Min Variance, Max Mean of Time Series

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1 Notation

We denote with capital letters random variables. A *multivariate* random variable X is just vector $[X_1, \ldots X_n]$ of univariate random variables $X_1, \ldots X_n$, that is, $X = [X_1, \ldots X_n]$.

Let X be an uni-dimensional real valued random variable and $f : \mathbb{R} \rightarrow \mathbb{R}$. We denote with $E_X[f(X)]$ the expected value of $f(X)$.

Let $X = [X_1, \ldots X_n]$ be an n-dimensional real valued random variable and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. We denote with $E_X[f(X)]$ the expected value of $f(X)$. That is, $E_X[f(X)] = [E_{X_1}[f_1(X_1)], \ldots E_{X_n}[f_1(X_n)]]$

Let $X_i(t)$ $(i = 1, \ldots n)$ be n real-valued *discrete-time* stochastic processes. Assume that the n-dimensional stochastic process $X(t) = [X_1(t), \ldots X_n(t)]$ is stationary. We denote with $\mu_X \in \mathbb{R}^n$ the expected value of $X(t)$ and with Γ_X its covariance matrix.

We denote with $Y(t)$ the discrete-time stochastic process:

$$Y(t) = \sum_{i=1}^{n} \alpha_i X_i(t) \tag{1}$$

Since $X(t)$ is stationary, so is $Y(t)$. We denote with $\mu_Y \in \mathbb{R}$ the expected value of $Y(t)$ and with Γ_Y its covariance matrix, that is its variance.

2 Problem statement

We want to select the values for the coefficients $\alpha_1, \ldots \alpha_n$ so as to minimize the variance of Y and maximize its expected value. We proceed as follows.

2.1 Minimize Variance of Y

We solve the following optimization problem.

Given B, find $\alpha_1, \ldots \alpha_n \in \mathbb{R}$, $b_1, \ldots b_n \in \{0, 1\}$ such that:

$$\begin{align}
\min \Gamma_Y \tag{2}\\
0 \leq \alpha_i \leq 1, \quad \text{for } i = 1, \ldots n \tag{3}\\
\alpha_i \leq b_i, \quad \text{for } i = 1, \ldots n \tag{4}\\
b_i \in \{0, 1\}, \quad \text{for } i = 1, \ldots n \tag{5}\\
\sum_{i=1}^{n} \alpha_i = 1 \tag{6}\\
\sum_{i=1}^{n} b_i <= B \tag{7}
\end{align}$$

Let $\alpha_1^1, \ldots \alpha_n^1 \in \mathbb{R}$, $b_1^1, \ldots b_n^1 \in \{0, 1\}$ be the solution to the above optimization problem and σ_1^2 be the value of its objective function when the minimum is attained. Thus σ_1^2 is the minimum value for the variance of Y.

2.2 Maximize Expected Value Y

We solve the following optimization problem.

Let B, and σ_1^2 be as in Section 2.1. Find $\alpha_1, \dots \alpha_n \in \mathbb{R}$, $b_1, \dots b_n \in \{0, 1\}$ such that:

$$\max \mu_Y \tag{8}$$

$$\Gamma_Y \leq \sigma_1^2 \tag{9}$$

$$0 \leq \alpha_i \leq 1, \quad \text{for } i = 1, \dots n \tag{10}$$

$$\alpha_i \leq b_i, \quad \text{for } i = 1, \dots n \tag{11}$$

$$b_i \in \{0, 1\}, \quad \text{for } i = 1, \dots n \tag{12}$$

$$\sum_{i=1}^{n} \alpha_i = 1 \tag{13}$$

$$\sum_{i=1}^{n} b_i <= B \tag{14}$$

Let $\alpha_1^2, \dots \alpha_n^2 \in \mathbb{R}$, $b_1^2, \dots b_n^2 \in \{0, 1\}$ be the solution to the above optimization problem and μ_2 be the value of its objective function when the maximum is attained. Thus μ_2 is the maximum value for the mean of Y. Finally, we denote with σ_2^2 the value of the LHS (*Left Hand Side*) of the constraint 9 above.