IB Paper 6: Information Engineering

COMMUNICATIONS

Examples Paper 9: Digitisation, Digital Modulation, Multiple Access

1. (a) An ADC with a -1 to +1 volt signal range and 5-bit resolution is connected to a matching DAC. Given input $x(kT)$ volts, the ADC outputs integer code value m $(-16 \le m \le +15)$ such that $m/16$ is the nearest multiple of $1/16$ to $x(kT)$ and the DAC output voltage is then $m/16$V. For example, input sample $x(kT) = 0.1$V gives ADC output code $m = 2$ and output $2/16 = 0.125$V.

   The system is first tested using the signal $x(kT) = 0.9\sin(0.1k\pi)$. [The values of $x(kT)$ for $k = 0, 1, \ldots, 5$ are therefore 0, 0.2781, 0.5290, 0.7281, 0.8560, 0.9000]. It is then tested with a second signal $x_2(kT) = 0.1x(kT)$.

   Compute the actual mean-squared quantisation error which results in each case.

   (b) Now suppose that the same signals are digitised instead using a companded ADC and matching DAC. These preserve the sign of each input sample $x(kT)$ but cause the *magnitude* of $x(kT)$ to be replaced by the nearest value from the following list:

   0, 0.0280, 0.0561, 0.0841, 0.1122, 0.1346, 0.1615, 0.1938, 0.2326, 0.2791, 0.3349,

   0.4019, 0.4823, 0.5787, 0.6944, 0.8333

   Again, compute the mean-squared quantisation error for the two test signals.

   (c) Re-express the results of (a) and (b) as Signal-to-Noise ratios in dB.

   [A note, for information only: the first five quantisation levels in part (b) are linearly spaced, with spacing 0.028, while the remaining values are in a geometric progression, with each value a factor of 1.2 larger than its predecessor].

2. Consider the unit-energy sinc pulse

$$p(t) = \sqrt{\frac{1}{T}}\,\mathrm{sinc}\left(\frac{\pi t}{T}\right).$$

   For any integer k, define $\phi_k(t) = p(t - kT)$. Show that the signals $\{\phi_k(t)\}$ are orthonormal, i.e.,

$$< \phi_\ell, \phi_m > := \int_{-\infty}^{\infty} \phi_\ell(t)\phi_m(t)\, dt = \begin{cases} 1 & \text{if } \ell = m, \\ 0 & \text{otherwise} \end{cases}$$

   Hint: Use the Multiplication Theorem of Fourier Transforms in Signal and Data Analysis Handout 4.

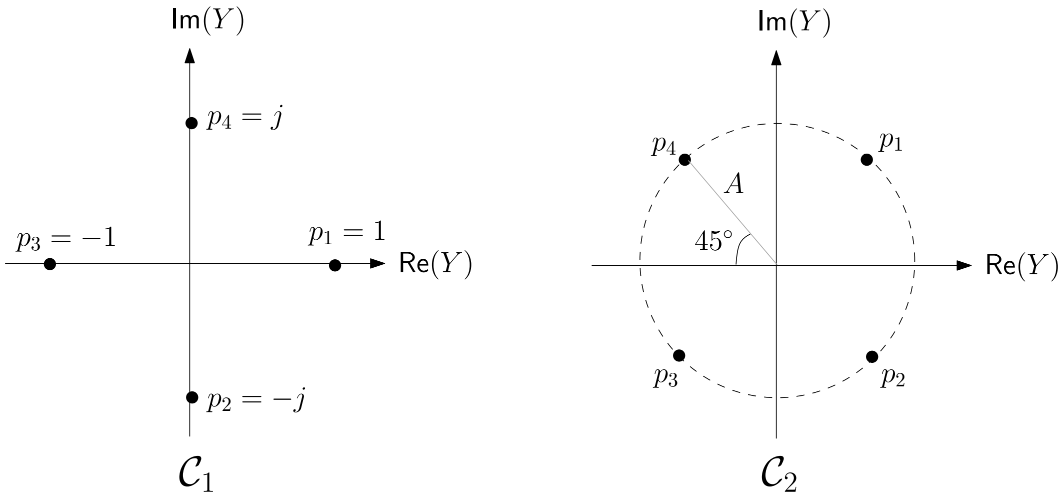


Figure 1: Two QPSK constellations

3. Consider Pulse Amplitude Modulation (PAM), where the information symbols $X_1, X_2, \ldots$ modulate a pulse $p(t)$ to produce the baseband waveform

$$X(t) = \sum_k X_k \, p(t - kT).$$

Suppose that each symbol X_k is drawn from the set $\{-3A, -A, \ A, \ 3A\}$. Assume that each X_k is equally likely to be any of the four symbols in the set. The waveform $X(t)$ is transmitted over a baseband AWGN channel, and the discrete-time received sequence is

$$Y_k = X_k + N_k,$$

where N_k is additive Gaussian noise with mean zero and variance σ^2.

(a) Sketch the decision regions that minimise the probability of detection error.

(b) Obtain the probability of detection error when the transmitted symbol is $-3A$. Note that the probability of detection error is the same when the symbol $+3A$ is transmitted.

(c) Obtain the probability of error when the transmitted symbol is $-A$ (or A). Combine this with part (b) to obtain an expression for the overall probability of error P_e.

(d) What is the average energy per symbol in terms of A? What is E_b, the average energy per bit?

(e) Express the probability of error in terms of the ratio E_b/σ^2.

4. *Quadrature Amplitude Modulation (QAM)*

Figure 1 shows two different QPSK constellations $\mathcal{C}_1$ and $\mathcal{C}_2$. For each constellation, pairs of bits are mapped to constellation symbols according to the rule

$$00 \to p_1, \quad 01 \to p_2, \quad 11 \to p_3, \quad 10 \to p_4.$$

The modulated complex-baseband waveform is $x_b(t) = \sum_k X_k p(t - kT)$, where X_k are symbols from the chosen constellation, and $p(t)$ is a unit-energy rectangular pulse (which is non-zero in $[0, T)$ and zero elsewhere). The passband QAM waveform is generated as

$$x(t) = \mathsf{Re}(x_b(t)) \cos(2\pi f_c t) - \mathsf{Im}(x_b(t)) \sin(2\pi f_c t).$$

(a) If the average energy per symbol of the two constellations are to be equal, specify the (complex) values of the constellation symbols in $\mathcal{C}_2$.

(b) For each constellation, sketch the (baseband) waveforms modulating the carriers $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$, for the following sequence of information bits: 11 00 10 01

(c) The passband waveform $x(t)$ is transmitted over an Additive White Gaussian Noise channel. Briefly describe the steps at the receiver to recover the transmitted symbols. Sketch the decision regions for constellations $\mathcal{C}_1$ and $\mathcal{C}_2$, assuming that the constellation symbols are equally likely.

5. *QAM Spectrum*

Fig. 2 shows the spectrum of a baseband signal $x_b(t)$. The passband QAM signal $x(t)$ is generated from $x_b(t)$ as follows.

$$x(t) = \mathsf{Re}\left[x_b(t) e^{j2\pi f_c t}\right].$$

(a) Use the spectrum $X_b(f)$ to show that $x_b(t)$ is a complex-valued signal in time-domain. (*Hint*: Recall that the spectrum of a real signal $s(t)$ must satisfy $S(-f) = S^*(f)$.)

(b) Find $X(f)$, the spectrum of $x(t)$, in terms of $X_b(f)$. Sketch $X(f)$ and verify that $X(-f) = X^*(f)$, confirming that $x(t)$ is real-valued.
(*Hint*: Note that for a complex number a, $\mathsf{Re}[a] = \frac{a+a^*}{2}$. Use this to express $x(t)$ in terms of $x_b(t)$ and its conjugate; then use properties of the Fourier transform.)

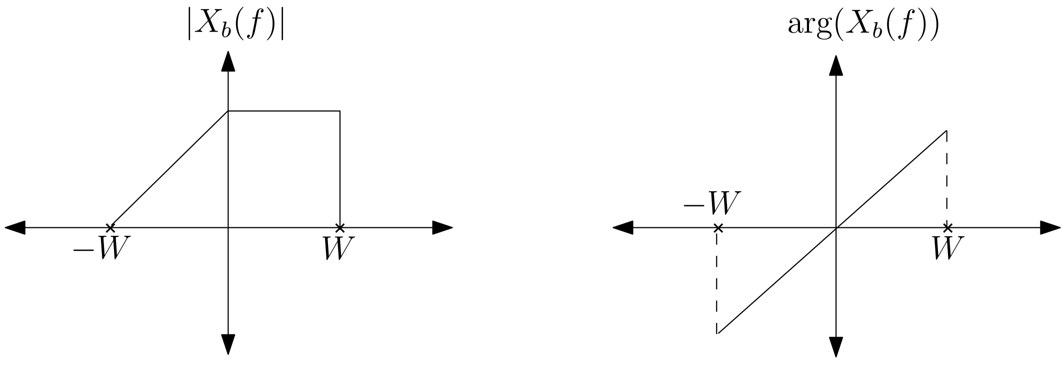


Figure 2: Spectrum $X_b(f)$

6. *Error-Correcting Codes*

(a) Consider a repetition code in which each information bit is repeated five times and transmitted over a binary symmetric channel (BSC). Assuming the BSC has crossover probability ϵ, what is the probability of bit error? What is the rate of the code?

(b) Now consider a $(7, 4)$ Hamming code which maps $k = 4$ information bits to a length $n = 7$ codeword.

i) Suppose that a codeword is transmitted over a BSC, and the received sequence is $\mathbf{r} = [1\ 1\ 0\ 1\ 0\ 1\ 1]$. Decode the received sequence to a codeword.

ii) Now suppose that the all-zeros codeword $[0\ 0\ 0\ 0\ 0\ 0\ 0]$ is transmitted and the received sequence is $[0\ 0\ 1\ 0\ 0\ 1\ 0]$, i.e., the channel has flipped two bits. Decode the received sequence to a codeword, and observe that the decoded codeword is not the transmitted one. This is because the Hamming code can only correct a single bit error.

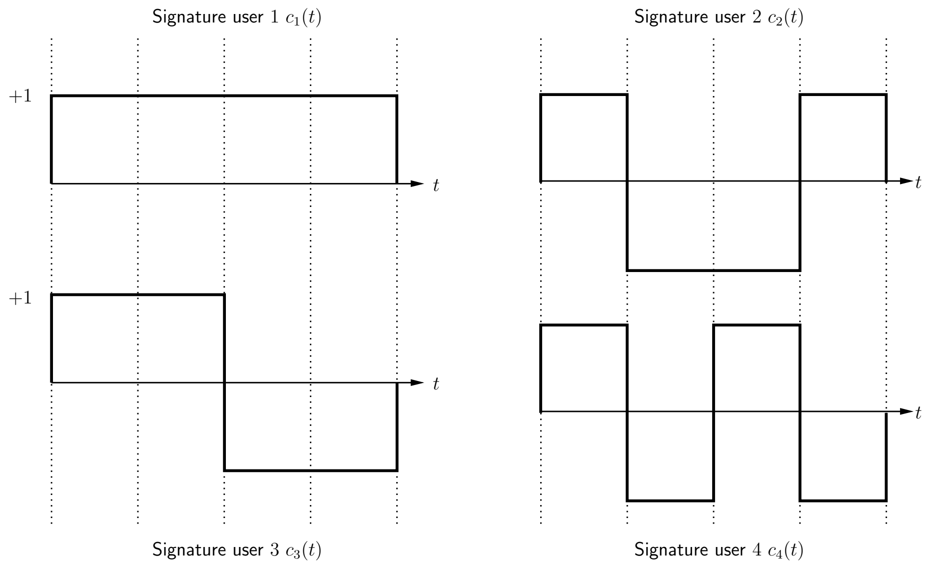
iii) When a Hamming code is used, a decoding error occurs if the channel flips two or more of the transmitted bits. Compute the probability of decoding error when a 7-bit Hamming codeword is transmitted over a BSC with crossover probability ϵ.

7. Consider a multiple-access channel with K users and a total bandwidth B.

(a) Explain how FDMA, TDMA and CDMA work, and outline the main differences between the three.

(b) How many users can be accommodated in an FDMA system with total bandwidth 20MHz, if each user employs binary Pulse Amplitude Modulation (± 1 symbols) with rectangular pulses at a rate of $R = 200$kbit/s? (assume that the carrier frequency is $\gg 20$ MHz, and that the band-pass spectrum of each user does not cause interference beyond the first side lobe).

(c) Show that the signature signals in the figure below are orthogonal in a CDMA system with $K = 4$ users.



Answers:

1. The exact numerical results will depend on whether you calculated $x(kT) = 0.9\sin(0.1k\pi)$ or used the rounded values given in the question;

   (a) MSE for $x(kT)$: 5.57×10^{-4}: MSE for $x_2(kT)$: 3.77×10^{-4}

   (b) MSE for $x(kT)$: 1.2×10^{-3}: MSE for $x_2(kT)$: 3.15×10^{-5}

   (c) SNRs: linear ADC: 28.6 dB, 10.3dB; companded ADC: 25.2 dB, 21.1 dB.

2.

3. b) $\mathcal{Q}\left(\frac{A}{\sigma}\right)$; c) $2\mathcal{Q}\left(\frac{A}{\sigma}\right)$, overall $P_e = \frac{3}{2}\mathcal{Q}\left(\frac{A}{\sigma}\right)$, d) $E_s = 5A^2$, $E_b = 2.5A^2$, e) $\frac{3}{2}\mathcal{Q}\left(\sqrt{\frac{2E_b}{5\sigma^2}}\right)$

4.

5. b) $X(f) = \frac{1}{2}\left[X_b(f - f_c) + X_b^*(-f - f_c)\right]$

6.

7. b) 25 users