TECHNICAL UNIVERSITY OF DENMARK

ASSIGNMENT 2

02417 Time Series Analysis

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1 Stability

Let the process $\{X_t\}$ be an $AR(2)$ given by: $X_t + \phi_1 X_{t-1} + \phi_2 X_{t-2} = \varepsilon_t$, where $\{\varepsilon_t\}$ is a white noise process with $\sigma_\varepsilon = 1$.

1.1

According to Theorem 5.9:

$$
\begin{aligned}
X_t + \phi_1 X_{t-1} + \phi_2 X_{t-2} &= \varepsilon_t \\
X_t + \phi_1 L X_t + \phi_2 L^2 X_t &= \varepsilon_t \\
X_t(1 + \phi_1 L + \phi_2 L^2) &= \varepsilon_t
\end{aligned}
$$

The characteristic equation, using $L = z^{-1}$, is now:

$$
\begin{aligned}
\phi(z) &= 0 \\
1 + \phi_1 z^{-1} + \phi_2 z^{-2} &= 0 \\
z^2 + \phi_1 z + \phi_2 &= 0 \quad \because \text{ multiplied by } z^2
\end{aligned}
$$

Substituting the given the values $\phi_1 = -0.7$ and $\phi_2 = -0.2$: $z^2 - 0.7z - 0.2 = 0$
The roots are:

$$\frac{0.7 \pm \sqrt{0.49 + 0.8}}{2} = \text{calculated in R} \rightarrow \{-0.2178908, 0.9178908\}$$

1.2

According to the same Theorem 5.9, an AR process is always invertible.

1.3

According to the book (Equation 5.22), the autocorrelation function (ACF) for a **stationary** process is defined as follows,

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)} = \frac{\gamma(k)}{\sigma^2}$$

where σ^2 is the variance of the process which is equal to 1. Rearranging the formula of our $AR(2)$ process, we get that, $X_t = \varepsilon_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}$. Using this, the autocovariance of k is,

$$
\begin{aligned}
\gamma(k) &= \mathrm{Cov}(X_t, X_{t-k}) \\
&= \mathrm{Cov}((\varepsilon_t - \phi_1 X_{t-1} - \phi_2 X_{t-2}), X_{t-k}) \\
&= \mathrm{Cov}(\varepsilon_t, X_{t-k}) - \phi_1 \mathrm{Cov}(X_{t-1}, X_{t-k}) - \phi_2 \mathrm{Cov}(X_{t-2}, X_{t-k}) \\
&= -\phi_1 \gamma(k-1) - \phi_2 \gamma(k-2)
\end{aligned}
$$

By substituting $\phi_1 = -0.7$ and $\phi_2 = -0.2$,

$$\therefore \rho(k) = 0.7\gamma(k-1) + 0.2\gamma(k-2)$$

1.4

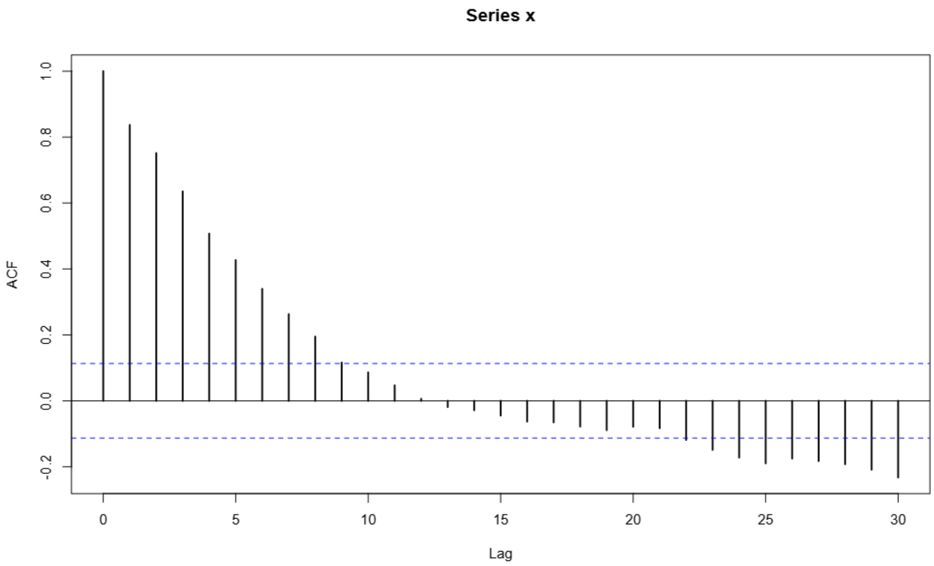

Figure 1: The ACF of a simulated AR(2) series with $\phi_1 = -0.7$ and $\phi_2 = -0.2$.

2 Simulating seasonal processes

2.1 $A(1,0,0) \times (0,0,0)_{12}$ model with parameter $\phi_1 = 0.6$

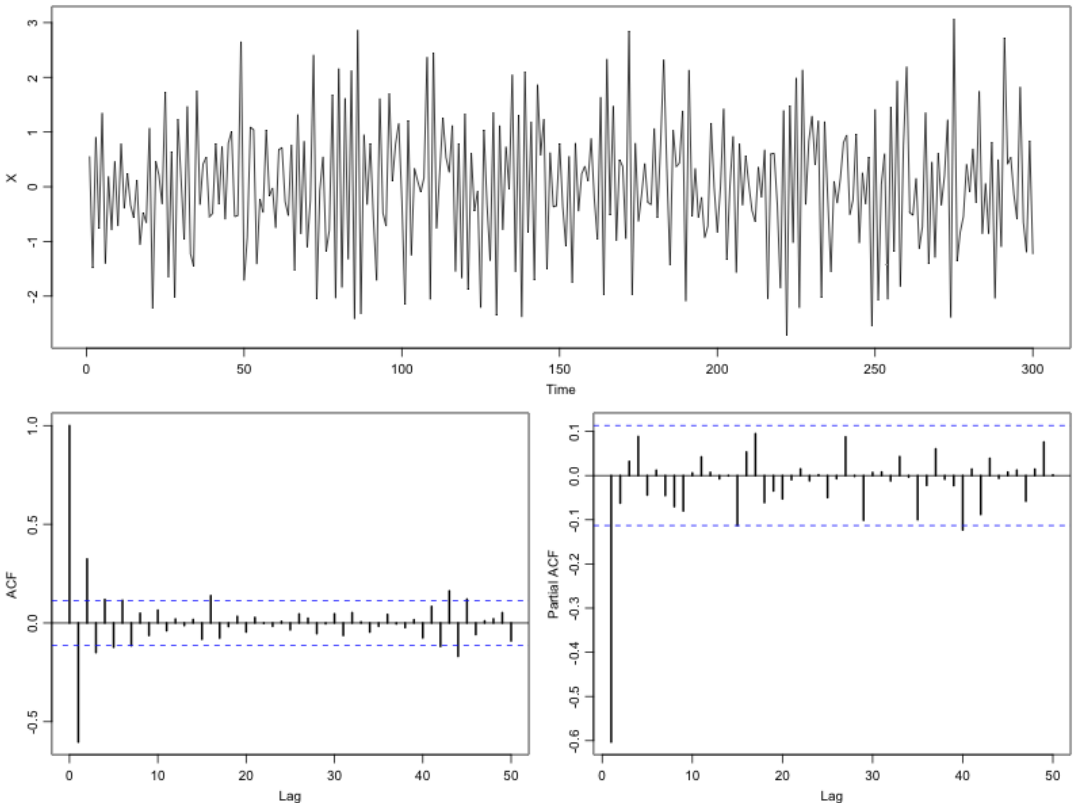

Figure 2: Plot of an $AR(1)$ series with no seasonality and $\phi_1 = 0.6$, its ACF, and its PACF.

The process alternates around 0. The ACF shows significant correlation with lag 1 and the PACF has a spike at 1 which is expected since this is an $AR(1)$.

2.2 $A(0,0,0) \times (1,0,0)_{12}$ model with parameter $\Phi_1 = -0.9$

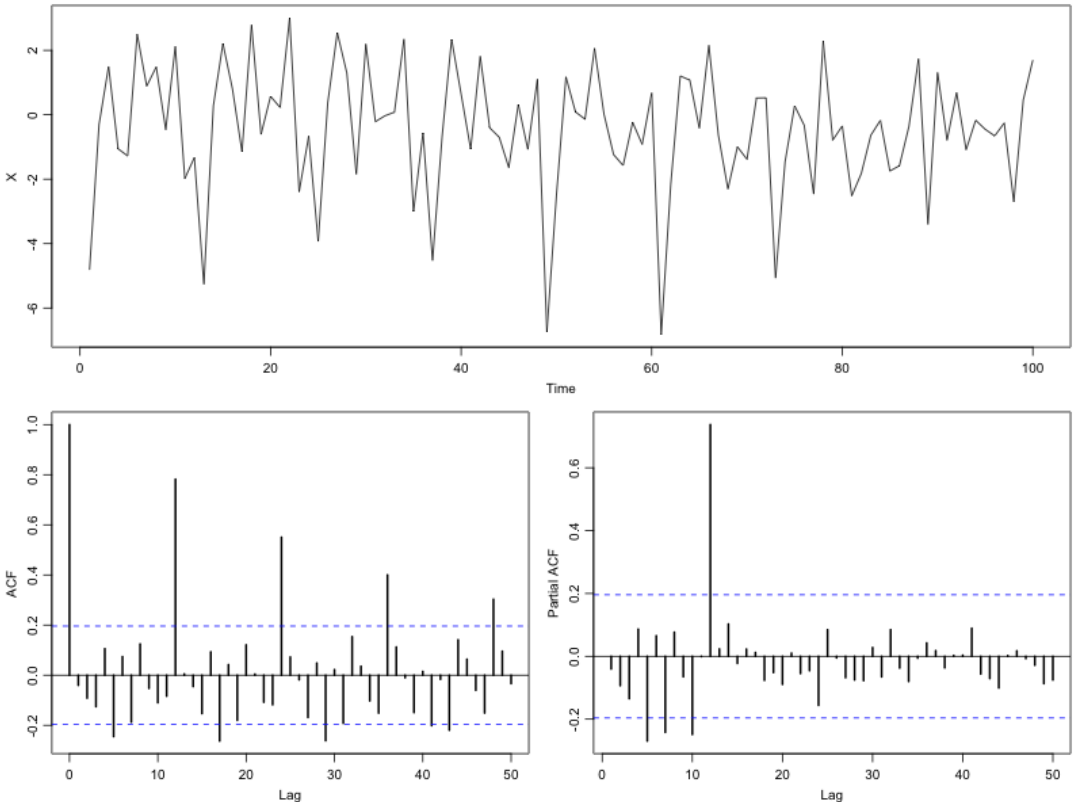

Figure 3: Plot of a white noise series with $AR(1)$ seasonality and $\Phi_1 = 0.9$, its ACF, and its PACF.

The ACF shows a strong correlation at lag 12, which gradually decreases for the following multiples of 12. This can also be seen by looking at the spike in lag 12 in the PACF.

2.3 $A(1,0,0) \times (0,0,1)_{12}$ with parameters $\phi_1 = 0.9$ and $\Theta_1 = -0.7$

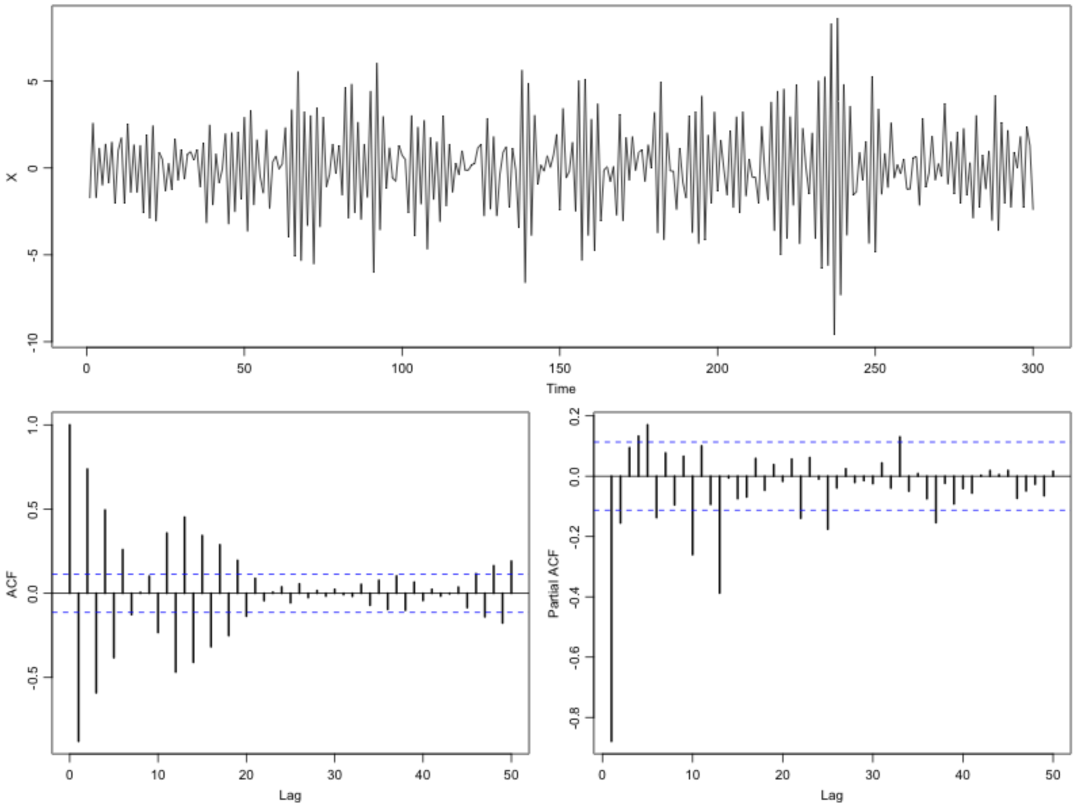

Figure 4: Plot of an $AR(1)$ series with $MA(1)$ seasonality, $\phi_1 = 0.9$ and $\Theta_1 = -0.7$, its ACF, and its PACF.

The ACF has a mix of different signals. It generally does get dampened as we would expect from an $AR(1)$ series but we also see significant peaks at lag 1 due to the underlying $MA(1)$ seasonality. The PACF shows a significant peak at lag 1 and also are repeated at multiples of 12 in a dampening fashion. Both ACF and PACF somewhat follow a sine curve indicating the use of both *AR* and *MA* series.

2.4 $A(1,0,0) \times (1,0,0)_{12}$ with parameters $\phi_1 = -0.6$ and $\Phi_1 = -0.8$

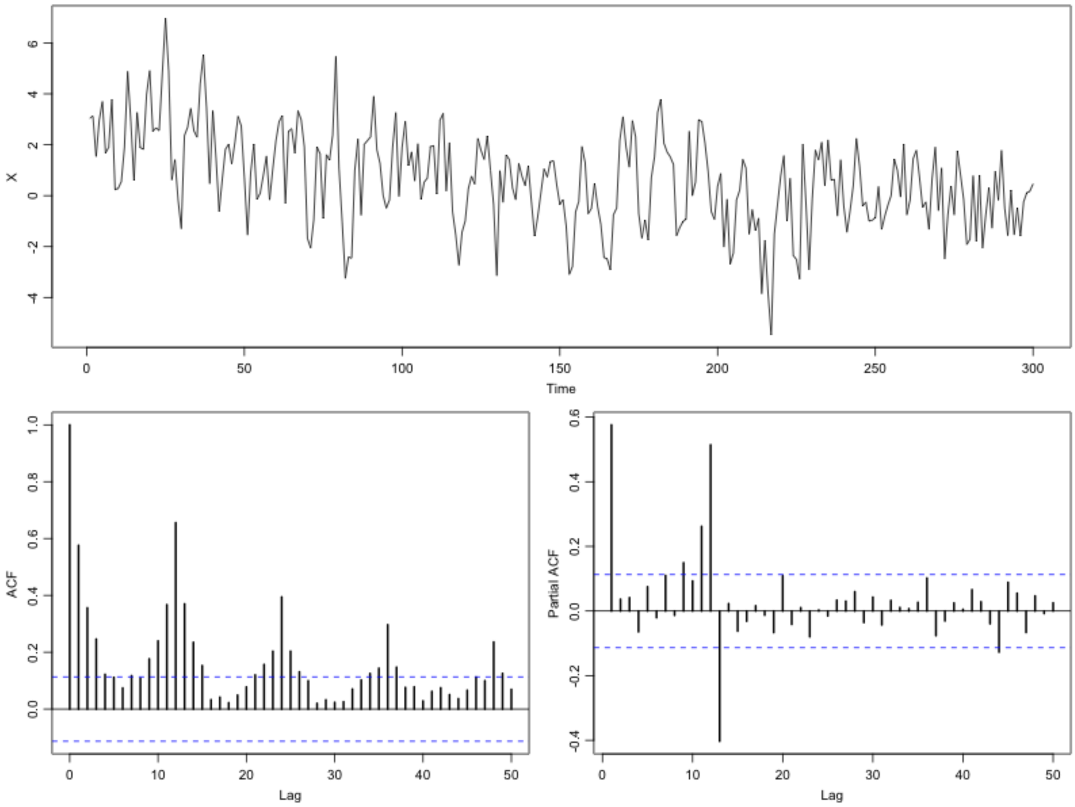

Figure 5: Plot of an $AR(1)$ series with $AR(1)$ seasonality, $\phi_1 = -0.6$ and $\Phi_1 = -0.8$, its ACF, and its PACF.

The ACF plot shows a decreasing trend with significant peaks at multiples of 12 and at lag 0. The PACF shows significant peaks at lag 1 and 12. These results are consistent with what we can expect from an $AR(1)$ series.

2.5 $A(0,0,1) \times (0,0,1)_{12}$ with parameters $\theta_1 = 0.4$ and $\Theta_1 = -0.8$

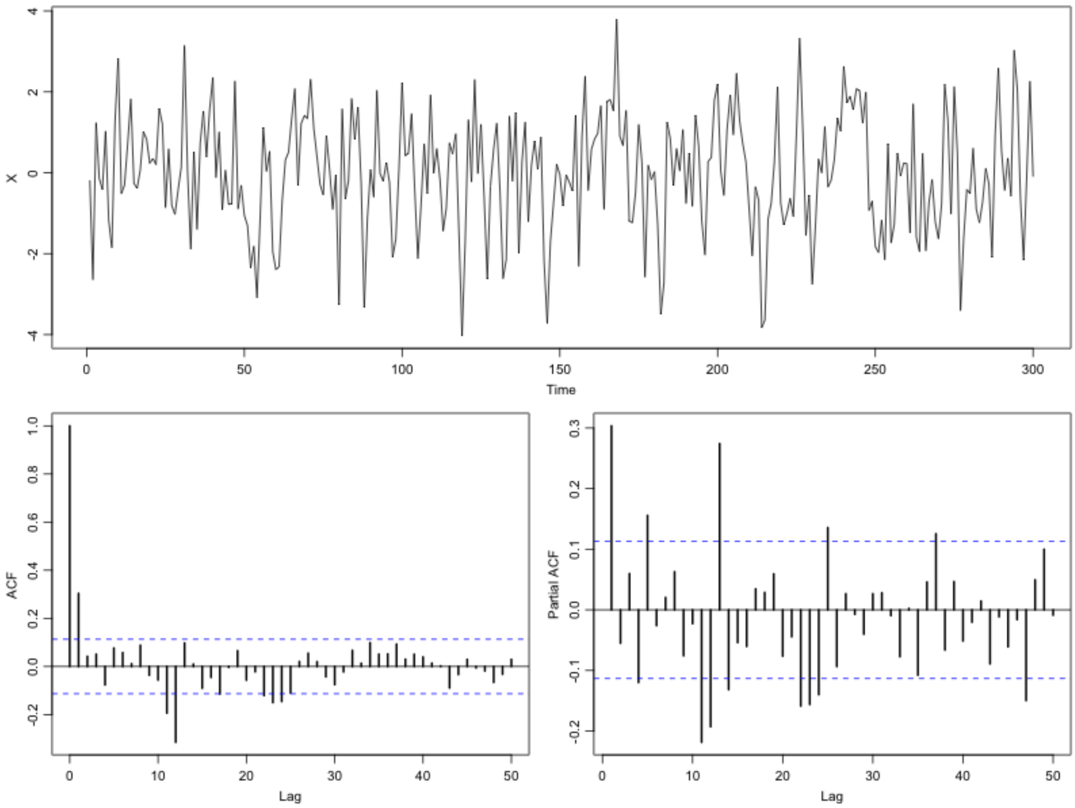

Figure 6: Plot of an $MA(1)$ series with $MA(1)$ seasonality, $\theta_1 = 0.4$ and $\Theta_1 = -0.8$, its ACF, and its PACF.

The ACF shows significant signals only at lags 1 and 12, which is as expected since it is an $MA(1)$ series. The PACF shows a decreasing sine trend with significant peaks at lag 1 and 12. These observations are consistent with $MA(1)$ series with a seasonal trend.

2.6 $A(0,0,1) \times (1,0,0)_{12}$ with parameters $\theta_1 = -0.4$ and $\Phi_1 = 0.7$

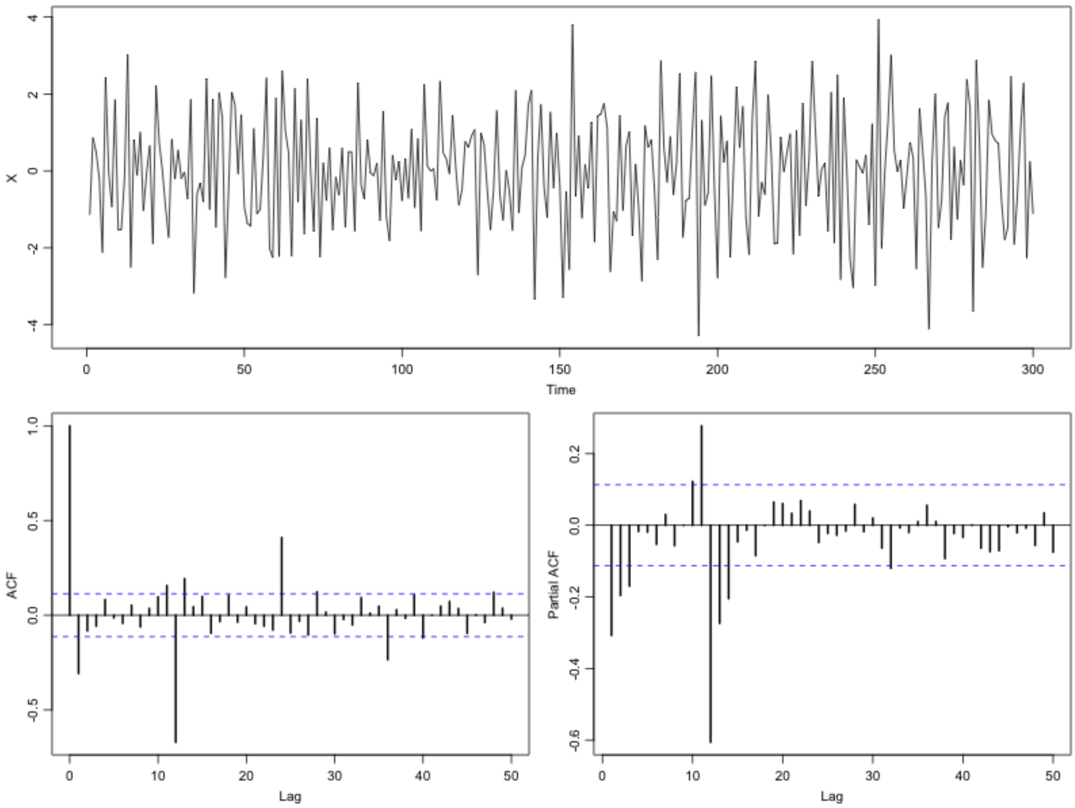

Figure 7: Plot of an $MA(1)$ series with $MA(1)$ seasonality, $\theta_1 = -0.4$ and $\Phi_1 = 0.7$, its ACF, and its PACF.

This was a little more confusing to analyze as ACF shows a peak at lag 12 and 24 in a somewhat sine wave function indicating a potential $AR(1)$ seasonality. It was hard to determine whether the signal at lag 1 is significant but knowing already that this is an $MA(1)$ series, we would have expected it. The PACF shows a another sine wave trend where there are significant peaks at lag 1 and 12.

2.7 Summary of observations

As mentioned in the textbook, the golden table shown in Table 1 helps identify whether a process is an MA or AR series based on identifying the trends in ACF and PACF plots. When introducing seasonality, the plots become more complicated.

For an $AR(1)$ seasonality, we see that in Figure 2, the ACF represents a dampened sine wave, however, we also now see significant peaks at each season, i.e., at multiples of 12. The PACF now only shows one significant peak at lag 12 instead of at lag 1.

Following this, we would expect a decreasing sine wave trend in the ACF of Section 2.4 along with a significant peak at lag 12. This seems to be true according to Figure 5! We also now see a significant peak at lag 1 in the PACF as we have an $AR(1)$ series.

For an $MA(1)$ series with $MA(1)$ seasonality, we would expect significant peaks at lag 1 and also at multiples of 12 in the ACF, and the PACF should show a dampening exponential or sine wave. This can be supported by the plots in Figure 6.

	$AR(p)$	$MA(q)$
ACF	Trails off (geometric decay)	Significant at lag q/ Cuts off after lag q
PACF	Significant at each lag p/Cuts off after lag p	Trails off (geometric decay)

Table 1: The golden table to identify AR and MA processes.

3 Identifying ARMA Models

3.1 Process 1:

The time series plot we see a pattern that appears random and has no clear structure. The ACF plot shows no autocorrelation at any lag - expect lag 0 which is always 1. The PACF also doesn't show any significant partial autocorrelation after the lag 0. This matches which noise which is an ARMA (0,0) process.

3.2 Process 2:

This time series plot shows more structure and pattern compared to the first process. The ACF plot shows an exponential decay, indicating an AR model. The partial ACF has only two significant lags, further defining the model as ARMA(2,0).

3.3 Process 3:

The third time series plot shows a pattern and smoother fluctuations. The ACF plot shows an exponential decay indicating that the process is AR. The PACF plot show an exponential alternating decay which indicates that the process in MA. So this process would be an ARMA (1,1).