

# ESC195 Notes

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## 1 Hyperbolic Functions

- Sometimes, combinations of  $e^x$  and  $e^{-x}$  are given certain names, for example:

- **Hyperbolic sine:**  $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$

- **Hyperbolic cosine:**  $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$

- They have the following properties:

$$\frac{d}{dx} \sinh x = \cosh x \quad (1)$$

$$\frac{d}{dx} \cosh x = \sinh x \quad (2)$$

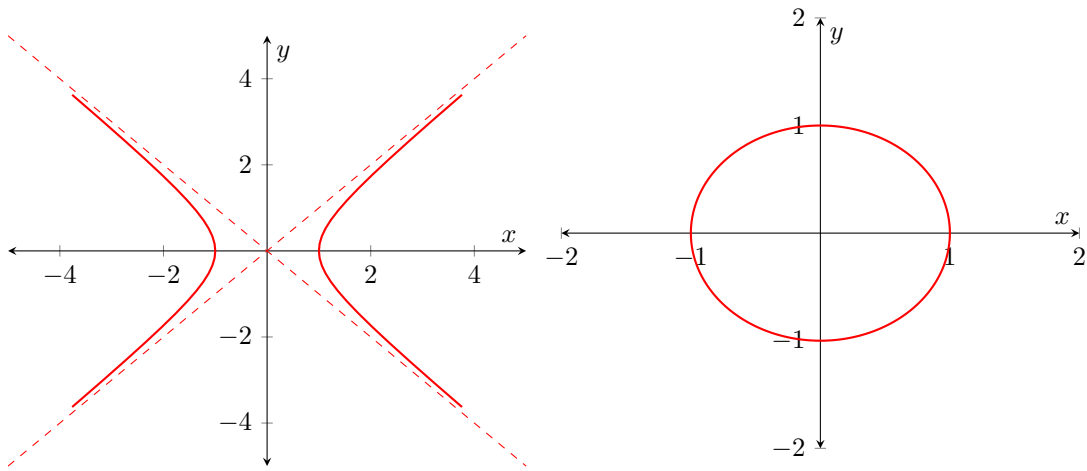
- They are related via:

$$\cosh^2 x - \sinh^2 x = 1 \quad (3)$$

- Both the area of a circular sector and that of a hyperbolic sector is described by:

$$A = \frac{1}{2}t \quad (4)$$

where  $t$  is the subtended angle, and the figures are parametrized by  $(\cos t, \sin t)$  and  $(\cosh t, \sinh t)$ .



- The catenary

$$y = a \cosh\left(\frac{x}{a}\right) + C \quad (5)$$

describes the shape of a free hanging rope between two walls separated by a width  $a$ .

- The hyperbolic tangent is given by  $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ . and its derivative is given by:

$$\frac{d}{dx} \tanh x = \operatorname{sech}^2 x \quad (6)$$

- The inverse of  $y = \sinh x$  is given by:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \quad (7)$$

**Tip:** A table of integrals and derivatives revolving around hyperbolic trig functions can be found in the textbook.

## 2 Indeterminate Forms

- A lot of the times, limits have an indeterminate form, where if we substitute in what  $x$  approaches to, we get it in the form of  $\frac{0}{0}$ , for example:

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} \quad (8)$$

**Theorem:** If  $f(x) \rightarrow 0$  and  $g(x) \rightarrow 0$  as  $x \rightarrow \pm, \infty$  or  $x \rightarrow c$  or  $x \rightarrow c^{+-}$  and if  $\frac{f'(x)}{g'(x)} \rightarrow L$ , then:

$$\frac{f(x)}{g(x)} \rightarrow L \quad (9)$$

**Example 1:** Solve:  $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

We can set  $f(x) = \sin x$ ,  $f'(x) = \cos x$ ,  $g(x) = x$  and  $g'(x) = 1$  such that:

$$\lim_{x \rightarrow 0} \frac{f'}{g'} = \lim_{x \rightarrow 0} \cos x = 1 \quad (10)$$

**Example 2:** Solve  $\lim_{x \rightarrow 0^+} \frac{\sin x}{\sqrt{x}}$ .

Set  $f = \sin x$ ,  $f' = \cos x$ ,  $g = \sqrt{x}$ ,  $g' = \frac{1}{2}x^{-1/2}$  and so:

$$\lim_{x \rightarrow 0^+} 2x^{1/2} \cos x = 0 \implies \lim_{x \rightarrow 0^+} = 0 \quad (11)$$

**Example 3:** Solve  $\lim_{x \rightarrow 0} \frac{e^x - x - 1}{3x^2}$ .

If we take the derivative, we get:

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{6x} \quad (12)$$

which is still  $\frac{0}{0}$ !. We can take derivatives again:

$$\lim_{x \rightarrow 0} \frac{e^x}{6} = \frac{1}{6} \quad (13)$$

so the original limit is  $\frac{1}{6}$ .

**Warning:** L'hospital's rule can *only* be used in indeterminate forms. Applying them to limits where

- To prove the L'hospital's rule, we first prove the **Cauchy Mean Value Theorem** as a lemma

**Theorem: Cauchy Mean Value Theorem:** Given  $f$  and  $g$  differentiable on  $(a, b)$ , continuous on  $[a, b]$  and  $g' \neq 0$  on  $(a, b)$ , there must exist some number  $r$  in  $(a, b)$  such that:

$$\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad (14)$$

- We then apply **Rolle's Theorem** to prove the Cauchy Mean Value Theorem:

*Proof.* Set:

$$G(x) = [g(b) - g(a)][f(x) - f(a)] - [g(x) - g(a)][f(b) - f(a)]$$

Note that  $G(a) = G(b) = 0$  so it satisfies the conditions of Rolle's Theorem. Taking the derivative, we get:

$$G'(x) = [g(b) - g(a)]f'(x) - g'(x)[f(b) - f(a)] \quad (15)$$

According to Rolle's, there must be some  $x = r$  such that  $G'(r) = 0$ , we can then substitute for this and solve:

$$G'(r) = 0 \implies [g(b) - g(a)]f'(r) = g'(r)[f(b) - f(a)] \quad (16)$$

Which is equivalent to:

$$\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad (17)$$

Furthermore, we have  $g'(c) = \frac{g(b) - g(a)}{b - a}$  from the mean value theorem. Since  $g' \neq 0$  we have  $g(b) - g(a) \neq 0$ .  $\square$

- Given  $x \rightarrow c^+$  and  $f(x), g(x) \rightarrow 0$  where:

$$\lim_{x \rightarrow c^+} \frac{f'(x)}{g'(x)} = L \quad (18)$$

we will now prove that  $\lim_{x \rightarrow c^+} \frac{f(x)}{g(x)} = L$ .

*Proof.* Consider the interval  $[c, c + h]$  and apply Cauchy MVT. There must be some number  $c_2$  in  $[c, c + h]$  such that:

$$\frac{f'(c_2)}{g'(c_2)} = \frac{f(c + h) - f(c)}{g(c + h) - g(c)} = \frac{f(c + h)}{g(c + h)} \quad (19)$$

The last step is a result of the given  $f(c) = g(c) = 0$ . The LHS can be rewritten as:

$$\lim_{h \rightarrow 0} \frac{f'(c_2)}{g'(c_2)} = \frac{f'(c)}{g'(c)} \quad (20)$$

since  $c_2$  lies in the interval  $[c, c + h]$  so if  $h \rightarrow 0$ , then the interval becomes smaller to contain just  $c$ . The RHS can be rewritten as:

$$\lim_{h \rightarrow 0} \frac{f(c + h)}{g(c + h)} = \lim_{x \rightarrow c^+} \frac{f(x)}{g(x)} \quad (21)$$

and therefore:

$$\lim_{x \rightarrow c^+} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} = L \quad (22)$$

□

- To prove the case for  $x \rightarrow \pm\infty$ , we can let  $x = \frac{1}{t}$  and take the limit as  $t \rightarrow \infty$ .

**Example 4:** Find  $\lim_{x \rightarrow \infty} \frac{\ln x}{x}$ .

Taking the derivative of top and bottom, we have:

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{1} = 0 \implies \lim_{x \rightarrow \infty} \frac{\ln x}{x} = 0 \quad (23)$$

**Idea:** The logarithm function grows very slowly. In fact, any positive power of  $x$  will grow faster than  $\ln x$ .

**Example 5:** Solve  $\lim_{x \rightarrow \infty} \frac{x^3}{e^x}$

This is indeterminate in the form of  $\frac{\infty}{\infty}$ . We apply L'hospital's rule multiple times:

$$\lim_{x \rightarrow \infty} \frac{x^3}{e^x} \stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{3x^2}{e^x} \left( = \frac{\infty}{\infty} \right) \quad (24)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{6x}{e^x} \left( = \frac{\infty}{\infty} \right) \quad (25)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{6}{e^x} = 0 \quad (26)$$

- Generally,  $\lim_{x \rightarrow \infty} \frac{x^m}{e^x} = 0$  where  $m$  is any positive integer.
- There are other indeterminate forms, such as  $0^0$ , for example:

$$\lim_{x \rightarrow 0} x^x \quad (27)$$

The central idea behind this is that  $a^b = e^{a \ln b}$ . Therefore, this limit is equal to:

$$\lim_{x \rightarrow 0} e^{x \ln x} \quad (28)$$

We can take the limit of the exponent to get:

$$\lim_{x \rightarrow 0} x \ln x = \lim_{x \rightarrow 0} \frac{\ln x}{1/x} \quad (29)$$

Note that the first equation is another indeterminate form with the  $0 \cdot \infty$  type, so we had to multiply top and bottom by  $\frac{1}{x}$  to get the quotient form. Then we have:

$$\lim_{x \rightarrow 0} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x \rightarrow 0} -x = 0 \quad (30)$$

Therefore:

$$\lim_{x \rightarrow 0} e^{x \ln x} = e^0 = 1 \quad (31)$$

so  $\lim_{x \rightarrow 0} x^x = 1$ .

**Example 6:** Solve  $\lim_{x \rightarrow \infty} (x+2)^{2/\ln x}$ .

This is of the type  $\infty^0$ . The approach is exactly the same as the previous example. We write it in exponential form:

$$= \lim_{x \rightarrow \infty} e^{\frac{2}{\ln x} \ln(x+2)} \quad (32)$$

and looking at the exponent gives:

$$\lim_{x \rightarrow \infty} \frac{2 \ln(x+2)}{\ln x} \quad (33)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{\left(\frac{2}{x+2}\right)}{\left(\frac{1}{x}\right)} = \lim_{x \rightarrow \infty} \frac{2x}{x+2} \left(= \frac{\infty}{\infty}\right) \quad (34)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{2}{1} = 2 \quad (35)$$

Therefore:

$$\lim_{x \rightarrow \infty} e^{\frac{2}{\ln x} \ln(x+2)} = e^2 \quad (36)$$

so:

$$\lim_{x \rightarrow \infty} (x+2)^{2/\ln x} = e^2 \quad (37)$$

**Example 7:** Solve  $\lim_{x \rightarrow \infty} \left[ \sin \left( \frac{\pi}{x} + \frac{\pi}{2} \right) \right]^x$

This is in the form of  $1^\infty$ . We rewrite it as:

$$\lim_{x \rightarrow \infty} \exp \left( x \ln \left( \sin \left( \frac{\pi}{x} + \frac{\pi}{2} \right) \right) \right) \quad (38)$$

and taking the limit of the exponent:

$$= \lim_{x \rightarrow \infty} x \ln \left( \sin \left( \frac{\pi}{x} + \frac{\pi}{2} \right) \right) \left(= \frac{0}{0}\right) \quad (39)$$

$$\stackrel{*}{=} \lim_{x \rightarrow \infty} \frac{\cos \left( \frac{\pi}{x} + \frac{\pi}{2} \right) \cdot \left( -\frac{\pi}{x^2} \right)}{\sin \left( \frac{\pi}{x} + \frac{\pi}{2} \right) \cdot \left( -\frac{1}{x^2} \right)} = \frac{0 \cdot \pi}{1} = 0 \quad (40)$$

Therefore:

$$\lim_{x \rightarrow \infty} \left[ \sin \left( \frac{\pi}{x} + \frac{\pi}{2} \right) \right]^x = \lim_{x \rightarrow \infty} \exp \left( x \ln \left( \sin \left( \frac{\pi}{x} + \frac{\pi}{2} \right) \right) \right) = 1 \quad (41)$$

### 3 Integration

#### 3.1 Recap of Integration

- The definite integral has the geometric interpretation as the area under the curve  $f(x)$  between  $x = a$  and  $x = b$  and the  $x$  axis:

$$\int_a^b f(x) dx \quad (42)$$

but can be rigorously defined using a Riemann sum:

$$\int_a^b f(x) dx \equiv \lim_{\|P\|} \sum_{i=1}^n f(x_i^*) \Delta x_i \quad (43)$$

Often, we have a uniform partition, such that  $\Delta x_i = \frac{b-a}{n}$  where  $n$  is the number of partitions. And if we choose to use the right hand endpoint, then:

$$f(x_i^*) = f(x_i) = f\left(a + \frac{b-a}{n}i\right) \quad (44)$$

**Example 8:** To solve  $\int_0^5 x^2 dx$ , we can choose a uniform partition with:

$$\Delta x = \frac{5-0}{n} = \frac{5}{n} \quad (45)$$

and:

$$x_i^* = x_i = i\Delta x \implies f(x_i^*) = (i\Delta x)^2 = \left(i\frac{5}{n}\right)^2 \quad (46)$$

The area approximation is:

$$A \simeq \sum_{i=1}^n \Delta x_i f(x_i^*) = \sum_{i=1}^n \left(\frac{5}{n}\right) \left(i\frac{5}{n}\right)^2 \quad (47)$$

$$= \frac{125}{n^2} \sum_{i=1}^n i^2 = \frac{125}{n^3} \frac{n(n+1)(2n+1)}{6} \quad (48)$$

Taking the limit as  $n \rightarrow \infty$ , we get:

$$\int_0^5 x^2 dx = \lim_{n \rightarrow \infty} \frac{125}{6} \left(2 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{5^3}{3}. \quad (49)$$

**Example 9:** To evaluate  $\int_1^2 x^{-2} dx$ , we can choose

$$x_i^* = \sqrt{x_{i-1}x_i} \quad (50)$$

and a uniform partition of:

$$\Delta x = \frac{2-1}{n} = \frac{1}{n} \quad (51)$$

such that:

$$x_i = 1 + i\Delta x = 1 + \frac{i}{n} = \frac{n+i}{n} \quad (52)$$

and

$$x_{i-1} = \frac{n+i-1}{n} \quad (53)$$

such that the area is:

$$\begin{aligned}
A &\simeq \sum_{i=1}^n \Delta x f(x_i^*) \\
&= \sum_{i=1}^n \frac{1}{n} \left( \frac{1}{x_i^*} \right)^2 \\
&= \sum_{i=1}^n \frac{1}{n} \frac{1}{x_{i-1} x_i} \\
&= \sum_{i=1}^n \frac{1}{n} \frac{n}{n+i-1} \cdot \frac{n}{n+i} \\
&= \sum_{i=1}^n \frac{1}{n+i-1} \cdot \frac{1}{n+i} \\
&= \sum_{i=1}^n \left( \frac{1}{n+i-1} - \frac{1}{n+i} \right) \\
&= n \left[ \sum_{i=1}^n \frac{1}{n+i-1} - \sum_{i=1}^n \frac{1}{n+i} \right] \\
&= n \left[ \sum_{i=0}^n \frac{1}{n+i} - \sum_{i=1}^n \frac{1}{n+i} \right] \\
&= n \left[ \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n-1} - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n} \right] \\
&= n \left( \frac{1}{n} - \frac{1}{2n} \right) \\
&= 1 - \frac{1}{2} = \frac{1}{2}
\end{aligned}$$

The part where we cancel out everything is called a **telescoping series**. Notice how the area doesn't depend on  $n$  so we get the exact area, even if we let  $n = 1$ !

- We need a better way to do integration, so we can define:

$$F(x) \equiv \int_a^x f(t) dt \quad (54)$$

such that  $F'(x) = f(x)$ . This is the definition of the antiderivative. This leads to the fundamental theorem of calculus:

$$\int_a^b f(t) dt = F(b) - F(a) \quad (55)$$

and the indefinite integral can be written as:

$$\int f(x) dx = G(x) + C \quad (56)$$

The main problem now becomes trying to *find antiderivatives*, which is much easier than Riemann sums, though still more difficult than calculating derivatives.

## 3.2 Techniques of Integration

- **Integration by Parts** attempts to reverse the product rule:

$$(fg)' = fg' + f'g \quad (57)$$

Taking the integral of both sides gives:

$$f(x)g(x) = \int f(x)g'(x) \, dx + \int f'(x)g(x) \, dx \quad (58)$$

$$\int f(x)g'(x) \, dx = \int f(x)g'(x) \, dx = f(x)g(x) - \int f'(x)g(x) \, dx \quad (59)$$

If the second integral is easier than the first, then we have made substantial progress.

**Idea:** Integration of parts tells us that:

$$\int u \, dv = uv - \int v \, du \quad (60)$$

**Example 10:** To solve  $\int xe^{2x}$ , we can let:

$$u = x \quad dv = e^{2x} \, dx \quad (61)$$

$$du = dx \quad v = \frac{1}{2}e^{2x} \quad (62)$$

which gives:

$$\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x} \, dx \quad (63)$$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \quad (64)$$

We can check:

$$\frac{d}{dx} \left( \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C \right) \quad (65)$$

$$= xe^{2x} + \frac{1}{2}e^{2x} - \frac{2}{4}e^{2x} \quad (66)$$

$$= xe^{2x} \quad (67)$$

**Example 11:** To solve  $\int x^2 \sin(2x) \, dx$ , we let:

$$u = x^2 \quad dv = \sin 2x \, dx \quad (68)$$

$$du = 2x \, dx \quad v = -\frac{1}{2} \cos(2x) \quad (69)$$

which gives:

$$= -\frac{1}{2}x^2 \cos 2x + \int x \cos(2x) \, dx \quad (70)$$

and we can apply integration by parts a second time, if we let:

$$u = x \quad dv = \cos 2x \, dx \quad (71)$$

$$du = dx \quad v = \frac{1}{2} \sin(2x) \quad (72)$$

which gives us:

$$= -\frac{1}{2}x^2 \cos(2x) + \frac{1}{2}x \sin(2x) - \int \frac{1}{2} \sin(2x) \, dx \quad (73)$$

$$= -\frac{1}{2}x^2 \cos(2x) + \frac{1}{2}x \sin(2x) + \frac{1}{4} \cos(2x) + C \quad (74)$$



**Example 12:** To solve  $I = \int e^x \sin x \, dx$ , we can let:

$$u = \sin x \quad dv = e^x \, dx \quad (75)$$

$$du = \cos x \, dx \quad v = e^x \quad (76)$$

to give us:

$$= e^x \sin x - \int e^x \cos x \, dx \quad (77)$$

We apply integration by parts a second time:

$$u = \cos x \quad dv = e^x \, dx \quad (78)$$

$$du = -\sin x \, dx \quad v = e^x \quad (79)$$

to get:

$$I = e^x \sin x - e^x \cos x - \underbrace{\int e^x \sin x \, dx}_I \quad (80)$$

$$2I = e^x (\sin x - \cos x) + C' \quad (81)$$

$$I = \frac{1}{2} e^x (\sin x - \cos x) + C \quad (82)$$

and we are done.

**Example 13:** We can also solve integrals that do not appear to have parts, such as  $\int \ln x \, dx$ . We choose:

$$u = \ln x \quad dv = dx \quad (83)$$

$$du = \frac{1}{x} \, dx \quad v = x \quad (84)$$

to give us:

$$\ln x - \int dx = x \ln x - x + C \quad (85)$$

- For a definite integral, we can write IBP as:

$$f(x)g(x) \Big|_a^b - \int_a^b f'(x)g(x) \, dx \quad (86)$$

**Example 14:** It is *possible* to apply integration of parts to find the integral of  $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$ . We can let:

$$u = \frac{1}{\cos x} = \sec x \quad dv = \sin x \, dx \quad (87)$$

$$du = \sec x \tan x \quad v = -\cos x \quad (88)$$

this gives us:

$$\int \tan x \, dx = -\frac{\cos x}{\cos x} + \int \tan x \, dx \quad (89)$$

Notice that we could try to subtract the original integral from both sides and get:

$$0 = -1 \quad (90)$$

which is clearly wrong! However, we forgot the constant of integration, so the correct statement would be:

$$0 + C' = -1 + C \quad (91)$$

which does not tell us anything interesting. This brings We can see this concretely by repeating the same steps but trying to evaluate the definite integral  $\int_a^b \tan x \, dx$  instead, which gives:

$$\int_a^b \tan x \, dx = (-1) \Big|_{x=a}^{x=b} + \int_a^b \tan x \, dx \implies 0 = (-1) - (-1) \implies 0 = 0 \quad (92)$$

which confirms our suspicion that this isn't anything useful, but it's also not an incorrect statement.

**Warning:** Sometimes it is possible to get more than one answer through various means that differ by a constant factor when solving indefinite integrals. When this happens, nothing is wrong: we simply need to consider the constant of integration.

**Idea:** But how do we know *which* values of  $u$  and  $dv$  we should pick? A common strategy is to use **LIATE**:

1. L: Logarithms
2. I: Inverse Trig
3. A: Algebraic
4. T: Trigonometric
5. E: Exponential

If a function consists of two terms, the term that is higher up (closer to L) usually gets differentiated and the term near the bottom (closer to E) usually gets integrated. See [this](#) for how it works, and this [video](#) for a tutorial.