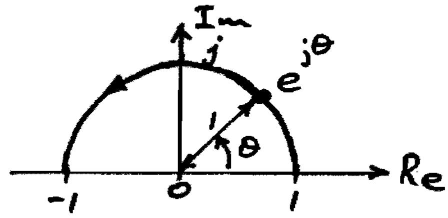
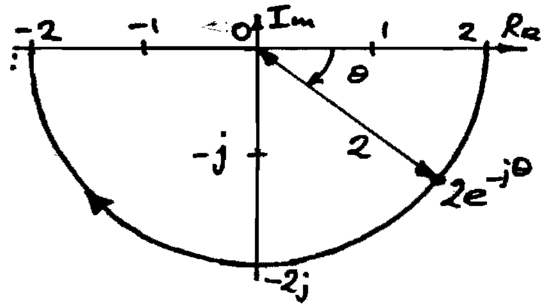
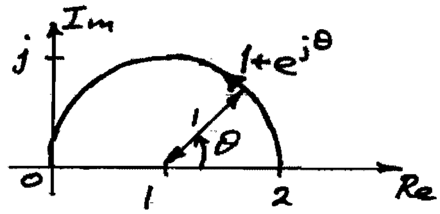
~~1 (a)~~ $|e^{j\theta}| = 1$, $\arg(e^{j\theta}) = \theta$:
(also note $e^{j0} = 1$, $e^{j\pi/2} = j$, $e^{j\pi} = -1$ etc)



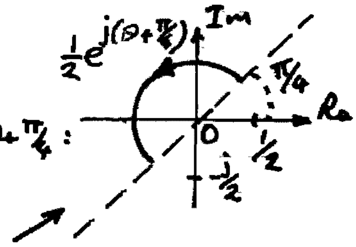
$|2e^{-j\theta}| = 2$, $\arg(2e^{-j\theta}) = -\theta$:
(also note $2e^{-j0} = 2$, $2e^{-j\pi/2} = -2j$,
$2e^{-j\pi} = -2$ etc)



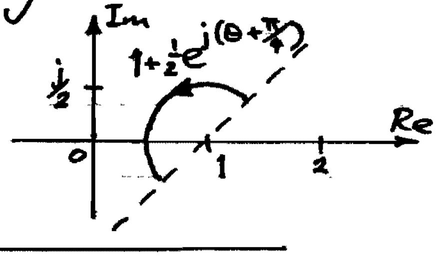
$1 + e^{j\theta}$ is just the same as $e^{j\theta}$, but shifted by +1 along the real axis.



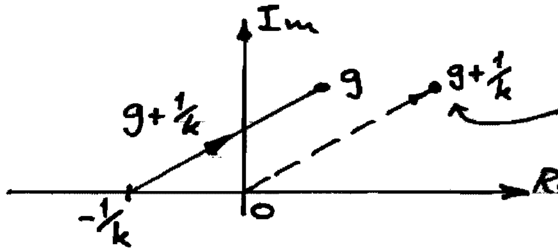
For $1 + \frac{1}{2}e^{j(\theta + \pi/4)}$, consider first $\frac{1}{2}e^{j(\theta+\pi/4)}$ alone: $\left|\frac{1}{2}e^{j(\theta+\pi/4)}\right| = \frac{1}{2}$, $\arg\left(\frac{1}{2}e^{j(\theta+\pi/4)}\right) = \theta + \frac{\pi}{4}$:



Now get $1 + \frac{1}{2}e^{j(\theta+\pi/4)}$ by shifting this by +1 along the real axis:



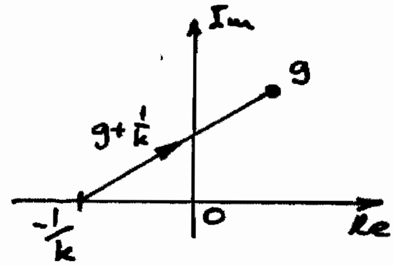
(b) (i)



This is the point $g + 1/k$. But for most purposes only the magnitude and argument matter (at least in this course).

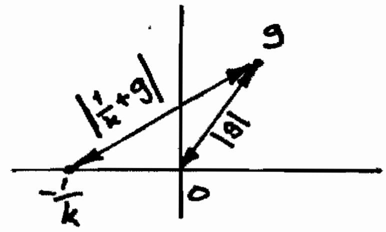
1(b) cont'd

(ii) $\left|\dfrac{k}{1+kg}\right| = \left|\dfrac{1}{\frac{1}{k}+g}\right| = \dfrac{1}{\left|\frac{1}{k}+g\right|}$

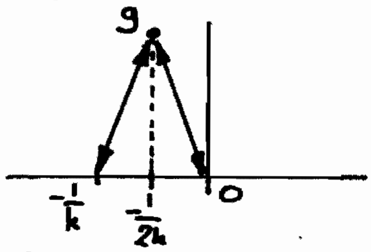


So just measure the length of the 'vector' $g+\frac{1}{k}$, then take its reciprocal.

(iii) $\left|\dfrac{kg}{1+kg}\right| = \left|\dfrac{g}{\frac{1}{k}+g}\right| = \dfrac{|g|}{\left|\frac{1}{k}+g\right|}$

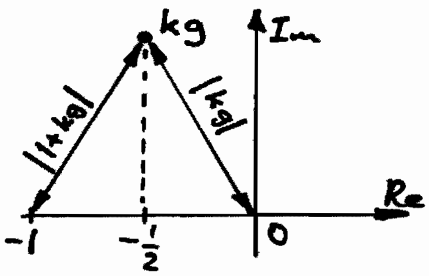


So the two lengths shown in the diagram must be equal. Hence the diagram must look like this:



Hence $\underline{Re(g) = -\frac{1}{2k}}$.

(Alternative: $\left|\dfrac{kg}{1+kg}\right| = \dfrac{|kg|}{|1+kg|}$, so draw the following diagram:

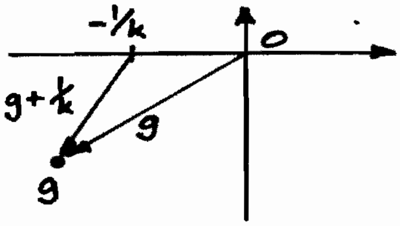


Hence

$\underline{Re(kg) = -\frac{1}{2}}$.)

(The two conditions are equivalent, since k is real and positive.)

Note: For control system analysis we usually have $Re(g)<0$ and $Im(g)<0$;



2. (a) To get Nyquist diagrams consider $G(j\omega)$.

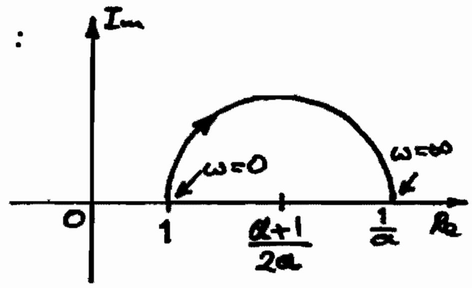
$\omega=0$: $G(j0) = 1$

$\omega = \frac{1}{T}$: $G(j\frac{1}{T}) = \dfrac{1+j}{1+ja}$. $\therefore \left|G(j\frac{1}{T})\right| = \dfrac{\sqrt{2}}{\sqrt{1+a^2}} > 1$ if $a<1$.

$\arg G(j\frac{1}{T}) = \arg(1+j) - \arg(1+ja) = \frac{\pi}{4} - \tan^{-1} a > 0$ if $a<1$.
↖ Phase lead

$\omega=\infty$: $G(j\infty) = \lim\limits_{\omega\to\infty} \dfrac{1+j\omega T}{1+j\omega aT} = \dfrac{1}{a} > 1$ if $a<1$.

Hence, using the 'note', we have:
(NB This transfer function provides 'phase lead', or 'phase advance'. Note that its gain increases with frequency.)

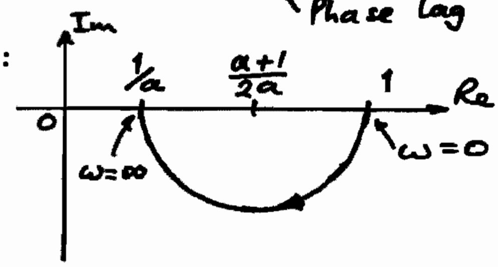


(b) With $a>1$ the calculations go as above, but we get:

$\left|G(j\frac{1}{T})\right| < 1$ if $a>1$, $\arg G(j\frac{1}{T}) < 0$ if $a>1$.
↖ Phase lag

$G(j\infty) = \frac{1}{a} < 1$ if $a>1$.

The diagram becomes:
(NB This is a 'phase lag'. Its gain decreases with frequency.)



(c) Since $0<c<1$, the roots of $1+2csT+s^2T^2$ are complex, so we can't write it as the product of 2 real factors.

$\omega=0$: $G(j0) = 1$. $\qquad G(j\omega) = \dfrac{1}{(1-\omega^2T^2)+2cj\omega T}$

$\omega = \frac{1}{T}$: $G(j\frac{1}{T}) = \dfrac{1}{0+2cj}$

so $\left|G(j\frac{1}{T})\right| = \dfrac{1}{2c}$ and $\arg G(j\frac{1}{T}) = -\frac{\pi}{2}$ (rad).
(>1 if $c<\frac{1}{2}$ ← RESONANCE)

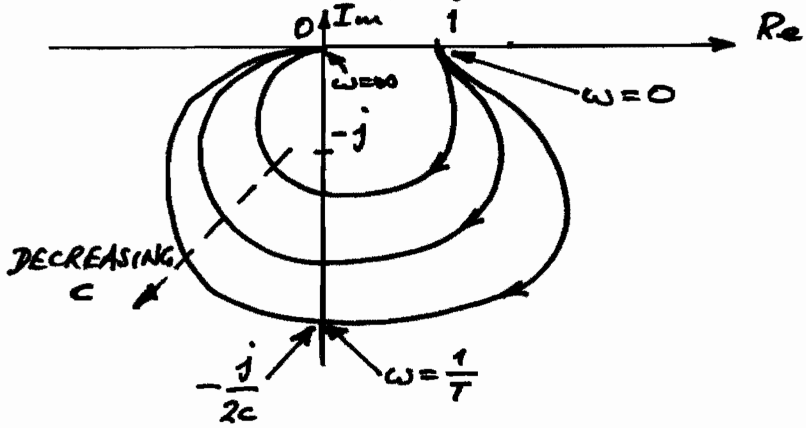
2(c) cont'd.

$G(j\infty) = 0$, but more usefully:

$$\lim_{\omega\to\infty} G(j\omega) = \lim_{\omega\to\infty} \frac{1}{(1-\omega^2T^2)+2cj\omega T} = \lim_{\omega\to T} \frac{1}{(-\omega^2T^2)}$$

so $\lim_{\omega\to\infty} \arg G(j\omega) = -\pi$ (i.e. diagram approaches 0 along −ve real axis.)

We get a whole family of curves, depending on c:



As c (damping factor) decreases, so the gain at resonance increases. Bode plots reveal that the phase change occurs over a smaller range of frequencies as c decreases.

(d) The main point here is not to repeat all of (c).

Note that $G(s) = \frac{1}{sT} \times$ (transfer function in (c))

so $|G(j\omega)| = \frac{1}{\omega T} \times$ (gain found in (c))

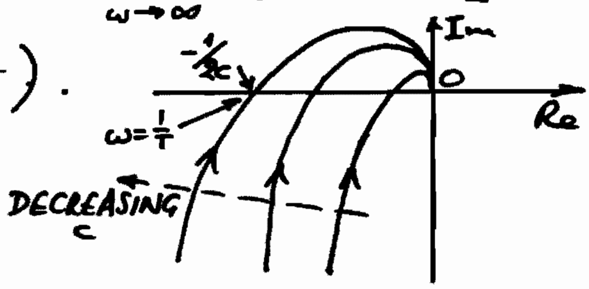
and $\arg G(j\omega) = -\frac{\pi}{2} +$ (arg found in (c)).

In particular, $|G(j0)| = \infty$, $\arg G(j0) = -\pi/2$.

$|G(j\frac{1}{T})| = 1 \times \frac{1}{2c} = \frac{1}{2c}$, $\arg G(j\frac{1}{T}) = -\frac{\pi}{2} - \frac{\pi}{2} = -\pi$.

$\lim_{\omega\to\infty} G(j\omega) = \lim_{\omega\to\infty} \frac{1}{-j\omega^3T^3}$, so $\lim_{\omega\to\infty} \arg G(j\omega) = -\frac{3\pi}{2}$

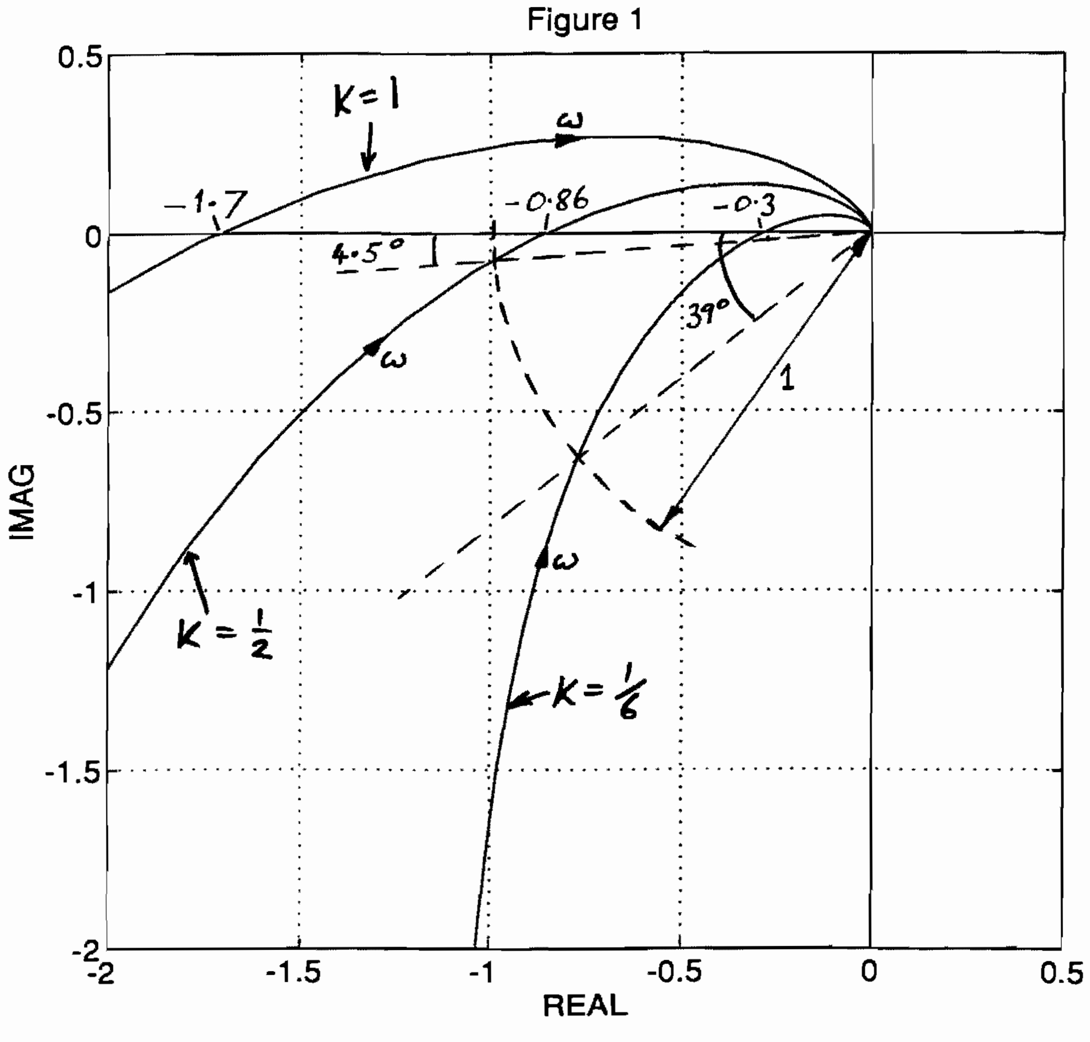
(and $|G(j\omega)| \xrightarrow[\omega\to\infty]{} \frac{1}{\omega^3T^3}$).



3. The diagram below shows fig. 1 with all pertinent measurements.

(a) Changing K just scales the Nyquist diagram, hence label as shown below. Arrows show increasing ω direction.

(b) The two diagrams which intersect the real axis between −1 and 0 ('leaving −1 on the left') give stable feedback systems. So $K = \frac{1}{2}$ and $K = \frac{1}{6}$ give stability.



(c) With $K = 1$ the Nyquist locus cuts the real axis at $-1{\cdot}7$ (by measurement from fig. 1 – could also find this exactly). So $K = \frac{1}{1{\cdot}7}$ is the gain for which stability is just lost.

(d) By measurement from fig. 1:

$K = \frac{1}{6}$ gives gain margin $= \frac{1}{0{\cdot}3}$, phase margin $= 39°$

$K = \frac{1}{2}$ gives gain margin $= \frac{1}{0{\cdot}86}$, phase margin $= 4{\cdot}5°$.

4. (a) Let $K(s) = \dfrac{0\cdot3(1+0\cdot083s)}{(1+0\cdot025s)}$.

$K(0) = 0\cdot3$, so $|K(0)| = 0\cdot3 = -10\cdot5\,dB$

$K(\infty) = \dfrac{0\cdot3\times0\cdot083\cancel{s}}{0\cdot025\cancel{s}} = 0\cdot996$, call it 1,

so $|K(\infty)| = 1 = 0\,dB$.

'Corner frequencies': $\dfrac{1}{0\cdot083} = 12$ (rad/sec)

$\dfrac{1}{0\cdot025} = 40$ (rad/sec).

Hence Bode plot of $K(s)$ as shown by chained lines on next page. Note the plot for phase is a very crude approximation. To get a reasonable sketch of the modified phase we shall calculate $\arg K(j\omega)$ at $\omega=12$ and $\omega=20$:

$\arg K(j12) = \tan^{-1}(\cdot083\times12) - \tan^{-1}(\cdot025\times12)$

$= \tan^{-1}(1) - \tan^{-1}(0\cdot3)$

$= 45^\circ - 17^\circ = 28^\circ$ (= $\arg K(j40)$, by symmetry).

$\arg K(j20) = \tan^{-1}(\cdot083\times20) - \tan^{-1}(\cdot025\times20)$

$= 59^\circ - 27^\circ = 32^\circ$

Hence note that $\arg K(j\omega) \approx 30^\circ$ for $12<\omega<40$.

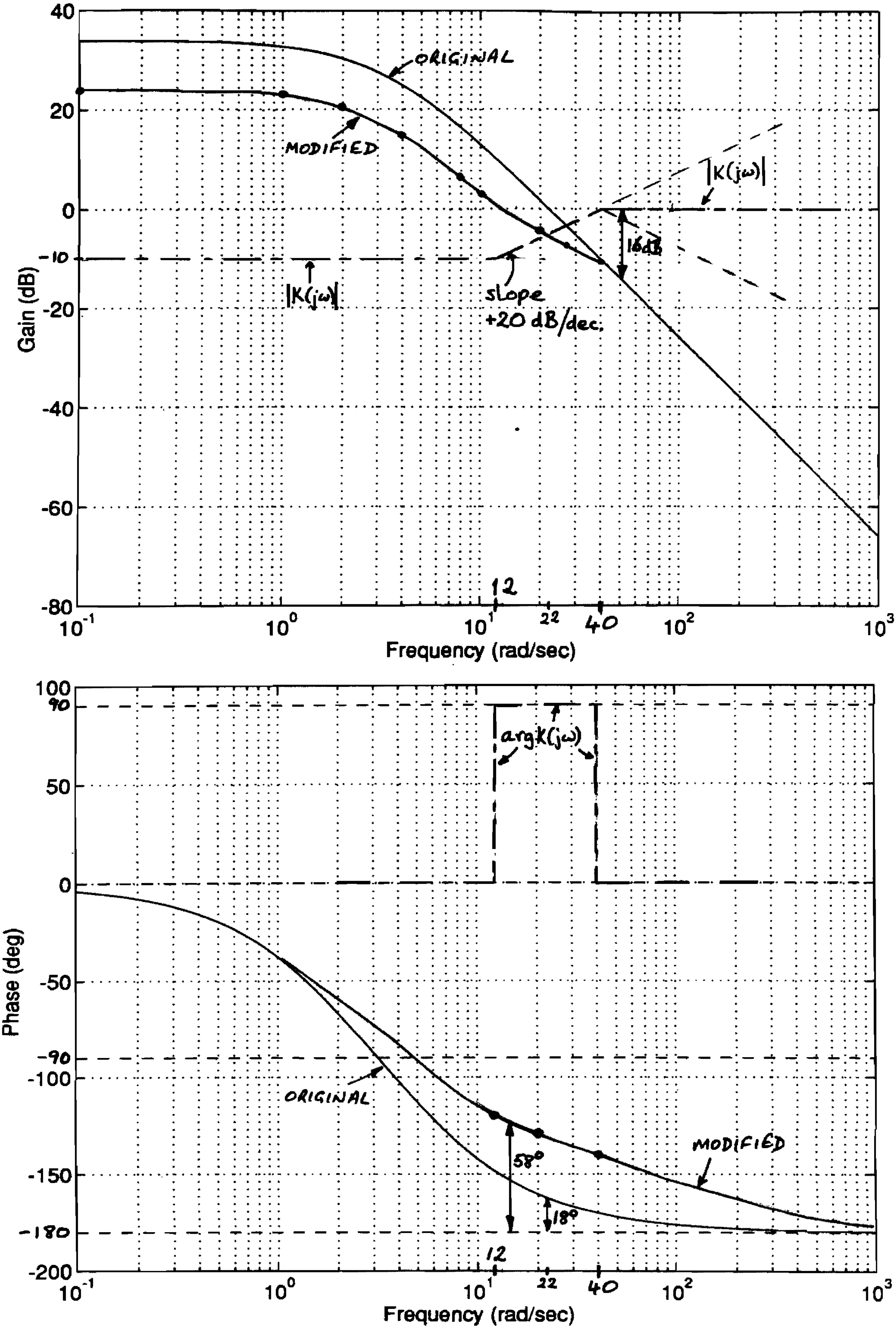
To get the modified Bode plot we just add the gain of the compensator (in dB) to the original gain, and we add the phases similarly. This is shown (approximately) by the continuous lines on the next page.

(b) The original gain plot crosses 0dB at 22 rad/sec. At this frequency the phase (of the original plot) is -162°, so the phase margin is $\underline{\underline{18^\circ}}$ (see next page).

To find the phase margin of the modified plot, we need to find the frequency at which the modified gain plot crosses 0dB. The sketch indicates approx. 12 rad/sec, but we should check the approximation:

$$|K(j12)| = \frac{0\cdot3\sqrt{1+(\cdot083\times12)^2}}{\sqrt{1+(\cdot025\times12)^2}} = \frac{0\cdot3\times\sqrt{2}}{1\cdot044} = 0\cdot406 = -7\cdot8\,dB.$$

Figure 2


ORIGINAL
MODIFIED
|K(jω)|
16dB
|K(jω)|
slope
+20 dB/dec.
12
22
40
Gain (dB)
Frequency (rad/sec)
arg K(jω)
ORIGINAL
MODIFIED
58°
18°
12
22
40
Phase (deg)
Frequency (rad/sec)

4(b) cont'd. But the sketch assumes that $|K(j12)| = -10\,dB$, so the actual gain is about $+2\,dB$ at $\omega = 12$. This error is small enough to proceed to evaluate the phase margin at 12 rad/sec. However, we shall locate the 0dB cross-over a little more accurately here:

Try $\omega = 15$: $$|K(j15)| = \frac{0.3\sqrt{1+(.083\times 15)^2}}{\sqrt{1+(.025\times 15)^2}} = \frac{0.3\times 1.6}{1.07} = 0.449$$
$$= -7\,dB.$$

The original gain at $\omega = 15$ is $+6\,dB$ (by measurement from fig. 1).

So the modified gain is $-1\,dB$.

Linear interpolation between 12 rad/sec ($+2\,dB$)
and 15 rad/sec ($-1\,dB$)

suggests taking the 0dB cross-over frequency as 14 rad/sec.

Now $$\arg K(j14) = \tan^{-1}(.083\times 14) - \tan^{-1}(.025\times 14)$$
$$= 49.3^\circ - 19.3^\circ = 30^\circ$$

and the original phase at $\omega = 14$ is -152° (by measurement)

so the modified phase at $\omega = 14$ is $-152+30 = -122^\circ$.

Hence the modified phase margin is <u>58°</u>

(c) At $\omega = 0$ the modified gain is $24\,dB$, or $10^{24/20} = 15.8$ (by measurement)

[or, from Q.1, sheet 6/3: original gain at $\omega = 0$ was 50, hence modified gain at $\omega = 0$ is $0.3\times 50 = 15$ — this value more accurate.]

Steady-state error $\approx \dfrac{1}{\text{loop gain at } \omega = 0} = \dfrac{1}{15.8} = \underline{6.3\%}.$

[or, using the value of 15: steady-state error $\approx \frac{1}{15} = \underline{6.67\%}.$]

To obtain 1% steady-state error we need gain, at $\omega = 0$, of 100, or $+40\,dB$. Therefore additional gain of $40 - 24 = 16\,dB$ is needed. With this additional gain the new 0dB cross-over frequency will be that frequency at which the existing gain is $-16\,dB$. From the Bode plot, this is $\omega = 50$ rad/sec, approx.

4(c) contd. Now $\arg K(j50) = \tan^{-1}(.083 \times 50) - \tan^{-1}(.025 \times 50)$

$$= 76.5^\circ - 51.3^\circ = +25.2^\circ$$

The original phase at $\omega = 50$ is -172°, so the modified phase at $\omega = 50$ is $-172 + 25 = -147^\circ$.

Hence the new phase margin is $180 - 147 = \underline{\underline{33^\circ}}$.

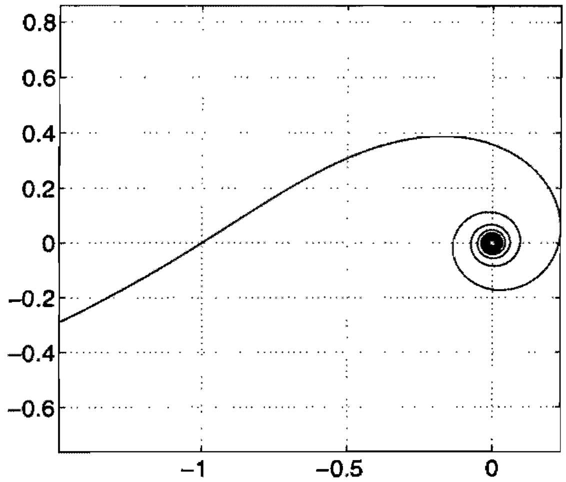
All the calculations could also be done by using the transfer function from Q.1, sheet 6/3, rather than by measurement from fig. 2.)

Examples Paper 6/4 Q5

(a) $L(j\omega) = 0.091 \times 1121 \times e^{-j\omega\tau} \times \dfrac{1+2j\omega}{-200\omega^2}$

and so $|L(j\omega)|^2 = (0.091 \times 1121)^2 \dfrac{1+4\omega^2}{200^2\omega^4}$.

Solving $(0.091 \times 1121)^2(1+4\omega^2) = (200^2\omega^4)$ as a quadratic in ω^2 gives $\omega = 1.1176$. At this frequency, $\arg L(j\omega) = -1.1176\tau + \arctan(2 \times 1.1176) - \pi$, but for marginal stability $L(j\omega) = -1$ and so $\arg L(j\omega) = -\pi$, equating these gives $\tau_{\text{crit}} = \arctan(2 \times 1.1176)/1.1176 = 1.0291$.



(b) For the Bode and Nyquist diagrams see Figures 1 and 2.

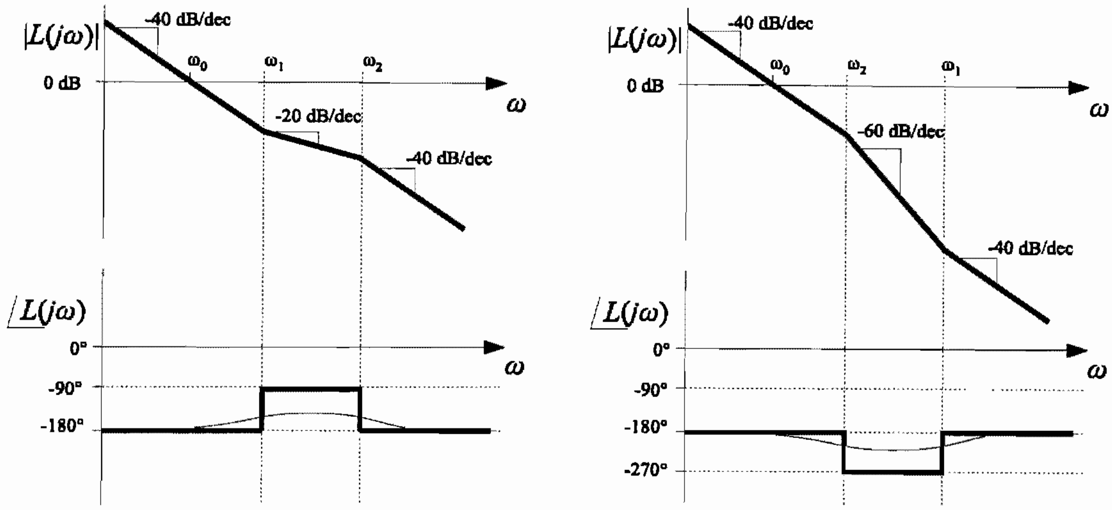
It is clear from the Bode diagrams that (a) is always stable (phase $> -180°$) and (b) is always unstable (phase $< -180°$). From the Nyquist diagrams, consider the image path of increasing ω: (a) leaves the '-1' point on its left so is stable, (b) leaves it on its right so is unstable. Therefore to ensure stability $\omega_1 < \omega_2$, i.e. $T_d > T$.

(c) With delay and lag, the return ratio becomes:

$$L(s) = \frac{H_0 k_p (1 + sT_d)e^{-s\tau}}{ms^2(1+sT)}$$

The Nyquist diagram is sketched in Figure 3.

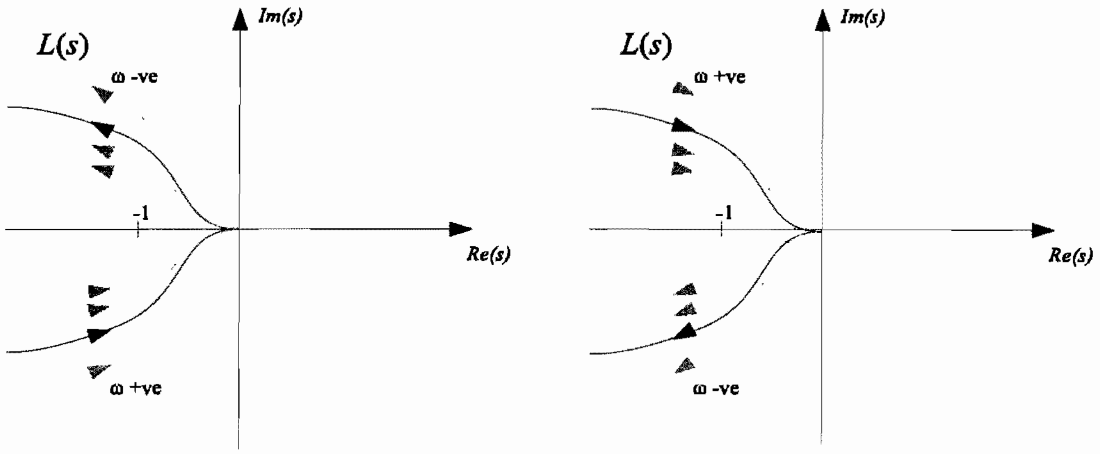
To determine the phase margin, the students should write the code necessary to draw the appropriate Bode diagram (Figure 4).



(a) $\omega_1 < \omega_2$

(b) $\omega_2 < \omega_1$

Figure 1: Bode diagrams, $\omega_0 = \sqrt{\frac{H_0 k_p}{m}}$, $\omega_1 = 1/T_d$ and $\omega_2 = 1/T$.



(a) $\omega_1 < \omega_2$

(b) $\omega_2 < \omega_1$

Figure 2: Nyquist diagrams.

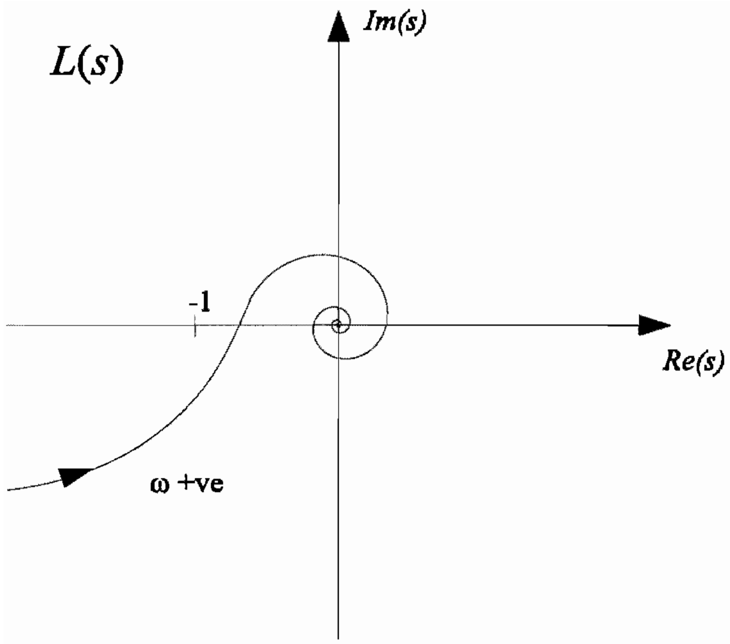


Figure 3: Nyquist diagram (with delay).

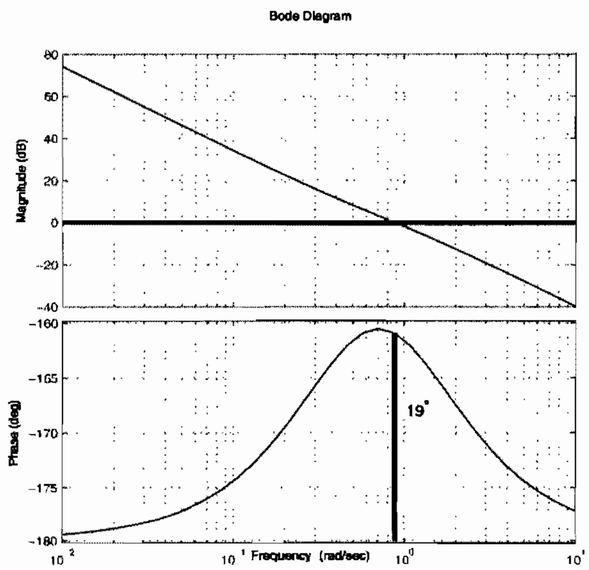


Figure 4: Bode diagram.

The critical delay is then:

$$\tau_{crit} = \frac{\text{Phase Margin}}{\omega_p} = \frac{19^\circ \times \frac{\pi}{180}}{0.88} = 0.377\,\text{s}.$$

(d) Simulations should agree. With the ENGINE_DELAY= $1.1\tau_{crit}$ initialising at 500 m, the lander oscillates with increasing amplitude until a limit cycle is reached (of about ±4m), caused by the system non-linearities (primarily from the throttle limits). This limit cycle is reached regardless of the initialisation altitude (unless the lander crashes!): initialising at 700 m results in decaying oscillation to begin with, even though the linearised system (about 500 m) is unstable. Similar behaviour is seen with the ENGINE_LAG and $k_d < k_p$ and $> k_p$

With lag and delay, simulations run using $\tau = 0.370$ s are stable, and $\tau = 0.380$ is unstable using `delta_t` $= 0.01$ s. To obtain agreement to a third decimal place requires using `delta_t` $=$ 0.001 s.

(e) When the delay $\tau = 0.4$ s, the students should obtain Figure 5. The region above the curve is unstable, therefore the stable range of K_D is $0.2 - 0.3$.

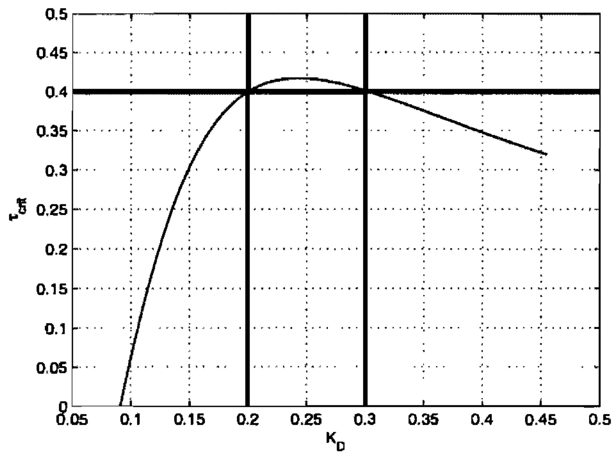
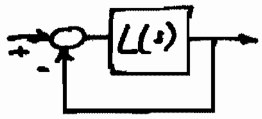


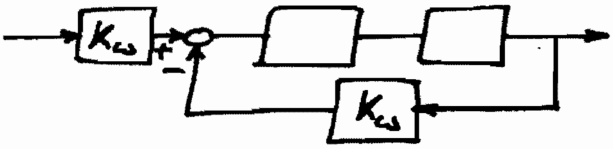
Figure 5: Critical delay as a function of K_D.

If the delay $\tau \geq 0.5$ s, no choice of K_D can stabilise the system. To stabilise the system you could reduce K_P to lower the overall gain.

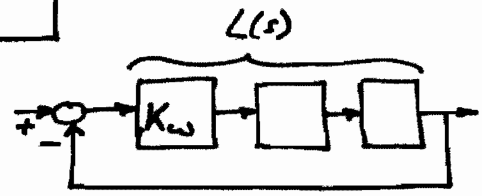
16 (a) The closed-loop transfer function from demanded speed to actual speed is $\dfrac{L(s)}{1+L(s)}$.



[Note that Q.1, Ex. Sht 6/3, shows the system to be



but this is equivalent to



]

So we are looking for the frequency (or frequencies) at which

$$\left|\frac{L(j\omega)}{1+L(j\omega)}\right| = \frac{|L(j\omega)|}{|1+L(j\omega)|} = 1 .$$

As shown in Q.1 (b)(iii), this requires $Re\{L(j\omega)\} = -\frac{1}{2}$.

This occurs on fig. 5(a) at $\underline{\underline{\omega = 73}}$, and on fig. 5(b) at $\underline{\underline{\omega = 16}}$.

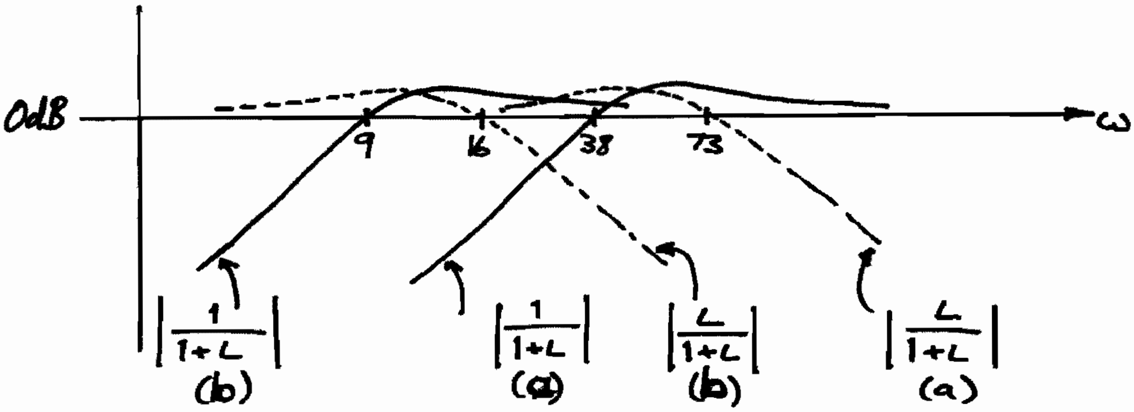
(b) $\left|\dfrac{1}{1+L(j\omega)}\right| = 1 \Rightarrow |1+L(j\omega)| = 1$. To find the frequency at which this happens, find the point on the locus which is distance 1 away from -1 (cf. Q.1(b)(ii)). By measurement, on fig. 5(a) this occurs for $\underline{\underline{\omega = 39}}$, and on fig. 5(b) for $\underline{\underline{\omega = 9}}$.

(c) The two designs have almost the same phase margins, so the shapes of transient responses (of the closed-loop) are going to be very similar.

The main difference is that design (a) gives a higher speed of response than design (b). The answer to (a) shows that design (a) can follow sinusoidal demand signals up to frequency 73 rad/sec without loss of gain, while design (b) can do this only up to 16 rad/sec. Roughly speaking, this means that design (a) is 73/16 = 4.5 times faster than design (b).

6(c) contd. The solution to part (b) shows that design (a) reduces the effects of sinusoidal output disturbances up to a frequency of 38 rad/sec, while design (b) does so only up to 9 rad/sec. Again, this shows that design (a) is about 4 times faster than design (b).

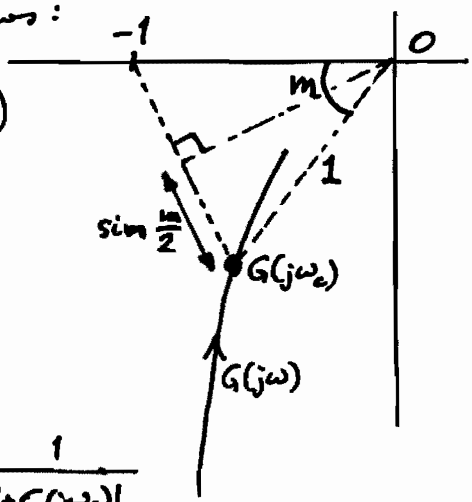
A more detailed answer could discuss the Bode plots of $\left|\frac{L}{1+L}\right|$ and of $\left|\frac{1}{1+L}\right|$. From parts (a) and (b) these have the following shapes:



(Note that faster speed of response is not necessarily a virtue, since it requires more power and implies more susceptibility to noise from sensors etc. It is only justified if the demand signal or the disturbances require it.)

---

7. (a) The geometry is as follows:



From the diagram, (and Q.1(b)(i))

$$|1+G(j\omega_c)| = 2\sin\left(\frac{m}{2}\right)$$

$$\therefore \frac{1}{|1+G(j\omega_c)|} = \frac{1}{2\sin(m/2)}.$$

Also,

$$\left|\frac{G(j\omega_c)}{1+G(j\omega_c)}\right| = \frac{|G(j\omega_c)|}{|1+G(j\omega_c)|} = \frac{1}{|1+G(j\omega_c)|}$$

since $|G(j\omega_c)| = 1$.

<u>Ex. Sht. 6/4</u>

7 b) If $m \leq 60°$ then $\sin(m/2) \leq \frac{1}{2}$, $2\sin(\frac{m}{2}) \leq 1$.

Hence $\left|\frac{1}{1+G(j\omega_c)}\right| \geq 1$ and $\left|\frac{G(j\omega_c)}{1+G(j\omega_c)}\right| \geq 1$

This means that the 'sensitivity' is greater than 1 at ω_c (and most frequencies above ω_c, usually). So at this frequency feedback is not beneficial: output disturbances are amplified rather than reduced, for example.

This means that the response from the demand to the output exhibits resonance. (The resonance peak usually occurs at a frequency a little higher than ω_c.)

(c) If m is very small then $\left|\frac{G(j\omega_c)}{1+G(j\omega_c)}\right| \gg 1$

so the closed-loop response is very resonant. This can only be due to at least one pair of very lightly damped closed-loop poles. So at least one pair of closed-loop poles approaches the imaginary axis as m approaches 0.
(This is consistent with the phase margin being a 'stability margin', of course.)