Active-Redundancy Allocation in Systems

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***Abstract*—An effective way of improving the reliability of a system is the allocation of active redundancies. Let $\mathbf{X_1}$, $\mathbf{X_2}$ be s-independent lifetimes of the components $\mathbf{C_1}$ and $\mathbf{C_2}$, respectively, which form a series system. Let us denote $\mathbf{U_1 = \min(\max(X_1, X), X_2)}$ and $\mathbf{U_2 = \min(X_1, \max(X_2, X))}$, where $\mathbf{X}$ is the lifetime of a redundancy (say $\mathbf{R}$) s-independent of $\mathbf{X_1}$ and X_2. That is, $\mathbf{U_1(U_2)}$ denote the lifetime of a system obtained by allocating $\mathbf{R}$ to $\mathbf{C_1(C_2)}$ as an active redundancy. Singh and Misra (1994) considered the criterion where $\mathbf{C_1}$ is preferred to $\mathbf{C_2}$ for the allocation of R as active redundancy if $\mathbf{P(U_1 > U_2) \geq P(U_2 > U_1)}$. In this paper, we use the same criterion of Singh and Misra (1994). We investigate the allocation of one active redundancy when it differs depending on the component with which it is to be allocated. We also compare the allocation of two active redundancies (say $\mathbf{R_1}$ and $\mathbf{R_2}$) in two different ways; that is, $\mathbf{R_1}$ with $\mathbf{C_1}$ & $\mathbf{R_2}$ with $\mathbf{C_2}$, and viceversa. For this case, the hazard rate order plays an important role. We furthermore consider the allocation of active redundancy to $\mathbf{k}$-out-of-$\mathbf{n}$: G systems.**

***Index Terms*—Active redundancy, hazard rate order, stochastic order.**

I. Acronyms and Abbreviations[1]

Cdf	cumulative distribution function
iff	if and only if
k-out-of-n : G	The system is good iff at least k of its n elements are good
Pdf	probability density function
$r.v$	random variable
s-	implies the statistical definition
Sf	survival function

II. Notation

$\vee(x, y)$	maximum of x and y
$\wedge(x, y)$	minimum of x and y
C_i	component i
X_i	lifetime of C_i
R_i	spare i
Y_i	lifetime of R_i
$f_i(t)$	$\mathrm{Pdf}(X_i)$
$F_i(t)$	$\mathrm{Cdf}(X_i)$
$\overline{F}_i(t)$	$1 - F_i(t)$: $Sf(X_i)$
$g_i(t)$	$\mathrm{Pdf}(Y_i)$
$G_i(t)$	$\mathrm{Cdf}(Y_i)$
$\overline{G}_i(t)$	$1 - G_i(t)$: $Sf(Y_i)$
$\lambda_i(t)$	hazard rate of X_i
$\leq_{pr}$	probability order
$\leq_{st}$	usual stochastic order
$\binom{n}{r}$	$n!/((n-r)!r!)$ binomial coefficient

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[1]The singular and plural of an acronym are always spelled the same.

III. Introduction

An EFFECTIVE way of improving the reliability of a system is the allocation of active redundancies. This problem has been studied by different authors using different criteria (see [1]–[4]). Let C_1 and C_2 form a series system with s-independent lifetimes X_1 and X_2. Let us denote $U_1 = \wedge(\vee(X_1, X), X_2)$, $U_2 = \wedge(X_1, \vee(X_2, X))$, where X is the lifetime of a redundancy R, s-independent of X_1 and X_2. That is, $U_1(U_2)$ denote the lifetime of a system obtained by allocating R to $C_1(C_2)$ as an active redundancy. In [4], the following criterion is considered to compare the lifetimes of these systems: it is better to allocate R as an active redundancy with C_1 instead of with C_2 if the following inequality holds

$$P(U_1 > U_2) \geq P(U_2 > U_1). \tag{1}$$

We will use throughout the paper the following definition.

Definition 1: We will say that a r.v X is greater than a r.v Y in the probability order, written $X \geq_{pr} Y$, if

$$P(X > Y) \geq P(Y > X).$$

Then we can write inequality (1) as $U_1 \geq_{pr} U_2$.

We will use in this paper the usual stochastic order.

Definition 2: A r.v X is said to be greater than a r.v Y in the stochastic order, written $X \geq_{st} Y$, if

$$P(X > t) \geq P(Y > t)$$

for all real value t.

In [1], it is shown that $X_1 \leq_{st} X_2$ implies $U_1 \geq_{st} U_2$. But if X and Y are s-dependent r.v, we may have $X \geq_{st} Y$ and $X \leq_{pr} Y$ [5]. Actually, the lifetimes U_1 and U_2 are s-dependent. For this reason, in [4] it is investigated if $X_1 \leq_{st} X_2$ implies $U_1 \geq_{pr} U_2$ also. They find out that this implication holds.

However, in some cases it is more realistic to consider that in a series system we may allocate one active redundancy that differs depending on the component with which it is to be allocated [1]. Suppose we have two redundancies, R_1 and R_2, and only one of them will be allocated. R_1 could be allocated with C_1, and R_2 could be allocated with C_2. It is of interest to decide which one of these two redundancies to allocate. It would be of interest also to compare the allocation of two redundancies in two different ways: R_1 with C_1 & R_2 with C_2, and viceversa. With the aim of studying these problems, we will consider the lifetimes U_1, U_2 and V_1, V_2 defined below.

Suppose Y_1 and Y_2 are s-independent r.v, and s-independent of X_1, X_2. Let us now redefine U_1 and U_2 as

$$U_1 = \wedge(\vee(X_1, Y_1), X_2), \quad U_2 = \wedge(X_1, \vee(X_2, Y_2)), \tag{2}$$

and denote

$$\begin{aligned} V_1 &= \wedge(\vee(X_1, Y_1), \vee(X_2, Y_2)), \\ V_2 &= \wedge(\vee(X_1, Y_2), \vee(X_2, Y_1)). \end{aligned} \tag{3}$$

If $X_1 \leq_{st} X_2$ and $Y_1 \geq_{st} Y_2$, then $U_1 \geq_{st} U_2$ and $V_1 \geq_{st} V_2$. It would be of interest to find out sufficient conditions for the lifetimes of components and redundancies such that the relations $U_1 \geq_{pr} U_2$ and $V_1 \geq_{pr} V_2$ hold. We will see that, in the case of the last relation, the hazard rate order plays an important role.

Definition 3: Suppose X & Y are nonnegative r.v, and let us denote by $\overline{F}(t)$ & $\overline{G}(t)$ its respective Sf. X is said to be greater than Y in the hazard rate ordering, written $X \geq_{hr} Y$, if $\overline{F}(t)/\overline{G}(t)$ is nondecreasing for all $t \geq 0$ where this quotient is defined.

If the Pdf of X and Y, say $f(t)$ and $g(t)$, exist, then the ordering $X \geq_{hr} Y$ can be equivalently expressed as

$$\frac{f(t)}{\overline{F}(t)} \leq \frac{g(t)}{\overline{G}(t)}.$$

For a reference in stochastic ordering, see [6].

The structure of this paper is as follows. In Section II, we establish some results which will be used in the proofs of Sections II and III. In Sections III and IV, we find sufficient conditions for $U_1 \geq_{pr} U_2$ and $V_1 \geq_{pr} V_2$ to hold. In both sections we consider the allocation of active redundancy to k-out-of-n: G systems. We also examine the decision between expanding a k-out-of-n: G system, and improving it by allocating active redundancy. Conclusions are presented in Section V where we briefly comment on directions of future research.

IV. Preliminary Results

For a set of r.v $\{Z_1, Z_2, \ldots, Z_n\}$, let $(Z_1, Z_2, \ldots, Z_n)_{[k]}$ denote the kth largest order statistics, so that $(Z_1, Z_2, \ldots, Z_n)_{[1]} \geq \cdots \geq (Z_1, Z_2, \ldots, Z_n)_{[n]}$. Let us consider the r.v $X_1, X_2, \ldots, X_n$, Y_1, Y_2, $n = 2, 3 \ldots$, and let us denote

$$\begin{aligned} U_1^{(k)} &= (\vee(X_1, Y_1), X_2, X_3, \ldots, X_n)_{[k]} \\ U_2^{(k)} &= (X_1, \vee(X_2, Y_2), X_3, \ldots, X_n)_{[k]}, \end{aligned} \tag{4}$$

$k = 2, \ldots, n$, $n = 3, 4, \ldots$.

Proposition 1: The following equivalencies hold:

a) $U_1 > U_2$ iff $X_1 < \wedge(X_2, Y_1)$.

b) For $n > 2$, $U_1^{(n)} > U_2^{(n)}$ iff

$$X_1 < \wedge(Y_1, X_2, X_3, \ldots, X_n).$$

c) For $n > 2$ and $1 < k < n$, $U_1^{(k)} > U_2^{(k)}$ iff one of the following $\binom{n-2}{k-1} + \binom{n-2}{k-2}$ excluding inequalities is satisfied:

$$\begin{aligned} &\vee(X_1, X_2, Y_2, X_{r_1}, \ldots, X_{r_{n-k-1}}) \\ &\qquad < \wedge(Y_1, X_{v_1}, \ldots, X_{v_{k-1}}), \\ &\vee(X_1, X_{j_1}, \ldots, X_{j_{n-k}}) \\ &\qquad < \wedge(X_2, Y_1, X_{i_1}, \ldots, X_{i_{k-2}}), \end{aligned} \tag{5}$$

where

$$\begin{aligned} \{r_1, \ldots, r_{n-k-1}\} &\subseteq \{3, \ldots, n\}, \\ \{v_1, \ldots, v_{k-1}\} &\subseteq \{3, \ldots, n\}, \\ \{j_1, \ldots, j_{n-k}\} &\subseteq \{3, \ldots, n\}, \\ \{i_1, \ldots, i_{k-2}\} &\subseteq \{3, \ldots, n\}, \end{aligned} \tag{6}$$

$$\{v_1, \ldots, v_{k-1}\} \bigcap \{r_1, \ldots, r_{n-k-1}\} = \varnothing \tag{7}$$

and

$$\{i_1, \ldots, i_{k-2}\} \bigcap \{j_1, \ldots, j_{n-k}\} = \varnothing. \tag{8}$$

Let us denote

$$\begin{aligned} V_1^{(k)} &= (\vee(X_1, Y_1), \vee(X_2, Y_2), X_3, \ldots, X_n)_{[k]}, \\ V_2^{(k)} &= (\vee(X_1, Y_2), \vee(X_2, Y_1), X_3, \ldots, X_n)_{[k]}, \end{aligned} \tag{9}$$

$k = 2, \ldots, n$, $n = 3, 4, \ldots$

Proposition 2: The following equivalences hold:

a) $V_1 > V_2$ iff one of the following two excluding inequalities is satisfied

$$\vee(X_1, Y_2) < \wedge(X_2, Y_1), \quad \vee(X_2, Y_1) < \wedge(X_1, Y_2).$$

b) For $n > 2$ and $1 < k \leq n$, $V_1^{(k)} > V_2^{(k)}$ iff one of the following $2\binom{n-2}{k-2}$ excluding inequalities is satisfied

$$\vee(X_1, Y_2, X_{j_1}, \ldots, X_{j_{n-k}}) < \wedge(X_2, Y_1, X_{i_1}, \ldots, X_{i_{k-2}}), \tag{10}$$

$$\vee(X_2, Y_1, X_{j_1}, \ldots, X_{j_{n-k}}) < \wedge(X_1, Y_2, X_{i_1}, \ldots, X_{i_{k-2}}), \tag{11}$$

where

$$\begin{aligned} \{i_1, \ldots, i_{k-2}\} &\subseteq \{3, \ldots, n\}, \\ \{j_1, \ldots, j_{n-k}\} &\subseteq \{3, \ldots, n\} \end{aligned} \tag{12}$$

and

$$\{i_1, \ldots, i_{k-2}\} \bigcap \{j_1, \ldots, j_{n-k}\} = \varnothing. \tag{13}$$

V. Allocation of an Active Redundancy

In this section, U_1, U_2, $U_1^{(k)}$, and $U_2^{(k)}$ are defined as in (2) & (4).

Lemma 1: Let X_1, X_2, Y_1, Y_2, and Z be nonnegative s-independent r.v. Suppose

i) X_1 and X_2 have probability densities, and

$$\lambda_1(x)\overline{G}_1(x) \geq \lambda_2(x)\overline{G}_2(x), \quad x \geq 0$$

or

ii) $X_1 \leq_{st} X_2$, and $\overline{F}_2(x)\overline{G}_1(x) \geq \overline{F}_1(x)\overline{G}_2(x)$, $x \geq 0$. Then

$$a) \quad P(X_1 < \wedge(Y_1, X_2)) \geq P(X_2 < \wedge(X_1, Y_2))$$

and

$$b) \quad P(X_1 < \wedge(Y_1, X_2, Z)) \geq P(X_2 < \wedge(X_1, Y_2, Z)).$$

Proposition 3: Let $X_1, X_2, \ldots, X_n$, Y_1, and Y_2 be s-independent lifetimes. Suppose

i) X_1 and X_2 have Pdf, and

$$\lambda_1(x)\overline{G}_1(x) \geq \lambda_2(x)\overline{G}_2(x), \quad x \geq 0,$$

or

ii) $X_1 \leq_{st} X_2$ and $\overline{F}_2(x)\overline{G}_1(x) \geq \overline{F}_1(x)\overline{G}_2(x)$, $x \geq 0$. Then

$$U_1 \geq_{pr} U_2 \text{ and } U_1^{(n)} \geq_{pr} U_2^{(n)}.$$

Conditions *i*) and *ii*) of Proposition 3 give us criteria for the optimal allocation in the sense of the probability order of a redundancy which differs depending on the component with which it is allocated. If $Y_1 \geq_{st} Y_2$, and it also holds that $X_1 \leq_{hr} X_2$ or $X_1 \leq_{st} X_2$; then it is optimal in the probability order to allocate the stronger redundancy (R_1) to the weaker component (C_1). If $G_1 = G_2$, condition *i*) reduces to hazard rate order between lifetimes X_1 and X_2, and condition *ii*) reduces to stochastic order between lifetimes X_1 and X_2.

Notice that $\overline{F}_i(x)\overline{G}_j(x)$, $i, j = 1, 2$, is the Sf of a series system formed by components with lifetimes X_i and Y_j. Then condition *ii*) can be stated in the following way. If the series system formed by C_2 and R_1 is stochastically greater than the series system formed by C_1 and R_2, and $X_1 \leq_{st} X_2$, then it is better to allocate R_1 in parallel with C_1 than to allocate R_2 in parallel with C_2.

The following lemma will be useful extending the result of Proposition 3 to k-out-of-n: G systems. Result *b*) in Lemma 2 is stated in Lemma 2.1 of [4].

Lemma 2: Let X_1, X_2, Y_1, Y_2, Z_1, and Z_2 be nonnegative s-independent r.v. Let Z_3 & Z_4 be nonnegative s-independent r.v, and s-independent of Y_1 & Y_2. Suppose that $X_1 \leq_{st} X_2$, and $Y_1 \geq_{st} Y_2$. Then

$$\begin{aligned} a) \quad & P(\vee(X_1, Z_1) < \wedge(X_2, Y_1, Z_2)) \\ & \quad \geq P(\vee(X_2, Z_1) < \wedge(X_1, Y_2, Z_2)). \\ b) \quad & P(\vee(Y_2, Z_3) < \wedge(Y_1, Z_4)) \\ & \quad \geq P(\vee(Y_1, Z_3) < \wedge(Y_2, Z_4)). \end{aligned}$$

Proposition 4: Let $X_1, \ldots, X_n, Y_1$, and Y_2 be s-independent lifetimes. Suppose that $X_1 \leq_{st} X_2$, and $Y_1 \geq_{st} Y_2$. Then for $1 < k < n$, $n > 2$,

$$U_1^{(k)} \geq_{pr} U_2^{(k)}.$$

VI. Allocation of More Than One Redundancy

In this section, we compare the allocation of redundancies R_1 and R_2 in two different ways; i.e., R_1 with C_1 & R_2 with C_2, and viceversa. We also consider the decision between expanding a k-out-of-n: G system, and improving the already existing system by means of component-wise redundancy.

In this section, V_1, V_2, $V_1^{(k)}$, and $V_2^{(k)}$ are defined as in (3) and (9).

Lemma 3: Let X_1, Y_1, X_2, Y_2, Z_1 and Z_2 be s-independent r.v. Suppose X_1, Y_1, X_2 and Y_2 have Pdf. Let $X_1 \leq_{hr} X_2$ and $Y_2 \leq_{hr} Y_1$. Then

$$\begin{aligned} a) \quad & P(\wedge(X_2, Y_1) > \vee(X_1, Y_2) \\ & \quad \text{or} \wedge (X_1, Y_2) > \vee(X_2, Y_1)) \\ & \quad \geq P(\wedge(X_2, Y_2) > \vee(X_1, Y_1) \\ & \quad\quad \text{or} \wedge (X_1, Y_1) > \vee(X_2, Y_2)) \end{aligned}$$

and

$$\begin{aligned} b) \quad & P(\wedge(X_2, Y_1, Z_2) > \vee(X_1, Y_2, Z_1) \\ & \quad \text{or} \wedge (X_1, Y_2, Z_2) > \vee(X_2, Y_1, Z_1)) \\ & \quad \geq P(\wedge(X_2, Y_2, Z_2) > \vee(X_1, Y_1, Z_1) \\ & \quad\quad \text{or} \wedge (X_1, Y_1, Z_2) > \vee(X_2, Y_2, Z_1)). \end{aligned}$$

Proposition 5: Let $X_1, \ldots, X_n, Y_1$, and Y_2 be s-independent lifetimes. Suppose X_1, Y_1, X_2, and Y_2 have Pdf. Let $X_1 \leq_{hr} X_2$, and $Y_2 \leq_{hr} Y_1$. Then

$$V_1 \geq_{pr} V_2 \text{ and } V_1^{(k)} \geq_{pr} V_2^{(k)}$$

for $1 < k \leq n$, $n > 2$.

Notice that this result has the following practical meaning. Suppose that there exists two options for allocating R_1 and R_2 as active redundancies to C_1 and C_2. One option is to allocate R_1 with C_1, and R_2 with C_2. Another option is to allocate R_1 with C_2, and R_2 with C_1. If the lifetime of R_1 is greater than the lifetime of R_2 in the hazard rate ordering, and the lifetime of C_2 is greater than the lifetime of C_1 in the hazard rate ordering, then it is better in the sense of the probability order to allocate the best redundancy with the weakest component; i.e., R_1 with C_1, and R_2 with C_2.

The decision between expanding a k-out-of-n: G system and improving the already existing system by means of a redundancy is studied in [7]. In the following proposition, we analyze this problem.

Proposition 6: Let $X_1, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_r$ $(r \leq n)$ be lifetimes. Then the following inequality always holds

$$\begin{aligned} & (X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_r)_{[k]} \\ & \quad \geq (\vee(X_1, Y_1), \vee(X_2, Y_2), \ldots, \\ & \quad\quad \vee (X_r, Y_r), X_{r+1}, \ldots, X_n)_{[k]}, \\ & 1 \leq k \leq n, \quad n = 1, 2, \ldots \end{aligned} \tag{14}$$

The result of this proposition has the following practical meaning. Suppose we have $r \leq n$ spares which can be used in two different ways. We can expand a k-out-of-n: G system to a k-out-of-$(n+r)$: G system. Alternatively, we can allocate each spare as an active redundancy to any component of the system (only one spare to each component of the system). Then it is better to expand the system to a k-out-of-$(n+r)$: G system than to allocate each spare in parallel with one component of the system.

VII. Conclusions and Extensions

We have discussed on the allocation of one or two active redundancies to a k-out-of-n: G system in order to improve the system in the sense of the probability order. As an extension of the research presented in this paper, we find of interest to study the problem of the optimal allocation of more than two active redundancies to a k-out-of-n: G system. For a series system, in the case that all components have the same lifetime distribution, this problem has been considered in [8] and [9], but still the questions concerning the optimal allocation in the sense of the probability order for k-out-of-n: G systems remain open.

Appendix

Let us denote by the corresponding lower letter z a value of a real random variable Z. Let us define, for a real number t, $z(t)$ as $z(t) = 1$ if $t \leq z$, and $z(t) = 0$ if $t > z$.

Consider now the values x and y of two r.v X and Y, respectively. Observe that the inequality $x > y$ is valid iff there exists a real number t such that $x(t) > y(t)$. This equivalence allows us to reduce the treatment of inequalities between real valued r.v to the treatment of inequalities between sums of variables with values 0, 1. In the following, in place of the functions of type $z(t)$, we will simply write z. That is, instead of $x_i(t)$ $(y_i(t))$, we will write x_i (y_i), $i = 1, 2, \ldots$.

A. Proof of Proposition 1

We will only prove b) and c), because a) follows in a similar fashion. Inequality

$$U_1^{(k)} > U_2^{(k)} \tag{15}$$

holds iff the following system of inequalities is satisfied

$$\vee(x_1, y_1) + x_2 + x_3 + \cdots + x_n \geq k, \tag{16}$$

$$x_1 + \vee(x_2, y_2) + x_3 + \cdots + x_n \leq -1. \tag{17}$$

Suppose $x_1 = 1$. Then $\vee(x_1, y_1) = 1$, and it is easy to see that in this case, (16) and (17) do not hold simultaneously; consequently, $x_1 = 0$. Subtracting now (17) from (16), we obtain $y_1 + x_2 \geq 1 + \vee(x_2, y_2)$. Because $x_2 \leq \vee(x_2, y_2)$, this implies $y_1 = 1$, and therefore

$$\vee(x_2, y_2) = x_2. \tag{18}$$

Substituting this last equality, and the values $x_1 = 0$ and $y_1 = 1$ in (16) and (17), we obtain

$$x_2 + x_3 + \cdots + x_n = k - 1. \tag{19}$$

Then the system given by inequalities in (16) and (17) is satisfied only if $x_1 = 0$, $y_1 = 1$, and the system given by (18) and (19) is satisfied. It is straightforward to verify that, conversely, if these conditions hold, the system given by the inequalities in (16) and (17) is satisfied.

For $k = n$, the system given by (18) and (19) has for all values of y_2 the unique solution $x_2 = x_3 = \cdots = x_n = 1$, and then ((15)) is equivalent to

$$X_1 < \wedge(Y_1, X_2, X_3, \ldots, X_n).$$

Let us consider now the case $k < n$. Observe that if $x_2 = 0$ from (18), we obtain $y_2 = 0$. Notice also that if $x_2 = 0$, (19) has $\binom{n-2}{k-1}$ solutions. These solution are $x_{r_1} = \cdots = x_{r_{n-k-1}} = 0$, $x_{v_1} = \cdots = x_{v_{k-1}} = 1$, where $\{v_1, \ldots, v_{k-1}\}$ and $\{r_1, \ldots, r_{n-k-1}\}$ satisfy (6) and (7). Then the condition in (5) follows.

For the case $x_2 = 1$, (19) has $\binom{n-2}{k-2}$ solutions; and the value of y_2 may be arbitrary. These solutions are $x_{j_1} = \cdots = x_{j_{n-k}} = 0$, $x_{i_1} = \cdots = x_{i_{k-2}} = 1$, where $\{j_1, \ldots, j_{n-k}\}$ and $\{i_1, \ldots, i_{k-2}\}$ satisfy (6) and (8). □

B. Proof of Proposition 2

We consider the case b), because a) follows in a similar manner. Inequality

$$V_1^{(k)} > V_2^{(k)} \tag{20}$$

holds iff the following system of inequalities holds

$$\vee(x_1, y_1) + \vee(x_2, y_2) + x_3 + \cdots + x_n \geq k, \tag{21}$$

$$\vee(x_1, y_2) + \vee(x_2, y_1) + x_3 + \cdots + x_n \leq k - 1. \tag{22}$$

If $\vee(x_1, y_2) = \vee(x_2, y_1) = 1$ from (22), we have $x_3 + \cdots + x_n \leq k - 3$; but then (21) is not satisfied. Let $\vee(x_1, y_2) = \vee(x_2, y_1) = 0$. In this case, from (21) and (22), we obtain the contradictory inequalities $x_3 + \cdots + x_n \geq k$ and $x_3 + \cdots + x_n \leq k - 1$.

Suppose now that $\vee(x_1, y_2) = 1$ and $\vee(x_2, y_1) = 0$, or that $\vee(x_1, y_2) = 0$ and $\vee(x_2, y_1) = 1$. In these cases, from (22), we have $x_3 + \cdots + x_n \leq k - 2$. Subtracting this inequality from (21), we obtain $\vee(x_1, y_1) + \vee(x_2, y_2) = 2$; and also, from (21), we have $x_3 + \cdots + x_n \geq k - 2$. Then the system given by inequalities in (21) and (22) is satisfied only if $\vee(x_2, y_1) = 0$, $\wedge(x_1, y_2) = 1$, and $x_3 + \cdots + x_n = k - 2$; or $\vee(x_1, y_2) = 0$, $\wedge(x_2, y_1) = 1$, and $x_3 + \cdots + x_n = k - 2$. Conversely, if these conditions hold, then the system given by inequalities in (21) and (22) is satisfied.

The $\binom{n-2}{k-2}$ solutions of equation $x_3 + x_4 + \cdots + x_n = k - 2$ are $x_{j_1} = \cdots = x_{j_{n-k}} = 0$, $x_{i_1} = \cdots = x_{i_{k-2}} = 1$, where $\{j_1, \ldots, j_{n-k}\}$ and $\{i_1, \ldots, i_{k-2}\}$ satisfy (12) and (13).

Then for the case in which $\vee(x_1, y_2) = 0$ and $\wedge(x_2, y_1) = 1$, we obtain the condition in (10), and for the case in which $\vee(x_2, y_1) = 0$ and $\wedge(x_1, y_2) = 1$, we obtain the condition in (11). □

C. Proof of Lemma 1

We only prove part b), because a) follows in a similar fashion. Let $H(x)$ denote the Cdf of Z, and

$$\begin{aligned}\Delta &= P(X_1 < \wedge(Y_1, X_2, Z)) - P(X_2 < \wedge(X_1, Y_2, Z))\\ &= \int_0^\infty \overline{F}_2(x)\overline{G}_1(x)\overline{H}(x)dF_1(x)\\ &\quad - \int_0^\infty \overline{F}_1(x)\overline{G}_2(x)\overline{H}(x)dF_2(x).\end{aligned}$$

From *ii*), it follows that

$$\Delta \geq \int_0^\infty \overline{F}_1(x)\overline{G}_2(x)\overline{H}(x)dF_1(x) - \int_0^\infty \overline{F}_1(x)\overline{G}_2(x)\overline{H}(x)dF_2(x) \geq 0$$

because $\overline{F}_1(x)\overline{G}_2(x)\overline{H}(x)$ is a nonincreasing function of x, and $F_1(x) \geq F_2(x)$ [6]. This proves b) from *ii*). Observe now that if X_1 and X_2 have Pdf,

$$\Delta = \int_0^\infty \overline{F}_1(x)\overline{F}_2(x)\overline{G}_1(x)\lambda_1(x)\overline{H}(x)dx - \int_0^\infty \overline{F}_1(x)\overline{F}_2(x)\overline{G}_2(x)\lambda_2(x)\overline{H}(x)dx.$$

Then *b*) follows from *i*). □

D. Proof of Proposition 3

Accordingly to Proposition 1, part b), $U_1^{(n)} \geq_{pr} U_2^{(n)}$ holds iff

$$P(X_1 < \wedge(Y_1, X_2, X_3, \ldots, X_n)) \geq P(X_2 < \wedge(X_1, Y_2, X_3, \ldots, X_n)).$$

But this inequality follows from part b) of Lemma 1 taking $Z = \wedge(X_3, \ldots, X_n)$. It is obvious that the case $U_1 \geq_{pr} U_2$ follows in a similar way. □

E. Proof of Lemma 2

Let $H_1(x)$ and $H_2(x)$ denote the Cdf of Z_1 and Z_2, respectively, and

$$\begin{aligned}\Delta &= P(\vee(X_1, Z_1) < \wedge(X_2, Y_1, Z_2))\\ &\quad - P(\vee(X_2, Z_1) < \wedge(X_1, Y_2, Z_2))\\ &= \int_0^\infty \int_0^\infty \overline{F}_2(\vee(x,y))\overline{G}_1(\vee(x,y))\\ &\quad \times \overline{H}_2(\vee(x,y))dF_1(x)dH_1(y)\\ &\quad - \int_0^\infty \int_0^\infty \overline{F}_1(\vee(x,y))\overline{G}_2(\vee(x,y))\\ &\quad \times \overline{H}_2(\vee(x,y))dF_2(x)dH_1(y).\end{aligned}$$

Because $\overline{G}_1(x) \geq \overline{G}_2(x)$ and $\overline{F}_2(x) \geq \overline{F}_1(x)$, it follows that

$$\begin{aligned}\Delta &\geq \int_0^\infty \int_0^\infty \overline{F}_2(\vee(x,y))\overline{G}_1(\vee(x,y))\\ &\quad \times \overline{H}_2(\vee(x,y))dF_1(x)dH_1(y)\\ &\quad - \int_0^\infty \int_0^\infty \overline{F}_2(\vee(x,y))\overline{G}_1(\vee(x,y))\\ &\quad \times \overline{H}_2(\vee(x,y))dF_2(x)dH_1(y).\end{aligned}$$

Using the same argument as in the proof of Lemma 1, it can be obtained that $\Delta \geq 0$, and then *a*) follows. □

F. Proof of Proposition 4

It is sufficient to use part c) of Proposition 1 with the same notation and conditions stated there, and to take

$$\begin{aligned}&Z_1 = \vee(X_{j_1}, \ldots, X_{j_{n-k}}), \quad Z_2 = \wedge(X_{i_1}, \ldots, X_{i_{k-2}}),\\ &Z_3 = \vee(X_1, X_2, X_{r_1}, \ldots, X_{r_{n-k-1}}),\\ &Z_4 = \wedge(X_{v_1}, \ldots, X_{v_{k-1}})\end{aligned}$$

in Lemma 2. □

G. Proof of Lemma 3

We will only prove b) because a) follows in a similar way. It is sufficient to prove that

$$\begin{aligned}\Delta &= P(\wedge(X_2, Y_1, Z_2) > \vee(X_1, Y_2, Z_1))\\ &\quad + P(\wedge(X_1, Y_2, Z_2) > \vee(X_2, Y_1, Z_1))\\ &\quad - P(\wedge(X_2, Y_2, Z_2) > \vee(X_1, Y_1, Z_1))\\ &\quad - P(\wedge(X_1, Y_1, Z_2) > \vee(X_2, Y_2, Z_1)) \geq 0.\end{aligned}$$

But

$$\begin{aligned}\Delta &= \int_0^\infty \int_0^\infty \int_0^\infty \overline{F}_2(\vee(x,y,z))\overline{G}_1(\vee(x,y,z))\\ &\quad \times \overline{H}_2(\vee(x,y,z))dG_2(x)dF_1(y)dH_1(z)\\ &\quad + \int_0^\infty \int_0^\infty \int_0^\infty \overline{F}_1(\vee(x,y,z))\overline{G}_2(\vee(x,y,z))\\ &\quad \overline{H}_2(\vee(x,y,z))dG_1(x)dF_2(y)dH_1(z)\\ &\quad - \int_0^\infty \int_0^\infty \int_0^\infty \overline{F}_2(\vee(x,y,z))\overline{G}_2(\vee(x,y,z))\\ &\quad \times \overline{H}_2(\vee(x,y,z))dG_1(x)dF_1(y)dH_1(z)\\ &\quad - \int_0^\infty \int_0^\infty \int_0^\infty \overline{F}_1(\vee(x,y,z))\overline{G}_1(\vee(x,y,z))\\ &\quad \times \overline{H}_2(\vee(x,y,z))dG_2(x)dF_2(y)dH_1(z),\end{aligned}$$

where $H_1(x)$ and $H_2(x)$ denote the Cdf of Z_1 and Z_2, respectively.

A sufficient condition for $\Delta \geq 0$ is

$$\begin{aligned}&\overline{F}_2(\vee(x,y,z))\overline{G}_1(\vee(x,y,z))g_2(x)f_1(y)\\ &\quad + \overline{F}_1(\vee(x,y,z))\overline{G}_2(\vee(x,y,z))g_1(x)f_2(y)\\ &\geq \overline{F}_2(\vee(x,y,z))\overline{G}_2(\vee(x,y,z))g_1(x)f_1(y)\\ &\quad + \overline{F}_1(\vee(x,y,z))\overline{G}_1(\vee(x,y,z))g_2(x)f_2(y),\end{aligned}$$

which can be rewritten as

$$\begin{aligned}&g_2(x)\overline{G}_1(\vee(x,y,z))\\ &\quad \times [f_1(y)\overline{F}_2(\vee(x,y,z))\\ &\quad - f_2(y)\overline{F}_1(\vee(x,y,z))]\\ &\geq g_1(x)\overline{G}_2(\vee(x,y,z))\\ &\quad \times [f_1(y)\overline{F}_2(\vee(x,y,z))\\ &\quad - f_2(y)\overline{F}_1(\vee(x,y,z))].\end{aligned} \tag{23}$$

Observe now that if $a \geq b \geq 0$, then

$$f_1(b)\overline{F}_2(a) - f_2(b)\overline{F}_1(a) \geq 0,$$

because from $X_1 \leq_{hr} X_2$ it follows that

$$f_1(b) \geq f_2(b)\frac{\overline{F}_1(b)}{\overline{F}_2(b)} \geq f_2(b)\frac{\overline{F}_1(a)}{\overline{F}_2(a)}.$$

Similarly, $Y_2 \leq_{hr} Y_1$ implies

$$g_2(b)\overline{G}_1(a) - g_1(b)\overline{G}_2(a) \geq 0.$$

Then (23) holds. □

H. Proof of Proposition 5

We only consider the case $1 < k \leq n$, $n > 2$, because the remaining case can be proved in a similar way. Then it is sufficient to use part b) of Proposition 2 with the same notation and conditions stated there, and to take

$$Z_1 = \vee(X_{j_1}, \ldots, X_{j_{n-k}}), \quad Z_2 = \wedge(X_{i_1}, \ldots, X_{i_{k-2}})$$

in Lemma 3. □

I. Proof of Proposition 6

Suppose the contrary, i.e., (14), does not hold. Using the notation presented at the beginning of Appendix, we can see that this means that the system

$$\begin{aligned} &\vee(x_1, y_1) + \vee(x_2, y_2) + \cdots \\ &\quad + \vee(x_r, y_r) + x_{r+1} + \cdots + x_n \geq k, \\ &x_1 + x_2 + \cdots + x_n + y_1 + y_2 + \cdots + y_r \leq k - 1 \end{aligned}$$

must be satisfied. However, this system has no solution, and hence (14) always holds. □

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