

ESC194 in a nutshell

QiLin Xue

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1 Delta-Epsilon Proofs

1.1 Brief Overview

The formal definition of the limit $\lim_{x \rightarrow c} f(x) = L$:

Definition: If for any $\epsilon > 0$, a $\delta > 0$ can be found such that for all $0 < |x - c| < \delta$, it can be proved that $|f(x) - L| < \epsilon$, then $\lim_{x \rightarrow c} f(x) = L$.

The *general steps* are as follows:

- Write: “For any $\epsilon > 0$, we want to pick a $\delta > 0$ such that $0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$ ”
- Start with $|f(x) - L| < \epsilon$ to start getting it under δ control (e.g. by expressing the LHS in terms of δ)
- Pick an arbitrary value of $\delta = a$ (if in doubt, choose $a = 1$) and modify $0 < |x - c| < a$ to write x in terms of a . Substitute this back into $|f(x) - L| < \epsilon$ to fully express the LHS in terms of δ .
- Solve for δ in terms of ϵ and pick $\delta = \min\{a, f(\epsilon)\}$.

A few *tips/tricks*:

- Apply the **Triangle Inequality**: $|a + b| \leq |a| + |b|$.
- Apply the identity: $|ab| = |a||b|$.
- Apply the inequality: $\frac{1}{x} > \frac{1}{x + a}$ for $x > 0$ given $a > 0$.
- Remember that $0 < |x - c| < \delta \implies c - \delta < x < c + \delta$.

Example 1: (2019 Midterm, Modified) Prove $\lim_{x \rightarrow 2} \frac{3x + 1}{(x + 1)^2} = 1$.

For any $\epsilon > 0$, we want to pick a $\delta > 0$ such that $0 < |x - 2| < \delta \implies \left| \frac{3x + 1}{(x + 1)^2} - 1 \right| < \epsilon$. We can start with:

$$\left| \frac{3x + 1}{(x + 1)^2} - 1 \right| < \epsilon \implies \left| \frac{3x + 1 - (x^2 + 2x + 1)}{(x + 1)^2} \right| \tag{1}$$

$$\implies \left| \frac{x - x^2}{(x + 1)^2} \right| < \epsilon \tag{2}$$

$$\implies \left| \frac{x(1 - x)}{(x + 1)^2} \right| < \epsilon \tag{3}$$

$$\implies \left| \frac{x}{(x + 1)^2} \right| |x - 1| < \epsilon \tag{4}$$

$$\implies \left| \frac{x}{(x + 1)^2} \right| |(x - 1 - 1) + (1)| < \epsilon \tag{5}$$

$$\implies \left| \frac{x}{(x + 1)^2} \right| (|x - 2| + |1|) < \epsilon \tag{6}$$

$$\implies \left| \frac{x}{(x + 1)^2} \right| (\delta + 1) < \epsilon \tag{7}$$

$$\tag{8}$$

We can set $\delta = 1$. If this is the case then:

$$0 < |x - 2| < 1 \implies 1 < x < 3 \iff 2 < x + 1 < 4 \tag{9}$$

We can bound the denominator $|(x+1)^2|$ by its lower bound $2^2 = 4$ and the numerator $|x|$ by its upper bound of 3, which we can substitute back in to get:

$$\left| \frac{x}{(x+1)^2} \right| (\delta+1) < \frac{3}{4}(\delta+1) \leq \epsilon \implies \delta \leq \frac{4}{3}\epsilon - 1 \quad (10)$$

Thus, we can pick:

$$\delta = \min\{1, \frac{4}{3}\epsilon - 1\} \quad (11)$$

and we are done. Note that we could also have applied the identity $\frac{1}{x} > \frac{1}{x+a}$ to bound the denominator by 1^2 instead.

1.2 Special Limits

For right handed limit, we have:

Definition: If for every $\epsilon > 0$, a $\delta > 0$ can be found such that $c < x < c + \delta \implies |f(x) - L| < \epsilon$, then $\lim_{x \rightarrow c^+} = L$.

For left handed limits:

Definition: If for every $\epsilon > 0$, a $\delta > 0$ can be found such that $c - \delta < x < c \implies |f(x) - L| < \epsilon$, then $\lim_{x \rightarrow c^-} = L$.

For infinite limits:

Definition: If for every $M > 0$, a $\delta > 0$ can be found such that $0 < |x - c| < \delta \implies f(x) > M$, then $\lim_{x \rightarrow c} = \infty$.

Here's an example using both:

Example 2: (2019 Quiz 2H, Modified) Prove the infinite limit $\lim_{x \rightarrow 2^+} \frac{x^{3/2}}{(x-2)^2} = \infty$.

For any $M > 0$, we want to pick a $\delta > 0$ such that $2 < x < 2 + \delta \implies \frac{x^{3/2}}{(x-2)^2} > M$. We can immediately start putting $\frac{x^{3/2}}{(x-2)^2} > M$ under δ control by minimizing the numerator and maximizing the denominator:

$$\frac{x^{3/2}}{(x-2)^2} > \frac{2^{3/2}}{(2+\delta-2)^2} \geq M \quad (12)$$

$$\implies \frac{2^{3/2}}{\delta^2} \geq M \quad (13)$$

$$\implies \frac{\delta^2}{2^{3/2}} \leq \frac{1}{M} \quad (14)$$

$$\implies \delta \leq \frac{2^{3/4}}{\sqrt{M}} \quad (15)$$

For horizontal asymptotes as $x \rightarrow \infty$:

Theorem: If for every $\epsilon > 0$, a $A > 0$ can be found such that $x > A \implies |f(x) - L| < \epsilon$, then $\lim_{x \rightarrow \infty} = L$.

Example 3: (Lecture 15, Assigned) Prove the limit $\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$ where $r > 0$.

For any $\epsilon > 0$, we want to pick a $A > 0$ such that $x > A \implies \left| \frac{1}{x^r} \right| < \epsilon$. We can place the LHS of $\left| \frac{1}{x^r} \right| < \epsilon$ straight away by minimizing the denominator by selecting the lower bound of x , which is A to get:

$$\frac{1}{x^r} < \frac{1}{A^r} \leq \epsilon \implies A \geq \epsilon^{1/r} \quad (16)$$

so choosing $A = \epsilon^{1/r}$ will always work.

2 Limit Theorems

Here are the limit theorems covered in class. Given $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$ are both well defined, then:

- **Constant Limit Theorem:** $\lim_{x \rightarrow c} A = A$
- **Additivity Limit Theorem:** $\lim_{x \rightarrow c} [f(x) + g(x)] = L + M$
- **Product Limit Theorem:** $\lim_{x \rightarrow c} [f(x)g(x)] = LM$
- **Polynomial Limit Theorem:** $\lim_{x \rightarrow c} P(x) = P(c)$ if $P(x)$ is a polynomial.
- **Rational Function Limit Theorem:** $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}$
- **Root Limit Theorem:** $\lim_{x \rightarrow c} f(x)^{1/n} = L^{1/n}$
- **Sandwich Limit Theorem:** If $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$ and $f(x) \leq g(x) \leq h(x)$ near c but not necessarily at c , then $\lim_{x \rightarrow c} g(x) = L$.

2.1 Limit Tips

To help with trigonometry limits, here are a few properties you should know (and understand how to derive):

- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$
- $\sin x \leq x \leq \tan x$ for $x \geq 0$. Since all these functions are odd, the inequality works in reverse for $x < 0$.
- $\sqrt{1 - x^2} \leq \cos x \leq 1$

When solving difficult trigonometry limits, try to break it up into $\sin x/x$ terms. If not possible, try to either bound the limit using the sandwich limit theorem, or bash through applying trig identities.

Other limits may involve terms that include e in both the problem and/or in the answer. The following properties of e may be helpful, and are derived in later sections:

- The **Taylor Expansion:** $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$
- $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$

3 Continuity Theorems

Here are the definitions for continuity at different points:

- **Continuity at a point:** $f(x)$ is continuous at c if $\lim_{x \rightarrow c} f(x) = f(c)$
- **Continuity on the right:** $f(x)$ is continuous on the right of c if $\lim_{x \rightarrow c^+} f(x) = f(c)$.
- **Continuity on the left:** $f(x)$ is continuous on the left of c if $\lim_{x \rightarrow c^-} f(x) = f(c)$.
- **Continuity on open interval:** $f(x)$ is continuous on (a, b) iff $f(x)$ is continuous at all $x \in (a, b)$.
- **Continuity on closed interval:** $f(x)$ is continuous on $[a, b]$ iff $f(x)$ is continuous at all $x \in (a, b)$ and $f(x)$ is continuous from the right of a and from the left of b .

There are also a few continuity theorems discussed in class:

- Given f, g , is continuous at a , then $f(x) + g(x)$ is continuous at a .
- If $g(x)$ is continuous at a and $f(x)$ is continuous at $g(a)$, then $f(g(x))$ is continuous at a .

4 Derivative Theorems

The derivative $f'(x)$ is defined as:

$$f'(x) \equiv \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (17)$$

where h is a dummy variable. A few definitions:

- **Differentiability at a point:** If $f'(a)$ exists, we say that $f(x)$ is differentiable at a .
- **Differentiability of function:** If $f'(x)$ is differentiable at all $x \in \text{domain of } f(x)$, then $f(x)$ is a differentiable function.
- **Differentiability on open interval:** $f(x)$ is differentiable on (a, b) if $f'(x)$ is defined for all $x \in (a, b)$
- **Differentiability on closed interval:** $f(x)$ is differentiable on $[a, b]$ if $f'(x)$ is defined for all $x \in (a, b)$ and the right hand derivative at a exists and the left hand derivative at b exists.
- **Relation to Continuity:** Given $f(x)$ is differentiable at a , then $f(x)$ is continuous at a .

When evaluating derivatives, there are a few theorems that we've learned. The following only apply if the derivatives of each function exists.

- **Constant DT:** If $f(x) = C$, then $f'(x) = 0$.
- **Additivity DT:** $(f + g)' = f' + g'$
- **Product DT:** $(fg)' = f'g + fg'$
- **Power DT:** If $f(x) = Cx^n$, then $f'(x) = nCx^{n-1}$.
- **Poly DT:** If $P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x^1 + a_0$, then $P'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \dots + a_1$.
- **Reciprocal DT:** $\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}$
- **Quotient DT:** $(f/g)' = \frac{f'g - fg'}{g^2}$.
- **Chain DT:** $\frac{d}{dx}f(g(x)) = g'(x)f'(g(x)) \iff \frac{df}{dx} = \frac{df}{dg} \frac{dg}{dx}$.

5 Features of a Graph

We can look at extrema points with derivatives:

- **Absolute Max:** $f(x)$ has an absolute maximum at c if $f(c) \geq f(x)$ for all $x \in \text{domain of } f(x)$.
- **Absolute Max in closed interval:** $f(x)$ has an absolute max on $[a, b]$ if $f(c) \geq f(x)$ for all $x \in [a, b]$.
- **Local Max:** $f(x)$ has a local max at c if $f(c) \geq f(x)$ for some open interval containing c .

Here are a few important theorems:

Theorem: Intermediate Value Theorem: Given that $f(x)$ is continuous on $[a, b]$ and C is some number such that $f(a) < C < f(b)$, there exists some C in $[a, b]$ such that $f(C) = C$.

Theorem: Extreme Value Theorem: Given $f(x)$ is continuous on $[a, b]$, then $f(x)$ has an absolute maximum $f(c)$ and an absolute minimum $f(d)$ for some $c, d \in [a, b]$.

Theorem: Rolle's Theorem: Given that f is continuous on $[a, b]$ and f is differentiable on (a, b) and $f(a) = f(b)$, then there exists some $c \in (a, b)$ such that $f'(c) = 0$. Note that there may be more than one c .

Theorem: Mean Value Theorem: Given that $f(x)$ is continuous on $[a, b]$ and $f(x)$ is differentiable on (a, b) , then there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

5.1 Estimation

We can approximate a function $f(x + \Delta x)$ as: $f(x + \Delta x) \approx f(x) + f'(x)\Delta x$. For example, this allows us to estimate something like $29^{1/3}$ as $27^{1/3} + \frac{d}{dx}x^{1/3} \Big|_{x=27} \cdot 2$.

An approximation by itself is useless without a bound. We can create lower and upper bounds by applying the MVT between $[x, x + \Delta x]$ and/or between $[x + \Delta x, x_1]$ and finding the minimum and maximum values for $f'(x)$.

6 Curve sketching

6.1 Formally Defining Horizontal Asymptotes

Horizontal asymptotes are formally defined as:

Definition: A horizontal asymptote occurs when $\lim_{x \rightarrow \infty} f(x) = L$. We can say that $f(x)$ goes to L as x goes to infinity if for any $\epsilon > 0$, a number A can be found s.t. for all $x > A$, $|f(x) - L| < \epsilon$.

The idea behind this revolves around finding f values as close to L as might be wanted by going to large enough x values.

An important theorem to determine horizontal asymptotes of reciprocal functions:

Theorem: The reciprocal horizontal asymptote limit:

$$\lim_{x \rightarrow \pm\infty} \frac{1}{x^r} = 0 \quad (18)$$

6.2 Prelims

We can use Fermat's theorem to determine critical points:

Definition: c is a critical point of $f(x)$ if $f'(c) = 0$ or $f'(c)$ DNE.

Here are some key features that might be seen on a graph:

- **Concavity:** If the graph of $y = f(x)$ lies above all its tangents in I , then $f(x)$ is concave up in I . If it lies below, then it is concave down.
- **Cusp:** A point c is a cusp if $f(x)$ is continuous at $x = c$ but $\lim_{x \rightarrow c^-} f'(x) = \pm\infty$ and $\lim_{x \rightarrow c^+} f'(x) = \mp\infty$.
- **Vertical Tangent:** A vertical tangent occurs when $\lim_{x \rightarrow c} |f'(x)| = \infty$ and $f(x)$ is continuous at c .
- **Slant Asymptote:** If $\lim_{x \rightarrow \infty} [f(x) - (mx + b)] = 0$, then $y = mx + b$ is a slant asymptote to $f(x)$ at $+\infty$.
- **Inflection point:** A point of inflection is at c if $f(x)$ is continuous at c and the sign of concavity changes at c .

A function is increasing on an interval I if $f(x_1) < f(x_2)$ for all $x_1 < x_2$ in I . Although we can use this definition to find local max/mins, there are a few cutie (QT/quick test) ways to do so:

- **QT1: Increasing/Decreasing Test.** If f is differentiable on the interval I , we show that if $f' > 0$, f is increasing. If $f' < 0$, f is decreasing. If $f' = 0$, f is constant.
- **QT2: First Derivative Test** Given that I contains a critical point and f is continuous at c_{crit} , and f is differentiable in I but not necessarily at c_{crit} . Then, if $f' > 0$ to the left of c_{crit} and $f' < 0$ to the right, then c_{crit} is a local max. If it's the opposite, we get the local minimum.
- **QT3: Concavity** Given that $f(x)$ is twice differentiable on I , then $f''(x)$ exists on I . As a result if $f''(x) > 0$, f is concave up. If $f'' < 0$, f is concave down.
- **QT4: Second Derivative Test** Given that $f''(x)$ is continuous near c and $f'(c) = 0$, then if $f''(c) > 0$, $f(c)$ is a local minimum. If $f''(c) < 0$, $f(c)$ is a local maximum. If $f''(c) = 0$, there is no verdict.

In general, the recipe to test for local max and min is to:

- Find all c_{crit} .
- If QT4 applies, use it.
- If it doesn't, and if QT2 applies, use it.
- If QT2 doesn't apply, use the basic definition of increasing/decreasing.

6.3 Curve Sketching Steps

1. Determine general behaviour:
 - Find Domain / Range / Limits at ∞ .
 - Determine endpoints if they exist.
 - Find vertical, horizontal, slant asymptotes if they exist:
2. Determine x and y intercepts.
3. Establish if $f(x)$ is symmetrical, even, odd, and/or periodic.
4. Find $f'(x)$ and use this to:
 - Find all critical points and $f(c_{\text{crit}})$.
 - Find when $f(x)$ is increasing/decreasing.

- Apply QT2.
 - Find vertical tangents / cusps if they exist.
5. Find $f''(x)$ and use it to:
- Find when $f(x)$ is concave up/down.
 - Find points of inflection if they exist.
 - Optional: Use QT4 to confirm local max/min
6. Determine the absolute maximum and min by choosing the largest and smallest values of f , if they exist.

7 Optimization

Here is a checklist for solving optimization problems. If we want to optimize f :

- Check critical points.
- Check for endpoints.
- Check for local max, min.
- Check $\lim_{x \rightarrow \infty}$ and $\lim_{x \rightarrow -\infty}$.
- Make a decision.

7.1 Numerical Methods

Theorem: The **method of successive bisections** can be performed if f is a continuous function and we can find values a and b such that $f(b) < 0 < f(a)$. These two values can be determined by trial and error. By IVT, the root must exist in between a and b . To use this method, we calculate the halfway point x_{h1} . If $f(x_{h1})$ is positive, we replace a with x_{h1} . If it's negative, we replace b with x_{h1} .

Theorem: Using **Newton's Method** is much faster computationally. However, there is the added restriction that $f(x)$ must be differentiable. It works in the following steps:

1. Make a first guess for the root, x_1
2. Find the equation for the tangent line at $(x_1, f(x_1))$
3. Find the x intercept of the tangent line, and let

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad (19)$$

and continue with x_2 . Note however, that this doesn't always work such as when it diverges away from the root such as $x^{1/3}$.

Here are the overall steps that are recommended:

1. Try Newton's Method first if function is differentiable.
2. If the x_n values converge, great!
3. If they do not, try another value.
4. If they still diverge, use the method of successive bisections.

8 Formal Definition of an Integral

The summation notation is denoted below:

Definition: If $a_m, a_{m+1}, a_{m+2}, \dots, a_n$ are real numbers and m and n are integers such that $m \leq n$, then:

$$\sum_{i=m}^n a_i = a_m + a_{m+1} + \dots + a_{n-1} + a_n \quad (20)$$

There are a few theorems:

- For a constant α :

$$\sum_{i=m}^n \alpha a_i = \alpha \sum_{i=m}^n a_i \quad (21)$$

- It is also linear:

$$\sum_{i=m}^n (a_i + b_i) = \sum_{i=m}^n a_i + \sum_{i=m}^n b_i \quad (22)$$

- $\sum_{i=1}^n \alpha = \alpha n$

- $\sum_{i=1}^n i = \frac{n(n+1)}{2}$

- $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

- $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2} \right)^2$

- $\sum_{i=1}^n i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

One way of defining an integral is thinking of the area under the curve. This introduces the concept of a **Riemann Sum**:

$$\sum_{i=1}^n f(x_i^*) \Delta x_i \quad (23)$$

where x_i represents points in the **partition** of the domain in which we want to approximate the area. The approximation gets more and more precise as the size Δx_i decreases. A few technical definitions to help:

Definition: A **partition** is a finite subset of the closed interval $[a, b]$, which contains the points a and b . Denoted by P .

Definition: The **norm** of $P = \|P\|$ which is the length of the longest subinterval:

$$\|P\| = \max(\Delta x_1, \Delta x_2, \dots, \Delta x_n) \quad (24)$$

Which can all be tied together to formally define the definite integral.

Definition: If f is a function defined on a closed interval $[a, b]$, let P be a partition of $[a, b]$ with partition $x_0, x_1, x_2, \dots, x_n$

where:

$$a = x_0 < x_1 < x_2 < \dots < x_n = b \quad (25)$$

Choose points x_i^* within each subinterval $[x_{i-1}, x_i]$ and let $\Delta x_i = x_i - x_{i-1}$, and $\|P\| = \max\{\Delta x_i\}$. Then the **definite integral** of f from a to b is:

$$\int_a^b f(x) dx \equiv \lim_{\|P\|} \sum_{i=1}^n f(x_i^*) \Delta x_i \quad (26)$$

if the limit exists. If the limit does exist, then f is called integrable on the interval $[a, b]$. The sign \int is called the integral sign. $f(x)$ is known as the **integrand**, and a, b are the limits of integration. The output is a single number that does not depend on x .

We can formally show that the definite integral can take on a specific value I with a $\delta - \epsilon$ statement:

Idea: If we have:

$$\int_a^b f(x) dx = I \quad (27)$$

then for ever $\epsilon > 0$, there exists a $\delta > 0$ such that:

$$\left| I - \sum_{i=1}^n f(x_i^*) \Delta x_i \right| < \epsilon \quad (28)$$

for all partitions P of $[a, b]$ with $\|P\| < \delta$ and all possible choices of x_i^* in $[x_{i-1}, x_i]$.

However, going through this proof would be a nightmare. Instead, we can show **integrability** via the following theorem:

Theorem: Continuous and/or piecewise continuous on $[a, b]$ guarantees integrability on $[a, b]$,

Definition: A function is **piecewise continuous** if it only has a finite number of jump discontinuities.

Now that we know when the integral exists, we can find ways of calculating it from scratch:

Idea: Going through with a full Riemann sum calculation is also tedious. As a result, here are a few conventions to make it easier:

- We usually select regular partitions:

$$\Delta x = \Delta x_1 = \Delta x_2 = \dots = \Delta x_n = \frac{b-a}{n} \quad (29)$$

- And we select x_i^* to be the RH end point such that:

$$x_i^* = x_i = a + i \Delta x = a + i \frac{b-a}{n} \quad (30)$$

Therefore, the integral can be written as:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(a + i \frac{b-a}{n}\right) \frac{b-a}{n} \quad (31)$$

9 Properties of Integration

9.1 Definite Integral Properties

There are a few properties:

- Constant:

$$\int_a^b c \, dx = c(b - a) \quad (32)$$

- Additivity:

$$\int_a^b (f(x) \pm g(x)) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx \quad (33)$$

- Constant Multiple:

$$\int_a^b c(f(x)) \, dx = c \int_a^b f(x) \, dx \quad (34)$$

- Changing Limits:

$$\int_a^b f(x) \, dx = \int_a^z f(x) \, dx + \int_z^b f(x) \, dx \quad (35)$$

There are also **order properties** of integrals. If $a < b$, then:

- If $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b f(x) \, dx \geq 0 \quad (36)$$

- If $f(x) \geq g(x)$ for $a \leq x \leq b$, then:

$$\int_a^b f \, dx \geq \int_a^b g(x) \, dx \quad (37)$$

- If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then:

$$m(b - a) \leq \int_a^b f \, dx \leq M(b - a) \quad (38)$$

- Absolute values:

$$\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx \quad (39)$$

9.2 Fundamental Theorem of Calculus

The **first fundamental theorem of calculus** states that:

Theorem: Let f be continuous on $[a, b]$. The function F is defined on $[a, b]$ by:

$$F(x) = \int_a^x f(t) \, dt \quad (40)$$

is continuous on $[a, b]$, differentiable on (a, b) , and has derivative:

$$F'(x) = f(x) \quad (41)$$

for all $x \in (a, b)$.

Rarely (never) will you get a simple question like this. Sometimes, the upper bound is a function $g(x)$ instead. If this is the case, then:

Idea: Assuming that f is continuous in $[a, b]$, then the function F is defined on $[a, b]$ by:

$$F(x) = \int_a^{g(x)} f(t) dt \quad (42)$$

has a derivative:

$$F'(x) = g'(x)f(g(x)) \quad (43)$$

for $x \in (a, b)$. To see why this is true, we can apply the chain rule:

$$F'(x) = \frac{d}{dx} f(g(x)) = g'(x)f(g(x)) \quad (44)$$

The **second fundamental theorem of calculus** states that:

Theorem: Let f be continuous on $[a, b]$. If G is any antiderivative for f on $[a, b]$, then:

$$\int_a^b f(t) dt = G(b) - G(a) \quad (45)$$

This can alternatively be written as:

$$\int_a^b F'(x) dx = F(b) - F(a) \quad (46)$$

which can be interpreted as the net change of $F(x)$. For example:

$$\Delta x = \int_a^b v(t) dt \quad (47)$$

gives the displacement from $t = a$ to $t = b$. The proofs for these two theorems are provided below:

Proof: (1st theorem) For x and $x + h$ in (a, b) ,

$$F(x + h) - F(x) = \int_a^{x+h} f(x) dt - \int_a^x f(x) dt \quad (48)$$

$$= \int_a^x f(t) dt + \int_x^{x+h} f(t) dt - \int_a^x f(t) dt \quad (49)$$

$$= \int_x^{x+h} f(t) dt \quad (50)$$

For $h \neq 0$, we have:

$$\frac{F(x + h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t) dt \quad (51)$$

We can separate it into cases. If $h > 0$, then we can write per the extreme value theorem the minimum value of f as $f(u) = m$ and the maximum value as $f(v) = M$ for $u, v \in [x, x + h]$ such that:

$$mh \leq \int_x^{x+h} f(t) dt \leq Mh \quad (52)$$

or:

$$f(u)h \leq \int_x^{x+h} f(t) dt \leq f(v)h \quad (53)$$

which we can rewrite, after dividing through by h :

$$f(u) \leq \frac{F(x + h) - F(x)}{h} \leq f(v) \quad (54)$$

As $h \rightarrow 0$, we have $u \rightarrow x$ and $v \rightarrow x$. Therefore:

$$\lim_{h \rightarrow 0} f(u) = \lim_{u \rightarrow x} f(u) = f(x) \quad (55)$$

$$\lim_{h \rightarrow 0} f(v) = \lim_{v \rightarrow x} f(v) = f(x) \quad (56)$$

which gives us:

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = f(x) \quad (57)$$

or:

$$\frac{d}{dx} \int_a^x f(t) dt = f(x) \quad (58)$$

Proof: (2nd theorem) Given that $F(x) = \int_a^x f(t) dt$ is an antiderivative of f and given that G is an antiderivative, then:

$$F'(x) = G'(x) \implies F(x) = G(x) + C \quad (59)$$

We know that $F(a) = 0$, so $G(a) + C = 0$ or $C = -G(a)$, which gives:

$$\int_a^b f(t) dt = F(b) = G(b) - G(a) \quad (60)$$

9.3 Integration Tricks

The **u-substitution** essentially reverses the chain rule.

Idea: Suppose we have an integral in the form:

$$\int f(g(x))g'(x) dx \quad (61)$$

If we let $u = g(x)$, then $du = g'(x)dx$. So we can simplify the integral to:

$$\int f(u) du = F(u) + C = F(g(x)) + C \quad (62)$$

Once we have the indefinite integral, we can use back substitution to find the definite integral. We can avoid this step using a change of variables.

Theorem:

$$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du \quad (63)$$

In general, here are a few tips, in no particular order:

- Refer to the table of integrals at the back of the book. You are allowed to use them.
- Look for symmetry and periodicity.
- Draw a picture. Sometimes, you can avoid a complicated integral and use plain old geometry this way!
- For u-substitution, look for derivative-function pairs.
- If there are not too many terms, you can sometimes expand functions into a polynomial.

- Check if the integral even exists!
- Apply the first theorem of calculus, if applicable.
- See if the integral (or a similar one) is in the book.

10 Areas and Volumes

10.1 Areas Between Curves

Suppose we wish to find the **area between two curves** $f(x)$ and $g(x)$. We can do this by partitioning the area into infinitesimally small rectangles:

$$\Delta A_i = [f(x_i^*) - g(x_i^*)] \Delta x_i \quad (64)$$

so that the area is given by:

$$A = \lim_{\|P\| \rightarrow 0} \sum_{i=1}^n [f(x_i^*) - g(x_i^*)] \Delta x_i \quad (65)$$

$$= \int_a^b [f(x) - g(x)] dx \quad (66)$$

If we let $f(x) \geq g(x)$ when $x \in [a, b]$, then this gives the difference in their areas $A_1 - A_2$. If the condition $f(x) \geq g(x)$ is not satisfied, then we must break up the integral into multiple parts (if we interpret the area as having a positive area only). We can modify the area formula to be:

$$A = \int_a^b |f(x) - g(x)| dx \quad (67)$$

Suppose we have a curve $x = f(y)$ and $x = g(y)$ instead. The area between $y = a$ and $y = b$ works in the same way:

$$A = \int_a^b |f(y) - g(y)| dy \quad (68)$$

10.2 Volumes

We can determine the **volume** of a solid by partitioning it into thin cylinders whose axes are parallel to the x axis. Then we can break up the volume into thin sections:

$$V_i \simeq A_i \Delta x_i \quad (69)$$

so the volume is:

$$V = \int_a^b A(x) dx \quad (70)$$

which is the general formula for the volume of any shape. If we can figure out $A(x)$ and the necessary bounds, we can find the volume for any change.

Idea: For **solids of revolution**, we rotate a curve $f(x)$ about the x axis. The volume of this solid using the **disk method** is:

$$V = \int_a^b \pi f(x)^2 dx \quad (71)$$

Similarly around the y axis:

$$V = \int_c^d \pi g(y)^2 dy \quad (72)$$

For the volume by rotating the region between two curves $f(x)$ and $g(x)$, we get:

$$V = \int_a^b \pi (f(x)^2 - g(x)^2) dx \quad (73)$$

which is known as the **washer method**.

Sometimes, the disk and washer method is too difficult to apply.

Idea: We can use the **shell method about the y-axis** to find the volume when a curve is rotated about the y axis. Suppose we wish to rotate a curve $f(x)$ from $x = a$ to $x = b$ around the y axis. Then the volume is:

$$V = \int_a^b 2\pi x f(x) dx \quad (74)$$

Similarly, if a curve is rotated about the x axis, we use the **shell method about the x-axis**:

$$V = \int_a^b 2\pi y f(y) dy \quad (75)$$

11 Misc

I honestly don't know where this section belongs, so I'm just copying and pasting from my notes (which I actually spent a decent amount of effort on):

- The average of a discrete set $\{a_1, a_2, \dots, a_N\}$ is given by:

$$a_{\text{avg}} = \frac{1}{N} \sum_i^N a_i \quad (76)$$

- For a continuous distribution, we can extend this to:

$$f_{\text{avg}} = \frac{1}{N} \sum_i^N f(x_i^*) \quad (77)$$

Taking the limit as $N \rightarrow \infty$, we get:

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx \quad (78)$$

Theorem: Mean Value Theorem for Integrals: If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that:

$$f(c) = f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx \quad (79)$$

Proof: Define $F(x) = \int_a^x f(t) dt$. If we apply the mean value theorem to F , then:

$$F'(c) = \frac{F(b) - F(a)}{b-a} \quad (80)$$

for some $c \in [a, b]$. Now since:

$$F'(x) = f(x) \quad (81)$$

it becomes apparent that:

$$f(c) = \frac{\int_a^b f(t) dt - \int_a^a f(t) dt}{b-a} = \frac{1}{b-a} \int_a^b f(t) dt \quad (82)$$

- We can also introduce **inverse functions**.

Definition: A function $f(x)$ is said to be one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Alternatively, we can say that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

- We can use the **horizontal line test**. If any horizontal line crosses the function more than one time, then it is not one-to-one.

Definition: Let f be a 1-1 function with domain A and range B . Then its inverse function, f^{-1} has domain B and range A , and is defined by:

$$f^{-1}(x) = y \iff f(y) = x \quad (83)$$

Therefore:

$$f^{-1}(f(x)) = f(f^{-1}(x)) = x \quad (84)$$

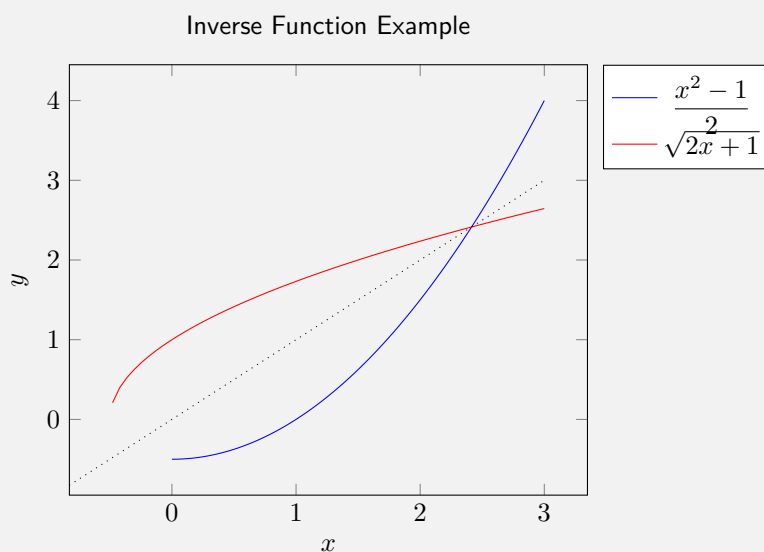
Warning: To prevent confusion, note that:

$$\frac{1}{f(x)} = [f(x)]^{-1} \neq f^{-1}(x) \quad (85)$$

- Geometrically, the inverse of a function represents a reflection of each point across the line $y = x$.

Example 4: If $g(x) = \sqrt{2x+1}$, it is implied that $x \geq -1/2$, so it is a one-to-one function. Therefore, the inverse function is:

$$g^{-1}(x) = \frac{x^2 - 1}{2} \quad (86)$$



Theorem: If f is either an increasing or decreasing function, then f is 1-1, and hence, has an inverse.

Proof. Say $f(x)$ is decreasing, then $x_1 < x_2 \implies f(x_1) > f(x_2)$ and if $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$. \square

Theorem: Let f be a 1-1 function defined on an interval I . If f is continuous, then f^{-1} is also continuous. (Proof provided in Appendix F)

- Let $g(x) = f^{-1}(x)$. Then:

$$f(g(x)) = x \quad (87)$$

$$\frac{d}{dx} f(g(x)) = 1 \quad (88)$$

$$f'(g(x))g'(x) = 1 \quad (89)$$

$$g'(x) = \frac{1}{f'(g(x))} \quad (90)$$

or:

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \quad (91)$$

which is equivalent to:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad (92)$$

Theorem: The inverse of composite functions is given by:

$$(f \circ g)^{-1} = g^{-1} \circ f^{-1} \quad (93)$$

Proof. Let $y = (f \circ g)^{-1}(x)$. Then:

$$x = (f \circ g)(y) = f(g(y)) \quad (94)$$

so we have:

$$g(y) = f^{-1}(x) \quad (95)$$

$$y = g^{-1}(f^{-1}(x)) \quad (96)$$

$$= (g^{-1} \circ f^{-1})(x) \quad (97)$$

□

12 Logarithms and Exponentials

Warning: Note that in this section, I make the assumption you are already familiar with general logarithm and exponential properties, so I won't spend time writing those down.

Definition: A **logarithm function** is a nonconstant differentiable function f defined for $x \in \{\mathbb{R}, (0, \infty)\}$ such that for all $a > 0$ and $b > 0$:

$$f(a \cdot b) = f(a) + f(b) \quad (98)$$

It has the following properties:

- $f(1) = 0$
- $f(1/x) = -f(x)$
- $f(x/y) = f(x) - f(y)$
- $f'(x) = \frac{1}{x} f'(1)$.

This leads to the definition of the **natural logarithm**:

Definition: The natural logarithm is defined as:

$$\ln(x) = \int_1^x \frac{dt}{t} \quad (99)$$

Note that $\ln(x)$ is not the antiderivative of $\frac{1}{t}$. We can instead write:

$$\int \frac{dt}{t} = \ln|x| + C \quad (100)$$

since x can be negative as well.

Theorem: **Feynman's trick of Differentiation^a** (otherwise known as logarithmic differentiation): The following was

popularized by Richard Feynman during the first of his Feynman Lectures. If we have a function:

$$g(x) = g_1(x)g_2(x)g_3(x) \cdots g_n(x) \quad (101)$$

Then taking the natural logarithm of both sides, applying the chain rule, and simplifying gives:

$$g'(x) = g(x) \left(\frac{g'_1}{g_1} + \frac{g'_2}{g_2} + \cdots + \frac{g'_n}{g_n} \right) \quad (102)$$

^aNote that this is not a formal name. I just chose it to name it after Feynman because I'm a huge Feynman stan and I first heard about it in the preface to the Feynman Lectures where he was talking about mathematical tricks.

Exponential functions can be introduced:

Definition: If z is a real number, then e^z is the number such that:

$$\ln(e^z) = z \quad (103)$$

More formally, we can write the exponential function as $\exp(x) = e^x$. The most useful property of e^x is that:

$$\frac{d}{dx} e^x = e^x \quad (104)$$

We can extend this to general logarithmic and exponential functions. If $x > 0$, then we can define:

Definition: The **general exponential** function is defined as

$$x^z = e^{z \ln x} \quad (105)$$

if $x > 0$.

Similarly:

Definition: The **general logarithm** can be defined as:

$$\log_p(x) = \frac{\ln x}{\ln p} \quad (106)$$

such that:

$$\frac{d}{dx} a^x = \ln(a) a^x \quad (107)$$

and

$$\frac{d}{dx} \log_p(x) = \frac{1}{x \ln p} \quad (108)$$

12.1 Bounding e

Idea: We can first bound e^x by setting a lower limit (which happens to be the Taylor series!). Notice that via integration:

$$e^x = 1 + \int_0^x e^t dt \quad (109)$$

Since e^x is always increasing, we can claim that $e^x > 1$ for $x > 0$ such that:

$$e^x = 1 + \int_0^x e^t dt > 1 + \int_0^x 1 dt = 1 + x \quad (110)$$

We can then repeat the previous step to show that

$$e^x = 1 + \int_0^x e^t dt > 1 + \int_0^x (1 + t) dt = 1 + x + \frac{x^2}{2} \quad (111)$$

Repeating the process, we eventually get:

$$e^x > 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \quad (112)$$

Instead of choosing to bound e^x , we can also choose to bound e . We have that:

$$\ln x = \int_1^x \frac{dt}{t} \quad (113)$$

such that:

$$\ln \left(1 + \frac{1}{n} \right) = \int_1^{1+1/n} \frac{dt}{t} < \int_1^{1+1/n} 1 dt \quad (114)$$

Since $\frac{1}{t} < \frac{1}{1}$ for $t > 0$. The upper bound then becomes:

$$1 + \frac{1}{n} - 1 = \frac{1}{n} \implies \ln \left(1 + \frac{1}{n} \right) < \frac{1}{n} \quad (115)$$

We can similarly repeat this process:

$$1 + \frac{1}{n} < e^{1/n} \implies \left(1 + \frac{1}{n} \right)^n < e \quad (116)$$

Note that if we take the limit as $n \rightarrow \infty$, *intuitively* we would expect the upper bound to become closer and closer to the true value. We shall explore this further, and we can write the lower bound as:

$$\ln \left(1 + \frac{1}{n} \right) = \int_1^{1+1/n} \frac{dt}{t} > \int_1^{1+1/n} \frac{dt}{1+1/n} \quad (117)$$

since $\frac{1}{t} > \frac{1}{1+1/n}$. We can write this in logarithm form to get:

$$\ln \left(1 + \frac{1}{n} \right) > \left(\frac{1}{1+1/n} \right) \left(1 + \frac{1}{n} - 1 \right) = \frac{1}{n+1} \implies \left(1 + \frac{1}{n} \right)^{n+1} > e \quad (118)$$

Putting it altogether, we have the following statement:

Idea: e can be estimated with its lower and upper bound with the following:

$$\left(1 + \frac{1}{n} \right)^n < e < \left(1 + \frac{1}{n} \right)^{n+1} \quad (119)$$

13 Inverse Trigonometric Functions

We can define the inverse function of trigonometric functions by restricting their domain, such as from $-\pi/2$ to $\pi/2$ for $\sin(x)$.

Definition: The inverse function for $\sin(x)$ is given by :

$$\sin^{-1}(x) = \arcsin(x) \quad (120)$$

Warning: You need to be very careful with the domain and range. Sometimes, if x falls out of the domain, it can lead to a different answer altogether, or it could be undefined.

There's a lot of formulas for this one, but to derive formula such as $\sin(\tan^{-1}(x))$, you just need to draw a picture of a right angled triangle with one of the legs as x and either the hypotenuse or the other leg as 1. If proofs are not needed, there is a formula sheet with all properties at the end of the book.

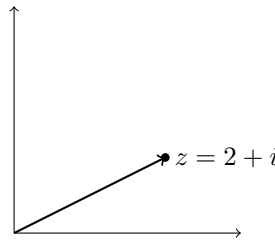
14 Complex Numbers

- We can introduce complex numbers to assign values to the solutions of algebraic equations such as:

$$x^2 = -1 \quad (121)$$

Definition: A complex number is defined as $z = a + ib$ where $a, b \in \mathbb{R}$ and $\text{Re}(z) = a$ and $\text{Im}(z) = b$.

- We can represent complex numbers on a plane:



- It is often helpful to write out a complex number using polar coordinates. The **modulus** of the number is:

$$|z| = |a + ib| = \sqrt{a^2 + b^2} \quad (122)$$

and the argument is the angle it makes with the real axis:

$$\arg(z) = \theta + 2k\pi \quad (123)$$

where k is an integer. This means that:

$$\begin{aligned} |z| \cos(\theta) &= a \\ |z| \sin(\theta) &= b \end{aligned}$$

Idea: The **polar representation** can be written as:

$$z = r (\cos \theta + i \sin \theta) \quad (124)$$

where $r = |z|$.

- The **complex conjugate** for a complex number $z = a + ib$ is:

$$\bar{z} = a - ib \quad (125)$$

- Let $z_1 = a + ib$ and $z_2 = c + id$. Then complex addition/subtraction has the following properties:

- $z_1 + z_2 = (a + c) + i(b + d)$
- $z_1 + z_2 = z_2 + z_1$ (commutative)
- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ (associative)
- $|z_1 + z_2| \leq |z_1| + |z_2|$ (triangle inequality)
- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

- Complex multiplication can be defined as:

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc) \quad (126)$$

It has the following properties:

- $z_1 \cdot z_2 = z_2 \cdot z_1$ (commutative)
- $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ (associative)
- $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$ (distributive)
- $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

Idea: When multiplying two complex numbers in their polar form, we get:

$$z_1 z_2 = r_1 r_2 (\cos(\theta_\phi) + i \sin(\theta + \phi)) \quad (127)$$

Note that:

$$\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2) \quad (128)$$

and the modulus is:

$$|z_1 z_2| = |z_1| |z_2| \quad (129)$$

What this means is that the magnitudes get multiplied like scalars and z_1 is rotated by the argument of z_2 .

- One direct consequence of this idea is that multiplying by i is equivalent to rotating counterclockwise a complex number by 90 degrees. Note that this is an important concept that will appear when dealing with phasors in the circuit course.

Theorem: De Moivre's Theorem: Let $z = \cos \theta + i \sin \theta$. We have $|z| = 1$ and $\arg(z) = \theta$. Then:

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta) \quad (130)$$

Definition: We can define division by multiplying the denominator by its conjugate:

$$\frac{1}{z} = \frac{1}{a + ib} = \frac{a - ib}{a^2 + b^2} + \frac{\bar{z}}{|z|^2} \quad (131)$$

Therefore:

$$\left| \frac{1}{z} \right| = \frac{1}{|z|} \quad (132)$$

and:

$$\arg\left(\frac{1}{z}\right) = -\arg(z) \quad (133)$$

- The most important tool in working with complex numbers is the complex exponential:

$$z = e^{ix} \quad (134)$$

We cannot define this by making the following observation. Note that the derivative of $f(x) = e^{ix}$ is:

$$f'(x) = ie^{ix} = if(x) \quad (135)$$

and $f(0) = 1$. If we define $g(x) = \cos(x) + i \sin(x)$, then:

$$g'(x) = -\sin(x) + i \cos(x) = ig(x) \quad (136)$$

and $g(0) = 1$ also. Therefore, it seems convincing that $f(x) = g(x)$ or:

$$e^{ix} = \cos(x) + i \sin(x) \quad (137)$$

This is not a complete proof however, but will be rigorously proved next semester by using a Taylor series.

15 Differential Equations

- A differential equation can be defined as:

Definition: A differential equation is an equation which contains an unknown function with one or more of its derivatives.

- A ordinary differential equation refers to one independent variable.
- The order of a differential equation refers to the highest derivative.

Definition: The general solution refers to an n parameter family of solutions if they include all solutions to the differential equation.

Definition: A particular solution refers to constants that are assigned particular values according to initial values, or boundary values.

- **Separable DEs** are the simplest, they are a first order homogeneous equation in the form of:

$$\frac{dy}{dx} = f(x)g(y) \quad (138)$$

and to solve this, we just need to solve the following:

$$\int \frac{dy}{g(y)} = \int f(x) dx \quad (139)$$

- **Orthogonal trajectories** refer to curves that pass through a family of curves such that they remain perpendicular to each other such that:

$$f' = \frac{1}{g'} \quad (140)$$

- There are many instances in physics and nature where the growth of a function is related to the function at that point, such as:

$$\frac{df}{dt} = kf(t) \implies k = \frac{1}{f} \frac{df}{dt} = \frac{d}{dt}(\ln f) \quad (141)$$

Separating, we get:

$$\ln(f) + kt + C \implies f = Ce^{kt} \quad (142)$$

where C is based off initial conditions.

- The **doubling time** refers to the time for a function to double:

$$2P_0 = P_0 e^{kt_2} \implies t_2 = \frac{\ln 2}{k} \quad (143)$$

- In many areas (such as radioactive decay), the half life gives the time necessary for the function to half. This occurs in functions where the DE looks like:

$$\frac{df}{dt} = -kN \quad (144)$$

where $k > 0$. Similarly, the half life is given by:

$$t_{1/2} = \frac{\ln 2}{k} \quad (145)$$

- For compound interest, the annual interest is given by:

$$V(t) = V_0(1+i)^t \quad (146)$$

If we compound the interest more and more often, we get:

$$V(t) = V_0 \left(1 + \frac{i}{n}\right)^{nt} \quad (147)$$

Taking the limit as $n \rightarrow \infty$, we get:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{i}{n}\right)^{nt} = V_0 \lim_{m \rightarrow \infty} \left(\left(1 + \frac{1}{m}\right)^m\right)^{it} \quad (148)$$

$$= V_0 e^{it} \quad (149)$$

where we made the substitution $m = n/i$.

- The **logistic model** is a realistic model for population growth:

$$\frac{dP}{dt} = kP \left(1 - \frac{P}{M}\right) \quad (150)$$

and the solution gives:

$$P(t) = \frac{M}{1 + Ae^{-kt}} \quad (151)$$

15.1 First Order Equations

- In general, the solution to a **linear first order equation**

$$y' + p(x)y = q(x) \quad (152)$$

is

$$y = e^{-H(x)} \left[\int e^{H(x)} q(x) dx + C \right] \quad (153)$$

where the **integrating factor** is $e^{H(x)}$ where:

$$H(x) = \int p(x) dx \quad (154)$$

with a constant of integration of zero.

- A **Bernoulli Equation** is a **nonlinear first order equation** that can be solved. They are in the form of:

$$y' + p(x)y = q(x)y^r \quad (155)$$

For $r \neq 0, 1$, we can make the substitution $u = y^{1-r}$ to simplify it to:

$$u' + (1-r)p(x)u = (1-r)q(x) \quad (156)$$

15.2 Homogeneous Second Order Equations

- A **homogeneous second order linear DE** takes on the form of:

$$y'' + ay' + by = 0 \quad (157)$$

To solve this, we need to solve the characteristic equation:

$$r^2 + ar + b = 0 \quad (158)$$

- There are three cases:

– Case One: $a^2 - 4b > 0$: Then r_1, r_2 are real and distinct so the general solution is:

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} \quad (159)$$

– Case Two: $a^2 - 4b = 0$: Then $r_1 = r_2 = -\frac{a}{2} = r$. Then the solution is:

$$y = e^{rx} + xe^{rx} \quad (160)$$

– Case Three: $a^2 - 4b < 0$: Then $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ where $\alpha = -\frac{a}{2}$ and $\beta = \frac{1}{2}\sqrt{4b - a^2}$. Using the complex identity, we can rewrite this as:

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} \quad (161)$$

$$= C_1 e^{\alpha x} (\cos \beta x + i \sin \beta x) + C_2 e^{\alpha x} (\cos \beta x - i \sin \beta x) \quad (162)$$

$$= e^{\alpha x} ((C_1 + C_2) \cos \beta x + i(C_1 - C_2) \sin \beta x) \quad (163)$$

$$= e^{\alpha x} (A \cos \beta x + B \sin \beta x) \quad (164)$$

where the coefficients could either be real or complex. Typically, we only look at the real part when dealing with boundary conditions that only look at the real part.

Theorem: If $y_1(x)$ and $y_2(x)$ are both solutions of a homogeneous second order linear differential equation and c_1, c_2 are any constants, then the linear combination:

$$y(x) = C_1 y_1(x) + C_2 y_2(x) \quad (165)$$

is also a solution.

Proof. We have:

$$(c_1 y_1 + c_2 y_2)'' + a(c_1 y_1 + c_2 y_2)' + b(c_1 y_1 + c_2 y_2) = 0 \quad (166)$$

$$c_1 (y_1'' + a y_1' + b y_1) + c_2 (y_2'' + a y_2' + b y_2) = 0 \quad (167)$$

$$c_1(0) + c_2(0) = 0 \quad (168)$$

□

Theorem: If $y_1(x)$ and $y_2(x)$ are linearly independent solutions to a homogeneous second order linear differential equation, then:

$$y(x) = C_1 y_1(x) + C_2 y_2(x) \quad (169)$$

is the general solution. Two solutions are linearly independent iff:

$$y_2(x) \neq C y_1(x) \quad (170)$$

15.3 Nonhomogeneous Second Order Differential Equations

- A **nonhomogeneous second order linear DE** is in the form of

$$y'' + ay' + by = \phi(x) \quad (171)$$

We can define the **complementary equation** to be:

$$y'' + ay' + by = 0 \quad (172)$$

Theorem: The general solution of a nonhomogeneous second order linear differential equation with constant coefficients is given by:

$$y(x) = y_p(x) + y_c(x) \quad (173)$$

where $y_p(x)$ is a particular solution of the complete differential equation and $y_c(x)$ is the general solution of the complementary homogeneous equation.

- The idea behind the **method of undetermined coefficients** is to assume that the undetermined function has the same form as $\phi(x)$. Suppose that $\phi(x)$ is in the form of:

$$\phi(x) = e^{kx} f(x) \quad (174)$$

with k possibly being equal to zero. We can proceed depending on what $f(x)$ is:

- If $f(x)$ is a polynomial $P(x)$, then guess a particular solution that is a quadratic with the same degree as $P(x)$. For example, if $P(x)$ is a quadratic, then guess:

$$y_p(x) = Ax^2 + Bx + C \quad (175)$$

- If $f(x)$ is in the form of $P(x) \sin(mx)$, then guess:

$$y_p(x) = e^{kx} (Q(x) \cos mx + R(x) \sin mx) \quad (176)$$

After guessing a solution, solve for the undetermined coefficients.

- Note that if the particular solution you guess is contained in the complementary solution, you need to prevent redundancy by multiplying it by x or x^2 .
- We can extend this to equations in the form of:

$$y'' + ay' + by = \phi_1(x) + \phi_2(x) \quad (177)$$

we can apply the superposition principle to determine the particular solution to be the particular solution to $\phi_1(x)$ added to the particular solution to $\phi_2(x)$.

- We can also use the methods of **variation of parameters** since guessing may not always be the most reliable. In general, if we have a differential equation in the form of:

$$y'' + ay' + by = \phi(x) \quad (178)$$

then the complementary solution is given as:

$$y_c = Ay_1(x) + By_2(x) \quad (179)$$

where $y_1(x) = e^{r_1x}$, $y_2(x) = e^{r_2x}$, and r_1, r_2 are the solutions to the quadratic:

$$r^2 + ar + b = 0 \quad (180)$$

except for the case of a double root.

- The particular solution is given by:

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \quad (181)$$

where $u_1'(x)$ and $u_2'(x)$ are given by:

$$u_1'(x) = \frac{-y_2\phi(x)}{y_1y_2' - y_2'y_1'} \quad (182)$$

$$u_2'(x) = \frac{y_1\phi(x)}{y_1y_2' - y_2'y_1'} \quad (183)$$

- Integrating and letting the constant of integration to be zero, we can solve for $u_1(x)$ and $u_2(x)$. The general solution is then:

$$y = y_c(x) + y_p(x) \quad (184)$$