Derivation of the Routh Hurvitz conditions.

Consider the degree n polynomial

$$p(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots a_1 s + a_0$$

with $a_n > 0$. We wish to know if $p(s)$ is stable, i.e. all its roots lie in the closed right half plane (i.e including the imaginary axis). It is easy to show that if any of the other coefficients a_i are negative then the polynomial must have a right half plane root, so we just need to deal with the case where all the a_i are positive.

Consider also the degree $n-1$ polynomial

$$\begin{aligned} q(s) &= p(s) - \frac{a_n}{a_{n-1}} s\Big(a_{n-1}s^{n-1} + a_{n-3}s^{n-3} + a_{n-5}s^{n-5} + \cdots\Big) \\ &= a_{n-1}s^{n-1} + \left(a_{n-2} - \frac{a_n a_{n-3}}{a_{n-1}}\right) s^{n-2} + a_{n-3}s^{n-3} + \left(a_{n-4} - \frac{a_n a_{n-5}}{a_{n-1}}\right) s^{n-4} + \cdots \end{aligned}$$

We will show that $p(s)$ is stable if, and only if, $q(s)$ (which will let us proceed to reducing our polynomial one degree at time until to we get to degree 1, when the answer is obvious).

To show this, consider the *family* of polynomials

$$q_\eta(s) = p(s) - \eta \frac{a_n}{a_{n-1}} s\Big(a_{n-1}s^{n-1} + a_{n-3}s^{n-3} + a_{n-5}s^{n-5} + \cdots\Big)$$

for

$$0 \le \eta \le 1$$

When $\eta = 0$, $q_\eta(s) = p(s)$ and when $\eta = 1$, $q_\eta(s) = q(s)$. We now consider values of s on the imaginary axis, $s = j\omega$. First consider the case where n is even, in this case the term $s\Big(a_{n-1}s^{n-1} + a_{n-3}s^{n-3} + a_{n-5}s^{n-5} + \cdots\Big)$ only contains even powers of s, and so will be purely real. So if $q_\eta(j\omega) = 0$ then the imaginary part of $p(j\omega)$ must be zero. But, the imaginary part of $p(j\omega)$ comes from the odd terms, $\Big(a_{n-1}s^{n-1} + a_{n-3}s^{n-3} + a_{n-5}s^{n-5} + \cdots\Big)$ (with $s = j\omega$) and so this bracketed term must be zero, which would mean $p(j\omega) = q_\eta(j\omega) = 0$. Thus, as η varies between 0 and 1, no roots of $q_\eta(s)$ can cross the imaginary axis. Since the roots of a polynomial move continuously with the coefficients, this means that if $q(s)$ is stable then so will $p(s)$ be. If n is odd, then the same argument works but with "real" and "imaginary" interchanged.

Applying this to the $n = 2$ case this gives, trivially,

$$a_2 s^2 + a_1 s + a_0$$

is stable if and only if

$$a_1 s + a_0$$

is, i.e. if and only if a_1 and a_o are both positive, which they are by assumption.

More interestingly, in the $n = 3$ case

$$a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

is stable if and only if

$$a_2 s^2 + (a_1 - a_3 a_0 / a_2) s + a_0$$

is i.e. if and only if

$$a_1 - a_3 a_0 / a_2 > 0 \text{ or } \boxed{a_1 a_2 > a_0 a_3}$$

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In the $n = 4$ case

$$a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

is stable if and only if

$$a_3 s^3 + (a_2 - a_4 a_1 / a_3) s^2 + a_1 s + a_0$$

is i.e. if and only if $a_2 - a_4 a_1 / a_3 > 0$ and $a_1 (a_2 - a_4 a_1 / a_3) > a_0 a_3$. The second condition guarantees the first (as all the a are positive) and can be rewritten as $\boxed{a_1 a_2 a_3 > a_1 a_4^2 + a_0 a_3^2}$.

Clearly the method extends up to higher n, in fact there is a standard tabular method for doing this (referenced in the notes).