Cambridge University Engineering Dept. Third year

Module 3F2: Systems and Control

LECTURE NOTES 1: STATE-SPACE SYSTEMS

Contents

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November 2001

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January 2009
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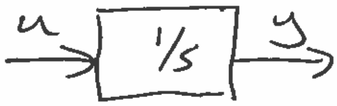
January 2015, 2019

1 State Space Descriptions Of Dynamical Systems

1.1 States

The essence of a dynamical system is its memory, i.e. the present output, $y(t)$ depends on past inputs, $u(\tau)$, for $\tau \leq t$. eg Integrator

$$y(t) = \int_{-\infty}^{t} u(\tau)\, d\tau$$



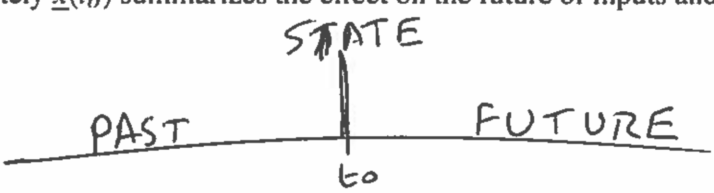
Three sets of variables define a dynamical system -

1. The inputs, $u_1(t), u_2(t), \ldots u_m(t)$ (input vector $\underline{u}(t)$)
2. The state variables, $x_1(t), \ldots x_n(t)$ (state vector $\underline{x}(t)$)
3. The outputs, $y_1(t), y_2(t), \ldots y_p(t)$, (output vector $\underline{y}(t)$)

The *states* are a subset of the system variables which satisfy the following two properties

1. for any $t_o < t_1$, $\underline{x}(t_1)$ can always be determined from $\underline{x}(t_o)$ and $\underline{u}(\tau), t_o \leq \tau \leq t_1$. STATE PROPERTY
2. The outputs $\underline{y}$ at time t, is a *memoryless* function of $\underline{x}(t)$ and $\underline{u}(t)$ (that is, $\underline{y}(t)$ depends only on $\underline{x}(t)$ and $\underline{u}(t)$).

That is completely $\underline{x}(t_o)$ summarizes the effect on the future of inputs and states prior to t_o.



eg Integrator

Let $x(t) = 3y(t)$

$$\Rightarrow x(t_1) = 3y(t_1) = 3\int_{-\infty}^{t_1} u(\tau)\, d\tau$$

$$= 3\int_{-\infty}^{t_0} u(\tau)\, d\tau + 3\int_{t_0}^{t_1} u(\tau)\, d\tau$$

$$= 3\,y(0) + 3\int_{t_0}^{t_1} u(\tau)\,d\tau$$

$$= x(t_0) + 3\int_{t_0}^{t_1} u(t)\,d\tau$$

$\Rightarrow$ STATE PROPERTY is Satisfied

NOT UNIQUE   eg $x(t) = y(t)$

or $x(t) = 5y(t)$ equally valid

The most general class of dynamical system that we will consider is described by the set of first order ordinary differential equations -

$$\mathcal{S}\left\{\begin{array}{rcll} \dfrac{dx_i}{dt} & = & f_i(x_1(t), x_2(t), ...x_n(t), u_1(t), ...u_m(t), t), & i = 1, 2, ...n. \\ y_j(t) & = & g_j(x_1(t), ...x_n(t), u_1(t), ...u_m(t), t) & j = 1, 2, ...p. \end{array}\right.$$

$\mathcal{S}$ is the standard form for a **state-space dynamical system model.** Or a vector form -

$$\mathcal{S}\left\{\begin{array}{rcl} \dot{\underline{x}}(t) & = & \underline{f}(\underline{x}(t), \underline{u}(t), t) \\ \underline{y}(t) & = & \underline{g}(\underline{x}(t), \underline{u}(t), t) \end{array}\right.$$

Note that $\mathcal{S}$ has only first order ode's, but we can use a standard technique to convert high order o.d.e's to first order vector o.d.e.'s, by the use of auxiliary variables.

Ex:

$$\dddot{y} + 6y\ddot{y} + 5(\dot{y})^3 + 12\sin(y) = \cos(t)$$

a state variable is given by: $x_1 = y,\ \ x_2 = \dot{y},\ \ x_3 = \ddot{y}$, when,

$$\left\{\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & x_3 \\ \dot{x}_3 & = & -6x_1x_3 - 5x_2^3 - 12\sin(x_1) + \cos(t) \end{array}\right.$$

which is in the form

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t), t)$$

where $\underline{f}$ is a vector valued function of a vector, i.e.

$$\underline{f}(\underline{x}, t) = \begin{bmatrix} x_2 \\ x_3 \\ -6x_1x_3 - 5x_2^3 - 12\sin(x_1) + \cos(t) \end{bmatrix}$$

For linear time-invariant dynamical systems we use the standard form:

$$S\begin{cases} \underline{\dot{x}}(t) &= \overset{n\times n}{A}\underline{x}(t) + \overset{n\times m}{B}\underline{u}(t) \\ \underline{y}(t) &= \underset{p\times n}{C}\underline{x}(t) + \underset{p\times m}{D}\underline{u}(t) \end{cases}$$

$\underline{x} \in \mathbb{R}^n$

$\underline{u} \in \mathbb{R}^m$

$\underline{y} \in \mathbb{R}^p$

1.2 General Guidelines For State-Space Modelling

Method 1: Choose system states, $x_1, x_2, \ldots x_n$ by considering the independent 'energy storage devices' or 'memory elements',

e.g. $E = \frac{1}{2}Li^2$ $E = \frac{1}{2}CV^2$

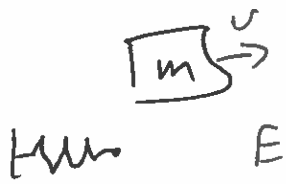
electrical circuits - current in L or voltage on C.

mechanics - positions and velocities of masses (linear and angular).

chemical engineering - temperature, pressure, volume, concentration.

Then use the basic physical laws derive expressions involving $\dot{x}_i$, e.g.

- $V = Ldi/dt, i = Cdv/dt$
- $m \times acc^n = \text{force};\quad \dfrac{d(\text{pos}^n)}{dt} = \text{velocity}.$
- $\dfrac{d(\text{volume})}{dt} = \text{flow} = \text{function of pressure}.$



$E = \frac{1}{2}mv^2$

$E = \frac{1}{2}kx^2$

Solve for $\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}, t)$.

Method 2: Choose state variables as successive time derivatives.

Example: Original ODE: $J\ddot{\theta} + B\dot{\theta} = M$.
Let $x_1 = \theta$, $x_2 = \dot{\theta}$. Then

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -\frac{B}{J}x_2 + \frac{1}{J}M$$

input

$$\underline{\dot{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -B/J \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} M$$

In general for n'th-order ODE in θ define

$$x_1 = \theta, \quad x_2 = \frac{d\theta}{dt}, \quad \ldots, \quad x_n = \frac{d^{n-1}\theta}{dt^{n-1}}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= \ldots \text{ (depends on ODE)} \end{aligned}$$

1.3 Ideal Operational Amplifier Circuits

In a negative feedback configuration the amplifier acts so as to make the +ve and -ve inputs have essentially equal voltages.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \overbrace{\begin{bmatrix} -\frac{1}{R_1C_1} & 0 \\ \frac{1}{R_2C_2} & 0 \end{bmatrix}}^{\text{"A"}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{1}{C_1R_1} \\ 0 \end{bmatrix}}^{\text{"B"}} v_i$$

$$V_o = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{\text{"C"}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\text{"D"}} v_i$$

$$\begin{cases} C_1\dot{x}_1 = (V_i - x_1)/R_1; \quad C_2\dot{x}_2 = x_1/R_2 \\ V_o = x_1 + x_2 \end{cases}$$

Linear operation until the amplifier *saturates* when quite different equations may hold.

If the amplifier output saturates at $\pm V_s$, then for $|x_1 + x_2| \geq V_s$

we will have: $C_2\dot{x}_2 = (\pm V_s - x_2)/R_2$.

1.4 Velocity Fields In The State-Space

Consider the free motion of the time invariant dynamical system -

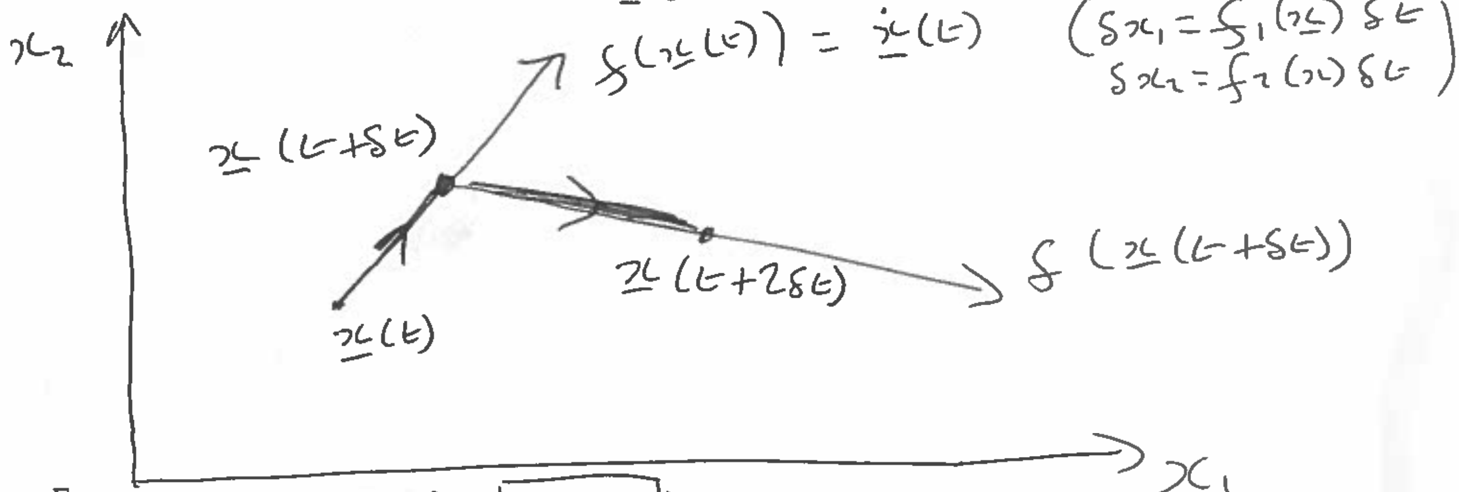
$$\underline{\dot{x}} = \underline{f}(\underline{x}) \implies \underline{x}(t+\delta t) \cong \underline{x}(t) + \underbrace{\underline{f}(\underline{x}(t))\delta t}_{\delta x} \qquad (\underline{u} = 0)$$

$\frac{dx}{dt}$

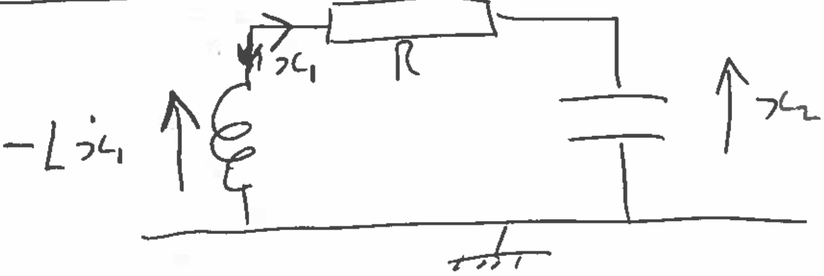
This implies a velocity field in the state space which can give a good qualitative idea of the system's behaviour in simple cases.

Ex n=2

The velocity field can be sketched by drawing an arrow in the direction $\underline{f}(\underline{x})$ for many values of $\underline{x}$. Equilibrium points (or *singular* points) where $\underline{f}(\underline{x}_e) = \underline{0}$, may be stable or unstable.



Ex.



$$-L\frac{dx_1}{dt} = x_1 R + x_2$$

$$C\frac{dx_2}{dt} = x_1$$

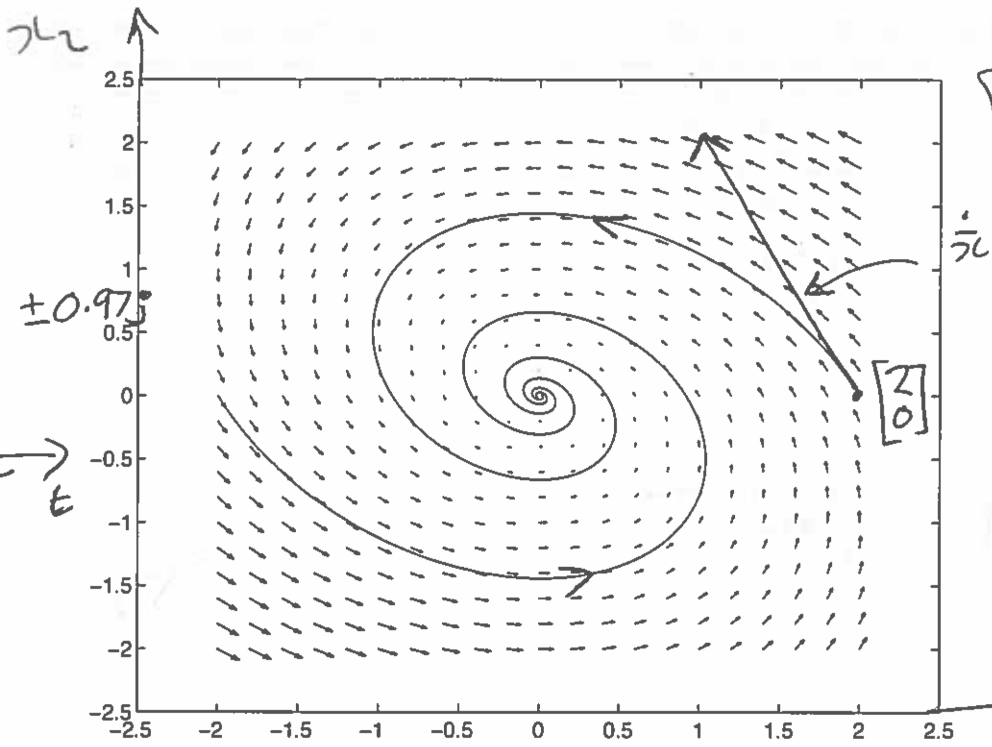
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\underline{x} + \underbrace{B\underline{u}}_{=0}$$

i) Let $R = 1/2,\ L = 1,\ C = 1$ (in consistent units) then

$$A = \begin{bmatrix} -\frac{1}{2} & -1 \\ 1 & 0 \end{bmatrix}$$

$\left[\delta t \approx 0.05 \text{ for}\right]$

eig(A) $= -0.25 \pm 0.97j$

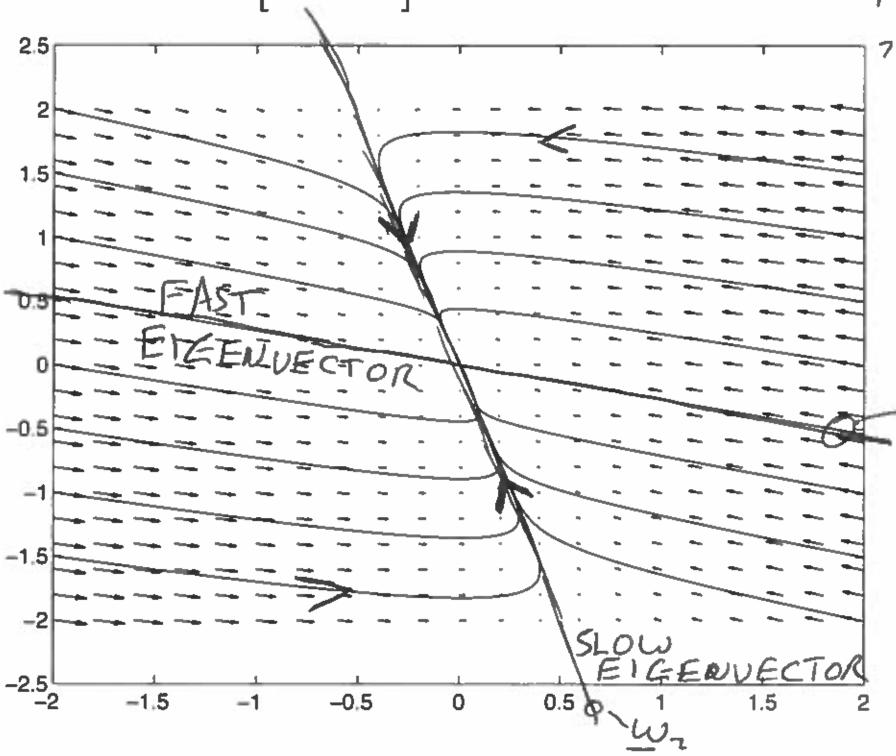


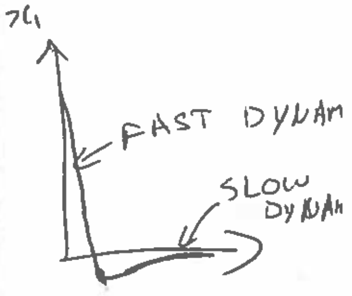
$$\dot{x} = \begin{bmatrix}-1/2 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$$

Figure 1: State space trajectories: underdamped RLC network

$$\underline{x}(t+\delta t) \approx \underline{x}(t) + \underline{\dot{x}}(t)\,\delta t = \underline{x}(t) + A\underline{x}(t)\,\delta t$$

(ii) Let $R = 4, L = 1, C = 1 \Rightarrow A = \begin{bmatrix}-4 & -1\\ 1 & 0\end{bmatrix}$ eig(A) $= \underset{\lambda_1}{-3.73}, \underset{\lambda_2}{-0.27}$





OVERDAMPED

Figure 2: State space trajectories: overdamped RLC network

Note: Changing sign of $A \Rightarrow \lambda_1 \to -\lambda_1$, $\lambda_2 \to -\lambda_2$ & Arrows change direction

Suppose $\quad A\underline{w}_1 = \lambda_1 \underline{w}_1$

& $\quad \underline{x}(0) = \underline{w}_1$

THEN $\quad \dot{\underline{x}} = A\underline{w}_1 = \lambda_1 \underline{w}_1 = \lambda_1 \underline{x}$

ie $\quad \dot{x}_1 = \lambda_1 x_1$

$\quad \dot{x}_2 = \lambda_1 x_2$

$\Rightarrow \underline{x}(t) = e^{\lambda_1 t}\underline{x}(0) \leftarrow$ FAST DECAY

If $\underline{x}(0) = \underline{w}_2 \Rightarrow \underline{x}(t) = e^{\lambda_2 t}\underline{x}(0) \leftarrow$ SLOW DECAY

Finding ev's

$$\det\begin{pmatrix} \lambda+4 & 1 \\ -1 & \lambda \end{pmatrix}$$

$$= \lambda^2 + 4\lambda + 1 = 0$$

$$= (\lambda+2)^2 - 3 = 0$$

$$\lambda = -2 \pm \underset{\substack{\uparrow \\ 1.73}}{\sqrt{3}} \quad \simeq -3.73, -0.27$$

$$\begin{pmatrix} 0.27 & 1 \\ -1 & -3.73 \end{pmatrix}\underline{w}_1 = 0$$

$$w_1 = \begin{bmatrix} -1 \\ 0.27 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 3.73 & 1 \\ -1 & -0.27 \end{bmatrix}\underline{w}_2 = 0 \Rightarrow \underline{w}_2 = \begin{bmatrix} 0.27 \\ -1 \end{bmatrix}$$

2 Linearizing Nonlinear Dynamical Systems

2.1 Linearizing the standard form

Suppose we have a nonlinear (time invariant) dynamical system in state space form -

$$S \begin{cases} \dot{\underline{x}} &= \underline{f}(\underline{x}, \underline{u}) \\ \underline{y} &= \underline{g}(\underline{x}) \end{cases}$$

Let $\underline{x}_e$ be an equilibrium state for the system when $\underline{u}(t) = \underline{u}_e$ (constant)

i.e. $\underline{f}(\underline{x}_e, \underline{u}_e) = \underline{0}$ and also let $\underline{y}_e = \underline{g}(\underline{x}_e)$.

Now consider small perturbations from this equilibrium, let

$$\underline{x}(t) = \underline{x}_e + \underline{\delta x}(t), \quad \underline{u}(t) = \underline{u}_e + \underline{\delta u}(t), \quad \underline{y}(t) = \underline{y}_e + \underline{\delta y}(t)$$

A Taylor Series expansion of the i-th equation of $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$ gives

$$\dot{x}_i = \overset{0}{\cancel{\dot{x}_{ei}}} + \delta\dot{x}_i \quad = f_i(x_{1e} + \delta x_1, x_{2e} + \delta x_2, \ldots, u_{1e} + \delta u_1, \ldots)$$

$$= f_i(x_{1e}, \ldots, u_{1e}, \ldots) + \delta x_1 \left.\frac{\partial f_i}{\partial x_1}\right|_e + \ldots$$

$$= f_i(\underline{x}_e, \underline{u}_e) + \left.\frac{\partial f_i}{\partial x_1}\right|_{\underline{x}_e, \underline{u}_e} \delta x_1 + \left.\frac{\partial f_i}{\partial x_2}\right|_{\underline{x}_e, \underline{u}_e} \delta x_2 + \ldots + \left.\frac{\partial f_i}{\partial x_n}\right|_{\underline{x}_e, \underline{u}_e} \delta x_n$$

$$+ \left.\frac{\partial f_i}{\partial u_1}\right|_{\underline{x}_e, \underline{u}_e} \delta u_1 + \ldots + \left.\frac{\partial f_i}{\partial u_m}\right|_{\underline{x}_e, \underline{u}_e} \delta u_m + \text{Remainder}$$

(HIGHER ORDER TERMS)

Thus for small $\underline{\delta x}$ and $\underline{\delta u}$ ($\Rightarrow$ v. small remainder) we get

$$\delta\dot{x}_i = [-A_i-]\,\underline{\delta x} + [-B_i-]\,\underline{\delta u}$$

rows

$$\dot{\underline{\delta x}} \cong A\underline{\delta x} + B\underline{\delta u}$$

where $A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \end{bmatrix}$

OR

$$A = \frac{\partial \underline{f}}{\partial \underline{x}}(\underline{x}_e, \underline{u}_e) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\underline{x}=\underline{x}_e, \underline{u}=\underline{u}_e}$$

$$B = \frac{\partial \underline{f}}{\partial \underline{u}}(\underline{x}_e, \underline{u}_e) = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{\underline{x}=\underline{x}_e, \underline{u}=\underline{u}_e}$$

Similarly $\delta \underline{y} = C\delta \underline{x}$ where $C = \frac{\partial g(\underline{x}_e)}{\partial \underline{x}}$.

The linearized system equations are thus

$$\begin{cases} \delta \dot{\underline{x}} & = & A\delta \underline{x} & + & B\delta \underline{u} \\ \delta \underline{y} & = & C\delta \underline{x} \end{cases}$$

which will accurately predict the system behaviour for $\underline{x}$ and $\underline{u}$ close to $\underline{x}_e$ and $\underline{u}_e$ respectively.

2.2 Linearizing when the State Equations are Implicit

Quite often a system's equations cannot easily be written as $\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u})$ but can be written as

$\underline{F}(\dot{\underline{x}}, \underline{x}, \underline{u}) = \underline{0}$ (n equations in the n unknowns $\dot{x}_1 \ldots \dot{x}_n$.)

The linearized model can be derived without solving for $\dot{\underline{x}}$ as follows. Since $(\underline{x}_e, \underline{u}_e)$ gives an equilibrium we have,

$\underline{F}(\underline{0}, \underline{x}_e, \underline{u}_e) = \underline{0}$

Now linearize $\underline{F}$ about $(\underline{0}, \underline{x}_e, \underline{u}_e)$ to get

$$\underbrace{\left.\frac{\partial \underline{F}}{\partial \dot{\underline{x}}}\right|_{(\underline{0}, \underline{x}_e, \underline{u}_e)}}_{L} \delta \dot{\underline{x}} + \underbrace{\left.\frac{\partial \underline{F}}{\partial \underline{x}}\right|_{(\underline{0}, \underline{x}_e, \underline{u}_e)}}_{M} \delta \underline{x} + \underbrace{\left.\frac{\partial \underline{F}}{\partial \underline{u}}\right|_{(\underline{0}, \underline{x}_e, \underline{u}_e)}}_{N} \delta \underline{u} \simeq \underline{0}$$

$$L\delta \dot{\underline{x}} + M\delta \underline{x} + N\delta \underline{u} \simeq \underline{0} \quad \Longrightarrow \quad \delta \dot{\underline{x}} \simeq \underbrace{-L^{-1}M}_{A}\delta \underline{x} \underbrace{- L^{-1}N}_{B}\delta \underline{u}$$

(if L is invertible)

(N.B. no nonlinear equations to solve except to obtain the equilibrium point)

2.3 Behaviour of Nonlinear Systems

As mentioned above the linearized equations will accurately predict the behaviour of the nonlinear system for $\underline{x}$ and $\underline{u}$ close to their equilibrium values. When the states and inputs are far away from the equilibrium values then the behaviour can be quite different, e.g.

- many equilibria with some stable and some unstable (e.g. inverted pendulum).
- limit cycles.
- divergence.

Example: Van der Pol oscillator

$$\ddot{x} - (1-x^2)\dot{x} + x_1 = 0$$

$$x_1 = x$$
$$x_2 = \dot{x}$$

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + (1 - x_1^2)x_2 \end{cases}$$

$\underline{x} = 0 \Rightarrow \underline{\dot{x}} = 0 \Rightarrow \underline{0}$ is an equilibrium

This has a stable *Limit Cycle* and an unstable equilibrium at $\underline{x} = \underline{0}$.

$x_1 = -2$
$x_2 = 3$
$\Rightarrow \dot{x}_1 = 3$
$\dot{x}_2 = 2 - 3.3$
$= -7$
✓

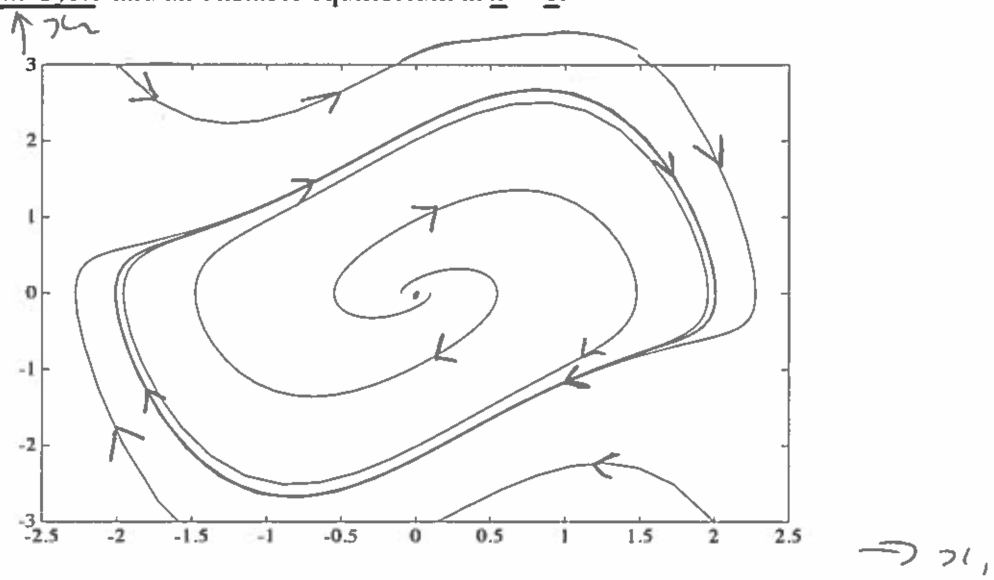


Figure 3: State space trajectories for Van der Pol Oscillator

Note that if we were to change the signs on the $\dot{x}_1$ and $\dot{x}_2$ terms then this will just change the directions of the arrows in the state space. Hence the origin would then be a stable equilibrium and the limit cycle would be unstable (i.e. if perturbed from the limit cycle then it would either decay to the origin or diverge to infinity).

3 Solutions of Linear State Equations

3.1 Using Laplace Transforms

Taking Laplace Transforms of

$$\begin{aligned}\dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t); \quad \underline{x}(0) = \underline{x}_0 \\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t)\end{aligned}$$

$$\underline{X}(s) = \mathcal{L}\,\underline{x}(t) = \begin{bmatrix} \bar{x}_1(s) \\ \bar{x}_2(s) \\ \vdots \end{bmatrix}$$

gives

$$\begin{aligned}s\underline{X}(s) - \underline{x}_0 &= A\underline{X}(s) + B\underline{U}(s) \\ (sI - A)\underline{X}(s) &= \underline{x}_0 + B\underline{U}(s) \\ \underline{X}(s) &= (sI - A)^{-1}\underline{x}_0 + (sI - A)^{-1}B\underline{U}(s) \\ \underline{Y}(s) &= \underbrace{C(sI - A)^{-1}\underline{x}_0}_{\text{initial condition response}} + \underbrace{(D + C(sI - A)^{-1}B)\underline{U}(s)}_{\text{input response}}\end{aligned}$$

(If $(sI - A)$ is invertible anywhere)

$$\mathcal{L}\{\dot{x}\} = s\begin{bmatrix} x_1(s) \\ x_2(s) \\ \vdots \end{bmatrix} - \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \end{bmatrix}$$

For $\underline{x}_0 = \underline{0}$, $\underline{Y}(s) = \underbrace{(D + C(sI - A)^{-1}B)}_{G(s)}\underline{U}(s)$

and

$$\boxed{\overset{p\times m}{G(s)} = \overset{p\times m}{D} + \underset{p\times n}{C}\overset{n\times n}{(sI - A)^{-1}}\overset{n\times m}{B}}$$

is called the **transfer function matrix**.

The i, j^{th} entry of $G(s)$ gives the transfer function from u_j to y_i.

3.2 Transfer function poles

Poles are values of s at which the transfer function becomes infinite:

$$\|G(p)\| = \infty \Rightarrow p \text{ is a pole of } G(s)$$

This can only happen when the matrix $(sI - A)$ becomes singular, ie when

$$\det(sI - A) = 0$$

namely at the eigenvalues of A.

[Later we will see that eigenvalues of A are not always poles of $G(s)$.]

Hence we have the important result:

$$\boxed{\text{Poles of } G(s) \subset \text{eigenvalues of } A}$$

Analytical expression for transfer function matrix

It can be shown that, for any matrix M,

$$M^{-1} = \frac{1}{\det M}\begin{bmatrix} M_{11} & M_{21} & \dots & M_{n1} \\ M_{12} & & & \\ \vdots & & & \\ M_{1n} & \dots & & M_{nn} \end{bmatrix} = \frac{\mathrm{adj}(M)}{\det(A)}$$

- 'ADJOINT' OR 'ADJUGATE' matrix

where M_{ij} is called the cofactor of m_{ij} given by

$$\begin{bmatrix} + & - & + & \dots \\ - & + & \dots & \\ + & \dots & & \\ \dots & & & \end{bmatrix} \quad M_{ij} = (-1)^{i+j}\det(M \text{ with } i\text{-th row and } j\text{-th column deleted})$$

← POLYNOMIAL MATRIX

$$\text{Hence}\quad (sI - A)^{-1} = \frac{1}{\alpha(s)}N(s) \quad \text{where} \quad \alpha(s) = \det(sI - A) = s^n + \alpha_1 s^{n-1} + \dots + \alpha_{n-1}s + \alpha_n$$

and $N(s) = N_1 s^{n-1} + N_2 s^{n-2} + \dots + N_{n-1}s + N_n$ ← $N_1, \dots, N_n$ are constant matrices

The transfer function can therefore be written as $G(s) = \frac{1}{\alpha(s)}(CN(s)B + D\alpha(s))$ with $(CN(s)B + D\alpha(s))$ being a matrix of polynomials in s. ← POLYNOMIAL MATRIX

$(n=3)$

eg if $N(s) = \begin{bmatrix} s^2+s+1 & s+1 \\ 2s+s & 6 \end{bmatrix} \Rightarrow N_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ N_2 = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix},\ N_3 = \begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}$

A square matrix satisfies its own characteristic polynomial

Cayley-Hamilton Theorem

$$\boxed{A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I = 0}$$

since

$$\alpha(s)I = (sI - A)\left(N_1 s^{n-1} + N_2 s^{n-2} + \cdots + N_n\right)$$

Equating coefficients of s^k and premultiplying by A^k for $k = n, .., 1, 0$ gives

$$\begin{array}{ccccc} s^n & : & A^n I & = & A^n N_1 \\ s^{n-1} & : & A^{n-1}\alpha_1 I & = & A^{n-1}N_2 - \cancel{A^n N_1} \\ \vdots & & & \vdots & \\ s & : & A\alpha_{n-1} I & = & AN_n - \cancel{A^2 N_{n-1}} \\ s^0 & : & \alpha_n I & = & -\cancel{AN_n} \end{array}$$

$\leftarrow I = N_1 \Rightarrow A^n = A^n N_1$

$\leftarrow \alpha_1 I = N_2 - AN_1$

Adding these equalities gives the result.

Hence *any* power of A is a linear combination of $I, A, A^2, \ldots, A^{n-1}$ — only!

$$A^n = -\alpha_1 A^{n-1} - \alpha_2 A^{n-2} - \cdots - \alpha_n I$$

$$\Rightarrow A^{n+1} = -\alpha_1 A^n - \alpha_2 A^{n-1} - \cdots - \alpha_n A$$

3.3 Initial Condition Response of the State

For $\underline{u}(t) = \underline{0}$ we have $\dot{\underline{x}}(t) = A\underline{x}(t)$ and hence,

$$\underline{X}(s) = (sI - A)^{-1}\,\underline{x}_0$$

$\Longrightarrow$

$$\underline{x}(t) = \mathcal{L}^{-1}\left((sI - A)^{-1}\right)\underline{x}_0 = \Phi(t)\underline{x}_0$$

where

$$\begin{aligned} \Phi(t) &= \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \mathcal{L}^{-1}\left(I s^{-1} + A s^{-2} + A^2 s^{-3} + \cdots\right) \\ &= \mathcal{L}^{-1}\left\{\sum_{k\geq 0} A^k s^{-(k+1)}\right\} \\ &= I + At + A^2t^2/2! + \cdots + A^k t^k/k! + \cdots \\ &\stackrel{def}{=} e^{At} \end{aligned}$$

"Proof" $(sI - A)(Is^{-1} + As^{-2} + \cdots) = I + \cancel{(As^{-1} - As^{-1})}^{\,0} + (\cancel{\ }) + \cdots$

Note that

$$\frac{d}{dt}\Phi(t) = Ae^{At} = e^{At}A$$

since $\frac{d}{dt}\left(I + At + \frac{(At)^2}{2} + \cdots\right)$

$= A + A^2 t + \cdots$

$= A(I + At + \cdots)$

(check by differentiating series expansion of $\Phi(t)$).

Hence with $\underline{x}(t) = \Phi(t)\underline{x}_0$,

$$\begin{aligned}
\frac{d}{dt}\underline{x}(t) = \frac{d}{dt}\left\{\Phi(t)\underline{x}_0\right\} &= Ae^{At}\underline{x}_0 = A\left\{\Phi(t)\underline{x}_0\right\} = A\underline{x}(t) \\
\Phi(0) &= I \\
\underline{x}(0) &= \underline{x}_0
\end{aligned}$$

and the differential equation and initial condition are satisfied as required.

$\Phi(t)$ is called the **state transition matrix.**

Properties of e^{At}

(1) **Change of state coordinates**

If

ie $TAT^{-1} = \bar{A}$

$$\begin{aligned}
A &= T^{-1}\bar{A}T \\
\text{then } A^2 &= T^{-1}\bar{A}TT^{-1}\bar{A}T = T^{-1}\bar{A}^2T \\
A^k &= T^{-1}\bar{A}^kT \\
\text{hence } e^{At} &= \sum_{k=0}^{\infty} A^k t^k/k! \\
&= \sum_{k=0}^{\infty} T^{-1}\bar{A}^kTt^k/k! \\
&= T^{-1}\left(\sum_{k=0}^{\infty}\bar{A}^kt^k/k!\right)T
\end{aligned}$$

$\leftarrow$ similarity transform - doesn't change eigenvalues

$$\Rightarrow \boxed{e^{At} = T^{-1}e^{\bar{A}t}T}$$

e.g. $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Why is this 'change of coordinates'?

If $\underline{\dot{x}}(t) = A\underline{x}(t)$ and $\underline{z} = T\underline{x}$, then

$$\begin{aligned}
\underline{\dot{z}}(t) &= T\underline{\dot{x}}(t) \\
&= TA\underline{x}(t) \\
&= TAT^{-1}\underline{z}(t) \\
&= \bar{A}\underline{z}(t) \\
\underline{z}(t) &= e^{\bar{A}t}\underline{z}(0) \\
\underline{x}(t) &= \underbrace{T^{-1}e^{\bar{A}t}T}_{e^{At}}\underline{x}(0)
\end{aligned}$$

$\underline{x} = T^{-1}\underline{z}$

Special case: Eigenvectors as coordinate axes

Recall that for W the matrix of eigenvectors of A, then the defining relations,

$$\begin{aligned}
A\underline{w}_1 &= \underline{w}_1\lambda_1 \\
A\underline{w}_2 &= \underline{w}_2\lambda_2 \\
&\vdots \\
A\underline{w}_n &= \underline{w}_n\lambda_n.
\end{aligned}$$

can be written as: $$A\underbrace{\left[\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_n\right]}_{W} = \underbrace{\left[\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_n\right]}_{W}\underbrace{\begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & \\ \vdots & & \ddots & \\ 0 & & \ldots & \lambda_n \end{bmatrix}}_{\Lambda}$$

Hence for *non-defective* A we have the eigenvalue/eigenvector decomposition:

$$\boxed{A = W\Lambda W^{-1}}$$

So that for $T = W^{-1}$ we have,

$$\bar{A} = TAT^{-1} = W^{-1}AW = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = \text{diag}\,\{\lambda_i\}$$

$$\begin{aligned} e^{\Lambda t} = \sum_{k=0}^{\infty} \Lambda^k t^k / k! &= \text{diag}\,\{\textstyle\sum \lambda_i^k t^k / k!\} \\ &= \text{diag}\,\{e^{\lambda_i t}\}. \end{aligned}$$

This gives one way of evaluating e^{At}.

(Another way is to evaluate $\mathcal{L}^{-1}(sI - A)^{-1}$.)

2) **Semigroup property** — Don't worry about the fancy name.

$$\begin{aligned} e^{A(t_1+t_2)} &= e^{At_1} e^{At_2} = e^{At_2} e^{At_1} \\ \text{since } \underline{x}(t_1 + t_2) &= \Phi(t_1 + t_2)\underline{x}(0) \\ &= \Phi(t_2)\underline{x}(t_1) \\ &= \Phi(t_2)\Phi(t_1)\underline{x}(0) \quad \text{for all } \underline{x}(0) \end{aligned}$$

NB: This only works because $(At_1)(At_2) = (At_2)(At_1)$.
For arbitrary matrices A and B, $e^{A+B} \neq e^A e^B$.

3) **Inverse**

$I = e^{A.0} = e^{A(t-t)} = e^{At} e^{-At} \quad \Rightarrow \quad \boxed{(e^{At})^{-1} = e^{-At}}$

4) **Derivative** — Repeated here for completeness.

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$

5) **Integral**

$$\begin{aligned}
\int_0^t e^{A\tau}d\tau &= \int_0^t \sum_{k=0}^{\infty} A^k\tau^k/k!\, d\tau \\
&= \sum_{k=0}^{\infty} \left[A^k\tau^{k+1}/(k+1)!\right]_0^t \\
&= A^{-1}\left\{\sum_{k=0}^{\infty} A^{k+1}t^{k+1}/(k+1)! - 0\right\} \\
&= \underline{A^{-1}e^{At} - A^{-1}} \quad \text{if} \quad \det(A) \neq 0.
\end{aligned}$$

If $\det(A) = 0$ then the above formula is not valid and the integration needs to be done directly. e.g. $A = 0$.

3.4 Example: Rotating Rigid Body

Let I_1, I_2, I_3 be the moments of inertia of a rigid body rotating in free space, about its 3 principal axes, and w_1, w_2, w_3 the corresponding angular velocities. Then in the absence of externally applied torques EULER'S EQUATIONS OF MOTION are:-

$$\left.\begin{aligned}
I_1\dot{w}_1 &= (I_2 - I_3)w_2w_3 \\
I_2\dot{w}_2 &= (I_3 - I_1)w_3w_1 \\
I_3\dot{w}_3 &= (I_1 - I_2)w_1w_2
\end{aligned}\right\}\text{nonlinear state space equations}$$

This is a lossless system since the Kinetic Energy,

$$\begin{aligned}
V(w_1, w_2, w_3) &= \frac{1}{2}I_1w_1^2 + \frac{1}{2}I_2w_2^2 + \frac{1}{2}I_3w_3^2 \\
\frac{dV}{dt} &= I_1w_1\dot{w}_1 + I_2w_2\dot{w}_2 + I_3w_3\dot{w}_3 \\
&= w_1w_2w_3\left[(I_2 - I_3) + (I_3 - I_1) + (I_1 - I_2)\right] \\
&= 0
\end{aligned}$$

$\Rightarrow$ <u>if the trajectory starts on an ellipsoid $V(w_1, w_2, w_3) =$ constant, it stays on it.</u>

In addition conservation of angular momentum implies that,

$$\begin{aligned}
\frac{d}{dt}\left\{I_1^2w_1^2 + I_2^2w_2^2 + I_3^2w_3^2\right\} &= 2w_1w_2w_3\left[I_1(I_2 - I_3) + I_2(I_3 - I_1) + I_3(I_1 - I_2)\right] \\
&= 0
\end{aligned}$$

Suppose $I_1 = 6, I_2 = 2, I_3 = 5$ then

$$\left.\begin{aligned} \dot{w}_1 &= -\tfrac{1}{2} w_2 w_3 \\ \dot{w}_2 &= -\tfrac{1}{2} w_3 w_1 \\ \dot{w}_3 &= \tfrac{4}{5} w_1 w_2 \end{aligned}\right\} \leftarrow f_1$$

Equilibrium Solutions satisfy: $\dot{w}_1 = \dot{w}_2 = \dot{w}_3 = 0 \;\Rightarrow\; w_2 w_3 = w_1 w_3 = w_1 w_2 = 0 \Rightarrow$

$$\begin{array}{llll} \textit{either} & (a) & w_2 = w_3 = 0 & \& \quad w_1 = \bar{w}_1 \\ \textit{or} & (b) & w_3 = w_1 = 0 & \& \quad w_2 = \bar{w}_2 \\ \textit{or} & (c) & w_1 = w_2 = 0 & \& \quad w_3 = \bar{w}_3 \\ \textit{or} & (d) & w_1 = w_2 = w_3 = 0. & \end{array}$$

The linearized equations are

$$\frac{\partial f_1}{\partial w_2}$$

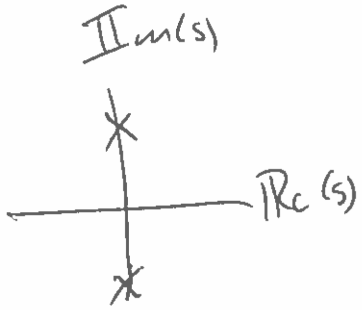
$$\dot{\underline{\delta w}} \simeq \frac{\partial f}{\partial \underline{w}} \underline{\delta w} = \begin{bmatrix} 0 & -\frac{1}{2} w_3 & -\frac{1}{2} w_2 \\ -\frac{1}{2} w_3 & 0 & -\frac{1}{2} w_1 \\ \frac{4}{5} w_2 & \frac{4}{5} w_1 & 0 \end{bmatrix} \underline{\delta w}.$$

case (a) $\quad w_2 = w_3 = 0$

$$\Rightarrow \dot{\underline{\delta w}} \cong \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\bar{w}_1 \\ 0 & \frac{4}{5}\bar{w}_1 & 0 \end{bmatrix} \underline{\delta w}.$$

$$\Rightarrow \delta w_1(t) \simeq \delta w_1(0) \& \frac{d}{dt}\begin{bmatrix} \delta w_2 \\ \delta w_3 \end{bmatrix} \cong \begin{bmatrix} 0 & -\frac{1}{2}\bar{w}_1 \\ \frac{4}{5}\bar{w}_1 & 0 \end{bmatrix} \begin{bmatrix} \delta w_2 \\ \delta w_3 \end{bmatrix}$$

and the state trajectories are ellipses (with the ratio of the principal axes = $\sqrt{8/5} \simeq 1.26$.



EIGENVALUES

$$\lambda^2 + \frac{4}{10}\bar{\omega}_1^2 = 0$$

$$\lambda = \pm j\, \bar{\omega}_1 \sqrt{\frac{4}{10}}$$

case (b) $\omega_3 = \omega_1 = 0$

$$\underline{\delta \dot{w}} \simeq \begin{bmatrix} 0 & 0 & -\frac{1}{2}\bar{w}_2 \\ 0 & 0 & 0 \\ \frac{4}{5}\bar{w}_2 & 0 & 0 \end{bmatrix} \underline{\delta w}$$

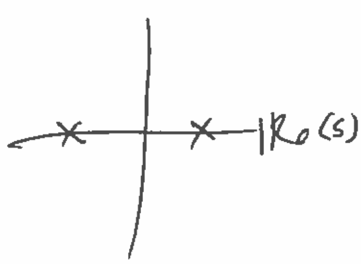
$$\delta w_2 \simeq \delta w_2(0) \& \frac{d}{dt}\begin{bmatrix} \delta w_1 \\ \delta w_3 \end{bmatrix} \simeq \begin{bmatrix} 0 & -\frac{1}{2}\bar{w}_2 \\ \frac{4}{5}\bar{w}_2 & 0 \end{bmatrix} \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix}$$

$\lambda = \pm j\, \bar{\omega}_2 \sqrt{\frac{4}{10}}$

case (c)

$$\underline{\delta \dot{w}} \simeq \begin{bmatrix} 0 & -\frac{1}{2}\bar{w}_3 & 0 \\ -\frac{1}{2}\bar{w}_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{\delta w}$$

$\lambda = \pm \frac{\bar{\omega}_3}{2}$



$$\delta w_3(t) \simeq \delta w_3(0),\ \frac{d}{dt}\begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix} = \underbrace{-\frac{1}{2}\bar{w}_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{"A"}} \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix}$$

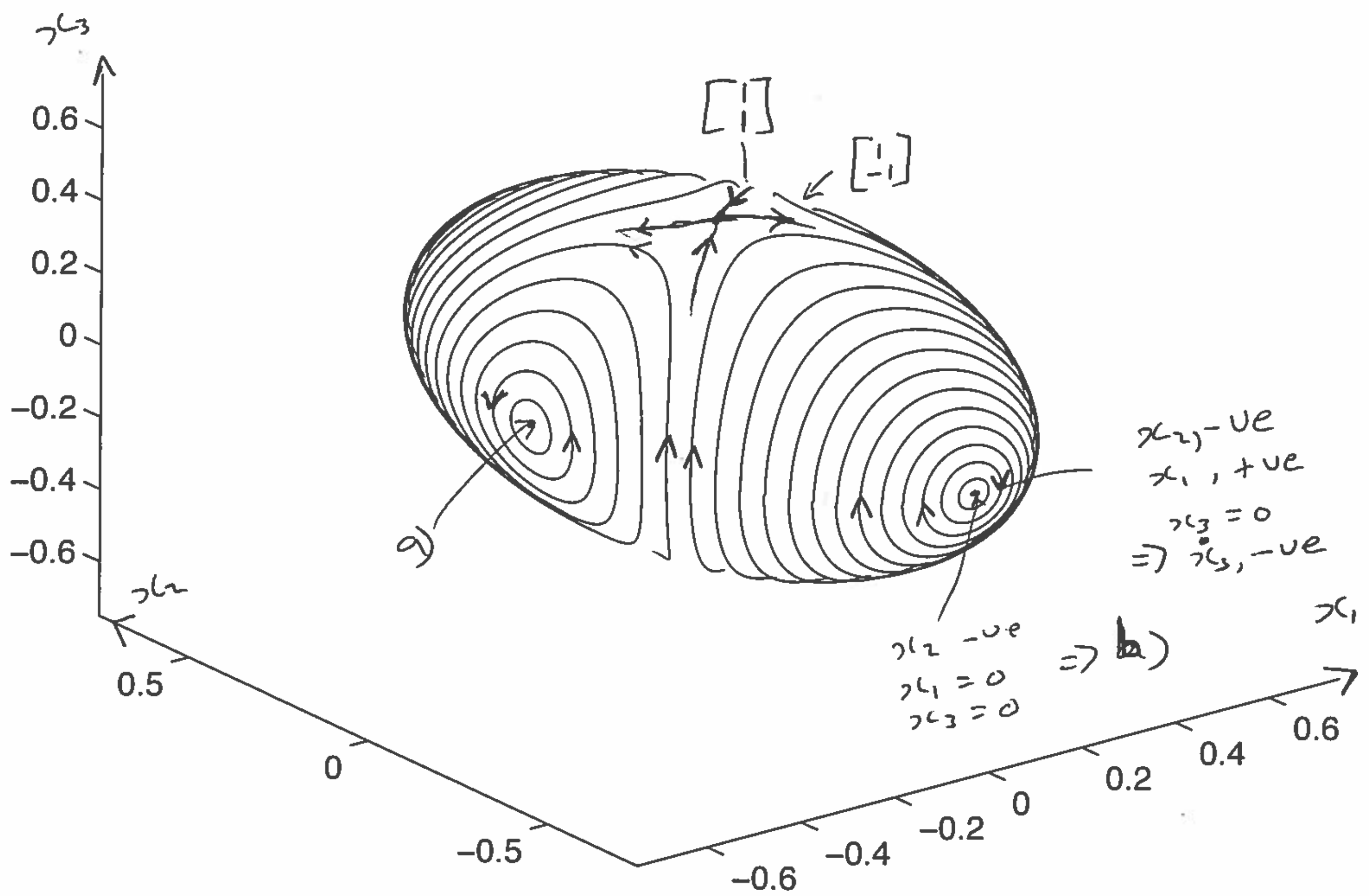
and

$$\begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-\frac{1}{2}\bar{w}_3 t}\left(\frac{\delta w_1(0) + \delta w_2(0)}{2}\right) + \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{\frac{1}{2}\bar{w}_3 t}\left(\frac{\delta w_1(0) - \delta w_2(0)}{2}\right)}_{\text{UNSTABLE}}$$

$\Bigg\} \; e^{At}\begin{bmatrix} \delta\omega_1(0) \\ \delta\omega_2(0) \end{bmatrix}$

$\neq 0$

$(A = W \Lambda W^{-1})$



x_3
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
x_2
0.5
0
−0.5
x_1
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
a)
$x_2, -ve$
$x_1, +ve$
$x_3 = 0$
$\Rightarrow \dot{x}_3, -ve$
x_2 -ve
$x_1 = 0$
$x_3 = 0$
$\Rightarrow$ b)

The trajectories in the 3-dimensional state space can thus be sketched as follows on a particular ellipsoid $V(w_1, w_2, w_3) = \text{const.}$

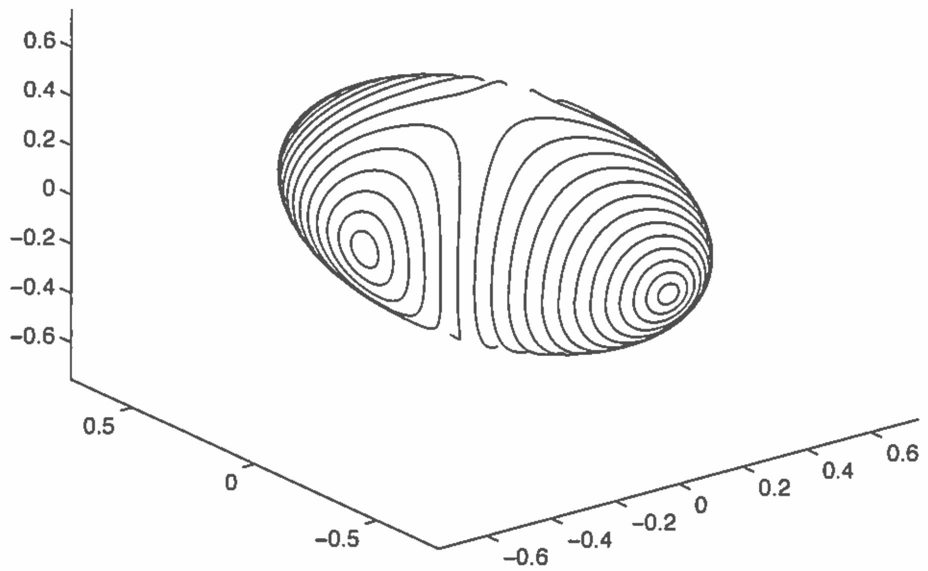


Figure 4: State space trajectories of a rotating rigid body

3.5 Convolution Integral

Consider

$$\underline{\dot{x}} - A\underline{x} = B\underline{u} \qquad (*)$$

Comparing with the scalar case note

$$\begin{aligned} \frac{d}{dt}\left\{e^{-At}\underline{x}(t)\right\} &= \frac{d}{dt}\left\{e^{-At}\right\}\underline{x}(t) + e^{-At}\frac{d\underline{x}}{dt} \\ &= -e^{-At}A\underline{x}(t) + e^{-At}\frac{d\underline{x}}{dt} \end{aligned}$$

now premultiply (*) by e^{-At} to give

$$\begin{aligned} e^{-At}\frac{d\underline{x}}{dt} - e^{-At}A\underline{x} &= e^{-At}B\underline{u} \\ \Rightarrow \frac{d}{dt}\left\{e^{-At}\underline{x}(t)\right\} &= e^{-At}B\underline{u}(t) \\ \Rightarrow e^{-At}\underline{x}(t) - \underline{x}_0 &= \int_0^t e^{-A\tau}B\underline{u}(\tau)d\tau \\ \Rightarrow \underline{x}(t) &= e^{At}\underline{x}_0 + e^{At}\int_0^t e^{-A\tau}B\underline{u}(\tau)d\tau \end{aligned}$$

$$\Rightarrow \boxed{\underline{x}(t) = \underbrace{e^{At}}_{\Phi(t)}\underline{x}_0 + \int_0^t e^{A(t-\tau)}B\underline{u}(\tau)d\tau}$$

and

$$\boxed{\underline{y}(t) = \underbrace{Ce^{At}\underline{x}_0}_{\text{initial condition response}} + \underbrace{D\underline{u}(t) + \int_0^t Ce^{A(t-\tau)}B\underline{u}(\tau)\,d\tau}_{\text{input response}}}$$

Let $H(t) = \begin{cases} D\delta(t) + Ce^{At}B & t \geq 0 \\ 0 & t < 0 \end{cases}$

then if $\underline{x}_0 = \underline{0}$,

CONVOLUTION

$$\underline{y}(t) = \int_0^t H(t-\tau)\underline{u}(\tau)d\tau = H(t) * \underline{u}(t)$$

$H(t)$ is called the **impulse response matrix**, since for multiple-input/multiple-output systems, if an impulse is applied at time 0^+ to the j-th input, with the other inputs at zero, then the i-th output will be

$$y_i(t) = \int_0^t h_{ij}(t-\tau)u_j(\tau)d\tau = h_{ij}(t)$$

Note that the transfer function, $G(s) = \mathcal{L}(H(t))$.

3.6 Frequency Response

Consider a linear time-invariant system that is asymptotically stable. What is the response due to sinusoidal input at each of the inputs? Let

$$u_j(t) = A_j\cos(\omega_o t + \theta_j),\ j = 1, 2, \cdots, m$$

then

$$y_i(t) \longrightarrow B_i\cos(\omega_o t + \phi_i) \text{ as } t \longrightarrow \infty$$

where

$$B_i e^{j\phi_i} = g_{i1}(j\omega_o)A_1e^{j\theta_1} + g_{i2}(j\omega_o)A_2e^{j\theta_2} + \cdots + g_{im}(j\omega_o)A_me^{j\theta_m}$$

i.e. the sum of the sinusoidal responses from each input. The rate at which the steady state is achieved depends on how quickly the impulse response tends to zero, which in turn depends on the pole positions.

3.7 Stability of $\underline{\dot{x}} = A\underline{x}$

Stability of systems is concerned with whether as $t \to \infty$,

$$\begin{aligned} \underline{x}(t) \quad &\to \quad 0 \\ &\to \quad \infty \\ &\text{or} \quad \text{remains bounded ?} \end{aligned}$$

but since $\underline{x}(t) = e^{At}\underline{x}_0$ the question becomes whether the elements of $e^{At} \to 0, \to \pm\infty$ or remain bounded as $t \to \infty$?

Consider

$$(sI - A)^{-1} = N(s)/\alpha(s)$$

where $\alpha(s)$ is the characteristic polynomial of A,

$$\alpha(s) = \det(sI - A) = s^n + \alpha_1 s^{n-1} + .. + \alpha_n$$

and

$$N(s) = N_1 s^{n-1} + N_2 s^{n-2} + .. + N_n$$

where N_i are $n \times n$ constant matrices. (Recall section 3.2.)

If we factor $\alpha(s)$ as

$$\alpha(s) = (s - \lambda_1)^{n_1}(s - \lambda_2)^{n_2}..(s - \lambda_r)^{n_r}$$

where λ_i will be the eigenvalues of A and $n_1 + n_2 + .. + n_r = n =$ dim (A), then the partial fraction expansion gives

$$(sI - A)^{-1} = \sum_{i=1}^{r}\sum_{k=1}^{n_i} \frac{C_{i,k}}{(s - \lambda_i)^k}$$

for suitable constant matrices $C_{i,j}$.

The inverse transform then gives

$$e^{At} = \sum_{i=1}^{r}\sum_{k=1}^{n_i} \frac{C_{i,k}t^{k-1}e^{\lambda_i t}}{(k-1)!}$$

Let $\lambda_i = \sigma_i + j\omega_i$ then $\left|e^{\lambda_i t}\right| = e^{\sigma_i t}\left|e^{j\omega_i t}\right| = e^{\sigma_i t}$

3 cases

(i) $\sigma_i < 0, \left|t^{k-1}e^{\lambda_i t}\right| \to 0, k = 1, 2, 3, \ldots$

(ii) $\sigma_i > 0, \left|t^{k-1}e^{\lambda_i t}\right| \to \infty, k = 1, 2, 3, \ldots$

(iii) $\sigma_i = 0$

(a) $\left|t^{k-1}e^{\lambda_i t}\right| = 1 \qquad k = 1$

(b) $\left|t^{k-1}e^{\lambda_i t}\right| \to \infty \quad k = 2, 3,.$

Hence $\underline{x}(t) \to \underline{0}$ as $t \to \infty$ for all $\underline{x}_0$ if and only if all λ_i satisfy $Re(\lambda_i) < 0$. (i.e. case i)).

Also $\underline{x}(t)$ remains bounded as $t \to \infty$ for all $\underline{x}_0$ if and only if $Re(\lambda_i) \leq 0$ and in partial fraction expansion of $(sI - A)^{-1}$ there are no terms of the form $C_{i,k}/(s - j\omega_i)^k$ with $k \geq 2$. (i.e. case (i) or (iii)(a).).

4 State Space Equations for Composite Systems

4.1 Cascade of Two Systems

$$\underline{u} \longrightarrow \boxed{G_1(s)} \xrightarrow{\underline{w}} \boxed{G_2(s)} \longrightarrow \underline{y}$$

Let $G_1(s)$ be realized by the state equation:

$$\begin{cases} \dot{\underline{x}}_1(t) &= A_1\underline{x}_1(t) + B_1\underline{u}(t) \\ \underline{w}(t) &= C_1\underline{x}_1(t) + D_1\underline{u}(t) \end{cases}$$

and $G_2(s)$ be realized by, (input $\underline{w}$, output $\underline{y}$)

$$\begin{cases} \dot{\underline{x}}_2(t) &= A_2\underline{x}_2(t) + B_2\underline{w}(t) \\ \underline{y}(t) &= C_2\underline{x}_2(t) + D_2\underline{w}(t) \end{cases}$$

$= A_2 \underline{x}_2 + B_2 (C_1 \underline{x}_1 + D_1 \underline{u})$

$= C_2 \underline{x}_2 + D_2 (\quad u \quad)$

then $\underline{Y}(s) = G_2(s)G_1(s)\underline{U}(s)$ is realized by

put $\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}$

$$\underbrace{\frac{d}{dt}\begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix}}_{\dot{\underline{x}}(t)} = \underbrace{\begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix}}_{\underline{x}(t)} + \underbrace{\begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix}}_{B} \underline{u}(t)$$

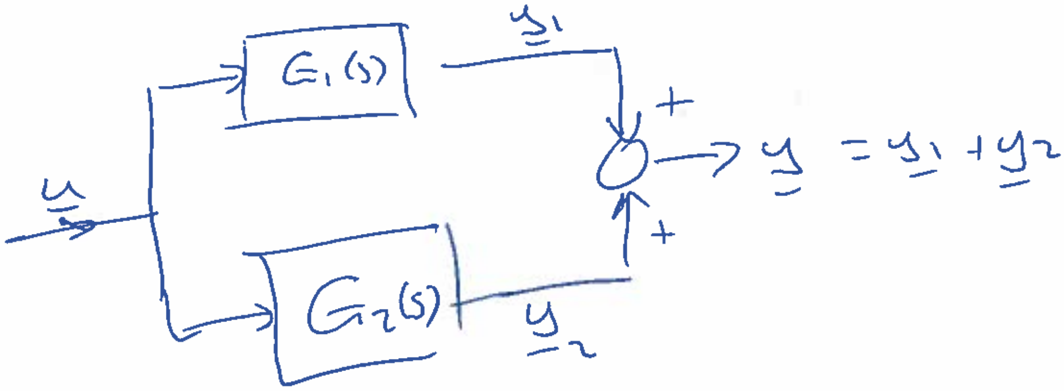
$$\underline{y}(t) = \underbrace{\begin{bmatrix} D_2C_1 & C_2 \end{bmatrix}}_{C} \underline{x}(t) + \underbrace{D_2D_1}_{D}\,\underline{u}(t)$$

The cascade realization of a single-input single-output system can hence be obtained if $G(s)$ is factored into second order factors such as

$$G(s) = A\frac{(s^2 + c_1s + d_1)\,(s^2 + c_2s + d_2)\dots}{(s^2 + a_1s + b_1)\,(s^2 + a_2s + b_2)\dots}$$

then the overall system can be realized as the cascade of these factors.

4.2 Parallel combination of two systems



Let $G_1(s)$ be realized by the state equation:

$$\begin{cases} \dot{\underline{x}}_1(t) &= A_1\underline{x}_1(t) + B_1\underline{u}(t) \\ \underline{y}_1(t) &= C_1\underline{x}_1(t) + D_1\underline{u}(t) \end{cases}$$

and $G_2(s)$ be realized by, (input $\underline{u}$, output $\underline{y}_2$)

$$\begin{cases} \dot{\underline{x}}_2(t) &= A_2\underline{x}_2(t) + B_2\underline{u}(t) \\ \underline{y}_2(t) &= C_2\underline{x}_2(t) + D_2\underline{u}(t) \end{cases}$$

then $\underline{Y}(s) = (G_1(s) + G_2(s))\underline{U}(s)$ is realized by

$$\underbrace{\frac{d}{dt}\begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix}}_{\dot{\underline{x}}(t)} = \underbrace{\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \underline{x}_1(t) \\ \underline{x}_2(t) \end{bmatrix}}_{\underline{x}(t)} + \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{B} \underline{u}(t)$$

$$\underline{y}(t) = \underbrace{\begin{bmatrix} C_1 & C_2 \end{bmatrix}}_{C} \underline{x}(t) + \underbrace{(D_1 + D_2)}_{D} \underline{u}(t)$$

IN GENERAL

1) WRITE DOWN STATE-SPACE EQUATIONS FOR ALL SUB-SYSTEMS

2) ASSEMBLE STATES INTO NEW STATE VECTOR

3) ELIMINATE ALL OTHER INTERNAL VARIABLES (e.g $\underline{w}$, $\underline{y}_1$, $\underline{y}_2$ ...)

Cambridge University Engineering Dept. Third year

Module 3F2: Systems and Control

LECTURE NOTES 2: 'CLASSICAL' METHODS

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1 Revision of Feedback Control

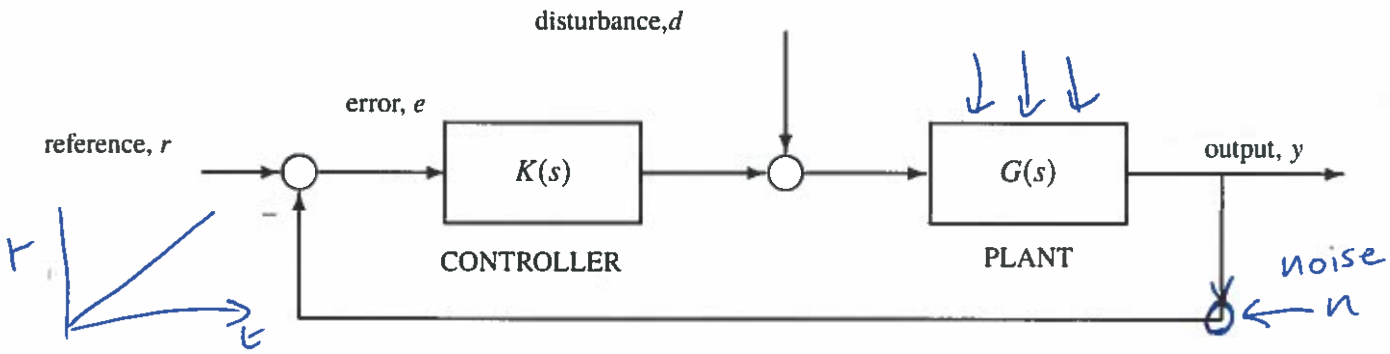
1.1 Why use Feedback?

- To reduce effects of uncertainty:
  - Disturbances — *Wind/waves, Friction, Impurities, ...*
  - Model errors — *Approximations, Tolerances, Ageing, ...*
- To stabilise unstable system:
  - Inverted pendulum, Bicycle
  - High-performance fighter aircraft *(Fly-by-wire)*
  - Helicopter, Submarine (depth)
  - Exothermic chemical reactor, Nuclear reactor

Problems with feedback:

- May destabilise system
- Sensors introduce noise

1.2 The Standard Feedback Loop



$L = GK$

ORDER OF MULTIPLICATION MATTERS !

1.3 Sensitivity and Complementary Sensitivity

Let $L(s)$ be the (open) loop transfer function: $L(s) = G(s)K(s)$ *"Return-ratio"*

Complementary Sensitivity: $T(s) = \frac{L(s)}{1+L(s)}$ (Multivariable: $T(s) = L(s)[I + L(s)]^{-1}$

$$\bar{y} = T\bar{r} = -T\bar{n} \qquad (\bar{n} = \text{sensor noise})$$

$(= (I+L)^{-1}L)$

$T \approx 1 \Rightarrow$ good "tracking" but no noise filtering.

Sensitivity: $S(s) = \frac{1}{1+L(s)}$ (Multivariable: $S(s) = [I + L(s)]^{-1}$)

$$\bar{e} = S\bar{r}, \quad S = \frac{dT/T}{dG/G}, \qquad \bar{y} = GS\bar{d}$$

Small $|S| \Rightarrow$ feedback is beneficial.

Fact: $\boxed{S(s) + T(s) = 1}$ (Multivariable: $S(s) + T(s) = I$)

Hence **trade-off**:

- Small $|S(j\omega)|$ at low frequencies
- Small $|T(j\omega)|$ at high frequencies.

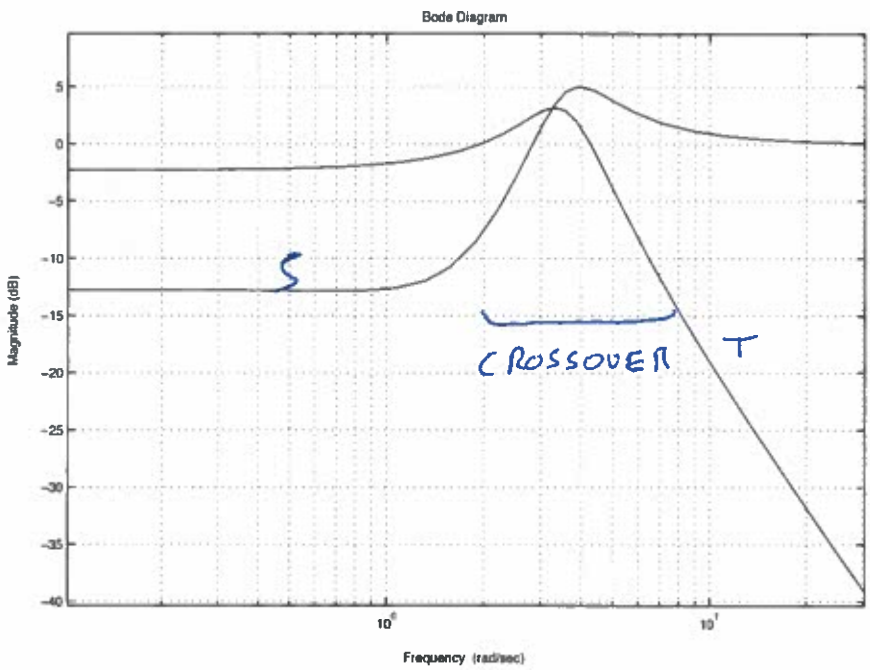


Figure 1.1: Sensitivity (S) and Complementary Sensitivity (T)

1.4 Steady-state Error

Constant reference: $r(t) = \alpha$, $\bar{r}(s) = \frac{\alpha}{s}$. Assume closed-loop is *asymptotically stable.*

$\lim_{t\to\infty} e(t) = S(0)\alpha = \frac{1}{1+L(0)}\alpha$ (Final Value Theorem or $S(j\omega)$ with $\omega = 0$)

$\lim_{t\to\infty} e(t) = 0$ if $|G(0)K(0)| = \infty \Leftarrow G(s)K(s)$ has pole at $s = 0$ — *integral action*

Constant disturbance: $d(t) = \beta$, $\bar{d}(s) = \frac{\beta}{s}$.

$\frac{\bar{e}}{\bar{d}} = -GS$. Zero steady-state error $\Leftarrow K(s)$ has pole at $s = 0$.

Ramp reference: $r(t) = \alpha t$, $\bar{r}(s) = \frac{\alpha}{s^2}$.

$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\bar{e}(s) = \lim_{s\to 0} \frac{\alpha}{s} S(s)$. (Final Value Theorem)

Hence:

- *Finite* steady-state error $\Leftarrow G(s)K(s)$ has a pole at $s = 0$.
- *Zero* steady-state error $\Leftarrow G(s)K(s)$ has *two* poles at $s = 0$.

Small steady-state error requires high gain at "DC".

↑
$\omega = 0$

1.5 The Nyquist Stability Theorem

Motivation:

- The frequency response can be determined experimentally.
- Or from transfer function or state-space model.
- Want a test for closed-loop stability that uses *open-loop* information.

Theorem:

- Plot $L(j\omega) = G(j\omega)K(j\omega)$ on the Argand diagram, for $-\infty < \omega < +\infty$ — the *Nyquist plot*.
- The closed loop is stable if and only if the Nyquist plot encircles the point $-1 + j0$ p_u times counterclockwise, where p_u is the number of *unstable poles* of $G(s)$ and $K(s)$.

2 The Root-Locus Method

2.1 An Example

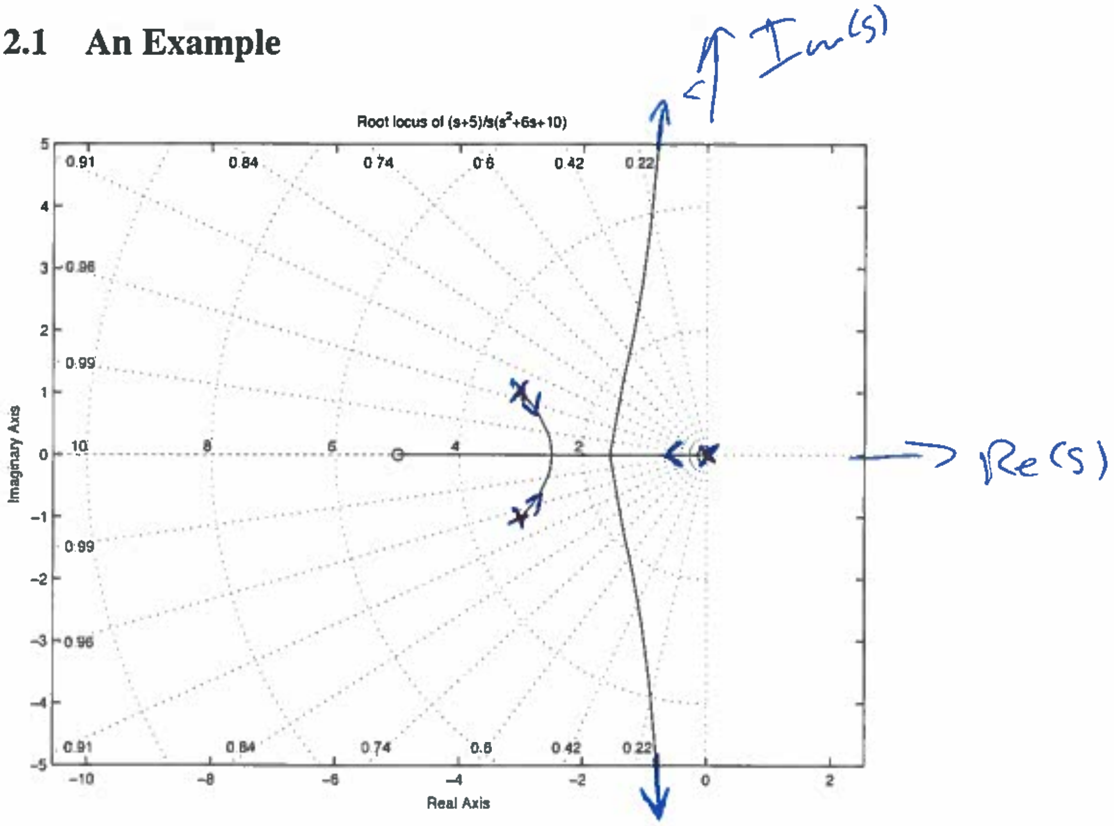


Figure 2.1:

Root-locus diagram for 'rlocus' *in Matlab*

$$L(s) = \frac{s+5}{s(s^2+6s+10)}$$

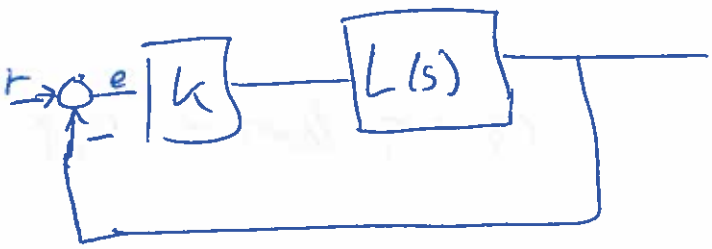
CLTF = ~ / $s(s^2+6s+10) + k(s+$

CLCE

This shows the locations of the roots of $1 + kL(s) = 0$ for $k > 0$.

In this case $0 \leq k \leq 2 \times 10^4$. $k = -\frac{1}{L(s)}$

Useful when the loop dynamics are fixed, and only the gain varies.



2.2 The Angle Condition

$$L(s) = \frac{n(s)}{d(s)} = \frac{c(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)} = \frac{c\,(s^m + \dots +\ )}{(s^n + \dots\ )}$$

monic

where any complex zeros or poles occur in conjugate pairs and $m \leq n$.

We assume for the moment that $c > 0$.

Suppose that s_0 is on the root-locus:

test point

$$1 + kL(s_0) = 0 \Rightarrow L(s_0) = -\frac{1}{k} \quad \text{real and negative} \tag{2.1}$$

(for some real +ve k)

Hence *angle condition* for s_0 to be on the root-locus:

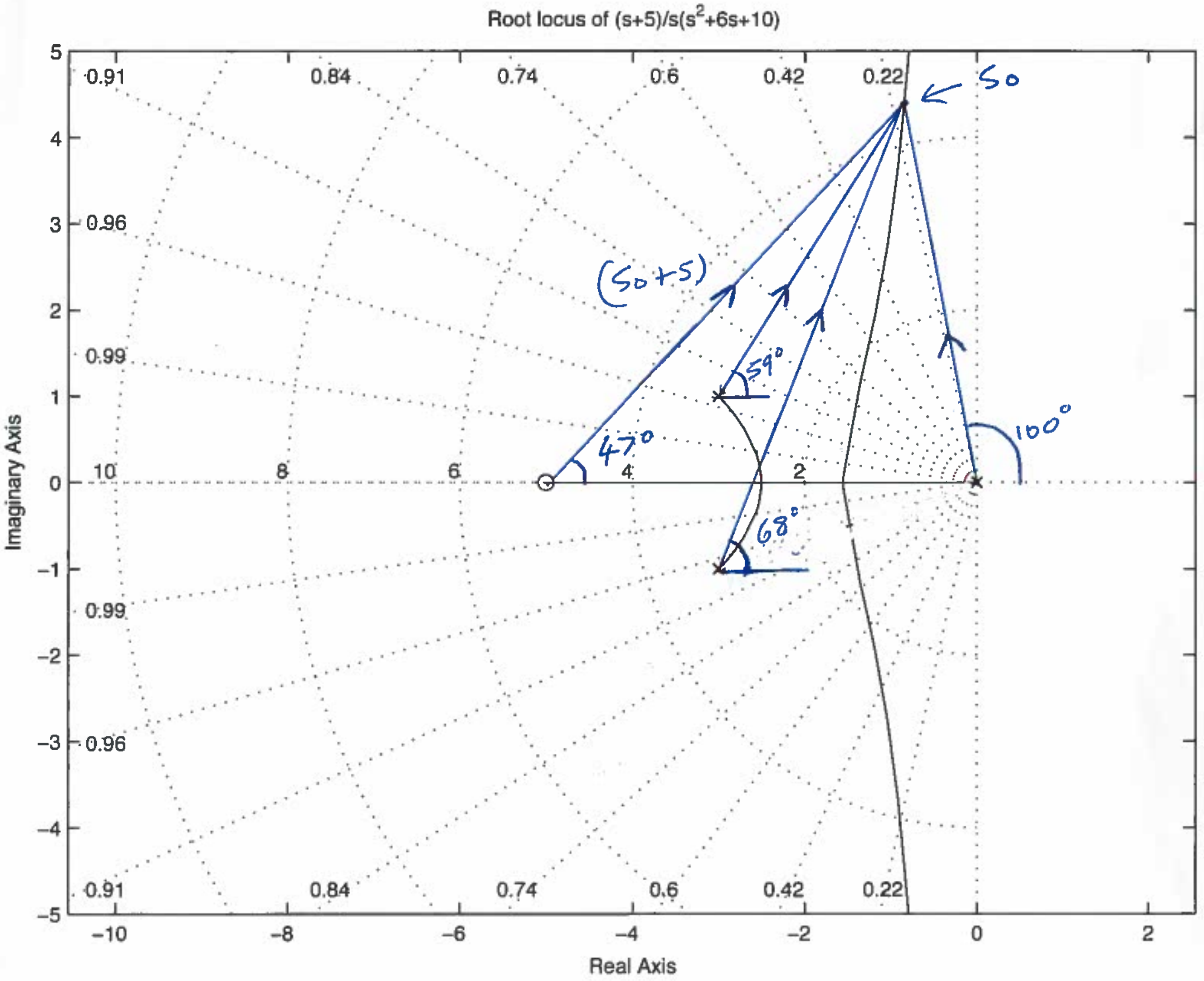
$$\angle L(s_0) = (2\ell + 1)\pi \tag{2.2}$$

$$\sum_{i=1}^{m} \angle(s_0 - z_i) - \sum_{i=1}^{n} \angle(s_0 - p_i) = (2\ell + 1)\pi \tag{2.3}$$

2.3 Finding Gain from the Root-Locus Plot

Once a root-locus plot has been obtained, it can be calibrated with k values. From (2.1) we have, at a point s_0 on the root-locus:

$$\begin{aligned} k &= \frac{1}{|L(s_0)|} \\ &= \frac{1}{c} \times \frac{|s_0 - p_1| \times |s_0 - p_2| \times \dots \times |s_0 - p_n|}{|s_0 - z_1| \times |s_0 - z_2| \times \dots \times |s_0 - z_m|} \end{aligned}$$



$$L = \frac{s+5}{s(s+3+j)(s+3-j)}$$

$$\angle L(s_0) = 47^\circ - 100^\circ - 59^\circ - 68^\circ$$
$$= -180^\circ \quad (!)$$

2.4 Constructing the Root-Locus Plot

Nowadays we can use software to draw root-locus diagrams (eg `rlocus` in *Matlab*).

But it is useful to have some understanding of how the form of the locus is determined. A set of about 15 'construction rules' has been developed. The 5 most important ones are given here. They are all consequences of (2.3) and properties of polynomials.

Rule 1. The root-locus diagram is symmetric with respect to the real axis and consists of n branches.

Rule 2. For $k = 0$ the n branches start at the open loop poles p_i. As $k \to \infty$, m branches tend to the zeros z_i and $n - m$ branches tend to infinity.

Rule 3. Points on the real axis which lie to the left of an odd number of poles *and* zeros are on the root-locus.

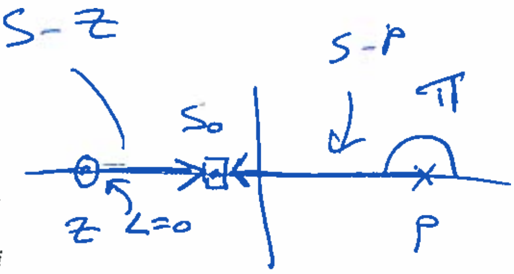
Rule 4. The breakaway points are those points on the root-locus for which $\frac{d}{ds}L(s) = 0$ (same as $dk/ds = 0$).

Rule 5. As $k \to \infty$, the $n - m$ branches which tend to infinity do so along straight line asymptotes at angles $(2\ell + 1)\pi/(n - m)$ to the +ve real axis ($\ell = 0, \ldots, n - m - 1$), and emanate from the point ('centre of gravity' — pole +ve mass, zero -ve mass):

$$\frac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{n - m}.$$

Proof of Rule 3:

Consider a point s_0, a pole p_i, and a zero z_i, all on the real axis.



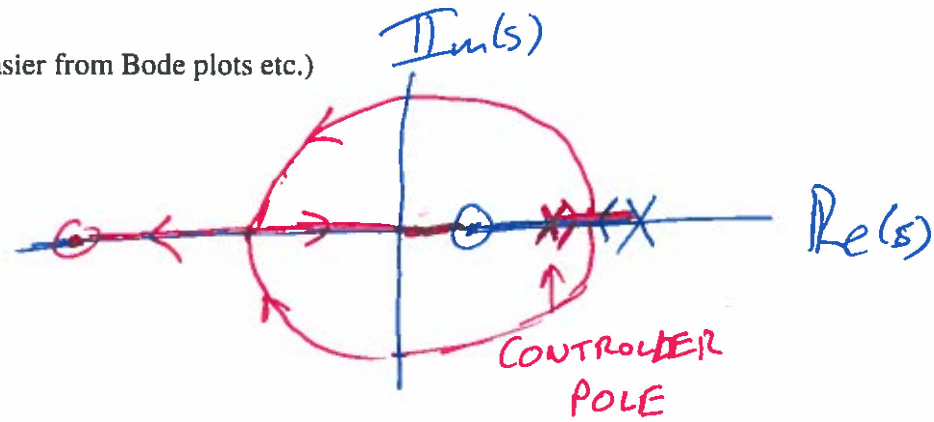
$$\angle(s_0 - p_i) = \begin{cases} 0 \text{ if } s_0 > p_i \\ \pi \text{ if } s_0 < p_i \end{cases}$$

The same holds for $\angle(s_0 - z_i)$. Rule 3 follows from (2.3).

Example of use of Rule 3:

Suppose that G(s) has one pole and one zero in the right half-plane, eg $p_1 = +5$, $z_1 = +2$. Rule 3 shows that $K(s)$ *must* have *at least* one pole to the right of +2.
— the controller must be *unstable*!

(Figuring out the details is often easier from Bode plots etc.)



Proof of Rule 2:

$$d(s_0) + kn(s_0) = 0 \qquad \text{from (2.1)}$$

So if $k = 0$ then $d(s_0) = 0 \Rightarrow$ Branches start at poles.

Also, for any fixed k,

$$(2.1) \iff \frac{1}{k}d(s_0) + n(s_0) = 0$$

So, as $k \to \infty$ the finite roots tend to zeros (i.e. finite branches end at zeros). There can be at most m of them.

(to see this, put $n(s) = (s - s_0)n'(s)$, then for s close to s_0 we have $s = s_0 - \frac{1}{k}\frac{d(s_0)}{n'(s_0)} \to s_0$.)

The remaining $n - m$ branches go to ∞, but how?

Proof of Rule 5: (Large S)

$$-k = \frac{1}{L(s)} = \frac{(s-p_1)(s-p_2)\ldots}{c(s-z_1)(s-z_2)\ldots}$$

$$-kc = \frac{s^n - \Sigma p_i s^{n-1} + \ldots}{s^m - \Sigma z_i s^{m-1} + \ldots}$$

$$\approx s^{n-m}\frac{(1 - \Sigma p/s)}{(1-\Sigma z/s)} \approx s^{n-m}\left(1 - \frac{\Sigma p}{s}\right)\left(1 + \frac{\Sigma z}{s}\right)$$

$$\approx s^{n-m}\left(1 - \left(\frac{\Sigma p}{s} - \frac{\Sigma z}{s}\right)\right)$$

$$\approx s^{n-m}\left(1 - \frac{\Sigma p - \Sigma z}{s(n-m)}\right)^{n-m}$$

$$= \left(s - \frac{\Sigma p_i - \Sigma z_i}{n-m}\right)^{n-m}$$

$$\Rightarrow \angle\, s - \frac{\Sigma p_i - \Sigma z_i}{n-m} = \frac{-180^\circ \pm 360^\circ \ell}{n-m}$$

Application to previous example:

$n = 3$, $m = 1$, so $n - m = 2$. Two asymptotes at angles $\pi/2$ and $3\pi/2$.

Asymptotes emanate from (Rule 5):

$$\frac{(0-3-3)-(-5)}{3-1} = -\frac{1}{2}$$

Proof of Rule 4: Repeated roots at $s = s_0$

$$\Rightarrow 1 + kL(s) = (s-s_0)^2(\ldots)$$

$$\Rightarrow \frac{d}{ds}(\text{''}) = kL'(s) = 2(s-s_0)(\ldots) + (s-s_0)^2\frac{d}{ds}(\ldots) = 0$$

Application of Rule 4 to example of Fig.2.1:

$$\begin{aligned}
\frac{d}{ds}\left(\frac{s+5}{s^3+6s^2+10s}\right) &= 0 \\
\Rightarrow \frac{1(s^3+6s^2+10s) - (s+5)(3s^2+12s+10)}{(s^3+6s^2+10s)^2} &= 0 \\
\Rightarrow \frac{-2s^3 - 21s^2 - 60s - 50}{(s^3+6s^2+10s)^2} &= 0 \\
\Rightarrow \frac{-2(s+1.5505)(s+2.5)(s+6.4495)}{(s^3+6s^2+10s)^2} &= 0
\end{aligned}$$

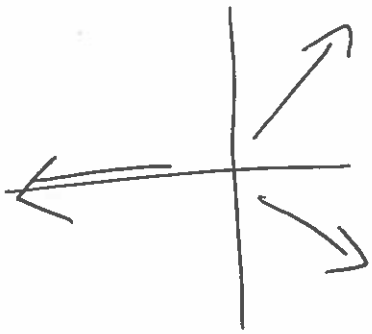
From Fig.2.1 it is seen that the root at -6.4495 is not on the root-locus.
The other two roots give the *breakaway points* (ie repeated roots).

2.5 Root-locus for negative k (or negative c)

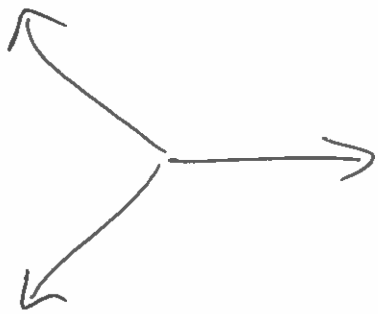
$$1 + kG(s_0) = 0 \quad \Rightarrow \quad G(s_0) = -\frac{1}{k} > 0$$
$$\Rightarrow \quad \angle G(s_0) = 2\ell\pi$$

- Rules 1,2,4 remain unchanged.
- Rule 3: Replace ‘odd’ by ‘even’.
- Angles of asymptotes become $2\ell\pi/(n-m)$.
  (Points from which asymptotes emanate remain unchanged.)

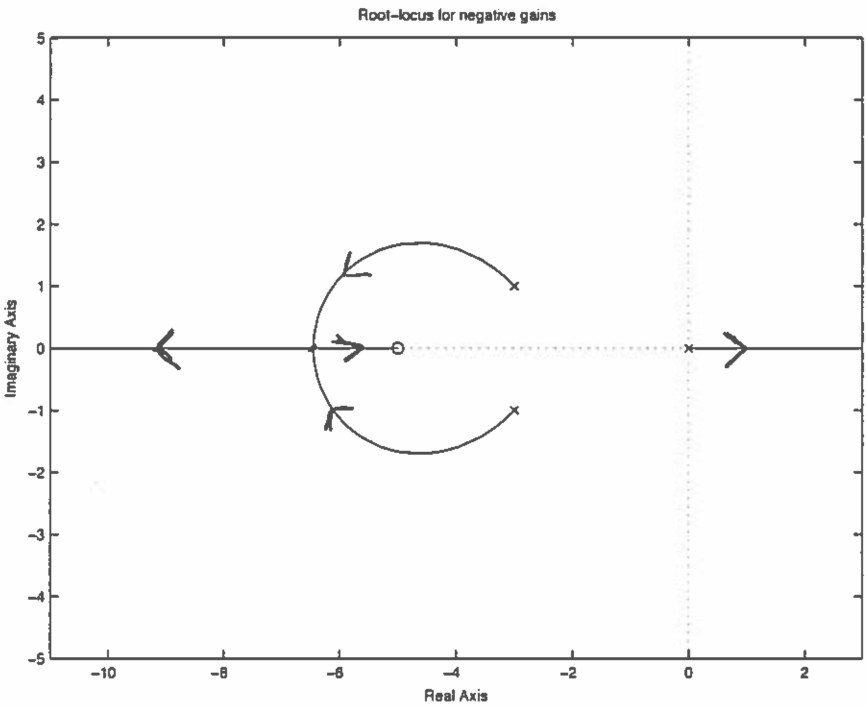
eg $n-m=3$



$k>0$



$k<0$



Root-locus diagram for negative gain when

$$L(s) = \frac{s+5}{s(s^2+6s+10)}$$

2.6 Studying Parameter Variations

Root-locus diagrams can be used to study the variation of closed-loop poles as other parameters vary — not just the loop-gain k.

All that is needed is to put the closed-loop characteristic equation into the form

$$1 + \lambda H(s) = 0 \tag{2.4}$$

where λ is the parameter that is varying, and $H(s)$ is a transfer function.

Example: Robot placing objects of varying mass

The 1-D equation of motion of a robot moving a mass m with viscous friction c and elastic tether is $m\ddot{x} = u - c\dot{x} - \alpha x$ where x is the mass position and u is the applied force. The use of a PI controller is proposed, with a transfer function $k(s+z)/s$.

With $m = 0.1$ kg, $c = 0.6$ N/(m/sec), $\alpha = 1$ N/m and $z = 5$ we have

$$G(s) = \frac{1}{0.1s^2 + 0.6s + 1} = \frac{10}{s^2 + 6s + 10} \qquad \text{and} \qquad K(s) = k\frac{s+5}{s}$$

Letting $L(s) = G(s)K(s)/k$ the closed-loop characteristic equation is $1 + kL(s) = 0$.

Using Fig.2.1, $\underline{10k = 1.395}$ places two closed-loop poles at -1.55 (one of the breakaway points) and the third pole at -2.9.

$$ck = \frac{\prod |s-p|}{\prod |s-z|} \qquad (\text{gain condition})$$

What if the mass varies?

The closed-loop characteristic equation is

$$1 + k\frac{(s+5)}{s(ms^2 + 0.6s + 1)} = 0$$

which has the same roots as

$$(ms^3 + 0.6s^2 + s) + k(s+5) = 0$$

divide by ms^3

or

$$1 + \underbrace{\frac{1}{m}}_{\lambda}\overbrace{\frac{0.6s^2 + [1+k]s + 5k}{s^3}}^{H(s)} = 0$$

which is in the form of (2.4) with $\lambda = 1/m$. The root-locus plot for this, with $k = 1.395/10$, is shown in Fig.2.2. The roots with $m = 0.1$ are marked. (as □)

Variations of closed-loop poles as $1/m$ varies can be clearly seen.

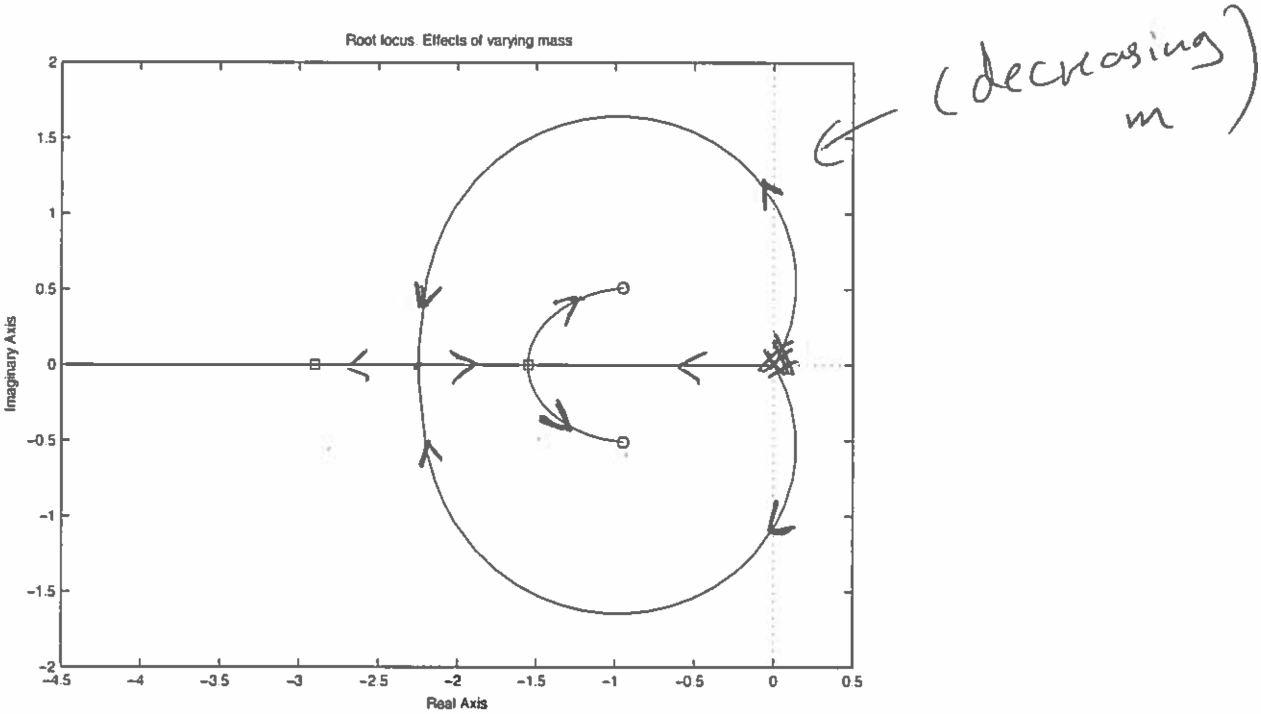


Figure 2.2: Root-locus diagram for variation of mass in robot problem.

3 The Routh-Hurwitz Criterion

The closed-loop characteristic equation

$$1 + G(s)K(s) = 0$$

has the same roots as

$$d_G(s)d_K(s) + n_G(s)n_K(s) = 0 \qquad \text{polynomial}$$

The Routh-Hurwitz criterion tests whether a polynomial has any roots with nonnegative real parts. So it tests for asymptotic stability.

Sometimes useful for finding value of k at which root-locus crosses imaginary axis.

Consider the polynomial (assume $a_0 > 0$):

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n \tag{3.5}$$

Easy to check that all roots have negative real parts *only if* $a_i > 0$ for each i.

A **Routh array** can be constructed for arbitrary n
— see Franklin, Powell and Emami-Naeini, 3rd edition, sec.4.4.3 (for example) for details.

For $n = 2, 3, 4$ simplifies as follows: *These are in Electrical and Information Data Book*
All the roots of (3.5) have negative real parts *if and only if*:

$$\begin{aligned} n = 2 &: \quad a_i > 0, \quad \text{No other conditions} \\ n = 3 &: \quad a_i > 0, \quad a_1 a_2 > a_0 a_3 \\ n = 4 &: \quad a_i > 0, \quad a_1 a_2 a_3 > a_0 a_3^2 + a_4 a_1^2 \end{aligned}$$