Solutions: Paper 5 – IB Information Engineering

Signal and Data Analysis: Revision of Fourier Series

1.

$$y = \begin{cases} x(\pi + x), & -\pi \le x \le 0 \\ x(\pi - x), & 0 \le x \le \pi \end{cases}$$

$$y' = \begin{cases} \pi + 2x, & -\pi \le x \le 0 \\ \pi - 2x, & 0 \le x \le \pi \end{cases}$$

$$y'' = \begin{cases} +2, & -\pi \le x \le 0 \\ -2, & 0 \le x \le \pi \end{cases}$$

Thus y and y' are continuous (over the interval $[-\pi, \pi]$), but y'' is discontinuous.

Now, coefficients $= O\left(\frac{1}{n^{r+2}}\right)$ where r is the order of the highest continuous derivative. So, for this function coeff$= O\left(\frac{1}{n^3}\right)$.

To evaluate the coefficients:
Function is *odd* so only *sine* terms are present in series.

Either :

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} y \sin nx \; dx = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin nx \; dx \\ &= \frac{2}{\pi}\left[-(\pi x - x^2)\frac{\cos nx}{n}\right]_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi} (\pi - 2x)\cos nx \; dx \\ &= \frac{2}{n\pi}\left[(\pi - 2x)\frac{\sin nx}{n}\right]_0^{\pi} - \frac{2}{n^2\pi}\int_0^{\pi} -2\sin nx \; dx \\ &= \frac{4}{n^2\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = \frac{4}{n^3\pi}(1 - \cos n\pi) \end{aligned}$$

$$= \begin{cases} \frac{8}{n^3\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Or :

$$y = \sum_n b_n \sin nx \quad \Longrightarrow \quad y' = \sum_n b_n \cos nx$$

$$\Longrightarrow \quad y'' = \sum_n -n^2 b_n \sin nx = \begin{cases} 2 & -\pi \le x \le 0 \\ -2, & 0 \le x \le \pi \end{cases}$$

Evaluating the Fourier coefficients $-n^2 b_n$ then gives us

$$-n^2 b_n = \frac{2}{\pi}\int_0^{\pi} -2\sin nx \; dx = \frac{4}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi} \quad \Longrightarrow \quad b_n = \frac{4}{n^3\pi}\left[1 - \cos nx\right]$$

$$\Longrightarrow \quad \sum_{n \; odd} \frac{8}{n^3\pi} \sin nx$$

2. $$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} + \sum_{n=1}^{\infty} b_n \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}$$

$$= \frac{a_0}{2} + a_1 \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} + \ldots + b_1 \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} + \ldots$$

$\therefore \; y = \sum_{-\infty}^{\infty} c_n e^{jn\omega_0 t}$ where $c_0 = \frac{a_0}{2}, \; c_1 = \frac{a_1}{2} - \frac{jb_1}{2}, \ldots$

$c_{-1} = \frac{a_1}{2} + \frac{jb_1}{2}, \ldots$

i.e. $c_0 = \frac{a_0}{2}$ $\quad c_n = \frac{a_n - jb_n}{2} \quad n>0$ $\quad c_n = \frac{a_{-n} + jb_{-n}}{2} \quad n<0$

Clearly $\quad c_n^* = \frac{a_n + jb_n}{2} = c_{-n}$

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3. $$y(t) = \begin{cases} e^{-\alpha t} & 0 \le t \le T/2 \\ 0 & T/2 \le t \le T \end{cases} \quad \& \quad y = \sum_n c_n e^{2\pi i n t/T}$$

$$\Rightarrow c_n = \frac{1}{T}\int_0^T y\, e^{-2\pi i n t/T} dt = \frac{1}{T}\int_0^{T/2} e^{-(\alpha + \frac{2\pi i n}{T})t} dt$$

$$= \frac{1}{T}\left[\frac{e^{-(\alpha+\frac{2\pi i n}{T})t}}{\alpha + \frac{2\pi i n}{T}}\right]_0^{T/2} = \frac{1 - e^{-\frac{\alpha T}{2} - \pi i n}}{\alpha T + 2\pi i n}$$

Now for $n \ge 0$ $\quad c_n = \frac{a_n - jb_n}{2}$ $\quad$ i.e. $\quad a_n = 2\,\mathrm{Re}\, c_n$
$\quad b_n = -2\,\mathrm{Im}\, c_n$

$$\therefore a_0 = \frac{2(1-e^{-\alpha T/2})}{\alpha T}$$

& $$c_n = \frac{1 - e^{-\frac{\alpha T}{2}}(\cos n\pi - i\overset{0}{\sin n\pi})}{\alpha T + 2\pi i n} = \frac{\alpha T - 2\pi i n}{\alpha^2 T^2 + 4\pi^2 n^2}\left(1 - e^{-\frac{\alpha T}{2}}\cos n\pi\right)$$

$$\therefore a_n = 2\,\mathrm{Re}\, c_n = \frac{2\alpha T(1 - e^{-\alpha T/2}\cos n\pi)}{\alpha^2 T^2 + 4n^2\pi^2}$$

& $$b_n = -2\,\mathrm{Im}\, c_n = \frac{4n\pi(1 - e^{-\alpha T/2}\cos n\pi)}{\alpha^2 T^2 + 4n^2\pi^2}$$

4. Odd fn $\Rightarrow$ $f(t) = \sum b_n \sin n\omega_0 t$ $\qquad \omega_0 = \frac{2\pi}{T}$

where $b_n = \frac{2}{T}\int_{-T/2}^{T/2} f \sin n\omega_0 t \, dt = \frac{4}{T}\int_0^{T/2} \sin n\omega_0 t \, dt$

$$= \frac{4}{T}\left[-\frac{\cos n\omega_0 t}{n\omega_0}\right]_0^{T/2} = \frac{4}{n\omega_0 T}\left(1 - \cos\frac{n\omega_0 T}{2}\right)$$

and $\cos\frac{n\omega_0 T}{2} = \cos n.\frac{2\pi}{2T}.T = \cos n\pi$

$$\therefore \quad b_n = \frac{8}{n.\frac{2\pi}{T}T} = \frac{4}{n\pi} \quad n \text{ odd} \qquad b_n = 0 \quad n \text{ even}.$$

$$C_n = \frac{a_n - jb_n}{2} \quad n \geqslant 0 \qquad C_n = \frac{a_n + jb_{-n}}{2} \quad n < 0$$

$$\therefore \quad C_n = -\frac{j}{2}\frac{4}{n\pi} = -\frac{2j}{n\pi} \text{ or } \frac{2}{jn\pi} \quad n > 0$$

$$C_n = +\frac{j}{2}\frac{4}{(-n)\pi} = -\frac{2j}{n\pi} \text{ or } \frac{2}{jn\pi} \quad n < 0$$

$$\therefore \quad C_n = \frac{2}{jn\pi} \quad n \text{ odd} \qquad C_n = 0 \quad n \text{ even}.$$

Directly

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T}\int_0^{T/2} e^{-jn\omega_0 t} dt - \frac{1}{T}\int_{-T/2}^{0} e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_0^{T/2} - \frac{1}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-T/2}^{0}$$

$$= -\frac{1}{jn\omega_0 T}\left\{e^{-jn\frac{\omega_0 T}{2}} - 1 - 1 + e^{+jn\frac{\omega_0 T}{2}}\right]$$

& $e^{-jn\frac{\omega_0 T}{2}} = e^{-jn\frac{2\pi}{T}\frac{T}{2}} = e^{-jn\pi} = (-1)^n$

$$\therefore \quad C_n = \frac{-2}{jn\frac{2\pi}{T}.T}\left\{(-1)^n - 1\right\} = \begin{cases} \frac{2}{jn\pi} & n \text{ odd} \\ 0 & n \text{ even}. \end{cases}$$

5. f_2 is a shift in time of f,

and f_2 "rises" at $t = -\frac{T}{4}$ f "rises" at $t = 0$

$\therefore$ $f_2(-T/4) = f(0)$ & in general $f_2(t) = f(t+\frac{T}{4})$

i.e. $\underline{a = \frac{T}{4}}$ (or $\frac{5T}{4}, \frac{9T}{4}, \dots, -\frac{T}{4}, \dots$)

Using 4, $f(t) = \sum c_n e^{2jn\pi t/T}$ $c_n = \begin{cases} \frac{2}{jn\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

$\therefore f_2(t) = f(t+\frac{T}{4}) = \sum c_n e^{2jn\pi\frac{t}{T} + 2jn\pi\frac{T}{4T}}$

$= \sum d_n e^{\frac{2jn\pi t}{T}}$ where $\underline{d_n = c_n e^{\frac{jn\pi}{2}}}$

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6. Coeffs $= O(\frac{1}{n^{r+2}})$ where r = highest order continuous derivative.

If coeff $= \frac{(-1)^n}{n^4}$ $\Rightarrow$ $r = 2$ i.e. function, first & 2nd derivatives c[ontinuous]

$f = \sum c_n e^{jn\omega_0 t}$, $\omega_0 = \frac{2\pi}{T}$ $c_n = \frac{(-1)^n}{n^4}$

(i) $f' = \sum jn\omega_0 c_n e^{jn\omega_0 t}$ $\Rightarrow$ coeffs $= jn\omega_0 \frac{(-1)^n}{n^4} = \frac{j\omega_0(-1)^n}{n^3}$

(ii) $f'' \to (jn\omega_0)^2 \frac{(-1)^n}{n^4} = \frac{\omega_0^2(-1)^{n+1}}{n^2}$

(iii) $\int f dt \to \frac{c_n}{jn\omega_0}$ $n \neq 0$ $\Rightarrow$ coeffs $= \frac{(-1)^n}{j\omega_0 n^5}$. $n \neq 0$.

c_0 is not determined. It is in fact determined by the const of integration.

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7. $$f * g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

For $t<0$ f is zero throughout the range $\Rightarrow$ $f*g=0$

For $t>0$ $$f*g = \int_0^t \tau e^{-\alpha(t-\tau)}\,d\tau = \left[\frac{\tau e^{\alpha(\tau-t)}}{\alpha}\right]_0^t - \frac{1}{\alpha}\int_0^t e^{\alpha(\tau-t)}\,d\tau$$
$$= \frac{t}{\alpha} - \frac{1}{\alpha}\left[\frac{e^{\alpha(\tau-t)}}{\alpha}\right]_0^t$$
$$= \frac{t}{\alpha} - \frac{1}{\alpha^2} + \frac{e^{-\alpha t}}{\alpha^2}$$

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8. Putting $\tau' = t-\tau$ $\Rightarrow$ $\displaystyle\int_{\tau=0}^t f(\tau)\,g(t-\tau)\,d\tau = \int_{\tau'=t}^0 f(t-\tau')\,g(\tau')\,-d\tau'$

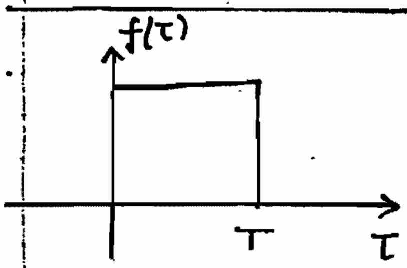
$\left(\Rightarrow \tau = t-\tau' \quad d\tau = -d\tau'\right)$

$$= \int_0^t f(t-\tau')\,g(\tau')\,d\tau'$$

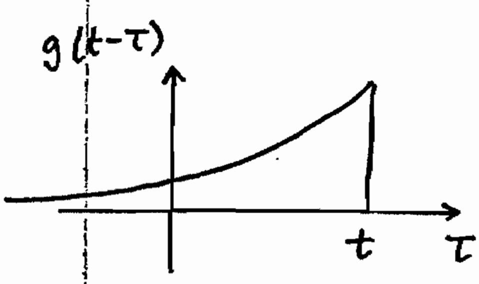
But τ' is a dummy variable and can thus be relabelled τ

$$\Rightarrow \underline{\int_0^t f(\tau)\,g(t-\tau)\,d\tau = \int_0^t f(t-\tau)\,g(\tau)\,d\tau.}$$

9.

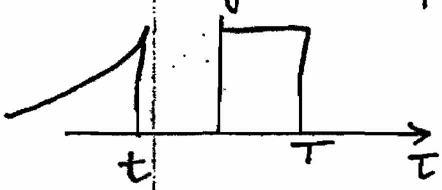


$$f(\tau) = \begin{cases} 1 & 0 \le \tau \le T \\ 0 & \text{otherwise} \end{cases}$$



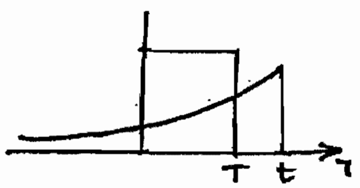
$$g(t-\tau) = \begin{cases} e^{-\alpha(t-\tau)} & t-\tau \ge 0 \quad \text{i.e. } \tau \le t \\ 0 & t-\tau < 0 \quad \text{i.e. } \tau > t \end{cases}$$

Convolution integrand is <u>product</u> of these two



$t<0$

$0<t<T$



$t>T$

There are three different forms for the convolution since

(i) $t < 0$ There is no overlap of the non-zero regions of f & g

$$\Rightarrow f * g = 0$$

(ii) $0 < t < T$ f is unity between $0 < \tau < 1$
but g "turns off" at $\tau = t$

i.e. $$f * g = \int_0^t e^{\alpha(\tau - t)}\, d\tau = \left[\frac{e^{\alpha(\tau - t)}}{\alpha} \right]_0^t$$

$$= \underline{\frac{1 - e^{-\alpha t}}{\alpha}}$$

(iii) $t > T$ The upper limit in integration is T since f is zero for values of τ bigger than this.

i.e. $$f * g = \int_0^T e^{\alpha(\tau - t)}\, d\tau = \left[\frac{e^{\alpha(\tau - t)}}{\alpha} \right]_0^T$$

$$= \underline{\frac{e^{-\alpha(t - T)} - e^{-\alpha t}}{\alpha}}$$