Part IB Paper 6: Information Engineering

LINEAR SYSTEMS AND CONTROL

Example Paper 6/4

Straightforward questions are marked †
*Tripos standard questions are marked ***

Revision of complex numbers

1. (a) † For $0 \leq \theta \leq \pi$ plot the curves in the complex plane corresponding to the following functions:

$$e^{j\theta}, \; 2e^{-j\theta}, \; 1 + e^{j\theta}, \; 1 + \frac{1}{2}e^{j(\theta + \pi/4)}.$$

   (b) Let g be a complex number, and let k be a real positive number.

   i. Show, on a picture of the complex plane, the points g and $-1/k$ and a vector (not starting at the origin) representing the the complex number $g + \frac{1}{k}$.

   ii. Show how $1 / \left| \frac{k}{1+kg} \right|$ can be obtained by measurement from your picture (if it were drawn to scale).

   iii. Show that, if $\left| \frac{kg}{1+kg} \right| = 1$, then $\operatorname{Re}(g) = -\frac{1}{2k}$ (or, equivalently, $\operatorname{Re}(kg) = -\frac{1}{2}$).

Nyquist (polar) diagrams

2. *Sketch* the Nyquist diagrams for the following transfer functions by considering the gain and phase when $\omega \to 0$, $\omega = 1/T$, and $\omega \to \infty$:

   (a) $G(s) = (1 + sT)/(1 + saT)$ for $a < 1$

   (b) $G(s) = (1 + sT)/(1 + saT)$ for $a > 1$

   (Note that these are semicircles centred on $(a+1)/2a$, because

$$\frac{1 + j\omega T}{1 + aj\omega T} = \frac{a+1}{2a} + \frac{a-1}{2a}\left[\frac{1 - aj\omega T}{1 + aj\omega T}\right],$$

   and the modulus of the term in square brackets is unity.)

(c) $G(s) = 1/(1 + 2\zeta sT + s^2T^2)$ for $0 < \zeta < 1$

(d) $G(s) = 1/\{sT(1 + 2\zeta sT + s^2T^2)\}$ for $0 < \zeta < 1$.

Recall that you drew Bode plots of similar functions in Examples Sheet 6/2. You may find those Bode plots helpful here.

Now compare Nyquist and Bode diagrams of these transfer functions in Matlab/Octave (see 'Simulation Notes' below).

Nyquist feedback stability theorem

3. † Figure 1 shows parts of the Nyquist diagrams of the transfer function

$$\frac{K}{s(1 + 3s)(1 + 4s)}$$

for $K = \frac{1}{6}$, $K = \frac{1}{2}$, and $K = 1$. A negative feedback loop is closed around a system with this transfer function.

(a) Label each Nyquist diagram with the appropriate value of K.

(b) Which diagrams represent stable feedback systems?

(c) What is the greatest value of K for which the closed loop is stable?

(d) For those diagrams which represent stable feedback systems, estimate the gain and phase margins.

4. * The open-loop Bode plot for the speed-control system of Example Sheet 6/3, Question 1, is shown in figure 2 (i.e. this is the Bode diagram of $\frac{K_a K_b K_\omega}{(1+sT_a)(1+sT_b)}$). A 'phase advance' compensator with transfer function

$$\frac{0.3(1 + 0.083s)}{(1 + 0.025s)}$$

is inserted into the loop, as shown in figure 3.

(a) Using straight line approximations, sketch the Bode plot of this compensator (on figure 2), and hence sketch the Bode plot for the modified loop.

(b) Estimate the phase margins of the original and the modified loops.

(c) The original feedback loop gave a steady-state error of 2% (see sheet 6/3, question 1). What is the steady-state error with the compensator inserted? If additional gain is inserted into the loop (constant at all frequencies), so as to reduce the steady-state error to 1%, what is the new phase margin?

5. * Until now, the engine of the Mars Lander has been assumed to give an instantaneous response. In this question we will consider different models for the thrust response to the throttle input.

2

(a) First consider the engine as a pure delay, $H(s) = H_0 e^{-s\tau}$. Write down the new return ratio $L(s) = P(s)H(s)K(s)$ for a controller of the form $K(s) = k_p(1 + sT_d)$ (i.e. a proportional and derivative controller with $k_d = k_p T_d$). Take $P = 1/(ms^2)$, $m = 200$, $H_0 = 1121.0$ as before.

Let $k_p = 0.091$ (from 6/3 question 7), $T_d = 2$, and find the frequency ω for which the gain of $L(j\omega)$ is equal to unity. Hence determine the smallest time delay τ_{crit} that will cause instability of the closed loop system. Sketch (roughly) the Nyquist diagram of $L(s)$ for this value of τ.

(b) Now replace the engine model of part (a) with the transfer function $H(s) = H_0/(1 + sT)$ with $T = 1$ (this corresponds to a 1 second lag in the engine dynamics).

Sketch the Bode and Nyquist diagrams for:

(i) $T_d > 1$

(ii) $T_d < 1$

Hence find the minimum value of T_d required to maintain stability of the feedback system.

(c) Use Matlab/Octave to *graphically* determine the phase margin of the system when $T_d = 2$, for k_p as above (see 'Simulation Notes' below). If a delay of τ seconds is present in addition to the lag, then what is the new return ratio, and what now is the critical minimum delay, τ_{crit}, before the feedback system becomes unstable?

(d) *Test:* Using $k_p = 0.0910$, $k_d = 2k_p$ set the variable ENGINE_DELAY in lander.h to 90% of the τ_{crit} you found in (a) and initialise the simulations at 500 m, 510 m and 700 m. Are the results as you would expect? Repeat with ENGINE_DELAY set to 110% of τ_{crit}

Repeat with the variable ENGINE_LAG in lander.h set to 1 second (and no ENGINE_DELAY) and with $k_d = 0.9k_p$ and $k_d = 1.1k_p$.

Finally, check your answer to (c) using both the lag and the delay.

(e) *Extension exercise:* The delay has been measured to be 0.4 seconds. Use Matlab/Octave to plot a graph of the critical delay as a function of k_d. Hence, or otherwise, find the range of k_d that stabilises the system (see 'Simulation Notes'). Do the simulations agree? By considering the Nyquist diagram of $L(s)$ with both lag and delay, what could you do to stabilise a system with a delay of 0.5 seconds or longer?

Performance of feedback systems

6. Figure 5(a) shows part of the open-loop Nyquist diagram (*ie* $L(j\omega)$ if $L(s)$ is the open-loop transfer function) for the speed-control system considered in question 4 (after adjustment of gain as required in question 4(c)). Figure 5(b) shows part of the Nyquist diagram which is obtained if a 'phase-lag' compensator with transfer function

$$\frac{1 + 2s}{1 + 6.667s}$$

is used *instead of* the phase advance compensator described in question 4. Both diagrams have frequency calibrations shown on them.

   (a) At which frequency is the *closed-loop* gain from demanded speed to actual speed equal to 1, for each design?

   (b) At which frequency is the gain of the sensitivity function equal to 1 (*ie* $|1/(1 + L(j\omega))| = 1$) for each design?

   (c) Comment on the comparative performance of the two designs, particularly as regards speed of response.

7. * Suppose that a feedback system has open-loop transfer function $G(s)$, phase margin m, and that ω_c is the only frequency at which $|G(j\omega_c)| = 1$.

   (a) Show that

$$\left| \frac{1}{1 + G(j\omega_c)} \right| = \left| \frac{G(j\omega_c)}{1 + G(j\omega_c)} \right| = \frac{1}{2\sin(m/2)}.$$

   (b) What can you deduce (from (a)) about closed-loop behaviour if the phase margin is smaller than 60 degrees?

   (c) What can you deduce about the locations of some of the closed-loop poles if the phase margin is very small?

Suitable Tripos questions: Q1 (c) & Q3 2017, Q2 & 3. 2016 ,Q1 & 2. 2015, Q1 & 2. 2013, Q2 & 3. 2012, Q2 & 3. 2011, Q2 & 3. 2010, Q2& 3. 2009, Q2&3. 2008.

Answers:

1. —

2. —

3. All answers approximate: (c) $\frac{1}{1.7} = 0.59$. (d) Gain margins: $\frac{1}{0.3} = 3.3$, $\frac{1}{0.86} = 1.16$. Phase margins: 39 deg, 4.5 deg.

4. All answers approximate: (b) 18 deg, 58 deg. (c) 6.7%, 33 deg.

5. a) $1121k_p(1+sT_d)e^{-s\tau}/(200s^2)$, $\omega = 1.12$, $\tau_{\text{crit}} = 1.03$ b) $T_d > 1$, c) $L(s) = 1121k_p(1+ sT_d)e^{-s\tau}/(200s^2(1 + s))$, Phase Margin $= 19.0°$, $\tau_{crit} \approx 0.377$ sec

6. —

7. —

Simulation Notes

The following Python code sets up an arbitrary transfer function G, and uses some common python-control functions to analyse the system. The vectors passed as arguments of `ctr.tf` correspond to the numerator and denominator polynomial coefficients in descending powers of s, so in the example $G = 1/(s^3 + 2s^2 + s)$:

```
import matplotlib.pyplot as plt
import control as ctr
import numpy as np


G = ctr.tf([1],[1,2,1,0])
fig1 = plt.figure()
mag, phase, omega = ctr.nyquist(G,[0.01,1000])
plt.grid()
fig2 = plt.figure()
mag, phase, omega = ctr.bode(G,[0.01,1000])
```

You can change the frequency range of these plots by modifying the arguments of `ctr.nyquist` and `ctr.bode` (type `help(ctr.nyquist)` and `help(ctr.bode)` for more details).

Python-control also provides `ctr.margin` as an automatic way of finding the gain and phase margins of the system. Try typing `help(ctr.margin)` to understand the output of this function in more detail. Note that phase is in degrees, and frequencies are in rad/sec. Test the function by double checking your graphically estimated phase margin.

When testing stability, first initialise the system at $500\,\mathrm{m}$, then test other initial altitudes. You may find that the *precise* boundary between stable and unstable systems does not seem to agree with the predictions, if so try a smaller time step (e.g. `delta_t` $= 0.001$ seconds) to limit numerical stability issues. Note that when running simulations close to instability, oscillations take a long time to decay so you will need to speed up the simulation to see whether the system is stable.

The following code will get you started for the extension exercise:

```
import matplotlib.pyplot as plt
import control as ctr
import numpy as np


counter = 0
Kmin = ...
kdrange = np.linspace(1,5,100)*Kmin
Tcrit = np.zeros(kdrange.size)
for kd in kdrange:
    G = ctr.tf([...],[...])
    gm, pm, Wcg, Wcp = ctr.margin(G)
    Tcrit[counter] = ...
    counter += 1
fig1 = plt.figure()
plt.plot(kdrange,Tcrit)
plt.grid()
```
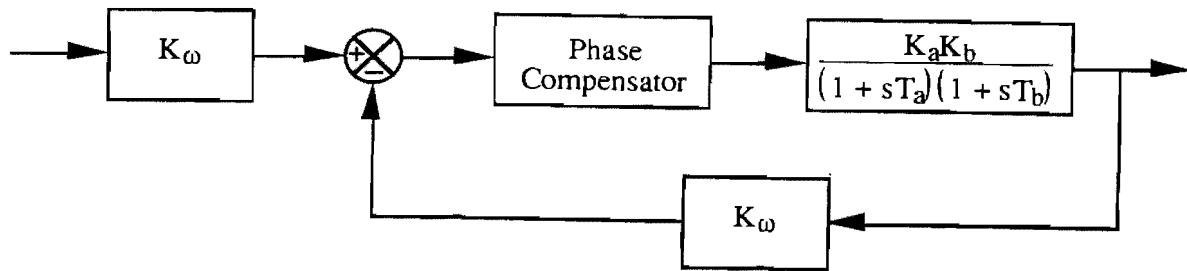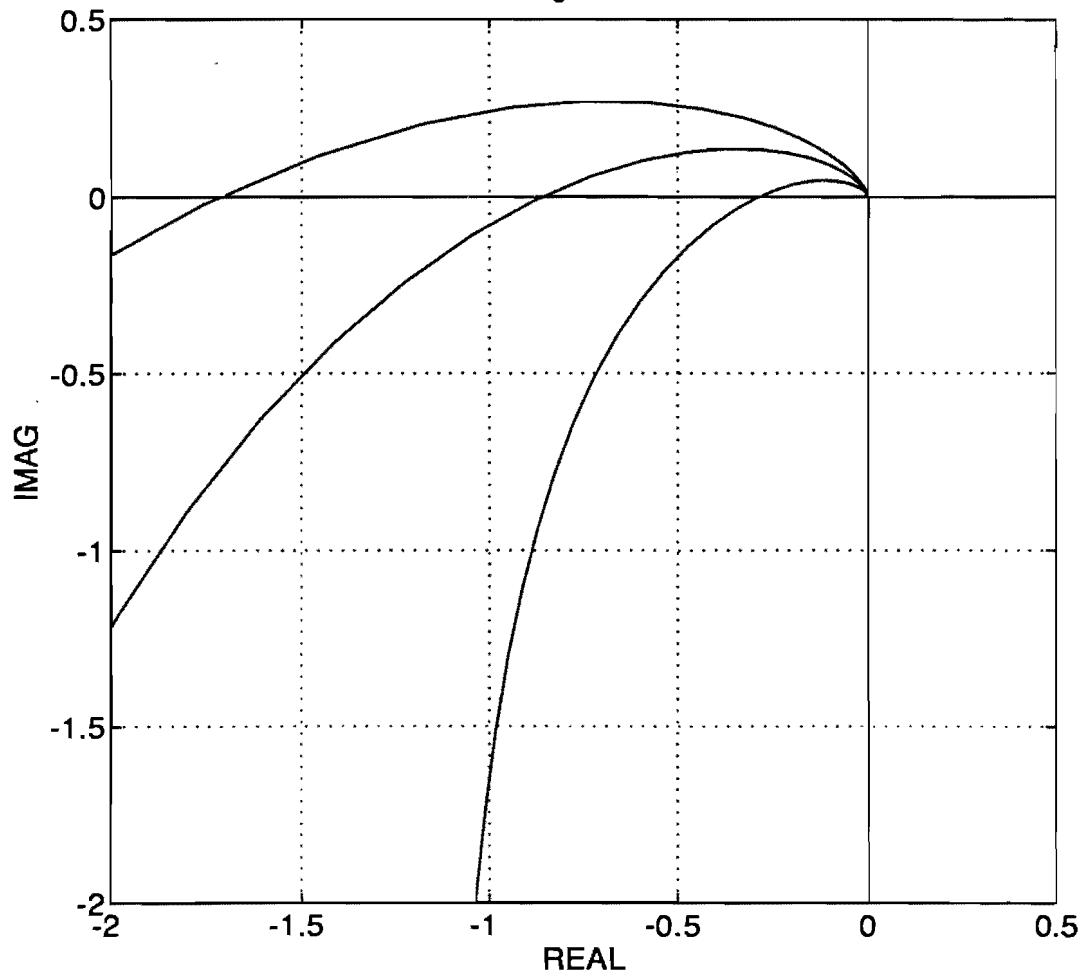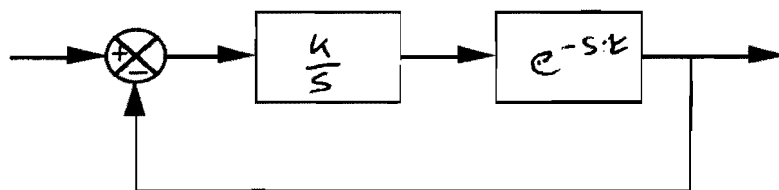
Figure 1

Figure 1

Fig. 3

Fig. 4

Figure 2

Figure 5(a)

Figure 5(b)

8