6/2 Solutions

Linear Systems & Control

1. For rational transfer functions ~~stability is~~ determined by the poles.

(a) Pole at -3. Hence asymptotically stable.

(b) Pole at -3. Hence asymptotically stable.
Zero at $+1$. (This irrelevant for stability)

(c) Pole at $+3$. Hence unstable.

(d) Pole at 0. This has Real part $=0$, and the pole is not repeated; hence marginally stable.

(e) Poles at $0, 0$. Both have Real part $=0$, but they are coincident; hence unstable. (Impulse response is t.)

(f) Poles at $\pm j2\sqrt{2}$. Both have Real part $=0$, and neither is repeated; hence marginally stable.
Zero at $-3/2$. (This irrelevant for stability.)

(g) Poles at $\pm 2\sqrt{2}$. One has Real part >0, hence unstable.
Zero at $-3/2$. (This irrelevant for stability.)

(h) Poles at $-5, \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$. Two poles have Real part >0; hence unstable.

(i) Consider this to be the transfer-function of two subsystems connected in series:

$$\longrightarrow \boxed{\frac{10}{(s+5)(s^2+s+1)}} \longrightarrow \boxed{e^{-2s}} \longrightarrow$$

The second one simply delays the output of the first one, and so has no effect on stability.
For the first one the poles are at $-5, -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$.
All of these have Real part <0, so the system is asymptotically stable.

1 cont. The pole-zero diagrams are shown on page 3. The only one which is not quite straightforward is (i), because of the term 'e^{-2s}' in the numerator. We can still define a 'zero' to be a point of the complex plane at which the transfer function becomes zero. But $e^{-2s} = 0$ only for $Re\{s\} = +\infty$, so in this case the transfer function has no zeros (except at infinity).

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2 (d) has poles in the right half-plane, so the system is unstable. The only impulse response which can correspond to this is (5).

Of the remaining stable systems, (a) has the lowest damping factor, (b) and (e) have a larger one, and (c) is overdamped, since it has 2 real poles. Of the impulse responses, (2) is the least damped, (3) and (4) equally damped, and (1) most heavily damped. So we have the correspondence

(a) $\leftrightarrow$ (2)

$\left.\begin{matrix}(b)\\(e)\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}(3)\\(4)\end{matrix}\right.$

(c) $\leftrightarrow$ (1)

In (b) the poles are at $-2 \pm j$; in (e) they are at $-60 \pm 30j$. Hence the response of (e) should be 30 times faster than that of (b).

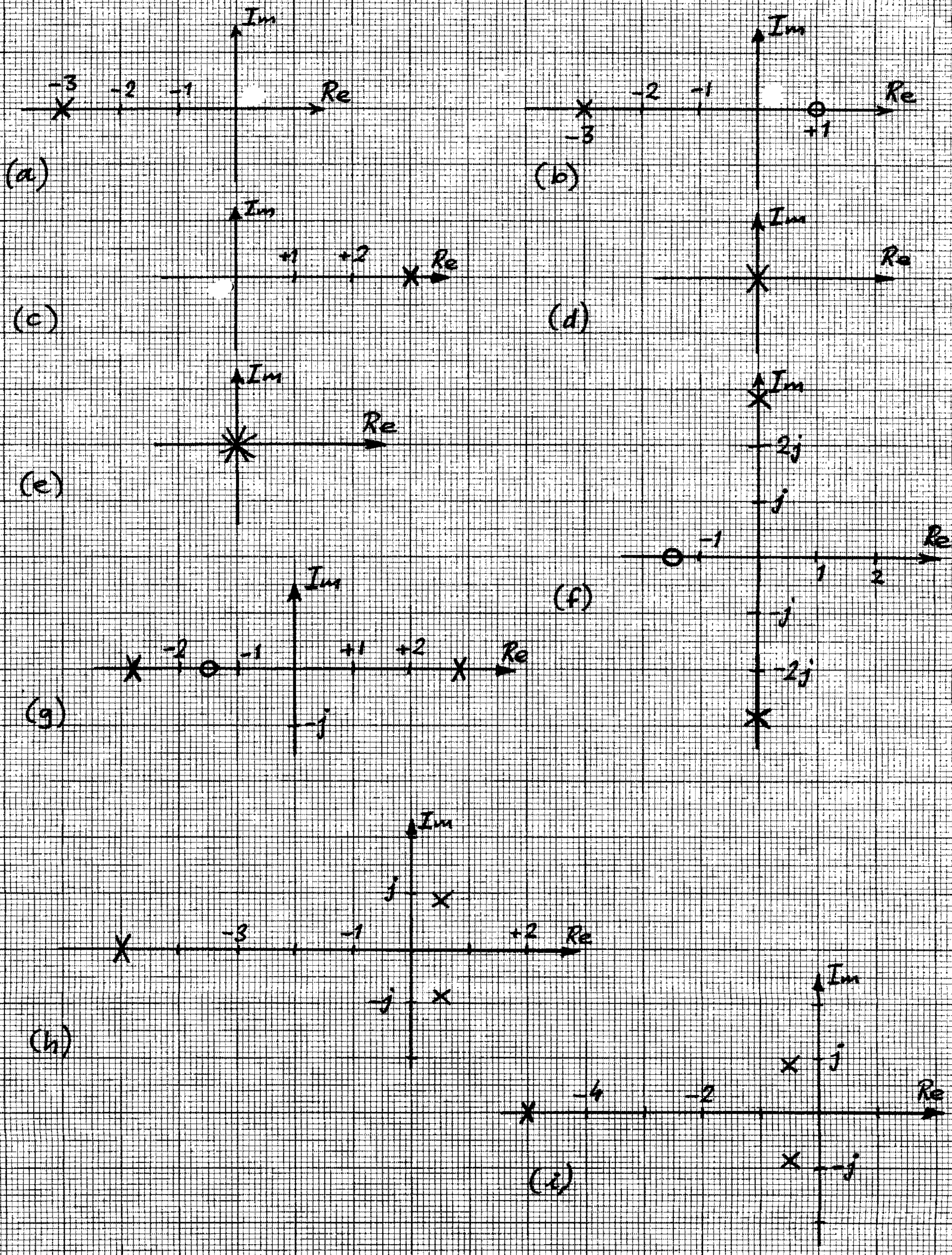
Thus we get (b) $\leftrightarrow$ (3)
and (e) $\leftrightarrow$ (4).

Summary	
LH	RH
a	2
b	3
c	1
d	5
e	4

6/2 Solutions

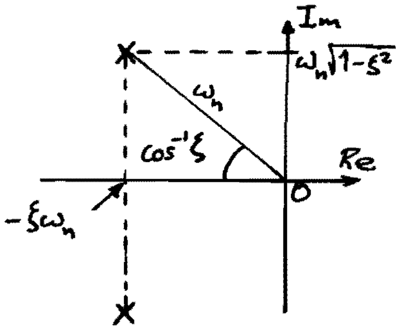
POLE – ZERO DIAGRAMS FOR Q. 1

(POLES MARKED AS 'x', ZEROS AS 'o')



3. Flight phase A: $\cos^{-1}(0{\cdot}4) = 66{\cdot}4^\circ$

Flight phase B: $\cos^{-1}(0{\cdot}08) = 85{\cdot}4^\circ$



The geometry of a pair of poles $p, \bar{p}$, such that

$$(s-p)(s-\bar{p}) = s^2 + 2\xi\omega_n s + \omega_n^2$$

is as shown in the figure:
(if $|\xi| < 1$)

Hence the permitted pole locations are as shown on page 7.

If $p = \sigma + j\omega$, $\bar{p} = \sigma - j\omega$, then

$$\sigma = Re\{p\} = -\xi\omega_n \quad (\text{since } p + \bar{p} = -2\xi\omega_n)$$

and $\omega = \sqrt{\omega_n^2 - \sigma^2} = \omega\sqrt{1-\xi^2}$ (since $p\bar{p} = \omega_n^2$).

<u>Flight phase A:</u> Note from the figure that the specification on $\xi\omega_n$ is redundant here, since the other 2 specs. ensure that $\xi\omega_n \geq 0{\cdot}4$.

The response dies away as fast as $e^{\sigma t} = e^{-\xi\omega_n t}$, so it dies away at least as fast as $\exp(-0{\cdot}4t)$, i.e. with a time constant no greater than 2·5 sec.

If both poles are real, the constraint $\omega_n \geq 1$ ensures that the response dies away at least as fast as te^{-t}. (This would be e^{-t} if the poles were distinct; te^{-t} allows for the possibility of repeated poles at -1.)

$\xi \geq 0{\cdot}4$ can be given the following interpretation: The period of the oscillatory term in the response is $\frac{2\pi}{\omega} = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$. But, over one period, the '$e^{\sigma t}$' term changes by

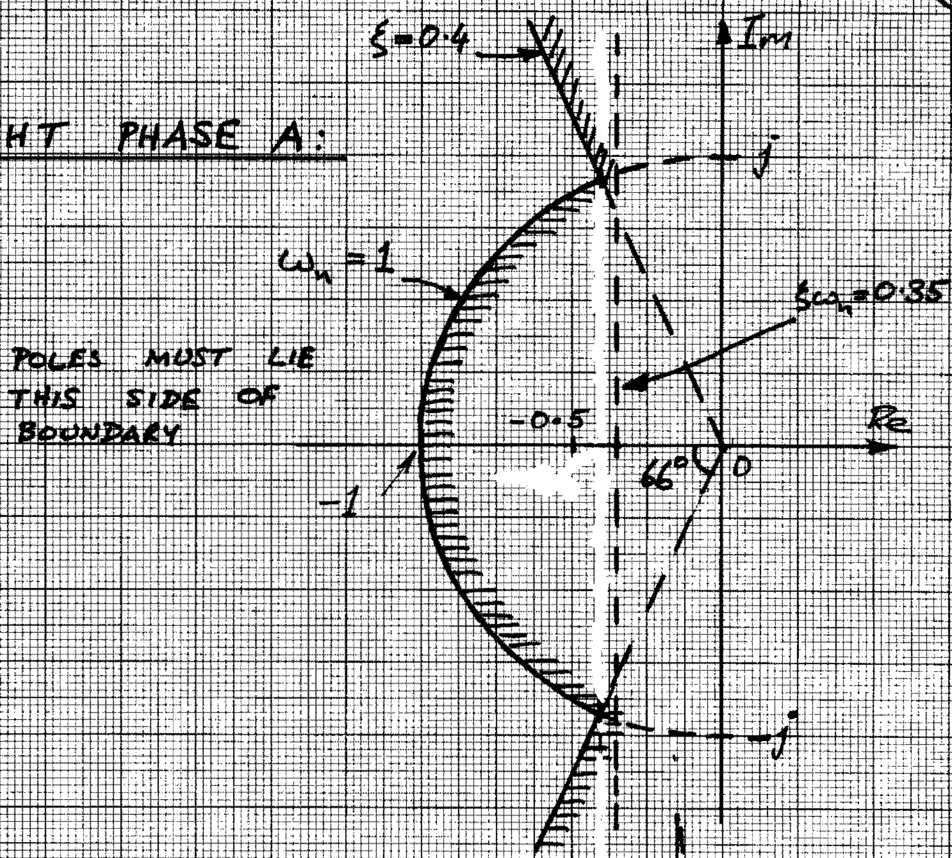
$$\exp\left(\frac{2\pi\sigma}{\omega}\right) = \exp\left(\frac{2\pi}{\omega_n\sqrt{1-\xi^2}} \times [-\xi\omega_n]\right)$$

$$= \exp\left(\frac{-2\pi\xi}{\sqrt{1-\xi^2}}\right).$$

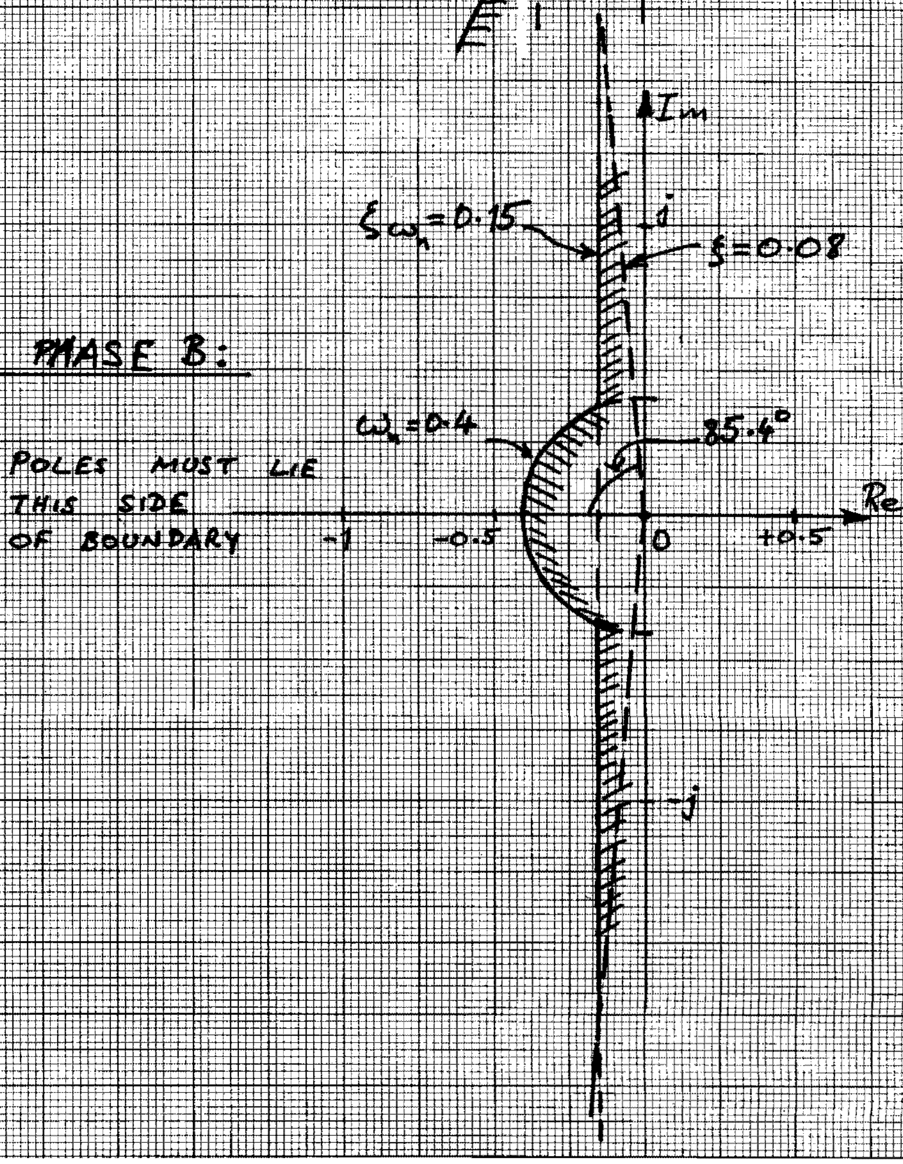
Since $\xi \geq 0{\cdot}4$, this factor is at most 0·064.

cont.

FLIGHT PHASE A:



FLIGHT PHASE B:



3 cont'd So the amplitude of oscillation is reduced by at least $\frac{1}{0.064} = 15.5$ during each cycle.

Flight phase B: The spec. $\zeta\omega_n \geq 0.15$ is active in this case.

The response is now guaranteed to die away at least as fast as $e^{-0.15t}$, i.e. with a time constant no greater than 6.7 sec.

If both poles are real, $\omega_n \geq 0.4$ ensures that the response dies away at least as fast as $e^{-0.4t}$.

$\zeta \geq 0.08$ ensures that the amplitude of oscillation is reduced by at least 1.66 during each cycle (since $\exp\left(\frac{2\pi \times 0.08}{\sqrt{1-0.08^2}}\right) = 1.66$).

Clearly the performance requirements are more stringent in Flight phase A than in B. ('A' includes activities such as reconnaisance and terrain following, while 'B' includes cruising and inflight refuelling.)

Note that it is quite awkward to describe the behaviour in words. Specifying the permitted pole locations conveys at least as much information, but more compactly and accurately.

4 $$\bar{g}(s) = \frac{1}{(s+1)(s+0.1)}$$

$$\therefore \quad \bar{g}(j\omega) = \frac{1}{(j\omega+1)(j\omega+0.1)}$$

$$= \frac{1}{(0.1-\omega^2)+1.1j\omega}$$

$$\angle \bar{g}(j\omega) = -\frac{\pi}{2} \quad \text{(rad)} \quad \text{when}$$

$$0.1-\omega^2 = 0$$

i.e. $\underline{\underline{\omega = 1/\sqrt{10}}}$ (rad/sec)

$$\left|\bar{g}(j\sqrt{10})\right| = \left|\frac{\sqrt{10}}{1.1j}\right| = \frac{\sqrt{10}}{1.1}.$$

So if the input amplitude is X, and the frequency is $\sqrt{10}$ rad/sec, then the output amplitude is

$$\underline{\underline{\frac{X\sqrt{10}}{1.1}}}$$

Q5. Substituting $s = j\omega$ in the given transfer function gives

$$G(j\omega) = e^{-l\sqrt{(j\omega+\mu)/a}}$$

and so

$$\log G(j\omega) = -l\sqrt{(j\omega + \mu)/a} \qquad (*).$$

Note that we can write

$$\log G(j\omega) = \log|G(j\omega)| + j\arg G(j\omega)$$

and so

$$(\log G(j\omega))^2 = \ldots + 2j\log|G(j\omega)|\arg G(j\omega).$$

But, from (*),

$$(\log G(j\omega))^2 = l^2(j\omega + \mu)/a.$$

Equating the imaginary part of these last two expressions gives the answer.

To use this result, simply vary the temperature of the end of the rod sinusoidally at a convenient frequency and measure the temperature both there and at a distance along the rod (which will also be sinsoidal but at a lower amplitude and lagged wrt the temperature at the end. The product of the natural logarithm of the amplitude ratio and the phase lag in radians is then $l^2\omega/2a$, where ω is the frequency of excitation. Heat losses do not affect the result.

6/2 Solutions

6. The Bode diagrams for these transfer functions are drawn on log-lin graph paper over the page with computer generated plots after that. The constructions to obtain these plots with a minimum of calculation are given below.

(a) $G_1(s)$ and $G_2(s)$.

For $G(s) = \dfrac{1+sT}{1+asT}$ we have

$G(0) = 1$ $(= 0\,dB)$; $G(j\infty) = 1/a$ $(= -20\log_{10} a \; dB)$.

"Break" frequencies at $\omega = 1/T$ and $1/aT$.

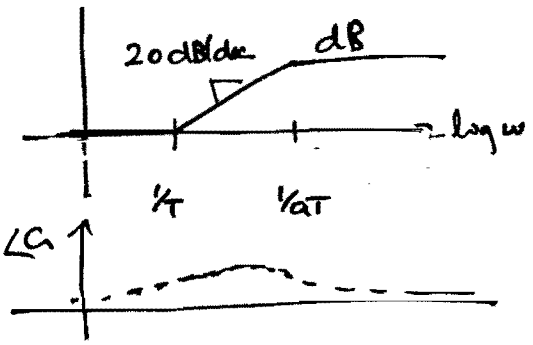
$\angle G(j/T) = \tan^{-1} 1 - \tan^{-1} a = \pi/4 - \tan^{-1} a$

$\angle G(j/aT) = \tan^{-1}(1/a) - \tan^{-1} 1 = \pi/4 - \tan^{-1} a$

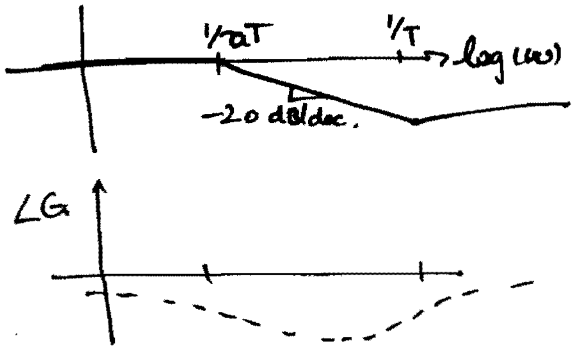
$\angle G(j/\sqrt{a}T) = \tan^{-1} 1/\sqrt{a} - \tan^{-1}\sqrt{a} = \pi/2 - 2\tan^{-1}\sqrt{a}$

$\angle G(j0) = 0^\circ$; $\angle G(j\infty) = 0^\circ$.

Hence for $a < 1$



for $a > 1$



for $a = 0.1$, $T = 1$

$\angle G(j/\sqrt{0.1}) = 55^\circ$

for $a = 4$, $T = 2.5$

$\angle G(j/_{2\times 2.5}) = -37^\circ$.

6 cont.

(b) For G_3 and G_4

Consider the term

$$G_3 = \frac{1}{1 + 2csT + s^2T^2}$$

$G_3(0) = 1$ as $\omega \to \infty$ $\omega^2 G_3(j\omega) \to -\frac{1}{T^2}$

$$G_3(j/T) = \frac{1}{1 + 2cj + j^2} = -\frac{j}{2c}$$

the peak is close to $\omega = 1/T$ if the value of c is small.

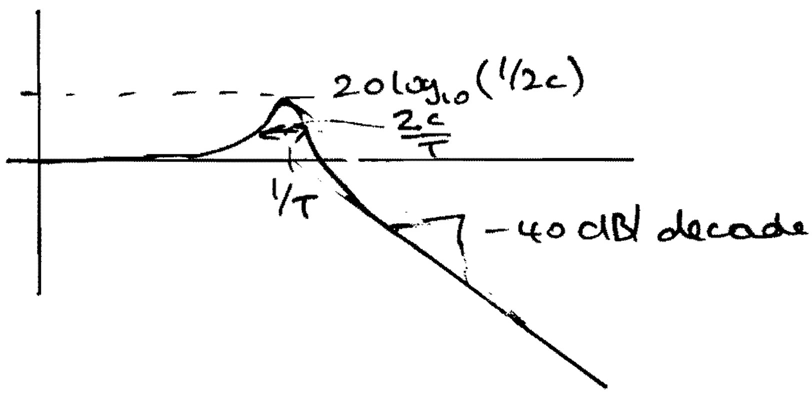
$\angle G_3(j0) = 0°$; $\angle G_3(j\infty) = -180°$

$\angle G_3(j/T) = -90°$

Then for ω close to $1/T$ we have

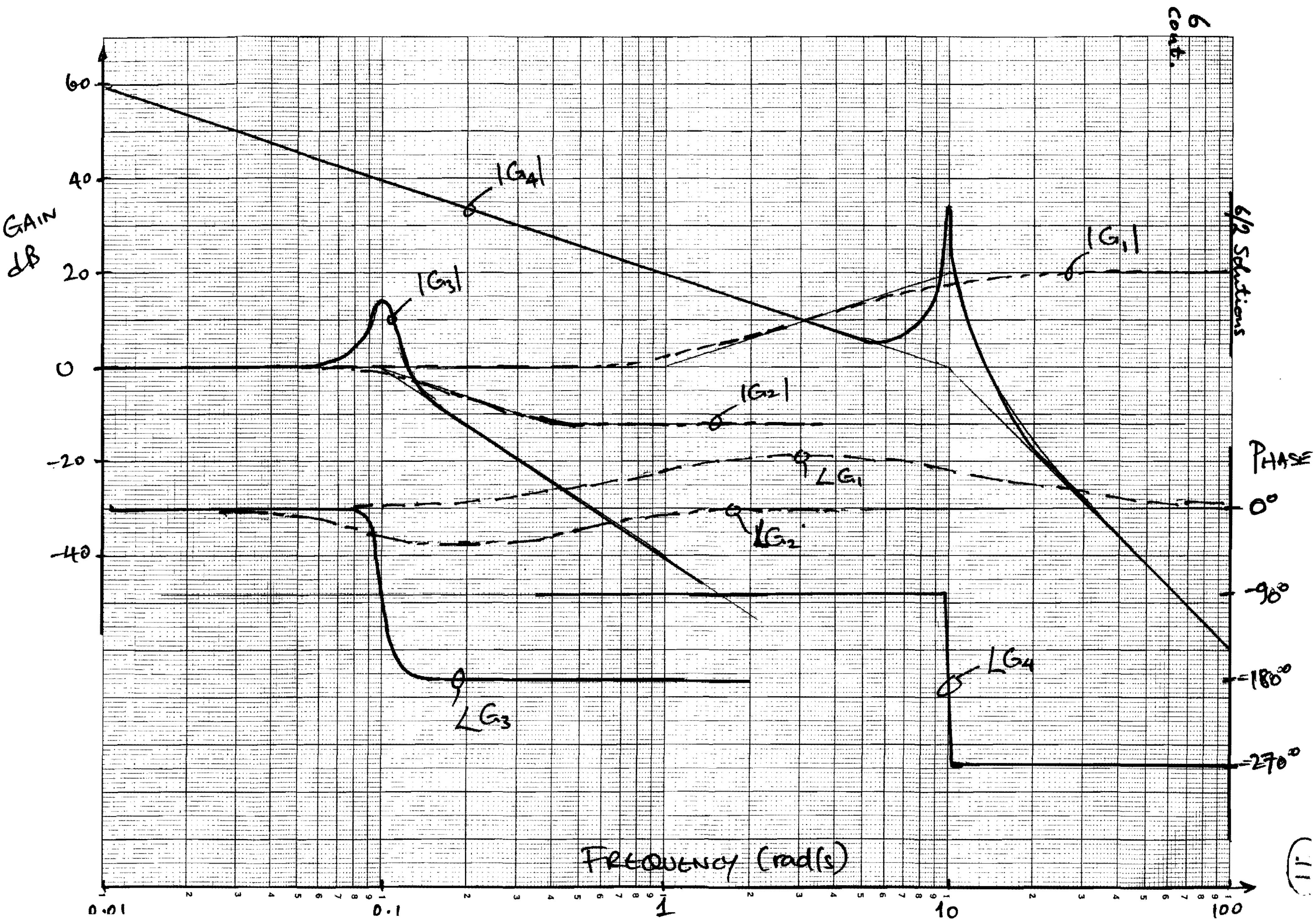
$$G_3\left(j\tfrac{1}{T}(1 \pm c)\right) = \frac{1}{(1 - (1 \pm c)^2 + 2cj(1 \pm c)]}$$

$$\simeq \frac{-j}{2c(1 \mp j)} = \frac{1}{2\sqrt{2}c} \angle -90° \pm 45°.$$



$$G_4 = \frac{1}{sT(1 + 2csT + s^2T^2)}$$ is the same except an additional gain of -20 dB/decade for ωT and an extra phase lag of $90°$.

Chartwell Graph Data Ref. 5541 Log 4 Cycles x mm, ½ and 1 cm



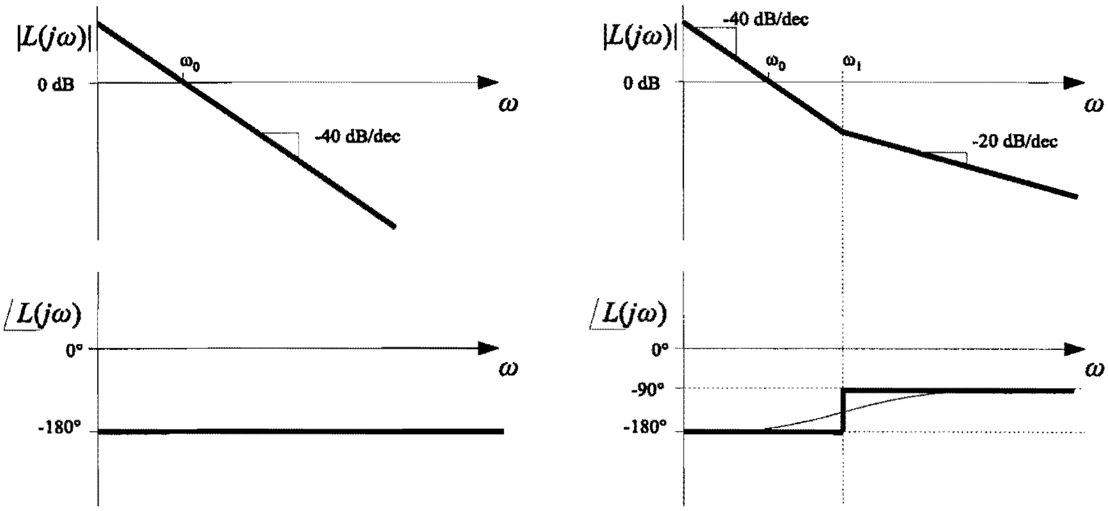
Examples Paper 6/2 Q7

$P(s) = 1/(200s^2 + 4.4142e - 4)$, so unstable pole at $s = 2.2e - 6$ which we shall neglect as it is so slow (time constant =126 hours).

(a) $L(s) = \frac{H_0 K_P}{ms^2}$. Simulations demonstrate marginal stability (but note that if you initialise the system at 700 m, non-linear drag slowly reduces the oscillation amplitude even though the linearised drag forces are zero).

(b) $L(s) = \frac{H_0(K_P + K_D s)}{ms^2}$. Simulations demonstrate a better damped system that converges to the target altitude.

The Bode diagrams are sketched in Figure 1.



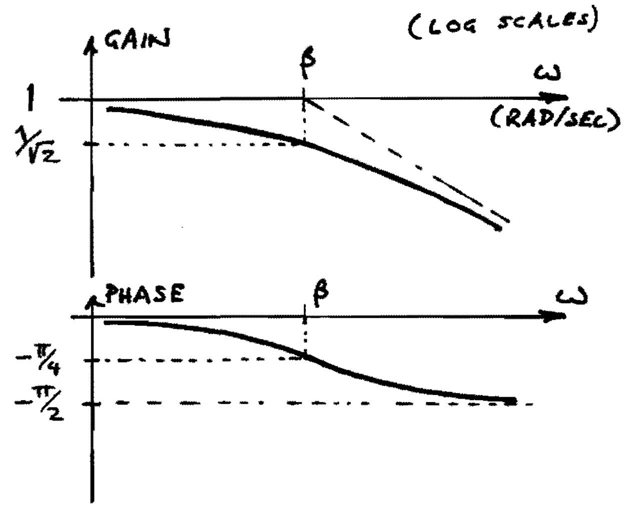
(a) (b)

Figure 1: Bode diagrams, $\omega_0 = \sqrt{\frac{H_0 k_P}{m}} = 0.24$, $\omega_1 = k_P/k_D = 1$

6/2 Solutions

8 **Bode diagrams:**

For the first-order model $g(t) = \beta e^{-\beta t}$, the transfer function is $\dfrac{\beta}{s+\beta}$. This has Bode diagram:



Attenuation is less than $1/\sqrt{2}$ (i.e. gain is greater than $1/\sqrt{2}$) for

$\underline{\underline{\omega < \beta}}$

For the second-order model, consider

$$\bar{g}(s) = s \times \frac{1}{s+300} \times \frac{22000}{s+22000}$$

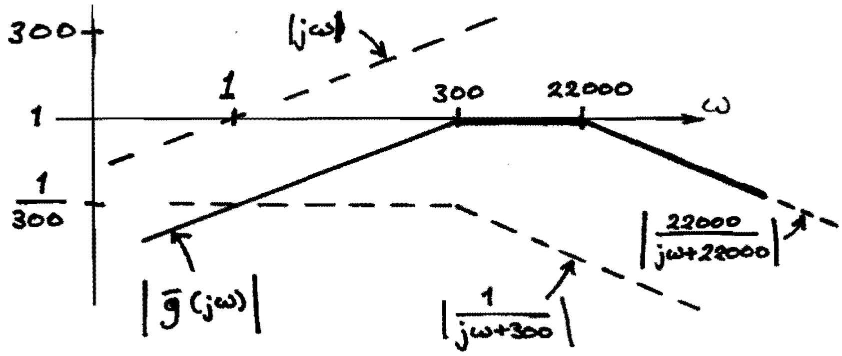
$$\bar{g}(j\omega) = \underbrace{(j\omega)} \times \underbrace{\frac{1}{j\omega+300} \times \frac{22000}{j\omega+22000}}$$

Bode diagram of each of these is as above.

$|j\omega| = \omega$

$\angle j\omega = +\pi/2$

Hence Bode diagram of $\bar{g}(j\omega)$ is constructed as shown.



For $300 < \omega < 22000$ the gain is ≈ 1.

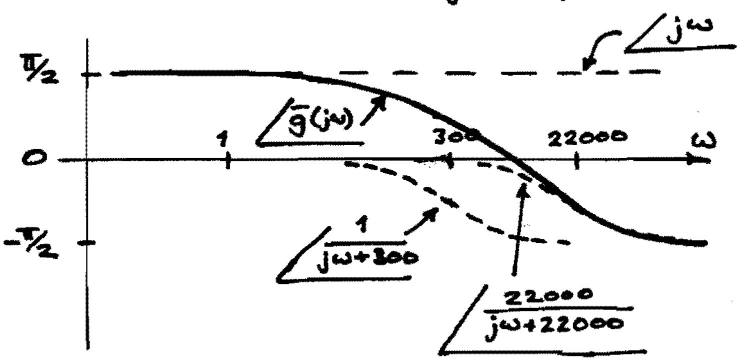
Checking more precisely,

$$|g(j300)| = \frac{300}{300\sqrt{2}} \times 0.9999$$

$$\approx \frac{1}{\sqrt{2}}.$$

$$|g(j22000)| = 0.9999 \times \frac{22000}{22000\sqrt{2}}$$

$$\approx \frac{1}{\sqrt{2}}.$$



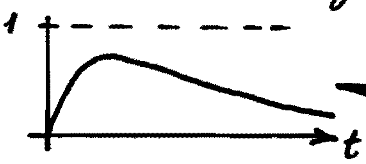
So the gain is greater than $1/\sqrt{2}$ for $\underline{300 < \omega < 22000}$ (RAD/SEC)

8 cont. **Optional extra:**

The initial part of the step response ($t < 10^{-4}$ sec) is almost identical for the two models. This is consistent with the fact that the two frequency responses are nearly the same at high frequencies ($\omega > 22000$ rad/sec, $f > 3.5$ kHz). The fact that the initial transient occurs so quickly is also consistent with the high bandwidth of the two models – consider time constant $\approx \frac{1}{22000} \approx 0.05$ msec.

exact for 1st-order model

The major difference is for $t > 10^{-4}$ sec. The response of the 1st-order models goes up to 1 :



while that of the 2nd order model decays to zero, with a time constant of about $\frac{1}{300} \approx 3$ msec. This is consistent with the frequency responses at $\omega = 0$:

$$\left.\frac{\beta}{j\omega + \beta}\right|_{\omega=0} = 1$$

$$\left.\frac{22000\, j\omega}{(j\omega+300)(j\omega+22000)}\right|_{\omega=0} = 0$$

The 2nd-order model has zero response (eventually) to a constant input. (In telephone networks this prevents DC currents from flowing between exchanges.)

9.
$$G_1(j\omega) = \frac{(j\omega+30)}{(j\omega+1)(j\omega+10)(j\omega+100)}$$

$$G_1(0) = \frac{30}{1000} = -30 \text{ dB}$$

$$\omega^2 G_1(j\omega) \rightarrow -1 = 1 \angle{-180^\circ} \quad \text{as } \omega \rightarrow \infty$$

and $\angle G_1(0) = 0$; $\angle G_1(j) \simeq -45^\circ$

$$G_2(j\omega) = \frac{j\omega}{(1-\omega^2 + 0.5j\omega)}$$

$G_2(j0) = 0$; $\omega G_2(j\omega) \rightarrow -j$

$$G_2(j) = \frac{j}{0.5j} = 2 = 6\text{dB}$$

$\angle G_2(j0) = 90^\circ$; $\angle G_2(j\infty) = -90^\circ$

$$G_3(j\omega) = \frac{1}{(1+0.01j\omega)} G_2(j\omega)$$

additional pole at $s = -100$ adding phase lag of 45° at $\omega = 100$.

$$G_4(j\omega) = \frac{(1-0.1j\omega)}{(1+0.1j\omega)} G_2(j\omega)$$

Now $\left|\frac{(1-0.1j\omega)}{1+0.1j\omega)}\right| = 1$ for all $\omega \Rightarrow$ same dB as G_2

$\angle\left(\frac{1-0.1j\omega)}{1+0.1j\omega)}\right) = -2\tan^{-1} 0.1\omega \Rightarrow$ extra phase lag of 180° at high frequency.

$$G_5(j\omega) = \left[1 + 0.001j\omega + 0.01(j\omega)^2\right] \frac{(1+10j\omega)}{(1+0.01j\omega)^4}$$

$$G(j10) = 0.01j\frac{(1+100j)}{(1+0.1j)^4} = \frac{-1}{(1+0.1j)^4}$$

$\angle G(j\infty) = -90^\circ$; phase change of $+180^\circ$ at $\omega \simeq 10$.

9 cont.

Figure 2

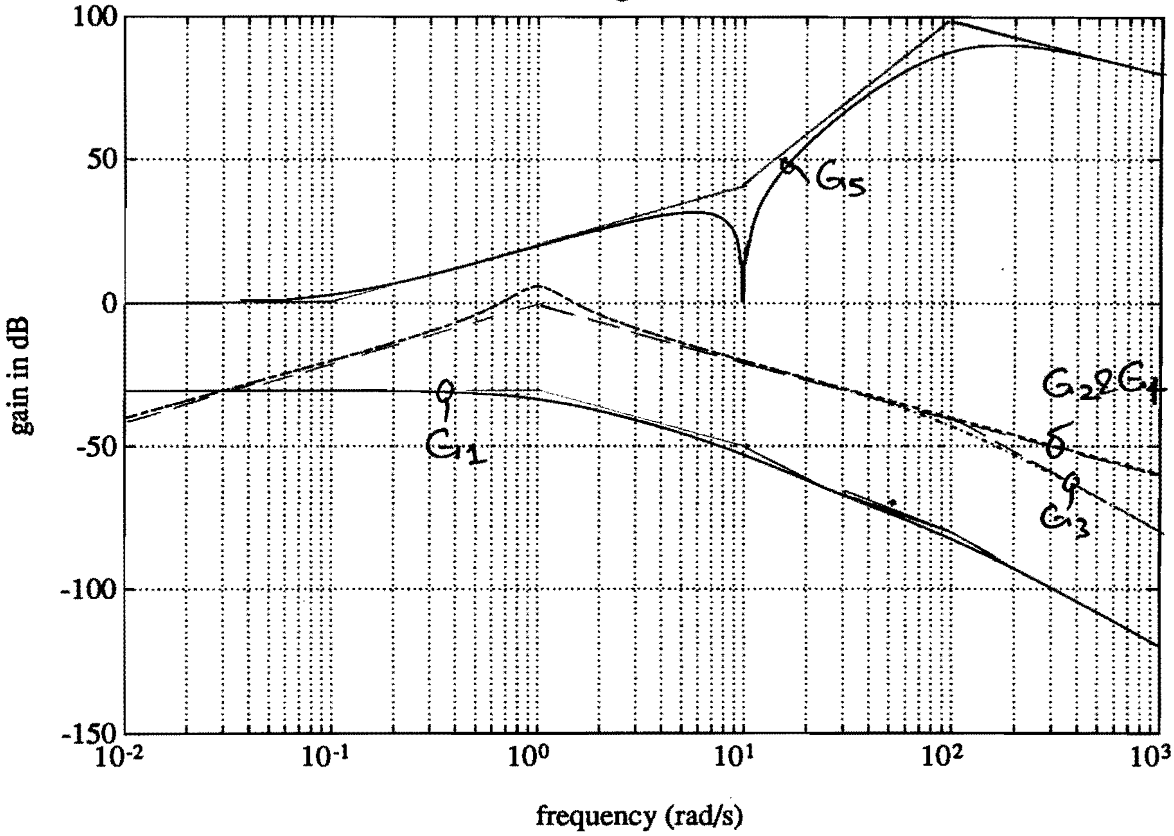
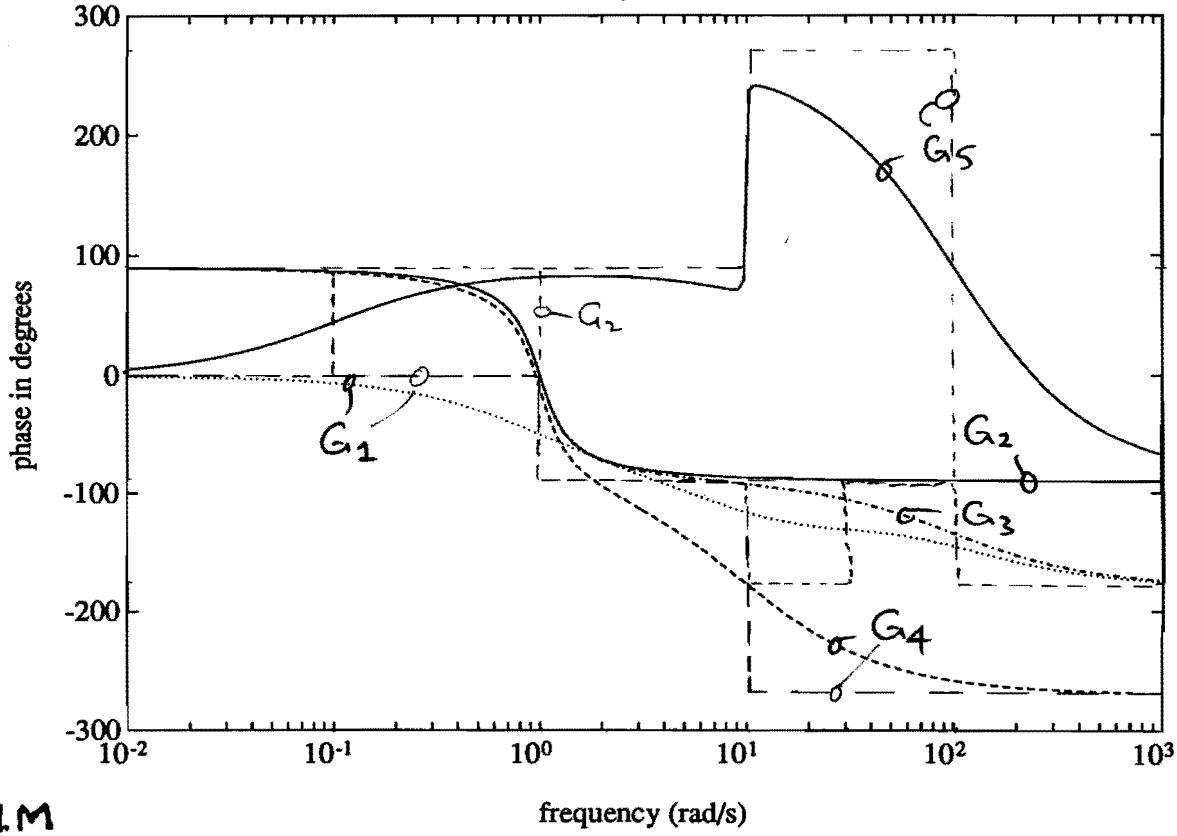


Figure 3



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