Part IB Paper 6: Information Engineering

SIGNAL AND DATA ANALYSIS

Examples paper 5

(Before starting this examples paper it is a good idea to revise your 1A Maths notes on Fourier Series and on Convolution)

Revision of Fourier Series

1 The function

$$y(x) = \begin{cases} x(\pi+x) & -\pi \le x \le 0 \\ x(\pi-x) & 0 \le x \le \pi \end{cases}$$

is represented by a Fourier series of period 2π. Which derivatives (first, second, etc.) of the function represented by this Fourier series are continuous for all values of x? How do the coefficients in the series vary with n? Verify your answer to the latter question by evaluating the coefficients.

2 The function $y(t)$ is represented by a Fourier Series of the form

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right\} \qquad \omega_0 = \frac{2\pi}{T}$$

over the range $0 < t < T$. By expressing the sines and cosines as complex exponentials, show that y can be represented by a complex Fourier Series of the form

$$y(t) = \sum_{n=-\infty}^{\infty} c_n \exp(jn\omega_0 t) ,$$

and find c_n in terms of the a's and b's. Pay particular attention to the cases $n = 0$ and n negative.

Show that $c_n^* = c_{-n}$.

3 A function $y(t)$ having period T is defined as

$$y(t) = \begin{cases} \exp(-\alpha t) & 0 \le t \le T/2 \\ 0 & T/2 \le t \le T \end{cases} .$$

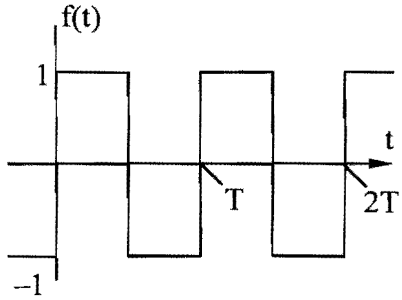
Obtain the coefficients c_n in the complex Fourier series for $y(t)$, where

$$y(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int/T} .$$

Hence obtain the coefficients a_n and b_n in the real Fourier series for $y(t)$, where

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nt}{T} .$$

4 Show that the Fourier Series representation of the square wave function shown is



$$f(t) = \frac{4}{\pi}\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} \sin n\,\omega_0 t$$

$$\text{where } \omega_0 = \frac{2\pi}{T}$$

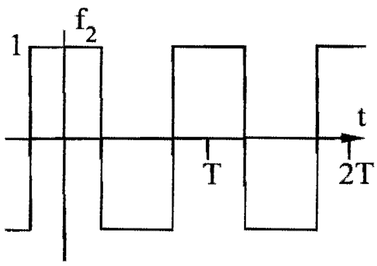
Hence find the coefficients c_n in the complex Fourier Series representation

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{j2\pi nt/T}$$

Verify that your answer is correct by finding c_n from the relationship

$$c_n = \frac{1}{T}\int_0^T f(t)\exp\left[-jn\omega_0 t\right]\ dt\ .$$

5 Show that the square wave $f_2(t)$,which is symmetric about $t = 0$, is related to f(t) of question 4 by $f_2(t) = f(t+a)$, and find the value of a .



Using the result of question 4, show that

$$f_2(t) = \sum_{n=-\infty}^{\infty} d_n\, e^{j2\pi nt/T}$$

$$\text{where } d_n = c_n \exp\left[\frac{jn\pi}{2}\right]$$

6 The coefficients in a complex Fourier Series representation of the function f(t) over the range $0 < t < T$ are $\frac{(-1)^n}{n^4}$.

Which derivatives of f are continuous for all values of t ? Derive expressions for the complex coefficients in the Fourier Series representations of

$$\text{(i) } \frac{df}{dt} \qquad \text{(ii) } \frac{d^2f}{dt^2} \qquad \text{(iii) } \int f\,dt$$

In the last case what can you say about c_0 ?

Convolution

7 By evaluating the convolution integral directly, find the convolution of f(t) and g(t), where

$$f(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad \text{and} \qquad g(t) = \begin{cases} \exp(-\alpha t) & t \geq 0 \\ 0 & t < 0 \end{cases} .$$

8 For any functions f and g show that

$$\int_{\tau=0}^{t} f(\tau)\, g(t-\tau)\, d\tau = \int_{\tau=0}^{t} f(t-\tau)\, g(\tau)\, d\tau .$$

9 It is desired to find the convolution of the functions f and g , where

$$f(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad g(t) = \begin{cases} \exp(-\alpha t) & t \geq 0 \\ 0 & t < 0 \end{cases} .$$

For a fixed value of t ($t > 0$), sketch $f(\tau)$ and $g(t-\tau)$ as functions of τ . Using your sketches, explain why the convolution of f and g will have three different forms in the ranges

(i) $t < 0$ (ii) $0 \leq t \leq T$ (iii) $T < t$

Evaluate the convolution of f and g, $\displaystyle\int_{\tau=0}^{t} f(\tau)\, g(t-\tau)\, d\tau$.

Answers

1 The value and first derivative are continuous: the coefficients are $O(1/n^3)$.

$$y(x) = \frac{8}{\pi} \sum_{n} \frac{\sin nx}{n^3} \quad (n = 1, 3, 5, ...)$$

2 $c_0 = \dfrac{a_0}{2}$, $\quad c_n = \dfrac{a_n - j b_n}{2}$ for $n > 0$, $\quad c_n = \dfrac{a_{-n} + j b_{-n}}{2}$ for $n < 0$

3 $c_n = \dfrac{1 - e^{-\alpha T/2 - in\pi}}{\alpha T + 2in\pi}$.

$$a_0 = \frac{2(1-e^{-\alpha T/2})}{\alpha T} \qquad a_n = \frac{2\alpha T(1-e^{-\alpha T/2}\cos n\pi)}{\alpha^2 T^2 + 4n^2\pi^2} \qquad b_n = \frac{4n\pi(1-e^{-\alpha T/2}\cos n\pi)}{\alpha^2 T^2 + 4n^2\pi^2}$$

4 $c_n = \dfrac{2}{jn\pi}$ for n odd (valid for positive <u>and</u> negative n), $c_n = 0$ n even .

5 $a = \dfrac{T}{4}$

6 The value, first and second derivatives are continuous.

(i) $j\,\omega_0 \dfrac{(-1)^n}{n^3}$ (ii) $\omega_0^2 \dfrac{(-1)^{n+1}}{n^2}$

(iii) $\dfrac{(-1)^n}{j\omega_0 n^5}$ c_0 depends on constant of integration.

7 $\dfrac{t}{\alpha} - \dfrac{1}{\alpha^2} + \dfrac{e^{-\alpha t}}{\alpha^2}$ for $t \geq 0$ (and 0 for $t < 0$).

9 (i) 0 (ii) $\dfrac{1 - e^{-\alpha t}}{\alpha}$ (iii) $\dfrac{e^{-\alpha(t-T)} - e^{-\alpha t}}{\alpha}$