Cambridge University Engineering Dept. Third year

Module 3F2: Systems and Control

LECTURE NOTES 4: CONTROLLABILITY AND STATE FEEDBACK

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1 Controllability

1.1 Controllability Gramian, Controllability matrix

A system:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

is said to be **controllable** if for all initial conditions $\underline{x}(0) = \underline{x}_0$, terminal conditions $\underline{x}_1$, and $t_1 > 0$ there exists an input $u(t)$, $0 \le t \le t_1$ such that

$$\underline{x}(t_1) = \underline{x}_1.$$

That is, given $\underline{x}_0$, $\underline{x}_1$ and $t_1 > 0$, we wish to find $\underline{u}(t), 0 < t < t_1$, such that

$$\underline{x}(t_1) = e^{At_1}\underline{x}_0 + \int_0^{t_1} e^{A(t_1-t)}B\underline{u}(t)\,dt = \underline{x}_1$$

Note that this equation can be solved for all $\underline{x}_o$ and $\underline{x}_1$ if and only if it can be solved for all $\underline{x}_1$ with $\underline{x}_o = \underline{0}$. So we will now just consider the zero initial condition response.

Define the **controllability Gramian**, $W_c(t_1) \stackrel{def}{=} \int_0^{t_1} e^{A\tau} BB^T e^{A^T\tau}\, d\tau.$

Now assume that $W_c(t_1)$ has an inverse and let $\underline{u}(t) = \underline{u}_o(t) = B^T e^{A^T(t_1-t)} W_c(t_1)^{-1}\underline{x}_1$ when

$$\begin{aligned}\underline{x}(t_1) &= \int_0^{t_1} e^{A(t_1-t)} BB^T e^{A^T(t_1-t)} W_c(t_1)^{-1}\underline{x}_1\, dt \\ &= W_c(t_1)W_c(t_1)^{-1}\underline{x}_1 = \underline{x}_1 \text{ as desired.}\end{aligned}$$

Hence if $\det W_c(t_1) \neq 0$ then we can reach any $\underline{x}(t_1)$ from $\underline{x}(0) = \underline{0}$ (and hence there exists $u(t)$ to go from any $\underline{x}(0)$ to any $\underline{x}(t_1)$).

(Recall from section 3.1 of Lecture Notes 3:
$W_o(t_1) = \int_0^{t_1} e^{A^T\tau} C^T C e^{A\tau} d\tau$ $\quad\leftarrow$ *Observability Gramian*)

If $W_c(t_1)$ is a singular matrix there exists $\underline{z} \neq \underline{0}$ such that

$$\underline{z}^T W_c(t_1) = \underline{0} \Rightarrow \underline{z}^T W_c(t_1)\underline{z} = 0 \Rightarrow \underline{z}^T e^{At} B = \underline{0} \text{ for all } t$$

and hence

$$\underline{z}^T \underline{x}(t_1) = \int_0^{t_1} \underline{z}^T e^{A(t_1-t)} B\underline{u}(t)\, dt = 0 \text{ for all } \underline{u}(t).$$

$\Rightarrow \underline{x}(t_1) \perp \underline{z}$ and the system is not controllable.

Hence: System is controllable if and only if $\det W_c(t_1) \neq 0$.

In section 3.1 of Lecture Notes 3 we showed

Null space of $W_o(t_1)$ = Null space of Q.

Similarly:

Null space of $W_c(t_1)$ = Null space of P^T

where the **controllability matrix** P is given by

$$P \stackrel{def}{=} \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

Hence $\boxed{\text{The system is controllable if and only if rank } P = n}$

1.1.1 Example

Figure 2 shows a design for a hydraulically actuated table for simulating earthquakes. The table is denoted as ABC, with the point C constrained to move horizontally. DA and EB denote hydraulic rams which are pin-jointed at each end and can produce forces F_1 and F_2, respectively. The equations of motion (which should *not* be verified) are:

$$M\ddot{z} = F_1 \cos\phi_1 + F_2 \sin\phi_2$$

and

$$\begin{aligned} \frac{2}{a}\left(I + \frac{1}{4}Ma^2\cos^2\theta\right)\ddot{\theta} &= M\cos\theta\left(\frac{1}{2}a\dot{\theta}^2\sin\theta - g\right) + \\ &+ \left[\sin(\theta+\phi_1) + \sin\phi_1\cos\theta\right]F_1 + \\ &+ (\cos\theta\cos\phi_2)\,F_2 \end{aligned}$$

where

$$\tan\phi_1 = \frac{a\sin\theta}{a + z - \frac{1}{2}a\cos\theta}, \quad \tan\phi_2 = \frac{z}{a + \frac{1}{2}a\sin\theta},$$

M and I are constants, a and z are the lengths shown in Fig. 2, and θ, ϕ_1, ϕ_2 are the angles shown in the figure.

(a) What conditions are satisfied at an equilibrium? Determine values F_{1e} and F_{2e} of the forces F_1 and F_2, which will give an equilibrium position $\theta = \theta_e$ and $z = z_e$, if $\theta_e = 0$ and $z_e = a$.

(b) The linearised equations about the equilibrium $(\theta_e = 0, z_e = a)$ are:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

where

$$\underline{x} = \left[\theta, z-a, \dot{\theta}, \dot{z}\right]^T, \ \underline{u} = \left[F_1 - F_{1e}, F_2 - F_{2e}\right]^T, \ A = \begin{bmatrix} 0 & I_2 \\ P & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ Q \end{bmatrix},$$

$$P = \begin{bmatrix} \dfrac{-13}{12\tau^2} & -\dfrac{1}{2a\tau^2} \\ -\dfrac{g}{4} & \dfrac{g}{2a} \end{bmatrix}, \ Q = \begin{bmatrix} 0 & \dfrac{1}{\sqrt{2}Mg\tau^2} \\ \dfrac{1}{M} & \dfrac{1}{\sqrt{2}M} \end{bmatrix}, \ \tau^2 = \frac{2I}{Mag} + \frac{a}{2g},$$ and I_2 is the 2×2 identity matrix.

Verify that the term $-\frac{g}{4}$, which appears in P, is correct. (Do *not* verify any other terms. Assume that the nonlinear equations are correct.)

(c) Is the linearised system of part (b) controllable from $\underline{u}$? Is it controllable from u_1 (the first element of $\underline{u}$) alone?

(d) Comment on the difference in the achievable behaviour of this system when only u_1 is available for control, and when the complete vector $\underline{u}$ is available.

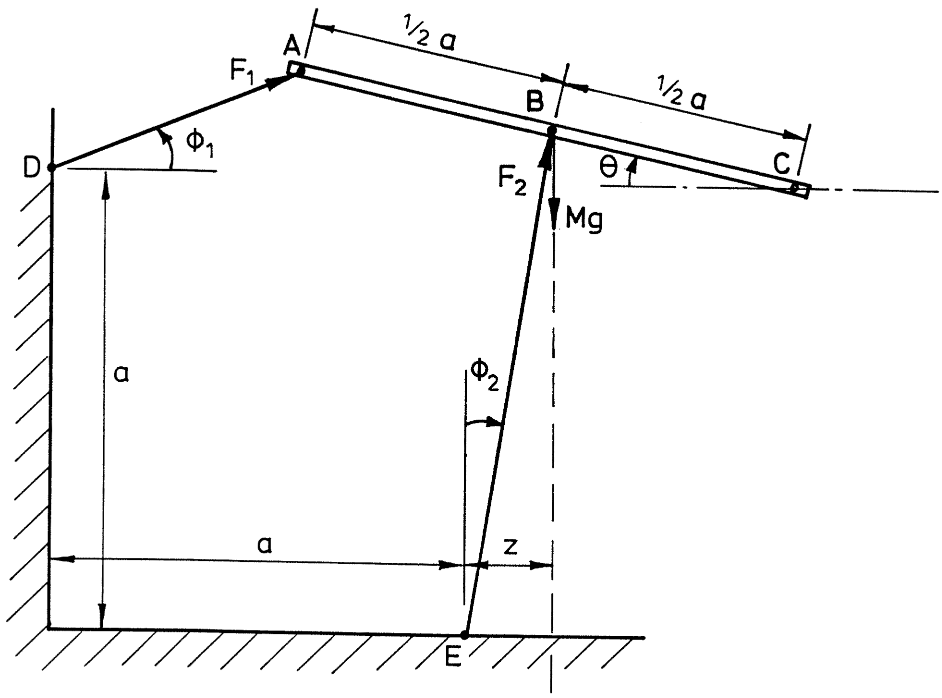


Fig. 2

Note that if the system is controllable this just implies that the state can pass through any value; it does not imply that there is an input to keep the state at this value (which depends on the equilibrium conditions):

Controllability from both inputs: $\dot{x} = Ax + Bu$,

Controllable? $\quad \text{rank}[B, AB, A^2B, A^3B] = 4$? $\qquad (n = 4)$

Controllability from u_1 alone: $\dot{x} = Ax + b_1u_1$, $\qquad (B = [b_1, b_2])$

Controllable? $\quad \text{rank}[b_1, Ab_1, A^2b_1, A^3b_1] = 4$?

Achievable steady-state (equilibrium) behaviour:

With both inputs:
$\dot{x} = Ax + Bu$, $\dot{x}_e = 0$ at equilibrium, so: $0 = Ax_e + Bu_e$ possible for some u_e?

With input u_1 only: $0 = Ax_e + b_1u_{1e}$ possible for some u_{1e}?

1.2 Minimum Energy Input

Theorem 1.1 *The input,* $\underline{u}(t) = \underline{u}_o(t) = B^T e^{A^T(t_1-t)} W_c(t_1)^{-1} \underline{x}_1$*, takes the state from* $\underline{x}(0) = \underline{0}$ *to* $\underline{x}(t_1) = \underline{x}_1$ *and in addition is the input with minimum energy that achieves this.*

Proof:

Let $\underline{u}(t) = \underline{u}_o(t) + \underline{u}_1(t)$ then $\underline{x}(t_1) = \underline{x}_1 + \int_0^{t_1} e^{A(t_1-t)} B \underline{u}_1(t)\, dt$ and hence $\underline{x}(t_1) = \underline{x}_1$ implies,

$$\int_0^{t_1} e^{A(t_1-t)} B \underline{u}_1(t)\, dt = \underline{0}$$

Energy in $\underline{u}(t)$ for $0 < t < t_1$ is defined as:

$$\begin{aligned} \int_0^{t_1} \|\underline{u}(t)\|^2 dt &= \int_0^{t_1} \underline{u}(t)^T \underline{u}(t)\, dt \\ &= \int_0^{t_1} \left(\underline{u}_o(t)^T \underline{u}_o(t) + \underline{u}_o(t)^T \underline{u}_1(t) + \underline{u}_1(t)^T \underline{u}_o(t) + \underline{u}_1(t)^T \underline{u}_1(t) \right) dt \end{aligned}$$

Now

$$\int_0^{t_1} \underline{u}_o(t)^T \underline{u}_1(t)\, dt = \int_0^{t_1} \underline{x}_1^T W_c(t_1)^{-1} e^{A(t_1-t)} B \underline{u}_1(t)\, dt = 0 = \int_0^{t_1} \underline{u}_1(t)^T \underline{u}_o(t)\, dt$$

and

$$\int_0^{t_1} \underline{u}_o(t)^T \underline{u}_o(t)\, dt = \underline{x}_1^T W_c(t_1)^{-1} \int_0^{t_1} e^{A(t_1-t)} B B^T e^{A^T(t_1-t)}\, dt\, W_c(t_1)^{-1} \underline{x}_1 = \underline{x}_1^T W_c(t_1)^{-1} \underline{x}_1$$

Hence

$$\int_0^{t_1} \underline{u}(t)^T \underline{u}(t)\, dt = \underline{x}_1^T W_c(t_1)^{-1} \underline{x}_1 + \int_0^{t_1} \underline{u}_1(t)^T \underline{u}_1(t) dt$$

Since both terms are ≥ 0 the minimum energy is achieved when $\underline{u}_1(t) = \underline{0}$ and hence $\underline{u}(t) = \underline{u}_o(t)$ when

$$\min \int_0^{t_1} \underline{u}(t)^T \underline{u}(t)\, dt = \underline{x}_1^T W_c(t_1)^{-1} \underline{x}_1.$$

Note that if $W_c(t_1)$ is nearly singular then a very large energy input is required to reach certain states.

Example

$$\begin{aligned}\underline{\dot{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u, \;\Rightarrow e^{At}B = \begin{bmatrix} t \\ 1 \end{bmatrix}, \\ \Rightarrow W_c(t_1) &= \int_0^{t_1} \begin{bmatrix} t^2 & t \\ t & 1 \end{bmatrix} dt = \begin{bmatrix} \frac{1}{3}t_1^3 & \frac{1}{2}t_1^2 \\ \frac{1}{2}t_1^2 & t_1 \end{bmatrix} \\ W_c(t_1)^{-1} &= \begin{bmatrix} 12/t_1^3 & -6/t_1^2 \\ -6/t_1^2 & 4/t_1 \end{bmatrix}\end{aligned}$$

Minimum energy to go from $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is hence $12/t_1^3$ achieved when

$$\underline{u}(t) = \left(e^{A(t_1-t)}B\right)^T W_c(t_1)^{-1}\underline{x}_1 = \begin{bmatrix} t_1 - t & 1 \end{bmatrix}\begin{bmatrix} 12/t_1^3 \\ -6/t_1^2 \end{bmatrix} = \frac{6}{t_1^3}(t_1 - 2t)$$

1.3 Reachable States and Minimal Realizations

We have seen in the previous section that if $W_c(t_1)$ is nearly singular then some directions in the state space are very difficult to reach, and if $W_c(t_1)$ is singular then some states cannot be reached and that $\underline{x}(t_1)$ is necessarily perpendicular to the null space of $W_c(t_1)$. It can in fact be shown *(details are omitted)* that the states that can be reached at time t_1 from $\underline{x}(0) = \underline{0}$ are precisely of the form:

$$\begin{aligned} \text{Reachable states} \quad &= \quad \text{Range space of } W_c(t_1) \\ &= \quad \text{Range space of } P = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \end{aligned}$$

(since null spaces of P^T and $W_c(t_1)$ are the same,

Definition 1.2 *A set of state equations given by (A, B, C, D) is called a minimal realization of its transfer function, $G(s) = D + C(sI - A)^{-1}B$, if there does not exist a state space realization of $G(s)$ with a lower state dimension.*

In section 3.2 of Lecture Notes 3 we saw that if a system was not observable then there was a change of state coordinates that gave an observable realisation of the transfer function with r states where $r = \text{rank}(Q)$.

If this system with r states is not controllable its state dimension could be further reduced in a similar manner and we are left with a state-space realisation of the transfer function that is both controllable and observable. It turns out that *(proof omitted)*:

Theorem 1.3 *A realization is minimal if and only if it is both controllable and observable.*

In single-input/single-output systems this means that if a system is either not controllable or not observable then there are pole/zero cancellations in the transfer function.

Example:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1,\ 1], \quad D = 0$$

Observability:

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \Rightarrow \text{rank}(Q) = 1 \Rightarrow \text{NOT observable}$$

$$Qx_0 = 0? \quad \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ so } x = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} \text{ is not observable.}$$

Controllability:

$$P = [B,\ AB] = \begin{bmatrix} & \\ & \end{bmatrix} \Rightarrow \text{rank}(P) = 2 \Rightarrow \text{controllable}$$

Example continued

Transfer function:

$$G(s) = C(sI - A)^{-1}B + D = [1,\ 1]\begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} =$$

$$= \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2} \qquad \text{pole-zero cancellation}$$

2 State Feedback

The response of a system is largely determined by the location of its closed loop poles. Can state feedback assign the closed loop poles?

System: $\dot{\underline{x}} = A\underline{x} + B\underline{u}$, with state feedback: $\underline{u} = -K\underline{x} + M\underline{r}$, giving closed loop:

$$\dot{\underline{x}} = (A - BK)\underline{x} + BM\underline{r}.$$

Theorem 2.1 *The closed loop poles will be the eigenvalues of* $(A - BK)$ *which can be placed arbitrarily by choice of* K *if and only if* (A, B) *is controllable.*

(This is entirely analogous to the statement in section 3.2 of Lecture Notes 3, that the eigenvalues of $(A - LC)$ can be arbitrarily assigned by choice of L —
if (A, C) is observable).

Ackerman's Formula: for K.

Let

$$r(s) = r_0 I + r_1 s + r_2 s^2 + \cdots + r_{n-1} s^{n-1} + s^n$$

and

$$r(A) = r_0 I + r_1 A + r_2 A^2 + \cdots + r_{n-1} A^{n-1} + A^n$$

In the single input case, let

$$K = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} P^{-1} r(A)$$

Then the eigenvalues of $A - BK \equiv$ roots of $r(s)$

proof (for interest): on moodle

In the multi-input case: If a system is controllable, it can be shown that it is always possible to choose a column of B, b_k, and a matrix N such that the pair $A + BN$, b_k is observable.

Where to place the poles?

- stable
- fast enough
- but not too fast since this might
  - saturate actuators
  - give poor stability margins.

2.1 Steady-State Gain

Servo-system. Suppose we want $\underline{y}(t) \to \underline{r}$.

Two approaches to obtain the correct DC gain:

(a) **Choice of** M

$$\begin{cases} \dot{\underline{x}} &= (A - BK)\underline{x} + BM\underline{r} \\ \underline{y} &= C\underline{x} \end{cases}$$

In steady-state: $\dot{\underline{x}} = \underline{0} \Rightarrow \underline{y} = C(-A + BK)^{-1}BM\underline{r}$.

Choose M such that $C(-A + BK)^{-1}BM = I$ and $\underline{y}(t) \to \underline{r}$ after a step change with speed given by eigenvalues of $(A - BK)$. [Such an M usually exists if $\dim(\underline{u}) \geq \dim(\underline{y})$ but not otherwise].

This requires *exact knowledge* of the system matrices. The steady-state error being zero is not *robust* to small changes in the system. Also need to know an equilibrium condition.

(b) **Integral Action**

Integral action can be incorporated by augmenting the state by the integral of the error, i.e.

$$\underline{\dot{e}} = \underline{r} - \underline{y} = \underline{r} - C\underline{x}$$

which gives

$$\begin{bmatrix} \underline{\dot{x}} \\ \underline{\dot{e}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \underline{u} + \begin{bmatrix} 0 \\ I \end{bmatrix} \underline{r}$$

with state feedback: $\underline{u} = -K_1\underline{x} - K_2\underline{e}$.

Choose K_1, K_2 to assign the closed-loop poles (possible if augmented system controllable) and then $\underline{e}(t) \to 0 \Rightarrow \underline{y}(t) \to \underline{r}$ after a step change.
Robust to small changes in A, B, C, K**.**
Does not require knowledge of the equilibrium condition.

3 Observers with State Feedback

$$\text{SYSTEM} \quad \begin{cases} \underline{\dot{x}} = A\underline{x} + B\underline{u} \\ \underline{y} = C\underline{x} \end{cases}$$

$$\text{OBSERVER} \quad \begin{cases} \underline{\dot{\hat{x}}} = A\underline{\hat{x}} + B\underline{u} + L(\underline{y} - \underline{\hat{y}}) = (A - LC)\underline{\hat{x}} + B\underline{u} + L\underline{y} \\ \underline{\hat{y}} = C\underline{\hat{x}} \end{cases}$$

$$\text{CONTROLLER} \quad \left\{ \underline{u} = -K\underline{\hat{x}} + M\underline{r} \right.$$

$$\begin{aligned} \text{Error:} \quad \underline{e} &= \underline{x} - \underline{\hat{x}} \\ \underline{\dot{e}} &= (A - LC)\underline{e} \\ \underline{u} &= -K(\underline{x} - \underline{e}) + M\underline{r} \\ \underline{\dot{x}} &= (A - BK)\underline{x} + BK\underline{e} + BM\underline{r} \end{aligned}$$

$$\Rightarrow \begin{cases} \begin{bmatrix} \dot{\underline{x}} \\ \dot{\underline{e}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix} + \begin{bmatrix} BM \\ 0 \end{bmatrix} \underline{r} \\ \underline{y} = [C \;\; 0] \begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix} \end{cases}$$

NB: Eigenvalues of $\begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ = {Eigenvalues of X} $\cup$ {Eigenvalues of Z}

So closed-loop poles are at the eigenvalues of $(A - BK)$ and those of $(A - LC)$.

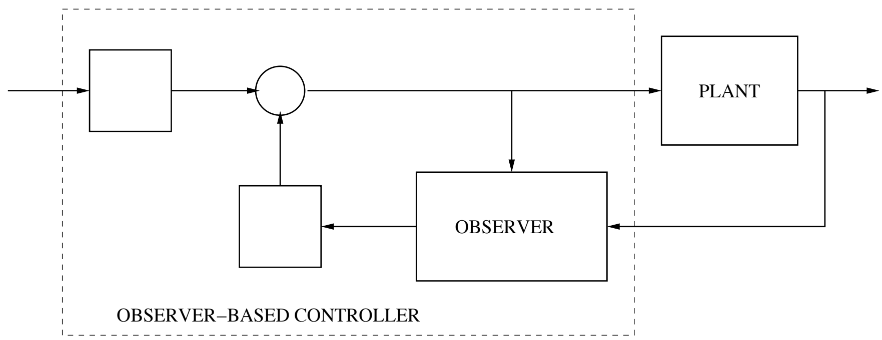
$\underline{e}$ is not affected by $\underline{r}$ so that $\underline{e}(t) \to \underline{0}$.

Separation of estimation and control.

Can this always be done?

If (A, B) is controllable and (A, C) is observable, then no problems —
we can place all eigenvalues anywhere we want.
If all uncontrollable and unobservable modes (states) are stable, may still be OK.
If any uncontrollable or unobservable modes are unstable, then NOT OK,
since they will remain in the closed-loop system.

Block diagram:



If $\underline{r} \equiv 0$ then this structure is the same as for a dynamic precompensator.

For $\underline{r} \not\equiv 0$ the structures are different.

4 State Feedback Design Example

Plant $G(s) = \frac{1}{s+2} \times \frac{1}{s+1}$

Design Spec

Response in y to a step command on r to have zero offset and small overshoot.

State Feedback Design

First formulate the state space equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \text{PLANT}$$

We will again need to add an integrator to ensure a zero offset. The state feedback formulation will now be

The extra state variable x_3 has been added to integrate the output error -

$$\dot{x}_3 = -x_1 + r$$

this gives an augmented set of state equations

$$\frac{d}{dt}\underline{x} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

and the proposed feedback scheme is given by $u = -\underline{k}^T\underline{x} = -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}\underline{x}$, so the closed loop state equations become

$$\underline{\dot{x}} = (A - B\underline{k}^T)\underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r = \begin{bmatrix} -2 & 1 & 0 \\ -k_1 & -1-k_2 & -k_3 \\ -1 & 0 & 0 \end{bmatrix}\underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

The closed loop characteristic equation becomes

$$\begin{aligned} \det[\lambda I - (A - B\underline{k}^T)] &= \det\begin{bmatrix} \lambda+2 & -1 & 0 \\ k_1 & \lambda+1+k_2 & +k_3 \\ 1 & 0 & \lambda \end{bmatrix} = (\lambda+2)(\lambda+1+k_2)\lambda + k_1\lambda - k_3 \\ &= \lambda^3 + (3+k_2)\lambda^2 + (2+k_1+2k_2)\lambda - k_3 \end{aligned}$$

Suppose we desired all the closed loop poles to be at -5, then the required characteristic equation would be:

$$(\lambda+5)^3 = \lambda^3 + 15\lambda^2 + 75\lambda + 125$$

Equating coefficients now gives

$$3 + k_2 = 15 \Rightarrow \underline{k_2 = 12}, \;\; 2 + k_1 + 2k_2 = 75 \Rightarrow \underline{k_1 = 49}, \;\; \underline{k_3 = -125}$$

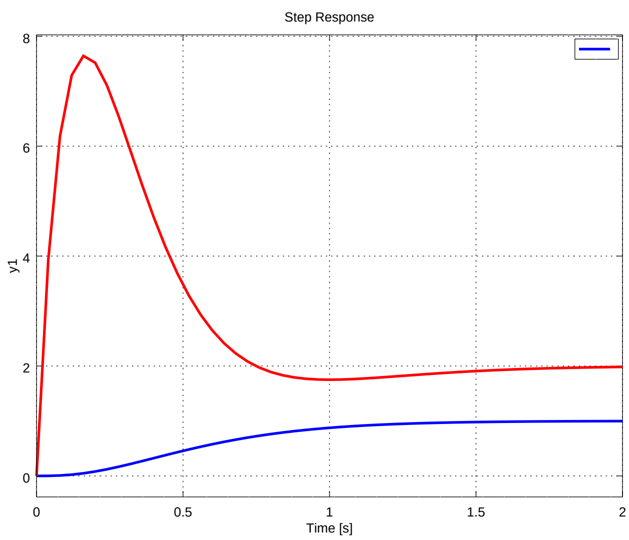
The transfer function from r to y can now be computed as:-

$$T_{R(s)\to Y(s)} = C[sI-(A-Bk^T)]^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = [1\ \ 0\ \ 0]\begin{bmatrix}s+2 & -1 & 0\\ 49 & s+13 & -125\\ 1 & 0 & s\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{125}{(s+5)^3}$$

Also the transfer function from r to u can be computed as:-

$$T_{R(s)\to U(s)} = -\frac{\underline{k}^T\underline{X}(s)}{R(s)} = [49\ \ 12\ \ -125]\begin{bmatrix}s+2 & -1 & 0\\ 49 & s+13 & -125\\ 1 & 0 & s\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{125(s+1)(s+2)}{(s+5)^3}$$

The step responses are thus:-



Note: If the second state is not measured, and we use an observer based on the output (the 1st state) and the error (3rd state), then the response to the reference is *identical* to this, as the error dynamics are not excited. If there are unmeasured disturbances however, the response with state feedback and with an observer based controller are different.

5 Lego Segway revisited

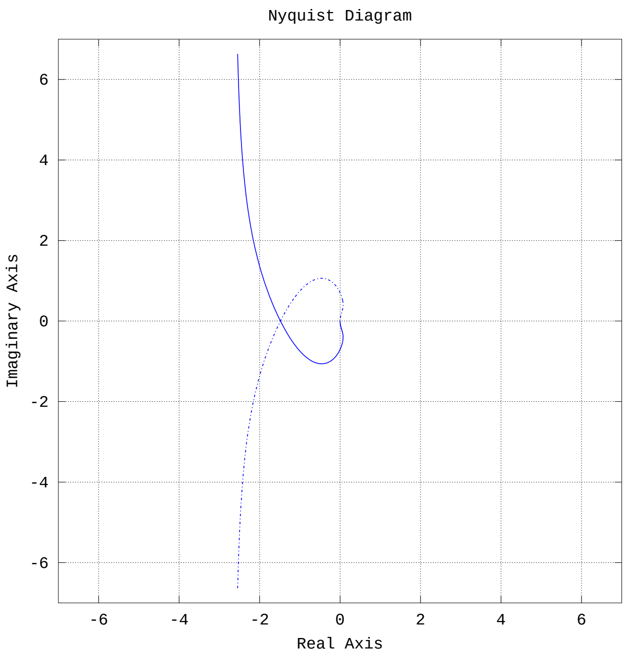
```
m1=.062;
m2=0.58;
L=.12;
r=.04;
I1=m1*.035^2;
I2=m2*0.2^2/12;
%x=[psidot;thetadot;psi;theta]

Jp=[I1+(m1+m2)*r^2   m2*L*r ;
     m2*r*L        I2+m2*L^2]
J=Jp*[1 1;0 1];
z=zeros(2,2);
A=[J\z J\[0 0;0 m2*9.8*L];eye(2) z];
B=[J\[1;-1];0;0];
A=A-B*[.15/(440*pi/180) 0 0 0] %motor back emf, 15Ncm at 440deg/s (50% power)
kp=35.0;kd=300.0;kp1=0.8;kd1=350.0;
cal=1000*0.23/370;%degs to errg for gyro
al=.995;




G=ss(A,B,eye(4),zeros(4,1));
F=ss(d2c(kd1*tf([1-al -(1-al)],[1 -al],1e-3)+kp1));
K=0.17*ss(F.a,[0 0 F.b 0],F.c,[0 kd/1000*cal F.d kp*cal]);
%0.17 is 30/100/100 * 180/pi, converts %power per deg to Nm per rad
% motor produces 30Ncm=30/100Nm at 100% power.

figure(1)
nyquist(-K*G);
axis(7*[-1 1 -1 1]);axis equal;grid on
```



5.1 Minimum energy translation