HD notes:

Plane-wave expansion:

$$h_{ij}(t,\vec{x}) = \int df \int d^2\Omega_{\hat{k}} \sum_A h_A(f,\hat{k})\, e^A_{ij}(\hat{k})\, e^{i2\pi f(t-\hat{k}\cdot\vec{x}/c)}$$

Ensemble average (unpolarized, isotropic, stationary-Gaussian background)

$$\langle h_A(f,\hat{k})\rangle = 0$$

$$\langle h_A(f,\hat{k})\, h^*_{A'}(f',\hat{k}')\rangle = \frac{1}{16\pi} S_h(f)\, \delta(f-f')\,\delta_{AA'}\,\delta^2(\hat{k},\hat{k}')$$

Detector response:

$$h_a(t) = \int df \int d^2\Omega_{\hat{k}} \sum_A h_A(f,\hat{k})\, R^A_a(f,\hat{k})\, e^{i2\pi f t}$$

$$\rightarrow \tilde{h}_a(f) = \int d^2\Omega_{\hat{k}} \sum_A h_A(f,\hat{k})\, R^A_a(f,\hat{k})$$

Cross-spectrum (1-sided):

$$\langle \tilde{h}_a(f)\, \tilde{h}^*_b(f')\rangle = \frac{1}{2}\delta(f-f')\, C_{ab}(f)$$

$$= \frac{1}{2}\delta(f-f')\, \Gamma_{ab}(f)\, S_h(f)$$

def'n of $\Gamma_{ab}(f)$

$$LHS = \langle \tilde{h}_a(f)\, \tilde{h}_b^*(f') \rangle$$

$$= \frac{1}{16\pi} S_h(f)\, \delta\delta\delta$$

$$= \int d^2\Omega_{\hat{k}} \int d^2\Omega_{\hat{k}'} \sum_A \sum_{A'} \langle h_A(f,\hat{k})\, h^*_{A'}(f',\hat{k}') \rangle\, R_a^A(f,\hat{k})\, R_b^{A'*}(f',\hat{k}')$$

$$= \int d^2\Omega_{\hat{k}} \sum_A \frac{1}{16\pi} S_h(f)\, \delta(f-f')\, R_a^A(f,\hat{k})\, R_b^{A*}(f,\hat{k})$$

$$= \frac{1}{2} \delta(f-f')\, S_h(f)\, \underbrace{\frac{1}{8\pi} \int d^2\Omega_{\hat{k}} \sum_A R_a^A(f,\hat{k})\, R_b^{A*}(f,\hat{k})}_{\Gamma_{ab}(f)}$$

$$= \frac{1}{2} \delta(f-f')\, S_h(f)\, \Gamma_{ab}(f)$$

where

$$\Gamma_{ab}(f) \equiv \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} \sum_A R_a^A(f,\hat{k})\, R_b^{A*}(f,\hat{k})$$

Simple example:

$$R_a^A(f,\hat{k}) = G\, e^{-i2\pi f \hat{k}\cdot\vec{x}_a/c}$$

$$\Gamma_{ab}(f) = \frac{1}{8\pi}\int d^2\Omega_{\hat{k}} \sum_A R_a^A(f,\hat{k})\, R_b^{A*}(f,\hat{k})$$

$$= \frac{1}{8\pi} 2G^2 \int d^2\Omega_{\hat{k}}\, e^{-i2\pi f \hat{k}\cdot(\vec{x}_a-\vec{x}_b)/c}$$

$$= \frac{1}{8\pi} 2G^2 \int d^2\Omega_{\hat{k}}\, e^{-i2\pi f \frac{D_{ab}}{c} \hat{k}\cdot\Delta\hat{x}_{ab}}$$

where $\vec{x}_a - \vec{x}_b \equiv \Delta\vec{x}_{ab} = D_{ab}\,\Delta\hat{x}_{ab}$

Define $\alpha_{ab} = 2\pi f \frac{D_{ab}}{c}$

Choose coord system so that $\hat{z} = \Delta\hat{x}_{ab}$, $\hat{k} = -\hat{r}$

$\rightarrow$ $\hat{k}\cdot\Delta\hat{x}_{ab} = -\hat{r}\cdot\hat{z} = \cos\theta$

$$\Gamma_{ab} = \frac{1}{4\pi} G^2 \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\, e^{+i\alpha_{ab}\cos\theta}$$

$$\Gamma_{ab}(f) = \frac{G^2}{4\pi} \cdot 2\pi \int_{-1}^{1} dx \, e^{i\alpha_{ab} x}$$

$$= \frac{G^2}{2} \frac{1}{i\alpha_{ab}} e^{i\alpha_{ab} x} \Big|_{-1}^{1}$$

$$= \frac{G^2}{\alpha_{ab}} \frac{1}{2i} \left(e^{i\alpha_{ab}} - e^{-i\alpha_{ab}} \right)$$

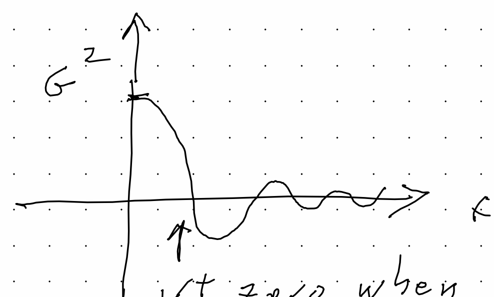
$$= \frac{G^2}{\alpha_{ab}} \sin(\alpha_{ab})$$

$$= G^2 \operatorname{sinc}(\alpha_{ab}) \quad \text{where} \quad \operatorname{sinc} x \equiv \frac{\sin x}{x}$$

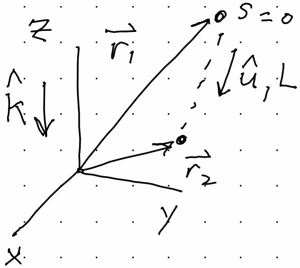
thus,

$$\Gamma_{ab}(f) = G^2 \operatorname{sinc}\left(\frac{2\pi f D_{ab}}{c} \right)$$

$$\cancel{2\pi} f_0 \frac{D_{ab}}{c} = \cancel{\pi} \leftrightarrow f_0 = \frac{c}{2D_{ab}}$$



1st zero when $f_0 = c/2D_{ab}$



$$h(t) \equiv \Delta T(t) = \frac{1}{2c} u^i u^j \int_0^L ds\, h_{ij}(t(s), \vec{x}(s))$$

where $t(s) = \left(t - \frac{L}{c}\right) + \frac{s}{c}$ $\quad \left\{ \begin{array}{l} s=0 \to t(0) = t - \frac{L}{c} \\ s = L \to t(L) = t \end{array} \right.$

$$\vec{x}(s) = \vec{r}_1 + (\vec{r}_2 - \vec{r}_1)\, s/L$$
$$= \vec{r}_1 + \hat{u}\, s$$

$$h_{ij}(t(s), \vec{x}(s)) = \int df \int d^2\Omega_k \sum_A h_A(f, \hat{k})\, e^A_{ij}(\hat{k})\, e^{i2\pi f\left(t(s) - \hat{k}\cdot\vec{x}(s)/c\right)}$$

$$e^{i2\pi f\left[\left(t - \frac{L}{c}\right) + \frac{s}{c} - \frac{\hat{k}}{c}\cdot(\vec{r}_1 + \hat{u}s)\right]}$$
$$= \underbrace{e^{i2\pi f\left[t - \frac{L}{c} - \frac{\hat{k}\cdot\vec{r}_1}{c}\right]}}_{\text{indep of } s \,\Rightarrow\, \text{comes out of integral}}\; e^{i2\pi f\left(1 - \hat{k}\cdot\hat{u}\right)\frac{s}{c}}$$

$$h(t) = \frac{1}{2c} \int df \int d^2\Omega_k \sum_A h_A(f, \hat{k})\, e^A_{ij}(\hat{k})\, u^i u^j\, e^{i2\pi f\left[t - \frac{L}{c} - \frac{\hat{k}\cdot\vec{r}_1}{c}\right]}$$
$$\times \int_0^L ds\, e^{i2\pi f\left(1 - \hat{k}\cdot\hat{u}\right)s/c}$$

$$\int_0^L ds\, e^{i2\pi f \frac{s}{c}(1-\hat{k}\cdot\hat{u})} = \left(\frac{c}{i2\pi f}\right)\left(\frac{1}{1-\hat{k}\cdot\hat{u}}\right)\left[e^{i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}-1\right]$$

Thus,

$$h(t)=\frac{1}{2c}\int df\int d^2\Omega_k \sum_A h_A(f,\hat{k})\left(\frac{c}{i2\pi f}\right)\frac{u^i u^j e^A_{ij}(\hat{k})}{(1-\hat{k}\cdot\hat{u})}\, e^{i2\pi f\left(t-\frac{L}{c}-\frac{\hat{k}\cdot\vec{r}_1}{c}\right)}\left[e^{i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}-1\right]$$

<u>Rewrite the exponential terms:</u>

$$e^{i2\pi f t}\, e^{-i2\pi f\frac{L}{c}\left(1+\frac{\hat{k}\cdot\vec{r}_1}{L}\right)}\left[e^{i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}-1\right]$$

$$= e^{i2\pi f t}\, e^{-i2\pi f\frac{L}{c}\left(1+\frac{\hat{k}\cdot\vec{r}_1}{L}\right)}\, e^{i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}\left[1-e^{-i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}\right]$$

$$= e^{i2\pi f t}\, e^{-i2\pi f\frac{L}{c}\hat{k}\cdot\left(\frac{\vec{r}_1}{L}+\hat{u}\right)}\left[1-e^{-i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}\right]$$

$$= e^{i2\pi f t}\, e^{-i2\pi f\,\hat{k}\cdot\vec{r}_2/c}\left[1-e^{-i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}\right]$$

Using $\hat{u}=\dfrac{\vec{r}_2-\vec{r}_1}{L}$

Thus

$$h(t) = \frac{1}{2\cancel{c}} \int df \int d^2\Omega_{\hat{k}} \sum_A h_A(f,\hat{k}) \left(\frac{\cancel{c}}{i2\pi f}\right) \frac{u^i u^j e^A_{ij}(\hat{k})}{(1-\hat{k}\cdot\hat{u})} e^{i2\pi f t} e^{-i2\pi f \hat{k}\cdot\vec{r}_2/c} \left[1 - e^{-i2\pi f \frac{L}{c}(1-\hat{k}\cdot\hat{u})}\right]$$

$$= \int df \int d^2\Omega_{\hat{k}} \sum_A h_A(f,\hat{k}) e^{i2\pi f t} e^{-i2\pi f \hat{k}\cdot\vec{r}_2/c} \left[1 - e^{-i2\pi f \frac{L}{c}(1-\hat{k}\cdot\hat{u})}\right] \left(\frac{1}{i2\pi f}\right) \frac{1}{2} \frac{u^i u^j e^A_{ij}(\hat{k})}{(1-\hat{k}\cdot\hat{u})}$$

so

$$R^A(f,\hat{k}) = \underset{\substack{\uparrow \\ 1 \text{ for } \frac{\Delta\nu}{\nu}}}{\left(\frac{1}{i2\pi f}\right)} \underbrace{\frac{1}{2} \frac{u^i u^j e^A_{ij}(\hat{k})}{(1-\hat{k}\cdot\hat{u})}}_{F^A(\hat{k})} \left[1 - \underbrace{e^{-i2\pi f \frac{L}{c}(1-\hat{k}\cdot\hat{u})}}_{\text{pulsar term}}\right] \underbrace{e^{-i2\pi f \hat{k}\cdot\vec{r}_2/c}}_{\text{this will be the same for radio measurements made from Earth}}$$

So for redshift measurements:

$$R^A_a(f,\hat{k}) \simeq F^A_a(\hat{k})\left[1 - e^{-i2\pi f \frac{L_a}{c}(1+\hat{k}\cdot\hat{p}_a)}\right] \qquad a = 1, 2, \ldots$$

where $F^A_a(\hat{k}) = \frac{1}{2}\frac{\hat{p}^i_a \hat{p}^j_a}{1+\hat{p}_a\cdot\hat{k}} e^A_{ij}(\hat{k})$, $\hat{u} = -\hat{p}$

$$\Gamma_{ab}(f) = \frac{1}{8\pi}\int d^2\Omega_{\hat{k}}\, R^A_a(f,\hat{k})\, R^{A*}_b(f,\hat{k})$$

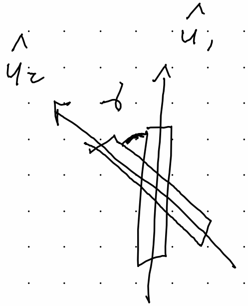
$$\simeq \frac{1}{8\pi}\int d^2\Omega_{\hat{k}}\, F^A_a(\hat{k})F^A_b(\hat{k})\left[1 - e^{-i2\pi f \frac{L_a}{c}(1+\hat{k}\cdot\hat{p}_a)}\right]\left[1 - e^{+i2\pi f \frac{L_b}{c}(1+\hat{k}\cdot\hat{p}_b)}\right]$$

1

except for autocorrelations

$$\rightarrow \Gamma_{ab}(f) \simeq \frac{1}{8\pi}\int d^2\Omega_{\hat{k}}\, F^A_a(\hat{k})F^A_b(\hat{k})\left[1 + \delta_{ab}\right]$$

HW problem



$$r_a(t) = \hat{u}_a \cdot \vec{E}(t, \vec{x}=\vec{0}) \quad , \quad a = 1,2$$

$$\vec{E}(t,\vec{x}) = \int df \int d^2\Omega_k \sum_{A=1,2} \tilde{E}_A(f,\hat{k})\, \hat{\epsilon}_A(\hat{k})\, e^{i2\pi f(t - \hat{k}\cdot\vec{x}/c)}$$

$$r_a(t) = \int df\, e^{i2\pi f t} \int d^2\Omega_k \sum_A \tilde{E}_A(f,\hat{k})\, \hat{u}_a \cdot \hat{\epsilon}_A(\hat{k})$$

$$= \int df\, e^{i2\pi f t} \int d^2\Omega_k \sum_A \tilde{E}_A(f,\hat{k})\, R_a^A(\hat{k})$$

So: $R_a^A(\hat{k}) = \hat{u}_a \cdot \hat{\epsilon}_A(\hat{k})$

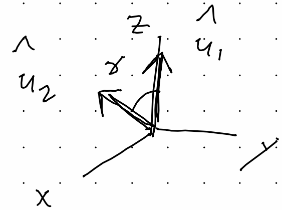
$$\hat{\epsilon}_1(\hat{k}) = -\hat{\phi} \quad , \quad \hat{\epsilon}_2(\hat{k}) = -\hat{\theta} \quad , \quad \hat{k} = -\hat{r}$$

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} = -\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} = -\hat{m}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} = -\hat{l}$$

$$\hat{u}_1 = \hat{z}$$



$$\hat{u}_2 = \sin\gamma\,\hat{x} + \cos\gamma\,\hat{z}$$

Thus,

$$\Gamma_{12} = \frac{1}{8\pi}\int d^2\Omega_{\hat{k}} \sum_A R_1^A(\hat{k})\, R_2^A(\hat{k})$$

$$= \frac{1}{8\pi}\int d^2\Omega_{\hat{k}} \sum_A \hat{u}_1\cdot\hat{\epsilon}_A(\hat{k})\ \hat{u}_2\cdot\hat{\epsilon}_A(\hat{k})$$

$$\hat{u}_1\cdot\hat{\epsilon}_1(\hat{k}) = -\hat{z}\cdot\hat{\phi} = 0$$

$$\hat{u}_1\cdot\hat{\epsilon}_2(\hat{k}) = -\hat{z}\cdot\hat{\theta} = \sin\theta$$

$$\hat{u}_2\cdot\hat{\epsilon}_1(\hat{k}) = -(\sin\gamma\,\hat{x} + \cos\gamma\,\hat{z})\cdot\hat{\phi} = +\sin\gamma\sin\phi$$

$$\hat{u}_2\cdot\hat{\epsilon}_2(\hat{k}) = -(\sin\gamma\,\hat{x} + \cos\gamma\,\hat{z})\cdot\hat{\theta} = -\sin\gamma\cos\theta\cos\phi + \cos\gamma\sin\theta$$

$$\Gamma_{12} = \frac{1}{8\pi}\int d^2\Omega_{\hat{k}}\ \sin\theta\left(-\sin\gamma\cos\theta\cos\phi + \cos\gamma\sin\theta\right)$$

$$= -\sin\gamma\left(\frac{1}{8\pi}\right)\int d^2\Omega_{k}\ \sin\theta\cos\theta\cos\phi + \cos\gamma\left(\frac{1}{8\pi}\right)\int d^2\Omega_{\hat{k}}\ \sin^2\theta$$

Now:

$$\int d^2\Omega_k \sin\theta\cos\theta\cos\phi = \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\sin\theta\cos\theta\cos\phi$$

$$= 0 \quad \left(\text{since } \int_0^{2\pi} d\phi\cos\phi = 0\right)$$

$$\int d^2\Omega_k \sin^2\theta = \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\sin^2\theta$$

$$= 2\pi \int_{-1}^{1} dx\,(1-x^2)$$

$$= 2\pi \left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$$

$$= 4\pi\left(1-\frac{1}{3}\right)$$

$$= \frac{8\pi}{3}$$

Thus

$$\Gamma_{12} = \cos\gamma\left(\frac{1}{8\pi}\right)\frac{8\pi}{3} = \boxed{\frac{1}{3}\cos\gamma}$$

Homework problem: Breathing mode / Scalar transverse

$$e^{B}_{ij} = \hat{l}_i \hat{l}_j + \hat{m}_i \hat{m}_j \qquad e^{+}_{ij} = \hat{l}_i \hat{l}_j - \hat{m}_i \hat{m}_j$$

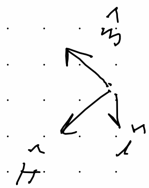
$$e^{\times}_{ij} = \hat{l}_i \hat{m}_j + \hat{m}_i \hat{l}_j$$

$$\Gamma_{ab}(t) = \frac{1}{8\pi}\int d^2\Omega_{\hat{k}}\; R^B_a(t,\hat{k})\, R_b^{B*}(t,\hat{k}) \simeq \frac{1}{8\pi}$$

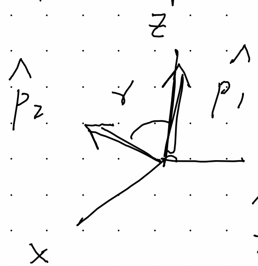
$$\simeq \frac{1}{8\pi}\int d^2\Omega_{\hat{k}}\; F^B_a(\hat{k})\, F^B_b(\hat{k})\,(1+\delta_{ab})$$

$$F^B_a(\hat{k}) = \frac{1}{2}\frac{p_a^i p_a^j}{(1+\hat{p}_a\cdot\hat{k})} e^B_{ij}(\hat{k}),$$

$$\hat{k} = -\hat{r}, \quad \hat{l} = -\hat{\phi}, \quad m = -\hat{\theta}$$



Take pulsars a and b along $\hat{z}$-axis and in xy-plane:



$$F^B_1(\hat{k}) = \frac{1}{2}\frac{\hat{z}^i \hat{z}^j}{(1+\hat{z}\cdot\hat{k})} e^B_{ij}(\hat{k})$$

$$\hat{z}^i \hat{z}^j e^B_{ij}(\hat{k}) = \hat{z}^i \hat{z}^j (\hat{l}_i \hat{l}_j + \hat{m}_i \hat{m}_j)$$

$$= (\hat{z}\cdot\hat{l})^2 + (\hat{z}\cdot\hat{m})^2$$

$\hat{z} \cdot \hat{l} = 0$ $\rightarrow \hat{z}^i \hat{z}^j e^B_{ij} = \sin^2\theta$

$\hat{z} \cdot \hat{m} = \sin\theta$ $\rightarrow F_1^B(\hat{n}) = \frac{1}{2} \frac{\sin^2\theta}{1-\cos\theta}$

$\hat{z} \cdot \hat{n} = -\cos\theta$

$$= \frac{1}{2}\left(\frac{1-\cos^2\theta}{1-\cos\theta}\right)$$

$$= \boxed{\frac{1}{2}(1+\cos\theta)}$$

$$F_2^B(\hat{n}) = \frac{1}{2} \frac{(\cos\gamma\, \hat{z}^i + \sin\gamma\, \hat{x}^i)(\cos\gamma\, \hat{z}^j + \sin\gamma\, \hat{x}^j)}{1 + (\cos\gamma\, \hat{z}\cdot\hat{n} + \sin\gamma\, \hat{x}\cdot\hat{n})} \underbrace{e^B_{ij}(\hat{n})}_{l_i l_j + m_i m_j}$$

<u>Now</u>: $\hat{z}\cdot\hat{l} = 0$, $\hat{z}\cdot\hat{m} = \sin\theta$, $\hat{z}\cdot\hat{n} = -\cos\theta$

$\hat{x}\cdot\hat{l} = \sin\phi$, $\hat{x}\cdot\hat{m} = -\cos\theta\cos\phi$, $\hat{x}\cdot\hat{n} = -\sin\theta\cos\phi$

<u>Denominator</u>: $1 - \cos\gamma\cos\theta - \sin\gamma\sin\theta\cos\phi$

$$\text{Numerator} = \left(\cos\gamma\,\hat{z}^i + \sin\gamma\,\hat{x}^i\right)\left(\cos\gamma\,\hat{z}^j + \sin\gamma\,\hat{x}^j\right)\left(l_i l_j + m_i m_j\right)$$

$$= \left(\cos^2\gamma\,\hat{z}^i\hat{z}^j + \sin^2\gamma\,\hat{x}^i\hat{x}^j + \sin\gamma\cos\gamma\left(\hat{z}^i\hat{x}^j + \hat{z}^j\hat{x}^i\right)\right)\left(l_i l_j + m_i m_j\right)$$

$$= \cos^2\gamma\,\cancel{\left(\hat{z}\cdot\hat{l}\right)^2}^{\,0} + \sin^2\gamma\,\underbrace{\left(\hat{x}\cdot\hat{l}\right)^2}_{\sin\phi} + 2\sin\gamma\cos\gamma\,\cancel{\left(\hat{z}\cdot\hat{l}\right)}^{\,0}\left(\hat{x}\cdot\hat{l}\right)$$

$$+ \cos^2\gamma\left(\hat{z}\cdot\hat{m}\right)^2 + \sin^2\gamma\left(\hat{x}\cdot\hat{m}\right)^2 + 2\sin\gamma\cos\gamma\left(\hat{z}\cdot\hat{m}\right)\left(\hat{x}\cdot\hat{m}\right)$$

$$= \sin^2\gamma\sin^2\phi + \cos^2\gamma\sin^2\theta + \sin^2\gamma\cos^2\theta\cos^2\phi$$

$$+ 2\sin\gamma\cos\gamma\sin\theta\left(-\cos\theta\cos\phi\right)$$

Thus,

$$F_z^B(\hat{k}) = \frac{1}{2}\,\frac{\sin^2\gamma\sin^2\phi + \cos^2\gamma\sin^2\theta + \sin^2\gamma\cos^2\theta\cos^2\phi - 2\sin\gamma\cos\gamma\sin\theta\cos\theta\cos\phi}{\left(1 - \cos\gamma\cos\theta - \sin\gamma\sin\theta\cos\phi\right)}$$

Now

Note:

$$(1-\cos\gamma\cos\theta-\sin\gamma\sin\theta\cos\phi)(1+\cos\gamma\cos\theta+\sin\gamma\sin\theta\cos\phi)$$

$$= 1 + \cancel{\cos\gamma\cos\theta} + \cancel{\sin\gamma\sin\theta\cos\phi} - \sin\gamma\cos\gamma\sin\theta\cos\theta\cos\phi$$
$$- \cancel{\cos\gamma\cos\theta} - \cos^2\gamma\cos^2\theta - \sin\gamma\cos\gamma\sin\theta\cos\theta\cos\phi$$
$$- \cancel{\sin\gamma\sin\theta\cos\phi} - \sin\gamma\cos\gamma\sin\theta\cos\theta\cos\phi - \sin^2\gamma\sin^2\theta\cos^2\phi$$

$$= 1 - \cos^2\gamma\underbrace{\cos^2\theta}_{(1-\sin^2\theta)} - \sin^2\gamma\underbrace{\sin^2\theta}_{(1-\cos^2\theta)}\cos^2\phi - 2\sin\gamma\cos\gamma\sin\theta\cos\theta\cos\phi$$

$$= \underline{1-\cos^2\gamma} + \cos^2\gamma\sin^2\theta \underline{- \sin^2\gamma\cos^2\phi} + \sin^2\gamma\cos^2\theta\cos^2\phi$$
$$- 2\sin\gamma\cos\gamma\sin\theta\cos\theta\cos\phi$$

$$= \sin^2\gamma(1-\cos^2\phi) + \cos^2\gamma\sin^2\theta + \sin^2\gamma\cos^2\theta\cos^2\phi$$
$$- 2\sin\gamma\cos\gamma\sin\theta\cos\theta\cos\phi$$

$$= \sin^2\gamma\sin^2\phi + \cos^2\gamma\sin^2\theta + \sin^2\gamma\cos^2\theta\cos^2\phi$$
$$- 2\sin\gamma\cos\gamma\sin\theta\cos\theta\cos\phi$$

= numerator

So $\quad F_2^B(\hat{k}) = \frac{1}{2}\left(1 + \cos\gamma \cos\theta + \sin\gamma \sin\theta \cos\phi\right)$

Recall: $\quad F_1^B(\hat{k}) = \frac{1}{2}\left(1+\cos\theta\right)$

Thus,

$$\Gamma_{12} = \frac{1}{8\pi}\int d^2\Omega_k \, \frac{1}{4}\left(1+\cos\theta\right)\left(1+\cos\gamma\cos\theta + \sin\gamma\sin\theta\cos\phi\right)$$

$$= \frac{1}{32\pi}\int_0^{2\pi} d\phi \int_{-1}^{1} dx \,(1+x)\left(1+\cos\gamma\, x + \underbrace{\sin\gamma\sqrt{1-x^2}\cos\phi}_{0}\right)$$

$x = \cos\theta$

0 when integrated $\int_0^{2\pi} d\phi$

$$= \frac{1}{32\pi} \cdot 2\pi \cdot \int_{-1}^{1} dx\,(1+x)(1+\cos\gamma\, x)$$

$$= \frac{1}{16}\int_{-1}^{1} dx\left(1 + \overbrace{(1+\cos\gamma)x}^{0} + \cos\gamma\, x^2\right)$$

$$= \frac{1}{16}\left[\, x\Big|_{-1}^{1} + \cos\gamma\,\frac{x^3}{3}\Big|_{-1}^{1}\right] = \frac{1}{8}\left[1 + \frac{1}{3}\cos\gamma\right]$$