ESC195 Notes

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1 Hyperbolic Functions

- Sometimes, combinations of e^x and e^{-x} are given certain names, for example:
  - **Hyperbolic sine:** $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$
  - **Hyperbolic cosine:** $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$
- They have the following properties:

$$\frac{d}{dx}\sinh x = \cosh x \tag{1}$$
$$\frac{d}{dx}\cosh x = \sinh x \tag{2}$$

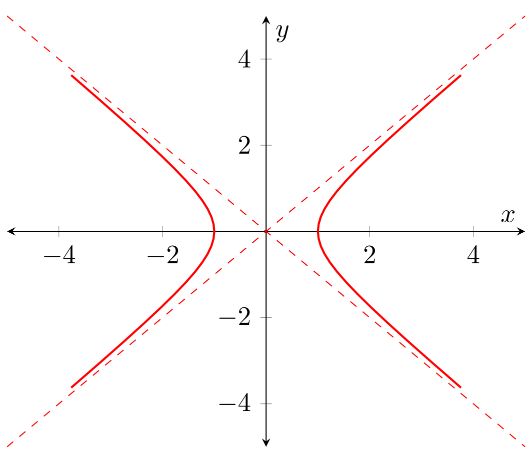
- They are related via:

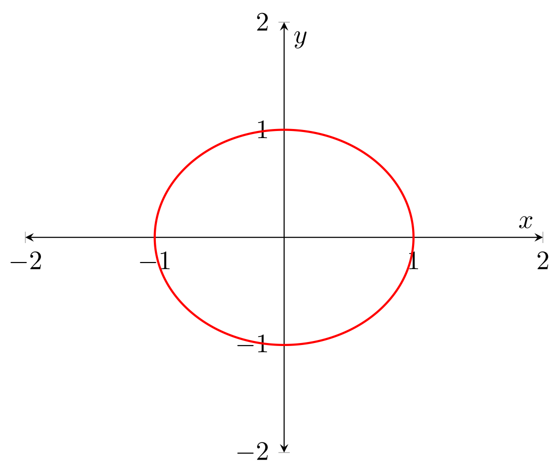
$$\cosh^2 x - \sinh^2 x = 1 \tag{3}$$

- Both the area of a circular sector and that of a hyperbolic sector is described by:

$$A = \frac{1}{2}t \tag{4}$$

  where t is the subtended angle, and the figures are parametized by $(\cos t, \sin t)$ and $(\cosh t, \sinh t)$.





- The catenary

$$y = a\cosh\left(\frac{x}{a}\right) + C \tag{5}$$

  describes the shape of a free hanging rope between two walls separated by a width a.

- The hyperbolic tangent is given by $\tanh x = \frac{\sinh x}{\cos hx} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$. and its derivative is given by:

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \tag{6}$$

- The inverse of $y = \sinh x$ is given by:
$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) \tag{7}$$

Tip: A table of integrals and derivatives revolving around hyperbolic trig functions can be found in the textbook.

2 Indeterminate Forms

- A lot of the times, limits have an indeterminate form, where if we substitute in what x approaches to, we get it in the form of $\frac{0}{0}$, for example:
$$\lim_{x\to 0}\frac{\sin x}{x} \tag{8}$$

Theorem: If $f(x)\to 0$ and $g(x)\to 0$ as $x\to\pm,\infty$ or $x\to c$ or $x\to c^{+-}$ and if $\frac{f'(x)}{g'(x)}\to L$, then:
$$\frac{f(x)}{g(x)}\to L \tag{9}$$

Example 1: Solve: $\lim_{x\to 0}\frac{\sin x}{x}$

We can set $f(x)=\sin x$, $f'(x)=\cos x$, $g(x)=x$ and $g'(x)=1$ such that:
$$\lim_{x\to 0}\frac{f'}{g'} = \lim_{x\to 0}\cos x = 1 \tag{10}$$

Example 2: Solve $\lim_{x\to 0^+}\frac{\sin x}{\sqrt{x}}$.

Set $f=\sin x$, $f'=\cos x$, $g=\sqrt{x}$, $g'=\frac{1}{2}x^{-1/2}$ and so:
$$\lim_{x\to 0^+} 2x^{1/2}\cos x = 0 \implies \lim_{x\to 0^+} = 0 \tag{11}$$

Example 3: Solve $\lim_{x\to 0}\frac{e^x-x-1}{3x^2}$.

If we take the derivative, we get:
$$\lim_{x\to 0}\frac{e^x-1}{6x} \tag{12}$$
which is still $\frac{0}{0}$!. We can take derivatives again:
$$\lim_{x\to 0}\frac{e^x}{6} = \frac{1}{6} \tag{13}$$
so the original limit is $\frac{1}{6}$.

Warning: L'hopital's rule can *only* be used in indeterminate forms. Applying them to limits where

- To prove the L'hopital's rule, we first prove the **Cauchy Mean Value Theorem** as a lemma

Theorem: Cauchy Mean Value Theorem: Given f and g differentiable on (a, b), continuous on $[a, b]$ and $g' \neq 0$ on (a, b), there must exist some number r in (a, b) such that:

$$\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)} \tag{14}$$

- We then apply **Rolle's Theorem** to prove the Cauchy Mean Value Theorem:

  Proof. Set:

  $$\begin{aligned} G(x) =& [g(b) - g(a)][f(x) - f(a)] \\ & -[g(x) - g(a)][f(b) - f(a)] \end{aligned}$$

  Note that $G(a) = G(b) = 0$ so it satisfies the conditions of Rolle's Theorem. Taking the derivative, we get:

  $$G'(x) = [g(b) - g(a)]f'(x) - g'(x)[f(b) - f(a)] \tag{15}$$

  Accoridng to Rolle's, there must be some $x = r$ such that $G'(r) = 0$, we can then substitute for this and solve:

  $$G'(r) = 0 \implies [g(b) - g(a)]f'(r) = g'(r)[f(b) - f(a)] \tag{16}$$

  Which is equivalent to:

  $$\frac{f'(r)}{g'(r)} = \frac{f(b) - f(a)}{g(b) - g(a)} \tag{17}$$

  Furthermore, we have $g'(c) = \dfrac{g(b) - g(a)}{b - a}$ from the mean value theorem. Since $g' \neq 0$ we have $g(b) - g(a) \neq 0$. □

- Given $x \to c^+$ and $f(x), g(x) \to 0$ where:

  $$\lim_{x \to c^+} \frac{f'(x)}{g'(x)} = L \tag{18}$$

  we will now prove that $\lim_{x \to c^+} \dfrac{f(x)}{g(x)} = L$.

  Proof. Consider the interval $[c, c + h]$ and apply Cauchy MVT. There must be some number c_2 in $[c, c + h]$ such that:

  $$\frac{f'(c_2)}{g'(c_2)} = \frac{f(c + h) - f(c)}{g(c + h) - g(c)} = \frac{f(c + h)}{g(c + h)} \tag{19}$$

  The last step is a result of the given $f(c) = g(c) = 0$. The LHS can be rewritten as:

  $$\lim_{h \to 0} \frac{f'(c_2)}{g'(c_2)} = \frac{f'(c)}{g'(c)} \tag{20}$$

  since c_2 lies in the interval $[c, c + h]$ so if $h \to 0$, then the interval becomes smaller to contain just c. The RHS can be rewritten as:

  $$\lim_{h \to 0} \frac{f(c + h)}{g(c + h)} = \lim_{x \to c^+} \frac{f(x)}{g(x)} \tag{21}$$

  and therefore:

  $$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} = L \tag{22}$$

  □

- To prove the case for $x \to \pm\infty$, we can let $x = \dfrac{1}{t}$ and take the limit as $t \to \infty$.

Example 4: Find $\lim_{x \to \infty} \dfrac{\ln x}{x}$.

Taking the derivative of top and bottom, we have:

$$\lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0 \implies \lim_{x \to \infty} \frac{\ln x}{x} = 0 \tag{23}$$

Idea: The logarithm function grows very slowly. In fact, any positive power of x will grow faster than $\ln x$.

Example 5: Solve $\lim_{x\to\infty} \frac{x^3}{e^x}$

This is indeterminate in the form of $\frac{\infty}{\infty}$. We apply L'hopital's rule multiple times:

$$\lim_{x\to\infty} \frac{x^3}{e^x} \stackrel{*}{=} \lim_{x\to\infty} \frac{3x^2}{e^x} \left(= \frac{\infty}{\infty}\right) \tag{24}$$

$$\stackrel{*}{=} \lim_{x\to\infty} \frac{6x}{e^x} \left(= \frac{\infty}{\infty}\right) \tag{25}$$

$$\stackrel{*}{=} \lim_{x\to\infty} \frac{6}{e^x} = 0 \tag{26}$$

- Generally, $\lim_{x\to\infty} \frac{x^m}{e^x} = 0$ where m is any positive integer.
- There are other indeterminate forms, such as 0^0, for example:

$$\lim_{x\to 0} x^x \tag{27}$$

The central idea behind this is that $a^b = e^{a \ln b}$. Therefore, this limit is equal to:

$$\lim_{x\to 0} e^{x\ lnx} \tag{28}$$

We can take the limit of the exponent to get:

$$\lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{1/x} \tag{29}$$

Note that the first equation is another indeterminate form with the $0 \cdot \infty$ type, so we had to multiply top and bottom by $\frac{1}{x}$ to get the quotient form. Then we have:

$$\lim_{x\to 0} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0} -x = 0 \tag{30}$$

Therefore:

$$\lim_{x\to 0} e^{x \ln x} = e^0 = 1 \tag{31}$$

so $\lim_{x\to 0} x^x = 1$.

Example 6: Solve $\lim_{x\to\infty} (x+2)^{2/\ln x}$.

This is of the type ∞^0. The approach is exactly the same as the previous example. We write it in exponential form:

$$= \lim_{x\to\infty} e^{\frac{2}{\ln x} \ln(x+2)} \tag{32}$$

and looking at the exponent gives:

$$\lim_{x\to\infty} \frac{2\ln(x+2)}{\ln x} \tag{33}$$

$$\stackrel{*}{=} \lim_{x\to\infty} \frac{\left(\frac{2}{x+2}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{2x}{x+2} \left(= \frac{\infty}{\infty}\right) \tag{34}$$

$$\stackrel{*}{=} \lim_{x\to\infty} \frac{2}{1} = 2 \tag{35}$$

Therefore:

$$\lim_{x\to\infty} e^{\frac{2}{\ln x} \ln(x+2)} = e^2 \tag{36}$$

so:

$$\lim_{x\to\infty}(x+2)^{2/\ln x}=e^2 \tag{37}$$

Example 7: Solve $\lim_{x\to\infty}\left[\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\right]^x$

This is in the form of 1^∞. We rewrite it as:

$$\lim_{x\to\infty}\exp\left(x\ln\left(\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\right)\right) \tag{38}$$

and taking the limit of the exponent:

$$=\lim_{x\to\infty}x\ln\left(\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\right)\left(=\frac{0}{0}\right) \tag{39}$$

$$\stackrel{*}{=}\lim_{x\to\infty}\frac{\cos\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\cdot\left(-\frac{\pi}{x^2}\right)}{\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\cdot\left(-\frac{1}{x^2}\right)}=\frac{0\cdot\pi}{1}=0 \tag{40}$$

Therefore:

$$\lim_{x\to\infty}\left[\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\right]^x=\lim_{x\to\infty}\exp\left(x\ln\left(\sin\left(\frac{\pi}{x}+\frac{\pi}{2}\right)\right)\right)=1 \tag{41}$$