ESC194: Midterm 2 Review

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Contents

Index

1 Curve sketching

1.1 Formally Defining Horizontal Aymptotes

Horizontal asymptotes are formally defined as:

Definition: A horizontal asymptote occurs when $\lim_{x\to\infty} f(x) = L$. We can say that $f(x)$ goes to L as x goes to infinity if for any $\epsilon > 0$, a number A can be found s.t. for all $x > A$, $|f(x) - L| < \epsilon$.

The idea behind this revolves around finding f values as close to L as might be wanted by going to large enough x values.

An important theorem to determine horizontal asymptotes of reciprocal functions:

Theorem: The reciprocal horizontal asymptote limit:

$$\lim_{x\to\pm\infty} \frac{1}{x^r} = 0 \tag{1}$$

1.2 Prelims

We can use Fermat's theorem to determine critical points:

Definition: c is a critical point of $f(x)$ if $f'(c) = 0$ or $f'(c)$ DNE.

Here are some key features that might be seen on a graph:

- **Concavity:** If the graph of $y = f(x)$ lies above all its tangents in I, then $f(x)$ is concave up in I. If it lies below, then it is concave down.
- **Cusp:** A point c is a cusp if $f(x)$ is continuous at $x = c$ but $\lim_{x\to c^-} f(x) = \pm\infty$ and $\lim_{x\to c^+} f(x) = \mp\infty$.
- **Vertical Tangent:** A vertical tangent occurs when $\lim_{x\to c} |f'(x)| = \infty$ and $f(x)$ is continuous at c.
- **Slant Asymptote:** If $\lim_{x\to\infty} [f(x) - (mx + b)] = 0$, then $y = mx + b$ is a slant asymptote to $f(x)$ at $+\infty$.
- **Inflection point:** A point of inflection is at c if $f(x)$ is continuous at c and the sign of concavity changes at c.

A function is increasing on an interval I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I. Although we can use this definition to find local max/mins, there are a few cutie (QT/quick test) ways to do so:

- **QT1: Increasing/Decreasing Test.** If f is differentiable on the interval I, we show that if $f' > 0$, f is increasing. If $f' < 0$, f is decreasing. If $f' = 0$, f is constant.
- **QT2: First Derivative Test** Given that I contains a critical point and f is continuous at c_{crit}, and f is differentiable in I but not necessarily at c_{crit}. Then, if $f' > 0$ to the left of c_{crit} and $f' < 0$ to the right, then c_{crit} is a local max. If it's the opposite, we get the local minimum.
- **QT3: Concavity** Given that $f(x)$ is twice differentiable on I, then $f''(x)$ exists on I. As a result if $f''(x) > 0$, f is concave up. If $f'' < 0$, f is concave down.
- **QT4: Second Derivative Test** Given that $f''(x)$ is continuous near c and $f'(c) = 0$, then if $f''(c) > 0$, $f(c)$ is a local minimum. If $f''(c) < 0$, $f(c)$ is a local maximum. If $f''(c) = 0$, there is no verdict.

In general, the recipe to test for local max and min is to:

- Find all c_{crit}.

- If QT4 applies, use it.
- If it doesn't, and if QT2 applies, use it.
- If QT2 doesn't apply, use the basic definition of increasing/decreasing.

1.3 Curve Sketching Steps

1. Determine general behaviour:
    - Find Domain / Range / Limits at ∞.
    - Determine endpoints if they exist.
    - Find vertical, horizontal, slant asymptotes if they exist:
2. Determine x and y intercepts.
3. Establish if $f(x)$ is symmetrical, even, odd, and/or periodic.
4. Find $f'(x)$ and use this to:
    - Find all critical points and $f(c_{\text{crit}})$.
    - Find when $f(x)$ is increasing/decreasing.
    - Apply QT2.
    - Find vertical tangents / cusps if they exist.
5. Find $f''(x)$ and use it to:
    - Find when $f(x)$ is concave up/down.
    - Find points of inflection if they exist.
    - Optional: Use QT4 to confirm local max/min
6. Determine the absolute maximum and min by choosing the largest and smallest values of f, if they exist.

2 Optimization

Here is a checklist for solving optimization problems. If we want to optimize f:

- Check critical points.
- Check for endpoints.
- Check for local max, min.
- Check $\lim_{x\to\infty}$ and $\lim_{x\to-\infty}$.
- Make a decision.

2.1 Numerical Methods

Theorem: The **method of successive bisections** can be performed if f is a continuous function and we can find values a and b such that $f(b) < 0 < f(a)$. These two values can be determined by trial and error. By IVT, the root must exist in between a and b. To use this method, we calculate the halfway point x_{h1}. If $f(x_{h1})$ is positive, we replace a with x_{h1}. If it's negative, we replace b with x_{h1}.

Theorem: Using **Newton's Method** is much faster computationally. However, there is the added restriction that $f(x)$ *must* be differentiable. It works in the following steps:

1. Make a first guess for the root, x_1
2. Find the equation for the tangent line at $(x_1, f(x_1))$
3. Find the x intercept of the tangent line, and let

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{2}$$

and continue with x_2. Note however, that this doesn't always work such as when it diverges away from the root such as $x^{1/3}$.

Here are the overall steps that are recommended:

1. Try Newton's Method first if function is differentiable.
2. If the x_n values converge, great!
3. If they do not, try another value.
4. If they still diverge, use the method of successive bisections.

3 Formal Definition of an Integral

The summation notation is denoted below:

Definition: If $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$ are real numbers and m and n are integers such that $m \leq n$, then:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_{n-1} + a_n \tag{3}$$

There are a few theorems:

- For a constant α:

$$\sum_{i=m}^{n} \alpha a_i = \alpha \sum_{i=m}^{n} a_i \tag{4}$$

- It is also linear:

$$\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i \tag{5}$$

- $\displaystyle\sum_{i=1}^{n} \alpha = \alpha n$
- $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$
- $\displaystyle\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

One way of defining an integral is thinking of the area under the curve. This introduces the concept of a **Riemann Sum**:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i \tag{6}$$

where x_i represents points in the **partition** of the domain in which we want to approximate the area. The approximation gets more and more precise at the size Δx_i decreases. A few technical definitions to help:

> **Definition**: A **partition** is a finite subset of the closed interval $[a, b]$, which contains the points a and b. Denoted by P.

> **Definition**: The **norm** of $P = \|P\|$ which is the length of the longest subinterval:
>
> $$\|P\| = \max\left(\Delta x_1, \Delta x_2, \ldots, \Delta x_n\right) \tag{7}$$

Which can all be tied together to formally define the definite integral.

> **Definition**: If f is a function defined on a closed interval $[a, b]$, let P be a partition of $[a, b]$ with partition $x_0, x_1, x_2, \ldots, x_n$ where:
>
> $$a = x_0 < x_1 < x_2 < \cdots < x_n = b \tag{8}$$
>
> Choose points x_i^* within each subinterval $[xi + 1, x_i]$ and let $\Delta x_i = x_i - x_{i-1}$, and $\|P\| = \max\{\Delta x_i\}$. Then the **definite integral** of f from a to b is:
>
> $$\int_a^b f(x)\,\mathrm{d}x \equiv \lim_{\|P\|} \sum_{i=1}^{n} f(x_i^*)\Delta x_i \tag{9}$$
>
> if the limit exists. If the limit does exist, then f is called integrable on the interval $[a, b]$. The sign $\displaystyle\int$ is called the integral sign. $f(x)$ is known as the **integrand**, and a, b are the limits of integration. The output is a single number that does not depend on x.

We can formally show that the definite integral can take on a specific value I with a $\delta - \epsilon$ statement:

> **Idea**: If we have:
>
> $$\int_a^b f(x)\,\mathrm{d}x = I \tag{10}$$
>
> then for ever $\epsilon > 0$, there exists a $\delta > 0$ such that:
>
> $$\left| I - \sum_{i=1}^{n} f(x_i^*)\Delta x_i \right| < \epsilon \tag{11}$$
>
> for all partitions P of $[a, b]$ with $\|P\| < \delta$ and all possible choices of x_i^* in $[x_{i-1}, x_i]$.

However, going through this proof would be a nightmare. Instead, we can show **integrability** via the following theorem:

> **Theorem**: Continuous and/or piecewise continuous on $[a, b]$ guarantees integrability on $[a, b]$,

> **Definition**: A function is **piecewise continuous** if it only has a finite number of jump discontinuities.

Now that we know when the integral exists, we can find ways of calculating it from scratch:

> **Idea**: Going through with a full Riemann sum calculation is also tedious. As a result, here are a few conventions to make it easier:

- We usually select regular partitions:

$$\Delta x = \Delta x_1 = \Delta x_2 = \cdots = \Delta x_n = \frac{b-a}{n} \tag{12}$$

- And we select x_i^* to be the RH end point such that:

$$x_i^* = x_i = a + i\Delta x = a + i\frac{b-a}{n} \tag{13}$$

Therefore, the integral can be written as:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n} \tag{14}$$

4 Properties of Integration

4.1 Definite Integral Properties

There are a few properties:

- Constant:

$$\int_a^b c\,\mathrm{d}x = c(b-a) \tag{15}$$

- Additivity:

$$\int_a^b (f(x) \pm g(x))\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x \tag{16}$$

- Constant Multiple:

$$\int_a^b c(f)x\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x \tag{17}$$

- Changing Limits:

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^z f(x)\,\mathrm{d}x + \int_z^b f(x)\,\mathrm{d}x \tag{18}$$

There are also **order properties** of integrals. If $a < b$, then:

- If $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,\mathrm{d}x \geq 0 \tag{19}$$

- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then:

$$\int_a^b f\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x \tag{20}$$

- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then:

$$m(b-a) \leq \int_a^b f\,\mathrm{d}x \leq M(b-a) \tag{21}$$

- Absolute values:

$$\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x \tag{22}$$

4.2 Fundamental Theorem of Calculus

The **first fundamental theorem of calculus** states that:

Theorem: Let f be continuous on $[a, b]$. The function F is defined on $[a, b]$ by:

$$F(x) = \int_a^x f(t)\,\mathrm{d}t \tag{23}$$

is continuous on $[a, b]$, differentiable on (a, b), and has derivative:

$$F'(x) = f(x) \tag{24}$$

for all $x \in (a, b)$.

Rarely (never) will you get a simple question like this. Sometimes, the upper bound is a function $g(x)$ instead. If this is the case, then:

Idea: Assuming that f is continuous in $[a, b]$, then the function F is defined on $[a, b]$ by:

$$F(x) = \int_a^{g(x)} f(t)\,\mathrm{d}t \tag{25}$$

has a derivative:

$$F'(x) = g'(x)f(g(x)) \tag{26}$$

for $x \in (a, b)$. To see why this is true, we can apply the chain rule:

$$F'(x) = \frac{d}{dx}f(g(x)) = g'(x)f(g(x)) \tag{27}$$

The **second fundamental theorem of calculus** states that:

Theorem: Let f be continuous on $[a, b]$. If G is any antiderivative for f on $[a, b]$, then:

$$\int_a^b f(t)\,\mathrm{d}t = G(b) - G(a) \tag{28}$$

This can alternatively be written as:

$$\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a) \tag{29}$$

which can be interpreted as the net change of $F(x)$. For example:

$$\Delta x = \int_a^b v(t)\,\mathrm{d}t \tag{30}$$

gives the displacement from $t = a$ to $t = b$. The proofs for these two theorems are provided below:

Proof: (1st theorem) For x and $x + h$ in (a, b),

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(x)\,\mathrm{d}t - \int_a^x f(x)\,\mathrm{d}t & (31)\\ &= \int_a^x f(t)\,\mathrm{d}t + \int_x^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t & (32)\\ &= \int_x^{x+h} f(t)\,\mathrm{d}t & (33) \end{aligned}$$

For $h \neq 0$, we have:

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t \tag{34}$$

We can separate it into cases. If $h > 0$, then we can write per the extreme value theorem the minimum value of f as $f(u) = m$ and the maximum value as $f(v) = M$ for $u, v \in [x, x+h]$ such that:

$$mh \leq \int_x^{x+h} f(t)\,\mathrm{d}t \leq Mh \tag{35}$$

or:

$$f(u)h \leq \int_x^{x+h} f(t)\,\mathrm{d}t \leq f(v)h \tag{36}$$

which we can rewrite, after dividing through by h:

$$f(u) \leq \frac{F(x+h)-F(x)}{h} \leq f(v) \tag{37}$$

As $h \to 0$, we have $u \to x$ and $v \to x$. Therefore:

$$\lim_{h\to 0} f(u) = \lim_{u\to x} f(u) = f(x) \tag{38}$$

$$\lim_{h\to 0} f(v) = \lim_{v\to x} f(v) = f(x) \tag{39}$$

which gives us:

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = f(x) \tag{40}$$

or:

$$\frac{d}{dx}\int_a^x f(t)\,\mathrm{d}t = f(x) \tag{41}$$

Proof: (2nd theorem) Given that $F(x) = \displaystyle\int_a^x f(t)\,\mathrm{d}t$ is an antiderivative of f and given that G is an antiderivative, then:

$$F'(x) = G'(x) \implies F(x) = G(x) + C \tag{42}$$

We know that $F(a) = 0$, so $G(a) + C = 0$ or $C = -G(a)$, which gives:

$$\int_a^b f(t)\,\mathrm{d}t = F(b) = G(b) - G(a) \tag{43}$$

4.3 Integration Tricks

The **u-substitution** essentially reverses the chain rule.

Idea: Suppose we have an integral in the form:

$$\int f(g(x))g'(x)\,\mathrm{d}x \tag{44}$$

If we let $u = g(x)$, then $du = g'(x)dx$. So we can simplify the integral to:

$$\int f(u)\,\mathrm{d}u = F(u) + C = F(g(x)) + C \tag{45}$$

Once we have the indefinite integral, we can use back substitution to find the definite integral. We can avoid this step using a change of variables.

Theorem:

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u \tag{46}$$

In general, here are a few tips, in no particular order:

- Refer to the table of integrals at the back of the book. You are allowed to use them.
- Look for symmetry and periodicity.
- Draw a picture. Sometimes, you can avoid a complicated integral and use plain old geometry this way!
- For u-substitution, look for derivative-function pairs.
- If there are not too many terms, you can sometimes expand functions into a polynomial.
- Check if the integral even exists!
- Apply the first theorem of calculus, if applicable.
- See if the integral (or a similar one) is in the book.

5 Areas and Volumes

5.1 Areas Between Curves

Suppose we wish to find the **area between two curves** $f(x)$ and $g(x)$. We can do this by partitioning the area into infinitesimally small rectangles:

$$\Delta A_i = [f(x_i^*) - g(x_i^*)]\Delta x_i \tag{47}$$

so that the area is given by:

$$A = \lim_{\|P\|\to 0} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\,\Delta x_i \tag{48}$$

$$= \int_a^b [f(x) - g(x)]\,\mathrm{d}x \tag{49}$$

If we let $f(x) \geq g(x)$ when $x \in [a, b]$, then this gives the difference in their areas $A_1 - A_2$. If the condition $f(x) \geq g(x)$ is not satisfied, then we must break up the integral into multiple parts (if we interpret the area as having a positive area only). We can modify the area formula to be:

$$A = \int_a^b |f(x) - g(x)|\,\mathrm{d}x \tag{50}$$

Suppose we have a curve $x = f(y)$ and $x = g(y)$ instead. The area between $y = a$ and $y = b$ works in the same way:

$$A = \int_a^b |f(y) - g(y)|\,\mathrm{d}y \tag{51}$$

5.2 Volumes

We can determine the **volume** of a solid by partitioning it into thin cylinders whose axes area parallel to the x axis. Then we can break up the volume into thin sections:

$$V_i \simeq A_i \Delta x_i \tag{52}$$

so the volume is:

$$V = \int_a^b A(x)\,\mathrm{d}x \tag{53}$$

which is the general formula for the volume of any shape. If we can figure out $A(x)$ and the necessary bounds, we can find the volume forany change.

Idea: For **solids of revolution**, we rotate a curve $f(x)$ about the x axis. The volume of this solid using the **disk method** is:

$$V = \int_a^b \pi f(x)^2 \, \mathrm{d}x \tag{54}$$

Similarly around the y axis:

$$V = \int_c^d \pi g(y)^2 \, \mathrm{d}x \tag{55}$$

For the volume by rotating the region between two curves $f(x)$ and $g(x)$, we get:

$$V = \int_a^b \pi(f(x)^2 - g(x)^2) \, \mathrm{d}x \tag{56}$$

which is known as the **washer method**.

Sometimes, the disk and washer method is too difficult to apply.

Idea: We can use the **shell method about the y-axis** to find the volume when a curve is rotated about the y axis. Suppose we wish to rotate a curve $f(x)$ from $x = a$ to $x = b$ around the y axis. Then the volume is:

$$V = \int_a^b 2\pi x f(x) \, \mathrm{d}x \tag{57}$$

Similarly, if a curve is rotated about the x axis, we use the **shell method about the x-axis**:

$$V = \int_a^b 2\pi y f(y) \, \mathrm{d}y \tag{58}$$

6 Misc

I honestly don't know where this section belongs, so I'm just copying and pasting from my notes (which I actually spent a decent amount of effort on):

- The average of a discrete set $\{a_1, a_2, \ldots, a_N\}$ is given by:

$$a_{\text{avg}} = \frac{1}{N} \sum_i^N a_i \tag{59}$$

- For a continuous distribution, we can extend this to:

$$f_{\text{avg}} = \frac{1}{N} \sum_i^N f(x_i^*) \tag{60}$$

  Taking the limit as $N \to \infty$, we get:

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) \, \mathrm{d}x \tag{61}$$

  Theorem: **Mean Value Theorem for Integrals:** If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that:

$$f(c) = f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) \, \mathrm{d}x \tag{62}$$

Proof: Define $F(x) = \displaystyle\int_a^x f(t)\,\mathrm{d}t$. If we apply the mean value theorem to F, then:

$$F'(c) = \frac{F(b) - F(a)}{b - a} \tag{63}$$

for some $c \in [a, b]$. Now since:

$$F'(x) = f(x) \tag{64}$$

it becomes apparent that:

$$f(c) = \frac{\int_a^b f(t)\,\mathrm{d}t - \cancel{\int_a^a f(t)\,\mathrm{d}t}}{b - a} = \frac{1}{b - a}\int_a^b f(t)\,\mathrm{d}t \tag{65}$$

- We can also introduce **inverse functions**.

Definition: A function $f(x)$ is said to be one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Alternatively, we can say that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

- We can use the **horizontal line test**. If any horizontal line crosses the function more than one time, then it is not one-to-one.

Definition: Let f be a 1-1 function with domain A and range B. Then its inverse function, f^{-1} has domain B and range A, and is defined by:

$$f^{-1}(x) = x \iff f(x) = y \tag{66}$$

Therefore:

$$f^{-1}(f(x)) = f(f^{-1}(x)) = x \tag{67}$$

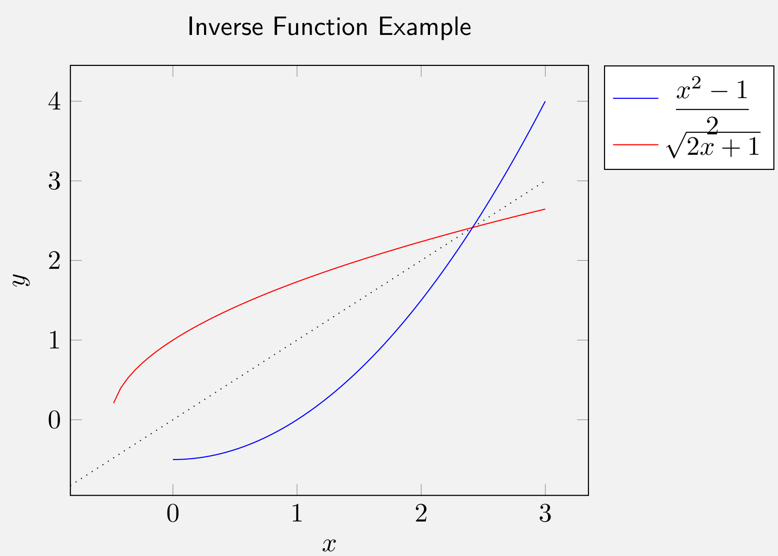
Warning: To prevent confusion, not that:

$$\frac{1}{f(x)} = [f(x)]^{-1} \neq f^{-1}(x) \tag{68}$$

- Geometrically, the inverse of a function represents a reflection of each point across the line $y = x$.

Example 1: If $g(x) = \sqrt{2x + 1}$, it is implied that $x \geq -1/2$, so it is a one-to-one function. Therefore, the inverse function is:

$$g^{-1}(x) = \frac{x^2 - 1}{2} \tag{69}$$



Theorem: If f is either an increasing or decreasing function, then f is $1-1$, and hence, has an inverse.

Proof. Say $f(x)$ is decreasing, then $x_1 < x_2 \implies f(x_1) > f(x_2)$ and if $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$. □

Theorem: Let f be a 1-1 function defined on an interval I. If f is continuous, then f^{-1} is also continuous. (Proof provided in Appendix F)

- Let $g(x) = f^{-1}(x)$. Then:

$$f(g(x)) = x \tag{70}$$

$$\frac{d}{dx} f(g(x)) = 1 \tag{71}$$

$$f'(g(x))g'(x) = 1 \tag{72}$$

$$g'(x) = \frac{1}{f'(g(x))} \tag{73}$$

or:

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \tag{74}$$

which is equivalent to:

$$\frac{dy}{dx} = \frac{1}{\frac{dy}{dx}} \tag{75}$$

Theorem: The inverse of composite functions is given by:

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1} \tag{76}$$

Proof. Let $y = (f \circ g)^{-1}(x)$. Then:

$$x = (f \circ g)(y) = f(g(y)) \tag{77}$$

so we have:

$$g(y) = f^{-1}(x) \tag{78}$$

$$y = g^{-1}(f^{-1})(x) \tag{79}$$

$$= (g^{-1} \circ f^{-1})(x) \tag{80}$$

□

7 Logarithms and Exponentials

Warning: Note that in this section, I make the assumption you are already familiar with general logarithm and exponential properties, so I won't spend time writing those down.

Definition: A **logarithm function** is a nonconstant differentiable function f defined for $x \in \{\mathbb{R}, (0, \infty)\}$ such that for all $a > 0$ and $b > 0$:

$$f(a \cdot b) = f(a) + f(b) \tag{81}$$

It has the following properties:

- $f(1) = 0$
- $f(1/x) = -f(x)$
- $f(x/y) = f(x) - f(y)$
- $f'(x) = \frac{1}{x} f'(1)$.

This leads to the definition of the **natural logarithm**:

Definition: The natural logarithm is defined as:

$$\ln(x) = \int_1^x \frac{\mathrm{d}t}{t} \tag{82}$$

Note that $\ln(x)$ is not the antiderivative of $\frac{1}{t}$. We can instead write:

$$\int \frac{\mathrm{d}t}{dt} = \ln|x| + C \tag{83}$$

since x can be negative as well.

Theorem: **Feynman's trick of Differentiation**[a] (otherwise known as logarithmic differention): The following was popularized by Richard Feynman during the first of his Feynman Lectures. If we have a function:

$$g(x) = g_1(x) g_2(x) g_3(x) \cdots g_n(x) \tag{84}$$

Then taking the natural logarithm of both sides, applying the chain rule, and simplifying gives:

$$g'(x) = g(x)\left(\frac{g_1'}{g_1} + \frac{g_2'}{g_2} + \cdots + \frac{g_n'}{g_n}\right) \tag{85}$$

[a] Note that this is not a formal name. I just chose it to name it after Feynman because I'm a huge Feynman stan and I first heard about it in the preface to the Feynman Lectures where he was talking about mathematical tricks.

Exponential functions can be introduced:

Definition: If z is a real number, then e^z is the number such that:

$$\ln(e^z) = z \tag{86}$$

More formally, we can write the exponential function as $\exp(x) = e^x$. The most useful property of e^x is that:

$$\frac{d}{dx} e^x = e^x \tag{87}$$

We can extend this to general logarithmic and exponential functions. If $x > 0$, then we can define:

Definition: The **general exponential** function is defined as

$$x^z = e^{z \ln x} \tag{88}$$

if $x > 0$.

Similarly:

Definition: The **general logarithm** can be defined as:

$$\log_p(x) = \frac{\ln x}{\ln p} \tag{89}$$

such that:

$$\frac{d}{dx} a^x = \ln(a) a^x \tag{90}$$

and

$$\frac{d}{dx} \log_p(x) = \frac{1}{x \ln p} \tag{91}$$

7.1 Bounding e

Idea: We can first bound e^x by setting a lower limit (which happens to be the Taylor series!). Notice that via integration:

$$e^x = 1 + \int_0^x e^t \, \mathrm{d}t \tag{92}$$

Since e^x is always increasing, we can claim that $e^x > 1$ for $x > 0$ such that:

$$e^x = 1 + \int_0^x e^t \, \mathrm{d}t > 1 + \int_0^x 1 \, \mathrm{d}t = 1 + x \tag{93}$$

We can then repeat the previous step to show that

$$e^x = 1 + \int_0^x e^t \, \mathrm{d}t > 1 + \int_0^x (1 + x) \, \mathrm{d}t = 1 + x + \frac{x^2}{2} \tag{94}$$

Repeating the process, we eventually get:

$$e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \tag{95}$$

Instead of choosing to bound e^x, we can also choose to bound e. We have that:

$$\ln x = \int_1^x \frac{\mathrm{d}t}{t} \tag{96}$$

such that:

$$\ln\left(1 + \frac{1}{n}\right) = \int_1^{1+1/n} \frac{\mathrm{d}t}{t} < \int_1^{1+1/n} 1 \, \mathrm{d}t \tag{97}$$

Since $frac1t < \frac{1}{1}$ for $t > 0$. The upper bound then becomes:

$$1 + \frac{1}{n} - 1 = \frac{1}{n} \implies \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n} \tag{98}$$

We can similarly repeat this process:

$$1 + \frac{1}{n} < e^{1/n} \implies (1 + \frac{1}{n})^n < e \tag{99}$$

Note that if we take the limit as $n \to \infty$, *intuitively* we would expect the upper bound to become closer and closer to the true value. We shall explore this further, and we can write the lower bound as:

$$\ln\left(1+\frac{1}{n}\right) = \int_1^{1+1/n} \frac{\mathrm{d}t}{t} > \int_1^{1+1/n} \frac{\mathrm{d}t}{1+1/n} \tag{100}$$

since $\dfrac{1}{t} > \dfrac{1}{1+1/n}$. We can write this in logarithm form to get:

$$\ln\left(1+\frac{1}{n}\right) > \left(\frac{1}{1+1/n}\right)\left(1+\frac{1}{n}-1\right) = \frac{1}{n+1} \implies \left(1+\frac{1}{n}\right)^{n+1} > e \tag{101}$$

Putting it altogether, we have the following statement:

Idea: e can be estimated with its lower and upper bound with the following:

$$\left(1+\frac{1}{n}\right)^{n} < e < \left(1+\frac{1}{n}\right)^{n+1} \tag{102}$$

8 Inverse Trigonometric Functions

We can define the inverse function of trigonometric functions by restricting their domain, such as from $-\pi/2$ to $\pi/2$ for $\sin(x)$.

Definition: The inverse function for $\sin(x)$ is given by :

$$\sin^{-1}(x) = \arcsin(x) \tag{103}$$

Warning: You need to be very careful with the domain and range. Sometimes, if x falls out of the domain, it can lead to a different answer altogether, or it could be undefined.

There's a lot of formulas for this one, but to derive formula such as $\sin\big(\tan^{-1}(x)\big)$, you just need to draw a picture of a right angled triangle with one of the legs as x and either the hypotenuse or the other leg as 1. If proofs are not needed, there is a formula sheet with all properties at the end of the book.