A lemma to improve weak continuity (提升弱连续性的一个引理)

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2025/9/1

定义

Let Y be a real Banach space, we say that a function $u : [a, b] \to Y$ is weakly continuous if u is continuous from $[a, b]$ to (Y, τ_{weak}) which is equivalent with for all $\psi \in Y'$, the function defined by $t \in [0, T] \mapsto \langle \psi, u(t) \rangle_{Y',Y} \in \mathbb{R}$ is continuous.
We denote all functions that are weakly continuous as $\mathcal{C}\left([0, T], Y_{\text{weak}}\right)$.

Note that like the strong continuity functions, an element $u \in \mathcal{C}\left([0, T], Y_{\text{weak}}\right)$ is uniquely determined(i.e. If we have two elements $u, v \in \mathcal{C}\left([0, T], Y_{\text{weak}}\right)$ such that $u = v$ almost everywhere, then $u = v$ on $[0, T]$)

引理

Let X, Y be Banach spaces, where X is reflexive. If there is an embedding (continuous linear injection) $J : X \to Y$, then we have

$$L^{\infty}(]0, T[, X) \cap \mathcal{C}\left([0, T], Y_{\textit{weak}}\right) \subset \mathcal{C}\left([0, T], X_{\textit{weak}}\right)$$

Note that when we say that X is continuously embedded in Y, we do not assume that X is a linear subspace of Y (consider the natural embedding $i : L^2 \to H^{-1}$)

When we say an elment $u \in L^\infty(]0,T[, X) \cap \mathcal{C}\left([0,T], Y_{\text{weak}}\right)$ the correct understanding is that we have an element $v(t) \in C([0,T], Y_{weak})$ and an element $u(t) \in L^\infty(]0,T[, X)$ such that $Ju = v$ almost everywhere.

We claim that there exists an element $\tilde{u} : \mathbb{R} \to X$ such that $J\tilde{u} = v$ on $[0, T]$ and there exists $M > 0$ such that $\|\tilde{u}(t)\| \leq M$ on $\mathbb{R}$

First, let us extend u and v to all of $\mathbb{R}$ (e.g., by successive reflections performed by setting $u(t) = u(-t)$ for $t \in [-T, 0]$, etc.). It is then obvious that $u \in L^\infty(\mathbb{R}, X)$ and $v \in \mathcal{C}\left(\mathbb{R}, Y_{\text{weak}}\right)$, $Ju = v$ almost everywhere on $\mathbb{R}$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a mollifying kernel. We set $u_n = u \star \eta_{1/n}$ which is defined for all t and takes its values in X .

Let $t_0 \in \mathbb{R}$ be fixed. For all $n \geq 1$, we have

$$\|u_n\left(t_0\right)\|_X = \left\|\left(u \star \eta_{1/n}\right)\left(t_0\right)\right\|_X \leq \|u\|_{L^\infty(\mathbb{R}, X)}$$

Then by the reflexivity of X, we can extract a subsequence of $u_n(t_0)$ which converges weakly to some $\tilde{u}(t_0) \in X$.

Then note that we also have $Ju_n(t)$ weakly converges to $v(t)$, $\forall t \in \mathbb{R}$. Indeed, let $\psi \in Y'$, we have

$$< \psi, \left(Ju \star \eta_{1/n}\right)(t_0) - V(t_0) >_{Y',Y} = \left(< \psi, v >_{Y',Y} \star \eta_{1/n}\right)(t_0) - \langle \psi, v \rangle_{Y',Y}(t_0) \xrightarrow[n\to\infty]{} 0,$$

where we used the weak continuity of v.
Since J is an embedding, then we have $J\tilde{u}(t) = v(t)$ on $\mathbb{R}$, also using the weak lower semicontinuity of the norm, we have that there exists $M > 0$ such that $\|\tilde{u}(t)\| \leq M$ on $[0, T]$.

To prove the weak continuity of $\tilde{u}$, let $\phi \in X'$, by Hahn-Banach Theorem, we have an extension $\psi \in Y'$ of $\phi \in X'$, then as $t_n \xrightarrow[n\to\infty]{} t$ we have

$$| < \phi, \tilde{u}(t_n) > - < \phi, \tilde{u}(t) > | = | < \psi, v(t_n) > - < \psi, v(t) > | \xrightarrow[n\to\infty]{} 0$$

Thus the conclusion follows.

商空间的定义

定义

设X是一个Banach空间, Y是X的闭子空间, 定义$X/Y := \{[x] : x \in X\}$

$$P = \left\{ h : D(h) \subset E \to \mathbb{R} \;\middle|\; \begin{array}{l} D(h) \text{ is a linear subspace of } E, \\ h \text{ is linear, } G \subset D(h), \\ h \text{ extends } g, \text{ and } h(x) \le p(x) \quad \forall x \in D(h) \end{array} \right\}. \tag{1}$$

牛顿第二定律可以表示为：

$$F = ma \tag{2}$$

相对论中的能量和动量关系：

$$E = mc^2 \tag{3}$$

$$p = mv \tag{4}$$

如公式 ?? 所示，力等于质量乘以加速度。结合公式 (??) 和 (??) 可以推导出更多结论。