IB PAPER 6 – INFORMATION ENGINEERING

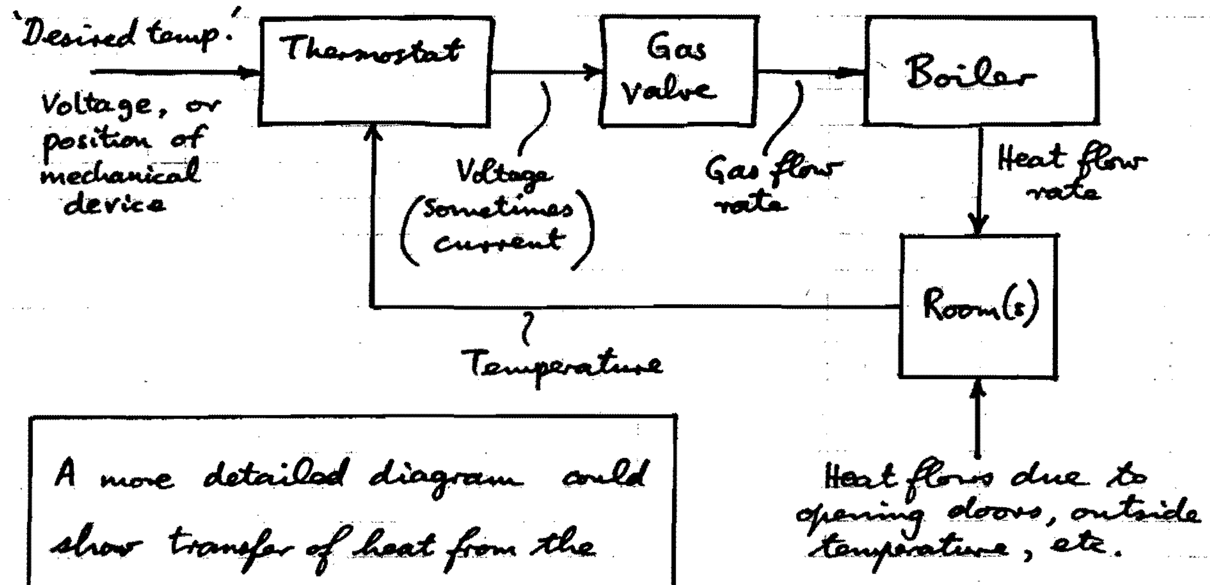
LINEAR SYSTEMS & CONTROL

EXAMPLES PAPER 1 – SOLUTIONS

1. In each case a good block diagram is sufficient to describe operation of the system. The block diagram may not be obvious, and there may be several equally good block diagrams.

(a) Domestic heating system

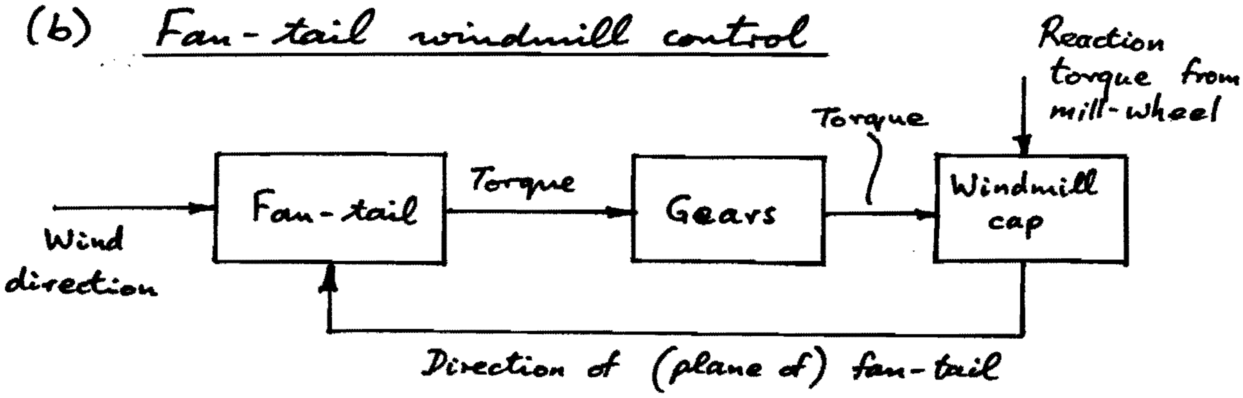
Assuming only one thermostat, a gas boiler, with the boiler controlled by the gas supply rate:



A more detailed diagram could show transfer of heat from the boiler to the circulating water, or the solenoid which operates the gas valve, etc.

1. contd.

(b) Fan-tail windmill control



Note that gears are used as 'torque amplifiers' here. The cap of the windmill carries both the main sail and the fan-tail. When the plane of the fan-tail lies parallel to the wind direction then the fan-tail is not driven by the wind. In this condition the main sail is at right angles to the wind, as required.

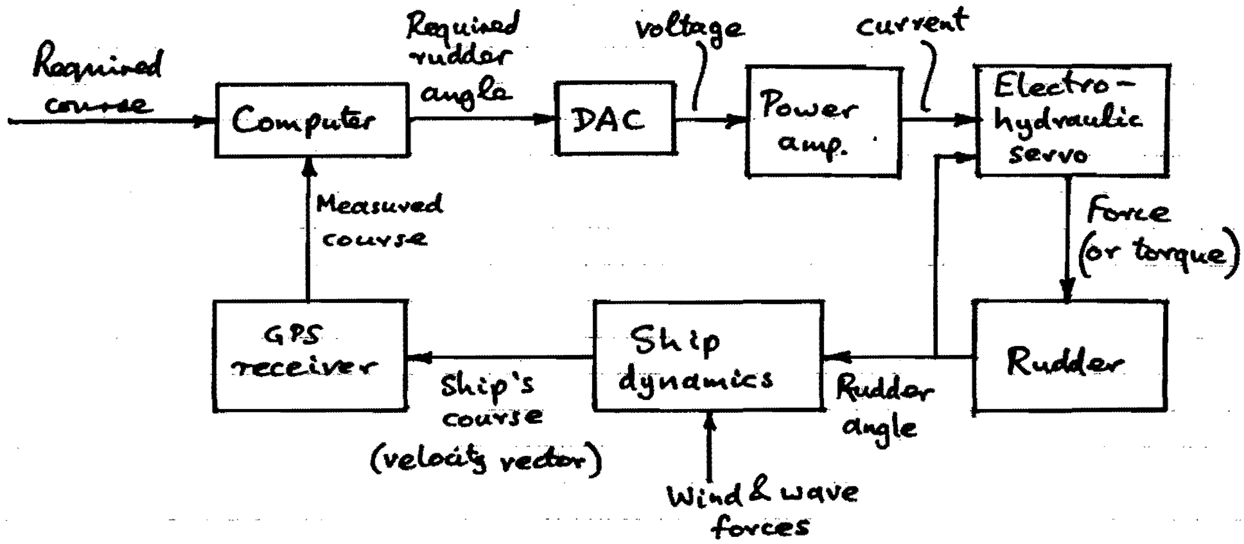
(c) Self-steering of ship

Assume steering is by a rudder (rather than bow-thrusters etc.)
A measurement of the ship's heading is required. Traditionally this is from a gyrocompass, usually corrected by estimates of current and wind forces (since the heading is not the true course). Nowadays very accurate measurements of the true course are available from the satellite-based 'Global Positioning System' (GPS).
Since the rudder has very large inertia, and its movements are opposed by very large forces, it is necessary to have a powerful hydraulic system (itself a feedback system, in fact) to move it.

1 contd.

The required rudder angle is nowadays worked out by a computer. The program may be sophisticated enough to avoid causing fuel wastage by unnecessary rudder movements.

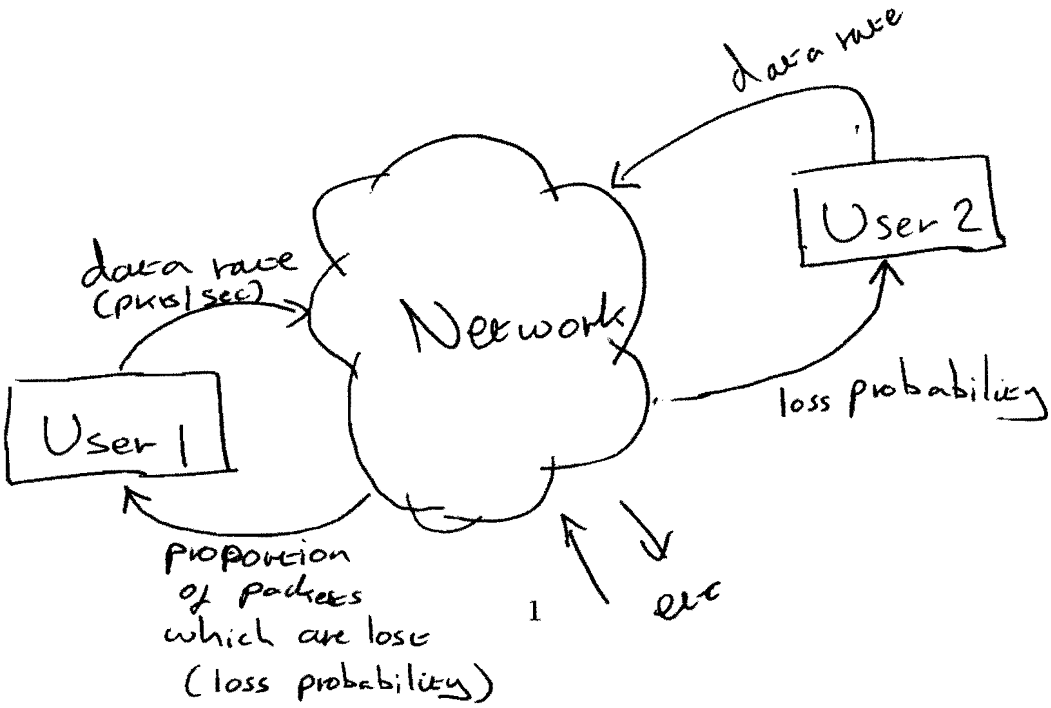
Here is a possible block diagram (in rather more detail than expected of students):



1. (d)

Files to be transferred across the Internet using the Transmission Control Protocol (TCP) - eg a download from the web - are broken into packets of typical size 1500bytes, with headers specifying the destination and the number of the packet amongst other information. These packets are sent one by one into the network, with the recipient sending acknowledgements back to the source whenever one is received. Routers in the network typically operate a drop tail queue. If a packet is received when the queue is full then it is simply discarded. Packet loss thus signals congestion. If a packet is received out of order, it is assumed that intervening packets have been lost. The recipient sends a duplicate acknowledgement to signal this and the source lowers its rate (in response to the congestion) and resends the lost packet(s). Whilst a steady stream of successive acknowledgements is being received the source gradually increases its sending rate. In normal operation sources are thus constantly increasing and decreasing their rates in an attempt to make use of the available bandwidth. Congestion (ie full queues and the resulting packet loss) can occurr anywhere in the network - at the edges (eg your adsl modem, or at the exchange), in the core (eg a big transatlantic link) or, very often, at peering points, which are the connections between the networks that make up the Internet.

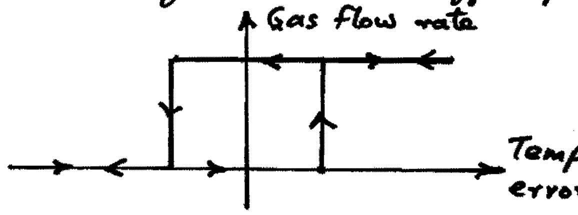
A high level description of this, from a control perpspective, might look like this:



2 a) i) & iii) are linear. ii) & iv) are not, as superposition of solutions doesn't hold.

2.b)(a) Both the boiler and the room(s) being heated can be modelled by differential equations. (Each probably modelled as a single heat capacity, leading to a 1st-order ODE for each.) These equations may be nonlinear (heat transfer coefficients & losses may depend on the temperature), but for small variations about a constant temperature a linear approximation is likely to be perfectly adequate.

However, the thermostat/gas valve may have a very non-linear, non-smooth characteristic. This will be the case if it only allows 'on-off' operation:



A system with this kind of thermostat cannot be analysed properly by using a linear approximation. If the gas flow rate can be varied continuously between limits, however, then a linear approximation is possible and probably adequate.

(b) The torque generated by the fan-tail probably depends on the sine of the angle between the wind direction and the plane of the fan-tail. For small changes this can be linearised. The cap of the windmill can be modelled as a rotating inertia, which gives a 2nd-order linear ODE (Newton's 2nd law). However, in a heavy, low-speed, relatively crudely manufactured mechanism of this kind, there is likely to be a large loss of torque due to friction. This is probably very nonlinear — almost constant friction torque, independent of speed. (Note that this is different from the linear 'viscous' friction which we often assume.)

The acceptability of a linear analysis will depend on how large this nonlinear frictional effect is. One can guess that it would not be acceptable for a traditionally-built windmill.

(c) For relatively small rudder and heading changes the slip probably behaves linearly, to a good approximation.

The internals of the rudder servo ~~is~~ are probably rather nonlinear — friction again, backlash in gears etc — but these are largely hidden by the operation of the servo. i.e. if we were designing the rudder servo we would

2 cont. probably need some nonlinear analysis, but if the servo already works than the rest of the system, and the external behaviour of the servo, can be modelled by linear equations.

The quantisation due to the use of a digital computer (which is a nonlinear effect) is probably negligible.

(d) The real system is highly nonlinear over short time scales (with rates ramping up and down) and packet losses are effectively stochastic. However, over longer timescales the network behaves in close to linear way provided it is stable (meaning that the fluctuations in rates are relatively small.

(6a)

Q3) The equation of motion of the lander is:

$$m\ddot{R} = -\frac{GMm}{R^2} - \frac{1}{2}\rho C_d A\dot{R}^2 \mathrm{sgn}(\dot{R}) + F_{thrust}$$

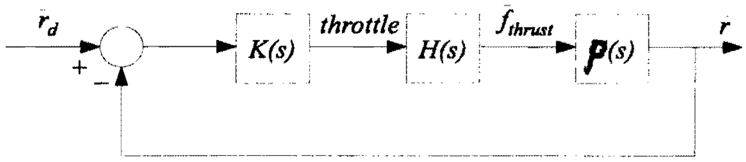
At equilibrium $R = R_0 = \texttt{MARS\_RADIUS} + 500$ (and $\ddot{R} = \dot{R} = 0$), so:

$$F_{eq} = \frac{GMm}{R_0^2} = 747.1 \text{ N}$$

To linearise, consider a small change from equilibrium, so that $R = R_0 + r$:

$$\begin{aligned}
F_G\left(R_0 + r\right) \approx& F_G(R_0) + r\left.\frac{\partial F_G}{\partial R}\right|_{R=R_0} = -\frac{GMm}{R_0^2} + 2\frac{GMm}{R_0^3}r \\
F_D(\dot{R}_0 + \dot{r}) \approx& F_D(\dot{R}_0) + \dot{r}\left.\frac{\partial F_D}{\partial \dot{R}}\right|_{\dot{R}=\dot{R}_0} = 0 + 0\dot{r} = 0. \\
F_{thrust} =& F_{eq} + f_{thrust} \Rightarrow m\ddot{r} = -\frac{GMm}{R_0^2} + 2\frac{GMm}{R_0^3}r + F_{eq} + f_{thrust} \\
m\ddot{r} =& 2\frac{GMm}{R_0^3}r + f_{thrust}.
\end{aligned}$$

The block diagram is:



To find the transfer function $G(s)$:

$$\begin{aligned}
m\ddot{r} =& 2\frac{GMm}{R_0^3}r + f_{thrust} \\
\Rightarrow ms^2\bar{r} + 2\frac{GMm}{R_0^3}\bar{r} =& \bar{f}_{thrust} \quad \text{(assuming zero IC's)} \\
\Rightarrow \bar{r} =& P(s)\bar{f}_{thrust} \\
\text{where } P(s) =& \frac{1}{ms^2 - 2GMm/R_0^3}
\end{aligned}$$

4

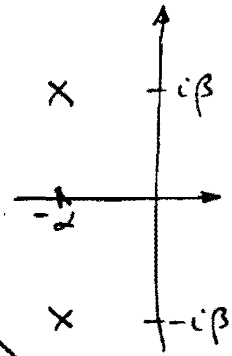
(a) $$f(t) = e^{-\alpha t}\cos(\beta t + \phi) \qquad t \geq 0$$

Let $$g(t) = \cos(\beta t + \phi) = \cos\beta t \cos\phi - \sin\beta t \sin\phi$$

$$\bar{g}(s) = \frac{\cos\phi \; s - \beta \sin\phi}{s^2 + \beta^2} \quad \text{by (A.14)}$$

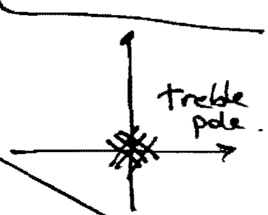
$$\bar{f}(s) = \bar{g}(s+\alpha) \quad \text{by } (\S 4.2)$$

$$= \frac{\cos\phi \; (s+\alpha) - \beta \sin\phi}{(s+\alpha)^2 + \beta^2}$$



(b) $$f(t) = 1 + t^2 \qquad t \geq 0$$

$$\bar{f}(s) = \frac{1}{s} + \frac{2}{s^3}$$



5 (a)

$$\bar{f}(s) = \frac{6}{(s+1)(s+2)(s+3)} = \frac{3}{s+1} - \frac{6}{s+2} + \frac{3}{s+3}$$

(by cover up rule)

$$f(t) = 3e^{-t} - 6e^{-2t} + 3e^{-3t}$$

(b) $$\bar{f}(s) = \frac{2}{(s+1)(s^2+2^2)} = \frac{2/5}{s+1} + \frac{As+B}{s^2+2^2} \quad \text{(by coverup)}$$

$$s=0 \Rightarrow B = 4\left(\tfrac{1}{2} - \tfrac{2}{5}\right) = 2/5$$

$$s\to\infty \{s\bar{f}(s)\} = 0 = 2/5 + A \Rightarrow A = -2/5$$

$$f(t) = 2/5\, e^{-t} - 2/5 \cos 2t + 1/5 \sin 2t$$

(c) $$\bar{f}(s) = \frac{3}{(s+1)(s+2)^2} = \frac{A}{(s+1)} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)}$$

coverup $\Rightarrow$ $A = 3$; $B = -3$ $\quad \lim_{s\to\infty}(s\bar{f}(s)) = 0 = A + C \Rightarrow C = -3$

$$f(t) = 3e^{-t} - 3te^{-2t} - 3e^{-2t}$$

5 The impulse response is $g(t) = \beta e^{-\beta t}$ $(t>0)$.

The transmitted signal is $x(t) = e^{-\alpha t}$ $(t>0)$.

By the convolution theorem, the received signal (the 'output') is

$$y(t) = \int_0^t g(t-\tau)\,x(\tau)\,d\tau \qquad (t>0)$$

$$= \int_0^t \left[\beta e^{-\beta(t-\tau)} H(t-\tau)\right]\left[e^{-\alpha\tau} H(\tau)\right] d\tau$$

where $H(t-\tau)$ and $H(\tau)$ have been used to represent the fact that $g(t)$ and $x(t)$ are zero for $t<0$.

Since $0 \le \tau \le t$, we have

$$y(t) = \int_0^t \beta e^{-\beta(t-\tau)} e^{-\alpha\tau}\,d\tau$$

$$= \beta e^{-\beta t} \int_0^t e^{(\beta-\alpha)\tau}\,d\tau.$$

If $\beta \neq \alpha$ we have

$$y(t) = \frac{\beta e^{-\beta t}}{\beta-\alpha}\left[e^{(\beta-\alpha)t} - 1\right]$$

$$= \underline{\underline{\frac{\beta}{\beta-\alpha}\left[e^{-\alpha t} - e^{-\beta t}\right]}}$$

If $\beta = \alpha$ we have

$$y(t) = \beta e^{-\beta t}\int_0^t d\tau = \underline{\underline{\beta t\, e^{-\beta t}}}$$

Using Laplace transforms:

$$\bar{x}(s) = \frac{1}{s+\alpha} \qquad \text{(from tables)}.$$

To find the transfer function: Input $\delta(t)$ gives output $\beta e^{-\beta t}$. Hence input transform 1 gives

6 cont'd

output transform $\dfrac{\beta}{s+\beta}$.

But the transfer function $= \dfrac{\text{output transform}}{\text{input transform}}$

$$= \frac{\beta}{s+\beta}.$$

[This illustrates general theorem:

transfer function = $\mathcal{L}\{$impulse response$\}$.]

Hence $\bar{y}(s) = \dfrac{\beta}{s+\beta} \times \dfrac{1}{s+\alpha}$.

If $\beta \neq \alpha$:

$$\bar{y}(s) = \frac{\beta}{(-\beta+\alpha)(s+\beta)} + \frac{\beta}{(\beta-\alpha)(s+\alpha)} \quad \text{(by 'cover-up' rule)}$$

$$= \frac{\beta}{\beta-\alpha}\left[\frac{1}{s+\alpha} - \frac{1}{s+\beta}\right]$$

and hence, from tables,

$$\underline{\underline{y(t) = \frac{\beta}{\beta-\alpha}\left[e^{-\alpha t} - e^{-\beta t}\right]}}.$$

If $\beta = \alpha$: $\bar{y}(s) = \dfrac{\beta}{(s+\beta)^2}$.

Hence, from tables, $\underline{\underline{y(t) = \beta t\, e^{-\beta t}}}$

7

In each case use the 'virtual-earth' assumption, and let i be the current flowing from the amplifier output through the feedback components:

(a) $i = C\dot{v}_o$ and $v_i = -iR$.

Hence $C\dot{v}_o = -\frac{1}{R}v_i$.

To get the transfer function take Laplace transforms and assume zero initial conditions:

$$Cs\bar{v}_o(s) = -\frac{1}{R}\bar{v}_i(s), \text{ so } \quad \bar{v}_o(s) = -\frac{1}{sCR}\bar{v}_i(s)$$

(b) $i = \frac{1}{R}v_o$ and $i = -C\dot{v}_i$

so $\frac{v_o}{R} = -C\dot{v}_i$. Proceeding as above,

$$\frac{1}{R}\bar{v}_o(s) = -Cs\bar{v}_i(s), \quad \text{so} \quad \bar{v}_o(s) = -sCR\,\bar{v}_i(s)$$

(c) $i = \frac{v_o}{R_2} + C_2\dot{v}_o$ and $i = -\frac{v_i}{R_1} - C_1\dot{v}_i$

so $\frac{v_o}{R_2} + C_2\dot{v}_o = -\left(\frac{v_i}{R_1} + C_1\dot{v}_i\right)$.

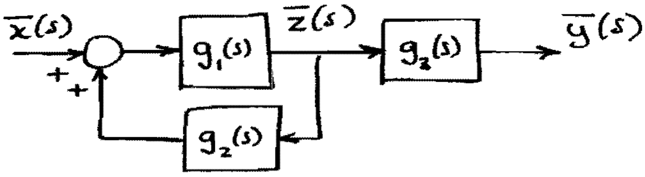
Hence $\left(\frac{1}{R_2} + sC_2\right)\bar{v}_o(s) = -\left(\frac{1}{R_1} + sC_1\right)\bar{v}_i(s)$

or
$$\bar{v}_o(s) = -\frac{R_2(1 + sC_1R_1)}{R_1(1 + sC_2R_2)}\bar{v}_i(s)$$

Note: (a) is an integrating circuit

(b) is a differentiating circuit

8 (a)



Introduce variable $\bar{z}(s)$ as shown.
Then

$$\bar{z}(s) = g_1(s)\left[\bar{x}(s) + g_2(s)\bar{z}(s)\right]$$

so $$\left[1 - g_1(s)g_2(s)\right]\bar{z}(s) = g_1(s)\bar{x}(s)$$

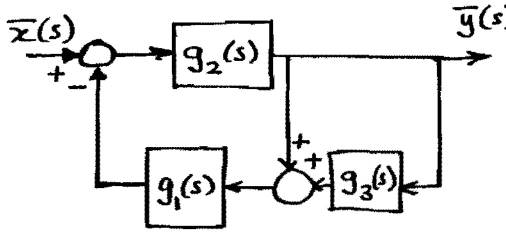
or $$\bar{z}(s) = \frac{g_1(s)}{1 - g_1(s)g_2(s)}\bar{x}(s)$$

Note this useful result for positive feedback loops

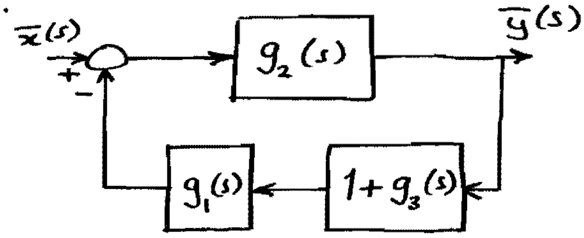
But $\bar{y}(s) = g_3(s)\bar{z}(s)$

so $$\underline{\underline{\bar{y}(s) = \frac{g_3(s)g_1(s)}{1 - g_1(s)g_2(s)}\bar{x}(s)}}$$

(b) Re-draw the figure as:



which is equivalent to:
(parallel connection rule)



Now apply the usual formula:

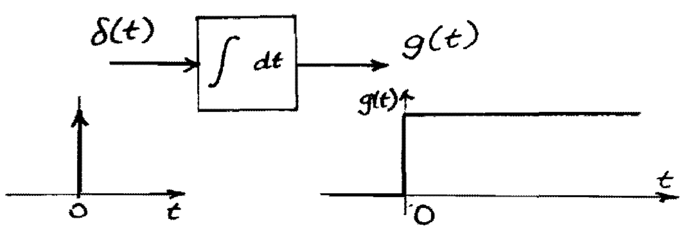
$$\underline{\underline{\bar{y}(s) = \frac{g_2(s)}{1 + g_2(s)g_1(s)\left[1 + g_3(s)\right]}\bar{x}(s)}}$$

6/1 Solutions

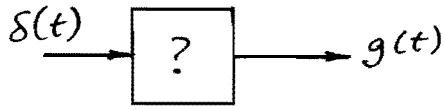
9 (a)

$$g(t) = \int_0^t \delta(\tau)\,d\tau$$

$$= \underline{\underline{H(t)}}$$



(b)



Since $\mathcal{L}\{\delta(t)\} = 1$,

$$g(t) = \mathcal{L}^{-1}\left\{ \frac{3s}{s^2+4} \times 1 \right\}$$

$$= \underline{\underline{3\cos(2t)}} \qquad \text{[from tables]}$$

10 $$\frac{\bar{y}(s)}{\bar{x}(s)} = \frac{1+as}{(1+s)(1+2s)}$$

If $x(t) = H(t)$ then $\bar{x}(s) = \frac{1}{s}$ (from tables)

so $$\bar{y}(s) = \frac{1+as}{s(1+s)(1+2s)}$$

$$= \frac{1}{s} + \frac{1-a}{1+s} + \frac{a-2}{\frac{1}{2}+s}. \quad \left(\begin{matrix}\text{using 'cover-up'}\\ \text{rule}\end{matrix}\right)$$

So read inverse transform off from tables:

$$\underline{\underline{y(t) = H(t)\left\{1 + (1-a)e^{-t} + (a-2)e^{-t/2}\right\}}}.$$

<u>Initial slope</u>: Differentiating gives

$$\dot{y}(t) = -(1-a)e^{-t} - \tfrac{1}{2}(a-2)e^{-t/2} \qquad (t>0)$$

so $$\dot{y}(0) = -(1-a) - \tfrac{1}{2}(a-2) = a/2.$$

Thus (i) $a>2 \Rightarrow \dot{y}(0) > 1$, (ii) $2>a>0 \Rightarrow 1>\dot{y}(0)>0$,

(iii) $a<0 \Rightarrow \dot{y}(0) < 0$.

<u>Turning points</u>: $\dot{y}(t)=0$ when $(1-a)e^{-t} = \frac{1}{2}(2-a)e^{-t/2}$

i.e. when $e^{t/2} = \dfrac{1-a}{1-a/2}$. For $t>0$, $e^{t/2} > 1$, so there

are turning points only if $\dfrac{1-a}{1-a/2} > 1$,

i.e. if $1 - \dfrac{a}{2-a} > 1$, i.e. if $\dfrac{a}{2-a} < 0$

which is so if <u>either</u> $a>0$, $2-a<0$, i.e. $a>2$

<u>or</u> $a<0$, $2-a>0$, i.e. $a<0$.

(i) If <u>$a>2$</u>, let t^* be the value of t at the turning point,

then $$y(t^*) = 1 + (1-a)\left(\frac{1-a/2}{1-a}\right)^2 + (a-2)\left(\frac{1-a/2}{1-a}\right) > 1$$ since $a>2$.

But $y(t) \xrightarrow[t\to\infty]{} 1$,

so the turning point must be a <u>maximum</u>.

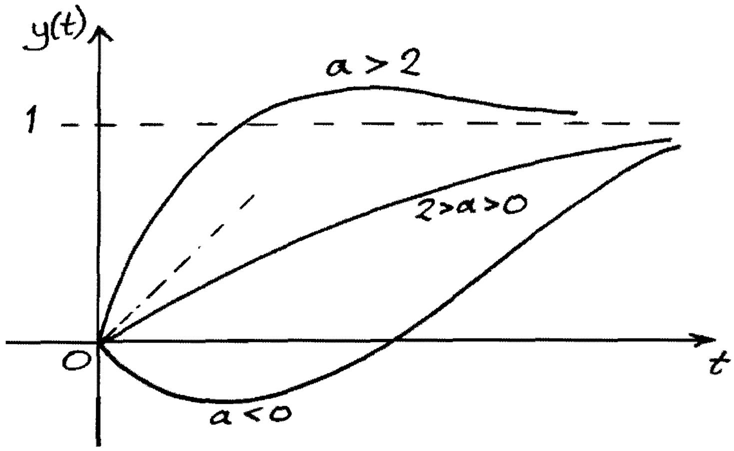
1 0 cont'd.

(ii) If $\underline{2 > a > 0}$ there is no turning point, and $\dot{y}(t) > 0$.

(iii) If $\underline{a < 0}$, we have $y(0)=0$, $\dot{y}(0) < 0$ and $y(t) \xrightarrow[t\to\infty]{} 1$,

so the turning point must be a <u>minimum.</u>

Hence sketch:



Note that an 'overshoot' is possible, even though all the poles are real.

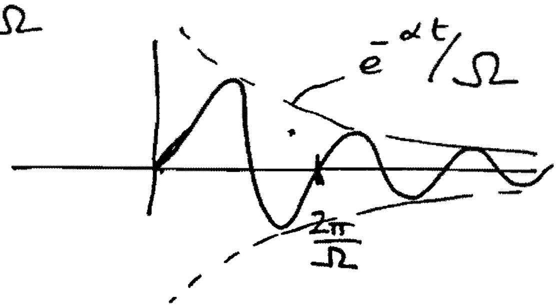
11 $$\ddot{y} + 2\alpha\dot{y} + (\Omega^2+\beta^2)y = x$$

$$G(s) = \frac{1}{s^2+2\alpha s+(\Omega^2+\beta^2)} \quad \text{(TRANSFER FUNCTION)}$$

impulse response is $\mathcal{L}^{-1}(G(s))$ (A26)

(a) $\alpha > 0$, $\beta = \alpha$, $\Omega^2 > 0$

$$\mathcal{L}^{-1}\frac{1}{(s+\alpha)^2+\Omega^2} = \frac{1}{\Omega}e^{-\alpha t}\sin\Omega t$$

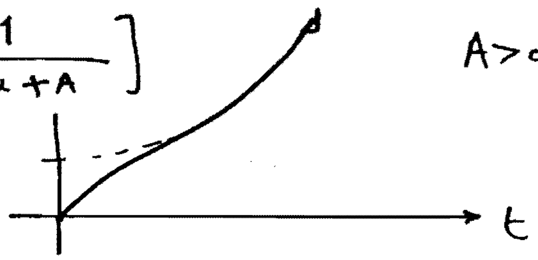


(b) $\beta = \alpha$, $\Omega^2 = -A^2 < 0$

$$\mathcal{L}^{-1}\frac{1}{(s+\alpha)^2-A^2} = \mathcal{L}^{-1}\frac{1}{(s+\alpha+A)(s+\alpha-A)}$$

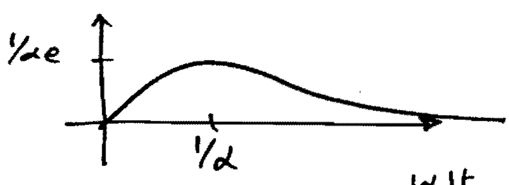
$$= \mathcal{L}^{-1}\frac{1}{2A}\left[\frac{1}{s+\alpha-A} - \frac{1}{s+\alpha+A}\right] \qquad A>\alpha$$

$$= \frac{e^{-(\alpha-A)t} - e^{-(\alpha+A)t}}{2A}$$



(c) $\Omega^2+\beta^2 = \alpha^2$

$$\mathcal{L}^{-1}\frac{1}{s^2+2\alpha s+\alpha^2} = \mathcal{L}^{-1}\frac{1}{(s+\alpha)^2} = te^{-\alpha t}$$



(d) $\alpha < 0$; $\beta = \alpha$; $\Omega^2 > 0$

$$\mathcal{L}^{-1}\frac{1}{(s+\alpha)^2+\Omega^2} = \frac{1}{\Omega}e^{|\alpha|t}\sin\Omega t$$