Cambridge University Engineering Dept. Third year

Module 3F2: Systems and Control

LECTURE NOTES 2: ‘CLASSICAL’ METHODS

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1 Revision of Feedback Control

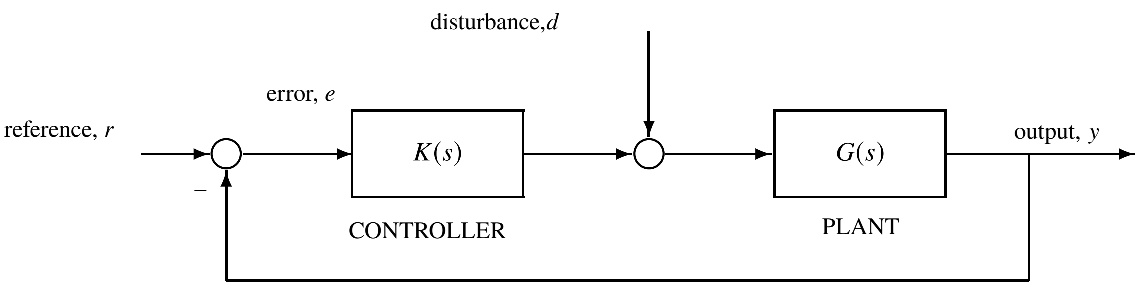
1.1 Why use Feedback?

- To reduce effects of uncertainty:
  - Disturbances *Wind/waves, Friction, Impurities, . . .*
  - Model errors *Approximations, Tolerances, Ageing, . . .*
- To stabilise unstable system:
  - Inverted pendulum,
  - High-performance fighter aircraft *(Fly-by-wire)*
  - Helicopter, Submarine (depth)
  - Exothermic chemical reactor, Nuclear reactor

Problems with feedback:

- May destabilise system
- Sensors introduce noise

1.2 The Standard Feedback Loop



1.3 Sensitivity and Complementary Sensitivity

Let $L(s)$ be the (open) loop transfer function: $L(s) = G(s)K(s)$ *"Return-ratio"*

Complementary Sensitivity: $T(s) = \frac{L(s)}{1+L(s)}$ (Multivariable: $T(s) = L(s)[I + L(s)]^{-1}$

$$\bar{y} = T\bar{r} = -T\bar{n} \qquad (\bar{n} = \text{sensor noise})$$

$T \approx 1 \Rightarrow$ good "tracking" but no noise filtering.

Sensitivity: $S(s) = \frac{1}{1+L(s)}$ (Multivariable: $S(s) = [I + L(s)]^{-1}$)

$$\bar{e} = S\bar{r}, \quad S = \frac{dT/T}{dG/G}, \qquad \bar{y} = GS\bar{d}$$

Small $|S| \Rightarrow$ feedback is beneficial.

Fact: $\boxed{S(s) + T(s) = 1}$ (Multivariable: $S(s) + T(s) = I$)

Hence **trade-off**:

- Small $|S(j\omega)|$ at low frequencies
- Small $|T(j\omega)|$ at high frequencies.

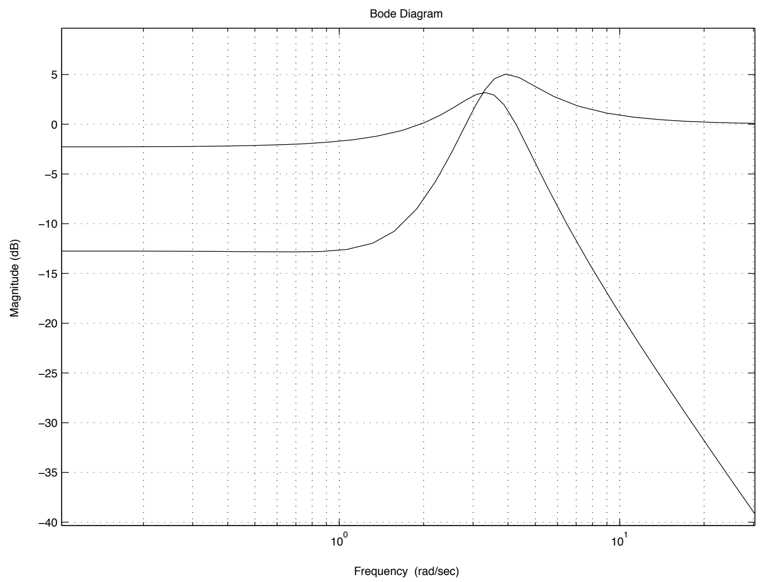


Figure 1.1: Sensitivity (S) and Complementary Sensitivity (T)

1.4 Steady-state Error

Constant reference: $r(t) = \alpha$, $\bar{r}(s) = \frac{\alpha}{s}$. Assume closed-loop is *asymptotically stable.*

$\lim_{t\to\infty} e(t) = S(0)\alpha = \frac{1}{1+L(0)}\alpha$ (Final Value Theorem or $S(j\omega)$ with $\omega = 0$)
$\lim_{t\to\infty} e(t) = 0$ if $|G(0)K(0)| = \infty \Leftarrow G(s)K(s)$ has pole at $s = 0$ — *integral action*

Constant disturbance: $d(t) = \beta$, $\bar{d}(s) = \frac{\beta}{s}$.

$\bar{e} = -GS\bar{d}$. Zero steady-state error $\Leftarrow K(s)$ has pole at $s = 0$.

Ramp reference: $r(t) = \alpha t$, $\bar{r}(s) = \frac{\alpha}{s^2}$.

$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\bar{e}(s) = \lim_{s\to 0} \frac{\alpha}{s} S(s)$. (Final Value Theorem)
Hence:

- *Finite* steady-state error $\Leftarrow G(s)K(s)$ has a pole at $s = 0$.
- *Zero* steady-state error $\Leftarrow G(s)K(s)$ has *two* poles at $s = 0$.

Small steady-state error requires high gain at "DC".

1.5 The Nyquist Stability Theorem

Motivation:

- The frequency response can be determined experimentally.
- Or from transfer function or state-space model.
- Want a test for closed-loop stability that uses *open-loop* information.

Theorem:

- Plot $L(j\omega) = G(j\omega)K(j\omega)$ on the Argand diagram, for $-\infty < \omega < +\infty$
— the *Nyquist plot*.
- The closed loop is stable if and only if the Nyquist plot encircles the point $-1 + j0$ p_u times counterclockwise, where p_u is the number of *unstable poles* of $G(s)$ and $K(s)$.

2 The Root-Locus Method

2.1 An Example

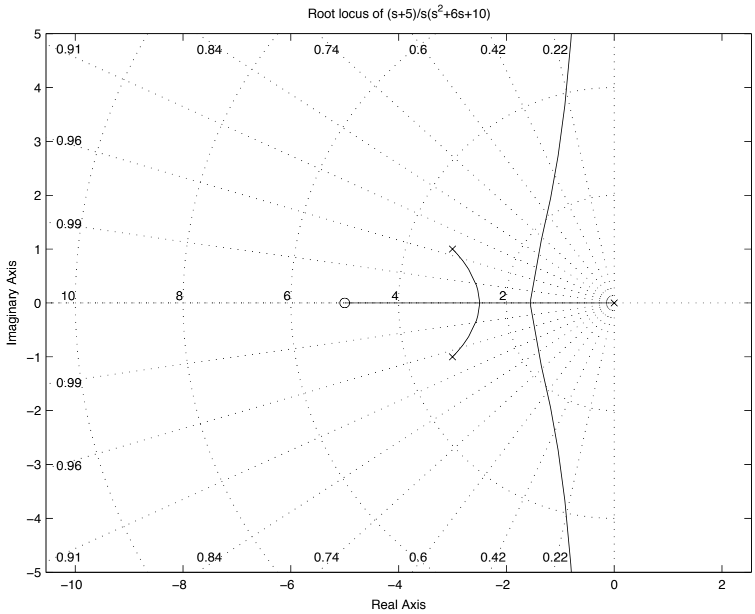


Figure 2.1:

Root-locus diagram for

$$L(s) = \frac{s+5}{s(s^2+6s+10)}$$

This shows the locations of the roots of $1 + kL(s) = 0$ for $k > 0$.

In this case $0 \le k \le 2 \times 10^4$.

Useful when the loop dynamics are fixed, and only the gain varies.

2.2 The Angle Condition

$$L(s) = \frac{n(s)}{d(s)} = \frac{c(s - z_1)(s - z_2)\dots(s - z_m)}{(s - p_1)(s - p_2)\dots(s - p_n)}$$

where any complex zeros or poles occur in conjugate pairs and $m \le n$.

We assume for the moment that $c > 0$.

Suppose that s_0 is on the root-locus:

$$1 + kL(s_0) = 0 \Rightarrow L(s_0) = -\frac{1}{k} \qquad \text{real and negative} \tag{2.1}$$

Hence *angle condition* for s_0 to be on the root-locus:

$$\angle L(s_0) = (2\ell + 1)\pi \tag{2.2}$$

$$\sum_{i=1}^{m} \angle(s_0 - z_i) - \sum_{i=1}^{n} \angle(s_0 - p_i) = (2\ell + 1)\pi \tag{2.3}$$

2.3 Finding Gain from the Root-Locus Plot

Once a root-locus plot has been obtained, it can be calibrated with k values. From (2.1) we have, at a point s_0 on the root-locus:

$$\begin{aligned} k &= \frac{1}{|L(s_0)|} \\ &= \frac{1}{c} \times \frac{|s_0 - p_1| \times |s_0 - p_2| \times \cdots \times |s_0 - p_n|}{|s_0 - z_1| \times |s_0 - z_2| \times \cdots \times |s_0 - z_m|} \end{aligned}$$

2.4 Constructing the Root-Locus Plot

Nowadays we can use software to draw root-locus diagrams (eg `control.rlocus` in *Python*).

But it is useful to have some understanding of how the form of the locus is determined. A set of about 15 'construction rules' has been developed. The 5 most important ones are given here. They are all consequences of (2.3) and properties of polynomials.

Rule 1. The root-locus diagram is symmetric with respect to the real axis and consists of n branches.

Rule 2. For $k = 0$ the n branches start at the open loop poles p_i. As $k \to \infty$, m branches tend to the zeros z_i and $n - m$ branches tend to infinity.

Rule 3. Points on the real axis which lie to the left of an odd number of poles *and* zeros are on the root-locus.

Rule 4. The breakaway points are those points on the root-locus for which $\frac{d}{ds}L(s) = 0$ (same as $dk/ds = 0$).

Rule 5. As $k \to \infty$, the $n - m$ branches which tend to infinity do so along straight line asymptotes at angles $(2\ell + 1)\pi/(n - m)$ to the +ve real axis ($\ell = 0, \ldots, n - m - 1$), and emanate from the point ('centre of gravity' — pole +ve mass, zero -ve mass):

$$\frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n - m}.$$

Proof of Rule 3:

Consider a point s_0, a pole p_i, and a zero z_i, all on the real axis.

$$\angle(s_0 - p_i) = \begin{cases} 0 \text{ if } s_0 > p_i \\ \pi \text{ if } s_0 < p_i \end{cases}$$

The same holds for $\angle(s_0 - z_i)$. Rule 3 follows from (2.3).

Example of use of Rule 3:

Suppose that G(s) has one pole and one zero in the right half-plane, eg $p_1 = +5$, $z_1 = +2$. Rule 3 shows that $K(s)$ *must* have *at least* one pole to the right of +2.
— the controller must be *unstable*!

(Figuring out the details is often easier from Bode plots etc.)

Proof of Rule 2:

$$d(s_0) + kn(s_0) = 0 \qquad \text{from (2.1)}$$

So if $k = 0$ then $d(s_0) = 0 \Rightarrow$ Branches start at poles.

Also, for any fixed k,

$$(2.1) \iff \frac{1}{k}d(s_0) + n(s_0) = 0$$

So, as $k \to \infty$ the finite roots tend to zeros (i.e. finite branches end at zeros). There can be at most m of them.

(to see this, put $n(s) = (s - s_0)n'(s)$, then for s close to s_0 we have $s = s_0 - \frac{1}{k}\frac{d(s_0)}{n'(s_0)} \to s_0$.)

The remaining $n - m$ branches go to ∞, but how?

Proof of Rule 5:

Application to previous example:

$n = 3$, $m = 1$, so $n - m = 2$. Two asymptotes at angles $\pi/2$ and $3\pi/2$.

Asymptotes emanate from (Rule 5):

$$\frac{(0 - 3 - 3) - (-5)}{3 - 1} = -\frac{1}{2}$$

Proof of Rule 4:

Application of Rule 4 to example of Fig.2.1:

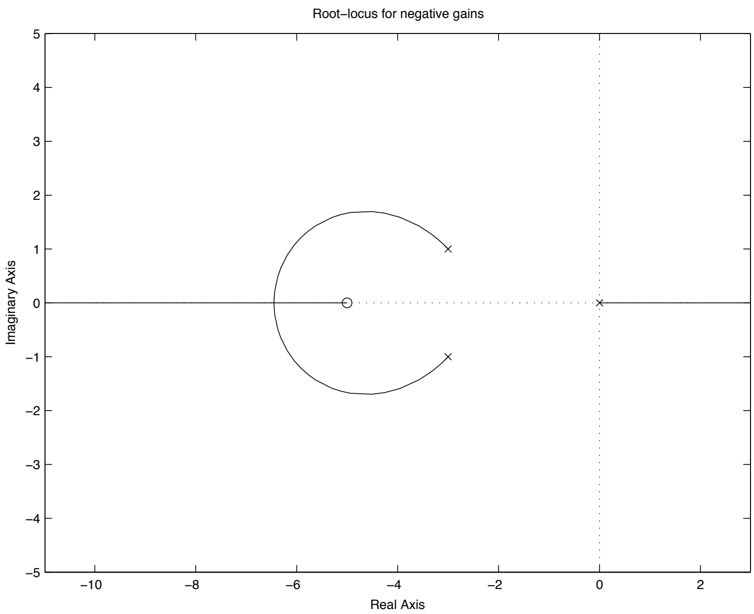
$$\begin{aligned}
\frac{d}{ds}\left(\frac{s+5}{s^3 + 6s^2 + 10s}\right) &= 0 \\
\Rightarrow \frac{1(s^3 + 6s^2 + 10s) - (s+5)(3s^2 + 12s + 10)}{(s^3 + 6s^2 + 10s)^2} &= 0 \\
\Rightarrow \frac{-2s^3 - 21s^2 - 60s - 50}{(s^3 + 6s^2 + 10s)^2} &= 0 \\
\Rightarrow \frac{-2(s + 1.5505)(s + 2.5)(s + 6.4495)}{(s^3 + 6s^2 + 10s)^2} &= 0
\end{aligned}$$

From Fig.2.1 it is seen that the root at -6.4495 is not on the root-locus.
The other two roots give the *breakaway points* (ie repeated roots).

2.5 Root-locus for negative k (or negative c)

$$\begin{aligned} 1 + kG(s_0) = 0 \quad &\Rightarrow \quad G(s_0) = -\frac{1}{k} > 0 \\ &\Rightarrow \quad \angle G(s_0) = 2\ell\pi \end{aligned}$$

- Rules 1,2,4 remain unchanged.
- Rule 3: Replace 'odd' by 'even'.
- Angles of asymptotes become $2\ell\pi/(n-m)$.
  (Points from which asymptotes emanate remain unchanged.)



Root-locus diagram for negative gain when

$$L(s) = \frac{s+5}{s(s^2+6s+10)}$$

2.6 Studying Parameter Variations

Root-locus diagrams can be used to study the variation of closed-loop poles as other parameters vary — not just the loop-gain k.

All that is needed is to put the closed-loop characteristic equation into the form

$$1 + \lambda H(s) = 0 \tag{2.4}$$

where λ is the parameter that is varying, and $H(s)$ is a transfer function.

Example: Robot placing objects of varying mass

The 1-D equation of motion of a robot moving a mass m with viscous friction c and elastic tether is $m\ddot{x} = u - c\dot{x} - \alpha x$ where x is the mass position and u is the applied force. The use of a PI controller is proposed, with a transfer function $k(s + z)/s$.

With $m = 0.1$ kg, $c = 0.6$ N/(m/sec), $\alpha = 1$ N/m and $z = 5$ we have

$$G(s) = \frac{1}{0.1s^2 + 0.6s + 1} = \frac{10}{s^2 + 6s + 10} \qquad \text{and} \qquad K(s) = k\frac{s+5}{s}$$

Letting $L(s) = G(s)K(s)/k$ the closed-loop characteristic equation is $1 + kL(s) = 0$.

Using Fig.2.1, $10k = 1.395$ places two closed-loop poles at -1.55 (one of the breakaway points) and the third pole at -2.9.

What if the mass varies?

The closed-loop characteristic equation is

$$1 + k\frac{(s+5)}{s(ms^2 + 0.6s + 1)} = 0$$

which has the same roots as

$$(ms^3 + 0.6s^2 + s) + k(s+5) = 0$$

or

$$1 + \frac{1}{m}\frac{0.6s^2 + [1+k]s + 5k}{s^3} = 0$$

which is in the form of (2.4) with $\lambda = 1/m$. The root-locus plot for this, with $k = 1.395/10$, is shown in Fig.2.2. The roots with $m = 0.1$ are marked.

Variations of closed-loop poles as $1/m$ varies can be clearly seen.

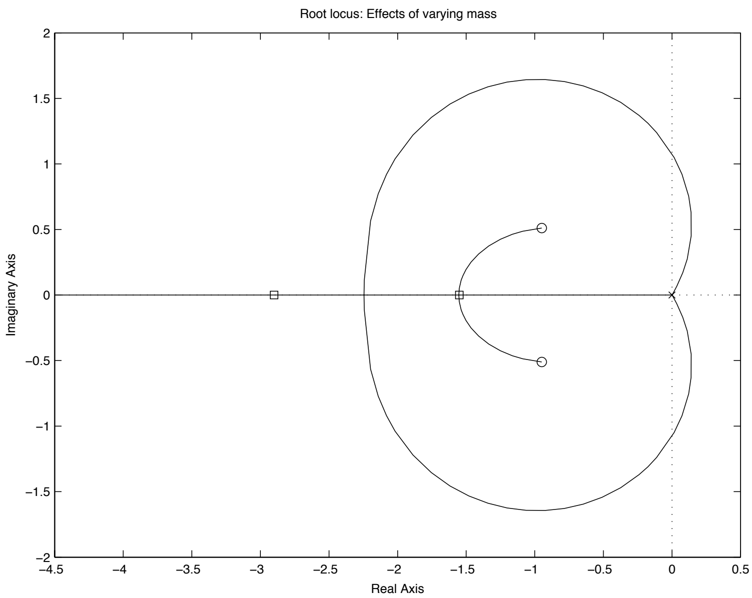


Figure 2.2: Root-locus diagram for variation of mass in robot problem.

3 The Routh-Hurwitz Criterion

The closed-loop characteristic equation

$$1 + G(s)K(s) = 0$$

has the same roots as

$$d_G(s)d_K(s) + n_G(s)n_K(s) = 0 \qquad \text{polynomial}$$

The Routh-Hurwitz criterion tests whether a polynomial has any roots with nonnegative real parts. So it tests for asymptotic stability.

Sometimes useful for finding value of k at which root-locus crosses imaginary axis.

Consider the polynomial (assume $a_0 > 0$):

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n \tag{3.5}$$

Easy to check that all roots have negative real parts *only if* $a_i > 0$ for each i.

A **Routh array** can be constructed for arbitrary n
— see Franklin, Powell and Emami-Naeini, 3rd edition, sec.4.4.3 (for example) for details.

For $n = 2, 3, 4$ simplifies as follows: *These are in Electrical and Information Data Book*
All the roots of (3.5) have negative real parts *if and only if*:

$$\begin{aligned} n = 2 &\quad : \quad a_i > 0, \quad \text{No other conditions} \\ n = 3 &\quad : \quad a_i > 0, \quad a_1 a_2 > a_0 a_3 \\ n = 4 &\quad : \quad a_i > 0, \quad a_1 a_2 a_3 > a_0 a_3^2 + a_4 a_1^2 \end{aligned}$$