VIPER Summer School on PTA GW Astrophysics

Introduction to gravitational waves (theory background)

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Skeleton

More or less everything I cover in this lecture comes out of Maggiore's textbook:

Maggiore, Michele. "Gravitational waves. Volume 1: Theory and experiments." Oxford University Press, 2008.

I did not type my notes because you are surely better off studying from the book.

I will assume some background knowledge in General Relativity. Given than, here below is a list of the key passages in the book that every aspiring gravitational-wave astronomer should study carefully (IMO at least). I won't be able to cover all of this in 1 hour, but I highly encourage you to start from my lecture and spend a few afternoons going through this selected material in the textbook.

- Sec. 1.1, p. 4 (quick recap).
- Sec. 1.2, p. 7 (quick recap).
- Sec. 1.4 (all subsections), p. 26.
- Sec. 3.1, p. 102.
- Sec. 3.2, p. 105.
- Sec. 3.3.1, p. 109.
- Sec. 3.3.2, p.113.
- Sec. 3.3.3, p. 114 (quickly).
- Sec. 3.3.4, p. 116.
- Problem 3.2, p. 158.
- Problem 3.3, p. 161 (quickly).
- Sec. 4.1 (intro), p. 167.
- Sec. 4.1.1, p. 169.
- Problem 4.1, p. 230.

After you've studied these sections, close the textbook and practice with the following two problems.

Problem 1: Gravitational radiation of the simple harmonic oscillator

Assume a gravitational system with reduced mass μ oscillating along the z direction with equation of motion

$$z_0(t) = a \cos \omega_s t \,. \tag{1}$$

The time-time component of the stress energy tensor is

$$T^{00}(t, \mathbf{x}) = \mu \delta(x) \delta(y) \delta(z - z_0(t)) \,. \tag{2}$$

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1. Predict the scalings of the radiated power P with μ, a, and ω_s using only the quadrupole formula $P \propto \dddot{Q}^2$.
2. Compute the second mass moment $M^{ij}(t)$.
3. Compute the gravitational-wave polarization $h_+(t,\theta,\phi)$ and $h_\times(t,\theta,\phi)$ where θ and ϕ are suitable spherical coordinates.
4. What is the frequency of the emitted waves?
5. Does the emission pattern reflect the cylindrical symmetry of the source?
6. The emission vanishes along the z axis. How is this related to the TT gauge projection tensor Λ_{ijkl}?
7. Compute the radiated power per unit angle $dP/d\Omega$.
8. Compute the total radiated power P. Does this agree with the estimate of point 1?
9. Compute the energy emitted during a single oscillation period $2\pi/\omega_s$.
10. Compute the back-reaction force acting on the source (hint: first predict the scalings).

Problem 2: Black hole binaries in elliptic orbits

Consider two masses m_1 and m_2 in elliptic orbits. As usual, the system is reduced to a particle of reduced mass $\mu = m_1 m_2/(m_1+m_2)$ subject to an acceleration $\ddot{\mathbf{r}} = -m/r^2 \hat{\mathbf{r}}$ where $m = m_1 + m_2$. Recall that Kelper's orbits are given by

$$r = \frac{a(1-e^2)}{1+e\cos\psi} \tag{3}$$

where a is the semi-major axis, e is the eccentricity, and ψ is the true anomaly. The motion along the orbit is described by

$$\dot{\psi} = \left(\frac{m}{a}\right)^{1/2} (1-e^2)^{-3/2} (1+e\cos\psi)^2 . \tag{4}$$

The orbital period is set by Kepler's law

$$T = 2\pi\sqrt{\frac{a^3}{m}} . \tag{5}$$

1. Compute the second mass moment $M^{ij}(t)$.
2. Compute the total power radiated P as a function of the orbital position ψ.
3. Average the result over the orbital period T. You should get

   $$P = \frac{32\mu^2 m^3}{5a^5} \frac{1}{(1-e^2)^{7/2}} \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right) \tag{6}$$

   This is a classic result by Peters and Mathews (1963).
4. Does this expression reduce to the circular case correctly?
5. What happens if $e \to 1$?
6. Hang on: I thought that an elliptic orbit reduces to a parabolic one for $e \to 1$. Does the power emitted during a parabolic encounter diverges? (hint: the answer is no). What's going on? (hint: think about the limit we are taking).