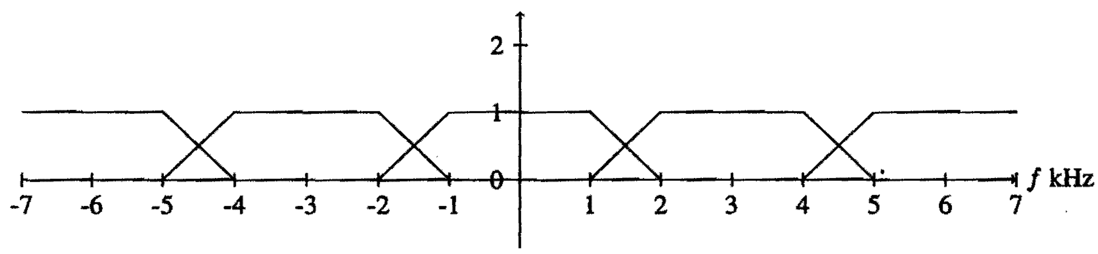
Solutions: Paper 7 – IB Information Engineering

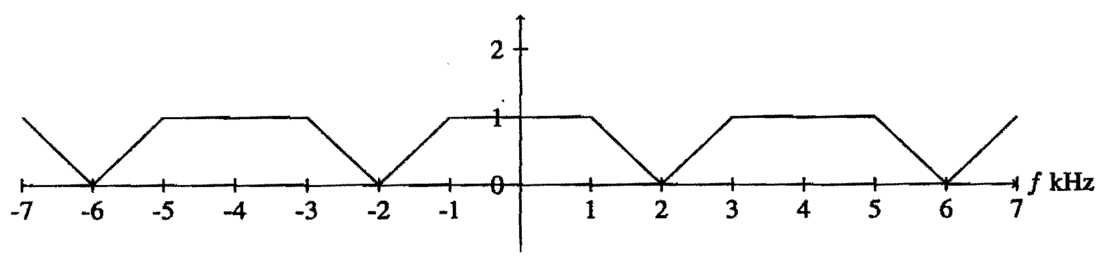
Signal and Data Analysis: Sampling, Discrete Signals and the DFT

1. (a) The sampled spectrum is $\frac{1}{T}\sum_{n=-\infty}^{\infty} f(\omega - n\omega_0)$. Hence

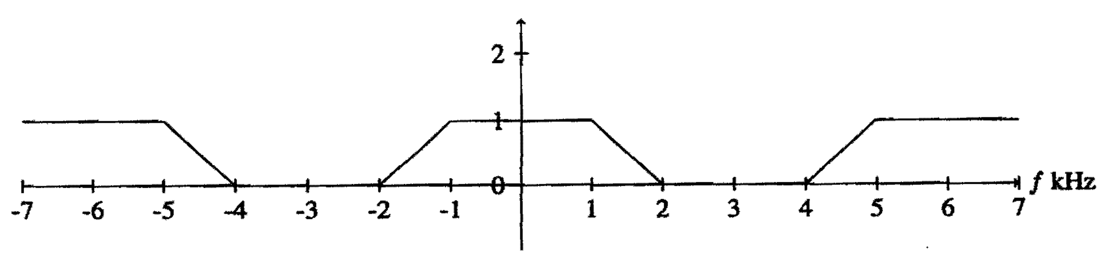
(i) 3kHz:



(i) 4kHz:



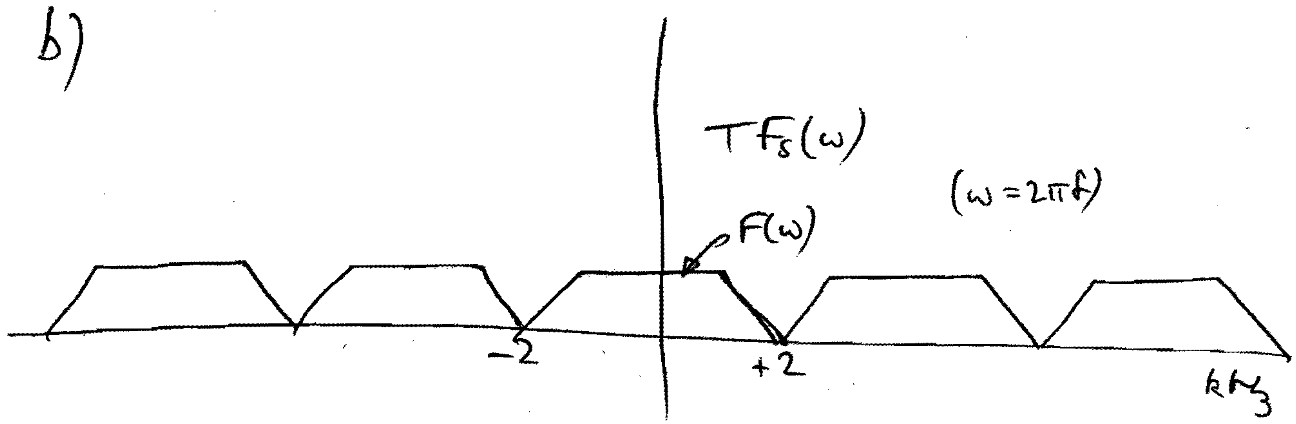
(i) 6kHz:



1.a) contd.

Minimum sampling rate $= 2 f_{max} = 4\,kHz$.

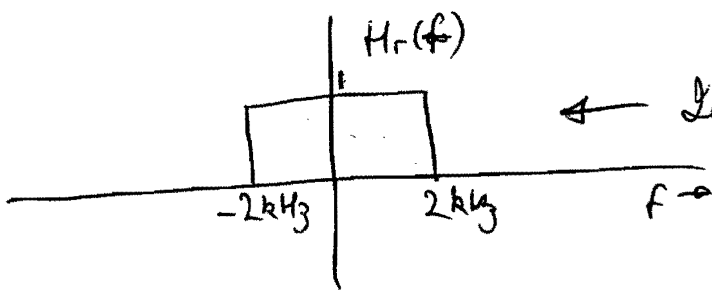
b)



Pass

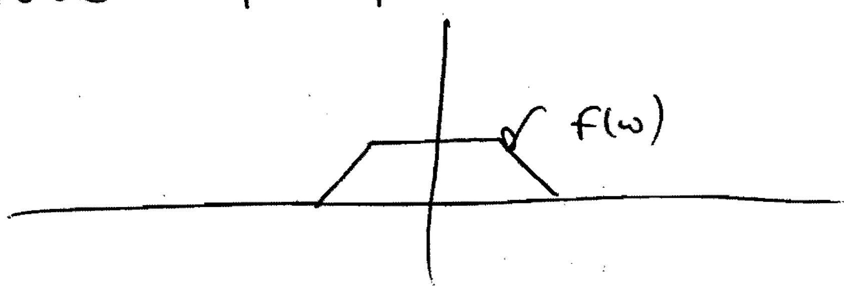
$$T\sum_{-\infty}^{+\infty}\delta(t-nT)\,f(nT)$$

thro' a lowpass filter:



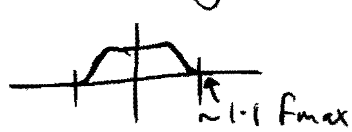
← Ideal reconstruction filter

Hence output spectrum is



and the signal is perfectly reconstructed.

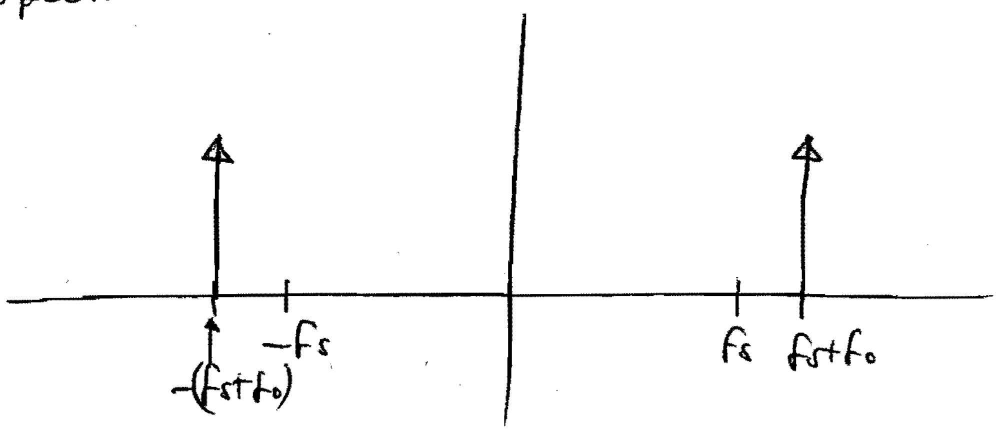
In practice, $H_r(f)$ is:



and the sampling rate must be increased proportionally.

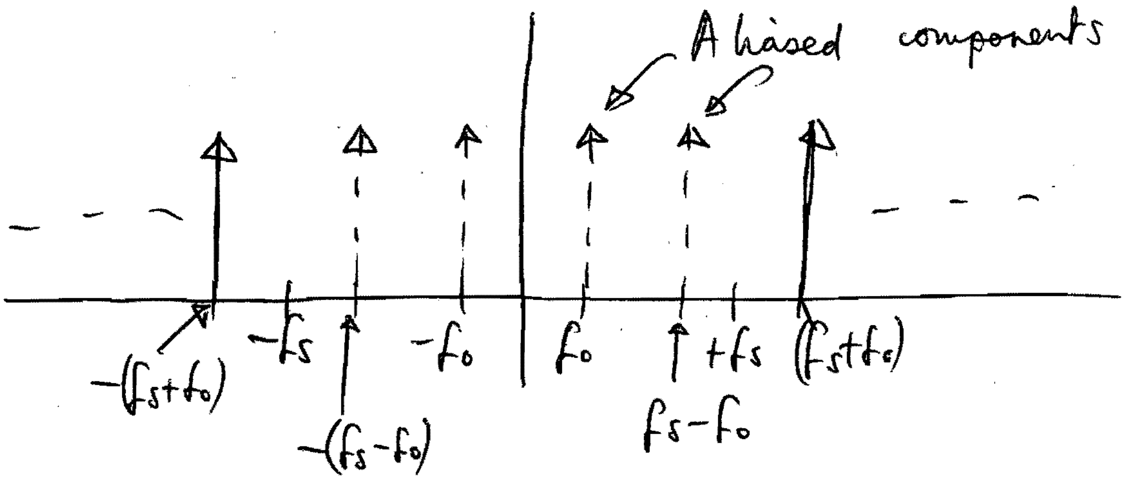
2. Consider e.g. $a\cos(2\pi(f_s+f_o)t)$

Spectrum is:



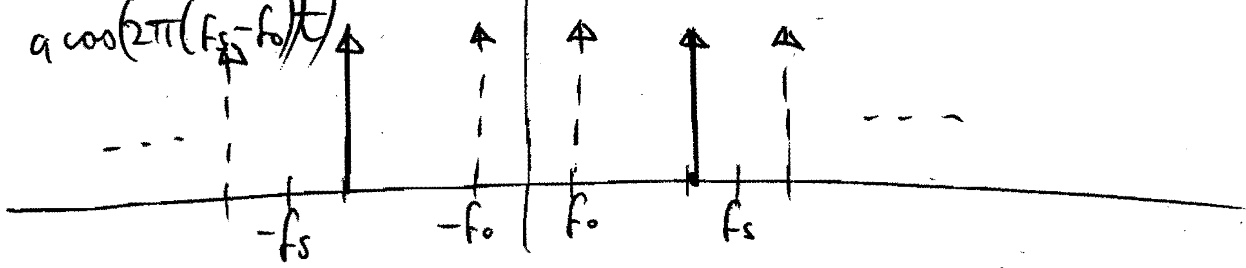
Sampled spectrum is therefore:

Aliased components shown dotted

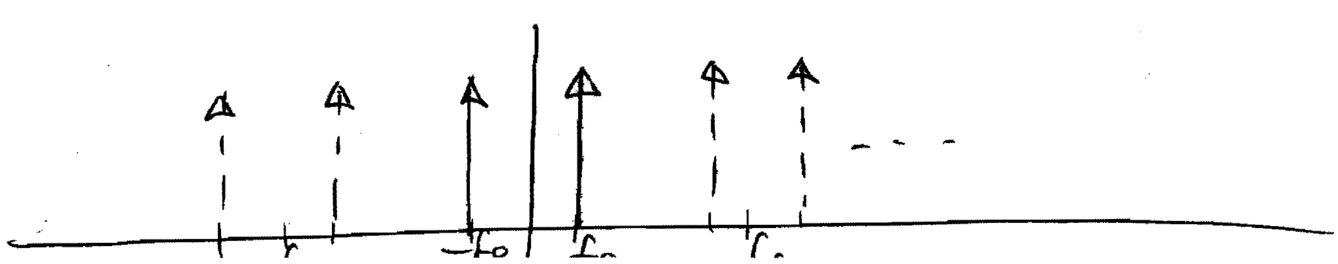


Similarly for other cases:

$a\cos(2\pi(f_s-f_o)t)$



$a\cos(2\pi f_o t)$



2. contd.

Comparison of the spectra shows all 3 are identical once sampled.

Verify:

$a \cos(2\pi(f_s + f_0)t)$:

$$f(n/f_s) = a \cos(2\pi(f_s + f_0) n/f_s)$$
$$= a \cos(2\pi n + 2\pi n f_0/f_s)$$
$$= \underline{\underline{a \cos(2\pi n f_0/f_s)}}$$

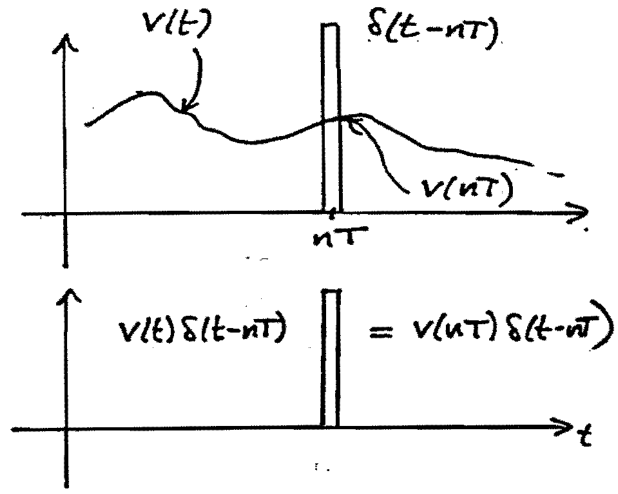
$a \cos(2\pi(f_s - f_0)t)$:

$$f(n/f_s) = a \cos(2\pi(f_s - f_0) n/f_s)$$
$$= a \cos(2\pi n - 2\pi n f_0/f_s)$$
$$= a \cos(-2\pi n f_0/f_s)$$
$$= \underline{\underline{a \cos(2\pi n f_0/f_s)}}$$

Hence all three are equal.

3.

Since the δ-function "picks out" the value of v at $t = nT$, this is the only value of v relevant



c.f. $\int \delta(x-a) f(x)\, dx = f(a)$

$\int \delta(x-a) f(a)\, dx = f(a)$ etc

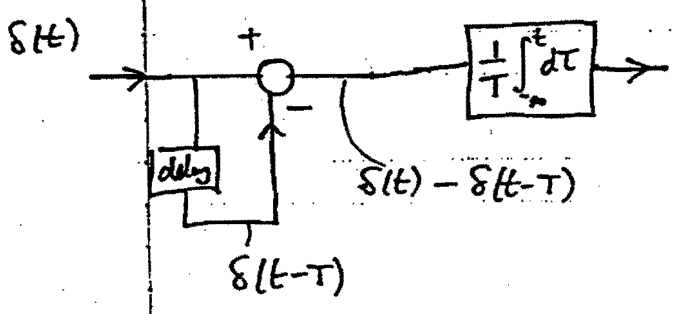
Thus $\quad v(t)\, \delta(t-nT) = v(nT)\, \delta(t-nT)$

Ideal sampled signal

$$v_s(t) = T v(t) \sum_n \delta(t-nT) = T \sum_n v(nT)\, \delta(t-nT)$$

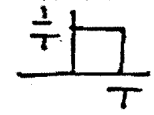
$$= T\{\ldots v(-2T)\,\delta(t+2T) + v(-T)\,\delta(t+T) + v(0)\,\delta(t) + \ldots\}$$

[The factor T is for normalisation purposes and is not essential]

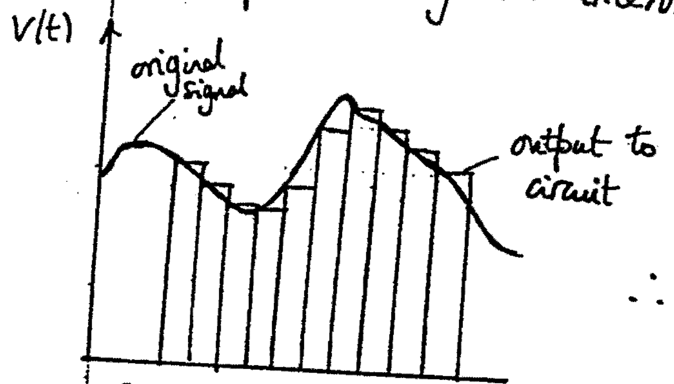


Integral of impulse $\delta(t)$ = unit step $= h(t)$

$\therefore$ output = impulse response $= \frac{1}{T}[h(t) - h(t-T)] =$



When $V_s(t)$ input to this circuit (V_s = sum of impulses) output during the interval $[nT, nT+1]$ is simply $V(nT)$.

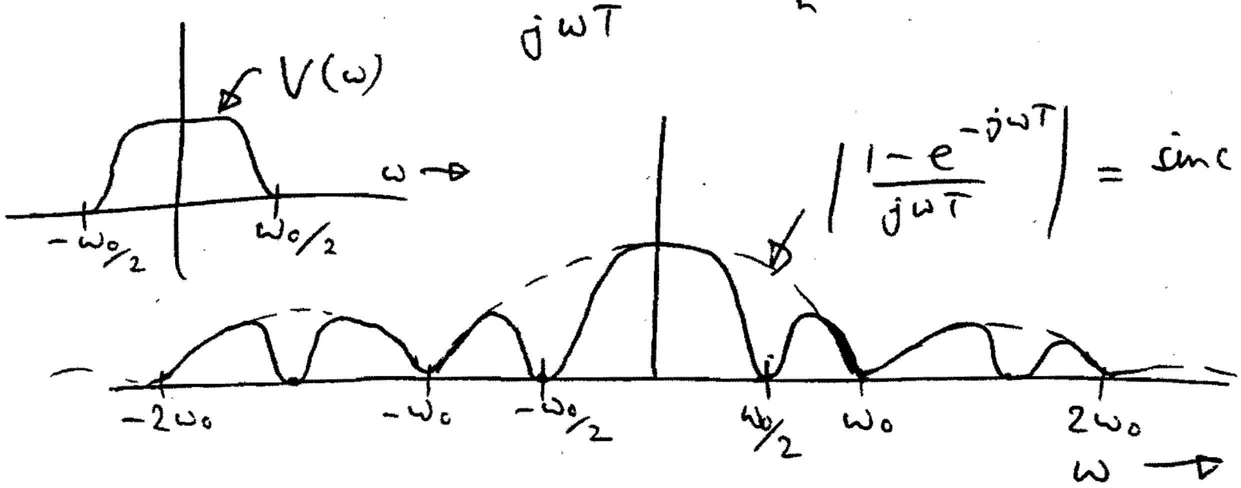


Transfer function of circuit $= \dfrac{1-e^{-j\omega T}}{j\omega T} = H(\omega)$

$\therefore$ Transform of output $= \dfrac{1-e^{-j\omega T}}{j\omega T} \sum_n V(\omega - n\omega_0)$

Thus this 'more natural' interpretation of the sampled signal still alias.

$$V_s(\omega) = \frac{1-e^{-j\omega T}}{j\omega T} \sum_n V(\omega - n\omega_0)$$



Hence can extract $V(\omega)$ with a filter:

$$H_r(\omega) = \begin{cases} \dfrac{j\omega T}{1-e^{-j\omega T}}, & -\omega_0/2 < \omega < +\omega_0/2 \\ 0, & \text{elsewhere.} \end{cases}$$

4. Due to sampling

$$\text{d.c.} \rightarrow n\omega_s = \frac{n\omega}{1+k}$$

$$\pm\omega \rightarrow \pm(\omega + n\omega_s) = \pm\left(\omega + \frac{n\omega}{1+k}\right) = \pm \frac{\omega(1+n+k)}{1+k}$$

$$\pm 2\omega \rightarrow \pm(2\omega + n\omega_s) = \pm\,\omega\left(\frac{2+n+2k}{1+k}\right).$$

If k is small and a low pass filter with cut-off $\simeq \omega/2$ is used then only small frequencies survive

$$\text{d.c.} \longrightarrow 0 \quad (\text{only } n=0 \text{ passes filter})$$

$$\pm\omega \longrightarrow \pm\frac{k\omega}{1+k} \quad (\text{" } n=-1 \text{ " "})$$

$$\pm 2\omega \longrightarrow \pm\frac{2k\omega}{1+k} \quad (\text{" } n=-2 \text{ " "})$$

At filter output, then, sampled signal contains

$$a_0 \;;\; a_1 \cos\frac{k\omega t}{1+k} \;;\; a_2 \cos\frac{2k\omega t}{1+k}$$

i.e. Output signal is

$$a_0 + a_1 \cos\omega b t + a_2 \cos 2\omega b t = x(bt)$$

where $b = \dfrac{k}{1+k}$.

5. Verify that:

$$f_m = \frac{T}{2\pi} \int_{-\pi/T}^{+\pi/T} F_s(\omega) e^{+j\omega m T} \, d\omega$$

Substitute for $F_s(\omega)$:

$$= \frac{T}{2\pi} \int_{-\pi/T}^{+\pi/T} \sum_{n=-\infty}^{+\infty} f_n e^{-jn\omega T} e^{+j\omega m T} \, d\omega$$

$$= \frac{T}{2\pi} \sum_{n=-\infty}^{+\infty} f_n \int_{-\pi/T}^{+\pi/T} e^{j\omega(m-n)T} \, d\omega$$

$$= \frac{T}{2\pi} \sum_{n=-\infty}^{+\infty} f_n \int_{-\pi}^{+\pi} e^{j\theta(m-n)} \, \frac{d\theta}{T}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} f_n \, 2\pi \, \delta[n-m]$$

$$\left(\text{where } \delta[n] = \begin{cases} 1, & n=0 \\ 0, & \text{elsewhere} \end{cases} \right)$$

$$= \frac{1}{2\pi} \times 2\pi f_m$$

$$= \underline{\underline{f_m}}.$$

6. With $x(0)=1$ $x(T)=0$ $x(2T)=0$ $x(3T)=1$ where T is the sampling time, the DFT is

$$X(k) = \sum_{n=0}^{N-1} x(nT)\, e^{-jk\Omega nT} \qquad \Omega = \frac{2\pi}{NT}$$

$$= \sum_{n=0}^{N-1} x(nT)\, e^{-jkn2\pi/N}$$

$$\therefore \quad X(0) = \sum_{n=0}^{3} x(nT)\, e^{-j0} = \sum x(nT) = 1+0+0+1 = \underline{2}$$

$$X(1) = \sum_{n=0}^{3} x(nT)\, e^{-j\frac{2\pi n}{N}} = 1+0+0+e^{-j\frac{2\pi 3}{4}} = \underline{1+j}$$

$$X(2) = \sum_{n=0}^{3} x(nT)\, e^{-\frac{4\pi n}{N}} = 1+0+0+e^{-j4\pi 3/4} = 1-1 = \underline{0}$$

$$X(3) = \sum_{n=0}^{3} x(nT)\, e^{-6\pi n/N} = 1+0+0+e^{-j6\pi 3/4} = \underline{\underline{1-j}}$$

$\therefore$ DFT is $\underline{(2,\ 1+j,\ 0,\ 1-j)}$ Magnitudes $(2, \sqrt{2}, 0, \sqrt{2})$ Phases $(0, 45^\circ, 0, -45^\circ)$

Use IDFT to check:

$$x(nT) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{jkn2\pi/N}$$

$$\therefore \quad x(0) = \frac{1}{4} \sum_{k=0} X(k) = \frac{1}{4}\left[2+1+j+0+1-j\right] = 1$$

$$x(T) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{jk\pi/2} = \frac{1}{4}\left[2+(1+j)e^{j\pi/2}+0+(1-j)e^{j3\pi/2}\right]$$

$$= \frac{1}{4}\left[2+j-1-j-1\right] = 0$$

$$x(2T) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{jk\pi} = \frac{1}{4}\left[2+(1+j)e^{j\pi}+0+(1-j)e^{3j\pi}\right]$$

$$= \frac{1}{4}\left[2-1-j+0-1+j\right] = 0$$

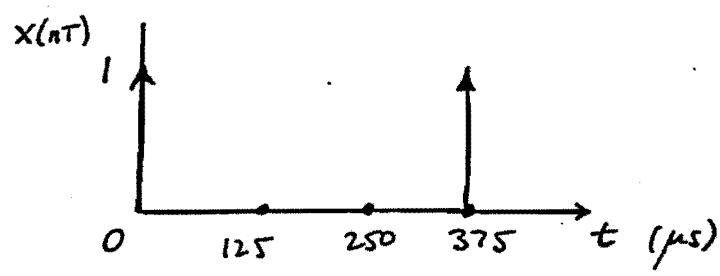
$$x(3T) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{3jk\pi/2} = \frac{1}{4}\left[2+(1+j)e^{3j\pi/2}+0+(1-j)e^{j9\pi/2}\right]$$

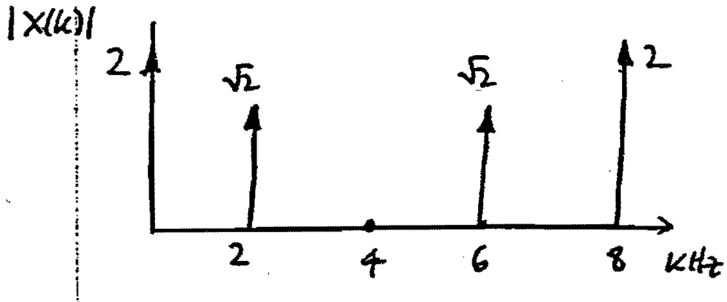
$$= \frac{1}{4}\left[2-j+1+j+1\right] = 1$$

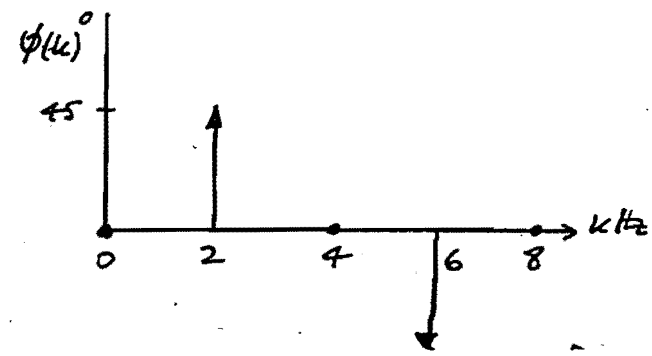
$\therefore$ $\underline{\text{Sequence is } \{1, 0, 0, 1\}}$.

Sampling frequency $= 8\,kHz \Rightarrow T = 125\,\mu s.$

$$\therefore \Omega = \frac{2\pi}{NT} = \Delta\omega = \frac{2\pi}{4 \times 125 \times 10^{-6}} = 2\ kHz$$







$$F_m = \sum_{n=0}^{N-1} f_n e^{-jnm 2\pi/N}$$

a) $$F_{-m} = \sum_{n=0}^{N-1} f_n e^{+jnm 2\pi/N}$$

$$F_{-m}^{*} = \sum_{n=0}^{N-1} f_n e^{-jnm 2\pi/N}$$

$$= \underline{\underline{F_m}}.$$

b) $$F_{m+N} = \sum_{n=0}^{N-1} f_n e^{-jn(m+N) 2\pi/N}$$

$$= \sum_{n=0}^{N-1} f_n e^{-jnm 2\pi/N} \underbrace{\cancel{e^{-jn2\pi}}}_{=1}$$

$$= \sum_{n=0}^{N-1} f_n e^{-jnm 2\pi/N} = \underline{\underline{F_m}}$$

7.6. DFT

$$F_K = \sum_{n=0}^{N-1} f(nT)\, e^{-j2\pi kn/N} \qquad \text{for } 0 \le k \le N-1$$

$$= \sum_{n=0} e^{-nT - j2\pi kn/N}$$

This is a G.P. with common ratio $e^{-T-j2\pi k/N}$

$$\therefore \quad F_K = \frac{1 - e^{-NT - j2\pi k}}{1 - e^{-j2\pi k/N}} \qquad \text{and } e^{-j2\pi k} = 1$$

$$= \frac{1 - e^{-NT}}{1 - e^{-j2\pi k/N}}$$

ω_s = Sampling frequency is $\frac{2\pi}{T}$, ω_K = frequency corresponding to F_K

$$= \frac{k}{N}\omega_s = \frac{2\pi k}{NT}$$

(More properly $\omega_K = \frac{k}{N}\omega_s \qquad 0 \le k \le \frac{N}{2} - 1$

$\omega_K = \frac{(k-N)\omega_s}{N} \qquad k \ge N/2$)