Part IB Paper 6: Information Engineering

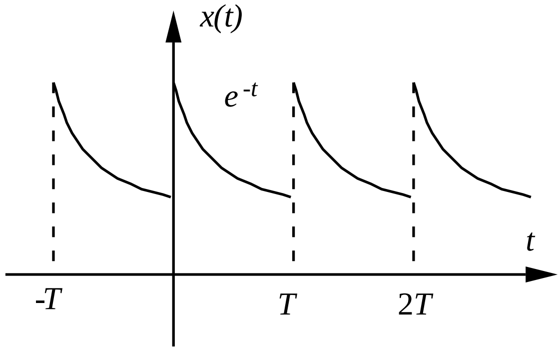
SIGNAL AND DATA ANALYSIS

Examples paper 2P6/6

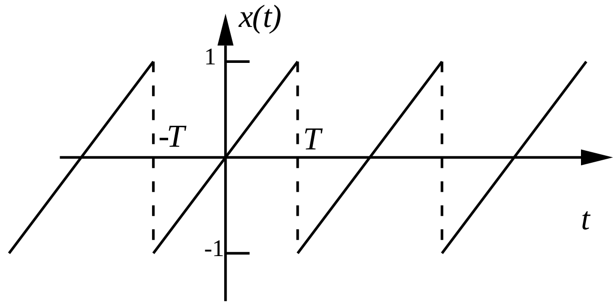
(Straightforward questions are marked †, problems of Tripos standard but not necessarily of Tripos length *).

Fourier Series and Systems

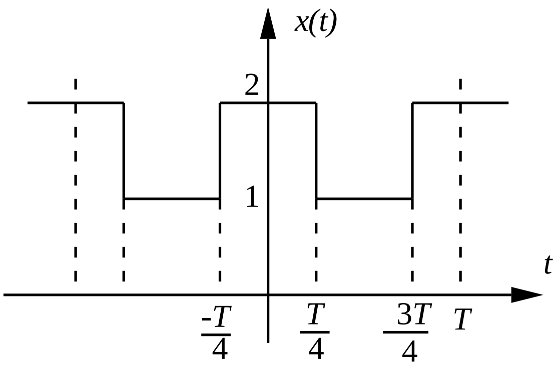
1. Determine the complex Fourier series expansion of each of the periodic signals shown. Do this either from first principles or, where appropriate, using time-shift, differentiation etc applied to simpler functions, series taken from the Data Sheet etc.. (Note that the period in case (b) is not T.)



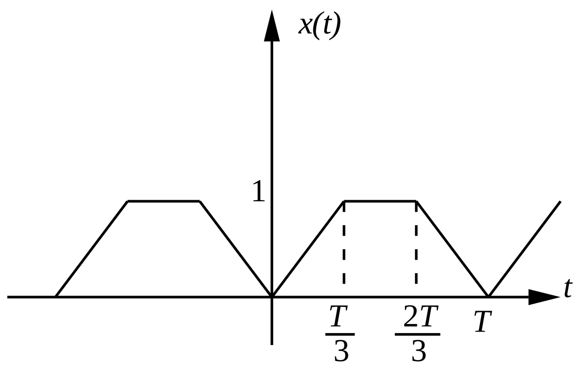
(a)



(b)



(c)



(d)

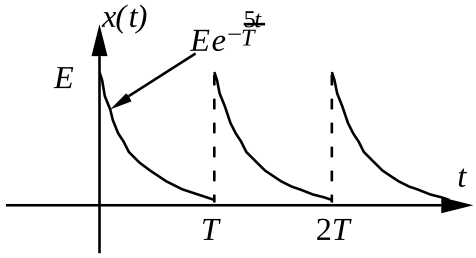
2.† Find the Fourier series representing an impulse train of period T, i.e.

$$x(t) \;=\; \ldots \;+\; \delta(t+2T) \;+\; \delta(t+T) \;+\; \delta(t) \;+\; \delta(t-T) \;+\; \delta(t-2T) \;+\; \ldots$$

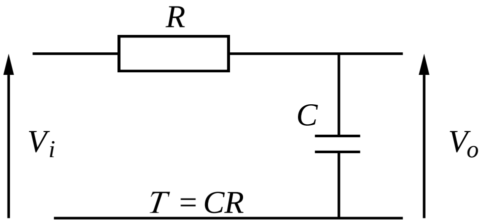
3. The periodic signal shown is defined by

$$x(t) = E \exp(-5t/T)$$

in the interval $0 \le t \le T$. Obtain the amplitudes of the d.c. component and the fundamental in this waveform in terms of E.



In order to reduce the amplitude of the fundamental, the signal is input to the low-pass filter shown. Show that the d.c. component is unaffected by the filter and that the amplitude of the fundamental at the output of the filter is $0.0389E$.



Fourier Transforms

4. A function $f(t)$ has Fourier transform $F(\omega)$. Show from the definition of the Fourier transform that,

a) † the Fourier Transform of $f(t-t_0)$ is $F(\omega)\exp(-j\omega t_0)$, where t_0 is a constant time offset;

b) the inverse Fourier transform of $\dfrac{dF(\omega)}{d\omega}$ is $-jt\,f(t)$;

c) $\displaystyle\int_{-\infty}^{+\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |f(\omega)|^2\,d\omega$;

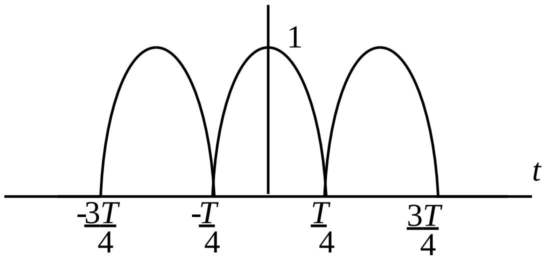
d) the inverse Fourier transform of $f(\omega)$ is $\dfrac{1}{2\pi}F(-t)$.

[Note that $f(\omega)$ is the same function as $f(t)$ but here considered as a function of frequency rather than of time.]

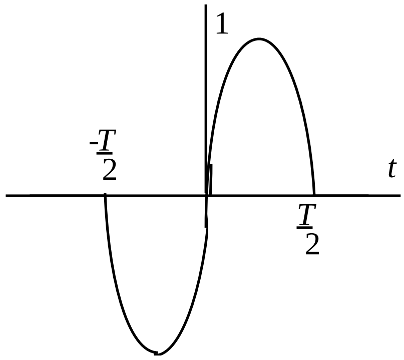
5.* Determine the Fourier transform of the half cosine pulse given by

$$
\begin{aligned}
x(t) &= \cos(2\pi t/T) \qquad -T/4 \le t \le T/4 \\
&= 0 \qquad\qquad\qquad \text{otherwise}
\end{aligned}
$$

Using the linearity and shift properties determine the transforms of the following signals:



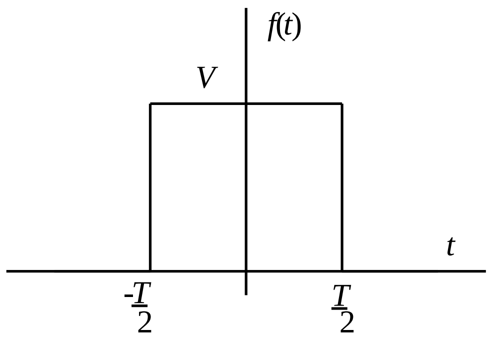
Triple half-cosine pulse



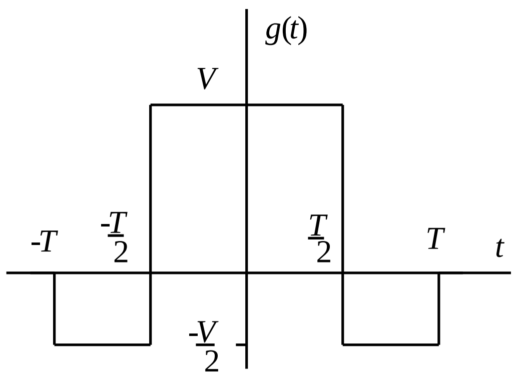
Sine pulse

6.* Show that the Fourier transform of the rectangular pulse $f(t)$ is given by

$$F(\omega) = V\,T\,\frac{\sin(\omega T/2)}{\omega T/2}.$$



Using this result and the relevant Fourier shift and/or scaling theorems obtain the Fourier transform of the signal $g(t)$.



Energy and Parseval's Theorem

7. Let $x(t)$ and $y(t)$ be two periodic signals with period T, and let x_n and y_n denote the complex Fourier *series* coefficients of these two signals. Show that

$$\frac{1}{T}\int_0^T x(t)y^*(t)\,dt = \sum_{n=-\infty}^{\infty} x_n y_n^*$$

[Hint: make sure you use the correct Fourier *series* coefficient definition and not the Fourier transform.]

8.* A system has a frequency response given by

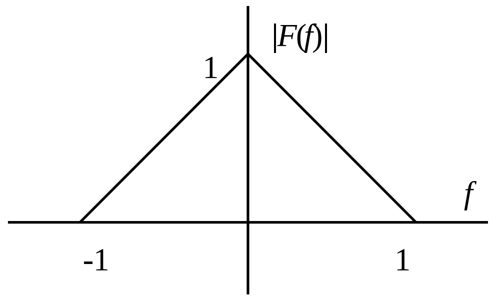
$$H(\omega) = \frac{1}{1+j\omega T}$$

If the input to such a system has a Fourier Transform given by

$$X(\omega) = \frac{1}{1+j\omega T_1}$$

what is the ratio of T_1/T such that 75% of the energy of the input signal will appear at the system output ?

9. A waveform has a Fourier Transform $F(f)$ whose magnitude is shown in the figure and where f is in Hz, i.e. $F(f) = \int_{-\infty}^{\infty} f(t)\exp(-2j\pi f t)dt$.



a) Find the energy of the waveform.
b) Calculate the frequency f_1 such that one half of the energy is in the frequency range $-f_1$ to f_1.

10.* Consider a signal consisting of two finite duration frequency components superimposed by summing as follows,

$$\begin{aligned} x(t) &= \cos pt + \cos qt, \qquad && -\tfrac{1}{2}T < t < \tfrac{1}{2}T ; \\ &= 0, && \text{otherwise} . \end{aligned}$$

Obtain the Fourier spectrum of this signal and sketch it for the cases where $p \gg q$ and $p \approx q$. What happens to the resolvability of these two frequency components as T increases? [Note that signal components are considered to be `resolved' in the frequency domain when each component can be clearly identified from the spectrum when they are superimposed.]

Answers

1. a) $c_n = \dfrac{1-e^{-T}}{T+j2\pi n}$ b) $c_n = \dfrac{(-1)^{n+1}}{j\pi n}$

c) $c_n = \dfrac{\sin n\pi/2}{n\pi}$ and $\dfrac{3}{2}$ for $n = 0$ d) $c_n = \dfrac{3}{2\pi^2 n^2}\left[\cos\dfrac{2\pi n}{3} - 1\right]$ and $\dfrac{2}{3}$ for $n = 0$

2. $c_n = \dfrac{1}{T}$ for all n.

3. Amp of dc = $0.199E$, Amp of fundamental $=0.247E$.

4.

5. a) $\dfrac{\sin(\omega-\omega_0)T/4}{\omega-\omega_0} + \dfrac{\sin(\omega+\omega_0)T/4}{\omega+\omega_0} = \dfrac{2\omega_0\cos\omega T/4}{\omega_0^2-\omega^2}$ with $\omega_0 = 2\pi/T$.

b) $\dfrac{2\omega_0}{\omega_0^2-\omega^2}\,(2\cos\omega T/4 + \cos 3\omega T/4)$.

c) $\dfrac{-2j\omega_0}{\omega_0^2-\omega^2}\sin\omega T/2$.

6. $V\,T\left(\dfrac{\sin\omega T/2}{\omega T/2} - \dfrac{1}{2}\dfrac{\sin\omega T/4}{\omega T/4}\cos 3\omega T/4\right)$.

8. Ratio=3.

9. Energy = 2/3, $f_1 = 0.21$.

10. $\dfrac{\sin((\omega-p)T/2)}{\omega-p} + \dfrac{\sin((\omega+p)T/2)}{\omega+p} + \dfrac{\sin((\omega-q)T/2)}{\omega-q} + \dfrac{\sin((\omega+q)T/2)}{\omega+q}$

Resolvability increases as T increases.

Suitable past tripos questions, all from 1B Paper 6:

2016 q.4, 2015 q.4, 2014 q.4, 2013 q.4, 2012 q.4, 2011 q.4, 2010 q.4, plus many additional questions from earlier years.

SJG Jan. 2018