Towards a Formal Framework for Emergence: Uncomputability, Fixed Points, and Diagonalization

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Abstract

Emergent phenomena across diverse systems exhibit coherent structures that cannot be reduced to their underlying rules. We propose a unified formal framework that connects undecidability, uncomputability, self-reference (fixed points), chaos, and diagonalization as fundamental mechanisms underlying emergence. Case studies from cellular automata and quantum decoherence are discussed, suggesting a deeper structural inevitability to emergence in complex systems.

1 Introduction

Emergence remains a central mystery in complex systems: how do novel, coherent structures arise from simple underlying rules? Traditional descriptions are often qualitative. Here, we propose a formal framework, grounded in logic, computation, and dynamical systems theory, that treats emergence as a mathematically inevitable consequence of limits on computability, fixed-point phenomena, and dynamical self-organization.

2 Core Proposal: Axioms of Emergence

We define a system as a tuple (S, R), where S is a set of states and $R : S \to S$ describes deterministic evolution.

We propose the following axioms:

1. **(Local Computability)**: Each step R is locally computable.
2. **(Global Uncomputability)**: The global behavior of iterated R is, in general, uncomputable (cf. Turing, Wolfram).
3. **(Fixed Points and Self-Reference)**: Sufficiently expressive systems necessarily contain self-referential structures (Lawvere's Fixed Point Theorem).
4. **(Incompleteness of Description)**: No finite axiomatization captures all global properties (cf. G"odel's Incompleteness Theorem).
5. **(Emergent Stability)**: Persistent patterns correspond to approximate fixed points, stabilized despite local unpredictability.

Emergence, under this view, is the appearance of robust structures not derivable from initial conditions alone, but inevitable due to formal structural constraints.

3 Case Studies

3.1 Cellular Automata

Rule 110 exhibits the emergence of mobile "particles" and stable structures within a background of chaotic activity. Despite fully local rules, long-range order develops. This reflects uncomputability (the system is Turing complete) and self-organized fixed points.

3.2 Quantum Decoherence

During decoherence, off-diagonal elements of the density matrix vanish, leaving a classical probabilistic mixture. Diagonalization here parallels the emergence of stable, information-preserving structures through interaction with an environment, matching our proposed formal structure.

4 Discussion and Future Work

This framework suggests that emergence is not merely accidental but formally necessary in sufficiently complex systems. Future work includes:

- Formalizing connections to Quantum Darwinism.
- Developing quantitative measures of "diagonalization" across systems.
- Exploring Lawvere Fixed Points in dynamic, non-static contexts.

References

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