(t)\,\mathrm{d}t \tag{320}$$

Letting $y = f(x) \implies y(t) = f(x(t))$ gives:

$$\begin{aligned} A &= \int_{t_1}^{t_2} f(x(t))x'(t)\,\mathrm{d}t && (321)\\ &= \int_{a=x(t_1)}^{b=x(t_2)} f(x)\,\mathrm{d}x && (322) \end{aligned}$$

- We can also determine the area of a closed curve:

> **Definition:** A curve is traversed in the positive sense as t increases, if the enclosed area is on the left (counterclockwise).

The area of a closed loop is then:

$$A = \int_{t_4}^{t_3} y(t)x'(t)\,\mathrm{d}t - \int_{t_4}^{t_5} y(t)x'(t)\,\mathrm{d}t + \int_{t_3}^{t_2} y(t)x'(t)\,\mathrm{d}t - \int_{t_1}^{t_2} y(t)x'(t)\,\mathrm{d}t \tag{323}$$

where t_1 represents the starting point on the bottom, t_2 is the rightmost point, t_3 is an arbitrary point on top, t_4 is the leftmost point, and t_5 is the endpoint that ends off as t_1. This gives:

$$-\int_{t_3}^{t_4} y(t)x'(t)\,\mathrm{d}t - \int_{t_4}^{t_5} y(t)x'(t)\,\mathrm{d}t - \int_{t_2}^{t_3} y(t)x'(t)\,\mathrm{d}t - \int_{t_1}^{t_2} y(t)x'(t)\,\mathrm{d}t \tag{324}$$

or:

$$A = -\int_{t_1}^{t_5} y(t)x'(t)\,\mathrm{d}t \tag{325}$$

where t_1 and t_5 are the starting and ending points respectively. They can be arbitarrily chosen so long as the points they coorespond to are on top of each other.

- Similarly, we can also write the area as:

$$A = \int_{t_1}^{t_5} x(t)y'(t)\,\mathrm{d}t \tag{326}$$

Example 46: Let us derive the area of an ellipse, parametized by:

$$\begin{aligned} x &= a\cos\theta && (327)\\ y &= b\sin\theta && (328) \end{aligned}$$

for $\theta \in [0, 2\pi]$. The area is then:

$$\begin{aligned} A &= -\int_0^{2\pi} b\sin\theta(-a\sin\theta)d\theta && (329)\\ &= ab\int_0^{2\pi} \sin^2\theta\,\mathrm{d}\theta && (330)\\ &= ab\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} && (331)\\ &= \pi ab && (332) \end{aligned}$$

- We can also determine the **arclength** of a parametric curve. We start with the summation:

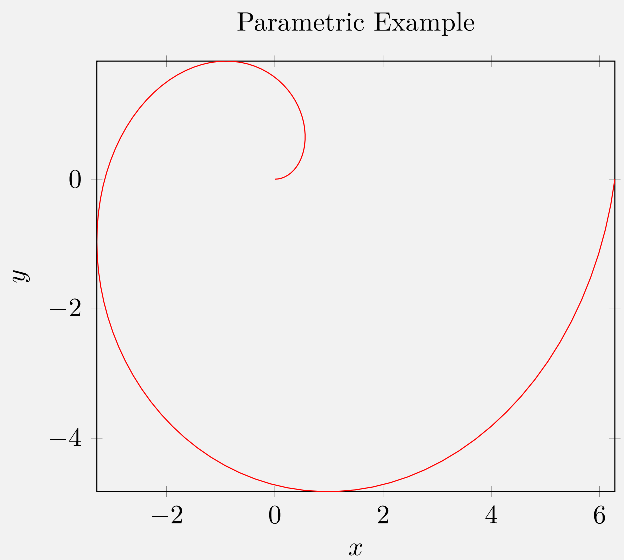
$$s \simeq \sum \sqrt{\Delta x^2 + \Delta y^2} \implies \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t \tag{333}$$

Note that if we let $x = t$ and $y = f(t) = f(x)$, then we get:

$$s = \int_\alpha^\beta \sqrt{1 + f'(x)^2}\,\mathrm{d}x \tag{334}$$

which is what we should expect.

Example 47: Suppose we have $x(\theta) = \theta\cos\theta$ and $y(\theta) = \theta\sin\theta$ for $\theta \in [0, 2\pi]$, this gives an Archimedes Spiral:



We can determine the arclength via:

$$s = \int_0^{2\pi} \sqrt{(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2}\,\mathrm{d}\theta \tag{335}$$

$$= \int_0^{2\pi} \sqrt{1 + \theta^2}\,\mathrm{d}\theta \tag{336}$$

$$= \left[\frac{1}{2}\theta\sqrt{1+\theta^2} + \frac{1}{2}\ln\left|\theta + \sqrt{1+\theta^2}\right|\right]\Big|_0^{2\pi} \tag{337}$$

$$= \pi\sqrt{1+4\pi^2} + \frac{1}{2}\ln\left(2\pi + \sqrt{1+4\pi^2}\right) \tag{338}$$

Idea: If we wish to find the speed, we have:

$$v = \frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2} \tag{339}$$

which comes from both the fundamental theorem of calculus, as well as from two-dimensional kinematics.

- The surface area can be written as:

$$A = \int_a^b 2\pi y\,\mathrm{d}s = \int_a^b 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t \tag{340}$$

- What is the *circumference of an ellipse?* If we try to carry out this calculation, we can parametize the ellipse as before:

$$x = a\sin\theta \tag{341}$$

$$y = b\cos\theta \tag{342}$$

with $0 \leq \theta \leq 2\pi$ and the arclength as:

$$s = \int_0^{2\pi} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,\mathrm{d}\theta \tag{343}$$
$$= \int_0^{2\pi} \sqrt{a^2(1-\sin 2\theta) + b^2\sin 2\theta}\,\mathrm{d}\theta \tag{344}$$
$$= \int_0^{2\pi} a\sqrt{1-\epsilon^2\sin^2\theta} \tag{345}$$

with $\epsilon \equiv \sqrt{\dfrac{a^2-b^2}{a^2}}$. Unfortunately, this is an elliptic integral of the second kind and has no analytic solution.

11 Polar Coordinates

- In polar coordinates, we can represent polar coordinates in terms of the distance r from the origin and the angle it makes with the positive horizontal axis:
$$(x,y) \iff [r,\theta] \tag{346}$$
- The magnitude of r gives the distance from the origin, and multiplying r by -1 rotates the point about the origin by π.
- Polar coordinates are not unique:
  - The pole is $[0,\theta]$ for all θ.
  - $[r,\theta] = [r,\theta+2n\pi]$ for any integer n.
  - $[r,\theta] = [-r,\theta+(2n+1)\pi]$ for any integer n.
- We can convert between cartesian and polar coordinates using the transformation:

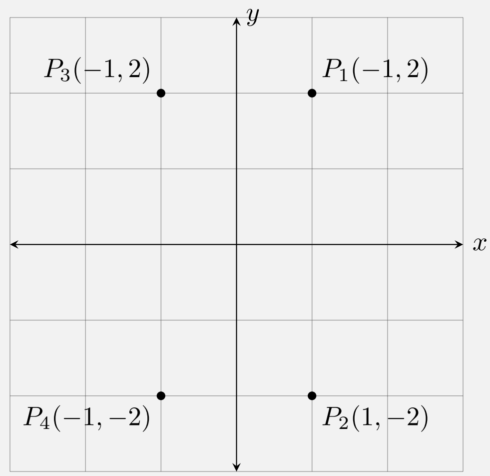
$$x = r\cos\theta \tag{347}$$
$$y = r\sin\theta \tag{348}$$

which gives:

$$r = \sqrt{x^2+y^2} \tag{349}$$
$$\theta = \arctan\left(\frac{y}{x}\right) \tag{350}$$

for $x \neq 0$.

Example 48: Suppose we have the following four points.



We can represent the four coordinates also as:

$$P_1[\sqrt{5}, 1.107] \tag{351}$$
$$P_2[\sqrt{5}, -1.107] \tag{352}$$
$$P_3[-\sqrt{5}, -1.107] \tag{353}$$
$$P_4[-\sqrt{5}, 1.107] \tag{354}$$

- We can represent straight lines as:
  - Straight lines: $y = mx$: $r = \alpha$ with $\alpha = \arctan(m)$.
  - Vertical lines $x = a$: We have $r\cos\theta = a \implies r = a\sec\theta$.
  - Horizontal lines: $y = b$. We have $r\sin\theta = b \implies r = b\csc\theta$
- We can represent circles in polar coordinates as:

$$x^2 + y^2 = 9 \iff r = 3 \tag{355}$$

- Converting *from* polar coordinates requires a bit of extra work. Suppose we have $r = 6\sin\theta$, then:

$$r^2 = 6r\sin\theta \tag{356}$$
$$x^2 + y^2 = 6y \tag{357}$$
$$x^2 + y^2 - 6y + 9 = 9 \tag{358}$$
$$x^2 + (y-3)^2 = 9 \tag{359}$$

which represents a circle with radius 3 centered at $(0, 3)$.
- Symmetry can also arise in many scenarios. For example:
  - Symmetry about x axis: $[r_1, \theta]$ and $[r_1, -\theta]$.
  - Symmetry about y axis: $[r, \pi - \theta]$ and $[r_1, \theta]$.
  - Symmetry about origin: $[r, \theta]$ and $[r, \theta + \pi]$.

which will help when sketching them.

Example 49: Suppose we wish to sketch the curve $r = \frac{1}{2} + \cos\theta$. Notice that this is periodic so we only need to look at values of θ where $0 \le \theta < 2\pi$.

1. Let us first find values of θ (if possible) that make $r = 0$:

$$0 = \frac{1}{2} + \cos\theta \implies \theta = \frac{2\pi}{3}, \frac{4\pi}{3} \tag{360}$$

2. Find local max and min values of r:

$$\frac{dr}{d\theta} = -\sin\theta = 0 \implies \theta = 0, \pi \tag{361}$$

At $\theta = 0$, we have $r = \frac{3}{2}$, at $\theta = \pi$ we have $r = -\frac{1}{2}$. If $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, then $r = \frac{1}{2}$.

3. We then loko at symmetry. Notice that:

$$\frac{1}{2} + \cos(-\theta) + \frac{1}{2} + \cos\theta \tag{362}$$

However:

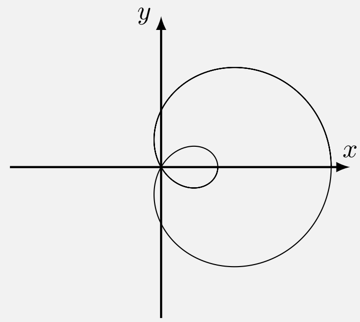
$$r(\theta - \pi) \neq r(\theta) \tag{363}$$
$$r(\pi + \theta) \neq r(\theta) \tag{364}$$

so it is not symmetryic about the y axis or origin.

4. We now look at the relevant intervals. From $0 \leq \theta < \frac{2\pi}{3}$, we have $\frac{dr}{d\theta} < 0$ so the radius is monotonically decreasing.

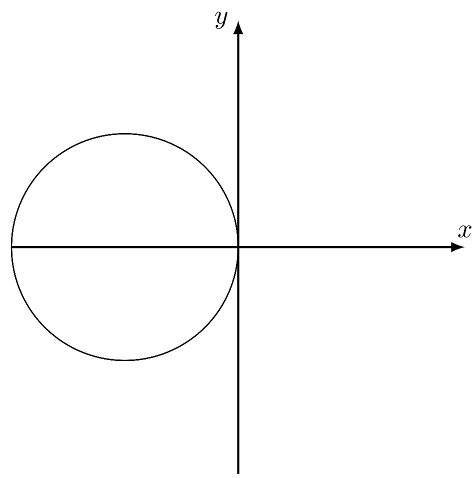
   We can also look at the interval $\frac{2\pi}{3} \leq \theta < \pi$ and see that r is negative and $\frac{dr}{d\theta} < 0$, so the manigutde of r *increases*.

   Noting that the function is differentiable at $\theta = \pi$, we can reflect the shape about the x axis to get a Limacon with inner loop.



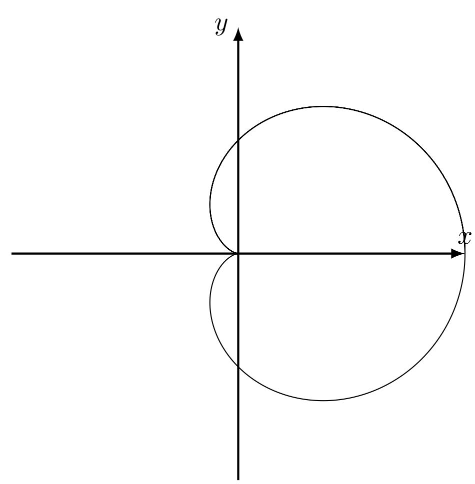
- There are a few common shapes. Each of these could be flipped or rotated by shifting the argument θ, or using negative numbers.
  - Circles:

$$r = -2\cos\theta \tag{365}$$



  - Cardioids:

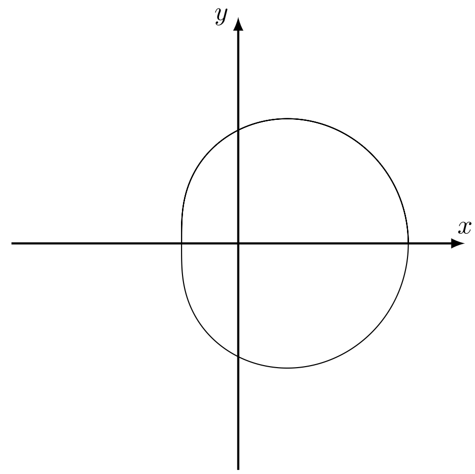
$$r = a + a\cos\theta \tag{366}$$



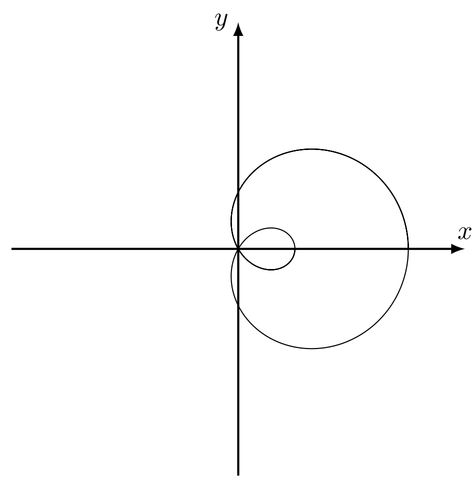
- Limacons:

$$r = a + b\sin\theta \tag{367}$$

There are two types, for $a > b$:

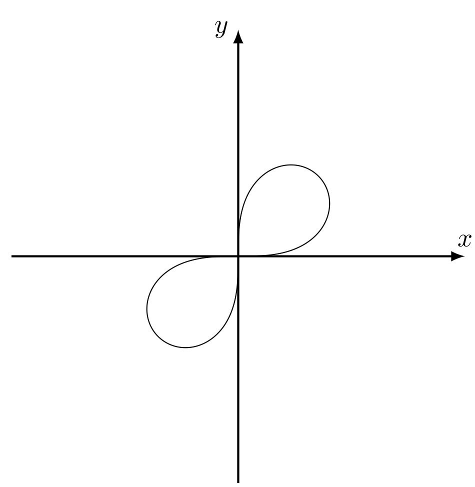


For $a < b$:



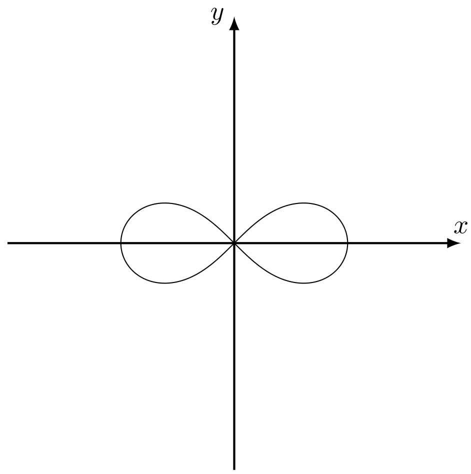
- Leminiscates. Again, there are two types. For:

$$r^2 = a\sin(2\theta) \tag{368}$$



and:

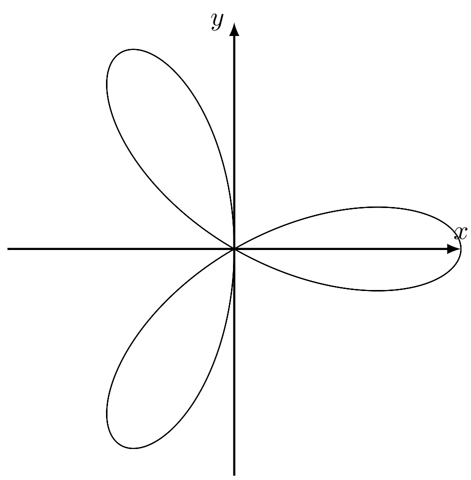
$$r^2 = a\cos(2\theta) \tag{369}$$



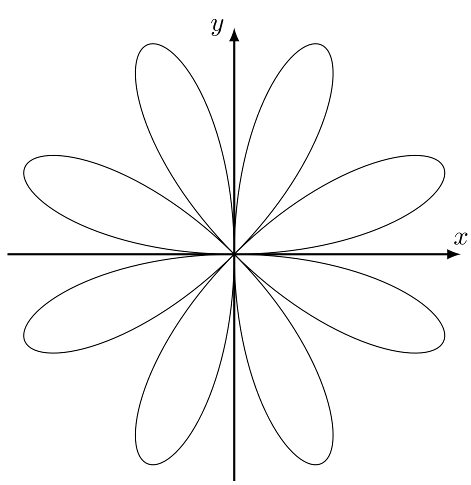
– Petal curves:

$$r = a\sin(n\theta) \tag{370}$$
$$r = a\cos(n\theta) \tag{371}$$

where n is an integer. There are n petals if n is odd and $2n$ petals if n is even. For example, the following is: $r = 2\cos 3\theta$:



and for $r = 2\sin(4\theta)$:

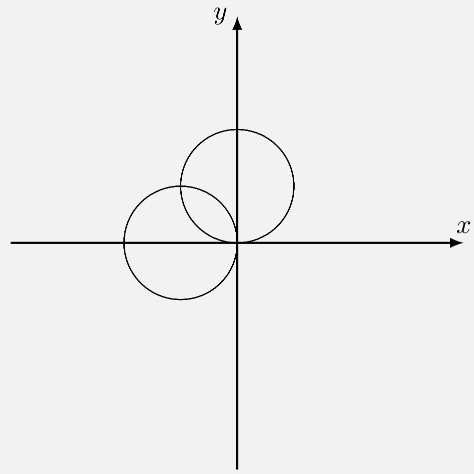


- We can also find the intersection of polar coordinates, but we have to be careful. The following example illustrates why.

Example 50: Suppose we have two curves $r = \sin\theta$ and $r = -\cos\theta$. Suppose we try to solve this via:

$$\sin\theta = -\cos\theta \implies \theta = \frac{3\pi}{4}, \frac{7\pi}{4} \tag{372}$$

Plugging this back into $x = r\cos\theta$ and $y = r\sin\theta$, we get $x = -\frac{1}{2}$ and $y = -\frac{1}{2}$. We can also use $\theta = \frac{7\pi}{4}$ to get: $x = -\frac{1}{2}$ and $y = \frac{1}{2}$ which is the same point. However, it represents the curves below:



There is actually two intersection points! The reason for this is that we assumed that the two curves intersect at the same value of θ, but this is not necessarily true for the origin, which can be obtained at any angle θ.

- As a result, we also have to check the origin.
- It is also possible to find the tangent:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \tag{373}$$

$$= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} \tag{374}$$

12 Areas and Lengths in Polar Coordinates

- Suppose we have a polar curve $r = \rho(\theta)$ for $\alpha \le \theta \le \beta$.
- We can determine the area by partitioning the curve into θ_i and approximating each subregion as a circular segment. The area of a circular segment is:

$$A = \frac{1}{2}a^2\Delta\theta \tag{375}$$

We can take the radius to be $r = \rho(\theta^*)$ where $\theta_{i-1} \le \theta^* \le \theta_i$. The area of each region is:

$$A_i = \frac{1}{2}\rho(\theta_i^*)^2\Delta\theta_i \tag{376}$$

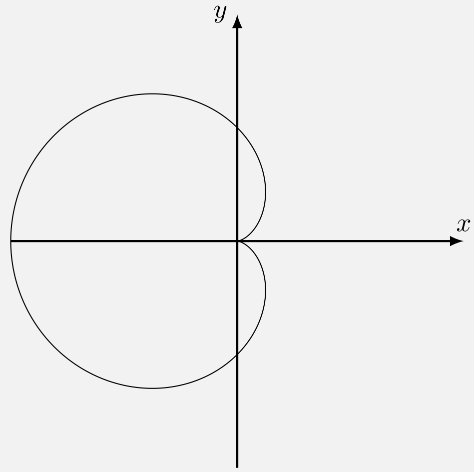
so the total area is:

$$A = \lim_{||P||}\sum_i \frac{1}{2}\rho(\theta_i^*)^2\Delta\theta_i \tag{377}$$

or

$$A = \frac{1}{2}\int_\alpha^\beta \rho(\theta)^2 \, d\theta \tag{378}$$

Example 51: Suppose we wish to find the area of $r = 1 - \cos\theta$.



The area is then:

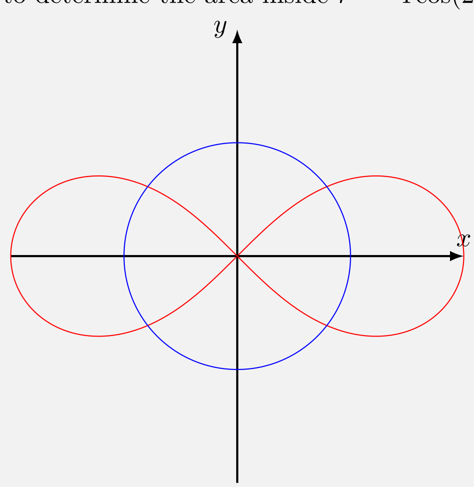
$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi}(1-\cos\theta)^2\,\mathrm{d}\theta && (379)\\
&= \frac{1}{2}\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)\,\mathrm{d}\theta && (380)\\
&= \frac{1}{2}\int_0^{2\pi}\left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos(2\theta)\right)\mathrm{d}\theta && (381)\\
&= \frac{1}{2}\left(\frac{3}{2}\theta-2\sin\theta+\frac{1}{4}\sin(2\theta)\right)\Bigg|_0^{2\pi} && (382)\\
&= \frac{3}{2}\pi && (383)
\end{aligned}
$$

- We can also find the area between two polar curves ρ_1 and ρ_2. We have:

$$A = \frac{1}{2}\int_\alpha^\beta \rho_1(\theta)^2\,\mathrm{d}\theta - \frac{1}{2}\int_\alpha^\beta \rho_2(\theta)^2\,\mathrm{d}\theta = \frac{1}{2}\int_\alpha^\beta (\rho_1^2-\rho_2^2)\,\mathrm{d}\theta \qquad (384)$$

Warning: Be careful when applying this formula as it is possible the two functions can overlap between $\alpha \le \theta \le \beta$. Therefore, we always need a good idea of what's happening.

Example 52: Suppose we want to determine the area inside $r^2 = 4\cos(2\theta)$ but outside $r = 1$. This gives:



We first find the four points of intersection:

$$4\cos(2\theta) = 1 \implies \cos(2\theta) = \frac{1}{4} \implies \theta = \pm 0.659 \tag{385}$$

or $\theta = \pi \pm 0.659$. Due to the symmetry, we only need to find the area of one half of the area we are interested in, which gives:

$$\frac{1}{2}A = \frac{1}{2}\int_{-0.659}^{0.659} (4\cos 2\theta - 1)\,\mathrm{d}\theta \tag{386}$$

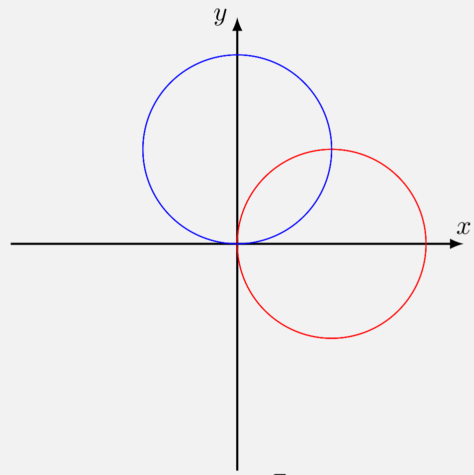
$$= \frac{1}{2}(2\sin 2\theta - \theta)\Big|_{-0.659}^{0.659} \tag{387}$$

$$= 1.277 \tag{388}$$

so the area is $A = 2.554$.

- There are a few challenging examples:

Example 53: Suppose we wish to find the area between $r = \sin\theta$ and $r = \cos\theta$:



We know from symmetry that the intersection is at $\theta = \frac{\pi}{4}$. We notice that the contribution to the area from each curve ρ is equal and *independent* from each other. Therefore:

$$A = A_1 + A_2 = \int_0^{\pi/4} \frac{1}{2}\sin^2\theta\,\mathrm{d}\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}\cos^2\theta\,\mathrm{d}\theta = \frac{\pi}{8} - \frac{1}{4} \tag{389}$$

- We can determine the arclength by working in parametric form. Let $x = r(\theta)\cos\theta$ and $y = r(\theta)\sin\theta$. Therefore:

$$s = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,\mathrm{d}\theta \tag{390}$$

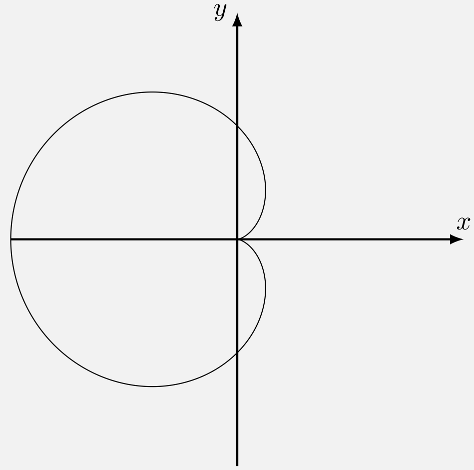
$$= \int_\alpha^\beta \sqrt{(r'\cos\theta - r\sin\theta)^2 + (r'\sin\theta + r\cos\theta)^2}\,\mathrm{d}\theta \tag{391}$$

$$= \int_\alpha^\beta \sqrt{(r'^2\cos^2\theta + r^2\sin^2\theta - \cancel{2rr'\cos\theta\sin\theta}) + (r'^2\sin^2\theta + r^2\cos^2\theta + \cancel{2r'r\cos\theta\sin\theta})}\,\mathrm{d}\theta \tag{392}$$

$$= \int_\alpha^\beta \sqrt{r^2(\cos^2\theta + \sin^2\theta) + r'^2(\cos^2\theta + \sin^2\theta)} \tag{393}$$

$$= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,\mathrm{d}\theta \tag{394}$$

Example 54: Suppose we want to find the arclength of $r = a(1-\cos\theta)$ from $0 \le \theta < 2\pi$. This looks like:



We have:

$$\begin{aligned}
s &= \int_0^{2\pi} \sqrt{r^2 + (r')^2}\,\mathrm{d}\theta && (395)\\
&= \int_0^{2\pi} \sqrt{a^2(1 - 2\cos\theta + \cos^2\theta) + a^2\sin^2\theta}\,\mathrm{d}\theta && (396)\\
&= a\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\,\mathrm{d}\theta && (397)\\
&= a\int_0^{2\pi} \sqrt{4\sin^2\left(\frac{\theta}{2}\right)}\,\mathrm{d}\theta && (398)\\
&= 2a\left[-2\cos\left(\frac{\theta}{2}\right)\right]\Bigg|_0^{2\pi} && (399)\\
&= 8a && (400)
\end{aligned}$$

13 Infinite Sequences and Series

- We use curly brackets to indicate a sequence, such as:

$$f(n) = \frac{1}{n} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\} \tag{401}$$

Alternatively, we can use a_n to represent a sequence.

Definition: A sequence $\{a_n\}$ is:
- increasing iff $a_n < a_{n+1}$
- non-decreasing iff $a_k \le a_{n+1}$
- decreasing iff $a_n > a_{n+1}$
- non-increasing iff $a_n \ge a_{n+1}$

A function that satisfies any of these are known as **monotonic** functions.

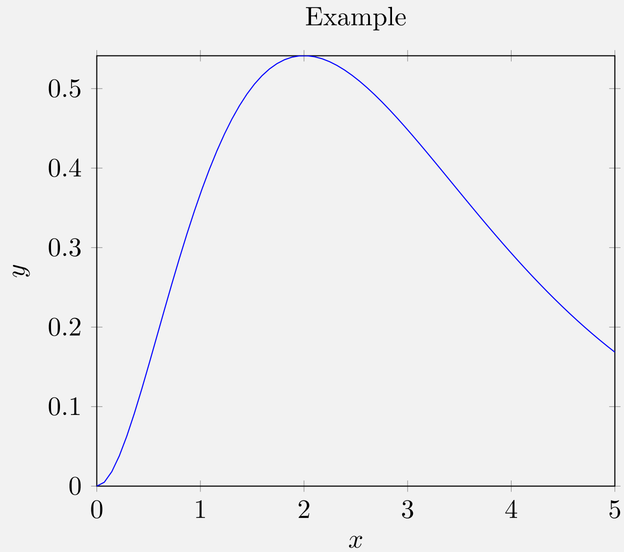
- Bounded functions have an upper or lower bound, while unbounded functions diverge to infinity or negative infinity.

Example 55: Suppose we wish to prove that 2^k is unbounded. We wish to find k such that $a_k > M$ or $2^k > M$. Taking the natural logarithm of both sides, we have:

$$k > \frac{\ln M}{\ln 2} \tag{402}$$

which is possible to do and we are done.

Example 56: Suppose we wish to find if $a_n = \frac{n^2}{e^n}$ is bounded or unbounded. This can be approached by working with derivatives through the function $f(x) = \frac{x^2}{e^x}$, represented in the following plot:



Taking the derivative $f'(x) = xe^{-x}(2-x)$, we see that f decreases for $x > 2$ so this means that a_n decreases for $n > 2$

Warning: Not everything in functions carries over to sequences. For example, $f(x) = \frac{1}{x-\sqrt{2}}$ is unbounded but $a_n = \frac{1}{n-\sqrt{2}}$ is bounded since $n \neq \sqrt{2}$ is impossible.

- We can only take the limit of a sequence as $n \to \infty$.

Definition: We can define $\lim_{n\to\infty} a_n = L$ iff for every $\epsilon > 0$, there exists an integer $k > 0$ such that if $n \geq k$, then $|a_n - L| < \epsilon$.

Example 57: Let us prove $\lim_{n\to\infty} \frac{n}{n+1} = 1$. We find k such that $\left|\frac{n}{n+1} - 1\right| < \epsilon$ for $n \geq k$. This can be rewritten as:

$$\left|\frac{1}{n+1}\right| < \epsilon \tag{403}$$

or $|n+1| > \frac{1}{\epsilon}$. Thus, if we choose $k = \frac{1}{\epsilon}$ such that if we choose $n > k = \frac{1}{\epsilon}$, then:

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{1}{n+1}\right| < \left|\frac{1}{n}\right| < \frac{1}{k} = \epsilon \tag{404}$$

Therefore, $\lim_{n\to\infty} \frac{n}{n+1} = 1$.

Theorem: Uniqueness of a Limit: If $\lim_{n\to\infty} = L$ and $\lim_{n\to\infty} a_n = M$, then $L = M$.

Definition: If a sequence has a limit, it is said to be convergent. Otherwise, it is divergent.

- This leads to the following:

1. If a sequence is convergent, it is bounded.
2. If a sequence is unbounded, it is divergent.
3. A bounded sequence is not necessarily convergent.

- For example, $a_n = \cos \pi n$ is bounded but not convergent.

> **Theorem: Monotonic Sequence Theorem**: A bounded nondecreasing sequence converges to its least upper bound. A bounded non increasing sequence converges to its greatest lower bound.

- The limit has a few properties. Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Then:

1. $\lim_{n\to\infty} (a_n + b_n) = L + M$
2. $\lim_{n\to\infty} \alpha a_n = \alpha L$ for $\alpha \in \mathbb{R}$.
3. $\lim_{n\to\infty} a_n b_n = L \cdot M$
4. $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{M}$ for $b_n \neq 0, M \neq 0$.
5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ for $b_n \neq 0, M \neq 0$.

> **Theorem: Pinching Theorem for Sequences**: If for large n, $a_n \leq b_n \leq c_n$ and if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} = L$, then $\lim_{n\to\infty} b_n = L$.

Example 58: Suppose we wish to find the limit $\lim_{n\to\infty} \frac{\sin(n\pi/6)}{n}$. We can let:

$$-\frac{1}{n} \leq \frac{\sin(n\pi/6)}{n} \leq \frac{1}{n} \tag{405}$$

Since $\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$, then the original limit must also be zero.

> **Theorem:** Supose we have a sequence: $c_n = g(f_n)$. Given $\lim_{n\to\infty} c_n = C$. If f is continuous at c, in the traditional way, then: $\lim_{n\to\infty} f(c_n) = f(c)$.

Example 59: Let us look at the function $\sin\left(\frac{1}{n^2+1}\right)$. We know that $\lim_{n\to\infty} \frac{1}{n^2+1} = 0$, so:

$$\lim_{n\to\infty} \sin\left(\frac{1}{n^2+1}\right) = \sin(0) = 0 \tag{406}$$

where we have applied the previous theorem.

14 Sequences

- We begin with some **important limits**:
  - For $x > 0$, $\lim_{n\to\infty} x^{1/n} = 1$.
  - If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

Proof. We know the function is decreasing: $|x|^{n+1} = |x||x^n| < |x|^n$. Alternatively, we want to show taht $|x^k - 0| < \epsilon$ for all $n > k$. We want to find k such that:

$$|x^k = 0| = |x^n| = |x|^k < \epsilon \tag{407}$$

or: $|x| < \epsilon^{1/n}$. We know that:

$$\lim_{n\to\infty} \epsilon^{1/k} = 1 \tag{408}$$

and since $|x| < \epsilon^{1/k}$, we must have $|x^n| < \epsilon$ for all $n > k$. □

- For $\alpha > 0$, $\lim_{n\to\infty} \frac{1}{n^\alpha} = 0$.

  Proof. Note that:

$$0 < \frac{1}{n^\alpha} = \left(\frac{1}{n}\right)^\alpha \tag{409}$$

  We can pick an odd positive integer p such that $1/p < \alpha$ such that:

$$\left(\frac{1}{n}\right)^\alpha \leq \left(\frac{1}{n}\right)^{1/p} \implies \lim_{n\to\infty}\left(\frac{1}{n}\right)^{1/p} = \left(\lim_{n\to\infty}\frac{1}{n}\right)^{1/p} = 0 \tag{410}$$

  □

- $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for $x \in \mathbb{R}$.
- $\lim_{n\to\infty} \frac{n!}{n^n} = 0$
- $\lim_{n\to\infty} \frac{\ln n}{n} = 0$.
- $\lim_{n\to\infty} n^{1/n} = 1$.
- $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

  Proof. First let's deal with the $x = 0$ case, which is trivial. Now:

$$\ln\left(1+\frac{x}{n}\right)^n = n\ln\left(1+\frac{x}{n}\right) \tag{411}$$

$$= \frac{x\ln(1+x/n)}{x/n} \tag{412}$$

$$= x\left(\frac{\ln(1+x/n) - \ln(1)}{x/n}\right) \tag{413}$$

  Taking the limit, we have:

$$\lim_{n\to\infty} \frac{\ln(1+x/h) - \ln 1}{x/h} = \lim_{h\to 0}\frac{\ln(1+h) - \ln 1}{h} \tag{414}$$

  which is the first principles definition of the derivative of $\ln(x)$ at $x = 1$, which gives:

$$\lim_{h\to\infty} \ln(1+x/n)^n = x \cdot 1 = x \implies \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \tag{415}$$

  □

- Sequences can also be defined recursively. We need a base term, e.g. $a_1 = 1$ and also a general relationship, such as:

$$a_n = \sqrt{6 + a_{n-1}} \tag{416}$$

  this gives the sequence $\{1, \sqrt{7}, \sqrt{6+\sqrt{7}}, \dots\}$

- How do we find the **limit** of such a recursively defined function? To do so, we first need to show that the limit actually exists. To do so, we must have both:

$$\lim_{n\to\infty} a_n = L \tag{417}$$

$$\lim_{n\to\infty} a_{n-1} = L \tag{418}$$

Therefore, we get:

$$L = \sqrt{6+L} \implies L = 3, -2 \tag{419}$$

Since it is increasing, we must have $L = 3$.

15 Series

- Suppose we wish to add the infinite series:

$$I = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \tag{420}$$

- We can define the partial sum to be:

$$s_0 = a_0 = \sum_{k=0}^{0} a_n \tag{421}$$

$$s_1 = a_0 + a_1 = \sum_{k=0}^{1} a_n \tag{422}$$

$$s_2 = a_0 + a_1 + a_2 = \sum_{k=0}^{2} a_n \tag{423}$$

$$\vdots \tag{424}$$

$$s_n = a_0 + a_1 + \cdots + a_n = \sum_{k=0}^{n} a_k \tag{425}$$

- We can then consider the sequence $\{s_n\} = \{a_0, a_0 + a_2, a_0 + a_1 + a_2, \cdots\}$. This sequence converges if the sum converges. Specifically, if $\lim_{n\to\infty}\{s_n\} = L$, then $\sum_{k=0}^{\infty} a_n = L$.

Example 60: Suppose we wish to evaluate:

$$\sum_{k=0}^{\infty} \frac{1}{(k+2)(k+3)} = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} \tag{426}$$

We can use partial fractions to write:

$$\frac{1}{(k+2)(k+3)} = \frac{1}{k+2} - \frac{1}{k+3} \tag{427}$$

so the sum becomes:

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} + \cdots - \frac{1}{n+3} = \frac{1}{2} - \frac{1}{n+3} \tag{428}$$

which is known as a telescoping sequence. Taking the limit as $n \to \infty$, we get that the sum converges to $\frac{1}{2}$.

- The sum of a geometric series is:

$$x^0 + x^1 + x^2 + x^3 + \cdots = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \tag{429}$$

which converges when $|x| < 1$.

Proof. Let $S_n = 1 + x + x^2 + \cdots + x^n$ and $xS_n = x + x^2 + x^3 + \cdots + x^{n+1}$. Then subtracting the two, we get:

$$S_n - xS_n = 1 - x^{n+1} \implies S_n = \frac{1 - x^{n+1}}{1 - x} \tag{430}$$

and for $|x| < 1$, the limit gives us $\dfrac{1}{1-x}$ and if $|x| > 1$, the limit diverges. □

- Suppose we wish to write the repeating fraction as a decimal: $0.\overline{285714}$. This is equal to:

$$= \frac{28574}{10^6} + \frac{285714}{10^12} + \cdots \tag{431}$$
$$= \frac{28574}{10^6}\left(1 + \frac{1}{10^6} + \frac{1}{10^{12}} + \cdots\right) \tag{432}$$

Evaluating this infinite series, we get:

$$\frac{2}{7} \tag{433}$$

Example 61: Suppose we wish to write out $\dfrac{x}{4-x^2}$ as a sum for $|x| < 2$. We have:

$$\frac{x}{4-x^2} = \frac{x}{4}\left(\frac{1}{1-x^2/4}\right) \tag{434}$$
$$= \frac{x}{4}\sum_{k=0}^{\infty}\left(\frac{x^2}{4}\right)^k \tag{435}$$
$$= \frac{x}{4}\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^{2k} \tag{436}$$
$$= \frac{1}{2}\left[\frac{x}{2} + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^5 + \cdots\right] \tag{437}$$

Theorem: Here are a few important properties that arise when applying limit laws:

- If $\sum_{k=0}^{\infty} a_k = n$ and $\sum_{k=0}^{\infty} b_k = M$, then $\sum_{k=0}^{\infty}(a_k + b_k) = L + M$.
- If $\sum_{k=0}^{\infty} a_k = L$, then $\sum_{k=0}^{\infty} \alpha a_k = \alpha L$ for $\alpha \in \mathbb{R}$.

Theorem: If $\sum_{k=0}^{\infty} a_k$ converges iff $\sum_{k=j}^{\infty} a_k$ converges where j is a positive integer.

Example 62: Suppose we are given that $\sum_{k=4}^{\infty} \frac{3^{k-1}}{3^{3k+1}}$ converges, then $\sum_{k=0}^{\infty} \frac{3^{k-1}}{3^{3k+1}}$ converges.

Theorem: If $\sum_{k=0}^{\infty} a_k$ converges, then $a_k \to 0$ as $k \to \infty 0$.

Theorem: (Test for Divergence:) This is the contrapositive of the previous theorem. If $a_k \not\to 0$ as $k \to \infty$, then $\sum_{k=0}^{\infty} a_k$ diverges.

16 Convergence Tests

- We start with the integral test:

Theorem: If f is continuous, decreasing, and positive on $[1,\infty)$, then: $\sum_{k=1}^{\infty} f(k)$ converges if and only if $\int_1^{\infty} f(x)\,\mathrm{d}x$ converges.

Example 63: Suppose we take the harmonic sum:

$$\sum_{k=1}^{\infty}\frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4} \tag{438}$$

However, the integral $\int_1^{\infty}\frac{\mathrm{d}x}{x} = \lim_{b\to\infty}\ln b$ diverges.

- The p-series is:

$$\sum_{k=1}^{\infty}\frac{1}{k^p} \tag{439}$$

which will converge if $p>1$ since $\int_1^{\infty}\frac{\mathrm{d}x}{x^p}$ converges iff $p>1$.

Example 64: Suppose we wish to look at $\sum_{n=5}^{\infty}\frac{1}{n^2+9}$. First, we notice that:

$$\lim_{t\to\infty}\int_5^t \frac{\mathrm{d}x}{x^2+9} = \lim_{t\to\infty}\left[\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]\Bigg|_5^t \tag{440}$$

which converges, and after checking the relevant conditions, this means that the sum converges too.

Definition: The **remainder** for a sequence $\{f_n\}$ is given as:

$$R_n = f(n) - f_n \tag{441}$$

where $f(n)$ denotes a continuous functoin while f_n is discrete.

- For a decreasing function, $R_n \le \int_n^{\infty} f(x)\,\mathrm{d}x$ and $R_n \ge \int_{n+1}^{\infty} f(x)\,\mathrm{d}x$. This means that:

$$S_n + \int_{n+1}^{\infty} f(x)\,\mathrm{d}x \le S \le S_n + \int_n^{\infty} f(x)\,\mathrm{d}x \tag{442}$$

- We can also use the **comparison test**

Theorem: Given $\sum a_k$ and $\sum b_k$ with $a_k>0$ and $b_k>0$:

1. If $\sum b_k$ is convergent, and if $a_k \le b_k$ for all sufficiently large k, then $\sum a_k$ converges.
2. If $\sum b_k$ is diverge and $a_k > b_k$ for all k sufficiently large, then $\sum a_k$ diverges.

Proof. Assume $a_k \le b_k$ for all k we can define:

$$S_n = \sum_{k=1}^{n} a_k \tag{443}$$

as the sequence of partial sums where:

$$b_k = \sum_{k=1}^{n} b_k \tag{444}$$

where $t = \sum_{k=1}^{\infty} b_k$ exists. This implies that $\{S_n\}$ is increasing since $a_k > 0$ and so:

$$S_n \leq t_n < t \tag{445}$$

where $\{S_n\}$ is a bounded sequence. By the monotonic sequence theorem, $\{S_n\}$ has a limit and $\sum_{k=1}^{\infty} a_k$ is defined to be equal to that limit. Therefore, $\sum a_k$ converges. ☐

Example 65: Suppose we wish to determine if $\sum_{n=1}^{\infty} \frac{7}{17n^2+3\sqrt{n}+5}$ converges. Notice that for $n \geq 1$, we have:

$$17n^2 + 3\sqrt{n} + 5 > 17n^2 \tag{446}$$

and so:

$$\frac{7}{17n^2+3\sqrt{n}+5} < \frac{7}{17n^2} \tag{447}$$

Since $\frac{7}{17}\sum \frac{1}{n^2}$ converges, then the original sum must also converge.

Example 66: Suppose we wish to determine if $\sum_{k=1}^{\infty} \frac{\ln(n/1000)}{n}$ converges. We want to find a k such that:

$$\frac{\ln(k/1000)}{k} > \frac{1}{k} \tag{448}$$

wihch means that we want to pick $k > 1000e > 2718$. Therefore, since $\sum_{k=2719}^{\infty} \frac{1}{k}$ is divergent, then the original sum is also divergent.

- Suppose we wish to determine if $\sum n = 2^{\infty}$ converges. This looks like $\frac{1}{n^3}$, but we notice that:

$$\frac{1}{n^3 - n} > \frac{1}{n^3} \tag{449}$$

and we run into trouble. This means we have to turn to the **limit comparison test**

Theorem: The limit comparison test: Given $\sum a_k$, $\sum b_k$ where $a_k > 0$ and $b_k > 0$:

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then both series converge or diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and if $\sum b_n$ converges, then $\sum_{a_n}$ converges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and if $\sum b_n$ diverges, then $\sum a_n$ diverges.

Proof. We are given that:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c \implies \left|\frac{a_n}{b_n} - c\right| < \epsilon \tag{450}$$

for $n < N$. We are working *backwards* here, so we are free to choose *any* value of ϵ and this will hold true. We can choose $\epsilon = \frac{c}{2}$ such that:

$$\frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2} \tag{451}$$

which gives:

$$\frac{c}{2}b_n < a_n < \frac{3c}{2}b_n \tag{452}$$

with $n > N$. If $\sum b_n$ converges, so does $\frac{3c}{2}\sum b_n$ since c is just a number. Thereforfe, $\sum a_n$ converges by the comparison test. If $\sum b_n$ diverges, so does $\frac{c}{2}\sum b_n$ and again by the comparison test, $\sum a_n$ diverges. $\square$

Example 67: We continue our previous discussion of $\sum_{n=2}^{\infty}\frac{1}{n^3-n}$. We let $a_n = \frac{1}{n^3-n}$ and $b_n = \frac{1}{n^3}$. Both a_n and b_n are convergent and:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{1-\frac{1}{n^2}} = 1 > 0 \tag{453}$$

so the original sequence is convergent.

Example 68: We can also revisit $\sum \frac{\ln(n/1000)}{n}$. We consider $a_n = \frac{\ln(n/1000)}{n}$ and $b_n = \frac{1}{n}$. Since $\sum b_n$ diverges and $\lim_{n\to\infty}\frac{a_n}{b_n}$ diverges too, then $\sum a_n$ diverges as well.

17 Alternating Series

- Some series have both positive and negative terms, such as:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \tag{454}$$

Theorem: The **Alternating Series Test**: Let $\{a_k\}$ be a sequence of positive numbers. If and only if $a_{k+1} < a_k$ and $a_k \to 0$ as $k \to \infty$, then:

$$\sum_{k=1}^{\infty}(-1)^{k-1}a_k \tag{455}$$

converges.

Proof. Let $S_2 = a_1 - a_2 > 0$ and $s_4 = s_2 + (a_3 - a_4)$. We can generalize this to:

$$S_{2n} = S_{2n-2} + (a_{2n-1} - a_{2n}) > S_{2n-2} \tag{456}$$

such that $\{S_{2n}\}$ is monotonically increasing. However, we also have:

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \tag{457}$$

Since $S_{2n} < a_1$ for all n, we can apply the monotonic limit theorem to show that the limit L exists. We then have:

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1} = L \tag{458}$$

$\square$

Example 69: Take the sum $1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{9} + \frac{1}{3} - \cdots$. Although $a_n \to 0$, the terms are not decreasing in mangitude, so it is divergent.

- For an alternating sequence, the limit will be between S_n and S_{n+1} so we can estimate the error as:

$$|L - S_n| \le a_{n+1} \tag{459}$$

- For example, the series expansion for e^{-1} is:

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots \tag{460}$$

If we continue to the $\frac{1}{5!}$ term, then we get:

$$e^{-1} \simeq 0.3666 \pm \frac{1}{6!} \tag{461}$$

- We introduce the absolute convergence and the ratio and root tests.

Definition: If $\sum |a_k|$ converges, we say that $\sum a_k$ is absolutely convergent. If $\sum a_k$ converges, but $\sum |a_k|$ does not, we say $\sum a_k$ is conditionally convergent.

Theorem: If $\sum |a_k|$ converges, then $\sum a_k$ converges.

Proof. Let:

$$-|a_n| \leq a_n \leq |a_n| \tag{462}$$
$$0 \leq a_n + |a_n| \leq 2|a_n| \tag{463}$$
$$0 \leq b_n \leq 2|a_n| \tag{464}$$

Note: Let $\sum a_n = \sum b_n - \sum |a_n|$. Since both $\sum b_n$ and $\sum |a_n|$ is convergent, then the original sum must be convergent as well. □

- For example, $\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}$ is conditionally convergent.

Theorem: The **Root Test:** Given $\sum a_k$, $a_k \geq 0$. If $(a_k)^{1/k} \to p$ as $k \to \infty$, then:

1. If $p < 1$, then $\sum a_k$ converges.
2. If $p > 1$, then $\sum a_k$ diverges.
3. If $p = 1$ the test is inconclusive.

Proof. Given $p < 1$, choose μ such that $p < \mu < 1$. Since $(a_k)^{1/k} \to p$, we have:

$$(a_k)^{1/k} < \mu \tag{465}$$

or

$$a_k < \mu^k \tag{466}$$

for k sufficiently large. But $\sum \mu^k$ converges (geometric series, $x < 1$), so $\sum a_k$ converges as well. □

Example 70: Take the series $\sum \left(\frac{n^2+1}{2n^1+1}\right)^n$. Note that $a_n^{1/k} = \frac{2k}{k+1} \to \frac{1}{2}$ so the series is convergent.

Theorem: The **ratio test:** Given $\sum a_k$, with $a_k > 0$. If $\frac{a_{k+1}}{a_k} \to \lambda$ as $k \to \infty$, then:

1. If $\lambda < 1$, $\sum a_k$ converges.
2. If $\lambda > 1$, $\sum a_k$ diverges.
3. If $\lambda = 1$, the test is inconclusive.

Proof. Given $\lambda < 1$, we can choose μ such that $\lambda < \mu < 1$. Thus:

$$\frac{a_{k+1}}{a_k} < \mu \tag{467}$$

for k sufficiently large, say $k > K$. We have:

$$a_{K+1} < \mu a_K \tag{468}$$
$$a_{K+2} < \mu a_{K+1} < \mu^2 a_K \tag{469}$$
$$\vdots \tag{470}$$
$$a_{K+j} < \mu^j a_K \tag{471}$$

for $j = 1, 2, 3, \ldots$. Let $n = K + j$. Then we can rewrite the last line as:

$$a_n < \mu^{n-k} a_K = \frac{a_K}{\mu^K}\mu^n \tag{472}$$

Since the factor $\frac{a_K}{\mu^K}$ is some constant and μ^n converges, then the original sum is convergent. $\square$

Tip: The ratio test is usually the most straightforward and the most useful test to employ.

Example 71: Suppose we take the sum $\sum \frac{k^2}{e^k}$. We have:

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2} = \frac{(k+1)^2}{k^2} \cdot \frac{1}{e} \tag{473}$$

As $k \to \infty$, we get $\frac{1}{e} < 1$ so the sum is convergent.

18 Power Series

- We can introduce the power series:

Definition: A power series is a series in the form:

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \tag{474}$$

- For example, if we let $c_n = 1$. Then for all n, we get:

$$\sum_{n=0}^{\infty} x_n = 1 + x + x^2 + \cdots = \frac{1}{1-x} \tag{475}$$

and converges if $|x| < 1$.

- A power series about a can be written as:

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \tag{476}$$

- Note that for $x = a$, the sum will always converge. However, we are interested for the entire range of values at which it converges..

Example 72: Suppose we have the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$. To test when it converges, we can apply the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n}\right| \tag{477}$$

$$= |x| \frac{n^2}{n+1}^{2} \tag{478}$$

As $n \to \infty$, we get $|x|$. Therefore, the series converges when $|x| < 1$. However, the test says nothing about the endpoints, so we have to test them separately. If $x = 1$, we have:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{479}$$

We can apply a p-series test to show it converges. For $x = -1$, we have:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \tag{480}$$

and apply the alternating series test to show that it converges. Therefore, the power series converges for:

$$-1 \leq x \leq 1 \tag{481}$$

Example 73: Suppose we have the power series $\sum_{n=0}^{\infty} \frac{(1+5^n)x^n}{n!}$. Using the ratio test, we have:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(1+5^{n+1})x^{n+1}}{(n+1)!} \cdot \frac{n!}{(1+5^n)x^n}\right| = \frac{1+5^{n+1}}{1+5^n} \cdot \left|\frac{x}{n+1}\right| \tag{482}$$

which approaches 0 as $n \to \infty$ so it is convergent for all $x \in \mathbb{R}$.

Example 74: Take the power series $\sum n!x^n$. The ratio test then gives:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{n!} \cdot \frac{x^{n+1}}{x^n}\right| = (n+1)|x| \tag{483}$$

This approaches ∞ as $n \to \infty$ so it diverges except for $x = 0$.

Theorem: For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are three possibilities with respect to convergence:

1. The series converges only when $x = a$
2. The series converges for all x
3. The series converges in some interval $|x-a| < R$ where R is the **radius of convergence**. However, the endpoints must be tested serparately.

Example 75: Take the power series $\sum_{n=0} \sum^{\infty} \frac{(-2)^n(x-1)^n}{n+2}$. The ratio test gives us:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}(x-1)^{n+1}}{n+3} \cdot \frac{n+2}{2^n(x-1)^n}\right| = 2\left(\frac{n+2}{n+3}\right)|x-1| \tag{484}$$

As $n \to \infty$, we get:

$$|x-1| < \frac{1}{2} \therefore R = \frac{1}{2} \tag{485}$$

We now need to check the endpoints. Test $x = \frac{1}{2}$. We get:

$$\sum_{n=0}^{\infty} \frac{(-2)^n \left(-\frac{1}{2}\right)^n}{n+2} = \sum_{n=0}^{\infty} \frac{1}{n+2} = \sum_{i=2}^{\infty} \frac{1}{i} \tag{486}$$

which diverges as it is the harmonic series. We now need to test $x = \frac{3}{2}$. We then get:

$$\sum_{n=0}^{\infty} \frac{(-2)^n \left(\frac{1}{2}\right)^n}{n+2} = \sum \frac{(-1)^n}{n+2} \tag{487}$$

Using the alternating series test, we see that this converges. Therefore, the interval of convergence is $\left(\frac{1}{2}, \frac{3}{2}\right]$.

- It is possible to represent functions as a power series. We saw that for $|x| < 1$, the infinite series:

$$\sum_{n=0}^{\infty} = 1 + x + x^2 + \cdots = \frac{1}{1-x} \tag{488}$$

If we let $f(x) = \frac{1}{1-x}$, then we can *approximate* it using a truncated power series representation for between $-1 < x < 1$.

Example 76: Suppose we have the function $\frac{x}{x-3}$. If we want to write it as a power series, we can write it as:

$$\begin{aligned} x \cdot \frac{1}{x-3} &= -x\frac{1}{3-x} && (489)\\ &= -\frac{x}{3}\frac{1}{1-\frac{x}{3}} && (490)\\ &= -\frac{x}{3}\left[1 + \frac{x}{3} + \left(\frac{x}{3}\right)^2 + \cdots\right] && (491)\\ &= -\frac{x}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n && (492)\\ &= -\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^{n+1} && (493)\\ &= -\sum_{n=1}^{\infty}\left(\frac{x}{3}\right)^n && (494) \end{aligned}$$

and it converges for $|x| < 3$.

Theorem: Term by Term Differentiation and Integration: Consider the pwoer series $\sum c_n(x-a)^n$ with $R = R_0 > 0$, then

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n \tag{495}$$

is differentiable and continuous on $(a - R_0, a + R_0)$ and:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \tag{496}$$

We can also take the integral:

$$\int f(x)\,\mathrm{d}x = C + c_0(x-a) + \frac{c_1(x-a)^2}{2} + \frac{c_2(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} \frac{c_n(x-a)^{n+1}}{n+1} \tag{497}$$

Notice that derivatives and infinite sums can be interchanged. Specifically:

$$\frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} \frac{d}{dx} c_n(x-a)^n \tag{498}$$

$$\int \sum_{n=0}^{\infty} c_n(x-a)^n\,\mathrm{d}x = \sum_{n=0}^{\infty} \int c_n(x-a)^n\,\mathrm{d}x \tag{499}$$

Warning: The radius of convergence between derivatives will always be the same, but the endpoints may change.

Example 77: Suppose we have the function $f(x) = \frac{1}{(1+x)^2}$. Note that:

$$\frac{d}{dx}\frac{-1}{1+x} = -\frac{1}{(1+x)^2} \tag{500}$$

so we can write it in terms of its derivative:

$$\frac{d}{dx} - \frac{1}{1+x} = \frac{d}{dx}\left[-\sum_{n=0}^{\infty}(-x)^n\right] = \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1} = \sum_{n=0}^{\infty}(-1)^n(n+1)x^n \tag{501}$$

Example 78: Let's find the power series representation of $\ln(1-x)$. We notice that it can be written as an integral:

$$\ln(1-x) = -\int \frac{\mathrm{d}x}{1-x} = -\int \sum_{n=0}^{\infty} x^n\,\mathrm{d}x = C - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \tag{502}$$

We can determine the constant of integration by setting $x = 0$, which gives $\ln(1) = 0 = C$. Therefore, we can write:

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \tag{503}$$

For $x = 1$, this diverges and for $x = -1$, it conditionally converges.

Example 79: Let us attempt to evaluate $\int_0^{0.1} \frac{\mathrm{d}x}{1+x^4}$ to 6 decimal places without a calculator. We first write it as a power series:

$$\frac{1}{1-(-x)^4} = \sum_{n=0}^{\infty}(-x^4)^n = 1 - x^4 - x^8 - \cdots \tag{504}$$

which converges for $|x| < 1$. Therefore, the integral is:

$$\int \frac{\mathrm{d}x}{1+x^4} = \sum_{n=0}^{\infty} \int (-x^4)^n\,\mathrm{d}x \tag{505}$$

$$= C + \sum_{n=0}^{\infty}(-1)^n \frac{x}{4n+1} \tag{506}$$

$$= C + x - \frac{x^5}{5} + \frac{x^9}{9} - \cdots \tag{507}$$

The integral is then:

$$\int_0^{0.1} \frac{\mathrm{d}x}{1+x^4} = 0.1 - \frac{0.1^5}{5} + \frac{0.1^9}{9} - \cdots = 0.099998 \pm 1.1 \times 10^{-10} \tag{508}$$

Example 80: Let us try to write the power series representaiton of the inverse tangent function $f(x) = \tan^{-1}(x)$. Note that:

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \tag{509}$$

We can write $f(x)$ as the integral:

$$\tan^{-1}(x) = \int \frac{\mathrm{d}x}{1+x^2} = \int (1 - x^2 + x^4 - x^6 + \cdots)\,\mathrm{d}x = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \tag{510}$$

We can calculate the constant of integration to be $C = \tan^{-1}(0) = 0$ such that we have:

$$\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} \tag{511}$$

with a radius of convergence of $R = 1$.

Remarks: If we substitute in $x = 1$, then we can a special series:

$$\tan^{-1}(1) = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{512}$$

and is known as Leibniz's formula for π.

19 Taylor and Maclaurin Series

- Recall that the power series can be written as:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots \tag{513}$$

for $|x-a| < R$, we note that $f(a) = c_0$. However, if we take the derivative:

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots \tag{514}$$

and we similarly get $f'(a) = c_1$. For the second derivative:

$$f''(x) = 2c_2 + 6c_3(x-a) + \cdots \tag{515}$$

we get $f''(a) = 2c_2$.

- In general:

$$f^(n)(a) = n!c_n \tag{516}$$

Theorem: If $f(x)$ has a power series representation about a:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \tag{517}$$

with $|x-a| < R$. Then the coefficients of the series are $c_n = \frac{f^{(n)(a)}}{n!}$

- For a Taylor series of f about a, we have:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \cdots \tag{518}$$

- For the Maclaurin Series, it is simply a Taylor series taken at $x = a$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots \tag{519}$$

Definition: A definition is called **analytic at a** if it can be represented as a power series about a.

Example 81: Let us attempt to write out the Maclaurin series of $f(x) = e^x$. First note that:

$$f'(x) = e^x = f''(x) = f'''(x) = f^{(n)}(x) \tag{520}$$

Therefore: $f^{(n)}(0) = e^0 = 1$. Therefore, we can write it as the series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{521}$$

We can check that this converges using the ratio test. Let $a_n = \dfrac{x^n}{n!}$. Then:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \tag{522}$$

which approaches zero as $n \to \infty$. As a result, $R = \infty$

- We ask ourselves the question: When is it true that $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

Definition: The nth degree Taylor polynomial of f about a can be written as:

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{523}$$

Example 82: Let us take a look at e^x about $a = 0$. Then the first, second, third degree series can be written as:

$$T_1(x) = 1 + x \tag{524}$$

$$T_2(x) = 1 + x + \frac{x^2}{2} \tag{525}$$

$$T_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \tag{526}$$

We can then define the remainder function as:

$$R_n(x) = f(x) - T_n(x) \tag{527}$$

Theorem: If $f(x) = T_n(x) + R_n(x)$ and $\lim\limits_{n\to\infty} R_n(x) = 0$ for $|x-a| < R$. Then f is equal to the sum of its Taylor series.

Given that f has $n+1$ continuous derivatives on an open interval I containing a, tyhen for all $x \in I$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} + R_n(x) \tag{528}$$

where

$$R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n \,\mathrm{d}t \tag{529}$$

Proof. Consider the fundamental theorem of calculus:

$$\int_a^b f'(t)\,\mathrm{d}t = f(b) - f(a) \tag{530}$$

Suppose we evaluate this via integration by parts:

$$u = f'(t) \qquad \mathrm{d}v = \mathrm{d}t$$
$$\mathrm{d}u = f''(t) \qquad v = t - b$$

This gives:

$$\int_a^b f'(t)\,\mathrm{d}t = [f'(t)(t-b)]_a^b - \int_a^b f''(t)(t-b)\,\mathrm{d}t \tag{531}$$
$$= (b-a)f'(a) + \int_a^b (b-t)fL''(t)\,\mathrm{d}t \tag{532}$$

We integrate by parts again:

$$u = f''(t) \qquad \mathrm{d}v = (b-t)\,\mathrm{d}t \tag{533}$$
$$\mathrm{d}u = f'''(t)\,\mathrm{d}t \qquad v = -\frac{(b-t)^2}{2} \tag{534}$$

which gives:

$$\int_a^b f^n(t)(b-t)\,\mathrm{d}t = \left[-\frac{(b-t)^2}{2}f''(t)\right]_a^b + \int_a^b \frac{(b-t)^2}{2}f'''(t)\,\mathrm{d}t \tag{535}$$

If we continue this a total of n times, then we eventually get:

$$\int_a^b f'(t)\,\mathrm{d}t = (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \frac{(b-a)^3}{3!}f''(a) + \cdots + \frac{(b-a)^n}{n!}f^{(n)}(a) + \int_a^b \frac{(b-t)^n}{n!}f^{n+1}(t)\,\mathrm{d}t \tag{536}$$

However, remember that this integration is equal to $f(b) - f(a)$. If we let $x = b$, then we get:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(x) \tag{537}$$

where from our previous work, we have

$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!}f^{n+1}(t)\,\mathrm{d}t \tag{538}$$

□

- For $|f^{(n+1)}(t)| \leq M$ for $a < t < x$ we can bound the remainder function by:

$$|R_n(x)| \leq \left|\int_0^x \frac{M(x-t)^n}{n!}\,\mathrm{d}t\right| = \left|M\left[\frac{(x-t)^{n+1}}{(n+1)!}\right]_a^x\right| = M\frac{|x-a|^{n+1}}{(n+1)!} \tag{539}$$

- If we instead use the MVT, we can obtain a slightly different expression for the reaminder:

$$R_n(x) = \frac{f^{n+1}(c)(x-a)^{n+1}}{(n+1)!} \tag{540}$$

with $a < c < x$.

Example 83: Suppose we wish to continue the proof that e^x is indeed equal to the sum of its Taylor series, we note again that $f^{(n+1)}(t) = e^t$. For $x > 0$, we can pick an x such that $0 < t < x$ where $e^t < e^x$. The remainder can then be written as:

$$R_n(x) < \frac{e^x x^{n+1}}{(n+1)!} \tag{541}$$

As $n \to \infty$, the remainder approaches zero and as a result:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{542}$$

for all x is a true statement.

Example 84: Let us now find the Maclaurin series for $\cos x$. We have:

$$f(x) = \cos x \qquad f(0) = 1 \tag{543}$$
$$f'(x) = -\sin x \qquad f'(0) = 0 \tag{544}$$
$$f''(x) = -\cos x \qquad f''(0) = -1 \tag{545}$$
$$f'''(x) = \sin x \qquad f'''(0) = 0 \tag{546}$$
$$f^{(4)}(x) = \cos x \qquad f(0) = 1 \tag{547}$$

and it repeats. Therefore, we propose that:

$$\cos x = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \cdots \tag{548}$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{549}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{550}$$

We can use the ratio test to show that the radius of convergence is $R = \infty$. Finally, we need to prove that this sum is $\cos x$. We note that:

$$|f^{n+1}(t)| = \pm\cos t \text{ or } \pm\sin t \leq 1 \tag{551}$$

so we can bound the remainder by:

$$|R_n(x)| \leq \left|\frac{Mx^{n+1}}{(n+1)!}\right| = \left|\frac{x^{n+1}}{(n+1)!}\right| \tag{552}$$

- An important idea is that it does not matter where the coefficients in the power series expansion comes from. There is only one unique set of coefficients so if one possible set is found, then it is the only set.

Example 85: Let us find the Maclaurin series for $\sin x$:

$$\sin x = -\frac{d}{dx} \cos x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{553}$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} 2n \frac{x^{2n-1}}{2n!} \tag{554}$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{555}$$

Example 86: Consider the function $x \sin x$. The power expansion is thus:

$$x \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)!} \tag{556}$$

Example 87: Let us try to find the Taylor series of $\cos x$ about $\frac{17\pi}{4}$. This gives:

$$f(x) = \cos x \qquad f\left(\frac{17\pi}{4}\right) = \frac{1}{\sqrt{2}} \tag{557}$$

$$f'(x) = -\sin x \qquad f'\left(\frac{17\pi}{4}\right) = -\frac{1}{\sqrt{2}} \tag{558}$$

$$f''(x) = -\cos x \qquad f''\left(\frac{17\pi}{4}\right) = -\frac{1}{\sqrt{2}} \tag{559}$$

$$f'''(x) = \sin x \qquad f'''\left(\frac{17\pi}{4}\right) = \frac{1}{\sqrt{2}} \tag{560}$$

$$f''''(x) = \cos x \qquad f''''\left(\frac{17\pi}{4}\right) = \frac{1}{\sqrt{2}} \tag{561}$$

Example 88: We have already seen that:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \tag{562}$$

fpr $-1 < x \leq 1$. We can verify this by verifying coefficients:

$$f(x) = \ln(1+x) \tag{563}$$

$$f'(x) = \frac{1}{1+x} \tag{564}$$

$$f''(x) = -\frac{1}{(1+x)^2} \tag{565}$$

$$f'''(x) = \frac{2}{(1+x)^3} \tag{566}$$

$$f''''(x) = \frac{-3!}{(1+x)^4} \tag{567}$$

$$f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{(1+x)^n} \tag{568}$$

Suppose we wish to bound the n^{th} derivative using the remainder function:

$$R_n = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x-t)^n \, \mathrm{d}t \tag{569}$$

$$= \frac{1}{n!}\int_0^x (-1)^{n+2}\frac{n!}{(1+t)^{n+1}}(x-t)^n \, \mathrm{d}t \tag{570}$$

$$= (-1)^n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}} \, \mathrm{d}t \tag{571}$$

Let us work with nonzero values of x: $0 \leq x \leq 1$. Then:

$$|R_n(x)| = \int_0^x \frac{(x-t)^n}{(1+t)^{n+1}} \, \mathrm{d}t \tag{572}$$

$$\leq \int_0^x (x-t)^n \, \mathrm{d}t \qquad (1+t) > 1 \tag{573}$$

$$= \frac{x^{n+1}}{n+1} \tag{574}$$

As $n \to \infty$, this approaches zero. Now let $-1 < x < 0$. Then:

$$|R_n(x)| = \left|\int_0^x \frac{(x-t)^n}{(1+t)^{n+1}} \, \mathrm{d}t\right| \tag{575}$$

$$= \int_x^0 \left(\frac{t-x}{1+t}\right)^n \frac{\mathrm{d}t}{1+t} \tag{576}$$

Note that from the mean value theorem, a number z exists, where $x < z < 0$ such that[a]:

$$\int_x^0 \left(\frac{t-x}{1+t}\right)^n \frac{\mathrm{d}t}{1+t} = \left(\frac{z-x}{1+z}\right)^n \left(\frac{-x}{1+z}\right) \tag{577}$$

Now $|x| < 1$ so $|x| - |z| < 1 - |z|$. This implies:

$$\frac{|x|-|z|}{1-|z|} < 1 \tag{578}$$

$$\frac{-x+z}{1+z} < 1 \tag{579}$$

We then have the limit:

$$\lim_{n\to 0} \left(\frac{z-x}{1+z}\right)^n = 0 \tag{580}$$

and therefore:

$$R_n(x) \to 0 \tag{581}$$

as $n \to \infty$.

[a] To see it explicitly, we can interpret the function as the area under the curve from x to 0 of: $\dfrac{(t-x)^n}{(1+t)^{n+1}}$. The mean value theorem tells us that the average height of this function has to occur at a value of $t = z$ where $x < z < 0$ and the total area can be represented as the average height (at $t = z$) multiplied by the width, which is $0 - x = -x$ (note that since x is negative, we can interpret this as multiplying it by -1 to get a positive area).

- It is possible to multiply and divide different power series

Example 89: Suppose we have the function $\dfrac{e^x}{1-x}$. Then the power series is given as:

$$= \left(1 + x + \frac{x^2}{2} + \cdots\right)(1 + x + x^2 + \cdots) \tag{582}$$

$$= 1 + 2x + \frac{5}{2}x^2 + \frac{16}{6}x^3 + \cdots \tag{583}$$

Example 90: We can determine the power expansion for $\tan x$ by using long division. We have:

$$\tan x = \frac{\sin x}{\cos x} \tag{584}$$

$$= \frac{x - x^3/3! + x^5/5! + \cdots}{1 - x^2/2 + x^4/4! + \cdots} \tag{585}$$

$$= x + \frac{x^3}{2} + \frac{2}{15}x^5 + \cdots \tag{586}$$

which for obvious reasons, I have omitted the long division steps. The radius of convergence is $|x| < R = \dfrac{\pi}{2}$ since the function diverges at $\tan \dfrac{\pi}{2}$.

20 Applications of Taylor Polynomials

- Recall that the n^{th} degree polynomial is:

$$T_n = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \tag{587}$$

The first degree polynomial is a linear approximation and the second degree is a quadratic approximation (at least, near $x = a$).

- We can use Taylor series to estimate erros:
    1. Alternating series: $|R_n(x)| < |a_{n+1}|$
    2. Taylor's formula: $|R_n| < \dfrac{M(x-a)^{n+1}}{(n+1)!}$

Example 91: Suppose we want to use the Taylor expansion of $f(x)=\sqrt{x}$ at $a=1$ and use it to evaluate $\sqrt{1.25}$. The first four derivatives are:

$$f(x) = x^{1/2} \qquad f(1) = 1 \tag{588}$$

$$f'(x) = \frac{1}{2}x^{-1/2} \qquad f'(1) = \frac{1}{2} \tag{589}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \qquad f''(1) = -\frac{1}{4} \tag{590}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} \qquad f'''(1) = \frac{3}{8} \tag{591}$$

$$f''''(x) = -\frac{15}{16}x^{-7/2} \qquad f''''(1) = -\frac{15}{16} \tag{592}$$

which gives:

$$\sqrt{x} \simeq T_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{4}\frac{(x-1)^2}{2!} + \frac{3}{8}\frac{(x-1)^3}{3!} \tag{593}$$

and the error is:

$$|R_3(x)| < |a_4| = \frac{15}{16}\frac{(x-1)^4}{4!} \tag{594}$$

so:

$$\sqrt{1.125} \simeq 1 + \frac{0.25}{2} - \frac{0.25^2}{8} + \frac{0.25^3}{16} \pm \frac{5}{128}0.25^4 \approx 1.11816 \pm 0.00015 \tag{595}$$

Example 92: Consider the maclaurin series of $\cos x$ about $a=0$. We want to find the error for $-\frac{\pi}{4} < x < \frac{\pi}{4}$, we have:

$$\cos x \simeq T_3(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \tag{596}$$

and the error would be:

$$|R_3(x)| < \left|\frac{x^8}{8!}\right| < \frac{(\pi/4)^8}{8!} \approx 3.6 \times 10^{-6} \tag{597}$$

Note that we can also use our alternating series result to get the same error.

Example 93: Suppose we wish to find $\ln(1.4)$ to within 0.001 using the series expansion for $\ln(1-x)$:

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \tag{598}$$

which converges for $|x| < 1$. Here, we want $x = -0.4$ and we get an alternating series. The remainders are then"

$$|R_1(x)| < \left|\frac{x^2}{2}\right| = 0.8 \tag{599}$$
$$|R_2(x)| < \left|\frac{x^3}{3}\right| = 0.02 \tag{600}$$
$$|R_3(x)| < \left|\frac{x^4}{4}\right| = 0.006 \tag{601}$$
$$|R_4(x)| < \left|\frac{x^5}{5}\right| = 0.002 \tag{602}$$
$$|R_5(x)| < \left|\frac{x^6}{6}\right| = 0.0007 < 0.001 \tag{603}$$

Therefore, we can take the first five terms of the expansion and get an answer within 0.0007 of the actual answer.

- The binomial theorem tells us:

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \cdots + \frac{k(k-1)(k-2)\cdots(k-n+1)}{k!}a^{k-n}b^n \tag{604}$$
$$= \sum_{n=0}^{k}\binom{k}{n}a^{k-n}b^n \tag{605}$$

- If we let $a = 1$ and $b = x$, we get:

$$(1+x)^k = \sum_{n=0}^{k}\binom{k}{n}x^n \tag{606}$$

which represents a power series. We can also get the coefficients using the Taylor series:

$$f(x) = (1+x)^k \qquad f(0) = 1 \tag{607}$$
$$f'(x) = k(1+x)^{k-1} \qquad f'(1) = k \tag{608}$$
$$f''(x) = k(k-1)(1+x)^{k-1} \qquad f''(0) = k(k-1) \tag{609}$$

such that:

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \tag{610}$$

which is equiavlent to before. We can compare the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{k(k-1)(k-2)\cdots(k-x)(x^{n+1})}{(n+1)!}\cdot\frac{n!}{k(k-1)\cdots(k-n+1)x^n}\right| = \left|\frac{k-n}{n+1}x\right| \tag{611}$$

which approaches $|x| < 1$ as $n \to \infty$. The interval convergence is:

$$k \leq -1 \qquad I = (-1,1) \tag{612}$$
$$-1 < k < 0: \qquad I = (-1,1] \tag{613}$$
$$k \geq 0: \qquad I = [-1,1] \tag{614}$$

Example 94: Suppose we have $f(x) = \dfrac{1}{\sqrt{2+x}}$. We then have:

$$(2+x)^{-0.5}n = \frac{1}{\sqrt{2}}\left(1+\frac{x}{2}\right)^{-1/2} \tag{615}$$
$$= \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{x}{2}\right)^n \tag{616}$$
$$= \frac{1}{\sqrt{2}}\left[1-\frac{1}{2}\frac{x}{2}+\frac{3}{4}\frac{(x/2)^2}{2!}-\cdots\right] \tag{617}$$

and the radius of convergence is 2 since we want $|x/2| < 1$.

Example 95: This is an extremely important linear approximation:

$$(1+x)^k \approx 1+kx \tag{618}$$

21 Fourier Series

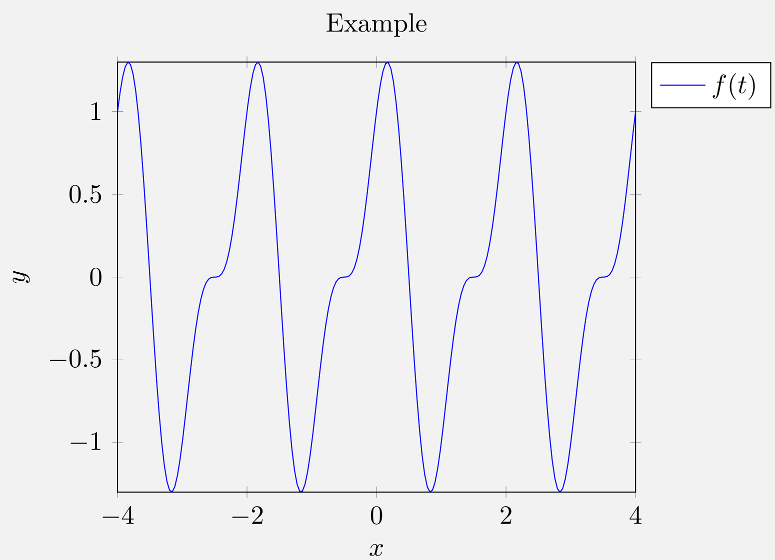
- Fourier Series is used to represent periodic functions using a series consisting of trigonometric functions.

Definition: A function is periodic if and only if there exists a constant T such that $f(t+T) = f(t)$. The smallest positive value T is known as the fundamental period.

Example 96: If we have a function in the form of:

$$f(t) = \cos(\pi t) + \frac{1}{2}\sin(2\pi t) \tag{619}$$

The period of the first term is 2 and the period of the second term is 1. The smallest period for f is 2:



Example 97:

Theorem: For $f(t)$ periodic, with fundamental period T, continuous and piecewise differentiable, then:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + b_n \sin(n\omega t) \tag{620}$$

where $\omega = \frac{2\pi}{T}$ is known as the Fourier series of f. a_n and b_n are Fourier coefficients.

- To determine the coefficients, we need the following integrals:

$$\int_{-T/2}^{T/2} \cos(n\omega t)\,\mathrm{d}t = \begin{cases} 0 & n \neq 0 \\ T & n = 0 \end{cases} \tag{621}$$

$$\int_{-T/2}^{T/2} \sin(n\omega t)\,\mathrm{d}t = 0 \tag{622}$$

$$\int_{-T/2}^{T/2} \cos(m\omega t)\cos(n\omega t)\,\mathrm{d}t = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases} \tag{623}$$

Example 98: Take the integral $\int_{-T/2}^{T/2} \cos(5\omega t)\cos(3\omega t)\,\mathrm{d}t$. We can write this as:

$$= \frac{1}{2}\int_{-T/2}^{T/2} (\cos(2\omega t) + \cos(8\omega t))\,\mathrm{d}t \tag{624}$$

$$= \frac{1}{2}\left[\frac{1}{2\omega}\sin(2\omega t) + \frac{1}{8\omega}\sin(8\omega t)\right]_{-T/2}^{T/2} \tag{625}$$

$$= 0 \tag{626}$$

- Consider the integrals:

$$\int_{-T/2}^{T/2} f(t)\cos(m\omega t)\,\mathrm{d}t = \frac{T}{2}a_m \tag{627}$$

$$\int_{-T/2}^{T/2} f(t)\sin(m\omega t)\,\mathrm{d}t = \frac{T}{2}b_m \tag{628}$$

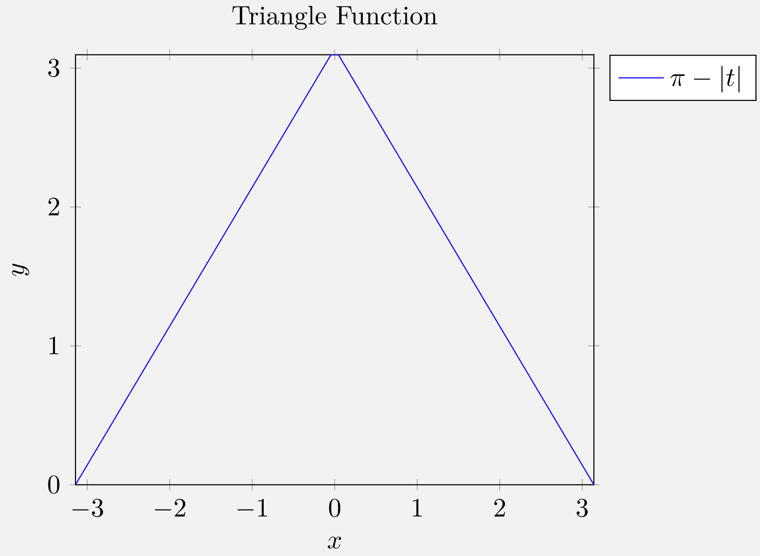
pi/ where we have substituted in equation 619 alongside the above integral identities. This then gives the coefficients as:

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,\mathrm{d}t \tag{629}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(n\omega t)\,\mathrm{d}t \tag{630}$$

for $n = 1, 2, 3, \ldots$.

Example 99: Suppose we have a triangle function: $f(t) = \pi - t$ from $[-\pi, \pi]$:



We start by calculating the b coefficient:

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,\mathrm{d}t \tag{631}$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \pi\sin(nt)\,\mathrm{d}t - \frac{1}{\pi}(-t)\int_{-\pi}^{\pi}\sin(nt)\,\mathrm{d}t \tag{632}$$

Notice that since this is an odd function, we have $b_n = 0$. Next, we calcluate a_n:

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,\mathrm{d}t \tag{633}$$

$$= 2\frac{1}{\pi}\int_{0}^{\pi}(\pi - t)\cos(nt)\,\mathrm{d}t \tag{634}$$

$$= \begin{cases} \pi & n = 0 \\ 0 & n \neq 0, \text{ n is even} \\ \dfrac{4}{\pi n^2} & \neq 0, \text{ n is odd} \end{cases} \tag{635}$$

Therefore, we have:

$$f(t) = \frac{\pi}{2} + \sum_{n=0}^{\infty}\frac{4}{\pi(2k-1)^2} \tag{636}$$

Idea: It is helpful to look if the function is even or odd at the beginning, allowing us to immediately set $a_n = 0$ or $b_n = 0$.

Example 100: Suppose we wish to write the Fourier series of $f(x) = x^2$ from $-\pi \leq x \leq \pi$. The period is $T = 2\pi$ and $\omega = \dfrac{2\pi}{T} = 1$. We then have:

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} t \qquad 2\cos(nt)\,\mathrm{d}t \tag{637}$$

For $n = 0$ we have:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,\mathrm{d}t = \frac{2\pi^2}{3} \tag{638}$$

For $n \neq 0$, we have:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 \cos(nt)\,\mathrm{d}t \tag{639}$$
$$= \frac{1}{\pi n^2}\int_{-\pi}^{\pi} (nt)^2 \cos(nt)\,\mathrm{d}(nt) \tag{640}$$
$$= \frac{1}{\pi n^2}\left[(nt)^2 \sin(nt)\right]\Bigg|_{t=-\pi}^{t=\pi} - \frac{2}{\pi n^2}\int_{-\pi}^{\pi} (nt)\sin(nt)\,\mathrm{d}(nt) \qquad \text{(reduction)} \tag{641}$$
$$= -\frac{2}{\pi n^3}(\sin(nt) - nt\cos(nt))\Bigg|_{t=-\pi}^{t=\pi} \qquad \text{(by parts)} \tag{642}$$
$$= \begin{cases} -\frac{4}{n^2} & \text{n odd} \\ +\frac{4}{n^2} & \text{n even} \end{cases} \tag{643}$$

Since the function is even, we have $b_n = 0$. Thus:

$$f = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nt \tag{644}$$

22 Vectors and the Geometry of Space

- Vectors can be written in the form of $\vec{a} = (1,1,1) = 1\hat{i} + 1\hat{j} + 1\hat{k}$. They do not typically represent a physical point in space.
- If we want to specify that the vector starts at the origin, we use the **radius vector** $\vec{r}$.
- Planes are written as:

$$ax + by + cz = d \tag{645}$$
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{646}$$

Let $\vec{n}$ be the normal vector, let $\vec{r}$ be a general vector whose head lies on the plane and let $\vec{r}_0$ be a generic vector that lies on the vector. Then: $\vec{r} - \vec{r}_0$ lies on the plane so:

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \tag{647}$$

Given $\vec{n} = (n_1, n_2, n_3)$, $\vec{r}_0 = (x_0, y_0, z_0)$, and $\vec{r} = (x, y, z)$. Then:

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0 \tag{648}$$

Example 101: Suppose we wish to represent the yz plane. We can let the normal vector be $\vec{n} = (2,0,0)$ and $\vec{r}_0 = (0,2,3)$. We can check that the solution to:

$$2(x - 0) + 0(y - 2) + 0(z - 3) = 0 \tag{649}$$

is $x = 0$.

- For a line, we follow a similar process. Let r_0 be a fixed point on a line and let $\vec{r}$ be a general vector that lies on the line. Let $t\vec{v} = \vec{r} - \vec{r}_0$ be a vector that lies on the line such that:

$$\vec{r} = \vec{r}_0 + t\vec{v} \tag{650}$$

where t is a parameter.

- We can write this in parametric form:

$$x = x_0 + tv_1 \tag{651}$$
$$y = y_0 + tv_2 \tag{652}$$
$$z = z_0 + tv_3 \tag{653}$$

Or we can write it in its equivalent symmetric form:

$$t = \frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3} \tag{654}$$

Example 102: Let $P_0 = (-3, 0, 0)$ and $\vec{v} = (2, 0, 0)$. This gives the equation:

$$x = -3 + 2t \tag{655}$$
$$y = 0 \tag{656}$$
$$z = 0 \tag{657}$$

23 Cylinders and Quadratic Cylinders

- The simplest *non-planar* function is in the form of:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

whicha re known as **quadratic surfaces.** There are nine distinct types of surfaces that can be represented. A list of shapes can be found in the Stewart textbook.

- There are several properties that are worth investigating to understand a quadratic surface:
  - Domain/Range
  - Intercepts with coordinate axes
  - Traces (intercepts with coordinates planes)
  - Sections (Intersections with other planes)
  - Center
  - Symmetry
  - Bounded / Unbounded

Example 103: Suppose we look at the hyperboloid of two sheets:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 \implies z = \pm c\sqrt{1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}} \tag{658}$$

This means the domain is $(x, y) \in (-\infty, \infty)$ and the range is $z \geq c$ or $z \leq c$. We now look at intercepts and we realize that at $x = 0$ and $y = 0$, we have $z = \pm c$. We then look at the traces:
- For the xy plane, we have nothing since $z \neq 0$.
- For the xz plane, we have $y = 0$ and we get a hyperbola:

$$z = \pm c\sqrt{1 + \frac{x^2}{a^2}} \tag{659}$$

- For the yz plane ($x = 0$, we have another hyperbola)

$$z = \pm c\sqrt{1 + \frac{y^2}{b^2}} \tag{660}$$

We then look at sections. We can start with the plane defined by $z = z_0$ with $|z_0| > c$. This gives:

$$\frac{a^2}{a^2} + \frac{y^2}{b^2} = \frac{z_0^2}{c^2} - 1 \tag{661}$$

since $z_0^2 > c^2$, the RHS is greater than zero so the section is an ellipse. The center is at the origin. No offset terms. There is also a lot of symmetry:

$$x \to -x \tag{662}$$
$$y \to -y \tag{663}$$
$$z \to -z \tag{664}$$

so it is symmetric about each coordinate axis and it is unbounded in all directions.

- Another important concept is the **projection**. If we have two three dimensional currves and they intersect, then we can define:

Definition: A curve of intersection: $C = (x, y, z)$ is defined such that $z = f(x, y)$ and $z = g(x, y)$. Then :$f(x, y) = g(x, y)$.

If we set $z = 0$, then the curve of intersection is given by $(x, y, z = 0)$. This is known as the projection and can be seen as the shadow of the points of intersection.

Example 104: Suppose we have a cone: $x^2 + y^2 = 2z^2$. We can rearrange this to:

$$z = \pm\sqrt{(x^2 + y^2)/2} \tag{665}$$

We can look at the plane $y + 4z = 5$. We then have:

$$\frac{5-y}{4} = \sqrt{\frac{x^2+y^2}{2}} \tag{666}$$

After solving, we get:

$$\frac{x^2}{25/7} + \frac{(y+5/7)^2}{200/49} = 1 \tag{667}$$

gives the projection onto the plane $z = 0$.

- Recall that a vector can be written like $\vec{r} = 3\hat{\imath} + 2\hat{\jmath} + 5\hat{k}$. However, instead of having constants, we can have functions instead:

$$\vec{f}(t) = f_1(t)\hat{\imath} + f_2(t)\hat{\jmath} + f_3(t)\hat{k} = (f_1(t), f_2(t), f_3(t)) \tag{668}$$

which is known as a vector-valued function or a **vector function**

- One simple example is that of a straight line, where we have:

$$\vec{f}(t) = (f_1, f_2, f_3) = (a_1 + b_1 t, a_2 + b_2 t, a_3 + b_3 t) \tag{669}$$

- Suppose we have a three dimensional function $y = g(x)$. We can let $f_1(t) = t = x$. This gives $f_2(t) = g(t)$, and $f_3(t) = 0$. This gives:

$$\vec{f}(t) = (t, g(t), 0) \tag{670}$$

and the same concept can be extended to three dimensions.

Definition: Let the vector function $\vec{f}$ be defined on some interval I containing the point t_0, except possibly at t_0 itself, and let $\vec{L}$ be a vector. Then:

$$\lim_{t\to t_0} \vec{f}(t) = \vec{L} \tag{671}$$

if

$$\lim_{t\to t_0} \|\vec{f}(t) - \vec{L}\| = 0 \tag{672}$$

- The simplest case is when $\lim\limits_{t\to t_0} \vec{f}(t) = \vec{L}$. Then we can immediately see that:

$$\lim_{t\to t_0} \|\vec{f}(t)\| = \|\vec{L}\| \tag{673}$$

> **Warning:** Note that this is not an if and only if statement. It is totally possible for two vectors to have the same magnitude, but are not equal to each other.

- Limit rules carry over from two dimensional calculus. Given $\vec{f}(t) \to \vec{L}$ and $\vec{g}(t) \to \vec{M}$ and $u(t) \to A$ as $t \to t_0$. Then:
  - $\vec{f} + \vec{g} \to \vec{L} + \vec{M}$
  - $\alpha\vec{f}(t) \to \alpha\vec{L}$
  - $u(t)\vec{f}(t) \to A\vec{L}$
  - $\vec{f}(t) \cdot \vec{g}(t) \to \vec{L} \cdot \vec{M}$
  - $\vec{f}(t) \times \vec{g}(t) \to \vec{L} \times \vec{M}$
- We have $\vec{f}(t) \to \vec{L}$ *if and only if* $f_1(t) \to L_1$, $f_2(t) \to L_2$, and $f_3(t) \to L_3$.

Example 105: Suppose we have a vector function:

$$f(t) = \frac{\sin t}{t}\hat{\imath} + (2+t^2)\hat{\jmath} + e^{t^2}\cos t\hat{k} \tag{674}$$

then:

$$\lim_{t\to 0}\vec{f}(t) = (1, 2, 1) \tag{675}$$

- A vector function $\vec{f}(t)$ is continuous at t_0 if $\lim\limits_{t\to t_0}\vec{f}(t) = \vec{f}(t_0)$.

24 Derivatives and Integrals of Vector Functions

- We can define derivatives as:

$$\vec{f}'(t) \equiv \lim_{h\to 0}\frac{\vec{f}(t+h) - \vec{t}}{h} \tag{676}$$

$$= f_1'(t)\hat{\imath} + f_2'(t)\hat{\jmath} + f_3'(t)\hat{k} \tag{677}$$

Proof. We have:

$$\vec{f}'(t) = \lim_{h\to 0}\frac{\vec{f}(t+h) - \vec{f}(t)}{h} \tag{678}$$

$$= \lim_{h\to 0}\left[\frac{f_1(t+h) - f_1(t)}{h}\hat{\imath} + \cdots\right] \tag{679}$$

$$= \lim_{h\to 0}\frac{f_1(t+h) - f_1(t)}{h}\hat{\imath} + \lim_{h\to 0}\cdots \tag{680}$$

$$= f_1'(t)\hat{\imath} + f_2'(t)\hat{\jmath} + f_3'(t)\hat{k} \tag{681}$$

□

Example 106: Suppose we have:

$$\vec{f}(t) = \frac{\sin t}{t}\hat{\imath} + (2+t^2)\hat{\jmath} + e^{t^2}\cos t\hat{k} \tag{682}$$

We can take the derivative of each term to get:

$$\vec{f}'(t) = \frac{t\cos t - \sin t}{t^2}\hat{\imath} + 2t\hat{\jmath} + (2te^{t^2}\cos t - e^{t^2}\sin t)\hat{k} \tag{683}$$

- Similarly we can take the integral of a vector function as:

$$\int_a^b \vec{f}(t)\,\mathrm{d}t = \hat{\imath}\int_a^b f_1(t)\,\mathrm{d}t + \hat{\jmath}\int_a^b f_2(t)\,\mathrm{d}t + \hat{k}\int_a^b f_3(t)\,\mathrm{d}t \tag{684}$$

For example, this is useful if we wish to get a velocity function from an acceleration function. The usual integration rules apply. For example:

- $\int_a^b \vec{c}\cdot\vec{f}(t)\,\mathrm{d}t = \vec{c}\cdot\int_a^b \vec{f}(t)\,\mathrm{d}t$
- $\left\|\int_a^b \vec{f}(t)\,\mathrm{d}t\right\| \leq \int_a^b \left\|\vec{f}(t)\right\|\mathrm{d}t$

- We can define a composite function as:

$$(\vec{f}\circ u)(t) = \vec{f}(u(t)) \tag{685}$$

Example 107: Let $u(t) = e^t$ and $\vec{f}(t) = (\cos t, \sin t, t^2)$. Then:

$$(\vec{f}\circ u)(t) = \cos\left(e^t\right)\hat{\imath} + \sin\left(e^t\right)\hat{\jmath} + e^{2t}\hat{k} \tag{686}$$

Warning: We can only take the composite of a vector function with a scalar function, not the other way around.

- We have the following differentiation rules:
  - $(\vec{f}+\vec{g})'(t) = \vec{f}'(t) + \vec{g}'(t)$
  - $(\alpha\vec{f})'(t) = \alpha f'(t)$
  - $(u\vec{f})'(t) = u(t)\vec{f}'(t) + u'(t)\vec{f}(t)$
  - $(\vec{f}\cdot\vec{g})'(t) = \left[\vec{f}(t)\cdot\vec{g}(t)\right] + \left[\vec{f}'(t)\cdot\vec{g}(t)\right]$
  - $(\vec{f}\times\vec{g})'(t) = \left[\vec{f}(t)\times\vec{g}(t)\right] + \left[\vec{f}'(t)\times\vec{g}(t)\right]$
  - $(\vec{f}\circ u)'(t) = \vec{f}'(u(t))u'(t)$

Example 108: Let us take the position vector $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ where $r \equiv \|\vec{r}\|$ or $\vec{r}\cdot\vec{r} = r^2$. If we differentiate this, we get:

$$\frac{d\vec{r}}{dt}\cdot\vec{r} + \vec{r}\cdot\frac{d\vec{r}}{dt} = 2r\frac{dr}{dt} \tag{687}$$

which confirms the dot product.

Example 109: Let us take the derivative $\frac{d}{dt}\frac{\vec{r}}{r}$. This is always the scalar function with a constant magnitude. However, the direction can change. The derivative then points to the direction in which the curve is changing in. We use the quotient rule:

$$\frac{d}{dt}\frac{\vec{r}}{r} = \frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r} \tag{688}$$

$$= \frac{1}{r^2}\left(r^2\frac{d\vec{r}}{dt} - r\frac{dr}{dt}\vec{r}\right) \tag{689}$$

$$= \frac{1}{r^3}\left((\vec{r}\cdot\vec{r})\frac{d\vec{r}}{dt} - \vec{r}\frac{d\vec{r}}{dt}\vec{r}\right) \tag{690}$$

$$= \frac{1}{r^3}\left((\vec{r}\times\frac{d\vec{r}}{dt})\times\vec{r}\right) \tag{691}$$

where we have applied the triple cross product:

$$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{c}\cdot\vec{a})\vec{b} - (\vec{c}\cdot\vec{b})\vec{a} \tag{692}$$

- Suppose we have a generic position vector $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$. This is a three dimensional curve where we can visualize an object moving across this curve. This gives us intuition for the following definition:

> **Definition**: Let C be parametized by $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ and be diffferentiable. Then $\vec{r}'(t) = x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{k}$ if not $\vec{0}$, is tangent to the curve C at the point $P(x(t), y(t), z(t))$ and $\vec{r}'(t)$ points in the direction of increasing t.

Example 110: Suppose we have a circle defined as $\vec{r} = a\cos t\hat{\imath} + a\sin t\hat{\jmath}$. The derivative is:

$$\vec{r}' = -a\sin t\hat{\imath} + a\cos t\hat{\jmath} \tag{693}$$

Since the direction of increasing t should be perpendicular to the position vector (radius), then the dot product should be zero:

$$\vec{r}\cdot\vec{r}' = -a^2\sin t\cos t + a^2\cos t\sin t = 0 \tag{694}$$

which we verified.

Example 111: Suppose we have a striahgt line with direction $\vec{d}$ through a point $P(x_0, y_0, z_0)$. This gives us: $\vec{r} = \vec{a} + t\vec{d}$ where $\vec{a} = (x_0, y_0, z_0)$. Taking the derivative, we get:

$$\vec{r}' = \vec{d} \tag{695}$$

as expected.

Example 112: Let us find the derivative of $\vec{r}(t) = (1+2t)\hat{\imath} + t^2\hat{\jmath} + \frac{t}{2}\hat{k}$ at $P(9, 64, 2)$. This occurs at $t = 4$. We have:

$$\vec{r}'(t) = 2\hat{\imath} + 3t^2\hat{\jmath} + \frac{1}{2}\hat{k} \tag{696}$$

$$(\vec{4}) = 2\hat{\imath} + 48\hat{\jmath} + \frac{1}{2}\hat{k} \tag{697}$$

The **tangent line** can be written as:

$$\vec{R}(q) = 9\hat{\imath} + 64\hat{\jmath} + 2\hat{k} + q(2\hat{\imath} + 48\hat{\jmath} + \frac{1}{2}\hat{k}) \tag{698}$$

- The **unit tangent vector** is given as:

$$\vec{T}(t) \equiv \frac{\vec{r}'(t)}{\|\vec{r'(t)}\|} \tag{699}$$

  Note that the dot product is:

$$\vec{T}(t)\cdot\vec{T}(t) = 1 \tag{700}$$

  Differentiating,

$$\vec{T}'(t)\cdot\vec{T}(t) + \vec{T}(t)\cdot\vec{T}'(t) = 0 \implies \vec{T}'(t)\cdot\vec{T}(t) = 0 \tag{701}$$

  This means the tangent vector is perpendicular to the derivative of the tangent vector.

25 Arc Length and Curvature

- The arclength can be extended to three dimensions:

$$s = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,\mathrm{d}t \tag{702}$$

Example 113: Suppose we have a circular helix (which looks like a screw):

$$\vec{r}(t) = 3\sin t\hat{i} + 3\cos t\hat{j} + 4t\hat{k} \tag{703}$$

for $t \in [0, 2\pi]$. The derivative is:

$$\vec{r'}(t) = 3\cos t\hat{i} - 3\sin t\hat{k} + 4\hat{k} \tag{704}$$

This gives:

$$\|\vec{r'}(t)\| = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = 5 \tag{705}$$

so:

$$s = \int_0^{2\pi} \|\vec{r'}(t)\| \, \mathrm{d}t = 5 \cdot 2\pi = 10\pi \tag{706}$$

- We can define the **function** $s(t)$ has the arclength from a time t_0 to a time t

$$s(t) = \int_{t_0}^{t} \|F'(\tau)\| \, \mathrm{d}\tau \tag{707}$$

- Sometimes it is helpful to parametrize a function in terms of s

Example 114: Suppose we have the curve $\vec{r}(t) = t^2\hat{i} + t^2\hat{j} - t^2\hat{k}$ from $(0, 0, 0)$. The arclength is then:

$$s = \int_0^t \sqrt{4\tau^2 + 4\tau^2 + 4\tau^2} \, \mathrm{d}\tau = \sqrt{3}t^2 \tag{708}$$

As a result, we can parametize it as:

$$t^2 = \frac{s}{\sqrt{3}} \tag{709}$$

so:

$$\vec{r}(s) = \vec{r}(t(s)) = \frac{s}{\sqrt{3}}\hat{i} + \frac{s}{\sqrt{3}}\hat{j} - \frac{s}{\sqrt{3}}\hat{k} \tag{710}$$

Definition: The curvature is defined as:

$$k = \left|\frac{d\phi}{ds}\right| \tag{711}$$

- Suppose we have some arbitrary function $y(x)$. We can define ϕ such that:

$$\frac{dy}{dx} = y' = \tan\phi \implies \phi = \tan^{-1}(y') \tag{712}$$

Recall that $\frac{ds}{dx} = \sqrt{1 + y'^2}$ so that we have:

$$\frac{d\phi}{dx} = \frac{y''}{1 + y'^2} \tag{713}$$

We can also use the chain rule:

$$\frac{d\phi}{dx} = \frac{d\phi}{ds} \cdot \frac{ds}{dx} = \frac{d\phi}{ds}\sqrt{1 + y'^2} \tag{714}$$

We can set our two expressions for $\frac{d\phi}{ds}$ equal to each other:

$$k = \left|\frac{d\phi}{ds}\right| = \frac{|y''|}{(1 + y'^2)^{3/2}} \tag{715}$$

- We can apply similar reasoning for a parametric curve. We have:

$$\frac{dy}{dx} = \frac{y'}{x'} \tag{716}$$

and:

$$\frac{d^2y}{dx^2} = \frac{x'y'' - y'x''}{x'^3} \tag{717}$$

so the curvature is given as:

$$k = \frac{y''}{(1+y')^{3/2}} \tag{718}$$

$$= \frac{\left|\frac{x'y''-y'x''}{x'^3}\right|}{1+\left(\frac{y'}{x'}\right)^2} \tag{719}$$

$$= \frac{|x'y''-y'x''|}{(x'^2+y'^2)^{3/2}} \tag{720}$$

Example 115: For a straight line $y = mx + B$, we have $y' = m$ and $y'' = 0$. This directly leads to $k = 0$.

Example 116: For a circle, the curve can be parametrized by:

$$\vec{r} = r\cos t\hat{i} + r\sin t\hat{j} \tag{721}$$

This gives:

$$k = \frac{|r^2\sin^2 t + r^2\cos^2 t|}{(r^2\sin^2 t + r^2\cos^2 t)^{3/2}} \implies = \frac{1}{r} \tag{722}$$

This leads to the following definition:

Definition: The radius of curvature is defined as $\rho = \frac{1}{k}$.

Example 117: An ellipse can be parametrized via:

$$x = a\cos t \qquad y = b\sin t \tag{723}$$

$$x' = -a\sin t \qquad y' = b\cos t \tag{724}$$

$$x'' = -a\cos t \qquad y'' = -b\sin t \tag{725}$$

and therefore:

$$k = \frac{|ab\sin^2 t + ab\cos^2 t|}{[a^2\sin^2 t + b^2\cos^2 t]^{3/2}} = \frac{ab}{[a^2\sin^2 t + b^2\cos^2 t]^{3/2}} \tag{726}$$

At $t = 0$ we have $x = a$ so the curvature reaches the maximum value of $k = \frac{a}{b^2}$. At $t = \frac{\pi}{2}$ we have $y = b$ and the curvature reaches the minimum value of $k = \frac{b}{a^2}$, assuming $a > b$.

- In three dimensions, we can define:

$$\vec{T} = \cos\phi\hat{i} + \sin\phi\hat{j} \tag{727}$$

such that:

$$\left\|\frac{d\vec{T}}{ds}\right\| = \left\|\frac{d\phi}{ds}\right\|\sqrt{\sin^2\phi + \cos^2\phi} = \left\|\frac{d\phi}{ds}\right\| = k \tag{728}$$

Definition: For three dimensional curves, we have $k = \left\|\frac{d\vec{T}}{ds}\right\|$

- For a curve $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, we have:

$$k = \left\| \frac{d\vec{T}}{dt} \cdot \frac{dt}{ds} \right\| = \frac{\|\vec{T}'\|}{\|\vec{r'}\|} \tag{729}$$

Example 118: Suppose we have a circular helix: $\vec{r}(t) = 3\sin t\hat{i} + 3\cos t\hat{j} + 4t\hat{k}$. We then have:

$$\frac{d\vec{r}}{dt} = 3\cos t\hat{i} - 3\sin t\hat{j} + 4\hat{k} \tag{730}$$

We have $\|\vec{r'}\| = 5$ from earlier, and:

$$\vec{T} = \frac{\frac{d\vec{r}}{dt}}{\|\frac{d\vec{r}}{dt}\|} = \frac{3}{5}\cos t\hat{i} - \frac{3}{5}\sin t\hat{j} + \frac{4}{5}\hat{k} \tag{731}$$

so:

$$\frac{d\vec{T}}{dt} = -\frac{3}{5}\sin t\hat{i} - \frac{3}{5}\cos t\hat{j} \implies \left\|\frac{d\vec{T}}{dt}\right\| = \frac{3}{5} \tag{732}$$

Therefore, we have $k = \frac{3}{25}$ and $\rho = \frac{25}{3} = 8\frac{1}{3}$ so stretching it longitudinally drastically increases the radius of curvature.

Idea: Another method of calculating the curvature is given by:

$$k = \frac{\|\vec{r'}(t) \times \vec{r'}'(t)\|}{\|\vec{r'}(t)\|^3} \tag{733}$$

- We can also introduce the idea of normal and binormal vectors. Let us define:

$$\vec{T} = \frac{\vec{r'}}{\|\vec{r}\|} \tag{734}$$

such that:

$$\vec{T} \cdot \vec{T}' = 0 \tag{735}$$

and we can define the principal unit normal to be:

$$\vec{N}(t) \equiv \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} \tag{736}$$

The osculating plane is then defined by $\vec{N}$ and $\vec{T}$. Intuitively, these two vectors define a plane that the curve is "on" at some time t.

Example 119: For a straight line, we have $\vec{T}'(t) = 0$, but there is no normal vector since there is no curvature.

Definition: The **binormal vector** is given by

$$\vec{B}(t) = \vec{T} \times \vec{N} \tag{737}$$

and gives the vector normal to the osculating plane.

Example 120: Let us go back to the circular helix example, with:

$$\vec{r}(t) = 3\sin t\hat{i} + 3\cos t\hat{j} + 4t\hat{k} \tag{738}$$

We have already worked through that:

$$\vec{T'} = -\frac{3}{5}\sin t\hat{i} - \frac{3}{5}\cos t\hat{j} \tag{739}$$

and

$$\|\vec{T'}(t)\| = \frac{3}{5} \tag{740}$$

and therefore the principal unit normal to be:

$$\vec{N'}(t) = -\sin t\hat{i} - \cos t\hat{j} \tag{741}$$

The binormal vector is then:

$$\vec{B} = \left(\frac{3}{5}\cos t, -\frac{3}{5}\sin t, \frac{4}{5}\right) \times (-\sin t, -\cos t, 0) = \left(\frac{4}{5}\cos t, -\frac{4}{5}\sin t, -\frac{3}{5}\right) \tag{742}$$

The equation of a plane is given by:

$$n_1(x-x_0) + n_2(y-y_0) + n_3(z-z_0) = 0 \tag{743}$$

Plugging in (n_1, n_2, n_3) and (x_0, y_0, z_0) gives:

$$4\cos tx - 4\sin ty - 3z = -12t \tag{744}$$

Idea: Sometimes, questions will ask for the osculating plane at a specific point. This is usually easier to do by first substituting in numbers, rather than finding a general solution.

- $\vec{T}$, $\vec{N}$, and $\vec{B}$ form a set of orthogonal unit vectors that span $\mathbb{R}^3$.

26 Motion in Space: Velocity and Acceleration

- If $\vec{r}(t)$ is used to describe a location in space, then we can define $\vec{r'}(t) = \vec{v}(t)$ is the velocity and $\vec{r''} = \vec{v'}(t) = \vec{a}(t)$ is the acceleration.
- Let us examine circular motion about the origin. We have:

$$\vec{r}(t) = a\cos(\theta)t\hat{i} + a\sin(theta)t\hat{j} \tag{745}$$

Note that a positive θ' represents a counterclockwise rotation.

Definition: The angular velocity is written as θ' and has units [/s]. The angular speed is the absolute value $|\theta'|$.

Let the angular speed be written as ω. Then:

$$\vec{r} = \cos\omega t\hat{i} + r\sin\omega t\hat{j} \tag{746}$$
$$\vec{v} = -r\sin\omega t + r\omega\cos\omega t\hat{j} \tag{747}$$
$$\vec{a} = -r\omega^2\cos\omega t\hat{i} - r\omega^2\sin\omega t\hat{j} \tag{748}$$
$$= -\omega^2\vec{r} \tag{749}$$

Warning: Note that we have to work with radians here. Using degrees, the math does not work out in the same way.

- We begin a look at vector mechanics. Newton's second law of motion can be written as:

$$\vec{F}(t) = m\vec{r''}(t) \tag{750}$$

Momentum is written as $\vec{p} = m\vec{v}$. In fact, Newton initially wrote out his law as:

$$\vec{F}(t) = \vec{p}'(t) \tag{751}$$

which is true for relativistic speeds *and* changing mass! (though Newton couldn't have possibly known this)

- Conservation of momentum is equivalent to:

$$\vec{p}' = \vec{F} = 0 \tag{752}$$

This means that if there is no external force, then the momentum is conserved.

Definition: Let us define angular momentum as:

$$\vec{L} \equiv \vec{r} \times \vec{p} = m\vec{r} \times \vec{v} \tag{753}$$

Example 121: Let $\vec{r}(t) = r\cos\omega t\hat{i} + r\sin\omega t\hat{j}$. LEt us calculate the angular momentum:

$$\begin{aligned}\vec{L} &= m\vec{r} \times \vec{v} &(754)\\ &= m(0, 0r^2\omega t + r^2\omega\sin^2\omega t) &(755)\\ &= (0, 0, mr^2\omega) &(756)\end{aligned}$$

Therefore, the angular momentum here is actually constant. We often write $\|\vec{L}\| = mr^2\omega = mrv$.

Example 122: Suppose we have uniform motion in a straight line. We have $\vec{r} = \vec{r}_0 + t\vec{v}$ and therefore the angular momentum is:

$$\vec{L} = m\vec{r} \times \vec{v} = m(\vec{r}_0 + t\vec{v}) \times \vec{v} = m\vec{r}_0 \times \vec{v} \tag{757}$$

which is a constant.

Definition: Torque is defined as the rate of change of the angular momentum:

$$\vec{\tau} \equiv \vec{L}' = \vec{r} \times \vec{F} \tag{758}$$

Definition: A force $\vec{F}$ is a central force if $\vec{F}(t)$ is always parallel to $\vec{r}$. As a result, $\vec{r} \times \vec{F} = 0$ and angular momentum is conserved (i.e. gravity)

- In general, the acceleration vector is composed of two components:

$$\vec{a} = \vec{a}_{\text{normal}} + \vec{a}_{\text{tangential}} \tag{759}$$

Recall that $\vec{T} = \frac{\vec{v}}{\frac{ds}{dt}}$ so we can write the velocity as:

$$\vec{v} = \frac{ds}{dt}\vec{T} \tag{760}$$

where $\vec{T}$ is the unit vector in the direction of motion. Taking the derivative of both sides:

$$\vec{v}' = \vec{a} = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\frac{d\vec{T}}{dt} \tag{761}$$

Recall that

$$\frac{dT}{dt} = \left\|\frac{d\vec{T}}{dt}\right\|\vec{N} \tag{762}$$

and

$$k = \frac{\|\frac{d\vec{T}}{dt}\|}{|ds/dt|} \tag{763}$$

and so the acceleration can be written as:

$$\vec{a} = \vec{a}_r + \vec{a}_N = \frac{d^2 s}{dt^2}\vec{T} + k\left(\frac{ds}{dt}\right)^2 \vec{N} \tag{764}$$

Idea: Note that:

$$k\left(\frac{ds}{dt}\right)^2 \vec{N} = kv^2 = \frac{v^2}{\rho} \tag{765}$$

where ρ is the radius of curvature. This is an equation that you should be familiar with.

27 Particle Motions in Electric and Magnetic Fields

- The Lorentz Force is given by:

$$\vec{F} = q\vec{v} \times \vec{B} \tag{766}$$

and the electric field is given by:

$$\vec{F} = q\vec{E} \tag{767}$$

where $\vec{E}$ and $\vec{B}$ is the electric and magnetic field, respectively.

Example 123: Let us analyze the confinement of hot plasma by a magnetic field (fusion energy). Matter above $10,000$K is hot enough to be ionized and turn into a "soup" of positively charged nucleons and negatively charged electrons, which is known as **plasma**. There are other properties of plasma, but it is not important. They will be expanded on in PHY293 and PHY294.

For fusion, we need $q = \pm 1.6 \times 10^{-19}$C, $v_{\text{electrons}} = 3 \times 10^7 \text{m}\,\text{s}^{-1}$, and $B_0 \approx 10$T. Let:

$$\vec{B} = B_0\hat{k} \tag{768}$$

And suppose the electron is moving along the positive x axis. From Newton's second law:

$$\vec{F} = m\vec{v}' = qB_0\vec{v} \times \hat{k} \tag{769}$$

or:

$$\vec{v'} = \omega_L \vec{v} \times \vec{k} \tag{770}$$

where $\omega_L = \dfrac{qB_0}{m}$ is the Larmor frequency. The velocity can be written in its general form:

$$\vec{v})t_{=} v_x(t)\hat{i} + v_y(t)\hat{j} + v_z(t)\hat{k} \tag{771}$$

Differentiating:

$$v_x'\hat{i} + v_y'\hat{j} + v_z'\hat{k} = \omega_L v_y\hat{i} - \omega_L v_x\hat{j} \tag{772}$$

Thus:

$$v_x'(t) = \omega_L v_y(t) \tag{773}$$
$$v_y'(t) = -\omega_L v_x(t) \tag{774}$$
$$v_z'(t) = 0 \tag{775}$$

This is known as coupled equations. The third equation is the easiest to solve, and the solution is $v_z =$ constant. To solve the other two equations, we can differentiate the first equation:

$$v_x'' = \omega_L v_y' \tag{776}$$

And substituting in v_y, we have:

$$v_x'' = -\omega_L^2 v_x \tag{777}$$

And the general solution is:

$$v_x(t) = A\sin(\omega_L t + \phi) \tag{778}$$

We can also solve for v_y:

$$v_y = \frac{1}{\omega_L} v_x' = A\cos(\omega_2 t + \phi) \tag{779}$$

Thus:

$$\vec{v}(t) = A\sin(\omega_L t + \phi)\hat{i} + A\cos(\omega_L t + \phi)\hat{j} + c\hat{k} \tag{780}$$

Integrating, we get:

$$\vec{r}(t) = \left[-\frac{A}{\omega_L t + \phi} + D_x\right]\hat{i} + \left[\frac{A}{\omega_L}\sin(\omega_L t + \phi) + D_y\right]\hat{j} + [ct + D_z]\hat{k} \tag{781}$$

We can set $\phi = D_x = D_y = D_z = 0$, and letting $r_L = \dfrac{A}{\omega_L}$, we get:

$$\vec{r}(t) = -r_L\cos(\omega_L t)\hat{i} + r_L\sin(\omega_L t)\hat{j} + ct\hat{k} \tag{782}$$

which describes a helix. For an electron, $\omega_L = 10^{13}$Hz and for a proton which corresponds to microwaves, $\omega_L = 6\times 10^9$Hz which corresponds to radio frequencies. Depending on the initial conditions, we can have D_x, D_y, D_z, ϕ take on different values.

- Suppose we have a magnetic field out of the page and an electric field pointing upwards, we have:

$$\vec{B} = (0, 0, B_z) \tag{783}$$
$$\vec{E} = (0, E_y, 0) \tag{784}$$

The differential equations then give us:

$$v_x'(t) = \frac{q}{m} v_y B_z \tag{785}$$
$$v_y'(t) = -\frac{q}{m} v_y B_z + \frac{qE_y}{m} \tag{786}$$
$$v_z'(t) = 0 \tag{787}$$

The solution to this gives:

$$v_x = v_\perp \sin\omega t + \frac{E_x}{B_z} \tag{788}$$
$$v_y = v_\perp \cos\omega t \tag{789}$$
$$v_z = \text{constant} \tag{790}$$

where we can define the drift velocity to be $v_d = \dfrac{E_y}{B_z}$ or:

$$\vec{v}_D = \frac{\vec{E}\times\vec{B}}{B^2} \tag{791}$$

28 Partial Derivatives

- There are a lot of things that need to be described by several variables (e.g. temperature as a function of x, y, z, and time t)

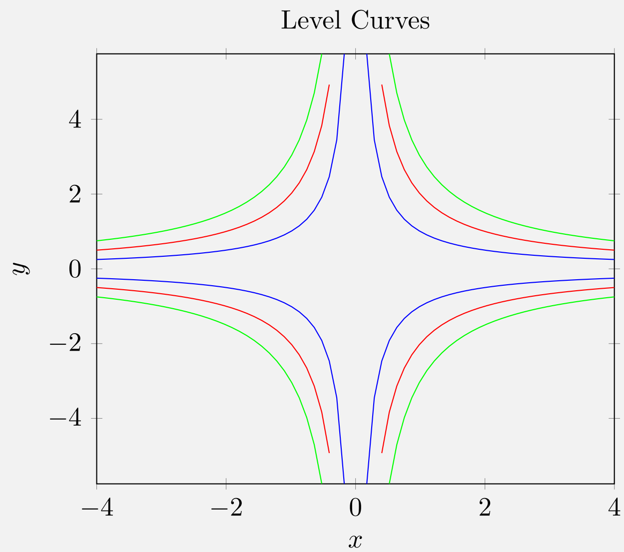
Example 124: Suppose that:

$$f(x, y) = \frac{1}{\sqrt{ax^2 - y^2}} \tag{792}$$

The domain is when $9x^2 > y^2 \implies |y| < 3|x|$. This represents a *region* in the xy plane, instead of a *line*. The range is when $f > 0$ and occurs for $f \in (0, \infty)$.

- Consider $z = \sqrt{a^2 - x^2 - y^2}$. We can define a **level curve**.

Example 125: Let $f(x,y) = xy$. Set $f = c$ such that: $y = \dfrac{c}{x}$. We can then plot out the function for different values of c:



- We can denote a function of three dimensions as $f(x,y,z) = f(\vec{x})$. For example:

$$f(x,y) = \frac{\sin x \sin y}{xy} \tag{793}$$

Note that $f(0,0), f(x,0), f(0,y)$ are not define. But does the limit of:

$$\lim_{(x,y)\to(0,0)} \frac{\sin x \sin y}{xy} = 1 \tag{794}$$

exist?

Definition: Let f be a function whose domain includes the region arbitrarily close to but not necessarily including $\vec{x}_0$. Then:

$$\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = L \tag{795}$$

if and only if for each $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < \|\vec{x} - \vec{x}_0\| < \delta$ then $|f(\vec{x}) - L| < \epsilon$.

Example 126: Again, let $f(x,y) = \dfrac{x^2y + y^2}{x + y^2}$. Suppose we wish to calculate:

$$\lim_{\vec{x}\to\vec{0}} f(x,y) \tag{796}$$

Our first path is when we set $x = 0$ and approach it from the y axis. Then $f(0,y) = \dfrac{y^2}{y^2}$ whose limit:

$$\lim_{y\to 0} f(0,y) = 1 \tag{797}$$

For our second path, we approach it from the x axis. Note that $f(x,0) = 0/x$ such that:

$$\lim_{x\to 0} f(x,0) = 0 \tag{798}$$

For our third path, we can choose an arbitrary path, say $y = \sqrt{x}$. Then we have:

$$f(x, x^{1/2}) = \frac{1}{2}(x^{3/2} + 1) \tag{799}$$

Note that for this:

$$\lim_{x\to 0^+} f(x, x^{1/2}) = \frac{1}{2} \tag{800}$$

We've picked three paths and they lead to different answers, so we can claim that the limit does not exist.

Example 127: Suppose we have the limit $\lim_{\vec{x}\to\vec{0}} \frac{x^2y^4}{x^4+y^8}$. For our path, let's let $y = mx$. Then:

$$f(x, mx) = \frac{m^4x^6}{x^4+m^8x^8} = \frac{m^4x^2}{1+m8x^4} \tag{801}$$

Letting $\vec{x} \to \vec{0}$ in a straight line, we get $f(x, mx) \to 0$. However, if we have a parabola and try $f(y^2, y)$, we get:

$$f(y^2, y) = \frac{y^4y^4}{y^8+y^8} = \frac{y^8}{2y^8} \to \frac{1}{2} \tag{802}$$

Since the two paths lead to different answers, the limit does not exist.

Warning: It's easy to determine if a function doesn't have a limit by finding counterexamples. However, it is much harder to prove a limit actually exists.

Example 128: Prove that $\lim_{\vec{x}\to\vec{0}} \frac{2xy^2}{x^2+y^2} = 0$.

1. Suppose an $\epsilon > 0$ is imposed.
2. It is required that $|f - L| < \epsilon$ or:

$$\left|\frac{2xy^2}{x^2+y^2} - 0\right| = \frac{2y^2|x|}{x^2+y^2} \tag{803}$$

$$< \epsilon \tag{804}$$

3. when

$$0 < \sqrt{x^2+y^2} < \epsilon \tag{805}$$

4. Note that:

$$y^2 \leq x^2 + y^2 \tag{806}$$

$$\frac{y^2}{y^2} \geq \frac{y^2}{x^2+y^2} \tag{807}$$

$$\frac{2y^2|x|}{x^2+y^2} \leq \frac{2y^2|x|}{y^2} \tag{808}$$

$$= 2|x| \tag{809}$$

Thus, we have:

$$2|x| = 2\sqrt{x^2} \leq 2\sqrt{x^2+y^2} < 2\delta \tag{810}$$

We thus choose $\delta = \frac{\epsilon}{2}$.

5. Given $\sqrt{x^2+y^2} < \frac{\epsilon}{2}$, we have:

$$\left|\frac{2y^2x}{x^2+y^2}\right| < \epsilon \tag{811}$$

and then:

$$\lim_{\vec{x}\to\vec{0}} \frac{2xy^2}{x^2+y^2} = 0 \tag{812}$$

- A multivariable function is continuous if:

$$\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) = f(\vec{x}_0) \tag{813}$$

Theorem: The continuity of composite functions: If g is continuous at $\vec{x}_0$ and f is continuous at the number $g(\vec{x}_0)$, then $f(g(\vec{x}))$ is continuous at $\vec{x}_0$.

- If $f(\vec{x})$ is continuous at $\vec{x}_0$, then:

$$\lim_{x\to x_0} f(x, y_0) = f(x_0, y_0) \tag{814}$$

$$\lim_{y\to y_0} f(x_0, y) = f(x_0, y_0) \tag{815}$$

Warning: This is not an if and only if statement. It's possible for the specific limit to exist when travelling across such paths but the overall limit doesn't exist.

29 Partial Derivatives

- Suppose we have the top half of a sphere with radius 5, then:

$$f = \sqrt{25 - x^2 - y^2} \tag{816}$$

Suppose we are interested in what happens if we move along the line $y = 2$.

Definition: The partial derivative of $f(x, y)$ is given by:

$$f_x(x, y) = \frac{\partial}{\partial x} f(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h} \tag{817}$$

or for the partial derivative with respect to y:

$$f_y(x, y) = \frac{\partial}{\partial y} f(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h} \tag{818}$$

This can be extended to an arbitrary number of dimensions.

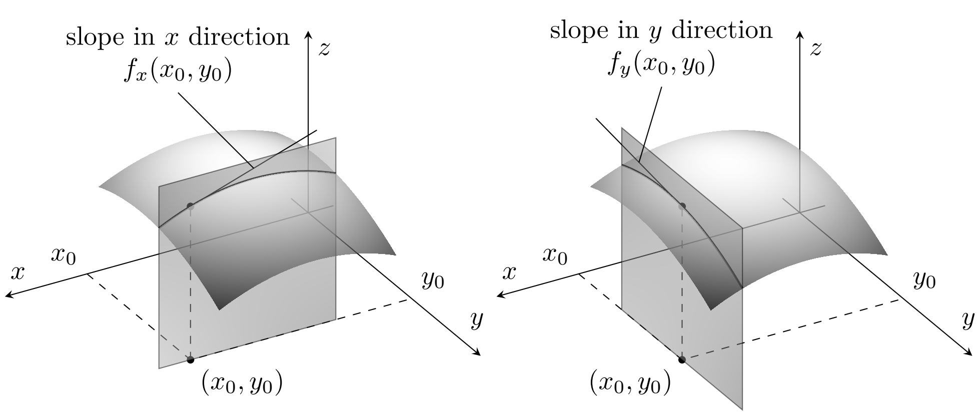
Example 129: Suppose we have $f(x, y) = e^{x^2y^3}$, then we have:

$$f_x = 2xy^2e^{x^2y^3} \tag{819}$$

and:

$$f_y = 3y^2x^2e^{x^2y^3} \tag{820}$$

- We can visualize using the diagram below:



Example 130: Suppose $f(x,y,z) = \ln\left(\frac{x}{y}\right) - ye^{xz}$. Then the partial derivatives are:

$$f_x = \frac{1}{x} - yze^{xz} \tag{821}$$
$$f_y = -\frac{1}{y} - e^{xz} \tag{822}$$
$$f_z = -xye^{xz} \tag{823}$$

Example 131: Suppose $f(r,\theta,\phi) = r^2\sin\theta\cos\phi$. Then:

$$h_r = 2r\sin\theta\cos\phi \tag{824}$$
$$h_\theta = r^2\cos\theta\cos\phi \tag{825}$$
$$h_\phi = -r^2\sin\theta\sin\phi \tag{826}$$

- We can also have mixed partials, such as:

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial x} \to \frac{\partial^2 f}{\partial x^2} \tag{827}$$
$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} \to \frac{\partial^2 f}{\partial y\partial x} \tag{828}$$

Theorem: Clairaut's Theorem says that:

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial^2 f}{\partial x\partial y} \tag{829}$$

on every open set on which f and its partials $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x\partial y}, \frac{\partial^2 f}{\partial y\partial x}$ are continuous.

- This can be extended to multiple variables.

Example 132: Let $f(x,y) = \cos(xy^2)$, we have:

$$\frac{\partial f}{\partial x} = -\sin(xy^2)y^2 \tag{830}$$
$$\frac{\partial f}{\partial y} = -\sin(xy^2)\cdot 2xy \tag{831}$$
$$\frac{\partial^2 f}{\partial y\partial x} = -2y\sin(xy^2) - y^2\cos(xy^2)\cdot 2xy \tag{832}$$
$$\frac{\partial^2 f}{\partial x\partial y} = -2y\sin(xy^2) - 2xy\cos(xy^2)y^2 = \frac{\partial^2 f}{\partial y\partial x} \tag{833}$$

- Partial differential equations describe differential equations with several variables. For example, Laplace's equation is given by:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \tag{834}$$

The one-dimensional wave equation is given by:

$$\frac{\partial^2 f}{\partial t^2} = a^2\frac{\partial^2 f}{\partial x^2} \tag{835}$$

where a represents the speed of the wave.

30 Directional Derivatives and the Gradient Vector

- We can attempt to define differentiability if the directional derivative exists:

$$\lim_{\vec{h}\to\vec{0}} \frac{f(\vec{x}_0+\vec{h})-f(\vec{x}_0)}{\vec{h}} = \lim_{\vec{h}\to\vec{0}} \frac{f(x_0+h_1, y_0+h+2, z_0+h_3)-f(x_0,y_0,z_0)}{\vec{h}} \tag{836}$$

However, we cannot simply divide two vectors. We also can't let the denominator be $\|\vec{h}\|$ because we then lose information about the direction. Therefore, we need to re-define the derivative:

Definition: If $\lim_{h\to 0} \frac{g(h)}{|h|} = 0$, then we can say that $g(h) = o(h)$ where $o(h)$ is "'little-o"

- We can start with the one-dimensional case, and write:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \tag{837}$$

$$\implies \lim_{h\to 0} \frac{f(x+h)-f(x)-f'(x)h}{h} = 0 \tag{838}$$

Therefore:

$$[f(x+h)-f(x)] - f'(x)h = o(h) \tag{839}$$

We want to look for terms that approach zero faster than $h \to 0$, so we can let them be $o(h)$ and leave them out of the final definition.

Example 133: Let $f(x) = x^2$. Then: $f(x+h)-f(x) = (x+h)^2 - x^2 = 2xh + h^2$. Note that:

$$\lim_{h\to 0} \frac{h^2}{h} = \lim_{h\to 0} h = 0 \tag{840}$$

so we can say that h^2 is $o(h)$, so from our re-definition, we look for the term:

$$(\text{term}) \cdot h \tag{841}$$

where the term gives the derivative. Therefore, $f'(x) = 2x$.

Idea: The overall idea is to write the derivative using only the numerator:

$$f(x+h) - f(x) \tag{842}$$

and looking for only the *important* terms and leaving out the unimportant terms. We can define the unimportant terms to be any terms that are $o(h)$. Once we have the important terms, they should (at least in the $1d$ case), be in the form of:

$$(\text{derivative}) \cdot h \tag{843}$$

and we can read out the derivative without having to divide.

Definition: We say that f is differentiable at $\vec{x}$ if and only if there exists a vector $\vec{y}$ such that: $f(\vec{x}+\vec{h})-f(\vec{x}) = \vec{y}\cdot\vec{h} + o(\vec{h})$ where:

$$\vec{y} = \nabla f(\vec{x}) \tag{844}$$

is known as the **gradient** of f.

Example 134: For example, let $f(x,y)=x+y^2$ and let $\vec{h}=(h_1,h_2)$. Then:

$$\begin{aligned} f(\vec{x}+\vec{h})-f(\vec{x}) &= f(x+h,y+h)-f(x,y) && (845)\\ &= x+h_1+(y+h_2)^2-x-y^2 && (846)\\ &= h_1+2yh_2+h_2^2 && (847)\\ &= (1\hat{i}+2y\hat{j})\cdot\vec{h}+h_2^2 && (848)\end{aligned}$$

To finish, we need to demonstrate the the excess term $h_2^2=o(h)$. To do this, let $g(\vec{h})=h_2^2=(h_2\hat{j})\cdot(h_1\hat{i}+h_2\hat{j})$. We write it as a dot product because we wish to divide by the magnitude later. Therefore, we can write that $g(\vec{h})=h_2\hat{j}\cdot\vec{h}$ and:

$$\begin{aligned} \frac{|g(\vec{h})|}{\|\vec{h}\|} &= \frac{\|h_2\hat{j}\|\|\vec{h}\|\cos\theta}{\|\vec{h}\|} && (849)\\ &\leq \frac{\|h_2\hat{j}\|\|\vec{h}\|}{\|\vec{h}\|} && (850)\\ &= |h_2| && (851)\end{aligned}$$

We know that $h_2\to 0$ as $\vec{h}\to\vec{0}$ so $g(\vec{h})=h_2^2$ is $o(\vec{h})$ and we can claim that:

$$\nabla f(\vec{x})=1\hat{i}+2y\hat{j} \tag{852}$$

Example 135: Let $f(x,y,z)=xyz$ and let $\vec{h}=(h_1,h_2,h_3)$. Then:

$$\begin{aligned} f(\vec{x}+\vec{h})-f(\vec{x}) &= (x+h_1)(y+h_2)(z+h_3)-xyz && (853)\\ &= xyz+xyh_3+xh_2z+xh_2h_3+h_1yz+h_1yh_3+h_1h_2z+h_1h_2h_3-xyz && (854)\\ &= (yz\hat{i}+xz\hat{j}+xy\hat{k})\cdot\vec{h}+xh_2h_3+yh_1h_3+zh_1h_2+h_1h_2h_3 && (855)\end{aligned}$$

Without loss of generality, we show that the first remainder term is $o(h)$. Consider $g(\vec{h})=xh_2h_3=(xh_2\hat{j})\cdot\vec{h}$. Therefore:

$$\frac{|g(\vec{h})|}{\|\vec{h}\|}=\frac{|x||h_2||\vec{h}\|\cos\theta}{\|\vec{h}\|}\leq|xh_2| \tag{856}$$

Now taking the limit:

$$\lim_{\vec{h}\to\vec{0}}|xh_2|=0 \tag{857}$$

so xh_2h_3 is $o(\vec{h})$, and so are the other terms. Therefore, we have:

$$\nabla f(\vec{x})=yz\hat{i}+xz\hat{j}+xy\hat{k} \tag{858}$$

Note the symmetry.

Theorem: In cartesian coordinates:

$$\nabla f(x,y)=\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j} \tag{859}$$

or equivalently written as:

$$\nabla f(\vec{x})=f_x\hat{i}+f_y\hat{j}+f_z\hat{i} \tag{860}$$

- Note that $\vec{x}$ is a vector, $f(\vec{x})$ is not a vector, but $\nabla f(\vec{x})$ is a vector.

Example 136: Let $f(\vec{x}) = xy^2z^3$. We have:

$$\frac{\partial f}{\partial x} = y^2z^3 \quad (861)$$

$$\frac{\partial f}{\partial y} = 2xyz^3 \quad (862)$$

$$\frac{\partial f}{\partial z} = 3xy^2z^2 \quad (863)$$

so the gradient is:

$$\nabla f = (y^2z^3, 2xy^3, 3xy^2z^2) \quad (864)$$

Example 137: Suppose we have a vector function $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. The magnitude is:

$$r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

and taking the gradient of this yields:

$$\nabla r = \frac{\vec{r}}{r} \quad (865)$$

- For a directional derivative, we can define it using:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(\vec{x}_0 + h\vec{i}) - f(\vec{x}_0)}{h} \quad (866)$$

$$(867)$$

and similar for the other partial derivatives.

Definition: The directional derivative of f at $\vec{0}$ in the direction $\hat{u}$ is given by:

$$f_{\hat{u}}(\vec{x}_0) = \lim_{h\to 0} \frac{f(\vec{x}_0 + h\hat{u}) - f(\vec{x}_0)}{h} \quad (868)$$

- The directional derivative can also be expressed as:

$$f_{\hat{u}}(\vec{x}_0) = \nabla f(\vec{x}_0) \cdot \hat{u} \quad (869)$$

Proof. We can write:

$$f(\vec{x} + \vec{h}) - f(\vec{x}) - \nabla f(\vec{x}) \cdot \vec{h} + o(\vec{h}) \quad (870)$$

here: $\vec{h} = h\hat{u}$. Using this, the above is also equal to:

$$\nabla f(\vec{x}) \cdot h\hat{u} + o(\vec{h}) \quad (871)$$

and therefore:

$$\frac{f(\vec{x} + h\hat{u}) - f(\vec{x})}{h} = \nabla f \cdot \hat{u} + \frac{o(\vec{h})}{h} \quad (872)$$

and taking the limit as $h \to 0$, we have the desired result. □

Example 138: Suppose we have some temperature gradient $T(x, y)$ where $\frac{\partial T}{\partial y} = 4°\,°\mathrm{C\,m^{-1}}$ and $\frac{\partial T}{\partial x} = 3°\,°\mathrm{C\,m^{-1}}$. Suppose we move in the direction $\hat{u} = \cos\theta\hat{i} + \sin\theta\hat{j}$. Then:

$$T_{\hat{u}} = \left(\frac{\partial T}{\partial x}\hat{i} + \frac{\partial T}{\partial y}\hat{j}\right)(\cos\theta\hat{i} + \sin\theta\hat{j}) = 3\cos\theta + 4\sin\theta \quad (873)$$

Example 139: Suppose we have a parabolic hill described by $z(x,y) = 20 - x^2 - y^2$ and we move straight up, or $\hat{u} = (0,-1)$. Then:

$$\frac{\partial f}{\partial x} = -2x \tag{874}$$

$$\frac{\partial f}{\partial y} = -2y \tag{875}$$

so:

$$z_{\hat{u}} = (-2x, -2y) \cdot (0,-1) = 2y \tag{876}$$

- Note that:

$$\begin{aligned} |f_{\hat{k}}(\vec{x})| &= |\nabla f \cdot \hat{u}| &(877)\\ &= \|\nabla f\| \|\hat{u}\| |\cos\theta| &(878)\\ &\leq \|\nabla f\| &(879) \end{aligned}$$

Example 140: Suppose that $z = f(x,y) = A + x + 2y - x^2 - 3y^2$ and we wish to find the steepest path down starting from $(0,0,A)$. We know that:

$$\frac{\partial f}{\partial x} = 1 - 2x \tag{880}$$

$$\frac{\partial f}{\partial y} = 2 - 6y \tag{881}$$

such that:

$$\nabla f = (1-2x)\hat{\imath} + (2-6y)\hat{\jmath} \implies -\nabla f = (2x-1)\hat{\imath} + (6y-2)\hat{\jmath} \tag{882}$$

The curve is given by:

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} \tag{883}$$

where $x'(t) = 2x(t) - 1$ and $y'(t) = 6y(t) - 2$. This is in parametric form and we can convert to cartesian form by writing the derivative as:

$$\frac{dy}{dx} = \frac{6y-2}{2x-1} \tag{884}$$

and solving this differential equation to get:

$$3y = (2x-1)^3 + 1 \tag{885}$$

31 The Chain Rule

- Suppose that we have a certain path $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$ travelling through a certain temperature field $T(x,y)$. How might we find:

$$\frac{d}{dt}T(\vec{r}(t)) \tag{886}$$

Theorem: The chain rule along a curve is given by:

$$\begin{aligned} \frac{d}{dt}f(\vec{r}(t)) &= \nabla f(\vec{r}(t)) = \vec{r}'(t) &(887)\\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dx}{dz} &(888)\\ &= \nabla f \cdot \vec{T} \cdot \left(\frac{ds}{dt}\right) &(889) \end{aligned}$$

Example 141: Suppose that $\vec{r}(t) = t^3\hat{i} + \cos t\hat{j}$ and $T(x,y) = xy^2$. Then:

$$\nabla T = (y^2, 2xy) \tag{890}$$
$$\vec{r}' = (3t^2, -\sin t) \tag{891}$$

such that:

$$\frac{dT}{dt} = \nabla T = F' \tag{892}$$
$$= y^2 \cdot 3t^2 - 2xy\sin t \tag{893}$$
$$= 3t^2\cos^2 t - 2t^3\cos t\sin t \tag{894}$$

and:

$$T = xy^2 = t^3\cos^2 t \tag{895}$$
$$\frac{dT}{dt} = 3t^2\cos^2 t - 2t^3\cos t\sin t \tag{896}$$

Example 142: Suppose we have a rectangular volume $V = \ell \cdot h \cdot d$. Say that ℓ, h, and d is increasing at $3\,\mathrm{m\,s^{-1}}$,$2\,\mathrm{m\,s^{-1}}$, and $5\,\mathrm{m\,s^{-1}}$, respectively. At $(\ell, h, d) = (2,3,4)$, how fast is the volume changing?

Let $\vec{q}(t) = (\ell, h, d)$. Then from the chain rule:

$$\frac{dV(t)}{dt} = \nabla V(\vec{q}(t)) \cdot \vec{q}'(t) \tag{897}$$
$$\implies \nabla V = (hd, \ell d, h\ell) \tag{898}$$
$$\implies \vec{q}'(t) = \left(\frac{d\ell}{dt}, \frac{dh}{dt}, \frac{dd}{dt}\right) = (3, -2, 5) \tag{899}$$
$$\frac{dV}{dt} = 3hd - 2\ell d + 5\ell h = 50\,\mathrm{m\,s^{-1}} \tag{900}$$

Note that we could also have solved this using single variable calculus by noting that:

$$V = (2+3t)(3-2t)(4+5t) \tag{901}$$

- We can take this idea even further. Say that:

$$x = x(t,s) \tag{902}$$
$$y = y(t,s) \tag{903}$$

then:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} \tag{904}$$
$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} \tag{905}$$

and in three dimensions:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t} \tag{906}$$

- We can also revisit implicit differentiation. Suppose that we have a function:

$$u(x,y) = 0 \tag{907}$$

How might we find $\frac{dy}{dx}$ =?. We can parametrize this with $x = t$ and $y = y(t)$ to get $u = u(t, y(t))$. Differentiating:

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \tag{908}$$

Now, we use the fact that $u(t, y(t)) = 0$ to get that $\frac{du}{dt} = 0$. Since $x = t$, we have $\frac{dx}{dt} = 1$ and therefore $\frac{dy}{dt} = \frac{dx}{dt}$. This then gives us:

$$0 = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} \tag{909}$$

$$\implies \frac{dy}{dx} = -\frac{\partial u/\partial x}{\partial u/\partial y} \tag{910}$$

Example 143: Suppose that we have $x^4 + 4x^3y + y^4 = 1$. We can write this as:

$$u = x^4 + 4x^3y + y^4 - 1 = 0 \tag{911}$$

The partial derivatives are:

$$\frac{\partial u}{\partial x} = 4x^3 + 12x^2y \tag{912}$$

$$\frac{\partial u}{\partial y} = 4x^3 + 4y^3 \tag{913}$$

such that:

$$\frac{dy}{dx} = -\frac{4x^3 + 12x^2y}{4x^2 + 4y^3} = \frac{x^2(x + 3y)}{x^3 + y^3} \tag{914}$$

32 Tangent Planes and Linear Approximations

- To set up our problem, suppose we have a three dimensional surface $z(x, y) = 20 - x^2 - y^2$ and we want to find a *level curve* such that the altitude does not change. If we start at $(1, 2)$, then we can solve for the height C to be:

$$C = 20 - 1^2 - 2^2 = 15 \tag{915}$$

and the equation of the level curve would be:

$$x^2 + y^2 = 5 \tag{916}$$

- The radius vector is given by $\vec{r} = (1, 2)$ and the tangent vector is $\vec{t}$. Since they are perpendicular:

$$\vec{t} \cdot \vec{r} = 0 \implies t_1 1 + t_2 2 = 0 \tag{917}$$

If we choose $t_1 = 2$ and $t_2 = -$, then $\vec{t} = (2, -1)$ is the tangent vector. Note that the gradient is actually perpendicular to the tangent vector:

$$\nabla f \cdot \vec{t} = (-2x_1, -2y) \cdot (2, -1) = -4x + 2y \tag{918}$$

and at $(1, 2)$, the dot product is zero.

Theorem: The gradient of a curve will be perpendicular to the tangent of the level curve at a specific point.

Proof. Let $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$. We know that $\vec{t} = \vec{r}'(t)$. We can write:

$$f(\vec{r}(t)) = C \tag{919}$$

and applying chain rule, we have:

$$\frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}) \cdot \vec{r}' \tag{920}$$

$$= \frac{dC}{dt} = 0 \tag{921}$$

Therefore, $\nabla f(\vec{r}) \cdot \vec{r} = 0$ so the two are perpendicular. Note that this means this property is true for any curve that can be expressed as $f(x, y) = C$. $\square$

- Let $\vec{t} = \left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right)$ which gives:
$$\nabla f \cdot \vec{t} = \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} = 0 \tag{922}$$
We can use this to determine the equations of tangent and normal lines.
- If (x, y) is a point on the tangent line at a certain point (x_0, y_0), then:
$$(x - x_0, y - y_0) \cdot \nabla f = 0 \tag{923}$$
$$(x - x_0)\frac{\partial f}{\partial x}(x_0, y_0) + (y - y_0)\frac{\partial f}{\partial y}(x_0, y_0) = 0 \tag{924}$$
- For the normal line, let (x, y) be a point on the normal line. Then:
$$(x - x_0, y - y_0) \cdot \vec{t} = 0 \tag{925}$$
and applying the same equation, we get:
$$(x - x_0)\frac{\partial f}{\partial y}(x_0, y_0) - (y - y_0)\frac{\partial f}{\partial x}(x_0, y_0) = 0 \tag{926}$$
- We can also find the equation for the normal line to a curve $f(x, y) = C$ at (x_0, y_0). We start with the two-dimensional case:

Example 144: Suppose we have a curve $x^2 + y^2 = 9$. Then we have:
$$\frac{\partial f}{\partial x} = 2x \tag{927}$$
$$\frac{\partial f}{\partial y} = 2y \tag{928}$$
The tangent line is given by:
$$(x - x_0)2x_0 + (y - y_0)2y_0 = 0 \tag{929}$$
Suppose that $(x_0, y_0) = \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$. Then the tangent line is given by:
$$\left(x - \frac{3}{\sqrt{2}}\right)2\frac{3}{\sqrt{2}} + \left(y - \frac{3}{\sqrt{2}}\right)2\frac{3}{\sqrt{2}} = 0 \tag{930}$$
this gives:
$$x + y = \frac{6}{\sqrt{2}} \tag{931}$$
Similarly, the normal line is given by:
$$\left(x - \frac{3}{\sqrt{2}}\right)2\frac{3}{\sqrt{2}} - \left(y - \frac{3}{\sqrt{2}}\right)2\frac{3}{\sqrt{2}} = 0 \implies y = x \tag{932}$$

- Extending it to three dimensions. A level surface is given by: $f(x, y, z) = C$.
- We know that the gradient $\nabla f(x_0, y_0, z_0)$ is perpendicular to the level surface curve $f(x, y, z) = C$.

Proof. Let $F(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$ be any curve on the surface such that:
$$f(x(t), y(t), z(t)) = f(\vec{r}(t)) = C \tag{933}$$
Again:
$$\frac{d}{dt}f(\vec{r}(t)) = \frac{dc}{dt} = 0 \tag{934}$$
Applying the chain rule, we get:
$$\nabla f(\vec{r}(t)) \cdot \vec{r}'(t) = 0 \tag{935}$$
□

Example 145: Suppose we have a sphere $x^2 + y^2 + z^2 = 25$. The gradient is then:

$$\nabla f = (2x, 2y, 2z) = 2(x, y, z) \tag{936}$$

The equation of the tangent plane to surface is:

$$(\vec{x} - \vec{x}_0) \cdot \nabla f = 0 \tag{937}$$

At $(0, 5, 0)$, we have:

$$(x - x_0)2x_0 + (y - y_0)2y_0 + (z - z_0)2z_0 = 0 \implies y - 5 \cdot 10 = 0 \implies y = 5 \tag{938}$$

The normal line is given by:

$$\vec{r}(q) = \vec{x}_0 + q\nabla f(\vec{x}_0) \tag{939}$$

and the equations for the normal line are:

$$x = x_0 + qf_x \tag{940}$$
$$y = y_0 + qf_y \tag{941}$$
$$z = z_0 + qf_z \tag{942}$$

Example 146: Find the normal line at $xy^2 + 2z^2 = 12$ at $(1, 2, 2)$. We have:

$$f_x = y^2 = 4 \tag{943}$$
$$f_y = 2xy = 4 \tag{944}$$
$$f_z = 4z = 8 \tag{945}$$

and so:

$$x = 1 + 4a \tag{946}$$
$$y = 2 + 4q \tag{947}$$
$$z = 2 + 8q \tag{948}$$

Example 147: Suppose we have an offset sphere given by:

$$f = x^2 + y^2 + z^2 - 8x - 8y - 6z + 24 = 0 \tag{949}$$

and an ellipsoid centered at the origin:

$$g = x^2 + 3y^2 + 2z^2 = 9 \tag{950}$$

and specifically at the point $(2, 1, 1)$. Let us attempt to show that the sphere is tangent to the ellipsoid at this specific point. We do this via:

$$\nabla f = (2x - 8, 2y - 8, 2z - 6) \implies \nabla f(2, 1, 1) = (-4, -6, -4) \tag{951}$$

and:

$$\nabla g = (2x, 6y, 4z) \implies \nabla f(2, 1, 1) = -\nabla f \tag{952}$$

The gradient is the same at that point and we can verify that they also touch at that point by showing $g(2, 1, 1) = f(2, 1, 1)$.

Example 148: Suppose we have another sphere and ellipsoid:

$$f = x^2 + y^2 + z^2 - 4y - 2z + 2 = 0 \tag{953}$$
$$g = 3x^2 + 2y^2 - 2z = 1 \tag{954}$$

at $P(1,1,2)$. To show that they are perpendicular at this point, we take their gradient vector and show that they are perpendicular (i.e. dot product is zero)

$$\nabla f(1,1,2) = (2,-2,2) \tag{955}$$
$$\nabla g(1,1,2) = (6,4,-2) \tag{956}$$

and the dot product is $12-8-4=0$ so they are perpendicular.

Example 149: Suppose we have a curve and an ellipsoid:

$$\vec{r}(t) = (\frac{3}{2}(t^2+1), t^4+1, t^3) \tag{957}$$
$$x^2+2y^2+3z^2 = 20 \tag{958}$$

at $(3,2,1)$. Notice that they intersect at $(3,2,1)$. The gradient of the ellipsoid is:

$$\nabla f(3,2,1) = (6,8,6) \tag{959}$$

and the tangent vector of the curve is: $\vec{r}(1) = (3,4,3) = \frac{1}{2}\nabla f$. The tangent to the curve is parallel to the gradient, so the normal vector must be perpendicular to the surface.

Example 150: Suppose someone's head is described by the ellipsoid $x^2+y^2+2z^2=7$, and there is a bee at $P(1,2,1)$. There is a bee-swatter at $2x+3y+z=49$. The bee moves with a speed of 4 along the normal. When does it hit the bee swatter?

The gradient is:

$$\nabla f = (2x,2y,4z) \implies \nabla f(1,2,1) = (2,4,4) \tag{960}$$

and the normal line is:

$$(x,y,z) = (1+2q, 2+4q, 1+4q) \tag{961}$$

such that:

$$\vec{r}(q) = (1+2q, 2+4q, 1+4q) \tag{962}$$

We want: $\|\vec{r}'(t)\| = 4$. We can pick $q = \frac{2}{3}t$ to get this. Therefore:

$$\vec{r}(t) = \left(1+\frac{4}{3}t, 2+\frac{8}{3}t, 1+\frac{8}{3}t\right) \tag{963}$$

To find when it intersects with the plane $2x+3y+z=49$, we set:

$$2(1+4t/3)+3(2+8t/3)+(1+8t/3) = 49 \implies t=3 \tag{964}$$

So the location of the bee once it hits the bee-swatter at $t=3$ is $(5,10,9)$.

33 Maximum and Minimum Values

- We can extend the idea of maximum and minimum values to multiple dimensions.

Definition: f is said to have a local maximum at $\vec{x}_0$ iff $f(\vec{x}_0) \geq f(\vec{x})$ fir $\vec{x}$ in some neighbourhood of $\vec{x}_0$. f is said to have a local minimum at $\vec{x}_0$ iff $f(\vec{x}_0) \leq f(\vec{x})$ for $\vec{x}$ in some neighbourhood of $\vec{x}_0$.

Theorem: If f has a local extreme values at $\vec{x}_0$, then either $\nabla f(\vec{x}_0) = \vec{0}$ or $\nabla f(\vec{x}_0)$ DNE.

Proof. Let $g(x) = f(x, y)$. Then:

$$\frac{dg}{dx}(x_0) = 0 = \frac{\partial f}{\partial x}(x_0, y_0) \tag{965}$$

Then if $z = f(x, y)$, we have:

$$g(x, y, z) = z \cdot f(x, y) = 0 \tag{966}$$

which implies $\nabla g = \hat{k}$ for $f_x = f_y = 0$. □

Definition: Points where $\nabla f = \vec{0}$ or DNE called critical points.

Definition: Points where $\nabla f = \vec{0}$ are called stationary points.

Definition: Stationary points which are not local extremes are called saddle points.

Definition: Let $f(x, y) = 20 - x^2 - y^2$. The gradient is:

$$\nabla f = (-2x, -2y) \tag{967}$$

which exists everywhere, but is zero at $(0, 0)$ which is a stationary point. To see what type of stationary point it is, set $x = h$ and $y = k$ where h and k are very small. Then:

$$f(0, 0) = 20 \tag{968}$$
$$f(h, k) = 20 - k^2 - h^2 \leq 20 \tag{969}$$

for all h, k. As a result, $f(0, 0)$ is a local maximum.

Example 151: Suppose we have $f(x, y) = xy$. The gradient is:

$$\nabla f = (y, x) \tag{970}$$

If we set the gradient to $\vec{0}$, we find that it is equal to zero at $(0, 0)$. Again, set $x = h$ and $y = k$. If both have the same sign, then it is positive. If they have opposite signs, then f is negative. This results in a saddle point.

Example 152: Suppose we have a function $f(x, y) = 2x^2 + y^2 - xy - 7y$. The gradient is:

$$\nabla f = (4x - y, 2y - x - 7) \tag{971}$$

which is zero at $(1, -4)$. To test if it's a maximum/minimum/saddle point, we can test points. However, this would require four test cases and I don't feel like writing it out. But if you do have the patience, you'll find that it's a local minimum.

Example 153: Suppose we have a cone given by $f(x, y) = -\sqrt{x^2 + y^2}$. The gradient is:

$$\nabla f = (-(x^2 + y^2)^{-1/2} \cdot 2x, -(x^2 + y^2)^{-1/2} \cdot 2y) \tag{972}$$

which does not exist at $(0, 0)$.

Theorem: **Second Derivatives Test:** For $f(x, y)$ with continuous second order partial derivatives, and $\nabla f(x_0, y_0) = \vec{0}$, set $A = \frac{\partial^2 f}{\partial x^2}(x_0, y_0)$, $B = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$, $C = \frac{\partial^2 f}{\partial y^2}(x_0, y_0)$. They form the discriminant[a]:

$$D = AC - B^2 \tag{973}$$

1. If $D < 0$, then (x_0, y_0) is a saddle point.
2. If $D > 0$, and $A, C > 0$, then (x_0, y_0) is a local minimum.
3. If $D > 0$, and $A, C < 0$ then (x_0, y_0) is a local maximum.

[a] This comes from more advanced calculus, but here's the intuition from where it'll come from (credit to Nathan): You can take linear approximations and they're planes, you can also take "quadric approximations" of surfaces at a stationary point and you would only get either a quadratic paraboloid (which is a max or min) or a hyperbolic paraboloid (has a saddle) - and this test basically looks at the quadric approximation and tells you which case it is.

Example 154: For $f(x,y) = xy$, at $(0,0)$ we have: $f_{xx} = A = 0$, $f_{yy} = C = 0$, and $f_{xy} = 1 = B$. Then:

$$D = AC - B^2 = -1 < 0 \tag{974}$$

so $(0,0)$ is a saddle point.

Example 155: For $f(X,y) = 2x^2 + y^2 - xy - 7y$, we have $\nabla f = \vec{0}$ at $(1,4)$. Then:

$$A = 4 \tag{975}$$
$$B = -1 \tag{976}$$
$$C = 2 \tag{977}$$

so $D = 8 - 1 = 7 > 0$ and we have a local minimum.

Theorem: If f is continuous on a bounded, closed set, then f takes on both an absolute minimum and an absolute maximum on that set.

Example 156: Let $f(x,y) = (x-4)^2 + y^2$ on the set $\{(x,y) : 0 \le x \le 2, x^3 \le y \le 4x\}$. The gradient is given by:

$$\nabla f = (2(x-4), 2y) \tag{978}$$

and the gradient is zero at $(4,0)$. To find the extrema, we have to look at the boundaries. The first boundary is $y = x^3$ from $0 \le x \le 2$. We can parametrize this with $x = t$ such that $y = t^3$ from $0 \le t \le 2$, and we have the vector function:

$$\vec{r}_1(t) = (t, t^3) \tag{979}$$

We then need to find when $f(\vec{r}_1(t)) = f_1(t)$ has an extrema. Using the chain rule:

$$f_1'(t) = \nabla f \cdot \vec{r}'(t) \tag{980}$$
$$= (2(t-4), 2t^3) \cdot (1, 3t^2) \tag{981}$$
$$= 2t - 8 + 6t^5 \tag{982}$$

Setting $f_1'(t) = 0$, we get $t = 1$ or $(1,1)$ where $f(1,1) = 10$. For this boundary, we can test the second derivative and get:

$$f_1''(t) = 2 + 30t^4 = 32 > 0 \tag{983}$$

so that *in this boundary*, we have a local minimum. For the second boundary, we have $y = 4x$. We can parametize it by setting $x = t$, $y = 4t$, $0 \le t \le 2$ such that:

$$f_2(t) = (t-4)^2 + (4t)^2 = 17t^2 - 8t + 16 \tag{984}$$

which is minimized at $t = \frac{4}{17}$, which corresponds to:

$$f\left(\frac{4}{17}, \frac{16}{17}\right) \approx 15.06 \tag{985}$$

Using the second derivative test, we see that this is indeed a local minimum. We also need to check the endpoints: $f(0,0) = 16$ and $f(2,8) = 68$. Therefore, $f(1,1) = 10$ is the absolute minimum and $f(2,8)$ is absolute maximum.

- In general, there are four steps we need to take to find the extrema:
  1. Check when ∇f does not exist.
  2. Check when $\nabla f = 0$.
  3. Check the boundaries.
  4. Check the end points of the boundaries.

Example 157: Let $f(x,y) = xy^2 - x$ in the region $\{(x,y)|x^2+y^2 \leq 3\}$. We take partial derivatives to find that:

$$f_x = y^2 - 1 \qquad f_{xx} = 0 \tag{986}$$
$$f_y = 2xy \qquad f_{yy} - 2x \tag{987}$$

and $f_{xy} = 2y$. Setting $\nabla f = \vec{0}$ gives us the two critical points $(0,1)$ and $(0,-1)$. For $(0,1)$, we have $A = 0, B = 2, C = 0$. The discriminant is $D = AC - B^2 = -4 < 0$, so this is a saddle point. For $(0,-1)$, we have $A = 0, C = 0, B = -2$. The discriminant is $D = AC - B^2 - 4 < 0$ which is also a saddle point.

We then look at the boundary $x^2 + y^2 = 3 \implies y^2 = 3 - x^2$. We can substitute this into the function to get:

$$f(x) = x(3-x^2) - x = 2x - x^3 \tag{988}$$

so we have:

$$f_1'(x) = 2 - 3x^2 \tag{989}$$
$$f_1'(x) = 0 \implies x = \pm\sqrt{\frac{2}{3}} \tag{990}$$

Testing cases and using the second derivative $f_1''(x) = -6x$, we can see that at $+\sqrt{\frac{2}{3}}$, we have a local max and at $-\sqrt{\frac{2}{3}}$, we have a local minimum.

However, we also need to take into account the "endpoints" of a (since we're going from $x = -\sqrt{3}$ to $x = \sqrt{3}$). At $f(-\sqrt{3},0) = \sqrt{3}$, we have the absolute max and at $f(\sqrt{3},0) = -\sqrt{3}$, we have the absolute minimum.

There is another approach to this problem by parameterize the curve. Let $\vec{r}(t) = \sqrt{3}\cos t\hat{\imath} + \sqrt{3}\sin t\hat{\jmath}$ for $0 \leq t \leq 2\pi$. Using this parametrization, we do not need to check endpoints.

34 Lagrange Multipliers

- Let us develop the basis behind Lagrange Multipliers.
- To maximize a function $f(x,y)$ subject to the boundary $g(x,y) = k$, we can draw in level curves $f(x,y) = C$. We want to find the largest value of C such that the level curve touches $g(x,y) = k$ (i.e. they share a common tangent line).
- Since the gradient vector points perpendicular to the tangent line, the gradient of $f(x,y) = C$ is parallel to the gradient of $g(x,y) = k$. This can be written as:

$$\nabla f = \lambda \nabla g \tag{991}$$

where λ is a constant, known as the **Lagrange Multiplier**. We have:

$$g(x_0,y_0) = k \tag{992}$$
$$f_x(x_0,y_0) = \lambda g_x(x_0,y_0) \tag{993}$$
$$f_y(x_0,y_0) = \lambda g_y(x_0,y_0) \tag{994}$$

which has three equations and three unknowns, so we can solve for x_0, y_0, λ. This can easily be extended to n dimensions.

Example 158: Suppose we wish to maximize $f(x,y)=x^2-y^2$ on the circle $x^2+y^2=1$. We have:

$$f_y = 2x \qquad g_x = 2x \tag{995}$$
$$f_y = -2y \qquad g_y = 2y \tag{996}$$

and our three equations are:

$$x_0^2+y_0^2=1 \tag{997}$$
$$2x_0 = \lambda 2x_0 \tag{998}$$
$$-2y_0 = \lambda 2y_0 \tag{999}$$

We have two cases. First, if $\lambda=1$, then $y_0=0$ and $x_0=\pm 1$. Second, if $\lambda=-1$, then $x_0=0$ and $y_0=\pm 1$. We can substitute these in:

$$f(1,0)=1 \tag{1000}$$
$$f(-1,0)=1 \tag{1001}$$
$$f(0,-1)=-1 \tag{1002}$$
$$f(0,1)=-1 \tag{1003}$$

which gives us the minimum and maximum values *at the boundary.*

Example 159: Let us revisit our example: $f(x,y)=xy^2-x$ constrained by the boundary $g=x^2+y^2=3$. We have:

$$\nabla f = (y^2-1, 2xy) \qquad \nabla g = (2x,2y) \tag{1004}$$

Our three equations are:

$$x^2+y^2=3 \tag{1005}$$
$$y^2-1=\lambda 2x \tag{1006}$$
$$2xy = \lambda 2y \tag{1007}$$

We can start from the third equation and consider two possibilities:
- Case 1: $y=0 \implies x=\pm\sqrt{3}$. This gives $f(\pm\sqrt{3},0)=\mp\sqrt{3}$.
- Case 2: $x_0=\lambda$. This requires a little bit of work and gives:

$$x=\pm\sqrt{\frac{2}{3}}, \quad y=\pm\sqrt{\frac{7}{3}} \tag{1008}$$

  which gives the same result as earlier.

- If there are two constraintsw, then we need to find the minimum and maximum of $f(x,y,z)$ with $g(x,y,z)=k$ and $h(x,y,z)=C$. We can define $\vec{T}=\nabla h\times\nabla g$ such that ∇f is perpendicular to $\vec{T}$. This leads to the relationship:

$$\nabla f(x_0)=\lambda\nabla g(\vec{x}_0)+\mu\nabla h(\vec{x}_0) \tag{1009}$$

which gives the following system of five equations:

$$g(\vec{x}_0)=k \tag{1010}$$
$$h(\vec{x}_0)=c \tag{1011}$$
$$f_y(\vec{x}_0)=\lambda g_x(\vec{x}_0)+\mu h_x(\vec{x}_0) \tag{1012}$$
$$f_x(\vec{x}_0)=\lambda g_y(\vec{x}_0)+\mu h_y(\vec{x}_0) \tag{1013}$$
$$f_z(\vec{x}_0)=\lambda g_z(\vec{x}_0)+\mu h_z(\vec{x}_0) \tag{1014}$$

Example 160: Suppose we have a function $f(x,y,z) = xy + 2z$, constrained by a plane $x+y+z=0$ and a sphere $x^2+y^2+z^2-24$. Our five equations are then:

$$x + y + z = 0 \tag{1015}$$
$$x^2 + y^2 + z^2 = 24 \tag{1016}$$
$$y = \lambda + \mu \cdot 2x \tag{1017}$$
$$x = \lambda + \mu \cdot 2x \tag{1018}$$
$$2 = \lambda + \mu \cdot 2x \tag{1019}$$
$$\tag{1020}$$

Combining, we get $(x-y)(1+3\mu) = 0$.

– Case 1: $x = y \implies z = -2x$. Solving for x, y, and z leads to $f(2,2,-4) = -4$ and $f(-2,-2,4) = 12$.
– Case 2: $\mu = -\frac{1}{2}$. This is harder to solve for and I would write out the whole solution if I woke up earlier, but solving it leads to:

$$f\left(\frac{1+3\sqrt{5}}{2}, \frac{1-3\sqrt{5}}{2}, -1\right) = -13 \tag{1021}$$
$$f\left(\frac{1-3\sqrt{5}}{2}, \frac{1+3\sqrt{5}}{2}, -1\right) = -13 \tag{1022}$$

35 Reconstructing a Function from its Gradient

- If we have the gradient, how can we go back to the original function?

Example 161: Suppose we have the gradient $\nabla f = (1 + y^2 + xy^2, x^2y + y + 2xy + 1)$. We can then recognize that:

$$\frac{\partial f}{\partial x} = 1 + y^2 + xy^2 \tag{1023}$$
$$\implies f = x + xy^2 + \frac{1}{2}x^2y^2 + \phi(y) \tag{1024}$$

where $\phi(y)$ is a function that does not depend on x. Similarly:

$$\frac{\partial f}{\partial y} = 2xy + x^2y + \phi'(y) \tag{1025}$$
$$= x^2y + y + 2xy + 1 \tag{1026}$$

Therefore: $\phi'(y) = y + 1 \implies \phi(y) = \frac{1}{2}y^2 + y + C$. As a result, we have:

$$f(x,y) = x + xy^2 + \frac{1}{2}x^2y^2 + \frac{1}{2}y^2 + y + C \tag{1027}$$

Example 162: In three dimensions, suppose that $\nabla f(x,y,z) = (\cos x - y \sin x)\hat{\imath} + (\cos x + z^2)\hat{\jmath} + (2yz)\hat{k}$. It is possible to recover f using the method from the previous example, but there is another method. We have:

$$f_y = \cos x - y \sin x \implies f = \sin x + y \cos x + \psi_1(y,z) \tag{1028}$$
$$f_y = \cos x + z^2 \implies f = y \cos x + yz^2 + \phi_2(x,z) \tag{1029}$$
$$f_z = 2yz \implies f = yz^2 + \phi_3(x,y) \tag{1030}$$

and the original function is:

$$f(x,y,z) = \sin x + y \cos x + yz^2 + C \tag{1031}$$

Example 163: Suppose we have $\nabla f(x,y) = y\hat{i} - x\hat{j}$. We have:

$$f_x = y \qquad f_y = -x \tag{1032}$$
$$f_{xy} = 1 \qquad f_{yx} = -1 \tag{1033}$$

We have:

$$\frac{\partial^2 f}{\partial y \partial x} \neq \frac{\partial^2 f}{\partial x \partial y} \tag{1034}$$

Since this contradicts Clairut's theorem, it cannot be a gradient!

Theorem: Let P and Q be functions of two variables, each continuously differentiable. The linear combination:

$$P(x,y)\hat{i} + Q(x,y)\hat{j} \tag{1035}$$

is a gradient if and only if:

$$\frac{\partial P(x,y)}{\partial y} = \frac{\partial Q(x,y)}{\partial x} \tag{1036}$$

Example 164: Suppose we have the function $(y^3 + x, x^2 + y)$. We have:

$$\frac{\partial P}{\partial y} = 3y^2 \neq \frac{\partial Q}{\partial x} = 2x \tag{1037}$$

so it is not a gradient.

Example 165: Suppose we have $(2\ln(3y) + 1/x, 2x/y + y^2)$. We have:

$$\frac{\partial P}{\partial y} = 2y = \frac{\partial Q}{\partial x} = \frac{2}{y} \tag{1038}$$

so it is a gradient. We can recover the original function:

$$f_x = 2\ln(3y) + 1/x \implies f = 2x\ln(3y) + \ln x + \phi(y) \tag{1039}$$
$$f_y = 2x/y + y^2 \implies f = 2x\ln(y) + \frac{y^3}{3} \tag{1040}$$

which gives:

$$f = 2x\ln(3y) + \ln x + \frac{y^3}{3} + C \tag{1041}$$