IB Paper 6: Information Engineering

COMMUNICATIONS

Solutions to Examples Paper 9: Digitisation, Digital Modulation, Multiple Access

1. See the last two pages.

2. We first observe that the Fourier transform of $p(t) = \sqrt{\frac{1}{T}}\,\mathrm{sinc}\left(\frac{\pi t}{T}\right)$ is

$$P(\omega) = \sqrt{T}\,\mathrm{rect}\left(\frac{\omega}{2\pi/T}\right) = \begin{cases} \sqrt{T} & -\frac{\pi}{T} \le \omega \le -\frac{\pi}{T} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

To see this, either verify that $p(t) = \frac{1}{2\pi}\int P(\omega)e^{j\omega t}dt$, or use the duality property: observe that

$$\mathrm{rect}\left(\frac{t}{A}\right) \leftrightarrow A\,\mathrm{sinc}\left(\frac{\omega A}{2}\right)$$

Using the Duality Property (SDA Handout 4), this implies that

$$A\,\mathrm{sinc}\left(\frac{tA}{2}\right) \leftrightarrow 2\pi\,\mathrm{rect}\left(\frac{\omega}{A}\right)$$

Substituting $A = \frac{2\pi}{T}$ yields (1).

Next, from the multiplication theorem, we have

$$\int_{-\infty}^{\infty} \phi_\ell(t)\phi_m(t)\,dt = \int_{-\infty}^{\infty} \phi_\ell(t)\phi_m^*(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_\ell(\omega)\Phi_m^*(\omega)\,d\omega \tag{2}$$

where $\Phi_\ell(\omega), \Phi_m(\omega)$ are the Fourier transforms of $\phi_\ell(t), \phi_m(t)$, respectively. As $\phi_\ell(t), \phi_m(t)$ are obtained by shifting $p(t)$ by ℓT and mT, respectively, we have

$$\Phi_\ell(\omega) = e^{-j\ell T\omega}P(\omega) \tag{3}$$
$$\Phi_m(\omega) = e^{-jmT\omega}P(\omega) \tag{4}$$

Substituting (3) and (4) in (2), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \phi_\ell(t)\phi_m(t)\,dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |P(\omega)|^2\, e^{j(m-\ell)T\omega}\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} T\, e^{j(m-\ell)T\omega}\,d\omega \\ &= \begin{cases} 1 & \text{if } \ell = m \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

3. (a) As the symbols are equally likely, the optimum MAP detector reduces to a minimum-distance rule, i.e., choose $\hat{X}$ to be the symbol that minimizes $(Y-\hat{X})^2$. Thus

$$\hat{X} = \begin{cases} -3A & \text{if } -\infty < Y \le -2A \\ -A & \text{if } -2A < Y \le 0 \\ A & \text{if } 0 < Y \le 2A \\ 3A & \text{if } 2A < Y < \infty \end{cases}$$

(b) The probability of error when $3A$ is sent can be computed as

$$\begin{aligned} P(\hat{X} \neq -3A \mid X = -3A) &= P(Y > -2A \mid X = -3A) \\ &= P(-3A + N > -2A \mid X = -3A) \\ &= P(N > A) \\ &= P\left(\frac{N}{\sigma} > \frac{A}{\sigma}\right) \\ &= \mathcal{Q}\left(\frac{A}{\sigma}\right). \end{aligned} \tag{5}$$

By symmetry, the probability of error when $3A$ is transmitted is the same.

(c) The probability of error when $-A$ is sent can be computed as

$$\begin{aligned} P(\hat{X} \neq -A \mid X = -A) &= P(\{Y \le -2A\} \cup \{Y > 0\} \mid X = -A) \\ &= P(\{-A + N \le -2A\} \cup \{-A + N > 0\} \mid X = -A) \\ &= P(\{N \le -A\} \cup \{N > A\}) \\ &= P\left(\{\tfrac{N}{\sigma} \le -\tfrac{A}{\sigma}\} \cup \{\tfrac{N}{\sigma} > \tfrac{A}{\sigma}\} \right) \\ &= 2\mathcal{Q}\left(\frac{A}{\sigma}\right). \end{aligned} \tag{6}$$

By symmetry, the probability of error when A is transmitted is the same. Using (5) and (6), the overall probability of error is

$$\begin{aligned} P_e &= \frac{1}{4}\left(\mathcal{Q}\left(\frac{A}{\sigma}\right) + 2\mathcal{Q}\left(\frac{A}{\sigma}\right) + 2\mathcal{Q}\left(\frac{A}{\sigma}\right) + \mathcal{Q}\left(\frac{A}{\sigma}\right)\right) \\ &= \frac{3}{2}\mathcal{Q}\left(\frac{A}{\sigma}\right). \end{aligned} \tag{7}$$

(d) The average energy per symbol is

$$E_s = \frac{1}{4}\left((-3A)^2 + (-A)^2 + (A)^2 + (3A)^2\right) = 5A^2$$

The average energy per bit is $E_b = E_s/\log_2 4 = \frac{5A^2}{2}$.

(e) From part (d), $A = \sqrt{\frac{2E_b}{5}}$. Substituting in (7), we obtain

$$P_e = \frac{3}{2}\mathcal{Q}\left(\sqrt{\frac{2E_b}{5\sigma^2}}\right).$$

4. (a) The average energy per symbol is 1 for $\mathcal{C}_1$, and A^2 for $\mathcal{C}_2$. Hence $A = 1$ and the constellation symbols in $\mathcal{C}_2$ are

$$p_1 = \frac{1}{\sqrt{2}}(1+j), \quad p_2 = \frac{1}{\sqrt{2}}(1-j), \quad p_3 = \frac{1}{\sqrt{2}}(-1-j), \quad p_4 = \frac{1}{\sqrt{2}}(-1+j)$$

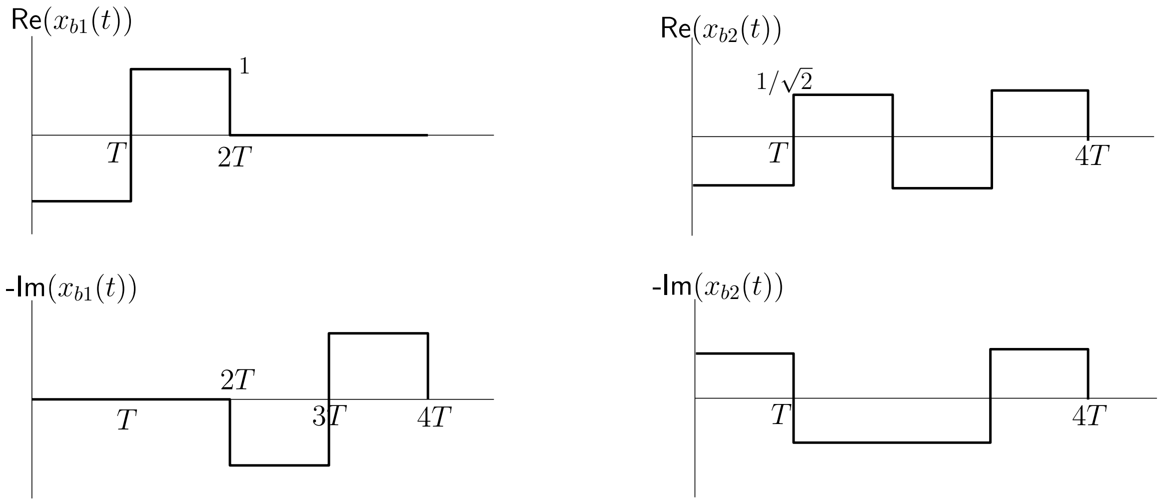
(b) For the information bits 11 00 10 01, the sequence of constellation symbols is p_3, p_1, p_4, p_2.
With $\mathcal{C}_1$, the modulated baseband waveform is

$$x_{b1}(t) = -1\,p(t) + 1\,p(t-T) + j\,p(t-2T) - j\,p(t-3T)$$

With $\mathcal{C}_2$, the modulated baseband waveform is

$$x_{b2}(t) = \frac{-1}{\sqrt{2}}(1+j)\,p(t) + \frac{1}{\sqrt{2}}(1+j)\,p(t-T) + \frac{1}{\sqrt{2}}(-1+j)\,p(t-2T) + \frac{1}{\sqrt{2}}(1-j)\,p(t-3T)$$

The baseband waveforms modulating the carriers $\cos(2\pi f_c t)$ and $\sin(2\pi f_c t)$ are $\mathsf{Re}(x_b(t))$ and $-\mathsf{Im}(x_b(t))$, which are sketched below for $x_{b1}(t)$ and $x_{b2}(t)$.

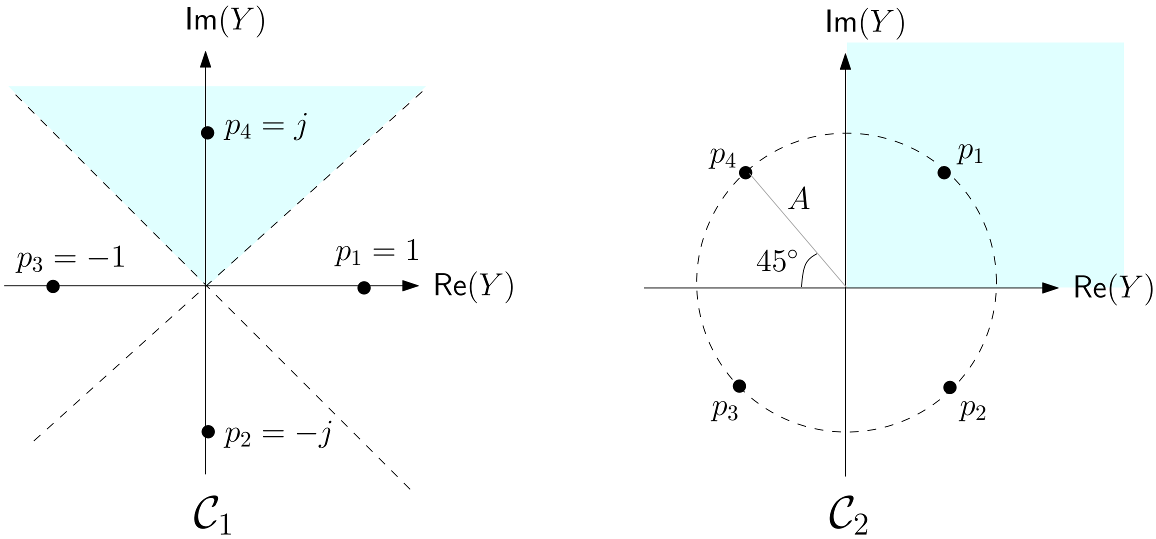


(c) The structure of the receiver is described in the lecture notes: Multiply the received waveform $y(t)$ by $\cos(2\pi f_c t)$ and low-pass filter to obtain the baseband waveform $y^r(t)$; also multiply $y(t)$ by $-\sin(2\pi f_c t)$ and low-pass filter to obtain the baseband waveform $y^i(t)$. Demodulate $y^r(t), y^i(t)$ by passing each through a matched filter with impulse response $p(-t)$, and sample the output at times mT, for $m = 0, 1, \ldots$. At the output of the demodulator, we have discrete-time samples $Y_m = (Y_m^r, Y_m^i)$, where for $m \geq 0$

$$Y_m^r = X_m^r + N_m^r, \quad Y_m^i = X_m^i + N_m^i,$$

where $X_m = (X_m^r, X_m^i)$ is the transmitted constellation symbol for time period m. The optimal decision rule is the "minimum-distance rule": pick the constellation symbol closest to the observed complex number Y. Mathematically, this is expressed as

$$\hat{X}_m = \underset{x \in \text{constellation}}{\arg\min} \; (Y_m - x)^2$$



The decision boundaries for $\mathcal{C}_1$ are shown by the dashed lines in the figure above, with the shaded region indicating the decision region for p_4. The decision regions for $\mathcal{C}_2$ are the four quadrants of the complex plane; the shaded region indicates the decision region for p_1.

5. (a) For $x_b(t)$ to be real, we need $X_b(-f) = X^*(f)$. In particular, this implies $|X_b(-f)| = |X_b(f)|$. But the figure clearly shows that this is not satisfied as $|X_b(f)|$ is not symmetric around the origin. Hence $x_b(t)$ is complex-valued.

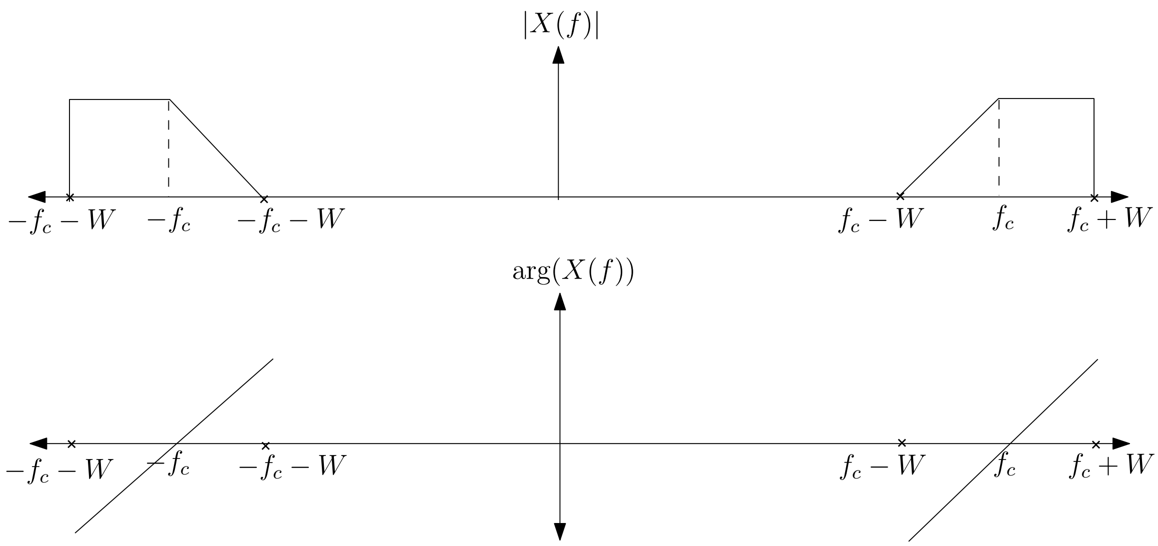
(b) We can write

$$x(t) = \frac{x_b(t)e^{j2\pi f_c t} + x_b^*(t)e^{-j2\pi f_c t}}{2}.$$

We have $\mathcal{F}\left[x_b(t)e^{j2\pi f_c t}\right] = X_b(f - f_c)$. Further,

$$\begin{aligned}\mathcal{F}\left[x_b^*(t)e^{-j2\pi f_c t}\right] &= \int_{-\infty}^{\infty} x_b^*(t)e^{-j2\pi f_c t}e^{-j2\pi f t}dt = \left(\int x_b(t)e^{j2\pi(f+f_c)t}dt\right)^* \\ &= \left(\int x_b(t)e^{-j2\pi(-f-f_c)t}dt\right)^* = X_b^*(-f - f_c).\end{aligned}$$

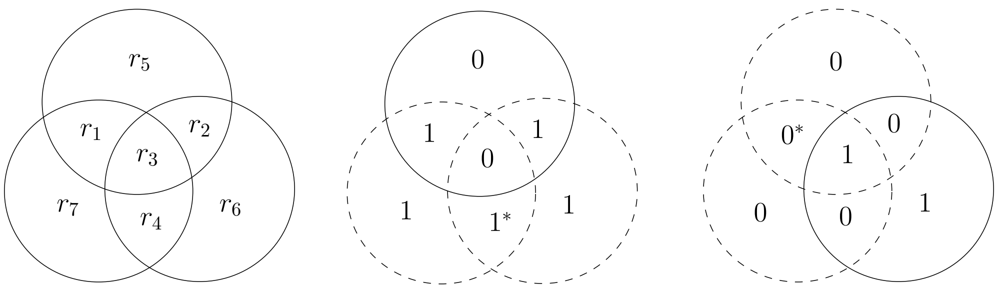
Therefore $X(f) = \frac{1}{2}\left[X_b(f - fc) + X_b^*(-f - f_c).\right]$. It can be easily verified from this expression that that $X(-f) = X^*(f)$, as it must be since $x(t)$ is real. The spectrum of $X(f)$ is shown below.



6. (a) The rate of the code is 1/5. The "majority-vote" decoder makes an error if the channel flips at least three of the five bits. The probability of error is therefore

$$P_e = \binom{5}{3}\epsilon^3(1-\epsilon)^2 + \binom{5}{4}\epsilon^4(1-\epsilon) + \binom{5}{5}\epsilon^5$$

(b) The Hamming code can be decoded using the "parity circles" in Fig. (a) below. If *no errors are made*, each of the three circles should have parity 0 – this is because $r_1, \ldots, r_4$ are the four source bits and $r_5 = r_1 \oplus r_2 \oplus r_3$, $r_6 = r_2 \oplus r_3 \oplus r_4$, and $r_7 = r_1 \oplus r_3 \oplus r_4$.



(a) **(b)** **(c)**

(i) To decode $\mathbf{r} = [1\ 1\ 0\ 1\ 0\ 1\ 1]$, arrange the bits in circles as shown in Fig. (b). The dashed circles indicate those for which the parity is 1. The decoder flips one bit so that the parity of the dashed circles becomes 0, *without* changing the 0 parity of the solid circle. Due to the properties of the Hamming code, such a bit can always be found, and is indicated by a star in Fig. (b) (bit r_4). Therefore the decoded codeword is $[1\ 1\ 0\ 0\ 0\ 1\ 1]$.

(ii) The received sequence $[0\ 0\ 1\ 0\ 0\ 1\ 0]$ is arranged as in Fig. (c). The decoder can ensure 0 parity to all three circles by flipping the starred bit r_1. Therefore the decoded codeword is $[1\ 0\ 1\ 0\ 0\ 1\ 0]$, which is different from the all-zeros codeword. The Hamming code can reliably correct any one bit flip in a transmitted codeword, but fails if there are two or more flipped bits.

(iii) The probability of decoding error is

$$P_e = \sum_{k=2}^{7}\binom{7}{k}\epsilon^k(1-\epsilon)^{7-k} = 1 - \binom{7}{0}(1-\epsilon)^7 - \binom{7}{1}\epsilon(1-\epsilon)^6.$$

7. (a) **FDMA** multiplexes the users in frequency. Each user is assigned a bandwidth $B_u = \frac{B}{K}$ and it is modulated to a carrier such that there are no overlaps among the spectrum of adjacent users. Each user transmits with power P.

TDMA multiplexes the users in the time domain. Each user is assigned a given time slot of duration $T_u = \frac{T_f}{K}$, where T_f is the duration of a frame, and each time it transmits, it uses the whole bandwidth B with power KP to keep an average power of P.

CDMA multiplexes the users assigning different signatures to different users. Each user transmits using the whole bandwidth B over the whole duration of the frame T_f with power P. Signatures should be orthogonal to enable recovery of the transmitted data by each user.

(b) Each user employs binary PAM modulation with rectangular pulses, and the spectrum is given by

$$S(f) = \frac{1}{2}\left[X(f - f_c) + X(f - f_c)\right]$$

where

$$X(f) = \sqrt{T}\text{sinc}(\pi f T)$$

is the spectrum of the unmodulated digital signal using a rectangular pulse. The main-lobe bandwidth is $\frac{2}{T}$ and the first-side-lobe bandwidth $\frac{4}{T}$ (note that each user transmits a band-pass signal). If each user transmits at a rate $R = 200$kbit/s, the symbol period is

$$T = \frac{1}{R} = 5\mu\text{s}.$$

The first-side-lobe bandwidth is

$$B_{\text{sl}} = \frac{4}{T} = 800kHz$$

and therefore the number of users is

$$K = \frac{20\text{MHz}}{800\text{kHz}} = 25$$

(c) To show orthogonality we have to show that

$$\frac{1}{T}\int_0^T c_i(t)c_j(t)dt = 0$$

for all pairs of different signatures.

$c_1(t), c_2(t)$

$$\frac{1}{T}\int_0^T c_1(t)c_2(t)dt = \frac{1}{T}\int_0^{\frac{T}{4}} 1dt + \frac{1}{T}\int_{\frac{T}{4}}^{\frac{3T}{4}} -1dt + \frac{1}{T}\int_{\frac{3T}{4}}^{T} 1dt = 0$$

$c_1(t), c_3(t)$

$$\frac{1}{T}\int_0^T c_1(t)c_2(t)dt = \frac{1}{T}\int_0^{\frac{T}{2}} 1dt + \frac{1}{T}\int_{\frac{T}{2}}^T -1dt = 0$$

$c_1(t), c_4(t)$

$$\frac{1}{T}\int_0^T c_1(t)c_2(t)dt = \frac{1}{T}\int_0^{\frac{T}{4}} 1dt + \frac{1}{T}\int_{\frac{T}{4}}^{\frac{T}{2}} -1dt + \frac{1}{T}\int_{\frac{T}{2}}^{\frac{3T}{4}} 1dt + \frac{1}{T}\int_{\frac{3T}{4}}^T -1dt = 0$$

$c_2(t), c_3(t)$

$$\frac{1}{T}\int_0^T c_1(t)c_2(t)dt = \frac{1}{T}\int_0^{\frac{T}{4}} 1dt + \frac{1}{T}\int_{\frac{T}{4}}^{\frac{T}{2}} -1dt + \frac{1}{T}\int_{\frac{T}{2}}^{\frac{3T}{4}} 1dt + \frac{1}{T}\int_{\frac{3T}{4}}^T -1dt = 0$$

$c_2(t), c_4(t)$

$$\frac{1}{T}\int_0^T c_1(t)c_2(t)dt = \frac{1}{T}\int_0^{\frac{T}{4}} 1dt + \frac{1}{T}\int_{\frac{T}{4}}^{\frac{T}{2}} 1dt + \frac{1}{T}\int_{\frac{T}{2}}^{\frac{3T}{4}} -1dt + \frac{1}{T}\int_{\frac{3T}{4}}^T -1dt = 0$$

$c_3(t), c_4(t)$

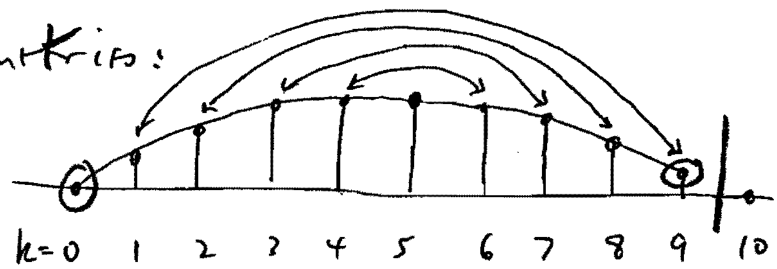
$$\frac{1}{T}\int_0^T c_1(t)c_2(t)dt = \frac{1}{T}\int_0^{\frac{T}{4}} 1dt + \frac{1}{T}\int_{\frac{T}{4}}^{\frac{T}{2}} -1dt + \frac{1}{T}\int_{\frac{T}{2}}^{\frac{3T}{4}} -1dt + \frac{1}{T}\int_{\frac{3T}{4}}^T 1dt = 0$$

1.) The method for doing this is :

for each sample in turn, identify the nearest quantised level q, compute the error $(x(kT)-q)$ and square it. Compute the mean value of this squared error over <u>one cycle</u> of the sinewave.

Note the symmetries:

$x(kT)$



and then the same negated (with therefore same squared error

Therefore to compute the <u>mean squared</u> error, find

$$\frac{1}{10}\left[\epsilon_0^2+2\epsilon_1^2+2\epsilon_2^2+2\epsilon_3^2+2\epsilon_4^2+\epsilon_5^2\right].$$

So for (a)

	x_0	x_1	x_2	x_3	x_4	x_5	...
x :	0	.2781	.529	.728	.856	.9	
q :	0	.25	.5	.75	.875	.875	
$\lvert err \rvert$:	[0	.0281	.029	.0219	.019	.025]	

<u>MSE = 5.57 $\times 10^{-4}$</u>

second signal

x :	0	0.02781	0.0529	0.0728	0.0856	0.09	...
q :	0	0	0.0625	0.0625	0.0625	0.0625	
$\lvert e \rvert$:	[0	.02781	.0096	0.0103	0.0231	0.0275]	

<u>MSE = 3.77 $\times 10^{-4}$</u>

1.) cont.

for (b)

	x	0	0.2781	.529	.728	.856	.9
	q :	0	.2791	.4823	.6944	.8333	.8333
(from list)	$\lvert e\rvert$:	[0	.001	.0468	.0337	.0226	.0667]

$\underline{MSE = 0.0012}$

second signal	x	0	.0278	.0529	.0728	.0856	.09
	q	0	.028	.0561	.0841	.0841	.0841
	$\lvert e\rvert$:	[0	.0002	.0032	.0113	.0015	.0059]

$\underline{MSE = 3.15 \times 10^{-5}}$

c) <u>SNR</u> Signal "power" for first signal $= \dfrac{0.9^2}{2}$

for 2nd signal $= \dfrac{0.09^2}{2}$

$$SNR = 10 \log_{10} \left(\frac{\text{Signal "power"}}{MSE} \right)$$

hence answers.