ESC195 Notes

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Contents

1 Alternating Series

- Some series have both positive and negative terms, such as:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \tag{1}$$

Theorem: The **Alternating Series Test**: Let $\{a_k\}$ be a sequence of positive numbers. If and only if $a_{k+1} < a_k$ and $a_k \to 0$ as $k \to \infty$, then:

$$\sum_{k=1}^{\infty} (-1)^{k-1} a_k \tag{2}$$

converges.

Proof. Let $S_2 = a_1 - a_2 > 0$ and $s_4 = s_2 + (a_3 - a_4)$. We can generalize this to:

$$S_{2n} = S_{2n-2} + (a_{2n-1} - a_{2n}) > S_{2n-2} \tag{3}$$

such that $\{S_{2n}\}$ is monotonically increasing. However, we also have:

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \tag{4}$$

Since $S_{2n} < a_1$ for all n, we can apply the monotonic limit theorem to show that the limit L exists. We then have:

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} a_{2n+1} = L \tag{5}$$

□

Example 1: Take the sum $1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{9} + \frac{1}{3} - \cdots$. Although $a_n \to 0$, the terms are not decreasing in mangitude, so it is divergent.

- For an alternating sequence, the limit will be between S_n and S_{n+1} so we can estimate the error as:

$$|L - S_n| \leq a_{n+1} \tag{6}$$

- For example, the series expansion for e^{-1} is:

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots \tag{7}$$

If we continue to the $\frac{1}{5!}$ term, then we get:

$$e^{-1} \simeq 0.3666 \pm \frac{1}{6!} \tag{8}$$

- We introduce the absolute convergence and the ratio and root tests.

Definition: If $\sum |a_k|$ converges, we say that $\sum a_k$ is absolutely convergent. If $\sum a_k$ converges, but $\sum |a_k|$ does not, we say $\sum a_k$ is conditionally convergent.

Theorem: If $\sum |a_k|$ converges, then $\sum a_k$ converges.

Proof. Let:

$$-|a_n| \leq a_n \leq |a_n| \tag{9}$$
$$0 \leq a_n + |a_n| \leq 2|a_n| \tag{10}$$
$$0 \leq b_n \leq 2|a_n| \tag{11}$$

Note: Let $\sum a_n = \sum b_n - \sum |a_n|$. Since both $\sum b_n$ and $\sum |a_n|$ is convergent, then the original sum must be convergent as well. □

- For example, $\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}$ is conditionally convergent.

Theorem: The **Root Test:** Given $\sum a_k$, $a_k \geq 0$. If $(a_k)^{1/k} \to p$ as $k \to \infty$, then:

1. If $p < 1$, then $\sum a_k$ converges.
2. If $p > 1$, then $\sum a_k$ diverges.
3. If $p = 1$ the test is inconclusive.

Proof. Given $p < 1$, choose μ such that $p < \mu < 1$. Since $(a_k)^{1/k} \to p$, we have:

$$(a_k)^{1/k} < \mu \tag{12}$$

or

$$a_k < \mu^k \tag{13}$$

for k sufficiently large. But $\sum \mu^k$ converges (geometric series, $x < 1$), so $\sum a_k$ converges as well. □

Example 2: Take the series $\sum \left(\frac{n^2+1}{2n^1+1}\right)^n$. Note that $a_n^{1/k} = \frac{2k}{k+1} \to \frac{1}{2}$ so the series is convergent.

Theorem: The **ratio test:** GTiven $\sum a_k$, with $a_k > 0$. If $\frac{a_{k+1}}{a_k} \to \lambda$ as $k \to \infty$, then:

1. If $\lambda < 1$, $\sum a_k$ converges.
2. If $\lambda > 1$, $\sum a_k$ diverges.
3. If $\lambda = 1$, the test is inconclusive.

Proof. Given $\lambda < 1$, we can choose μ such that $\lambda < \mu < 1$. Thus:

$$\frac{a_{k+1}}{a_k} < \mu \tag{14}$$

for k sufficiently large, say $k > K$. We have:

$$a_{K+1} < \mu a_K \tag{15}$$
$$a_{K+2} < \mu a_{K+1} < \mu^2 a_K \tag{16}$$
$$\vdots \tag{17}$$
$$a_{K+j} < \mu^j a_K \tag{18}$$

for $j = 1, 2, 3, \ldots$. Let $n = K + j$. Then we can rewrite the last line as:

$$a_n < \mu^{n-k} a_K = \frac{a_K}{\mu^K} \mu^n \tag{19}$$

Since the factor $\frac{a_K}{\mu^K}$ is some constant and μ^n converges, then the original sum is convergent. $\square$

Tip: The ratio test is usually the most straightforward and the most useful test to employ.

Example 3: Suppose we take the sum $\sum \frac{k^2}{e^k}$. We have:

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{e^{k+1}} \cdot \frac{e^k}{k^2} = \frac{(k+1)^2}{k^2} \cdot \frac{1}{e} \tag{20}$$

As $k \to \infty$, we get $\frac{1}{e} < 1$ so the sum is convergent.