Cambridge University Engineering Dept. Third year

Module 3F2: Systems and Control

LECTURE NOTES 3: OBSERVABILITY & OBSERVERS

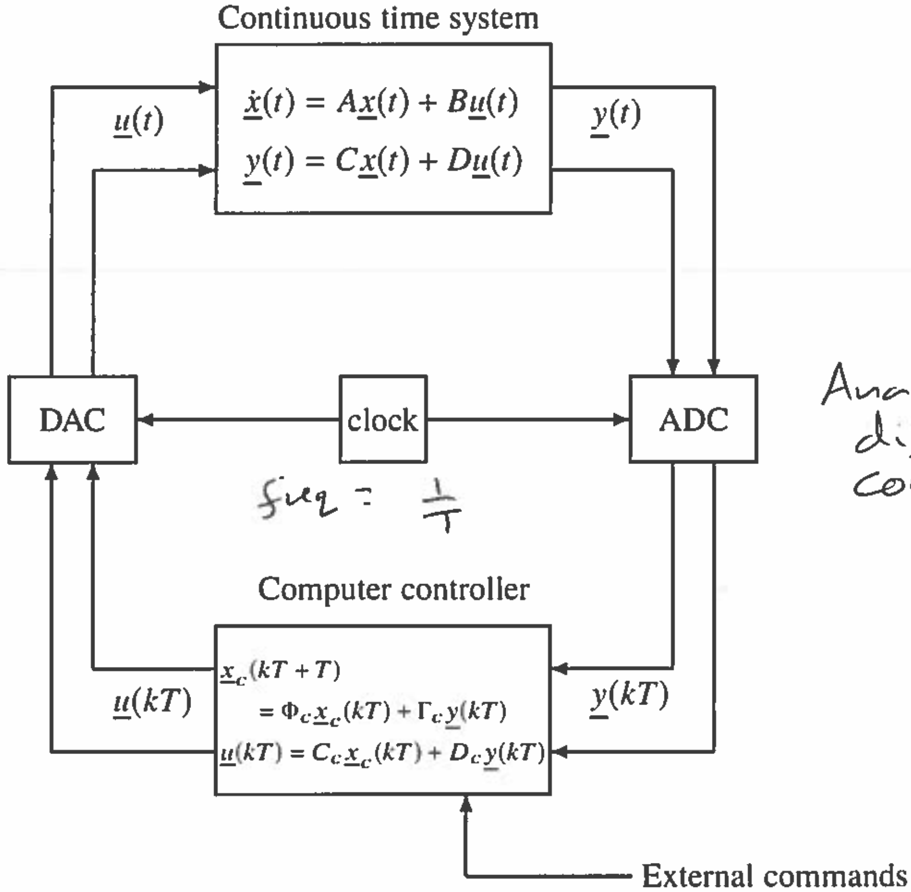
Contents

Contents

K. Glover Revised by G. Vinnicombe
Jan 2002 February 2018

1 Sampled Data Control System

Zero order hold →



Analogue to digital converter

The sampled data system satisfies:

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t), \quad \text{with } \underline{u}(t) = \underline{u}(kT), \text{ for } kT \le t < (k+1)T. \quad \Leftarrow \text{ZOH}$$

Apply result from Handout 1, section 4.5 (Convolution integral):

$$\begin{aligned}
\underline{x}((k+1)T) &= \underbrace{e^{AT}}_{\Phi} \underline{x}(kT) + \underbrace{\int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)} B d\tau}_{\Gamma} \underline{u}(kT) \\
&= \Phi \underline{x}(kT) + \Gamma \underline{u}(kT)
\end{aligned}$$

u(t) is constant in the interval

where

$$\begin{aligned}
\Gamma &= \int_0^T e^{A\tau'} d\tau' B \ = A^{-1}\left(e^{AT} - I\right) B, \text{ if } \det(A) \neq 0. \\
\underline{y}(kT) &= C\underline{x}(kT) + D\underline{u}(kT)
\end{aligned}$$

This gives the standard state-space model in discrete time. Entirely analogous results can be obtained for the discrete time case as in the continuous time case:

- Solution of vector difference equations,
- Discrete-time convolution,
- z−transform for frequency response caculations etc,
- Notions of controllability and observability — coming next.

- Identical tests for these

2 Solving Linear Equations

For convenience we will repeat some results and definitions from linear algebra.

Definition 2.1 *Let A be an $m \times n$ matrix then,*

(a) the set of all $\underline{x} \neq \underline{0}$ such that $A\underline{x} = \underline{0}$ is called the **Null Space** *of A (null(A)). This is sometimes referred to as the* **Kernel** *of A.*

(b) the set of all $\underline{y}$ such that $\underline{y} = A\underline{x}$ for some $\underline{x}$ is called the **Range Space** *of A (or the range of A, range(A)); This is sometimes referred to as the* **Column Space** *or* **Image** *of A.*

(c) A is said to have full row *rank if* range$(A) = \mathbb{R}^m$ *(i.e. $\underline{z}^T A \neq \underline{0}$ for all $\underline{z} \neq \underline{0}$);*

(d) A is said to have full column *rank if* null$(A) = \emptyset$ *(i.e. $A\underline{x} \neq \underline{0}$ for all $\underline{x} \neq \underline{0}$.)*

Theorem 2.2 *For any matrix A the row rank and the column rank are equal, and denoted* rank(A).

Given an $m \times n$ matrix A and an $m \times 1$ vector $\underline{b}$, consider the equation:

$$A\underline{x} = \underline{b},$$

in the unknown $\underline{x}$ in $\mathbb{R}^n$. Two natural questions are:

(a) Does there exist a solution, $\underline{x}$? $\Leftrightarrow$ $\underline{b} \in$ range(A)

(b) If so, is it unique? $\Leftrightarrow$ Null$(A) = \underline{0}$

Fact 2.3 *For the case $m = n$:*

(a) If $\det(A) \neq 0$ *then for any $\underline{b}$ there exists a solution, $\underline{x}$, such that $A\underline{x} = \underline{b}$, and this solution is unique (Indeed it is given by $\underline{x} = A^{-1}\underline{b}$).*

(b) If $\det(A) = 0$ *then there exists $\underline{x} \neq \underline{0}$ such that $A\underline{x} = \underline{0}$.* $\Rightarrow$ any solutions are not unique

Fact 2.4 *For any $m \times n$ matrix, M,*

$$M^T M\underline{x} = \underline{0} \quad \Leftrightarrow \quad M\underline{x} = \underline{0}.$$

$\Leftarrow$ obvious

$\Rightarrow$ $M^T M x = 0 \Rightarrow x^T M^T M x = (Mx)^T Mx = 0$

$\Rightarrow Mx = 0$

Fact 2.5 *For the case $m \leq n$,* $A = [\quad\quad]$ (wide)

(a) If $\det(AA^T) \neq 0$ *then* $\underline{x} = A^T\left(AA^T\right)^{-1}\underline{b}$, *solves* $A\underline{x} = \underline{b}$ *for any* $\underline{b}$.

(not unique)

(b) If $\det(AA^T) = 0$ *then there exists a* $\underline{b} \neq \underline{0}$ *such that* $\underline{b} \perp A\underline{x}$ *(i.e.* $\underline{b}^T A\underline{x} = \underline{0}$*) for all* $\underline{x}$.

For the case $m \geq n$, $A = [\;]$ (tall)

(c) If $\det(A^T A) \neq 0$ *then there may not be a solution to* $A\underline{x} = \underline{b}$, *but if there is then it is unique.*

(d) If $\det(A^T A) = 0$ *then there exists* $\underline{x} \neq \underline{0}$ *such that* $A\underline{x} = \underline{0}$.

For hand calculations it is generally easiest to use the following observations:

(a) If you can find a set of n rows of A such that the determinant of the $n \times n$ submatrix given by these rows is nonzero, then A has full column rank. ($m \geq n$)

(b) If you can find a nonzero vector, $\underline{x}$, such that $A\underline{x} = \underline{0}$ then clearly A does not have full column rank.

(c) If you can find a set of m columns of A such that the determinant of the $m \times m$ submatrix given by these columns is nonzero, then A has full row rank. ($m \leq n$)

(d) If you can find a nonzero vector, $\underline{z}$, such that $\underline{z}^T A = \underline{0}$ then clearly A does not have full row rank.

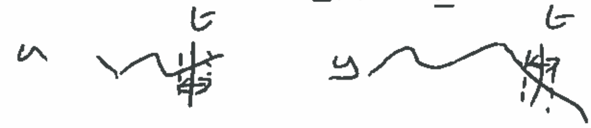
3 Observability

A system:

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + B\underline{u} \\ \underline{y} &= C\underline{x} \end{aligned}$$

e.g. $t - \epsilon < \tau < t + \epsilon$

is called **observable** if we can deduce the state, $\underline{x}(t)$, from measurements of $\underline{u}(\tau)$ and $\underline{y}(\tau)$ over some time interval. (including t) eg



Now consider differentiating $\underline{y}(t)$ to give

$$\underbrace{\begin{bmatrix} \underline{y}(t) \\ \dot{\underline{y}}(t) \\ \ddot{\underline{y}}(t) \\ \vdots \\ \underline{y}^{(n-1)}(t) \end{bmatrix}}_{\text{known}} = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{Q} \underbrace{\underline{x}(t)}_{?} + \underbrace{\begin{bmatrix} \underline{0} \\ CB\underline{u}(t) \\ CAB\underline{u}(t) + CB\dot{\underline{u}}(t) \\ \vdots \\ CA^{n-2}B\underline{u} + \cdots + CB\underline{u}^{(n-2)} \end{bmatrix}}_{\text{known}}$$

$\dot{\underline{y}} = C\dot{\underline{x}}$
$= CA\underline{x} + CB\underline{u}$

$\ddot{\underline{y}} = C\ddot{\underline{x}} = CA\dot{\underline{x}} + CB\dot{\underline{u}}$
$= CA^2\underline{x} + CAB\underline{u} + CB\dot{\underline{u}}$ etc

We can solve the above equation uniquely for $\underline{x}(t)$ if and only if rank $Q = n$. Hence, defining the **observability matrix**

$$Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

we obtain the **Observability test:**

$$\boxed{\text{The system is observable if and only if rank } Q = n}$$

If a system is *not* observable, there will exist a vector $\underline{x}_o \neq 0$ for which $Q\underline{x}_o = 0$. This is called an **unobservable state**, for the following reason.

$$
\begin{aligned}
Q\underline{x}_o = \underline{0} \quad &\Rightarrow \quad CA^k\underline{x}_o = \underline{0} \text{ for } k = 0, \cdots n-1 \\
&\Rightarrow \quad CA^n\underline{x}_o = C\left(-\alpha_1 A^{n-1} \cdots - \alpha_{n-1}A - \alpha_n I\right)\underline{x}_o \text{ by Cayley-Hamilton Theorem} \\
&= \quad \underline{0} \\
&\Rightarrow \quad CA^k\underline{x}_o = \underline{0} \text{ for all } k \geq 0 \text{ by repeated use of Cayley-Hamilton theorem.} \\
&\Rightarrow \quad Ce^{At}\underline{x}_o = \underline{0} \text{ for all } t \text{ by the power series expansion of } e^{At}
\end{aligned}
$$

$\left(e^{At} = I + At + A^2\frac{t^2}{2} + \cdots\right)$

Conversely, $Ce^{At}\underline{x}_o = 0$ for all t implies $\frac{d^n}{dt^n}Ce^{At}\underline{x}_o = CA^n e^{At}\underline{x}_o = \underline{0}$ and so $Q\underline{x}_o = \underline{0}$.

[Hence $Ce^{At}\underline{x}_0 = \underline{0}$ for all $t \iff Q\underline{x}_0 = \underline{0}$.]

Recall that

$$
\underline{y}(t) = \underbrace{Ce^{At}\underline{x}(0)}_{\text{initial condition response}} \quad \underbrace{+ D\underline{u}(t) + \int_0^t Ce^{A(t-\tau)}B\underline{u}(\tau)\,d\tau}_{\text{input response}}
$$

and so if two initial states $\underline{x}_1 \neq \underline{x}_2$ give the same outputs then $\underline{0} = \underline{y}_2 - \underline{y}_1 = Ce^{At}(\underline{x}_2 - \underline{x}_1)$. In this case, $\underline{x}_o = \underline{x}_1 - \underline{x}_2$ is an unobservable state.

3.1 Effect of Initial Condition on Output

Now consider the difference between two initial condition responses:

$$
\underline{y}_o(t) = Ce^{At}\underline{x}_o \text{ and } \underline{y}(t) = Ce^{At}\left(\underline{x}_o + \underline{d}\right) \quad \text{so} \quad \underline{y}(t) - \underline{y}_o(t) = Ce^{At}\underline{d}
$$

Can $\left(\underline{y}(t) - \underline{y}_o(t)\right)$ be small in spite of $\underline{d}$ being large? Measure the size of $\left(\underline{y}(t) - \underline{y}_o(t)\right)$ over the time interval $0 < t < t_1$ by

$$
\begin{aligned}
\int_0^{t_1} \left\|\underline{y}(t) - \underline{y}_o(t)\right\|_2^2 dt &= \int_0^{t_1} \left(\underline{y}(t) - \underline{y}_o(t)\right)^T \left(\underline{y}(t) - \underline{y}_o(t)\right) dt \\
&= \int_0^{t_1} \underline{d}^T e^{A^T t} C^T C e^{At}\, \underline{d}\, dt = \underline{d}^T\, W_o(t_1)\, \underline{d} \text{ where } W_o(t_1) = \int_0^{t_1} e^{A^T t} C^T C e^{At}\, dt
\end{aligned}
$$

(observability grammian)

Clearly this difference must be ≥ 0 so $W_o(t_1)$ is a positive semi-definite matrix. The system will be observable if $\underline{d}^T W_o(t_1)\underline{d} > 0$ for all $\underline{d} \neq \underline{0}$, i.e. if $W_o(t_1)$ is a positive definite matrix.

Also,

$$
\begin{aligned}
\underline{d} \text{ in Null Space of } W_o(t_1) \quad &\Leftrightarrow \quad W_o(t_1)\underline{d} = \underline{0} \Leftrightarrow \underline{d}^T W_o(t_1)\underline{d} = \underline{0} \Leftrightarrow Ce^{At}\underline{d} = \underline{0} \text{ for all } t < t_1 \\
&\Leftrightarrow \quad \underline{d} \text{ is an unobservable state.} \\
\Rightarrow \text{Null Space of } W_o(t_1) \quad &= \quad \text{Null Space of } Q.
\end{aligned}
$$

Example $\dot{x} = Ax + \cancel{Bu}$ $u=0$
$y = Cx$

$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$

$$\underline{\dot{x}} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \underline{x}, \quad y = \begin{bmatrix} 1 & 1 \end{bmatrix} \underline{x} \quad \Rightarrow Ce^{At} = \begin{bmatrix} e^{-t} & e^{-2t} \end{bmatrix}$$

$$W_o(t_1) = \int_0^{t_1} \begin{bmatrix} e^{-2t} & e^{-3t} \\ e^{-3t} & e^{-4t} \end{bmatrix} dt = \begin{bmatrix} \frac{1}{2}\left(1-e^{-2t_1}\right) & \frac{1}{3}\left(1-e^{-3t_1}\right) \\ \frac{1}{3}\left(1-e^{-3t_1}\right) & \frac{1}{4}\left(1-e^{-4t_1}\right) \end{bmatrix} \xrightarrow[\text{as } t_1\to\infty]{} \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

$= \int [Ce^{At}]^T [Ce^{At}] dt = \int \begin{bmatrix} e^{-t} \\ e^{-2t} \end{bmatrix} \begin{bmatrix} e^{-t} & e^{-2t} \end{bmatrix} dt$

$n=2$

$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

$\det(\cdot) = \frac{1}{8} - \frac{1}{9} \neq 0$
$\Rightarrow$ Full rank
$\Rightarrow$ Observable

$\det(Q) = -2+1 \neq 0$
$\Rightarrow$ Observable

3.2 Change of State Coordinates when System is not Observable

If (A, C) is not observable then we can make a change of state coordinates to isolate the unobservable states as follows.

If the rank $Q = r < n$ then there exists a nonsingular $n \times n$ matrix T and a $pn \times r$ matrix $\tilde{Q}_1$ of rank r, such that *(Recall QR factorization)*

$$Q = \begin{bmatrix} \tilde{Q}_1 & 0 \end{bmatrix} T$$

$\tilde{Q}_1$ has r columns

Now change the state coordinates to $\underline{\tilde{x}} = T\underline{x}$:

$$\underline{\dot{\tilde{x}}} = \underbrace{TAT^{-1}}_{\tilde{A}} \underline{\tilde{x}} + \underbrace{TB}_{\tilde{B}} \underline{u}, \quad \underline{y} = \underbrace{CT^{-1}}_{\tilde{C}} \underline{\tilde{x}}.$$

Theorem 3.1 *In these coordinates if we partition the state,* $\underline{\tilde{x}} = \begin{bmatrix} \underline{\tilde{x}}_1 \\ \underline{\tilde{x}}_2 \end{bmatrix}$ *with* $\underline{\tilde{x}}_1$ *of dimension* r,

and compatibly partition:

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \underbrace{\tilde{A}_{12}}_{} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}; \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}; \quad \tilde{C} = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}$$

$\tilde{A}_{12} = 0$ $\quad$ $\tilde{C}_2 = 0$ ($n-r$ columns) then

$$\tilde{C}_2 = 0, \quad \tilde{A}_{12} = 0, \quad \textit{and } (\tilde{A}_{11}, \tilde{C}_1) \textit{ is observable}$$

Proof: Firstly $\tilde{C}\tilde{A}^k = CT^{-1}TA^kT^{-1} = CA^kT^{-1}$ and so the observability matrix in the transformed coordinates is given by

$$\tilde{Q} = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} = \begin{bmatrix} CT^{-1} \\ CAT^{-1} \\ \vdots \\ CA^{n-1}T^{-1} \end{bmatrix} = QT^{-1} = \begin{bmatrix} \tilde{Q}_1 & 0 \end{bmatrix}$$

Hence

$$\tilde{Q}\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} = \begin{bmatrix} \tilde{Q}_1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} = 0$$

From which it follows that

$$\tilde{C}\tilde{A}^k\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} = 0 \text{ for all } k.$$

Since $\tilde{C} = [\tilde{C}_1 \ \tilde{C}_2]$

In particular, $\tilde{C}_2 = 0$. Furthermore

$$\tilde{Q}\tilde{A}\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} = \begin{bmatrix} \tilde{C}\tilde{A} \\ \tilde{C}\tilde{A}^2 \\ \vdots \\ \tilde{C}\tilde{A}^n \end{bmatrix}\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} = 0$$

But

$$\tilde{Q}\tilde{A}\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} = \begin{bmatrix} \tilde{Q}_1 & 0 \end{bmatrix}\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}\begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} = \tilde{Q}_1\tilde{A}_{12} \quad = 0$$

which implies that $\tilde{A}_{12} = 0$ since $\tilde{Q}_1$ is full column rank.

Hence in these state coordinates we have,

$$\dot{\tilde{\underline{x}}}_1 = \tilde{A}_{11}\tilde{\underline{x}}_1 + \tilde{B}_1\underline{u}, \quad \underline{y} = \tilde{C}_1\tilde{\underline{x}}_1$$

and the input/output response (i.e. the transfer function) depends only on $\tilde{\underline{x}}_1$ and the states $\tilde{\underline{x}}_2$ are all unobservable.

3.2.1 A subspace interpretation

As before, we start by factorising Q as $Q = \begin{bmatrix} \tilde{Q}_1 & 0 \end{bmatrix} T$.

Now put $T^{-1} = [X\ Y]$.

Y in $\mathbb{R}^{n \times r}$ is a basis for null(Q), which we shall call $\bar{O}$, the unobservable subspace. (i.e. whenever $a = Yb$, $Qa = 0$)

and X complements Y

(i.e. range$[X\ Y] = \mathbb{R}^n$ and, whenever $a_1 = Yb_1$ and $a_2 = Xb_2$, then $a_1^T a_2 = 0$.)

Note that $A\bar{O} \subseteq \bar{O}$ and $\bar{O} \subseteq$ null(C).

Since $AT^{-1} = T^{-1}\hat{A}$, we have

$$A[X\ Y] = [X\ Y]\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}$$

or

$$AY = [X\ Y]\begin{bmatrix} \hat{A}_{12} \\ \hat{A}_{22} \end{bmatrix} = \overset{0}{\cancel{X\hat{A}_{12}}} + Y\hat{A}_{22}$$

and so $\hat{A}_{12} = 0$.

Also $CT^{-1} = \hat{C}$, i.e.

$$C[X\ Y] = [\hat{C}_1\ 0]$$

4 Observers

4.1 Differentiating signals is a bad idea

Typically the state is not available for measurement,
but we can estimate $\underline{x}(t)$ from $\underline{y}$ and $\underline{u}$

In the section on observability we saw how to exactly deduce $\underline{x}(t)$ from

$$y, \dot{y}, .., y^{(n-1)}, u, \dot{u}, ..u^{(n-2)}$$

but differentiating signals has bad noise amplification problems:

$$\begin{aligned} y(t) &= \sin \omega t + \epsilon \sin \omega_n t & \text{S/N ratio } &= 1/\epsilon \\ \dot{y}(t) &= \omega \cos \omega t + \epsilon \omega_n \cos \omega_n t & \text{S/N ratio } &= (\omega/\epsilon\omega_n) \\ \ddot{y}(t) &= -\omega^2 \sin \omega t - \epsilon \omega_n^2 \sin \omega_n t & \text{S/N ratio } &= \frac{1}{\epsilon}\left(\frac{\omega}{\omega_n}\right)^2 \end{aligned}$$

ϵ small

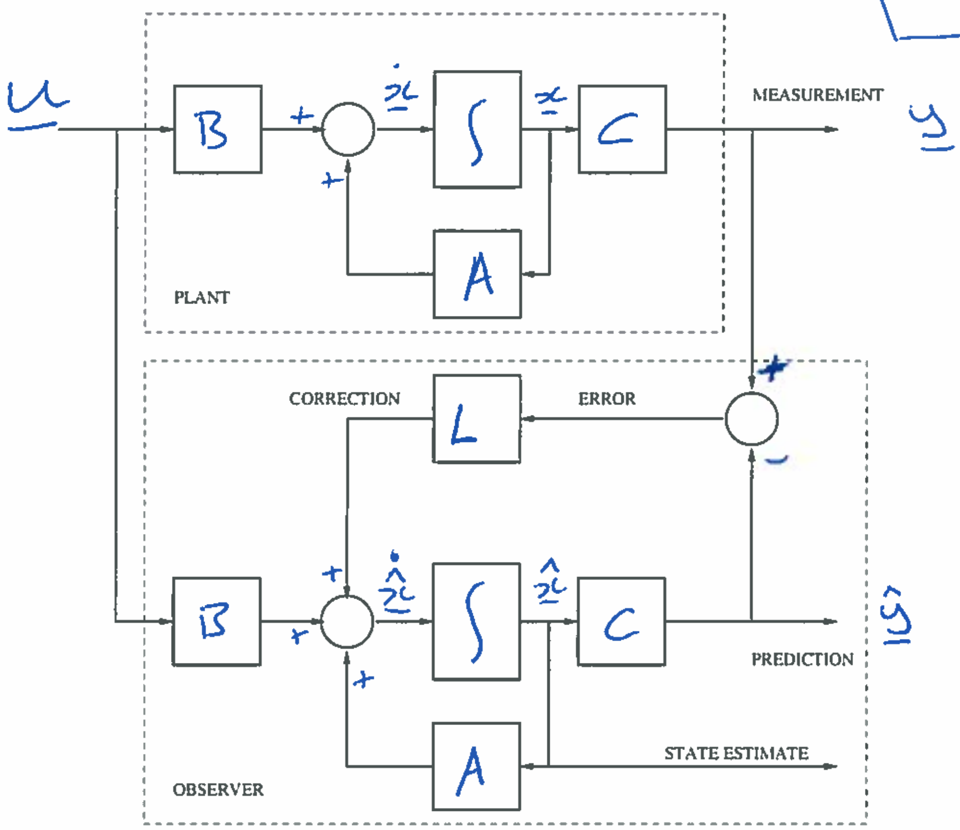
$\omega_n > \omega$

4.2 Observer structure

Instead we will use a *state observer* (Luenberger Observer) which contains a dynamic model of the system and whose state, $\hat{\underline{x}}(t)$, approaches $\underline{x}(t)$ as $t \to \infty$.

$$\begin{cases} \dot{\hat{\underline{x}}} = A\hat{\underline{x}} + B\underline{u} + L(\underline{y} - \hat{\underline{y}}) \\ \hat{\underline{y}} = C\hat{\underline{x}} \end{cases}$$

ASSUME
D = 0



L: OBSERVER GAIN MATRIX

Consider the error $\underline{e}(t) = \underline{x}(t) - \hat{\underline{x}}(t)$

$$\begin{aligned}\dot{\underline{e}} &= \dot{\underline{x}} - \dot{\hat{\underline{x}}} = (A\underline{x} + B\underline{u}) - (A\hat{\underline{x}} + B\underline{u} + L(\underline{y} - \hat{\underline{y}})) \\ &= A(\underline{x} - \hat{\underline{x}}) - LC(\underline{x} - \hat{\underline{x}}) = (A - LC)\underline{e}\end{aligned}$$

$e(t) = e^{(A-LC)t} e(0)$ $\boxed{\dot{\underline{e}} = (A - LC)\underline{e}}$ error dynamics

We want $e^{(A-LC)t} \to 0$ quickly as t increases.

This is achieved if the eigenvalues of $(A - LC)$ are large and negative, for example.

ASYMTOTICALLY STABLE OBSERVER!
All n eigenvalues of A-LC
must have -ve real parts

Can we assign the eigenvalues of $(A - LC)$ by choice of L?

Suppose (A, C) is **not** observable then in section 3.2 we found a change of coordinates, $\underline{\tilde{x}} = T\underline{x}$ such that,

$$\begin{bmatrix} \dot{\underline{\tilde{x}}}_1 \\ \dot{\underline{\tilde{x}}}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} \underline{\tilde{x}}_1 \\ \underline{\tilde{x}}_2 \end{bmatrix} + \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \underline{u}, \quad \underline{y} = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix} \underline{\tilde{x}} + D\underline{u}$$

Hence

$$T(A - LC)T^{-1} = \tilde{A} - \tilde{L}\tilde{C} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} - \begin{bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{bmatrix} \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix} = \begin{bmatrix} (\tilde{A}_{11} - \tilde{L}_1\tilde{C}_1) & 0 \\ (\tilde{A}_{21} - \tilde{L}_2\tilde{C}_1) & \tilde{A}_{22} \end{bmatrix},$$

and the eigenvalues of the observer,

$$\lambda_i(A - LC) = \lambda_i(\tilde{A} - \tilde{L}\tilde{C}) = \lambda_i(\tilde{A}_{11} - \tilde{L}_1\tilde{C}_1) \cup \lambda_i(\tilde{A}_{22}),$$

and $\lambda_i(\tilde{A}_{22})$ are not changed by $\tilde{L}$.

However *it can* & will *be shown that*

We can arbitrarily assign the eigenvalues of $(A - LC)$ by choice of L if and only if the system is observable.

- We can thus make the error, $\underline{e}(t) \longrightarrow 0$ arbitrarily quickly.

- But high gains might imply very large transient errors and noisy estimates.

4.3 Tracking disturbances, ignoring noise

Imagine tracking aircraft by radar (1-D). Aircraft position z is affected by random turbulence. Take $\underline{x} = [z, \dot{z}]^T$:

$$\dot{\underline{x}}(t) = A\underline{x}(t) + Bd(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d(t)$$

The radar measurement is corrupted by noise:

$$y(t) = C\underline{x}(t) + n(t) = [1 \quad 0]\underline{x}(t) + n(t)$$ $y = x_1 = z$

Observer: $\dot{\hat{\underline{x}}}(t) = A\hat{\underline{x}}(t) + L[y(t) - C\hat{\underline{x}}(t)]$ NB: $d(t)$ not known, so not used.

d large, n small: Believe the measurements. Use large L. *React quickly.*

d small, n large: Don't trust measurements, believe model. Use small L.

— *Smooth the measurements.* (using knowledge of the system)

4.4 Kalman Filter

Assume we have measurements of $\underline{u}(t)$ and $\underline{y}(t)$ and the model

$$\dot{\underline{x}} = A\underline{x} + B(\underline{u} + \underline{d})$$
$$\underline{y} = C\underline{x} + \underline{n}$$

What are the *smallest* $\underline{d}$ and $\underline{n}$, in terms of $(\int_0^\infty \underline{d}^T \underline{d}\, dt)^2 + (\int_0^\infty \underline{n}^T \underline{n}\, dt)^2$, which make the measurement consistent with the model, and what is the corresponding estimate of the state?

The solution is given by *Kalman Filter* theory, which gives an optimal trade-off between tracking d and rejecting n. The solution is a Luenberger observer with $L = \Sigma C^T$ where $\Sigma > 0$ solves the quadratic matrix equation

$$A\Sigma + \Sigma A^T + BB^T - \Sigma C^T C\Sigma = 0$$ Ricatti equation

(if the system is observable, then it can be shown that such a solution exists, is unique, and that the resulting observer is stable).

Generalises to arbitrary disturbance/noise spectra. Very widely used *Navigation & guidance, Telecomms, Control, Finance, ...*

Especially in discrete time. Software implementation *Matlab*: `kalman`, `dkalman`, `estim` etc.

4.5 Application to sensor fusion

Satellite, 1 axis of rotation: $J\ddot{\theta} = u + d$ (u = control torque, d = disturbance torque).

Two noisy sensors: Star sensor: $y_1 = \theta + n_\theta$, Rate gyro: $y_2 = \dot{\theta} + n_\omega$

Let $\underline{x} = [\theta, \dot{\theta}]^T$. State-space model:

$$\begin{aligned}
\dot{\underline{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ 1/J & 1/J \end{bmatrix} \begin{bmatrix} u \\ d \end{bmatrix} \\
\underline{y} &= \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} n_\theta \\ n_\omega \end{bmatrix} = I\underline{x} + \begin{bmatrix} n_\theta \\ n_\omega \end{bmatrix}
\end{aligned}$$

Observable? Yes. ($C = I$, so rank $C = 2$, so rank$\begin{bmatrix} C \\ CA \end{bmatrix} = 2$.)

Observer:

$$\begin{aligned}
\dot{\hat{\underline{x}}} &= A\hat{\underline{x}} + B\begin{bmatrix} u \\ 0 \end{bmatrix} + L(\underline{y} - C\hat{\underline{x}}) \qquad (d \text{ not known}) \\
&= (A - LC)\hat{\underline{x}} + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} u + \overset{2\times 2}{L}\underline{y} \qquad \text{but } C = I \text{ so:} \\
&= \begin{bmatrix} -\ell_{11} & 1 - \ell_{12} \\ -\ell_{21} & -\ell_{22} \end{bmatrix} \hat{\underline{x}} + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} u + \underbrace{\begin{bmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{bmatrix}}_{L} \underline{y}
\end{aligned}$$

Place both eigenvalues at -10 (say): Using trace$(A - LC) = \sum_i \lambda_i$ and det$(A - LC) = \prod_i \lambda_i$: $-\ell_{11} - \ell_{22} = -20$ and $\ell_{11}\ell_{22} + \ell_{21}(1 - \ell_{12}) = 100$. This leaves some design freedom.

$n_\theta \ll n_\omega$: Make $\ell_{11} \gg \ell_{12}$ and $\ell_{21} \gg \ell_{22}$.

$n_\theta \gg n_\omega$: Make $\ell_{11} \ll \ell_{12}$ and $\ell_{21} \ll \ell_{22}$.

Optimal trade-off: *Kalman Filter* again.

4.6 Application to sensor bias estimation

Satellite, as before: $J\ddot{\theta} = u$

Sensors: Star tracker measures angular position: $y_1 = \theta$
Rate gyro measures angular velocity with bias: $y_2 = \dot{\theta} + b_\omega$.

Augment state vector: $\underline{x} = [\theta, \dot{\theta}, b_\omega]^T$, and assume bias is constant: $\dot{b}_\omega = 0$.

State-space model:

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{b}_\omega \end{bmatrix} = \quad \dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1/J \\ 0 \end{bmatrix} u$$

$$\underline{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \underline{x}$$

Is the state observable?

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

LINEARLY INDEPENDENT

$\Rightarrow$ Rank $= 3$

$\Rightarrow$ Observable

First 3 rows are linearly independent (Or: All three columns are linearly independent).

So rank = 3. Hence: **Observable**. So can use observer to estimate $\underline{x}$:

$$\dot{\hat{\underline{x}}} = A\hat{\underline{x}} + Bu + L(y - C\hat{\underline{x}})$$

$A - LC$ stable $\Rightarrow \hat{x}_3 \longrightarrow b_\omega$ as $t \longrightarrow \infty$. Rate of convergence depends on eigenvalues of $A - LC$.

Cambridge University Engineering Dept. Third year

Module 3F2: Systems and Control

LECTURE NOTES 4: CONTROLLABILITY AND STATE FEEDBACK

K. Glover later revisions by J.Maciejowski, G. Vinnicombe

Jan 2002 current version: February 2019

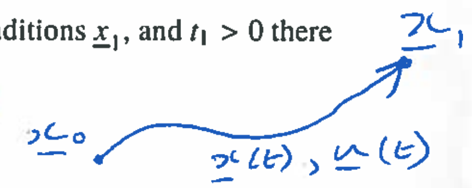
1 Controllability

1.1 Controllability Gramian, Controllability matrix

A system:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

is said to be **controllable** if for all initial conditions $\underline{x}(0) = \underline{x}_0$, terminal conditions $\underline{x}_1$, and $t_1 > 0$ there exists an input $u(t)$, $0 \le t \le t_1$ such that



$$\underline{x}(t_1) = \underline{x}_1.$$

That is, given $\underline{x}_0$, $\underline{x}_1$ and $t_1 > 0$, we wish to find $\underline{u}(t), 0 < t < t_1$, such that

$$\underline{x}(t_1) = e^{At_1}\underline{x}_0 + \int_0^{t_1} e^{A(t_1-t)}B\underline{u}(t)\,dt = \underline{x}_1$$

Note that this equation can be solved for all $\underline{x}_o$ and $\underline{x}_1$ if and only if it can be solved for all $\underline{x}_1$ with $\underline{x}_o = \underline{0}$. So we will now just consider the zero initial condition response.

Define the **controllability Gramian**, $W_c(t_1) \stackrel{def}{=} \int_0^{t_1} e^{A\tau} B B^T e^{A^T\tau}\, d\tau$.

Now assume that $W_c(t_1)$ has an inverse and let $\underline{u}(t) = \underline{u}_o(t) = B^T e^{A^T(t_1-t)} W_c(t_1)^{-1} \underline{x}_1$ when

$$
\begin{aligned}
\underline{x}(t_1) &= \int_0^{t_1} e^{A(t_1-t)} B \underbrace{B^T e^{A^T(t_1-t)} W_c(t_1)^{-1} \underline{x}_1}_{\underline{u}_o} dt \\
&= W_c(t_1) W_c(t_1)^{-1} \underline{x}_1 = \underline{x}_1 \text{ as desired.}
\end{aligned}
$$

Hence if $\det W_c(t_1) \neq 0$ then we can reach any $\underline{x}(t_1)$ from $\underline{x}(0) = \underline{0}$ (and hence there exists $u(t)$ to go from any $\underline{x}(0)$ to any $\underline{x}(t_1)$).

det ~~...~~ $W_c(t_1) \neq 0$ for all t_1

$\Rightarrow$ CONTROLLABLE

(Recall from section 3.1 of Lecture Notes 3:
$W_o(t_1) = \int_0^{t_1} e^{A^T\tau} C^T C e^{A\tau} d\tau \quad \leftarrow$ *Observability Gramian*)

$\Leftrightarrow W_c(t_1)\underline{z} = 0$

If $W_c(t_1)$ is a singular matrix there exists $\underline{z} \neq \underline{0}$ such that

$$\underline{z}^T W_c(t_1) = \underline{0} \Rightarrow \underline{z}^T W_c(t_1) \underline{z} = 0 \Rightarrow \underline{z}^T e^{At} B = \underline{0} \text{ for all } t$$

$\int (B^T e^{A^T\tau} \underline{z})^T (B^T e^{A^T\tau} \underline{z})\, d\tau$

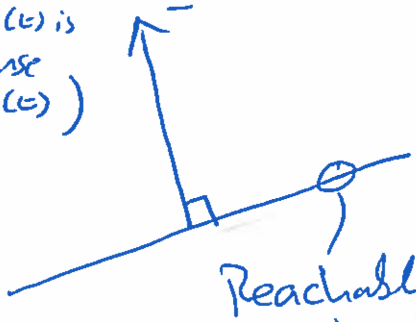
and hence

$$\underline{z}^T \underline{x}(t_1) = \int_0^{t_1} \underline{z}^T e^{A(t_1-t)} B \underline{u}(t)\, dt = 0 \text{ for all } \underline{u}(t).$$

(where $x(t)$ is the response to $u(t)$)

$\Rightarrow \underline{x}(t_1) \perp \underline{z}$ and the system is not controllable.

Hence: System is controllable if and only if $\det W_c(t_1) \neq 0$.



In section 3.1 of Lecture Notes 3 we showed

Null space of $W_o(t_1)$ = Null space of Q.

Similarly ~~we can show:~~

Null space of $W_c(t_1)$ = Null space of P^T

where the **controllability matrix** P is given by

$$P \stackrel{def}{=} \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

$P^T = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \end{bmatrix}$ (call x(t...

Hence $\boxed{\text{The system is controllable if and only if rank } P = n}$

1.2 Minimum Energy Input

Theorem 1.1 *The input,* $\underline{u}(t) = \underline{u}_o(t) = B^T e^{A^T(t_1-t)} W_c(t_1)^{-1}\underline{x}_1$, *takes the state from* $\underline{x}(0) = \underline{0}$ *to* $\underline{x}(t_1) = \underline{x}_1$ *and in addition is the input with minimum energy that achieves this.*

Proof:

Let $\underline{u}(t) = \underline{u}_o(t) + \underline{u}_1(t)$ then $\underline{x}(t_1) = \underline{x}_1 + \int_0^{t_1} e^{A(t_1-t)}B\underline{u}_1(t)\,dt$ and hence $\underline{x}(t_1) = \underline{x}_1$ implies,

$$\int_0^{t_1} e^{A(t_1-t)}B\underline{u}_1(t)\,dt = \underline{0}$$

Energy in $\underline{u}(t)$ for $0 < t < t_1$ is defined as:

$$\begin{aligned}\int_0^{t_1} \|\underline{u}(t)\|^2 dt &= \int_0^{t_1} \underline{u}(t)^T\underline{u}(t)\,dt \qquad = \int (\underline{u}_o+\underline{u}_1)^T(\underline{u}_o+\underline{u}_1)\,dt \\ &= \int_0^{t_1}\left(\underline{u}_o(t)^T\underline{u}_o(t) + \underline{u}_o(t)^T\underline{u}_1(t) + \underline{u}_1(t)^T\underline{u}_o(t) + \underline{u}_1(t)^T\underline{u}_1(t)\right) dt\end{aligned}$$

Now

$$\int_0^{t_1} \underline{u}_o(t)^T\underline{u}_1(t)\,dt = \int_0^{t_1} \underline{x}_1^T W_c(t_1)^{-1} e^{A(t_1-t)}B\underline{u}_1(t)\,dt = \underline{0} = \int_0^{t_1} \underline{u}_1(t)^T\underline{u}_o(t)\,dt$$

and

$$\int_0^{t_1} \underline{u}_o(t)^T\underline{u}_o(t)\,dt = \underline{x}_1^T W_c(t_1)^{-1}\int_0^{t_1} e^{A(t_1-t)}BB^Te^{A^T(t_1-t)}\,dt\; W_c(t_1)^{-1}\underline{x}_1 = \underline{x}_1^T W_c(t_1)^{-1}\underline{x}_1$$

Hence

$$\int_0^{t_1} \underline{u}(t)^T\underline{u}(t)\,dt = \underline{x}_1^T W_c(t_1)^{-1}\underline{x}_1 + \int_0^{t_1} \underline{u}_1(t)^T\underline{u}_1(t)dt$$

Since both terms are ≥ 0 the minimum energy is achieved when $\underline{u}_1(t) = \underline{0}$ and hence $\underline{u}(t) = \underline{u}_o(t)$ when

$$\min \int_0^{t_1} \underline{u}(t)^T\underline{u}(t)\,dt = \underline{x}_1^T W_c(t_1)^{-1}\underline{x}_1.$$

Note that if $W_c(t_1)$ is nearly singular then a very large energy input is required to reach certain states.

Not analogy will large $x(0)$ can give small $\int_0^t \|y\|^2 dt$ if $W_o(t_1)$ is close to singular. - Lecture notes 3

In single-input/single-output systems this means that if a system is either not controllable or not observable then there are pole/zero cancellations in the transfer function.

Example:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1,\ 1], \quad D = 0$$

Observability:

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \Rightarrow \text{rank}(Q) = 1 \Rightarrow \text{NOT observable}$$

$$Qx_0 = 0? \quad \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{so } x = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} \text{ is not observable.}$$

Controllability:

$$P = [B,\ AB] = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \Rightarrow \text{rank}(P) = 2 \Rightarrow \text{controllable}$$

Example continued

Transfer function:

$$G(s) = C(sI - A)^{-1}B + D = [1,\ 1]\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \frac{[1\ \ 1]\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{s(s+3) + 2}$$

$$= \frac{[1\ \ 1]\begin{bmatrix} 1 \\ s \end{bmatrix}}{(s+1)(s+2)}$$

$$= \frac{(\cancel{s+1})}{(\cancel{s+1})(s+2)} = \frac{1}{s+2} \qquad \text{pole-zero cancellation}$$

1.3 Reachable States and Minimal Realizations

We have seen in the previous section that if $W_c(t_1)$ is nearly singular then some directions in the state space are very difficult to reach, and if $W_c(t_1)$ is singular then some states cannot be reached and that $\underline{x}(t_1)$ is necessarily perpendicular to the null space of $W_c(t_1)$. It can in fact be shown *(details are omitted)* that the states that can be reached at time t_1 from $\underline{x}(0) = \underline{0}$ are precisely of the form:

$$\begin{aligned} \text{Reachable states} &= \text{Range space of } W_c(t_1) \\ &= \text{Range space of } P = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \end{aligned}$$

(since null spaces of P^T and $W_c(t_1)$ are the same, – lec notes 3

Example ($n=3$)

$$P = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{Range (or column span)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \alpha' \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \beta' \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Definition 1.2 *A set of state equations given by (A, B, C, D) is called a minimal realization of its transfer function, $G(s) = D + C(sI - A)^{-1}B$, if there does not exist a state space realization of $G(s)$ with a lower state dimension.*

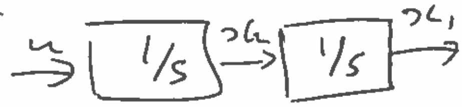
In section 3.2 of Lecture Notes 3 we saw that if a system was not observable then there was a change of state coordinates that gave an observable realisation of the transfer function with r states where $r = \text{rank}(Q)$.

If this system with r states is not controllable its state dimension could be further reduced in a similar manner and we are left with a state-space realisation of the transfer function that is both controllable and observable. It turns out that *(proof omitted)*:

Theorem 1.3 *A realization is minimal if and only if it is both controllable and observable.*

Example

DOUBLE-INTEGRATOR



$$\underbrace{\underline{\dot{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u}, \quad \Rightarrow e^{At}B = \begin{bmatrix} t \\ 1 \end{bmatrix},$$

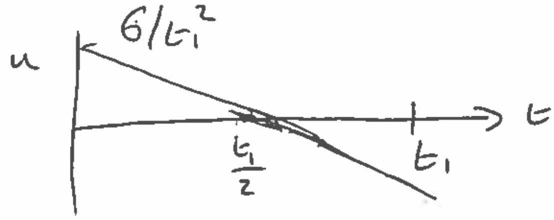
$$\begin{bmatrix} t \\ 1 \end{bmatrix} \begin{bmatrix} t & 1 \end{bmatrix}$$

$$\Rightarrow W_c(t_1) = \int_0^{t_1} \begin{bmatrix} t^2 & t \\ t & 1 \end{bmatrix} dt = \begin{bmatrix} \frac{1}{3}t_1^3 & \frac{1}{2}t_1^2 \\ \frac{1}{2}t_1^2 & t_1 \end{bmatrix}$$

$$W_c(t_1)^{-1} = \begin{bmatrix} 12/t_1^3 & -6/t_1^2 \\ -6/t_1^2 & 4/t_1 \end{bmatrix}$$

Minimum energy to go from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is hence $12/t_1^3$ achieved when

$$\underline{u}(t) = \left(e^{A(t_1-t)}B\right)^T W_c(t_1)^{-1}\underline{x}_1 = \begin{bmatrix} t_1 - t & 1 \end{bmatrix} \begin{bmatrix} 12/t_1^3 \\ -6/t_1^2 \end{bmatrix} = \frac{6}{t_1^3}(t_1 - 2t)$$



$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow A^2 = 0 \quad \Rightarrow A^3 = 0, \text{ etc}$$

$$\Rightarrow e^{At} = I + At + \frac{(At)^2}{2!} + \dots$$

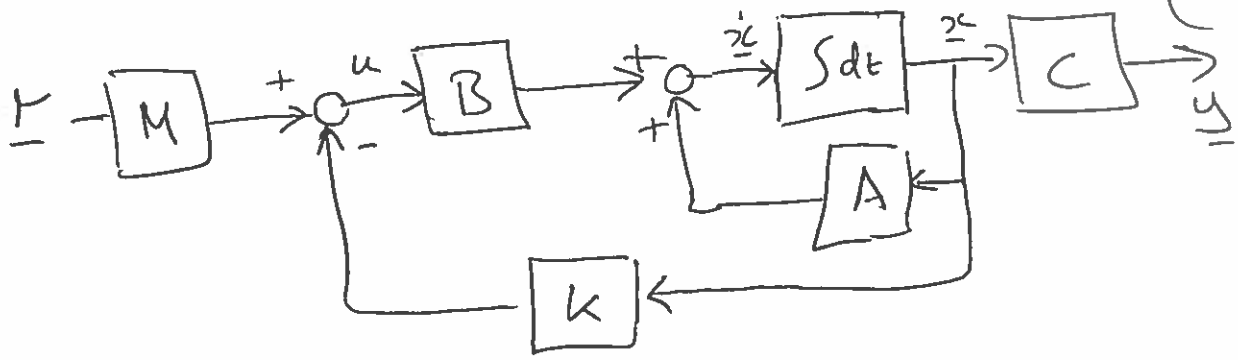
$$= I + At$$

$$= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

2 State Feedback

The response of a system is largely determined by the location of its closed loop poles. Can state feedback assign the closed loop poles?

(ASSUME D=0)



System: $\dot{\underline{x}} = A\underline{x} + B\underline{u}$, with state feedback: $\underline{u} = -K\underline{x} + M\underline{r}$, giving closed loop:

$$\dot{\underline{x}} = (A - BK)\underline{x} + BM\underline{r}.$$

Theorem 2.1 *The closed loop poles will be the eigenvalues of $(A - BK)$ which can be placed arbitrarily by choice of K if and only if (A, B) is controllable.*

(This is entirely analogous to the statement in section 3.2 of Lecture Notes 3, that the eigenvalues of $(A - LC)$ can be arbitrarily assigned by choice of L —
if (A, C) is observable).

*

Ackerman's Formula: for K.

Let

$$r(s) = r_0 + r_1 s + r_2 s^2 + \cdots + r_{n-1} s^{n-1} + s^n$$

and

$$r(A) = r_0 I + r_1 A + r_2 A^2 + \cdots + r_{n-1} A^{n-1} + A^n$$

In the single input case, let

$$K = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} P^{-1} r(A)$$

Then the eigenvalues of $A - BK \equiv$ roots of $r(s)$

proof (for interest): on moodle

In the multi-input case: If a system is controllable, it can be shown that it is always possible to choose a column of B, b_k, and a matrix N such that the pair $A + BN$, b_k is observable.

Where to place the poles?

- stable
- fast enough
- but not too fast since this might
  - saturate actuators
  - give poor stability margins.

Optimal - LQR

2.1 Steady-State Gain

Servo-system. Suppose we want $\underline{y}(t) \to \underline{r}$.

Two approaches to obtain the correct DC gain:

(a) **Choice of M**

$$\begin{cases} \dot{\underline{x}} &= (A - BK)\underline{x} + BM\underline{r} \\ \underline{y} &= C\underline{x} \end{cases}$$

In steady-state: $\dot{\underline{x}} = \underline{0} \Rightarrow \underline{y} = C(-A + BK)^{-1}BM\underline{r}$.

Choose M such that $C(-A + BK)^{-1}BM = I$ and $\underline{y}(t) \to \underline{r}$ after a step change with speed given by eigenvalues of $(A - BK)$. [Such an M usually exists if $\dim(\underline{u}) \geq \dim(\underline{y})$ but not otherwise].

This requires *exact knowledge* of the system matrices. The steady-state error being zero is not *robust* to small changes in the system. Also need to know an equilibrium condition.

(b) Integral Action

Integral action can be incorporated by augmenting the state by the integral of the error, i.e.

$$\underline{\dot{e}} = \underline{r} - \underline{y} = \underline{r} - C\underline{x} \qquad \underline{e} = \int_0^t r(\tau) - y(\tau)\, d\tau$$

which gives

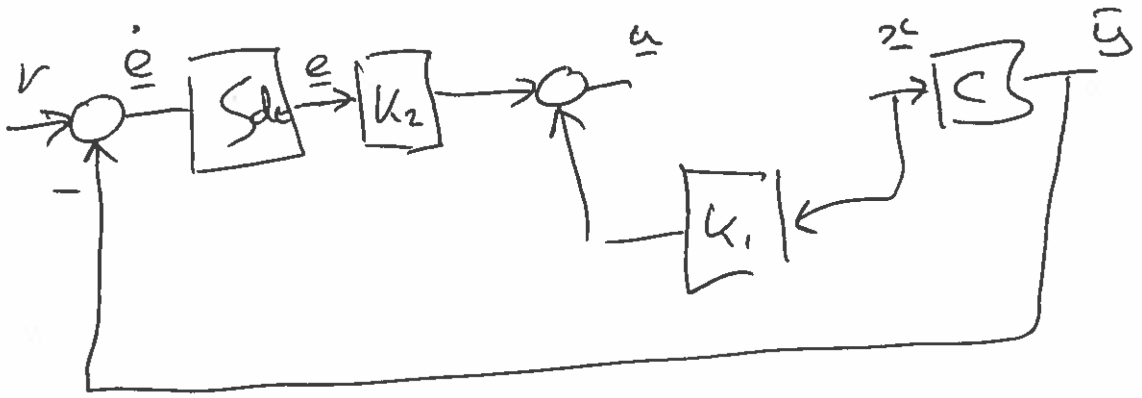
$$\begin{bmatrix} \underline{\dot{x}} \\ \underline{\dot{e}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \underline{u} + \begin{bmatrix} 0 \\ I \end{bmatrix} \underline{r}$$

with state feedback: $\underline{u} = -K_1\underline{x} - K_2\underline{e}$. $= -\begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix}$

Choose K_1, K_2 to assign the closed-loop poles (possible if augmented system controllable) and then $\underline{e}(t) \to 0 \Rightarrow \underline{y}(t) \to \underline{r}$ after a step change.

Robust to small changes in A, B, C, K.

Does not require knowledge of the equilibrium condition.



3 Observers with State Feedback

$$\text{SYSTEM} \quad \begin{cases} \underline{\dot{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} \end{cases}$$

$$\text{OBSERVER} \quad \begin{cases} \underline{\dot{\hat{x}}} = A\underline{\hat{x}} + B\underline{u} + L(\underline{y} - \underline{\hat{y}}) = (A - LC)\underline{\hat{x}} + B\underline{u} + L\underline{y} \\ \underline{\hat{y}} = C\underline{\hat{x}} \end{cases}$$

$$\text{CONTROLLER} \quad \left\{ \underline{u} = -K\underline{\hat{x}} + M\underline{r} \right.$$

$$\begin{aligned} \text{Error:} \quad \underline{e} &= \underline{x} - \underline{\hat{x}} \\ \underline{\dot{e}} &= (A - LC)\underline{e} \\ \underline{u} &= -K(\underline{x} - \underline{e}) + M\underline{r} \\ \underline{\dot{x}} &= (A - BK)\underline{x} + BK\underline{e} + BM\underline{r} \end{aligned}$$

$$\Rightarrow \begin{cases} \begin{bmatrix} \underline{\dot{x}} \\ \underline{\dot{e}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix} + \begin{bmatrix} BM \\ 0 \end{bmatrix} \underline{r} \\ \underline{y} = [C \;\; 0] \begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix} \end{cases}$$

NB: Eigenvalues of $\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ = {Eigenvalues of X} ∪ {Eigenvalues of Z}

So closed-loop poles are at the eigenvalues of $(A - BK)$ and those of $(A - LC)$.

$\underline{e}$ is not affected by $\underline{r}$ so that $\underline{e}(t) \rightarrow \underline{0}$.

Separation of estimation and control.

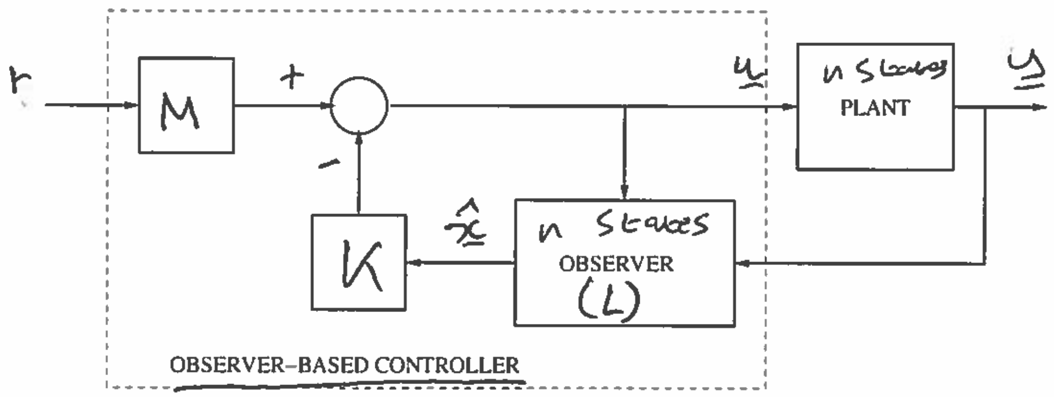
Can this always be done?

If (A, B) is controllable and (A, C) is observable, then no problems —
we can place all eigenvalues anywhere we want.
If all uncontrollable and unobservable modes (states) are stable, may still be OK.
If any uncontrollable or unobservable modes are unstable, then NOT OK,
since they will remain in the closed-loop system.

$$\det\left[\lambda I - \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\right] = \det \begin{bmatrix} \lambda I - X & -Y \\ 0 & \lambda I - Z \end{bmatrix}$$

$$= \det(\lambda I - X) \det(\lambda I - Z)$$

Block diagram:



If $\underline{r} \equiv 0$ then this structure is the same as for a dynamic precompensator.

For $\underline{r} \not\equiv 0$ the structures are different.

4 State Feedback Design Example

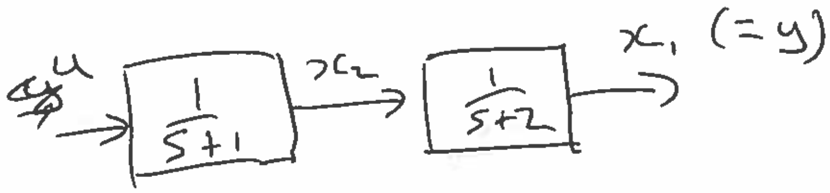
Plant $G(s) = \frac{1}{s+2} \times \frac{1}{s+1}$

Design Spec

Response in y to a step command on r to have zero offset and small overshoot.

State Feedback Design

First formulate the state space equations



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \text{PLANT}$$

We will again need to add an integrator to ensure a zero offset. The state feedback formulation will now be

$$u = -K_1 x_1 - K_2 x_2 - K_3 x_3$$

$$= -\begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where

The extra state variable x_3 has been added to integrate the output error -

$$\dot{x}_3 = -x_1 + r$$

this gives an augmented set of state equations

$$\frac{d}{dt}\underline{x} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

and the proposed feedback scheme is given by $u = -\underline{k}^T \underline{x} = -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \underline{x}$, so the closed loop state equations become

$$\underline{\dot{x}} = (A - B\underline{k}^T)\underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r = \begin{bmatrix} -2 & 1 & 0 \\ -k_1 & -1-k_2 & -k_3 \\ -1 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

The closed loop characteristic equation becomes

$$\begin{aligned} \det[\lambda I - (A - B\underline{k}^T)] &= \det \begin{bmatrix} \lambda + 2 & -1 & 0 \\ k_1 & \lambda + 1 + k_2 & +k_3 \\ 1 & 0 & \lambda \end{bmatrix} = (\lambda+2)(\lambda+1+k_2)\lambda + k_1\lambda - k_3 \\ &= \lambda^3 + (3+k_2)\lambda^2 + (2 + k_1 + 2k_2)\lambda - k_3 \end{aligned}$$

Suppose we desired all the closed loop poles to be at -5, then the required characteristic equation would be:

$$(\lambda + 5)^3 = \lambda^3 + 15\lambda^2 + 75\lambda + 125$$

Equating coefficients now gives

$$3 + k_2 = 15 \Rightarrow \underline{k_2 = 12}, \;\; 2 + k_1 + 2k_2 = 75 \Rightarrow \underline{k_1 = 49}, \;\; \underline{k_3 = -125}$$

Shorthand

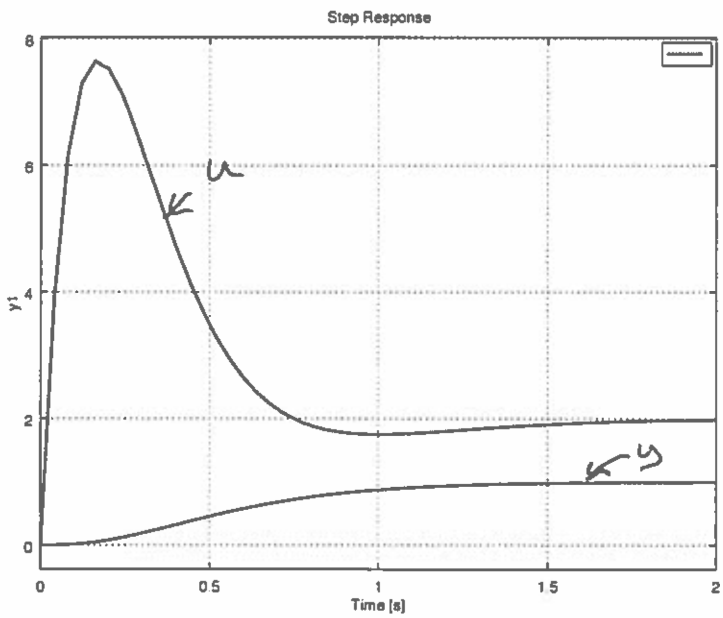
The transfer function from r to y can now be computed as:-

$$T_{R\to Y} = \left(\frac{Y(s)}{R(s)}\right) = C[sI-(A-Bk^T)]^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = [1\ \ 0\ \ 0]\begin{bmatrix}s+2 & -1 & 0\\ 49 & s+13 & -125\\ 1 & 0 & s\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{125}{(s+5)^3}$$

Also the transfer function from r to u can be computed as:-

$$T_{R\to U} = \left(\frac{U(s)}{R(s)}\right) = -\frac{\underline{k}^T \underline{X}(s)}{R(s)} = [49\ \ 12\ \ -125]\begin{bmatrix}s+2 & -1 & 0\\ 49 & s+13 & -125\\ 1 & 0 & s\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{125(s+1)(s+2)}{(s+5)^3}$$

The step responses are thus:-



Note: If the second state is not measured, and we use an observer based on the output (the 1st state) and the error (3rd state), then the response to the reference is *identical* to this, as the error dynamics are not excited. If there are unmeasured disturbances however, the response with state feedback and with an observer based controller are different.