Ackerman's Formula for Pole Placement

Let

$$r(A) = r_0 I + r_1 A + r_2 A^2 + \cdots + r_{n-1} A^{n-1} + A^n$$

Put $F = Lr(A)$. We wish to show, for a suitable choice of L, that

$$r(A - BF) = r(A - BLr(A)) = 0$$

or, equivalently,

$$r(A - r(A)BL) = 0$$

(using Cayley-Hamilton).

Writing $Z = A - r(A)BL$, we can partially (and cleverly) expand Z^k as follows.

$$\begin{aligned}
(A - r(A)BL)^2 &= A^2 - Ar(A)BL - r(A)BLZ \\
(A - r(A)BL)^3 &= A^3 - A^2 r(A)BL - Ar(A)BLZ - r(A)BLZ^2 \\
&\vdots \\
(A - r(A)BL)^n &= A^n - A^{n-1} r(A)BL - A^{n-2} r(A)BLZ - \cdots - r(A)BLZ^{n-1} \qquad (1)
\end{aligned}$$

and so

$$\begin{aligned}
r(A - r(A)BL) = r_0 I + r_1(A - r(A)BL) + r_2(A^2 - Ar(A)BL - r(A)BLZ) + \\
r_3(A^3 - A^2 r(A)BL - Ar(A)BLZ - r(A)BLZ^2) + \cdots + \\
(A^n - A^{n-1} r(A)BL - A^{n-2} r(A)BLZ - \cdots - r(A)BLZ^{n-1})
\end{aligned}$$

$$\begin{aligned}
= r(A) - r(A)\big(r_1 BL + r_2(ABL + BLZ) + r_3(A^2 BL + ABLZ + BLZ^2) + \cdots + \\
(A^{n-1}BL + A^{n-2}BLZ + \cdots + BLZ^{n-1})\big)
\end{aligned}$$

$$= r(A)(I - \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} r_1 L + r_2 LZ + r_3 LZ^2 + \cdots LZ^{n-1} \\ r_2 L + r_3 LZ + \cdots + LZ^{n-2} \\ r_3 L + \cdots + LZ^{n-3} \\ \vdots \\ L \end{bmatrix} \qquad (2)$$

Single input case

Since $P = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$ is square the expression (2) equals 0 iff

$$\begin{bmatrix} r_1L + r_2LZ + r_3LZ^2 + \cdots LZ^{n-1} \\ r_2L + r_3LZ + \cdots + LZ^{n-2} \\ r_3L + \cdots + LZ^{n-3} \\ \vdots \\ L \end{bmatrix} P = I$$

Clearly, it is necessary that $LP = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$, and this has a unique solution since P is square and full rank.

To conclude the proof, we need to show that the products of the other rows with P equal the other rows of I. This can be shown using

$$LZ^{k+1}P = LZ^kP \begin{bmatrix} 0 & 0 & \cdots & 0 & r_0 \\ 1 & 0 & \cdots & 0 & r_1 \\ 0 & 1 & \cdots & 0 & r_2 \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & r_{n-1} \end{bmatrix}$$

(which follows from the formulae(1)). Start with showing $(r_{n-1}L + LZ)P = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$ and proceed recursively.

Multi Input Case

If a system is controllable, it can be shown that it is always possible to choose a column of B, b_k, and a matrix N such that the pair $A + BN$, b_k is observable – the proof then proceeds as above.

The Kalman Decomposition and Minimality

Consider a state transformation $x = T\tilde{x}$ (or $\tilde{x} = T^{-1}x$)

Construct T as follows.

- Let $T_{R\bar{O}}$ be a basis for null$(Q) \cap$ range(P), where null$(Q) = \bar{\mathcal{O}}$, the unobservable subspace and range$(P) = \bar{\mathcal{R}}$, the reachable subspace.
- Let T_{RO} complement $T_{R\bar{O}}$ in the reachable subspace (so $\mathcal{R}$ = range$[T_{R\bar{O}}\ T_{RO}]$).
- Let $T_{\bar{R}\bar{O}}$ complement $T_{R\bar{O}}$ in the unobservable subspace (so $\mathcal{O}$ = range$[T_{R\bar{O}}\ T_{\bar{R}\bar{O}}]$).
- Let $T_{\bar{R}O}$ complement $[T_{R\bar{O}}\ T_{RO}T_{\bar{R}\bar{O}}]$ in $\mathfrak{R}^n$.

and put $T = [T_{R\bar{O}}\ T_{RO}T_{\bar{R}\bar{O}}T_{\bar{R}O}]$ (which must, by construction, be invertible).

Using the same logic as the unobservable case (eg $A\mathcal{R} \subseteq \mathcal{R}$),

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} \\ 0 & \tilde{A}_{12} & 0 & \tilde{A}_{14} \\ 0 & 0 & \tilde{A}_{13} & \tilde{A}_{14} \\ 0 & 0 & 0 & \tilde{A}_{14} \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} 0 & \tilde{C}_2 & 0 & \tilde{C}_4 \end{bmatrix}$$

This is the Kalman decomposition. If a system is not observable or not controllable, or both, it can be used to extract the observable and controllable part of the system, which will necessarily be of lower degree. If the system is observable and controllable, then write

$$QP = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} =$$

$$\begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \\ CAB & & \cdots & \\ \vdots & & & \\ CA^{n-1}B & & \cdots & CA^{2n-1}B \end{bmatrix}$$

which will be rank n, and which can be determined from $G(s)$, since

$$G(s) = C(sI - A)^{-1}B = CB + \frac{1}{s}CAB + \frac{1}{s^2}CA^2B + \cdots .$$

It follows that no state system of smaller dimension can represent $G(s)$.

The Kalman decomposition can be represented graphically as: