IB Paper 6: Information Engineering

COMMUNICATIONS

Solutions to Examples Paper 8: Analogue Modulation and Digitisation

1. (a) We have that

$$\begin{aligned}
\mathcal{F}[f(t) * g(t)] &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau \right) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\tau) g(t-\tau) e^{-j\omega t} dt \right) d\tau \\
&= \int_{-\infty}^{\infty} f(\tau) \left(\int_{-\infty}^{\infty} g(t-\tau) e^{-j\omega t} dt \right) d\tau \\
&= \int_{-\infty}^{\infty} f(\tau) \left(\int_{-\infty}^{\infty} g(x) e^{-j\omega (x+\tau)} dx \right) d\tau \\
&= G(\omega) \int_{-\infty}^{\infty} f(\tau) e^{-j\omega\tau} d\tau \\
&= F(\omega) G(\omega)
\end{aligned}$$

where the first step follows from the definitions of convolution and Fourier transform, the second step follows from a change of integration order and the fourth step follows from the change of variables $x = t - \tau$. The other steps are straightforward.

(b) We have that

$$\begin{aligned}
\mathcal{F}[f(t)\cos(\omega_0 t)] &= \int_{-\infty}^{\infty} f(t) \cos(\omega_0 t) e^{-j\omega t} dt && (1) \\
&= \int_{-\infty}^{\infty} f(t) \frac{e^{-j\omega_0 t} + e^{j\omega_0 t}}{2} e^{-j\omega t} dt && (2) \\
&= \frac{1}{2} \int_{-\infty}^{\infty} f(t) e^{-j\omega_0 t} e^{-j\omega t} dt + \frac{1}{2} \int_{-\infty}^{\infty} f(t) e^{j\omega_0 t} e^{-j\omega t} dt && (3) \\
&= \frac{1}{2} F(\omega + \omega_0) + \frac{1}{2} F(\omega - \omega_0) && (4)
\end{aligned}$$

where the first step follows from the definition of Fourier transform, the second from the cosine expansion in exponentials, and the fourth by identifying the transformed-domain variables $\omega + \omega_0$ and $\omega - \omega_0$.

(c) We have that

$$\begin{aligned}
\int |f(t)|^2 dt &= \int_{-\infty}^{\infty} f(t) f^*(t) dt && (5)\\
&= \int_{-\infty}^{\infty} f(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) e^{-j\omega t} d\omega \right) dt && (6)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) \left(\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right) d\omega && (7)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) F(\omega) d\omega && (8)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega && (9)
\end{aligned}$$

where the second step follows from the conjugate of the inverse Fourier transform:

$$f^*(t) = \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \right)^* = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) e^{-j\omega t} d\omega$$

2.) Bandwidth of an AM signal =

2 × Bandwidth of modulating signal = 2 × 4 kHz = 8 kHz

∴ Spacing of carriers = Bandwidth + Gap = 8 + 3 = 11 kHz

∴ Max no. of simultaneous transmissions =

$$\frac{\text{Total freq range of MW Band}}{\text{Spacing of carriers}} = \frac{1500-500}{11}$$

= 90.9 or $\underline{\underline{91}}$ since no gap is needed at outer edges

Bandwidth of an SSB signal =

Bandwidth of modulating signal = 4 kHz

∴ Spacing of carriers = 4 + 3 = 7 kHz

∴ Max no. of simultaneous transmissions =

$$\frac{1500-500}{7} = 142.8 \text{ or } 143 \text{ with no gaps at edges}$$

∴ No. of extra transmissions = 143 − 91 = $\underline{\underline{52}}$

3.)

$$s(t) = [10 + 3\underbrace{\cos(3.10^3\pi t)}_{\text{signal}}]\underbrace{\cos(18.10^6\pi t)}_{\text{carrier}}$$

carrier amplitude — signal amplitude

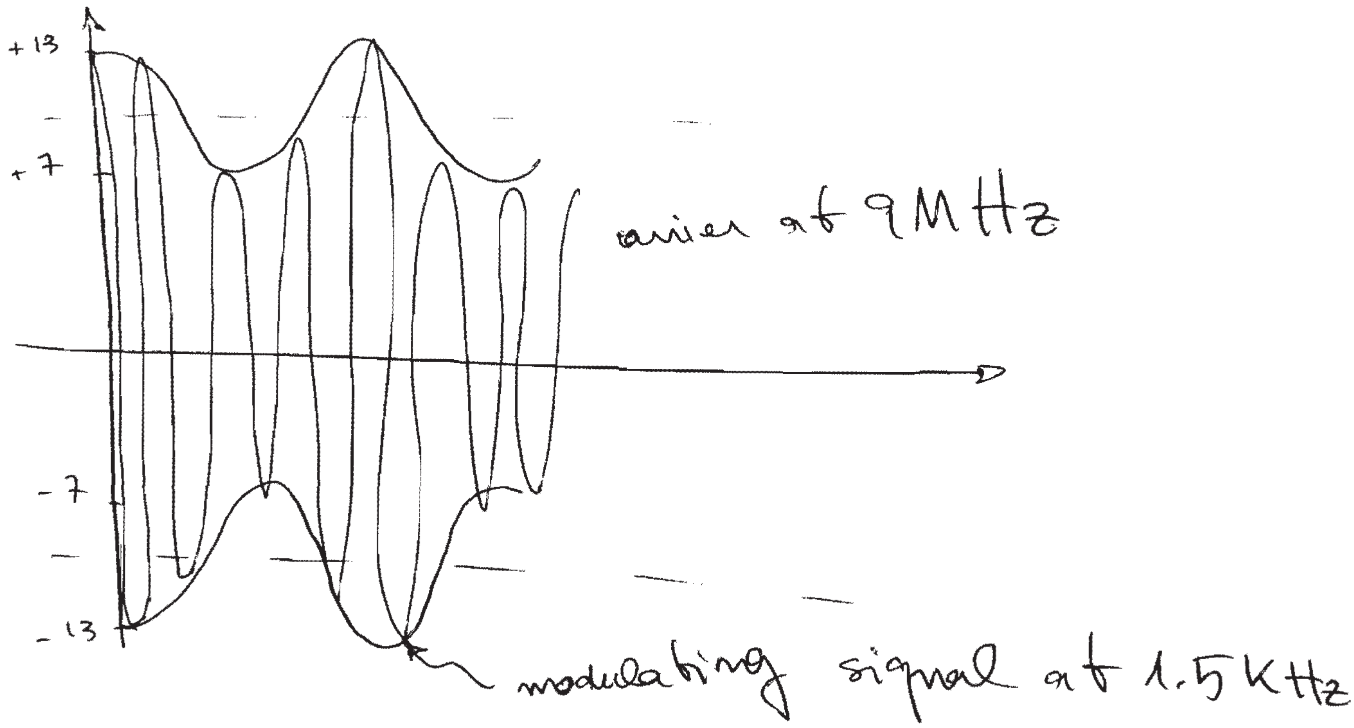
Unmodulated carrier term = $10\cos(18.10^6\pi t)$

∴ Unmod. carrier ampl = $\underline{\underline{10 \text{ volts}}}$

Unmod. carrier freq = $18.10^6\pi$ rad/s. = 9.10^6 Hz = $\underline{\underline{9 \text{ MHz}}}$

3.) continued

(b) Modulation index = $\frac{\text{peak signal amp}}{\text{carrier amp}} = \frac{3}{10} = 0.3$



4. The output of the product modulator is

$$
\begin{aligned}
v(t) &= x(t)\cos(2\pi f_c t)\cos(2\pi f_c t + \phi) \\
&= \frac{x(t)}{2}\left[\cos(4\pi f_c t + \phi) + \cos(\phi)\right] \\
&= \underbrace{\frac{1}{2}x(t)\cos(\phi)}_{\text{low freq. component}} + \underbrace{\frac{1}{2}x(t)\cos(4\pi f_c t + \phi)}_{\text{high freq. component}}
\end{aligned}
$$

where we have used the trigonometric identity

$$2\cos(\theta_1)\cos(\theta_2) = \cos(\theta_1 + \theta_2) + \cos(\theta_1 - \theta_2)$$

with $\theta_1 = 2\pi f_c t + \phi$, and $\theta_2 = 2\pi f_c t$

If the bandwidth of $x(t)$ is W, passing $v(t)$ through an ideal low-pass filter for which $H(f) = 2$ for $-W \leq f \leq W$ and 0 otherwise, we obtain:

$$\hat{x}(t) = x(t)\cos(\phi).$$

When $\phi = 90^\circ$, the demodulated signal is zero.

Note that we *cannot* compensate for $\cos\phi$ by adjusting the gain of the low-pass filter as ϕ is unknown at the receiver. Thus phase mismatch reduces the amplitude of the demodulated signal. This problem can be addressed by more complex receivers, which is the price of suppressing the carrier wave to save transmitted power.

5) $$y = \left(2 + b\cos(2\pi f_m t) + \cos(2\pi f_c t)\right)^2$$

$$= 4 + 4\left(b\cos(2\pi f_m t) + \cos(2\pi f_c t)\right)$$
$$+ b^2\cos^2(2\pi f_m t) + \cos^2(2\pi f_c t)$$
$$+ 2b\cos(2\pi f_m t)\cos(2\pi f_c t)$$

* using $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$$= \left[\begin{array}{l} 4 + \\ 4b\cos(2\pi f_m t) + \\ \frac{b^2}{2}\left(\cos(4\pi f_m t) + 1\right) + \\ \frac{1}{2}\left(\cos(4\pi f_c t) + 1\right) + \\ \cos(2\pi f_c t)\left[4 + 2b\cos(2\pi f_m t)\right] \end{array}\right] \quad \begin{array}{l} \text{frequencies} \\ 0 \\ f_m \\ 2f_m,\ 0 \\ 2f_c,\ 0 \\ \\ \end{array}$$

The last term is of the form:

$$A\cos(2\pi f_c t)\left[1 + m_A\cos(2\pi f_m t)\right]$$

with $A = 4$ and $\underline{m_A = b/2}$. This occupies frequency range $(f_c - f_m)$ to $(f_c + f_m)$ which is separable from the other frequencies above (given $f_c \gg f_m$ as usual)

6. (a)

$$
\begin{aligned}
s_{\mathrm{FM}}(t) = A_c \cos\left(2\pi f_c t + \beta \sin(2\pi f_x t)\right) &= \mathsf{Re}\left(A_c e^{j(2\pi f_c t + \beta \sin(2\pi f_x t))}\right) \\
&= \mathsf{Re}\left(e^{j2\pi f_c t} \underbrace{A_c e^{j\beta \sin(2\pi f_x t))}}_{\tilde{s}(t)}\right)
\end{aligned}
\tag{10}
$$

(b) As $\sin(2\pi f_x t)$ is periodic with fundamental frequency f_x, $\tilde{s}(t)$ is also periodic with fundamental frequency f_x.

(c) The Fourier coefficients c_n can be calculated as

$$
\begin{aligned}
c_n &= f_x \int_{-1/(2f_x)}^{1/(2f_x)} \tilde{s}(t) e^{-j2\pi n f_x t}\, dt \\
&= f_x A_c \int_{-1/(2f_x)}^{1/(2f_x)} e^{j\beta \sin(2\pi f_x t) - j2\pi n f_x t}\, dt \\
&= \frac{A_c}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin u - nu)}\, du \qquad\qquad \text{(using } u = 2\pi f_x t\text{)} \\
&= A_c\, J_n(\beta)
\end{aligned}
$$

where $J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin u - nu)}\ du$.

(d) To show that that $J_n(\beta)$ is real, we calculate $J_n^*(\beta)$

$$
J_n^*(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-j(\beta \sin u - nu)}\ du.
$$

Substitute $u = -v$, and simplify the integral to show $J_n* = J_n$. Hence $J_n(\beta) = \mathsf{Re}(J_n^*(\beta)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\beta \sin u - nu)\ du$

(e) Substituting $\tilde{s}(t) = \sum_n c_n e^{j2\pi n f_x t}$ in (10), we obtain

$$
s_{\mathrm{FM}}(t) = \mathsf{Re}\left(\sum_n c_n e^{j2\pi f_c t} e^{j2\pi n f_x t}\right) = A_c \sum_n J_n(\beta)\ \cos(2\pi(f_c + n f_x)t) \tag{11}
$$

(f) Expressing $\cos(2\pi(f_c + n f_x)t) = \frac{1}{2}(e^{j2\pi(f_c + n f_x)t} + e^{-j2\pi(f_c + n f_x)t})$, and taking Fourier transforms, we get

$$
S_{\mathrm{FM}}(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) \left[\delta(f - f_c - n f_x) + \delta(f + f_c + n f_x)\right].
$$

At $f = f_c$, we have

$$
S_{\mathrm{FM}}(f_c) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) \left[\delta(-n f_x) + \delta(2f_c + n f_x)\right].
$$

The only non-zero term in the first set of impulses at $\{nf_x\}$ is at $n = 0$. This impulse has strength $A_c J_0(5) = -0.18A_c$.

For the second set of impulses at $\{2f_c + nf_x\}$, we have a non-zero term at $n = -2f_c/f_x$ only if $-2f_c/f_x$ if it is an integer. Even so, , the value of $J_n(5)$ for this impulse is negligible since $n = -2f_c/f_x \approx 0$ when $f_c \gg f_x$.

7) Gain of freq. modulator = 50 kHz/volt

a) Peak freq. deviation = $1 \times 50 = 50$ kHz

Modulation index, $m_f = \dfrac{\text{Peak freq deviation}}{\text{Modulation freq.}}$

$= 50/5 = \underline{\underline{10}}$

Bandwidth (using Carson's Rule)

$= 2(\text{freq. deviation} + \text{modulation freq.}) = 2(50+5) = \underline{\underline{110\,k}}$

b) Freq. deviation $= 1 \times 50 = 50$ kHz

$m_f = 50/10 = \underline{\underline{5}}$

Bandwidth $= 2(50+10) = \underline{\underline{120 \text{ kHz}}}$

c) Freq. deviation $= 0.2 \times 50 = 10$ kHz

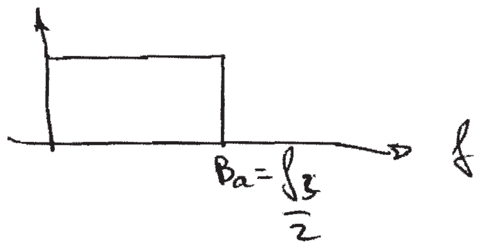
$m_f = 10/10 = \underline{\underline{1}}$

Bandwidth $= 2(10+10) = \underline{\underline{40 \text{ kHz}}}$

8) To avoid aliasing, the sampling frequency must be

$$f_s \geq 2 B_x$$

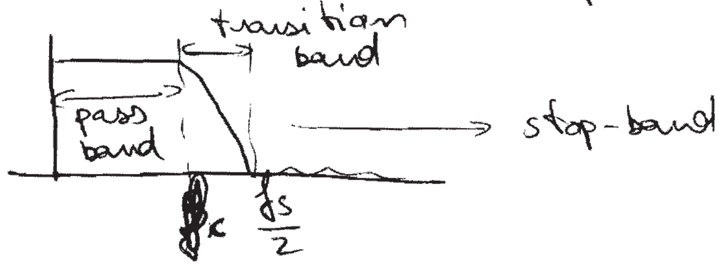
where B_x is the signal bandwidth.
Then, the ideal anti-aliasing filter will have bandwidth $B_a = \frac{f_s}{2}$



Then, the minimum sampling frequencies are:

a) $f_s = 2 \cdot 100\ Hz = 200\ Hz$
b) $f_s = 2 \cdot 3400\ Hz = 6.8\ kHz$
c) $f_s = 2 \cdot 16\ kHz = 32\ kHz$
d) $f_s = 2 \cdot 5.5\ MHz = 11\ MHz$

If we use non-ideal filters,



the stop band must start at half of the sampling frequency. If the passband cutoff frequency is f_c and the transition bandwidth is $0.2 f_c$ we have that $f_s = 2.4 f_c$

8 ~~7~~ cont

Then we have that the new minimum sampling frequencies are

a) $f_s = 2.4 \cdot 100 = 240$ Hz
b) $f_s = 2 \cdot 4 \cdot 3400 = 8160$ Hz
c) $f_s = 2 \cdot 4 \cdot 16$ kHz $= 38.4$ kHz
d) $f_s = 2 \cdot 4 \; 5.5$ MHz $= 13.2$ MHz

~~The~~ So far we have that we are sampling at a rate of f_s samples/second. Using an n-bit quantiser, the bit rate is

$$R = n \cdot f_s \; \frac{\text{bits}}{\text{second}}$$

a) Using 12 bits: $R = 2880$ bits/s
b) Using 12 bits: $R = 97920$ bits/s
c) Using 16 bits: $R = 614.4$ k bit/s
d) Using 8 bits: $R = 1{,}056 \cdot 10^8$ bit/s $=$
$= 105.6$ Mbit/s

9) In an ideal quantiser (A-D converter) with a quantising step size of δv, the mean square quantising noise voltage is given by (see lecture notes):

$$\frac{1}{\delta v}\int_{-\frac{\delta v}{2}}^{\frac{\delta v}{2}} x^2\,dx = \frac{\delta v^2}{12}$$

$\therefore$ RMS noise voltage $= \delta v/\sqrt{12}$

Now $\delta v = \dfrac{\text{Peak-to-peak input voltage range}}{\text{No. of quantising levels}}$

a) $\delta v = \dfrac{5+5}{2^8} = 39.06\text{ mV}$

$\therefore$ RMS noise $= 39.06/\sqrt{12} = \underline{11.28}\text{ mV}$

(i) Max RMS sinusoidal signal volts $= \dfrac{5+5}{2\sqrt{2}} = 3.536\text{ V}$

$\therefore$ Max signal/noise power ratio $= \left(\dfrac{3.536}{0.01128}\right)^2 = 98304$

In dB, this becomes $10\log_{10}(98304) = \underline{\underline{49.9\text{ dB}}}$

(ii) Compound signal. rms volts $= \dfrac{\text{peak}}{2\sqrt{2}} = \dfrac{5}{2\sqrt{2}} = 1.768\text{ V}$ (half previous

SNR (power) $= \left(\dfrac{1.768}{0.01128}\right)^2 = 24576 \Rightarrow \underline{\underline{43.9\text{ dB}}}$ (6dB lower

b) $\delta v = \dfrac{20}{2^{12}} = 4.883\text{ mV}$

$\therefore$ RMS noise $= 4.883/\sqrt{12} = \underline{1.410\text{ mV}}$

(i) Max RMS sinusoidal signal volts $= \dfrac{20}{2\sqrt{2}} = 7.071\text{ V}$

$\therefore$ Max signal/noise power ratio $= \left(\dfrac{7.071}{0.00141}\right)^2 = 25{,}165{,}824$

$= \underline{\underline{74.0\text{ dB}}}$

(ii) compound signal rms volts $= \dfrac{\text{peak}}{2\sqrt{2}} = \dfrac{10}{2\sqrt{2}} = 3.5355\text{ V}$

SNR $= \underline{\underline{68\text{ dB}}}$

9. contd.

For a square wave signal, the quantisation error is not randomly distributed, but is itself a square wave.