Issued on: FRIDAY 16th JANUARY 2015

Solutions: Paper 6/6 – IB Information Engineering

Signal and Data Analysis: Fourier Series/Fourier Transforms/Energy and Parseval's Theorem

1. (a) Signal is periodic with period $T \implies x(t) = \sum_{-\infty}^{\infty} c_n e^{j2\pi nt/T}$ where

$$c_n = \frac{1}{T}\int_0^T e^{-t}e^{-j2\pi nt/T}dt = \frac{1}{T}\left[\frac{e^{-(1+j2\pi n/T)t}}{1+j2\pi n/T}\right]_0^T$$

$$= \frac{1-e^{-T}}{T+j2\pi n}$$

(b) $x(t) = \sum_{-\infty}^{\infty} c_n e^{j\pi nt/T}$ since function is periodic and period is $2T$,

$$c_0 = \frac{1}{2T}\int_{-T}^{T} x(t)dt = 0, \quad c_n = \frac{1}{2T}\int_{-T}^{T}\frac{t}{T}e^{-jn\pi t/T}dt$$

$$\implies c_n = \frac{1}{2T^2}\left[-\frac{e^{-jn\pi t/T}}{jn\pi/T}t\right]_{-T}^{T} + \frac{1}{2T^2}\int_{-T}^{T}\frac{e^{-jn\pi t/T}}{jn\pi/T}dt$$

$$= \frac{1}{2jn\pi T}\left\{-Te^{-jn\pi} - Te^{jn\pi}\right\} + \frac{1}{2jn\pi T}\left[-\frac{e^{-jn\pi t/T}}{jn\pi/T}\right]_{-T}^{T}$$

$$= -\frac{2T(-1)^n}{2j\pi nT} + \frac{1}{2(jn\pi)^2}\left\{-(-1)^n + (-1)^n\right\} = \frac{(-1)^{n+1}}{jn\pi}$$

Note: This function is on the data sheet (except that there the period is T), but coefficients on the data sheet are independent of T. Hence

$$x(t) = \frac{1}{j\pi}\sum_{-\infty}^{\infty}\frac{(-1)^{n+1}}{n}e^{jn\pi t/T}$$

(c) Function is 1+ pulse wave with $a = T/2$ (or $a/T = 1/2$). Therefore

$$x(t) = \frac{3}{2} + \frac{1}{2}\sum_{n=-\infty\ n\neq 0}^{\infty}\frac{\sin n\pi/2}{n\pi/2}e^{jn2\pi t/T} = \frac{3}{2} + \sum_{n=-\infty\ n\neq 0}^{\infty}\frac{\sin n\pi/2}{n\pi}e^{jn2\pi t/T}$$

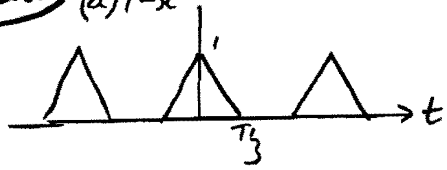
Note:to do this directly take $x(t) = 1 + f(t)$, so that for $f(t)$ we have

$$c_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{1}{2}, \quad c_n = \frac{1}{T}\int_{-T/4}^{T/4} e^{-j2\pi nt/T}dt$$

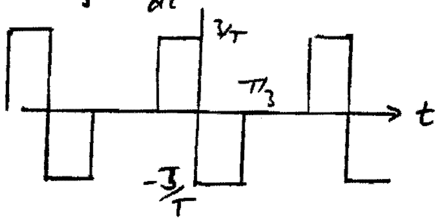
$$\implies c_n = \frac{1}{T}\left[-\frac{T}{j2\pi n}e^{-j2\pi nt/T}\right]_{-T/4}^{T/4} = \frac{1}{j2\pi n}\left[e^{jn\pi/2} - e^{-jn\pi/2}\right] = \frac{\sin n\pi/2}{n\pi}$$

from which the result follows.

Q1 cont (d) $1-x$



$f = \frac{d}{dt}(1-x)$



$f(t) = \frac{d}{dt}(1-x)$ is as shown. The series for f can be found by splitting into two pulse waves and found using suitably time-shifted series from the data sheet, or

$$f(t) = \sum_{-\infty}^{\infty} d_n e^{j2\pi n t/T} \qquad d_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-j2\pi n t/T} dt$$

$$\Rightarrow d_n = \frac{3}{T^2}\int_{-T/3}^{0} e^{-j2\pi n t/T} dt - \frac{3}{T^2}\int_{0}^{T/3} e^{-j2\pi n t/T} dt \qquad \& \quad \underline{d_0 = 0}$$

$$= \frac{3}{T^2}\left[-\frac{T}{j2\pi n} e^{-j2\pi n t/T}\right]_{-T/3}^{0} - \frac{3}{T^2}\left[-\frac{T}{j2\pi n} e^{-j2\pi n t/T}\right]_{0}^{T/3}$$

$$= \frac{3}{j2\pi n T}\left\{ e^{2\pi n j/3} - 1\right\} + \frac{3}{j2\pi n T}\left\{ e^{-2\pi n j/3} - 1\right\}$$

$$= \frac{3}{j\pi n T}\left\{\cos\frac{2n\pi}{3} - 1\right\}$$

Integrating $\Rightarrow$ $$1 - x = \sum_{\substack{-\infty \\ n\neq 0}}^{\infty} \frac{d_n}{j\frac{2\pi n}{T}} e^{j2\pi n t/T} = \sum_{\substack{-\infty \\ n\neq 0}}^{\infty} -\frac{3}{2n^2\pi^2}\left(\cos\frac{2n\pi}{3} - 1\right) e^{j2\pi \ldots} + \text{co}\ldots$$

For $n=0$ $$\frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{2}{3} \quad \Rightarrow \quad \underline{x = \frac{2}{3} + \sum_{\substack{-\infty \\ n\neq 0}}^{\infty} \frac{3}{2n^2\pi^2}\left(\cos\frac{2n\pi}{3} - 1\right) e^{-j2n\pi t \ldots}}$$

Aliter:

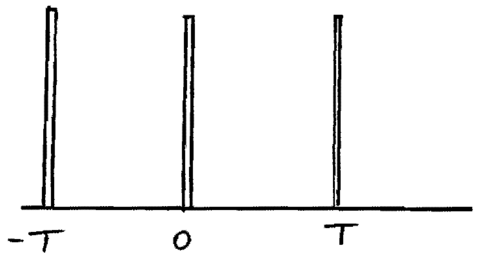
$$x_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{2}{3}$$

If $n \neq 0$ then

$$\begin{aligned}
x_n &= \frac{1}{T}\int_0^T x(t)e^{-j2\pi\frac{n}{T}t}dt = \frac{1}{T}\int_0^{\frac{T}{3}} \frac{3}{T} t e^{-j2\pi\frac{n}{T}t}dt \\
&+\frac{1}{T}\int_{\frac{T}{3}}^{\frac{2T}{3}} e^{-j2\pi\frac{n}{T}t}dt + \frac{1}{T}\int_{\frac{2T}{3}}^{T} \left(-\frac{3}{T}t + 3\right)e^{-j2\pi\frac{n}{T}t}dt \\
&= \frac{3}{T^2}\left(\frac{jT}{2\pi n} t e^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2 n^2} e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_0^{\frac{T}{3}} \\
&-\frac{3}{T^2}\left(\frac{jT}{2\pi n} t e^{-j2\pi\frac{n}{T}t} + \frac{T^2}{4\pi^2 n^2} e^{-j2\pi\frac{n}{T}t}\right)\Bigg|_{\frac{2T}{3}}^{T} \\
&+\frac{j}{2\pi n} e^{-j2\pi\frac{n}{T}t}\Bigg|_{\frac{T}{3}}^{\frac{2T}{3}} + \frac{3}{T}\frac{jT}{2\pi n} e^{-j2\pi\frac{n}{T}t}\Bigg|_{\frac{2T}{3}}^{T}
\end{aligned}$$

3 , $2\pi n$,

Q2 This was derived in lectures as the limit of a sequence of pulses. Directly:–



$$f(t) = \delta_p(t) = \sum_{-\infty}^{\infty} c_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T}\cdot 1$$

(Recall $\int \delta(x-a) f(x)\, dx = f(a)$)

$$\therefore \quad \delta_p(t) = \frac{1}{T}\sum_{-\infty}^{\infty} e^{jn\omega_0 t}$$

Q3 D.C. Component $= c_0 = \dfrac{E}{T}\displaystyle\int_0^T e^{-t/T}\, dt = \dfrac{E}{T}\left[-\dfrac{T e^{-5t/T}}{5}\right]_0^T$

$$= \frac{E}{5}\left[1 - e^{-5}\right] = \underline{\underline{0.199\,E}}$$

The amplitude of the fundamental $= 2|c_1|$ (N.B. factor 2)

where $c_1 = \dfrac{1}{T}\displaystyle\int_0^T E e^{-5t/T - j\omega_0 t}\, dt = \dfrac{E}{T}\left[-\dfrac{e^{-\frac{5}{T}t - j\omega_0 t}}{j\omega_0 + \frac{5}{T}}\right]_0^T \qquad \omega_0 = \dfrac{2\pi}{T}$

$$= \frac{E\,(1 - e^{-5})}{5 + j2\pi} \Rightarrow \underline{\underline{\text{Amplitude} = 0.247\,E}}$$

Frequency Response of filter $= \dfrac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \dfrac{1}{1 + j\omega CR} = \dfrac{1}{1 + j\omega T}$

$$\Rightarrow \left|\frac{V_O(j\omega)}{V_I(j\omega)}\right| = \frac{1}{\sqrt{1+\omega^2 T^2}} \Rightarrow \text{D.C. } \left|\frac{V_O}{V_I}\right| = 1 \Rightarrow \underline{\text{unaffected}}$$

Fundamental $\omega = \dfrac{2\pi}{T} \Rightarrow \left|\dfrac{V_O}{V_I}\right| = \dfrac{1}{\sqrt{1+(2\pi)^2}} = .157$

$\therefore$ Amplitude at output $= .247 \times .157E = \underline{\underline{0.0389E}}$

Q4

(a) If $f(t)$ has Fourier transform $F(\omega)$, what is the Fourier transform of $f(t-t_0)$?

$$f(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{F(\omega)e^{-j\omega t_0}\}e^{j\omega t}d\omega$$

which therefore tells us that

$$f(t-t_0) \stackrel{FT}{\longleftrightarrow} F(\omega)e^{-j\omega t_0} \tag{1}$$

(b) If $f(t)$ has the Fourier transform $F(\omega)$, what is the function which has Fourier transform $F(\omega)' = \frac{dF}{d\omega}$?

$$\begin{aligned} F'(\omega) &= \int_{-\infty}^{\infty} f(t)\frac{d}{d\omega}\{e^{-j\omega t}\}\,dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\{-jtf(t)\}\,e^{-j\omega t}dt \end{aligned}$$

i.e.

$$(-jt)f(t) \stackrel{FT}{\longleftrightarrow} F'(\omega) \tag{2}$$

(c) Rewrite $f(t)$ in terms of $F(\omega)$:

$$\begin{aligned} \int |f(t)|^2\,dt &= \int f(t)f^*(t)\,dt \\ &= \int f^*(t)\frac{1}{2\pi}\int F(\omega)\exp(+j\omega t)d\omega\,dt \\ &= \frac{1}{2\pi}\int F(\omega)\int f^*(t)\exp(+j\omega t)\,dt\,d\omega \\ &= \frac{1}{2\pi}\int F(\omega)F^*(\omega)\,d\omega \\ &= \frac{1}{2\pi}\int |F(\omega)|^2\,d\omega \end{aligned}$$

(d) From definition of inverse transform:

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\omega)\exp(+j\omega t)d\omega &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\omega)\exp(j\omega(-t))d\omega \\ &= \frac{1}{2\pi}F(-t) \end{aligned}$$

(by inspection).

Q5

$$X(\omega) = \int_{-T/4}^{T/4} \cos\omega_0 t\, e^{-j\omega t}\,dt \qquad \omega_0 = \frac{2\pi}{T}$$

$$= \int_{-T/4}^{T/4} \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\, e^{-j\omega t}\,dt = \frac{1}{2}\left[-\frac{e^{-j(\omega-\omega_0)t}}{j(\omega-\omega_0)} - \frac{e^{-j(\omega+\omega_0)t}}{j(\omega+\omega_0)}\right.$$

$$= \frac{\sin(\omega-\omega_0)T/4}{\omega-\omega_0} + \frac{\sin(\omega+\omega_0)T/4}{\omega+\omega_0}$$

Now, noting that $\sin\frac{(\omega-\omega_0)T}{4} = \sin\frac{\omega T}{4}\cos\frac{\omega_0 T}{4} - \cos\frac{\omega T}{4}\sin\frac{\omega_0 T}{4}$

& $\frac{\omega_0 T}{4} = \frac{2\pi}{T}\frac{T}{4} = \frac{\pi}{2} \Rightarrow \qquad = -\cos\frac{\omega T}{4}$

& Similarly, $\sin\frac{(\omega+\omega_0)T}{4} = +\cos\frac{\omega T}{4}$

$$\Rightarrow \quad X(\omega) = \cos\frac{\omega T}{4}\left(\frac{1}{\omega+\omega_0} - \frac{1}{\omega-\omega_0}\right) = \underline{\frac{-2\omega_0}{\omega^2-\omega_0^2}\cos\frac{\omega T}{4}}.$$

Q5 cont The triple cosine pulse = Half cosine + time delayed + time advanced

$$= x(t) + x(t - \tfrac{T}{2}) + x(t + T/2)$$

$$\therefore \quad X_1(\omega) = X(\omega) + e^{-\frac{j\omega T}{2}} X(\omega) + e^{j\omega T/2} X(\omega)$$

$$= \frac{2\omega_0}{\omega_0^2 - \omega^2} \cos\frac{\omega T}{4}\left[1 + 2\cos\frac{\omega T}{2}\right]$$

$$= \frac{2\omega_0}{\omega_0^2 - \omega^2}\left[2\cos\frac{\omega T}{4} + \cos\frac{3\omega T}{4}\right]$$

The sine pulse can be expressed as $x_2(t) = x(t - T/4) - x(t + T/4)$

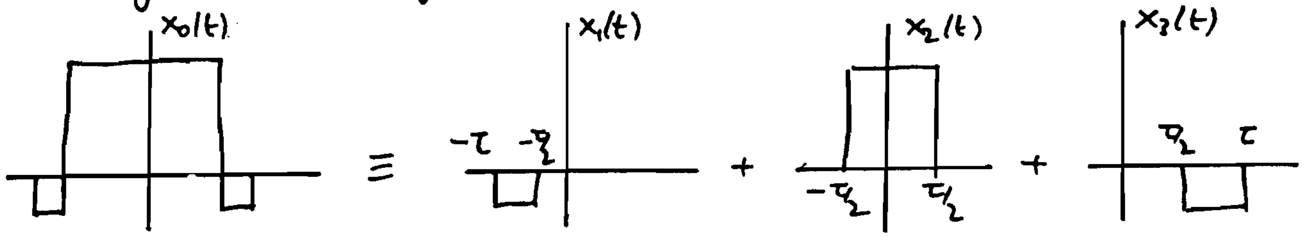
$$\Rightarrow \quad X_2(\omega) = X(\omega)\left(e^{-j\omega T/4} - e^{j\omega T/4}\right)$$

$$= \frac{2\omega_0}{\omega_0^2 - \omega^2} \cos\frac{\omega T}{4}\left(-2j\sin\frac{\omega T}{4}\right)$$

$$= \frac{-2j\omega_0 \sin\frac{\omega T}{2}}{\omega_0^2 - \omega^2}$$

Q6 The first part of the question is derived in lectures.

The signal can be regarded as the sum of 3 signals



$$x_0 = x_1(t) + x_2(t) + x_3(t)$$

The signals x_1 and x_3 are scaled and time shifted versions of x_2:

$$x_1(t) = -\tfrac{1}{2}\, x_2\left(2(t - 3\tau/4)\right)$$

$$x_3(t) = -\tfrac{1}{2}\, x_2\left(2(t + \tfrac{3\tau}{4})\right)$$

$$\therefore \quad F(\omega) = V\tau\frac{\sin\omega\tau/2}{\omega\tau/2} - e^{j\frac{3\omega\tau}{4}}\,\frac{1}{2}\cdot\frac{1}{2}\, V\tau\frac{\sin\omega\tau/4}{\omega\tau/4} - e^{-\frac{3j\omega\tau}{4}}\,\frac{1}{2}\cdot\frac{1}{2}\, V\tau\frac{\sin\omega\tau/4}{\omega\tau/4}$$

$$= V\tau\left(\frac{\sin\omega\tau/2}{\omega\tau/2} - \frac{1}{2}\,\frac{\sin\omega\tau/4}{\omega\tau/4}\cos\frac{3\omega\tau}{4}\right)$$

7. $$\frac{1}{T_0}\int_0^{T_0} x(t)\,y^*(t)\,dt = \frac{1}{T_0}\int_0^{T_0}\left\{\sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t}\sum_{m=-\infty}^{\infty} y_n^* e^{-jm\omega_0 t}\right\}dt \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x_n y_m^* \left\{\frac{1}{T_0}\int_0^{T_0} e^{j\frac{2\pi}{T_0}(n-m)t}\,dt\right\}$$

The integral in brackets is zero for $n \neq m$ and unity for $n = m$.

$$\therefore \quad \frac{1}{T_0}\int_0^{T_0} x(t)\,y^*(t)\,dt = \sum_{n=-\infty}^{\infty} x_n y_n^*$$

8. Energy of input signal is $\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega}{1+(\omega T_1)^2}$

and $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} \Rightarrow$ Energy $= \frac{1}{2\pi T}\left[\tan^{-1}\omega T_1\right]_{-\infty}^{\infty}$

$$= \frac{1}{2T_1}$$

F.T. of output signal $Y(\omega) = H(\omega)\,X(\omega) = \frac{1}{1+j\omega T_1}\cdot\frac{1}{1+j\omega T}$

$$\therefore |Y(\omega)|^2 = \frac{1}{1+(\omega T_1)^2}\cdot\frac{1}{1+(\omega T)^2}$$

$\therefore$ Output Energy $= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{1+(\omega T_1)^2}\cdot\frac{1}{1+(\omega T)^2}\,d\omega$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{T_1^2-T^2}\left\{\frac{T_1^2}{1+(\omega T_1)^2} - \frac{T^2}{1+(\omega T)^2}\right\}d\omega$$

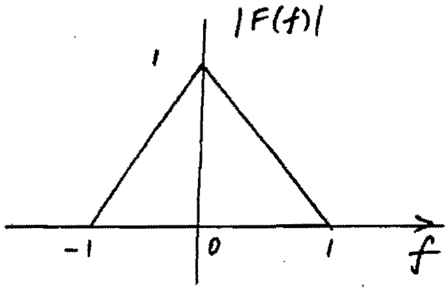
$$= \frac{1}{2\pi}\cdot\frac{1}{T_1^2-T^2}\cdot\left\{\frac{T_1^2}{T_1} - \frac{T^2}{T}\right\}\pi$$

$$= \frac{1}{2}\,\frac{1}{T+T_1}$$

$$\therefore \frac{\text{Out}}{\text{In}} = \frac{1}{1+T/T_1} = .75 \quad \text{if} \quad \frac{T_1}{T} = 3$$

9.

a) Energy $= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |F(f)|^2 df$



$$= 2\int_0^1 (1-f)^2 df = 2\left[f - f^2 + \frac{f^3}{3}\right]_0^1$$

$$\underline{E = \frac{2}{3}}$$

b) $\frac{E}{2} = \frac{1}{3} = \int_{-f_1}^{f_1} |F(f)|^2 df = 2\int_0^{f_1} |F(f)|^2 df = 2\left[f - f^2 + \frac{f^3}{3}\right]_0^{f_1}$

$$= 2\left(f_1 - f_1^2 + \frac{f_1^3}{3}\right) \Rightarrow f_1 = \cdot 21 \text{ (numerically).}$$

[Integration perhaps better done as $2\int_0^{f_1} (1-f)^2 df = 2\left[-\frac{(1-f)^3}{3}\right]_0^{f_1}$

$= \frac{2}{3}\left(1 - (1-f_1)^3\right)$. Then $\frac{1}{3} = \frac{2}{3}\left(1 - (1-f_1)^3\right) \Rightarrow f_1 = 1 - \frac{1}{\sqrt[3]{2}}$]

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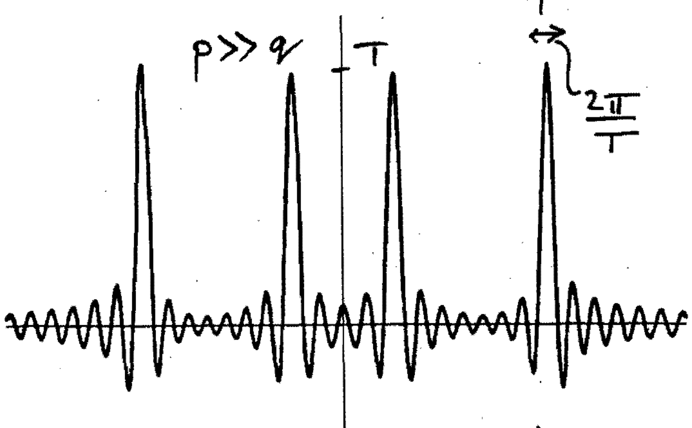
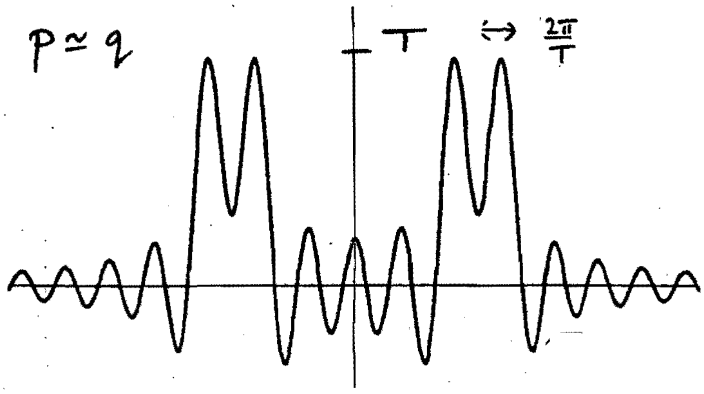
10. $\int_{-T/2}^{T/2} \cos pt \, e^{j\omega t} dt = \frac{1}{2}\int_{-T/2}^{T/2} \left[e^{j(p+\omega)t} + e^{j(\omega-p)t}\right] dt$

$$= \frac{1}{2}\left[\frac{e^{j(\omega+p)}}{j(\omega+p)} + \frac{e^{j(\omega-p)}}{j(\omega-p)}\right]_{-T/2}^{T/2} = \frac{1}{2}\frac{\sin(\omega+p)T/2}{\omega+p} + \frac{1}{2}\frac{\sin(\omega-p)T/2}{\omega-p}$$

Thus Spectrum

$$X(\omega) = \frac{1}{2}\frac{\sin(\omega+p)T/2}{\omega+p} + \frac{1}{2}\frac{\sin(\omega-p)T/2}{\omega-p} + \frac{1}{2}\frac{\sin(\omega+q)T/2}{\omega+q} + \frac{1}{2}\frac{\sin(\omega-q)T/2}{\omega-q}$$





as T increases, peaks become sharper ⇒ resolvability increases