Section 9: Satellite Dynamics (and other examples)

We now have a range of tools at our disposal that can be applied to understanding and predicting the motion of particles:

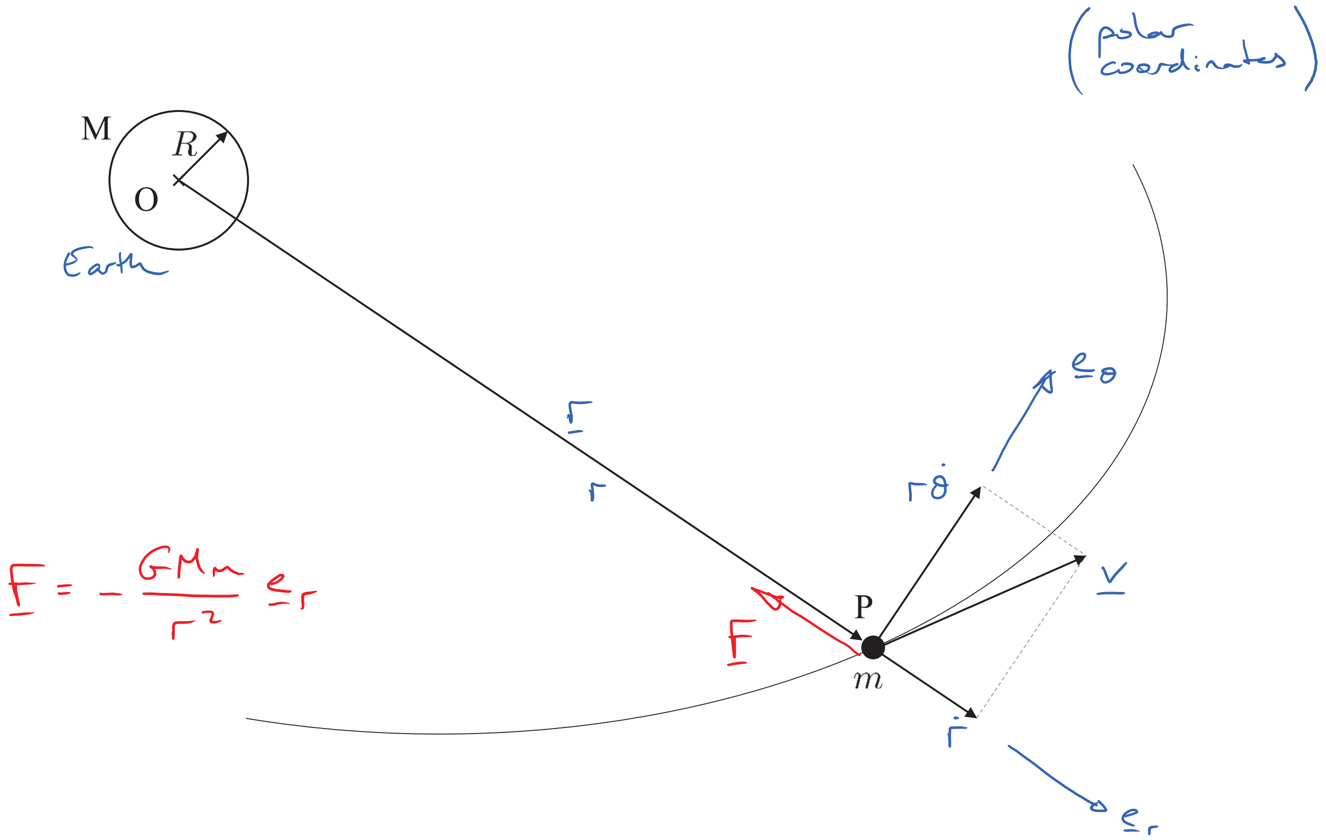
- Newton's Second Law $\sum \underline{F} = m\underline{a}$
- Energy $\frac{dT}{dt} = \underline{F} \cdot \underline{v}$, $\int_A^B \underline{F} \cdot d\underline{r} = T_B - T_A$
- Linear momentum $\frac{d\underline{p}}{dt} = \underline{F}$, $\int_A^B \underline{F}\, dt = \underline{p}_B - \underline{p}_A$
- Angular momentum $\frac{d\underline{h}_o}{dt} = \underbrace{\underline{r} \times \underline{F}}_{\underline{q}_o}$, $\int_A^B \underbrace{(\underline{r} \times \underline{F})}_{\underline{q}_o} dt = \underline{h}_B - \underline{h}_A$

These concepts will be extended to rigid bodies in the next part of this course; but first we will look at how these principles can be combined to understand satellite motion (and other central force examples).

We will consider the case of a small mass orbiting a large mass, such that the large mass can be considered a fixed point: for example a satellite orbiting Earth.

There are several things we may wish to know: the equations of motion (so we can numerically simulate the system); the shape of the orbit; and the energy required to put a satellite into a particular orbit.

A key feature of the gravitational force exerted by a planet on a satellite is that it always acts towards the centre of the planet: it is an example of a central field. This immediately identifies the planet as a point about which angular momentum of the satellite must be conserved.



The angular momentum about the point O is constant:

$$\underline{h}_o = \underline{r} \times (m\underline{v}) = r\underline{e}_r \times m(\dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta)$$

$$= mr^2\dot{\theta}\underline{k}$$

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$$= \text{constant}$$

Now from $h_0 = mr^2\dot{\theta}$:

★ $$\dot{\theta} = \frac{h_o}{mr^2}$$ (scalar h_o means magnitude) ★

which is useful later.

There are now two ways to derive the equations of motion:

(1) by energy;

(2) from Newton's Second Law;

Both ways need the angular momentum result highlighted above.

(1) Energy method

We know that gravitational force is conservative, so:

$$T + V = E \text{ (constant)}$$

From Section 6 we know that the potential energy expression of the gravitational force field is:

$$V = -\frac{GMm}{r}$$

and the kinetic energy is:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2\right]$$

Putting these together gives:

$$\frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2\right] - \frac{GMm}{r} = E \text{ (constant)} \qquad \star\left(\dot{\theta} = \frac{h_0}{mr^2}\right)\star$$

But this equation still has $\dot{\theta}$ in it, so we need to use the conservation of angular momentum result to get the equation of radial motion:

$$\frac{1}{2}m\left[\dot{r}^2 + \frac{h_0^2}{m^2r^2}\right] - \frac{GMm}{r} = E \text{ (constant)}$$

To simplify notation we define the 'specific angular momentum' and call it $\hat{h}_0$:

$$\hat{h}_0 \equiv \frac{h_0}{m}$$

so that the equation becomes:

$$\frac{1}{2}m\left[\dot{r}^2 + \frac{\hat{h}_0^2}{r^2}\right] - \frac{GMm}{r} = E \text{ (constant)}$$

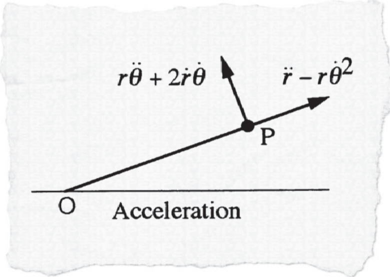
Then differentiate to get:

$$\cancel{\frac{1}{2}}\cancel{m}\left[\cancel{2}\cancel{\dot{r}}\,\ddot{r} - \frac{\cancel{2}\hat{h}_0^2\cancel{\dot{r}}}{r^3}\right] + \frac{GM\cancel{m}\cancel{\dot{r}}}{r^2} = 0$$

Then simplify to obtain the equation of motion:

$$\boxed{\ddot{r} - \frac{\hat{h}_0^2}{r^3} + \frac{GM}{r^2} = 0}$$

(2) Newton's Second Law method



Applying Newton's Second Law:

$$\underline{F} = m\underline{a}$$

$$-\frac{GMm}{r^2}\underline{e}_r = m\left[\left(\ddot{r}-r\dot{\theta}^2\right)\underline{e}_r + \left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)\underline{e}_\theta\right]$$

In the radial direction:

$$-\frac{GMm}{r^2} = m\left(\ddot{r}-r\dot{\theta}^2\right)$$

$$\star\left(\dot{\theta} = \frac{h_0}{mr^2}\right)\star$$

Now we again use the angular momentum result and the 'specific angular momentum' definition which gives the same equation of motion as before.

Actually it is possible to get to the equation of motion without reference to momentum, by considering the tangential direction of Newton's Second Law expression: see proof below.

In the tangential direction:

$$0 = m\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right) \tag{1}$$

Notice that:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) &= 2r\dot{r}\dot{\theta}+r^2\ddot{\theta}\\ &= r\left(2\dot{r}\dot{\theta}+r\ddot{\theta}\right)\end{aligned}$$

Comparing with (1) gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 0$$

So $r^2\dot{\theta}$ is constant: this is proportional to the angular momentum, which is the desired result.

Shape of the orbit

To find an equation for the shape of the orbit then we need an expression that relates r directly to θ. The equation of motion is:

$$\ddot{r} - \frac{\hat{h}_0^2}{r^3} + \frac{GM}{r^2} = 0$$

The solutions would give us r as a function of time, so we somehow need to eliminate time from this second-order nonlinear differential equation. It turns out that this particular equation can be solved relatively easily (only after you know how). First define a new variable u for convenience:

$$u(r) = \frac{1}{r}$$

To put it into the equations of motion then we need $\ddot{r}$:

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{u}\right) = \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{u}\right)\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{1}{u^2}\dot{\theta}\frac{\mathrm{d}u}{\mathrm{d}\theta} = -(r^2\dot{\theta})\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\hat{h}_0\frac{\mathrm{d}u}{\mathrm{d}\theta}$$

$$\ddot{r} = -\hat{h}_0\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) = -\hat{h}_0\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\hat{h}_0\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\cdot\frac{\hat{h}_0}{r^2} = -\hat{h}_0^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}$$

Substituting this and $u(r)$ into the equations of motion gives:

$$-\hat{h}_0^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} - u^3\hat{h}_0^2 + GMu^2 = 0$$

$$\hat{h}_0^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + \hat{h}_0^2u = GM \qquad \left(u = 1/r \ , \ \hat{h}_0 = r^2\dot{\theta} = \text{constant}\right)$$

This is (surprisingly) now a linear second order differential equation, with standard general solution:

$$\begin{aligned} u &= A\cos(\theta+\alpha) + \frac{GM}{\hat{h}_0^2} \\ &= \frac{GM}{\hat{h}_0^2}(1 + e\cos\theta) \end{aligned}$$

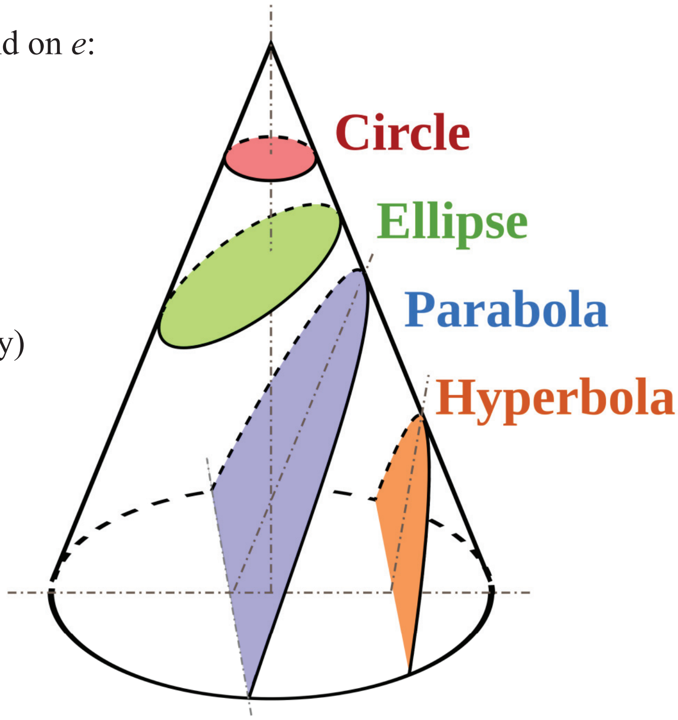
and transforming back to r and choosing $\alpha = 0$ (as a datum) gives:

$$\frac{1}{r} = \frac{GM}{\hat{h}_0^2}\left(1 + e\cos\theta\right)$$

This equation looks complicated and hard to visualise, but it turns out it is the equation for a conic section (the shape you get when you slice a cone).

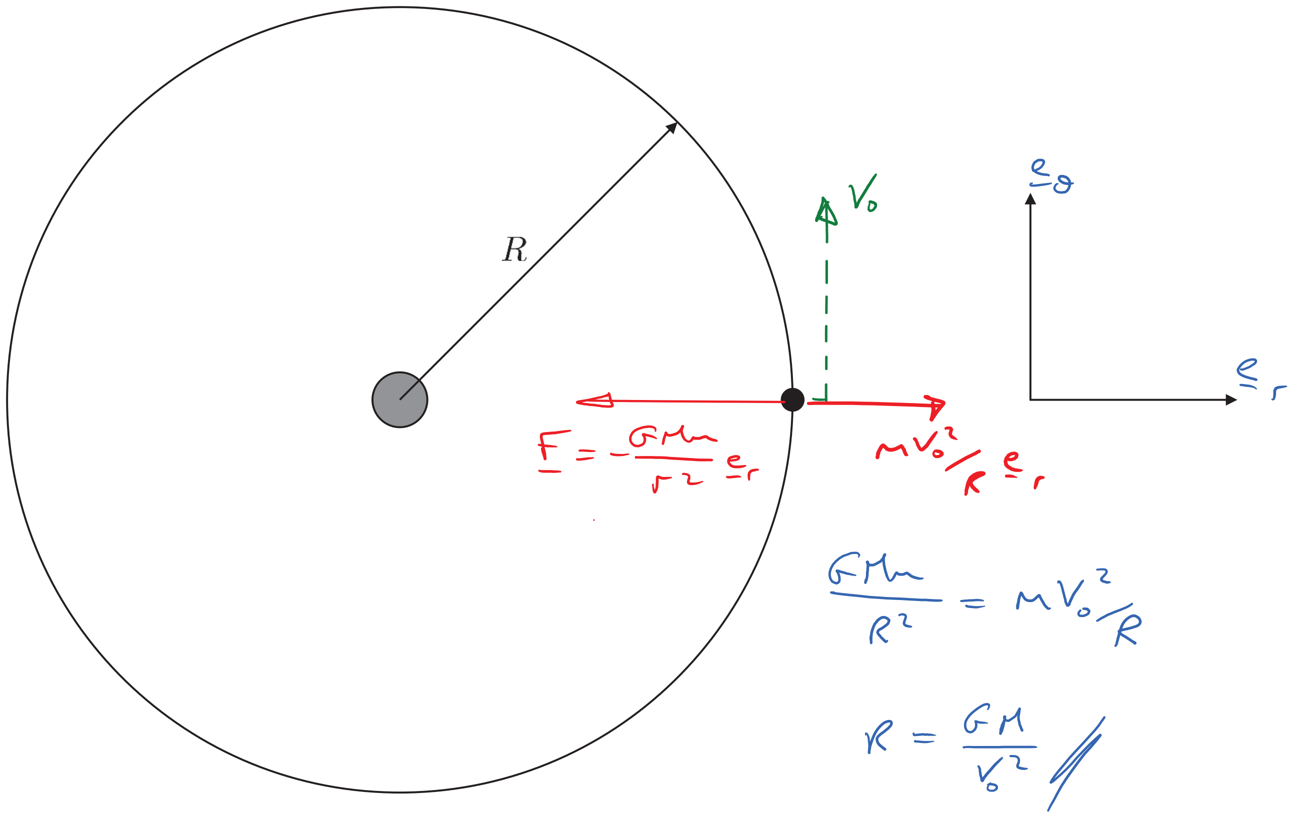
There are four categories of solutions that depend on e:

- $e = 0$ (circle – constant radius)
- $e < 1$ (ellipse – takes some thought to see why)
- $e = 1$ (parabola)
- $e > 1$ (hyperbola)



We will just consider the first two cases, where the satellite is in a periodic orbit.

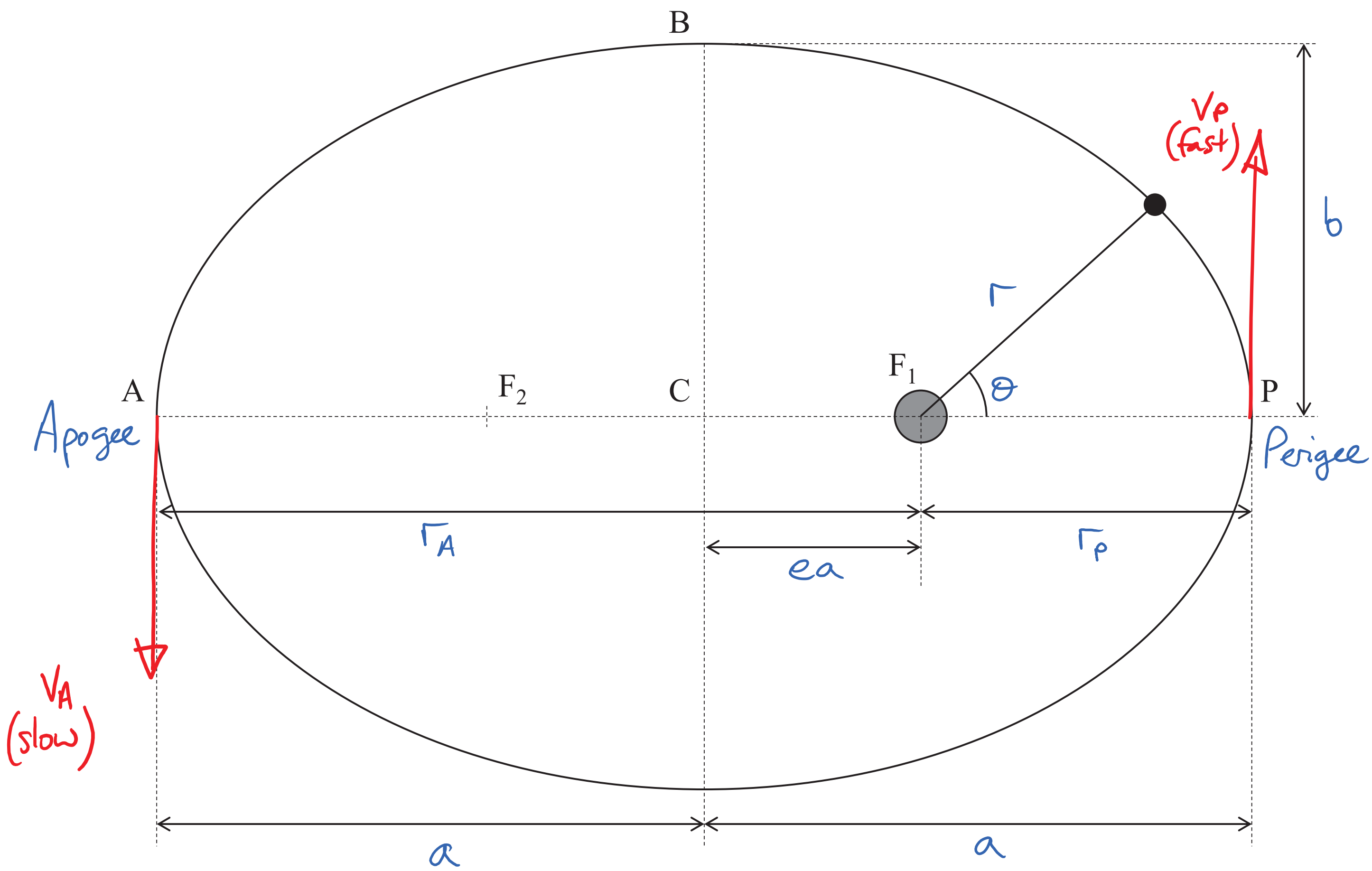
The first case $e = 0$ corresponds to circular motion, and we can analyse this without reference to the general solution:



$$\frac{GMm}{R^2} = mV_0^2/R$$

$$R = \frac{GM}{V_0^2}$$

The second case $e < 1$ is one way of writing the equation of an ellipse. The geometry is as follows:



The ellipse can be defined in terms of its <u>major semi-axis</u> 'a' and its <u>minor semi-axis</u> 'b'

Point P is the <u>perigee</u> of the orbit: the closest point on the path to the Earth:

$$\theta = 0, \; r_P = (1-e)a$$

Point A is the <u>apogee</u> of the orbit: the furthest point on the path to the Earth:

$$\theta = \pi, \; r_A = (1+e)a$$

At the perigee and apogee the satellite's angular momentum is particularly easy to find:

$$h_p = m V_p r_p$$
$$h_a = m V_A r_A$$

and because angular momentum is conserved about the Earth, then:

$$m V_p r_p = m V_A r_A$$

$$\frac{V_A}{V_p} = \frac{r_p}{r_A} = \frac{(1-e)}{(1+e)}$$

We can obtain an expression for 'a' by rearranging the equation of a conic:

$$\frac{1}{r} = \frac{GM}{\hat{h}_0^2}(1 + e\cos\theta)$$

$$\frac{1}{r_P} = \frac{GM}{\hat{h}_0^2}(1 + e) = \frac{1}{(1-e)a}$$

Giving:

$$a = \frac{\hat{h}_0^2}{GM(1-e^2)}$$

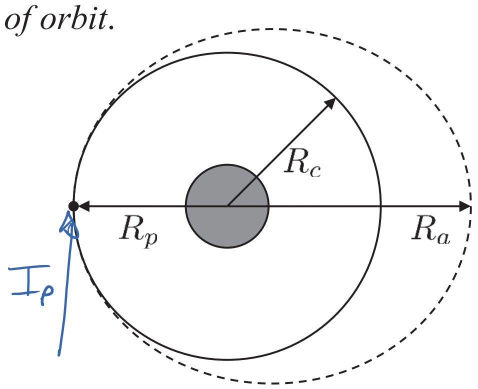
And to obtain 'b' we can use the relationship that:

$$\frac{b}{a} = \sqrt{1-e^2}$$

This may seem a lot to remember: the emphasis in this course is not on the detailed mathematics of an ellipse (or conic sections more generally), but rather to show how the principles of energy and angular momentum are applied. The Mechanics Databook (section 2.2) provides the key geometric information required.

Example: A satellite of mass 500 kg is in a circular orbit 6×10^6 m above the surface of the earth. This is to be changed to an elliptical orbit with perigee 6×10^6 m above the earth and apogee 14×10^6 m above the earth. Determine the magnitude and direction of the impulse required to initiate the change of orbit.

$$\begin{aligned} R_{\text{earth}} &= 6.4 \times 10^6 \text{ m} \\ M_{\text{earth}} &= 6 \times 10^{24} \text{ kg} \\ G &= 6.7 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2} \end{aligned}$$



Circular : $R_c = R_{earth} + 6 \times 10^6 = 12.4 \times 10^6$ m

new orbit : $R_p = R_c = 12.4 \times 10^6$ m.

$R_A = R_{earth} + 14 \times 10^6 = 20.4 \times 10^6$ m

$I_p = \Delta mv = m(v_p - v_c)$

v_c — circular velocity

v_p — perigee velocity.

need v_c & v_p :

v_c //

$$\frac{GMm}{R_c^2} = \frac{mV_c^2}{R_c} \Rightarrow V_c = \sqrt{\frac{GM}{R_c}}$$
$$= 5.69 \times 10^3 \, m\, s^{-1}$$

v_p //

conservation angular momentum :
$$\frac{V_P}{V_A} = \frac{r_A}{r_P}$$
$$V_P = \frac{r_A}{r_P} V_A = 1.65 \, V_A$$

energy

$$\frac{1}{2} m V_P^2 - \frac{GMm}{r_P} = \frac{1}{2} m V_A^2 - \frac{GMm}{r_A}$$

$$V_P^2 - V_A^2 = 2GM\left(\frac{1}{r_P} - \frac{1}{r_A}\right) \Rightarrow V_P = 6.35 \times 10^3 \, m s^{-1}$$

$$I_P = 500(6.35 - 5.69) \times 10^3 = 328 \times 10^3 \, kg \, m s^{-1}$$ //

https://www.youtube.com/watch?v=xrGAQCq9BMU

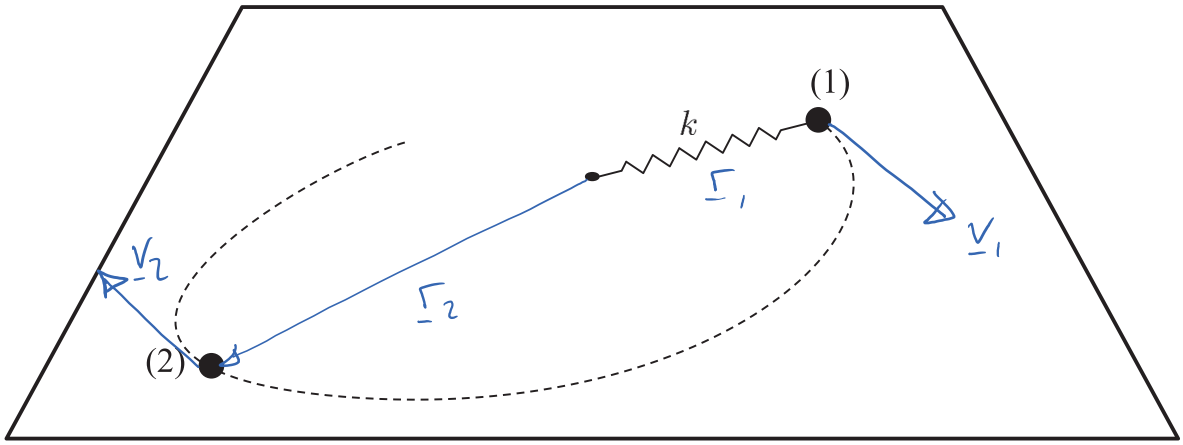
Other central forces

The gravitational inverse square law is just one kind of force field that always acts towards a fixed centre. It was this ‘central’ property that allowed conservation of angular momentum about the centre to be applied, and conservation of energy could also be applied because gravitational force is also conservative.

The full equations of motion and equation for the path were complicated: but these two principles allowed fast calculation of the behaviour at specific points in the path. In the case of orbits, the perigee and apogee are where the velocities are orthogonal to the position vector from the Earth.

There are other kinds of idealised cases that can be solved this way: these represent idealised mathematical examples with less immediately obvious application, but they help understand how the principles can be applied in different situations.

Example: a particle of mass m slides on a frictionless surface and is connected to a fixed point O by a linear spring of stiffness k. At position (1) its angular momentum is mv_1r_1 and its kinetic energy is $\frac{1}{2}mv_1^2$. What will be the minimum and maximum distances from O during its motion?



$\underline{v}_1 \perp \underline{r}_1$ because $KE = \frac{1}{2}mv_1^2$ & $h = mr_1v_1$

C.o.A.M. $\quad m v_1 r_1 = m v_2 r_2 \Rightarrow v_2 = \frac{r_1}{r_2} v_1$

C.o.E. $\quad \frac{1}{2} m v_1^2 + \frac{1}{2} k r_1^2 = \frac{1}{2} m v_2^2 + \frac{1}{2} k r_2^2$

$$m\left(v_1^2 - v_2^2\right) = k\left(r_2^2 - r_1^2\right)$$

$$m v_1^2 \cancel{\left[1 - \left(\frac{r_1}{r_2}\right)^2\right]} = r_2^2 k \cancel{\left[1 - \left(\frac{r_1}{r_2}\right)^2\right]}$$

$$r_2 = v_1 \sqrt{m/k}$$

min & max $r = r_1$ & r_2 (can't say which way round)

It is essential to notice that when we use the same velocity for both conservation of angular momentum and conservation of energy, then we are implicitly solving the equations for when the actual velocity is orthogonal to the point or axis (depending on whether we are taking angular momentum about a point or axis).

Summary

- The equation of motion for orbiting satellites can be derived from Newton's Laws, or by conservation of angular momentum and conservation of energy;

- The path of an orbit can be circular; elliptical; parabolic or hyperbolic;

- For elliptic orbits we can use the principles or energy and angular momentum straightforwardly at the apogee (furthest point) and perigee (closest point);

- Conservation of energy and angular momentum apply to all central force fields;

- The points at which the expressions for angular momentum and kinetic energy use the same velocity term are important, because that happens when the velocity is orthogonal to the moment point or axis.