Engineering Tripos Part IB SECOND YEAR

Part IB Paper 6: Mathematics

SIGNAL AND DATA ANALYSIS

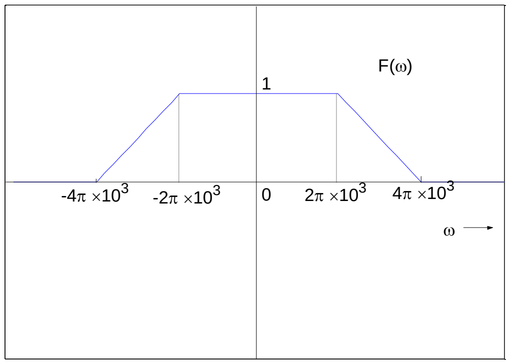
Examples paper 2P6/7

(Straightforward questions are marked †, problems of Tripos standard but not necessarily of Tripos length *)

Sampling, Discrete Signals, the DFT

1.† a) A signal f(t) is sampled. The spectrum of f(t), which is real valued, is shown in the figure.

Sketch the spectrum of the sampled signal when the sampling rate is i)3 kHz, ii) 4 kHz and iii) 6 kHz. What is the minimum sampling rate that will ensure perfect reconstruction of f(t) from its sampled sequence f(nT)?



b) Explain with the aid of sketches how f(t) in a) can be perfectly reconstructed from its sampled values f(nT) when the minimum sampling rate for perfect reconstruction is used. Sketch the ideal form of the reconstruction filter. How would this filter and the minimum sampling rate be modified in a practical scheme?

2. † Determine and sketch the spectrum of the following signals:

$$a \cos[2\pi(f_s + f_o)t], \quad a \cos[2\pi(f_s - f_o)t] \quad \text{and} \quad a \cos[2\pi f_o t]$$

where and f_o and f_s are constant frequencies in Hz.

Hence show that all three signals have identical spectra if they are sampled with a sampling frequency of f_s

Verify this fact directly by consideration of the sampled sequence $f(n/ f_s)$ in each of the three cases.

3.* Explain why, for any signal v(t),

$$v(t)\ \delta(t-nT) = v(nT)\ \delta(t-nT)\ .$$

If we sample v(t) with sampling interval T, then the sampled signal multiplied by T, which we call $v_s(t)$, can be written:

$$v_s(t) = T\ \{... + v(-2T)\,\delta(t+2T) + v(-T)\,\delta(t+T) + v(0)\,\delta(t) + v(T)\,\delta(t-T) + ...\}.$$

Show that the pulse broadening circuit shown has the impulse response shown in figure 2

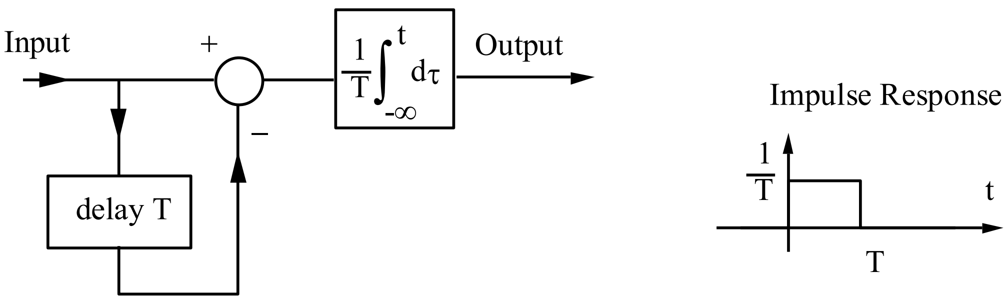


Figure 2

Sketch the output signal $w_s(t)$ which results from passing the signal $v_s(t)$ through this circuit, and find its spectrum in terms of the spectrum V(ω) of the original signal v(t).

Determine the ideal frequency response of a filter which can reconstruct v(t) from the pulse-broadened signal $w_s(t)$ (assuming that the maximum frequency component in v(t) is less than 1/(2T) Hz, where T is a known constant).

4.* A signal x(t) consists of a d.c. level plus two sinusoids

$$x(t) = a_0 + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t,$$

and it is sampled, without the use of an anti-aliasing filter, at a sampling frequency ω_s given by

$$\omega_s = \frac{\omega_0}{1+k}$$

where k is a constant much smaller than 1.

List the frequencies present in the sampled signal.

The sampled signal is passed through an ideal low pass filter with cut-off frequency ω_c equal to half the sampling frequency. Show that, since k is small, the output signal is proportional to a `stretched' version x(bt) of the orginal signal, and determine b.

[Background: A sampling oscilloscope uses this method to display periodic signals having bandwidths much larger than the bandwidth of a conventional oscilloscope amplifier.

The signal to be displayed x(t) is sampled once per period but with a sampling time that is much larger than the period of the signal. Passing the sampled signal through a low pass filter will produce an output signal proportional to x(bt) where b < 1, i.e. the original signal is time stretched and is now within the bandwith of the oscilloscope amplifier.]

5.* The DTFT of a data sequence f_n is defined as

$$F_s(\omega) = \sum_{-\infty}^{+\infty} f_n e^{-jn\omega T}$$

where T is the sampling interval.

Show that the sampled sequence f_m may be obtained from the DTFT spectrum $F_s(\omega)$ using the following formula:

$$f_m = \frac{T}{2\pi} \int_{-\pi/T}^{+\pi/T} F_s(\omega) e^{+jm\omega T} d\omega$$

Hint: substitute the definition for $F_s(\omega)$ into the formula and rearrange. You may use the following result:

$$\int_{-\pi}^{\pi} e^{jk\theta} d\theta = 2\pi \quad \text{if } k=0\text{, and zero for any other integer } k.$$

6.† The data sequence (1,0,0,1) has been obtained by sampling a signal at 8 kHz. Calculate the DFT of this sequence, and use the inverse DFT to verify your answer.

Plot the magnitude and phase of the DFT as a function of frequency, and comment on their symmetry properties.

Use the DFT formulae to show that for any length N sequence f_n:

a) $F_m = F_{-m}^*$ (for real valued signals)

b) $F_{m+N} = F_m$

7. Show that the DFT of the sampled sequence corresponding to the function

$$f(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases} \quad \text{is} \quad F_k = \frac{1 - e^{-NT}}{1 - e^{-T - j2\pi k/N}}$$

where T is the sampling period and N is the number of samples used for the DFT.

What is the sampling frequency? Determine also the frequency to which the k-th DFT component F_k corresponds.

Answers

1 a) 4 kHz.

2 sampled at times $t_k = k / f_s$ $\quad a\cos(2\pi k f_o / f_s)$ $\quad a\sin(2\pi k f_o / f_s)$

3. $\dfrac{1-e^{-jwT}}{jwT}\sum_n V(w - nw_0)$ where $\omega_0 = \dfrac{2\pi}{T}$. $H(\omega)=\dfrac{j\omega T}{1-e^{-jwT}}$ for $-\omega_0/2<\omega<\omega_0/2$ and 0 elsewhere

4. $\dfrac{n\omega}{1+k}$, $\pm\dfrac{\omega\,(1+n+k)}{1+k}$, $\pm\dfrac{\omega\,(2+n+2k)}{1+k}$

$b = \dfrac{k}{1+k} \approx k.$

5.

6. DFT = 2, 1 + j , 0, 1 − j

7. $\omega_s = \dfrac{2\pi}{T}$, $\omega_k = \dfrac{k}{N}\,\omega_s$

SJG 2018

Suitable past tripos questions, all from 1B Paper 6:
2014 q.5, 2013 q.5, 2012 q.5, 2011 q.5, 2010 q.5, plus many additional questions from earlier years.