$$p(A \mid B) = \frac{p(B \mid A)p(A)}{p(B)}$$

$$p(A \mid B, C) = \frac{p(B \mid A, C)\,p(A \mid C)}{p(B \mid C)}$$

$$p(\theta \mid D, M) = \frac{p(D \mid \theta, M)\,p(\theta \mid M)}{p(D \mid M)}$$

$$\text{Posterior} = \frac{\text{Likelihood} \quad \text{Prior}}{\text{Evidence}}$$

What we design: $p(D \mid \theta, M),\ p(\theta \mid M)$

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What we want: $p(\theta \mid D, M),\ p(D \mid M)$

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It is straightforward to compute $p(D \mid \theta, M)p(\theta \mid M)$ anywhere in $\{\theta\}$, but that doesn't give us much insight

$p(\theta \mid D, M)$ is a **probability density function**

$p(\theta\,|\,D, M)$ is a **probability density function**

One way to get insight into a PDF is by computing **moments**

$$\text{e.g., } \langle \theta \rangle = \int \theta \; p(\theta\,|\,D, M)\; d\theta$$

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Even if we wanted to look at just one parameter

$$p(\theta^i \,|\, D) = \int p(\theta \,|\, D, M) \; d\theta^j d\theta^k d\theta^l \ldots$$

But if somebody gave you a set of θ values **drawn from** $p(\theta^i \mid D, M)$

$$\{\theta^i\} \rightarrow \{\theta^i_0, \theta^i_1, \theta^i_2, \ldots, \theta^i_N\}$$

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we can compute moments etc. the old fashioned way

$$\langle \theta^i \rangle \simeq \frac{\sum_n \theta^i_n}{N}$$

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Sets from RNGs are a subset of a more general class called **Markovian processes**

$$p(\theta_N | \theta_0, \theta_1, \ldots, \theta_{N-1}) = p(\theta_N | \theta_{N-1})$$

Markov Chain Monte Carlo (MCMC):

Markov Chain Monte Carlo (MCMC):

A class of algorithms for generating (approximately) i.i.d. samples (i.e., a *Markov chain*) from some **target distribution**

The *Markov chain* must satisfy the **detailed balance** condition

$$p(y \mid x)p(x) = p(x \mid y)p(y)$$

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Proposal Distribution

e.g. $N[\mu, \sigma^2]$, $U[x_{\min}, x_{\max}]$

Our detailed balance equation becomes in inequality

$$q(y|x)p(x) > q(x|y)p(y)$$

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probability of transitioning from x to y

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Hasting's Ratio

Now reverse the inequality

$$H(x, y) > 1 \quad \text{but} \quad \alpha(x, y) \in [0,1]$$

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To generalize:

$$\boxed{\alpha(x, y) = \min\left[1,\ H(x, y)\right]}$$

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Markov Chain Monte Carlo (MCMC):

Repeat

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Now map to our inference problem:

State: Model parameters θ

Target Distribution: Posterior $p(\theta \mid D, M)$

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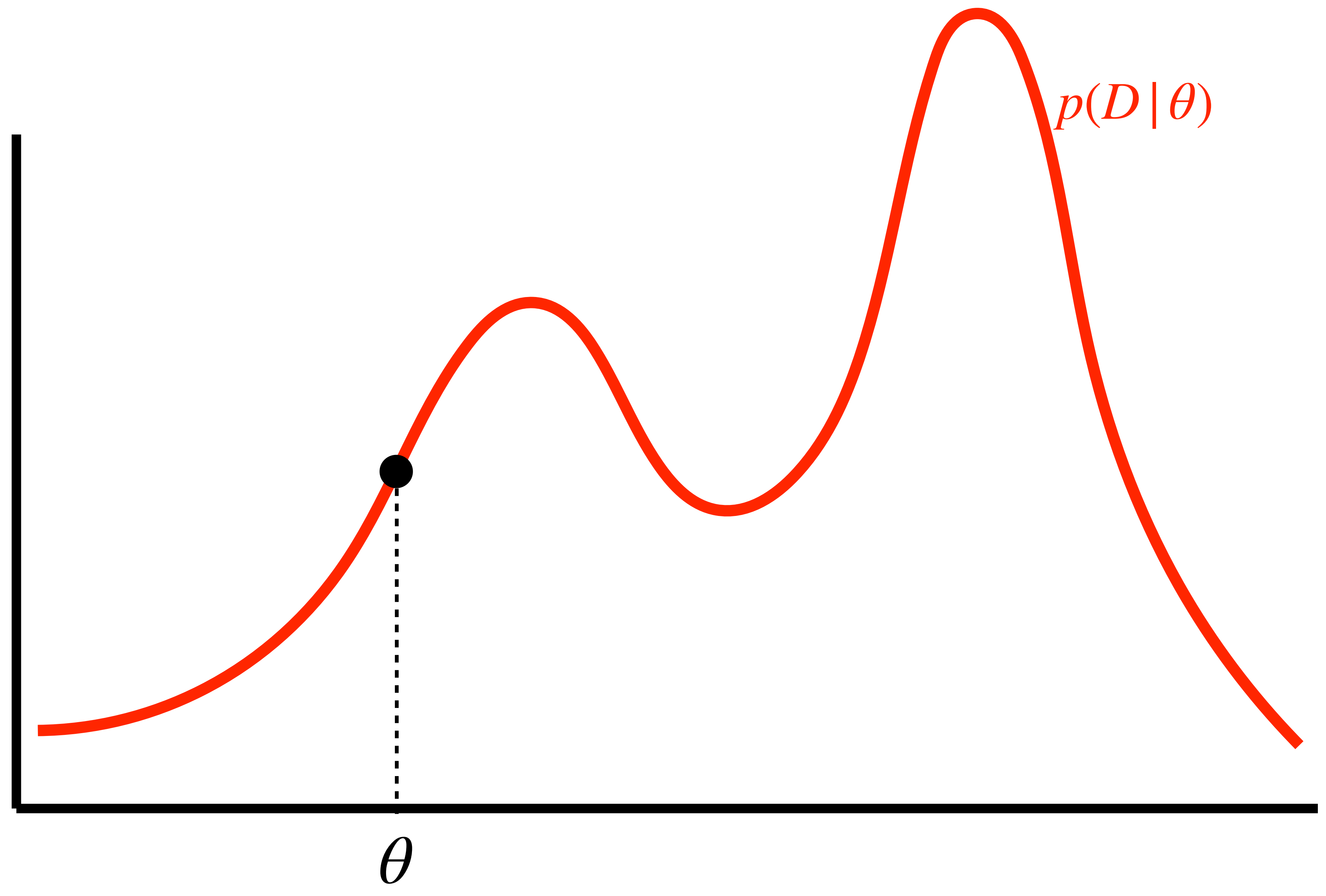
$$\alpha(\theta, \theta') = \frac{p(D\,|\,\theta', M)}{p(D\,|\,\theta, M)}$$



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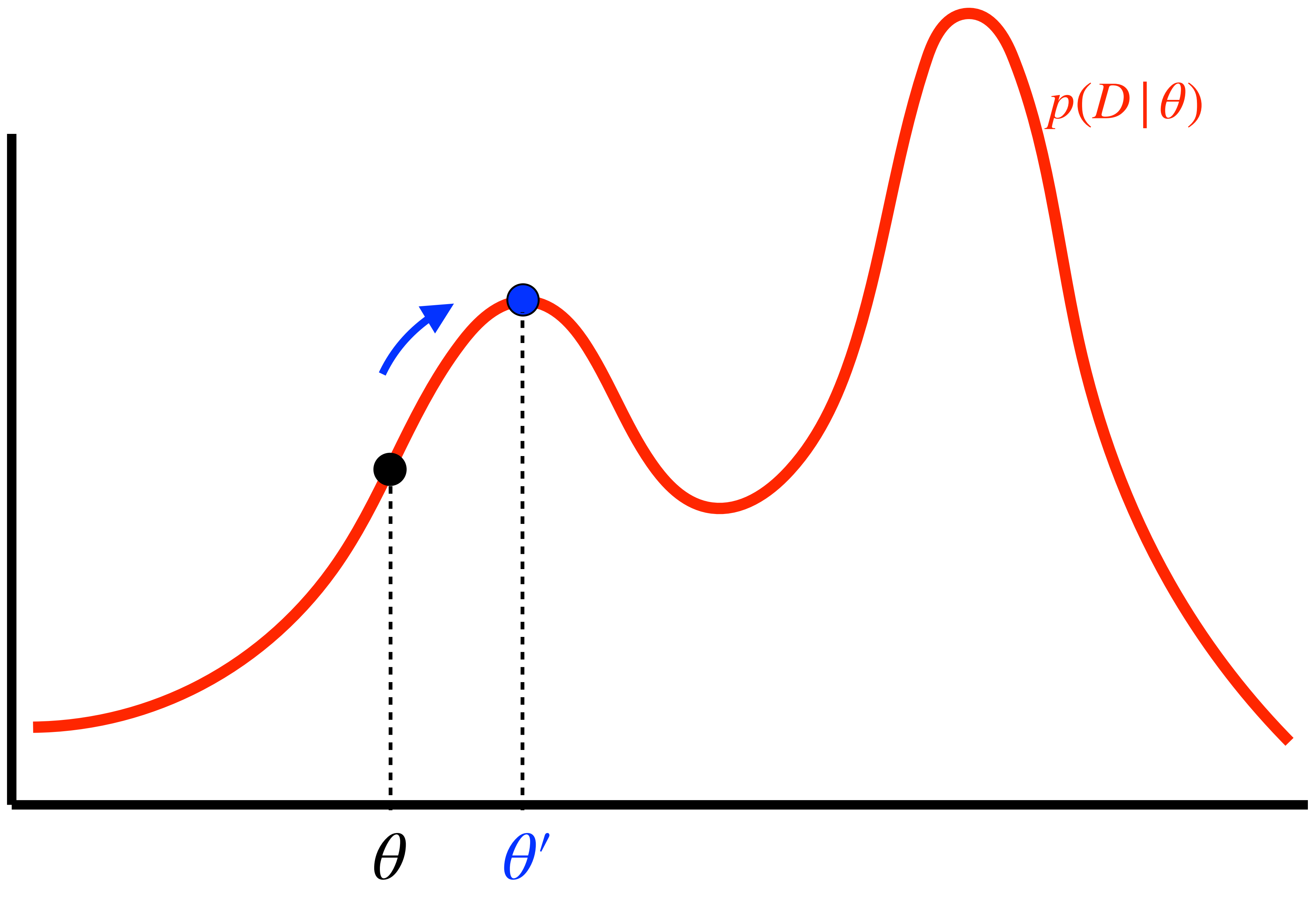

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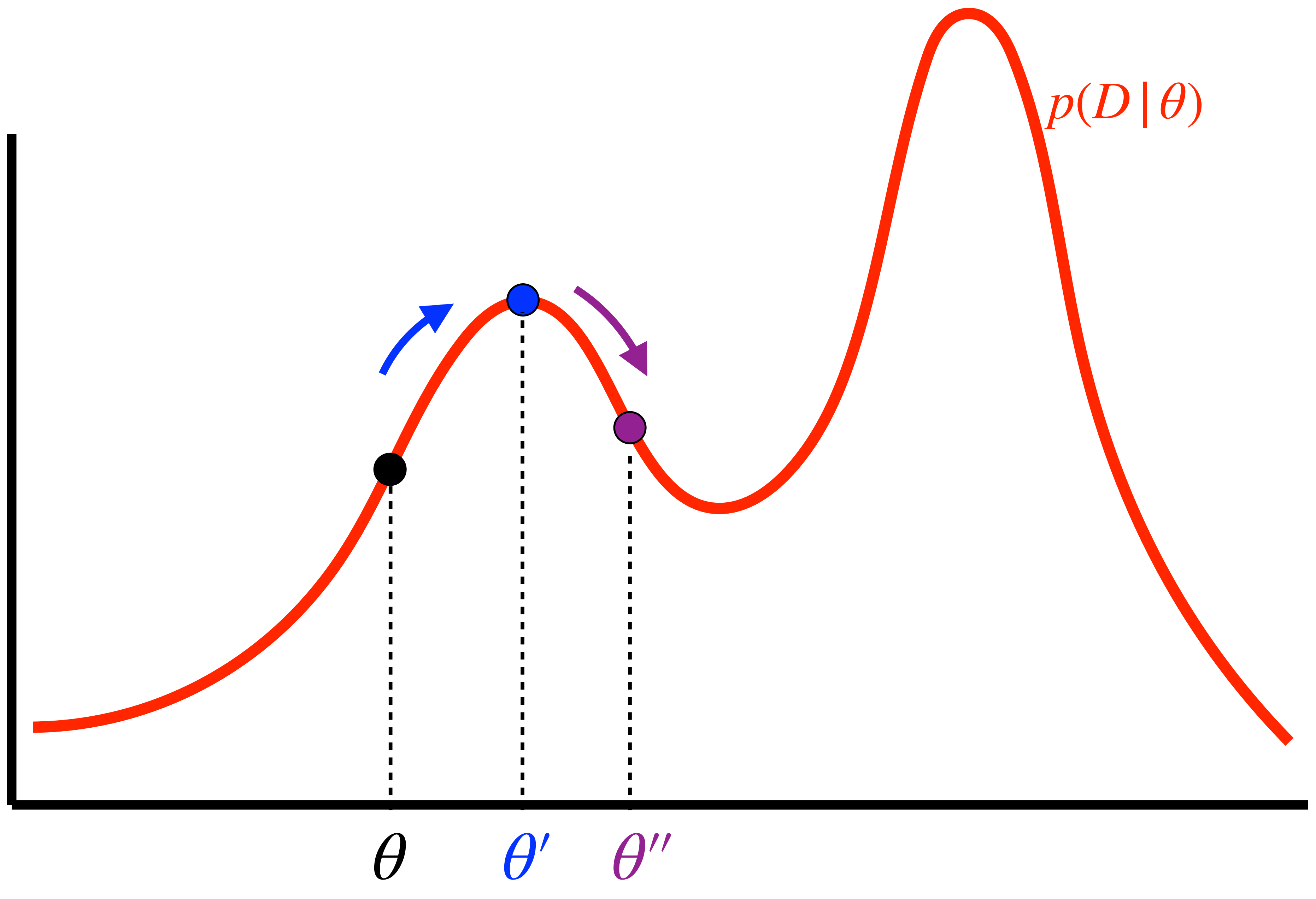

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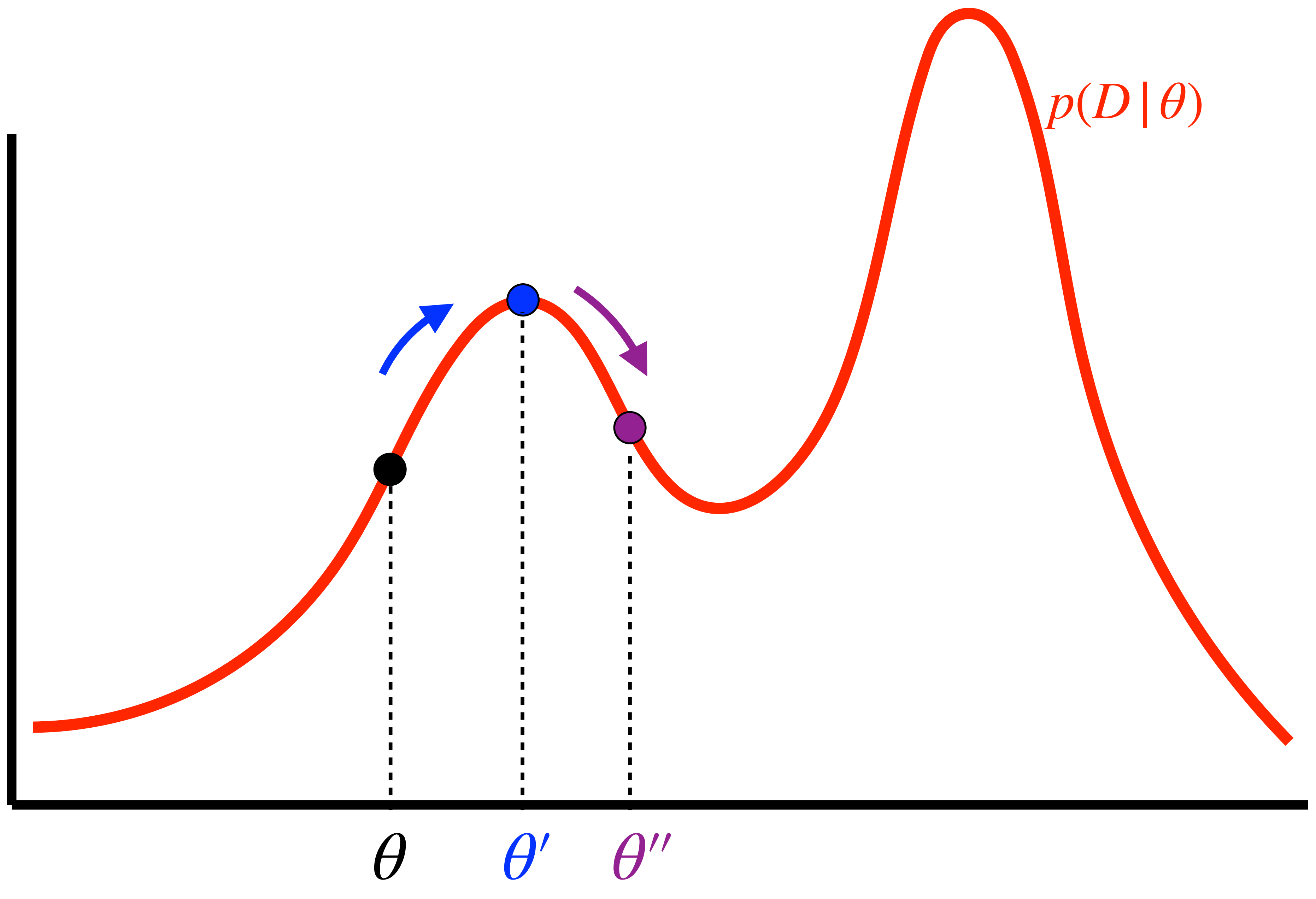

This guarantees that MCMCs *eventually* converge

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$$\alpha(\theta, \theta') = \frac{p(D \,|\, \theta', M) q(\theta \,|\, \theta')}{p(D \,|\, \theta, M) q(\theta' \,|\, \theta)},\ q(\theta \,|\, \theta') = p(\theta \,|\, D)$$

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$$\alpha(\theta, \theta') = 1$$

Everything gets accepted!
Maximally efficient

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Designing good proposal distributions is where the work hides

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The proposal is a teenager: The harder you push the harder it pushes back.

$$\alpha(\theta, \theta') = \frac{\boxed{p(D \mid \theta', M)\; p(\theta' \mid M)}\,\boxed{q(\theta \mid \theta')}}{\boxed{p(D \mid \theta, M)\; p(\theta \mid M)}\,\boxed{q(\theta' \mid \theta)}}$$

Always go uphill, but try going down as well

Counterbalance how hard you're pushing

INTERMISSION

What about $p(D \mid M)$?

$$p(D \mid M) = \int d\theta \; p(D \mid \theta, M) p(\theta \mid M)$$

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Large-dimensional
integral == HARD

Recast as 1D integral

$$Z = \int L \, dX$$

Evidence

Likelihood

$dX = p(\theta | M) d\theta$

Prior volume element

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Prior volume element

$$X \equiv X(\lambda) = \int_{L(\theta)>\lambda} p(\theta | M) d\theta$$

Recast as 1D integral

$$Z = \int L \, dX$$

Evidence

Likelihood

$dX = p(\theta \,|\, M) d\theta$

Prior volume element

$$X \equiv X(\lambda) = \int_{L(\theta) > \lambda} p(\theta \,|\, M) d\theta$$

Cumulant prior mass covering all likelihood values $> \lambda$

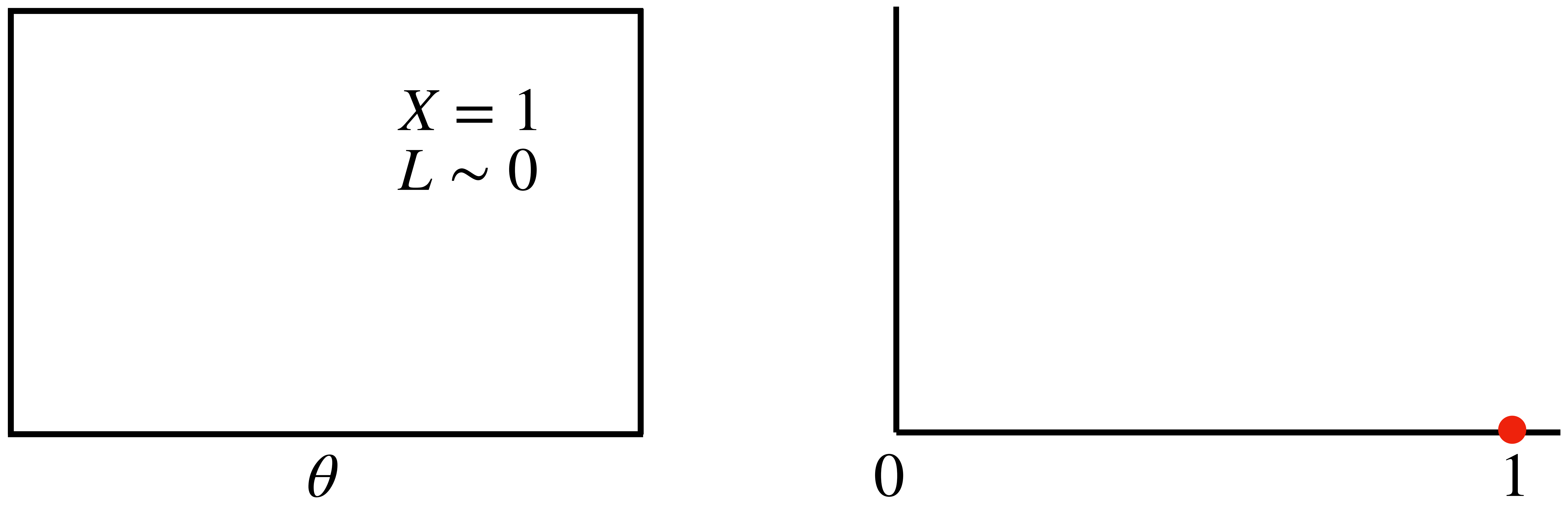
As λ increases, X goes from $1 \rightarrow 0$

λ is small: likelihood is low -> prior

so, $$Z = \int_0^1 L(X)\, dX$$

λ is large: likelihood is high

$L(X)$ is positive and monotonically decreasing. We can do this.



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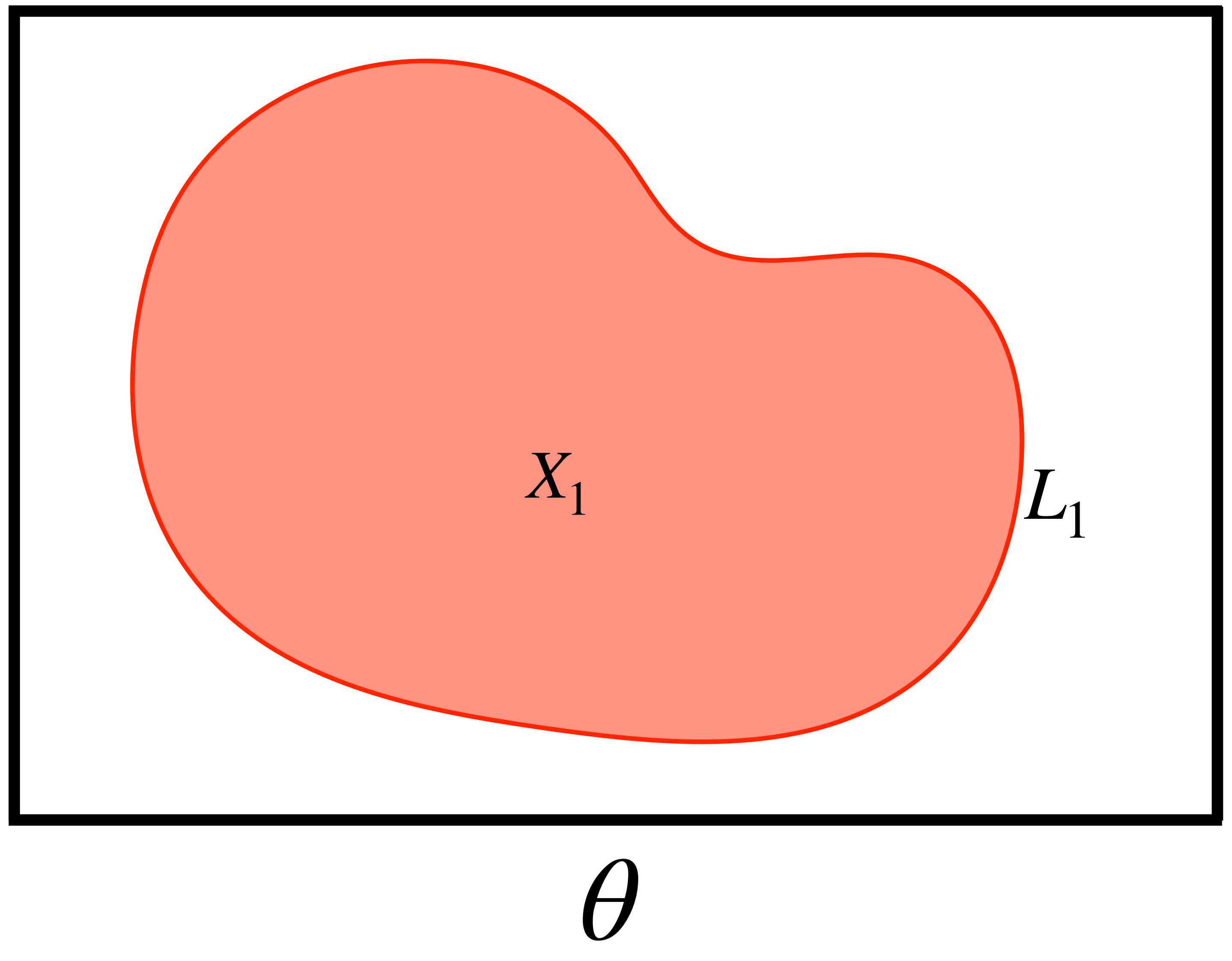

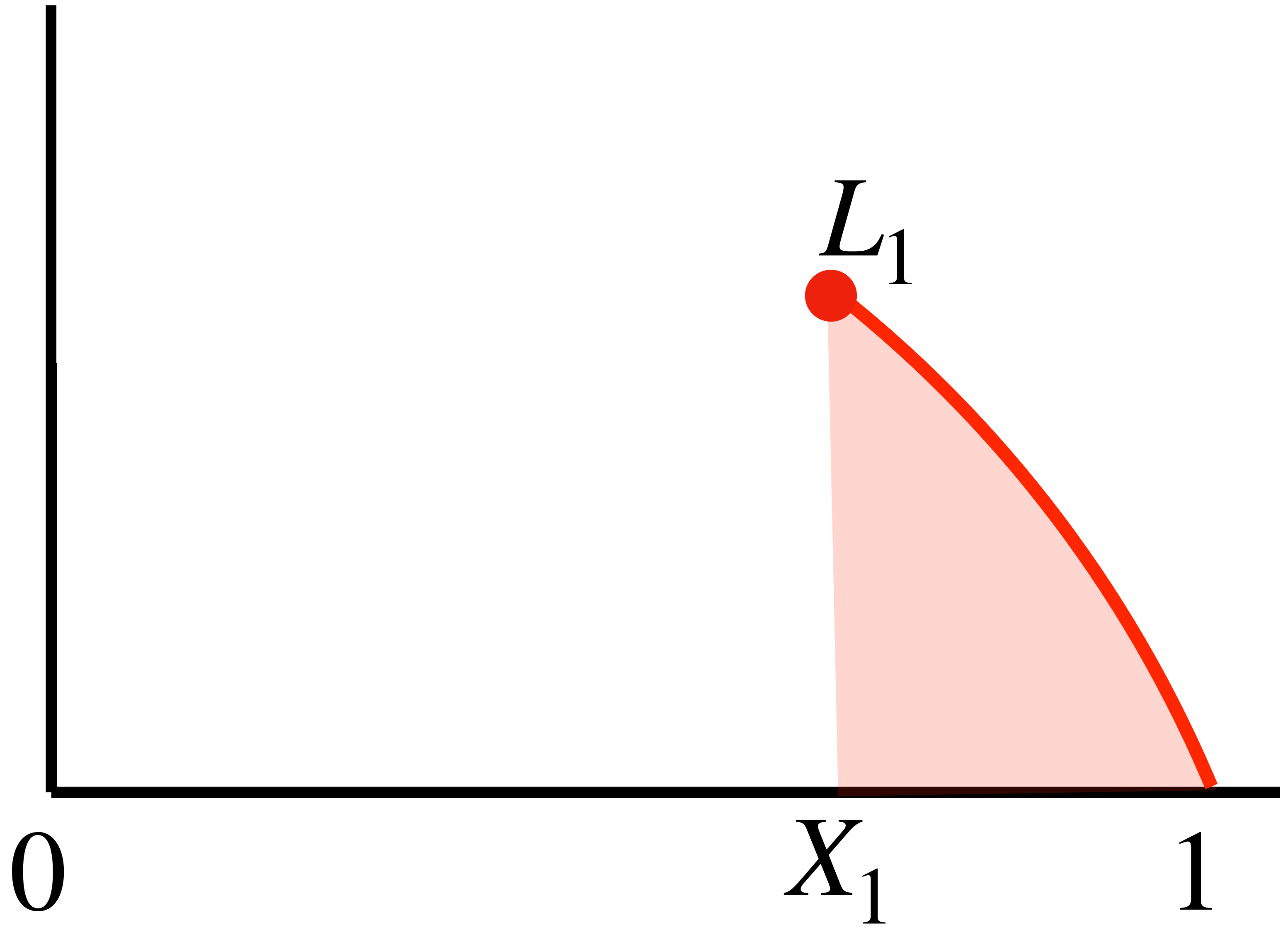

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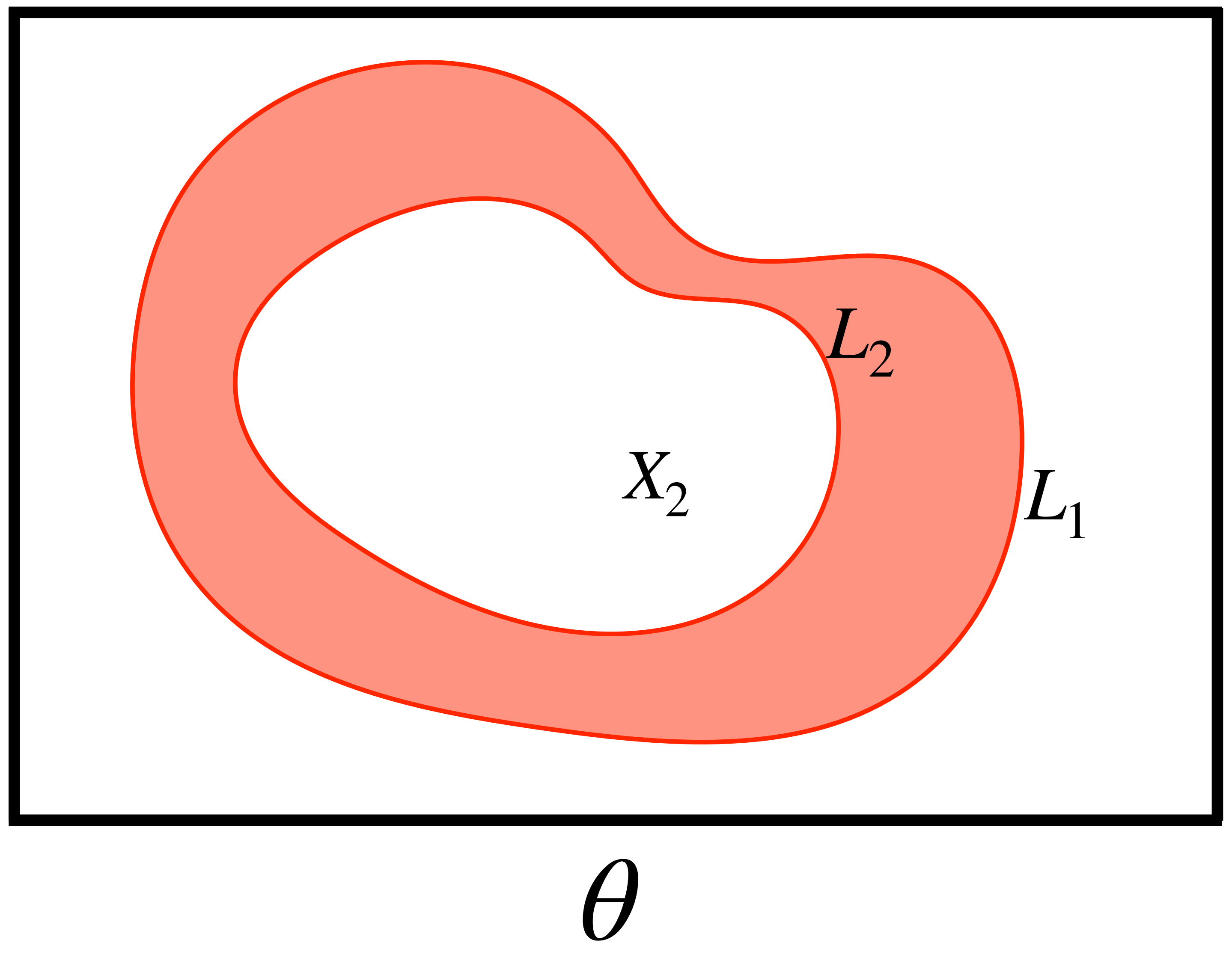

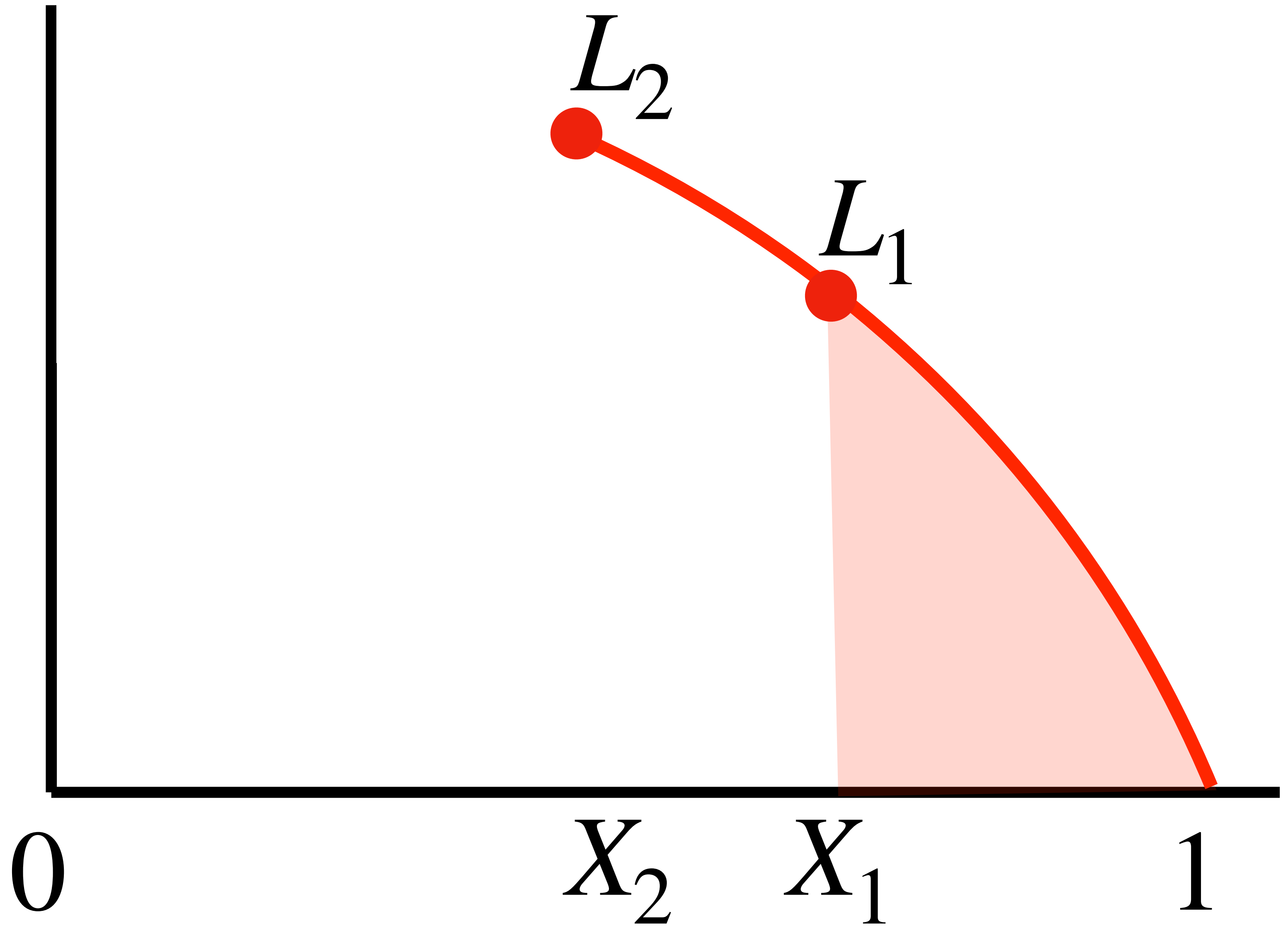

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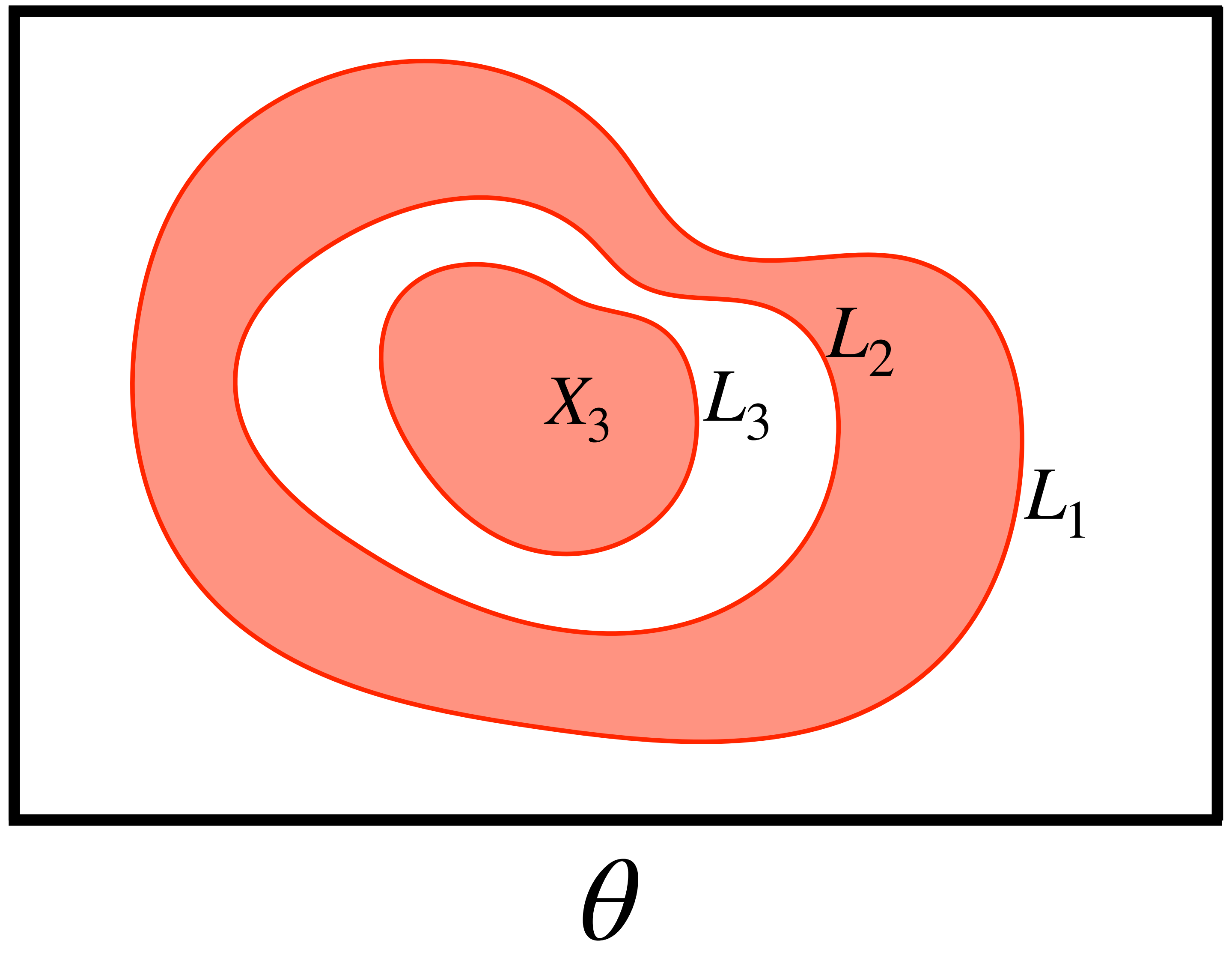

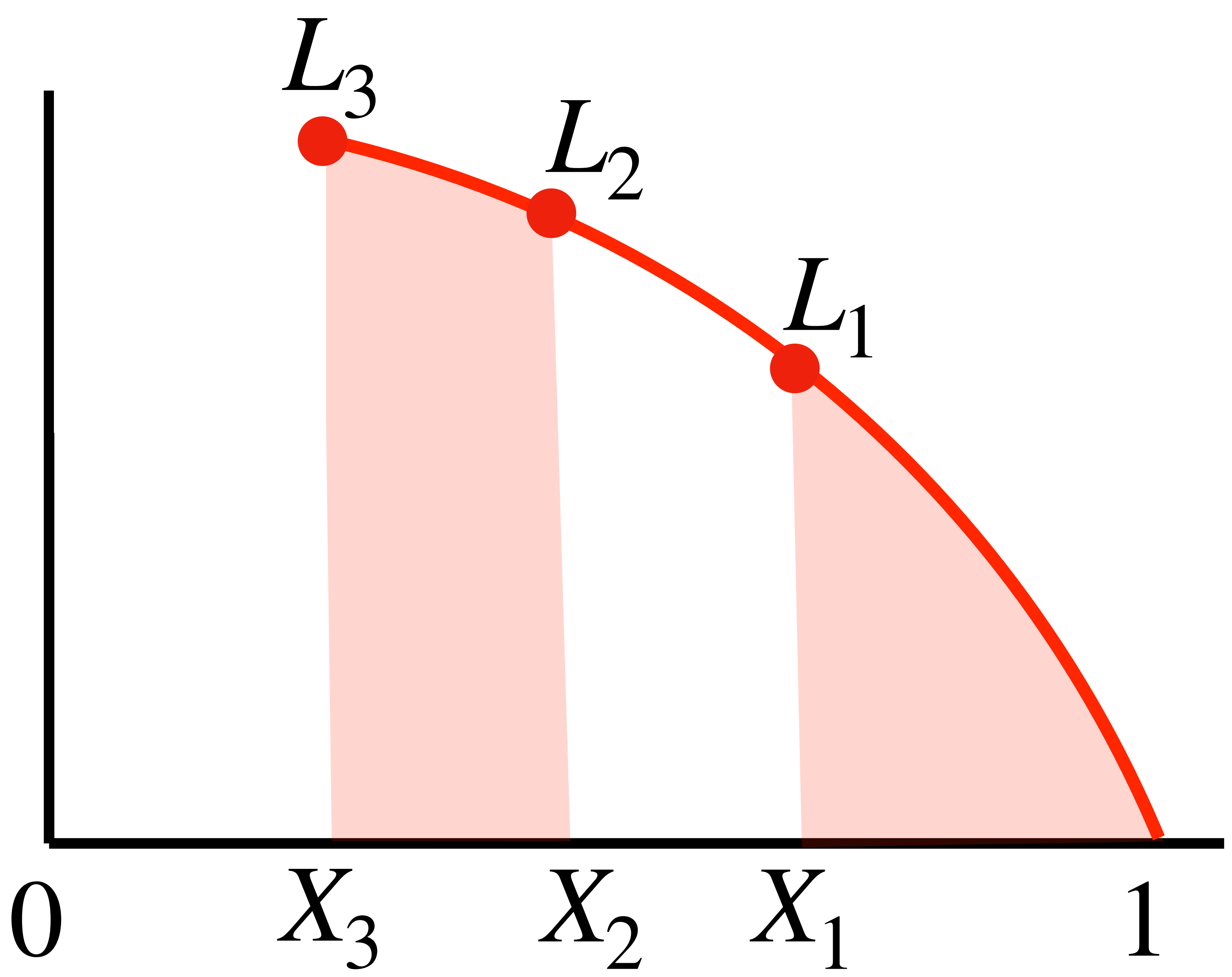

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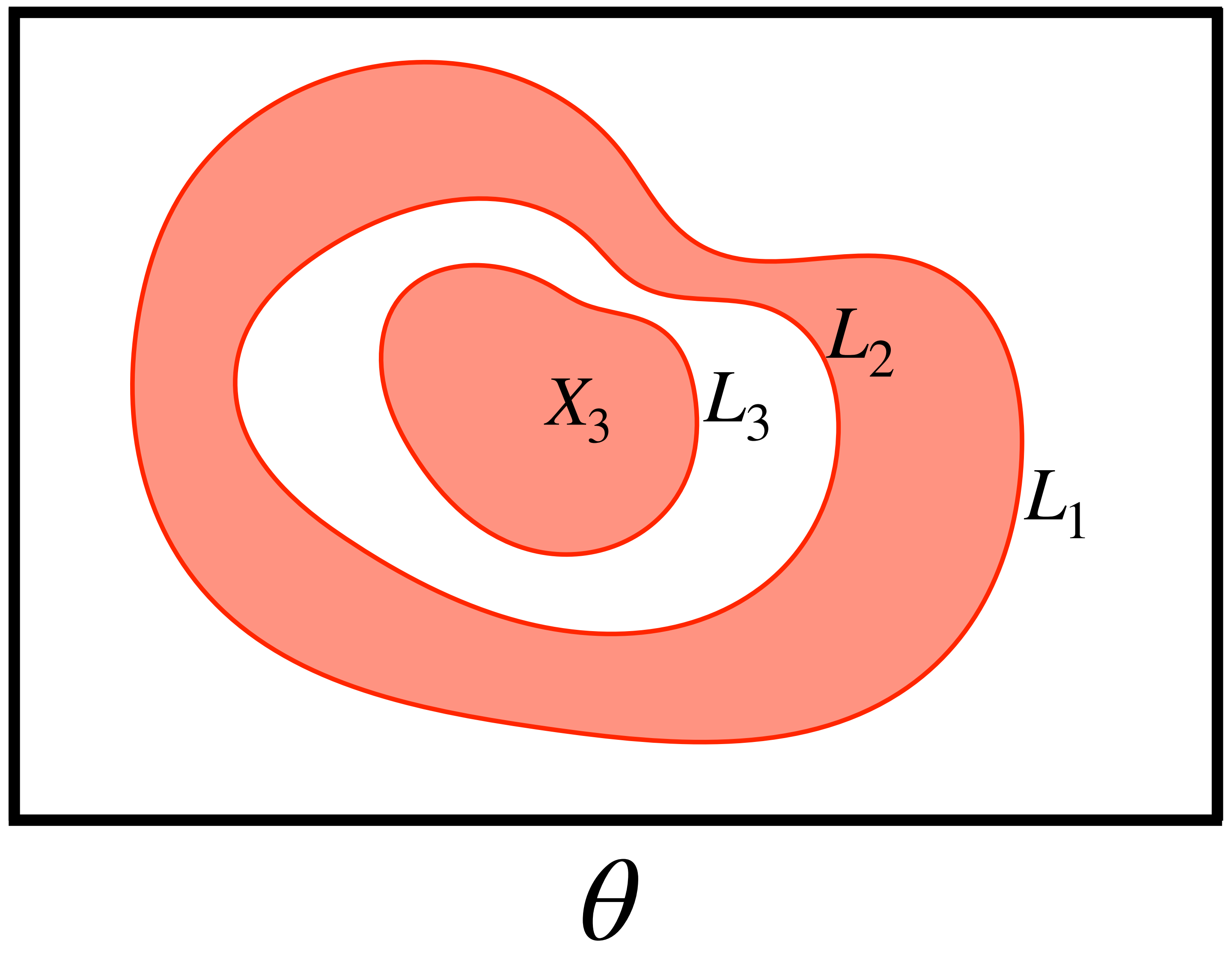

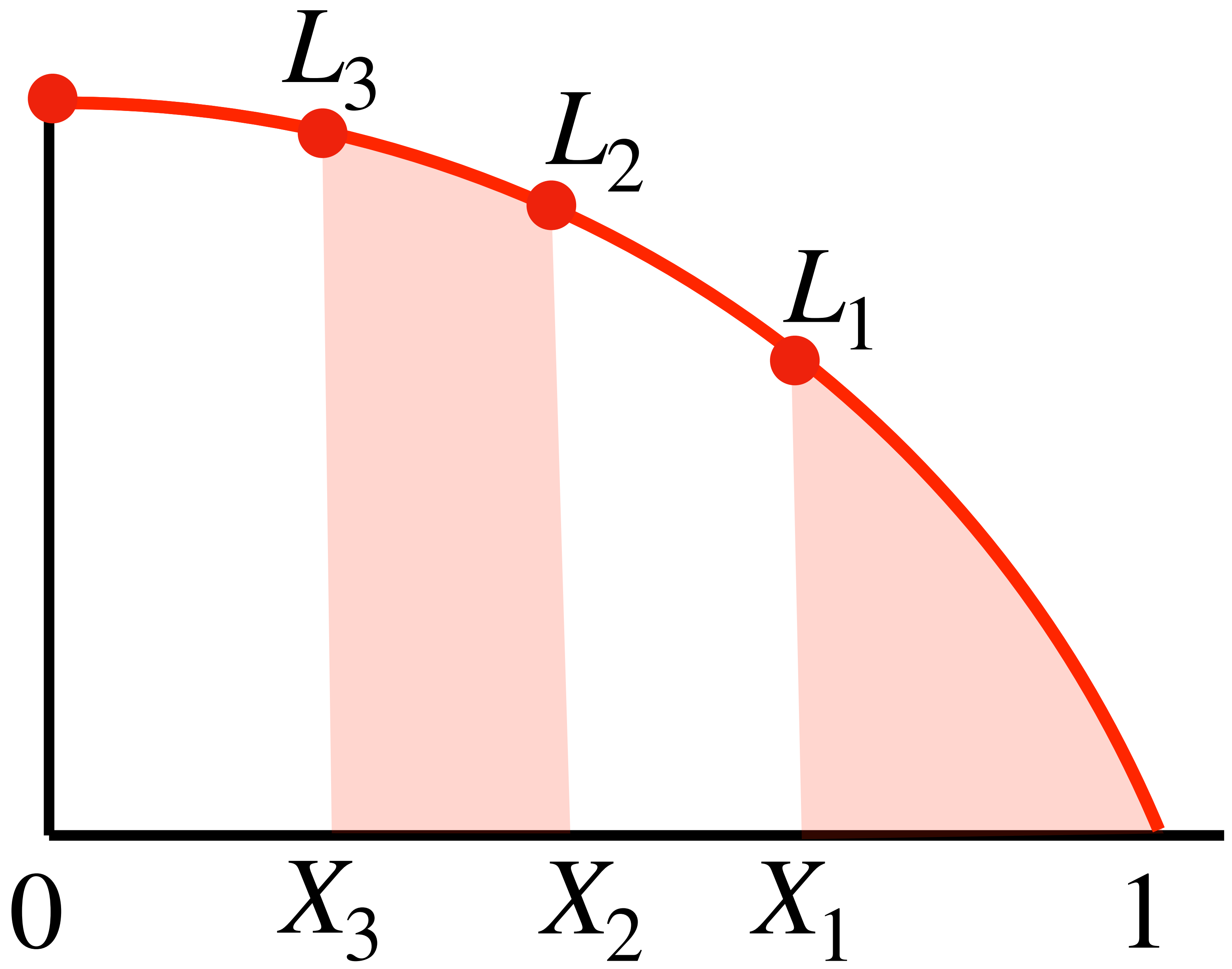

Nested Sampling:

- Start w/ N points sampling prior $\theta_1, \ldots, \theta_N$

$$Z = 0, \; X_0 = 1, \; i = 1$$

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- $Z = Z + \sum_n L(\theta_n) X_i / N$

Nested Sampling:

Generate posterior samples by reweighting $\{\theta_n\}$

$$p_i = L_i(X_{i-1} - X_i)/Z$$

ENCORE

[Other evidence calculators/estimates]

[Other sampling algorithms]