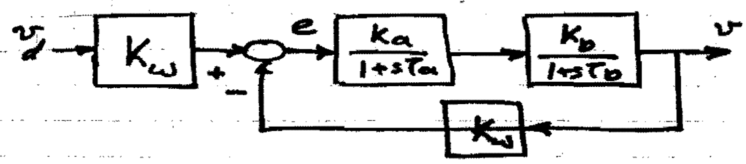
Example Sheet 6/3 - Solutions

SECOND YEAR

1. (a)



$$\bar{v}(s) = \frac{K_a K_b}{(1+sT_a)(1+sT_b)}\,\bar{e}(s)$$

$$= \frac{K_a K_b}{(1+sT_a)(1+sT_b)}\,K_\omega(\bar{v}_d - \bar{v})$$

$$\therefore \left[1 + \frac{K_a K_b K_\omega}{(1+sT_a)(1+sT_b)}\right]\bar{v}(s) = \frac{K_a K_b K_\omega}{(1+sT_a)(1+sT_b)}\,\bar{v}_d$$

so $$\frac{\bar{v}(s)}{\bar{v}_d(s)} = \frac{K_a K_b K_\omega}{(1+sT_a)(1+sT_b)+K_a K_b K_\omega}$$

$$= \frac{50}{(1+0.2s)(1+0.5s)+50}$$

$$= \frac{50}{0.1s^2+0.7s+51} = \underline{\underline{\frac{500}{s^2+7s+510}}}$$

(b) error $= v_d - v$.

$$\frac{\bar{v}_d(s) - \bar{v}(s)}{\bar{v}_d(s)} = 1 - \frac{\bar{v}(s)}{\bar{v}_d(s)} = 1 - \frac{500}{s^2+7s+510}$$

Get steady-state gain of this by setting $s=0$:

$$\frac{\bar{v}_d(0) - \bar{v}(0)}{\bar{v}_d(0)} = 1 - \frac{500}{510} = \frac{1}{51} \simeq \underline{\underline{2\%}}$$

($\underline{\text{NB}}$ $\frac{\text{Error}}{\text{Demand}} \approx \frac{1}{\text{'loopgain'}} = \frac{1}{K_a K_b K_\omega}$.)

(c) $$\frac{\bar{v}(s)}{\bar{v}_d(s)} = \frac{500}{s^2 + 7s + 510}$$, so 2nd order system

with undamped natural frequency $\omega_n = \sqrt{510} = 22.6$ rad/sec

and damping $c = \frac{7}{2\omega_n} = 0.155$

So underdamped response, with frequency

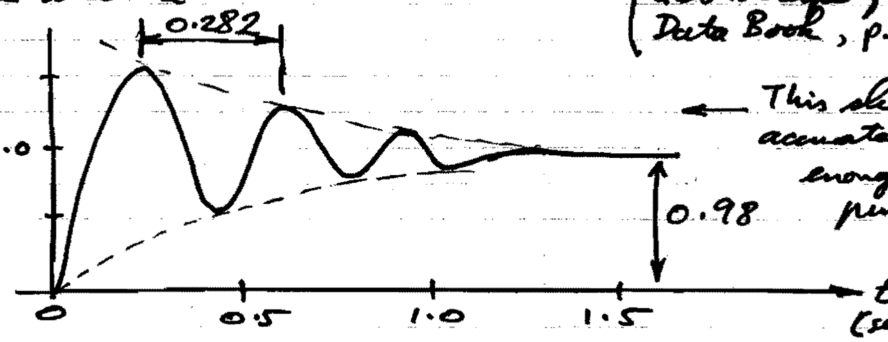
$\omega_d = \omega_n\sqrt{1-c^2} = 22.3$ rad/sec $= 3.55$ Hz, $\therefore$ period $= 0.282$ sec

and decay time constant $\frac{1}{c\omega_n} = 0.285$ sec

Steady-state gain $\frac{\bar{v}(0)}{\bar{v}_d(0)} = \frac{500}{510} = 0.98$

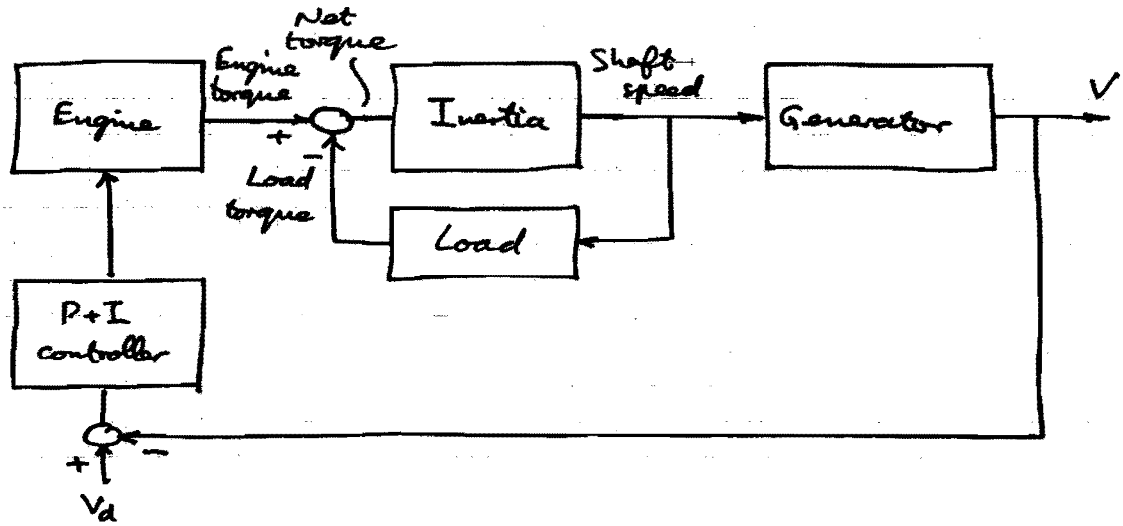
Hence sketch:

(Get shape from Mechanics Data Book, p. 4.)



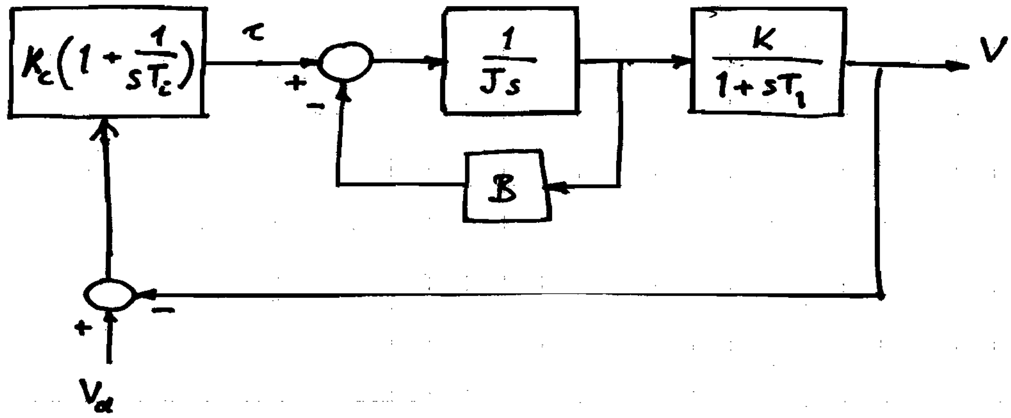
This sketch not very accurate, but good enough for most purposes.

2.

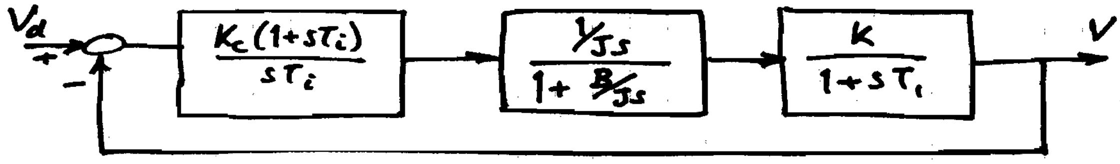


Putting in details of transfer functions:

2 contd.



This is equivalent to:

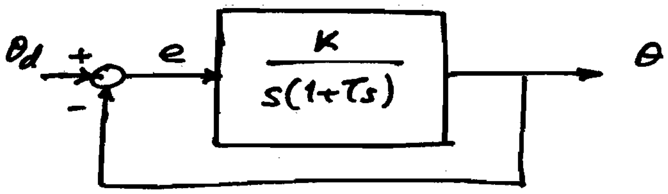


Hence (using '$\frac{g}{1+g}$'):

$$\frac{\bar{V}(s)}{\bar{V}_d(s)} = \frac{\frac{K_c(1+sT_i)}{sT_i} \times \frac{1}{Js+B} \times \frac{K}{1+sT_1}}{1 + \frac{K_c(1+sT_i)}{sT_i} \times \frac{1}{Js+B} \times \frac{K}{1+sT_1}}$$

$$= \underline{\underline{\frac{KK_c(1+sT_i)}{sT_i(Js+B)(1+sT_1) + KK_c(1+sT_i)}}}$$

3. (a)



$$\frac{\bar{\theta}(s)}{\bar{\theta}_d(s)} = \frac{\frac{K}{s(1+Ts)}}{1+\frac{K}{s(1+Ts)}} = \frac{K}{Ts^2+s+K} = \underline{\underline{\frac{K/T}{s^2+\frac{1}{T}s+K/T}}}$$

Hence $\omega_n^2 = \frac{K}{T}$ and $2c\omega_n = \frac{1}{T}$

$\therefore\ 100 = \frac{K}{T}$ and $10 = \frac{1}{T}$

$\therefore\ \underline{\underline{T = \frac{1}{10}}}$ and $\underline{\underline{K = 10}}$

3 cont'd. (b) (i) From Mechanics Data Book p.6 ('Case (a)'):

Max. gain occurs at (resonant frequency)

$$\omega_n\sqrt{1-2c^2} = \underline{\frac{10}{\sqrt{2}} \text{ rad/sec}} \quad (= 1.125\text{ Hz}).$$

Its value is $\dfrac{1}{2c\sqrt{1-c^2}} = \dfrac{2}{\sqrt{3}} = \underline{\underline{1.155}}$.

(NB These can also be read off the 'Y/X' curve for $c = 0.5$.)

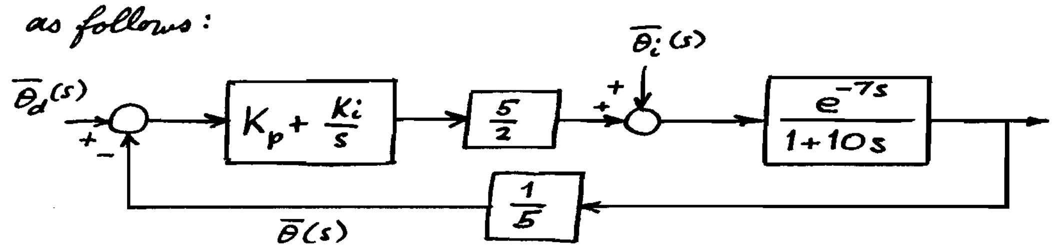
(ii) From Mechanics Data Book p.4, reading off the graph for $c=0.5$, the max. overshoot is about $\underline{\underline{0.16^\circ}}$.

4 (a) The 7-second delay has transfer function e^{-7s}

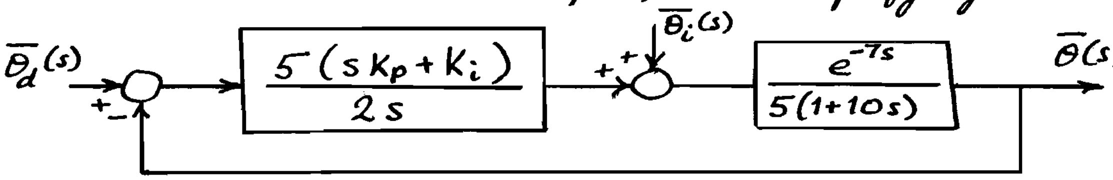
[Standard result, derived in lectures. Proof: $y(t) = x(t-7)$.

$$\bar{y}(s) = \int_0^\infty e^{-st}\,x(t-7)\,dt = \int_0^\infty e^{-s(t-7)}e^{-7s}\,x(t-7)\,dt = e^{-7s}\,\bar{x}(s).\ ]$$

Hence the block diagram can be re-drawn and simplified as follows:



Re-draw with $\bar{\theta}(s)$ as output, and simplify again:



Now let $g_1(s) = \dfrac{5(sK_p + K_i)}{2s}$ and $g_2(s) = \dfrac{e^{-7s}}{5(1+10s)}$.

Then $\bar{\theta}(s) = g_2(s)\left[\bar{\theta}_i(s) + g_1(s)\left(\bar{\theta}_d(s) - \bar{\theta}(s)\right)\right]$

so $\left[1 + g_2(s)g_1(s)\right]\bar{\theta}(s) = g_2(s)\,\bar{\theta}_i(s) + g_2(s)g_1(s)\,\bar{\theta}_d(s)$,

$$\text{and}\quad \bar{\theta}(s) = \underbrace{\frac{g_2(s)}{1+g_2(s)g_1(s)}}_{G_2(s)}\,\bar{\theta}_i(s) + \underbrace{\frac{g_2(s)g_1(s)}{1+g_2(s)g_1(s)}}_{G_1(s)}\,\bar{\theta}_d(s).$$

Transfer functions

$$G_2(s) = \ldots = \frac{0.2\,s\,e^{-7s}}{s(1+10s) + 0.5e^{-7s}(sK_p + K_i)}$$

$$G_1(s) = \ldots = \frac{0.5\,e^{-7s}(sK_p + K_i)}{s(1+10s) + 0.5\,e^{-7s}(sK_p + K_i)}$$

<u>NB</u> These transfer functions are 'irrational' i.e. ~~they~~ are not just ratios of polynomials.

6/3 Solutions

4 (b) Since the system reaches a steady state, we can
cont'd use the Final Value Theorem:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\bar{x}(s).$$

As shown in lectures, this implies that the steady-state gain of a (stable) system with transfer function $\bar{g}(s)$ is given by $\bar{g}(0)$.

(i) From part (a) we get, if $K_i \neq 0$:

$$G_1(0) = \frac{0\cdot 5K_i}{0\cdot 5K_i} = 1$$

But if $K_i = 0$ we have:

$$G_1(0) = \left.\frac{0\cdot 5e^{-7s}K_p}{1+10s+0\cdot 5e^{-7s}K_p}\right|_{s=0} = \frac{0\cdot 5K_p}{1+0\cdot 5K_p}$$

i) If $K_i \neq 0$:

$$G_2(0) = \frac{0}{K_i} = 0$$

But if $K_i = 0$:

$$G_2(0) = \left.\frac{0\cdot 2e^{-7s}}{1+10s+0\cdot 5e^{-7s}K_p}\right|_{s=0} = \frac{0\cdot 2}{1+0\cdot 5K_p} \quad \text{(volts/degC)}$$

(The significance of these results should be clear!)

4. (c) cont'd Period of 30 sec $\Rightarrow f = 0.0333\ldots$ Hz

$\Rightarrow \omega = 0.21$ rad/sec.

$\therefore \theta_i(t) = 2\sin(0.21t)$ (phase angle unimportant).

Consequently we need the frequency response of θ to θ_i at frequency 0.21, namely $G_2(j0.21)$.

Since we only want to know the amplitude, we need only work out $|G_2(j0.21)|$:

$$\frac{\text{Amplitude of } \theta}{\text{Amplitude of } \theta_i} = |G_2(j0.21)|$$

$$= \frac{0.2 \times |e^{-7\times0.21j}| \times |0.21j|}{|0.21j(1+2.1j) + 0.5e^{-7\times0.21j}(j\times0.21\times2+0.2)|}$$

$$= \frac{0.2\times1\times0.21}{|0.21j - 0.441 + 0.5(0.1006 - 0.9949j)(0.42j+0.2)|}$$

$$= \frac{0.042}{|-0.222 + 0.132j|}$$

$$= 0.163$$

$\therefore$ Amplitude of $\theta = 0.163 \times 2 = \underline{\underline{0.326}}$ (°C)

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Examples Paper 6/3 Q6

(a) The return ratio $L(s) = \frac{H_0(K_P+K_D s)}{ms^2}$, so the closed-loop transfer function is given by:

$$\bar{r}(s) = \frac{L(s)}{1+L(s)}\bar{r}_d(s) = \frac{H_0(K_P + K_D s)}{ms^2 + H_0(K_P + K_D s)}\bar{r}_d(s).$$

For the closed-loop poles with $K_D = 0$:

$$\begin{aligned} ms^2 + H_0(K_P + K_D s) &= 0 \\ ms^2 + H_0 K_P &= 0 \\ s &= \pm i\sqrt{\frac{H_0 K_P}{m}} \end{aligned}$$

Marginally stable (as found empirically in Examples Paper 6/2), with natural frequency $\omega_n = \sqrt{\frac{H_0 K_P}{m}}$. Simulations agree except for non-linear drag forces reducing oscillation amplitude slowly (and not exponentially).

(b) To design to specification, compare with second-order system characteristic equation:

$$s^2 + \frac{H_0 K_D}{m}s + \frac{H_0 K_P}{m} = s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$\begin{aligned} \omega_n^2 &= \frac{H_0 K_P}{m} &\Rightarrow K_P &= \frac{m\omega_n^2}{H_0} = 0.0910 \\ 2\zeta\omega_n &= \frac{H_0 K_D}{m} &\Rightarrow K_D &= \frac{2m\zeta\omega_n}{H_0} = 0.178 \end{aligned}$$

(c)

$$\bar{r}(s) = \frac{P(s)}{1+L(s)}\bar{f}_{\text{dist}}(s) = \frac{1}{ms^2 + (k_p + k_d s)H_o}\bar{f}_{\text{dist}}(s)$$

This transfer function has a modulus of 0.0098 at $s = 0.1j$, and so the amplitude is $100 \times 0.0098 = 0.98$. Without control, $k_p = k_d = 0$, the modulus is 1/2, giving an amplitude of 50m. (Or just solve $m\frac{d^2x}{dt^2} = 100\cos 0.1t$).

(d) Simulations should agree with predictions when initialising the lander close to the target altitude (500 m or 510 m) for both (i) and (ii). For (ii), when initialised at 700 m, the response is more oscillatory as the thrust saturates and the lander overshoots. Once the throttle becomes within the linear regime the behaviour is as expected.

(iii) With control you should see the altitude oscillate between 499m and 501m, as expected. Without it initially oscillates between 500m and 600m (which is an oscillation of amplitude 50 around 550m), but drifts due to the destabilising nonlinear gravity term. The offset is because the initialization is at 500m with zero velocity, which must then be an extreme point.

(iv) To make the lander hover at 500 m when initialising at 10 km (Scenario 1, but remember to turn the autopilot on!) you need to increase the damping sufficiently ($\zeta = 3$, $\zeta\omega_n$ unchanged).