HD notes:

Plane-wave expansion:

$$h_{ij}(t,\vec{x}) = \int df \int d^2\Omega_{\hat{k}} \sum_A h_A(f,\hat{k})\, e^A_{ij}(\hat{k})\, e^{i2\pi f(t-\hat{k}\cdot\vec{x}/c)}$$

Ensemble average (unpolarized, isotropic, stationary-Gaussian background)

$$\langle h_A(f,\hat{k})\rangle = 0$$

$$\langle h_A(f,\hat{k})\, h^*_{A'}(f',\hat{k}')\rangle = \frac{1}{16\pi} S_h(f)\, \delta(f-f')\,\delta_{AA'}\,\delta^2(\hat{k},\hat{k}')$$

Detector response:

$$r_a(t) = \int df \int d^2\Omega_{\hat{k}} \sum_A h_A(f,\hat{k})\, R^A_a(f,\hat{k})\, e^{i2\pi f t}$$

$$\rightarrow \tilde{r}_a(f) = \int d^2\Omega_{\hat{k}} \sum_A h_A(f,\hat{k})\, R^A_a(f,\hat{k})$$

Cross-spectrum (1-sided):

$$\langle \tilde{r}_a(f)\, \tilde{r}^*_b(f')\rangle = \frac{1}{2}\delta(f-f')\, C_{ab}(f)$$

$$= \frac{1}{2}\delta(f-f')\, \Gamma_{ab}(f)\, S_h(f)$$

def'n of $\Gamma_{ab}(f)$

$$\mathrm{LHS} = \langle \tilde{r}_a(f)\, \tilde{r}_b^*(f') \rangle$$

$$= \int d^2\hat{\Omega}_{\hat{k}} \int d^2\Omega_{\hat{k}'} \sum_A \sum_{A'} \underbrace{\langle h_A(f,\hat{k})\, h^*_{A'}(f',\hat{k}') \rangle}_{= \frac{1}{16\pi} S_h(f)\,\delta\delta\delta} R_a^A(f,\hat{k})\, R_b^{A'*}(f',\hat{k}')$$

$$= \int d^2\Omega_{\hat{k}} \sum_A \frac{1}{16\pi} S_h(f)\, \delta(f-f')\, R_a^A(f,\hat{k})\, R_b^{A*}(f,\hat{k})$$

$$= \frac{1}{2}\delta(f-f')\, S_h(f)\, \underbrace{\frac{1}{8\pi}\int d^2\Omega_{\hat{k}} \sum_A R_a^A(f,\hat{k})\, R_b^{A*}(f,\hat{k})}_{\Gamma_{ab}(f)}$$

$$= \frac{1}{2}\delta(f-f')\, S_h(f)\, \Gamma_{ab}(f)$$

where

$$\Gamma_{ab}(f) \equiv \frac{1}{8\pi}\int d^2\Omega_{\hat{k}} \sum_A R_a^A(f,\hat{k})\, R_b^{A*}(f,\hat{k})$$

Simple example:

$$R_a^A(f,\hat{k}) = G \; e^{-i2\pi f \hat{k}\cdot\vec{x}_a/c}$$

$$\Gamma_{ab}(f) = \frac{1}{8\pi}\int d^2\Omega_{\hat{k}} \sum_A R_a^A(f,\hat{k})\, R_b^{A*}(f,\hat{k})$$

$$= \frac{1}{8\pi}\, 2G^2 \int d^2\Omega_{\hat{k}}\, e^{-i2\pi f \hat{k}\cdot(\vec{x}_a-\vec{x}_b)/c}$$

$$= \frac{1}{8\pi}\, 2G^2 \int d^2\Omega_{\hat{k}}\, e^{-i2\pi f \frac{D_{ab}}{c} \hat{k}\cdot\Delta\hat{x}_{ab}}$$

where $\vec{x}_a - \vec{x}_b \equiv \Delta\vec{x}_{ab} = D_{ab}\,\Delta\hat{x}_{ab}$

Define $\alpha_{ab} = 2\pi f \frac{D_{ab}}{c}$

Choose coord system so that $\hat{z} = \Delta\hat{x}_{ab}$, $\hat{k} = -\hat{r}$

$\rightarrow \hat{k}\cdot\Delta\hat{x}_{ab} = -\hat{r}\cdot\hat{z} = \cos\theta$

$$\Gamma_{ab} = \frac{1}{4\pi} G^2 \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\, e^{+i\alpha_{ab}\cos\theta}$$

$$\Gamma_{ab}(f) = \frac{G^2}{4\pi} \cdot 2\pi \int_{-1}^{1} dx \; e^{i\alpha_{ab} x}$$

$$= \frac{G^2}{2} \frac{1}{i\alpha_{ab}} e^{i\alpha_{ab} x} \Big|_{-1}^{1}$$

$$= \frac{G^2}{\alpha_{ab}} \frac{1}{2i} \left(e^{i\alpha_{ab}} - e^{-i\alpha_{ab}} \right)$$

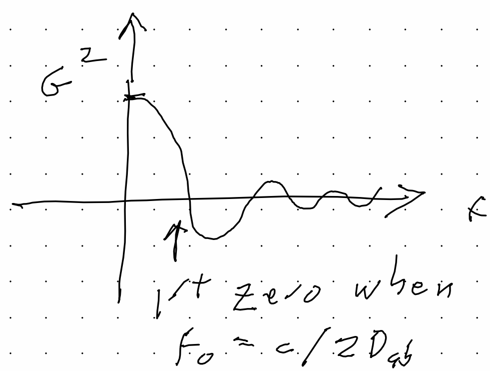
$$= \frac{G^2}{\alpha_{ab}} \sin(\alpha_{ab})$$

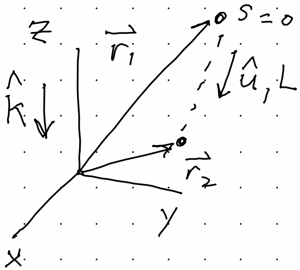
$$= G^2 \operatorname{sinc}(\alpha_{ab}) \quad \text{where} \quad \operatorname{sinc} x \equiv \frac{\sin x}{x}$$

thus,

$$\Gamma_{ab}(f) = G^2 \operatorname{sinc}\left(\frac{2\pi f D_{ab}}{c} \right)$$

$$\cancel{2\pi} f_0 \frac{D_{ab}}{c} = \cancel{\pi} \leftrightarrow f_0 = \frac{c}{2 D_{ab}}$$





$$r(t) \equiv \Delta T(t) = \frac{1}{2c} u^i u^j \int_0^L ds\, h_{ij}(t(s), \vec{x}(s))$$

where $t(s) = \left(t - \frac{L}{c}\right) + \frac{s}{c}$ $\quad \begin{cases} s=0 \to t(0) = t - \frac{L}{c} \\ s = L \to t(L) = t \end{cases}$

$$\vec{x}(s) = \vec{r}_1 + (\vec{r}_2 - \vec{r}_1)\, s/L$$
$$= \vec{r}_1 + \hat{u}\, s$$

$$h_{ij}(t(s), \vec{x}(s)) = \int df \int d^2\Omega_k \sum_A h_A(f, \hat{k})\, e^A_{ij}(\hat{k})\, e^{i2\pi f\left(t(s) - \hat{k}\cdot\vec{x}(s)/c\right)}$$

$$e^{i2\pi f\left[\left(t - \frac{L}{c}\right) + \frac{s}{c} - \frac{\hat{k}}{c}\cdot(\vec{r}_1 + \hat{u}s)\right]}$$
$$= \underbrace{e^{i2\pi f\left[t - \frac{L}{c} - \frac{\hat{k}\cdot\vec{r}_1}{c}\right]}}_{\text{indep of } s \Rightarrow \text{ comes out of integral}}\; e^{i2\pi f(1 - \hat{k}\cdot\hat{u})\frac{s}{c}}$$

$$r(t) = \frac{1}{2c}\int df \int d^2\Omega_k \sum_A h_A(f, \hat{k})\, e^A_{ij}(\hat{k})\, u^i u^j\, e^{i2\pi f\left[t - \frac{L}{c} - \frac{\hat{k}\cdot\vec{r}_1}{c}\right]}$$
$$\times \int_0^L ds\, e^{i2\pi f(1 - \hat{k}\cdot\hat{u})s/c}$$

$$\int_0^L ds\; e^{i2\pi f \frac{s}{c}(1-\hat{k}\cdot\hat{u})} = \left(\frac{c}{i2\pi f}\right)\left(\frac{1}{1-\hat{k}\cdot\hat{u}}\right)\left[e^{i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})} - 1\right]$$

Thus,

$$r(t) = \frac{1}{2c}\int df \int d^2\Omega_k \sum_A h_A(f,\hat{k})\left(\frac{c}{i2\pi f}\right)\frac{u^i u^j e^A_{ij}(\hat{k})}{(1-\hat{k}\cdot\hat{u})} e^{i2\pi f\left(t-\frac{L}{c}-\frac{\hat{k}\cdot\vec{r}_1}{c}\right)} \left[e^{i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})} - 1\right]$$

<u>Rewrite the exponential terms:</u>

$$e^{i2\pi f t}\, e^{-i2\pi f\frac{L}{c}\left(1+\frac{\hat{k}\cdot\vec{r}_1}{L}\right)}\left[e^{i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})} - 1\right]$$

$$= e^{i2\pi f t}\, e^{-i2\pi f\frac{L}{c}\left(1+\frac{\hat{k}\cdot\vec{r}_1}{L}\right)}\, e^{i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}\left[1-e^{-i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}\right]$$

$$= e^{i2\pi f t}\, e^{-i2\pi f\frac{L}{c}\hat{k}\cdot\left(\frac{\vec{r}_1}{L}+\hat{u}\right)}\left[1-e^{-i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}\right]$$

$$= e^{i2\pi f t}\, e^{-i2\pi f\,\hat{k}\cdot\vec{r}_2/c}\left[1-e^{-i2\pi f\frac{L}{c}(1-\hat{k}\cdot\hat{u})}\right]$$

Using $\hat{u} = \frac{\vec{r}_2-\vec{r}_1}{L}$

Thus

$$r(t) = \frac{1}{2\cancel{c}} \int df \int d^2\Omega_k \sum_A h_A(f,\hat{k}) \left(\frac{\cancel{c}}{i2\pi f}\right) \frac{u^i u^j e^A_{ij}(\hat{k})}{(1-\hat{k}\cdot\hat{u})} e^{i2\pi f t}$$

$$e^{-i2\pi f \hat{k}\cdot\vec{r}_2/c} \left[1 - e^{-i2\pi f \frac{L}{c}(1-\hat{k}\cdot\hat{u})}\right]$$

$$= \int df \int d^2\Omega_k \sum_A h_A(f,\hat{k})\, e^{i2\pi f t}\, e^{-i2\pi f \hat{k}\cdot\vec{r}_2/c}$$

$$\left(\frac{1}{i2\pi f}\right) \frac{1}{2} \frac{u^i u^j e^A_{ij}(\hat{k})}{(1-\hat{k}\cdot\hat{u})} \left[1 - e^{-i2\pi f \frac{L}{c}(1-\hat{k}\cdot\hat{u})}\right]$$

so

$$R^A(f,\hat{k}) = \underbrace{\left(\frac{1}{i2\pi f}\right)}_{1 \text{ for } \frac{\Delta\nu}{\nu}} \underbrace{\frac{1}{2} \frac{u^i u^j e^A_{ij}(\hat{k})}{(1-\hat{k}\cdot\hat{u})}}_{F^A(\hat{k})} \left[1 - \underbrace{e^{-i2\pi f \frac{L}{c}(1-\hat{k}\cdot\hat{u})}}_{\text{pulsar term}}\right] \underbrace{e^{-i2\pi f \hat{k}\cdot\vec{r}_2/c}}_{\text{this will be the same for radio measurements made from Earth}}$$

So for redshift measurements:

$$R_a^A(f,\hat{k}) \simeq F_a^A(\hat{k})\left[1 - e^{-i2\pi f \frac{L_a}{c}(1+\hat{k}\cdot\hat{p}_a)}\right] \qquad a=1,2,\ldots$$

where $F_a^A(\hat{k}) = \frac{1}{2}\frac{\hat{p}_a^i\,\hat{p}_a^j}{1+\hat{p}_a\cdot\hat{k}}\,e_{ij}^A(\hat{k})$, $\hat{u}=-\hat{p}$

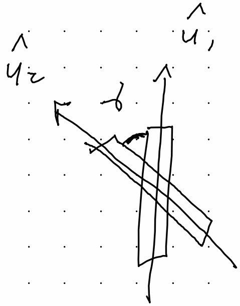
$$\Gamma_{ab}(f) = \frac{1}{8\pi}\int d^2\Omega_{\hat{k}}\, R_a^A(f,\hat{k})\, R_b^{A*}(f,\hat{k})$$

$$\simeq \frac{1}{8\pi}\int d^2\Omega_{\hat{k}}\, F_a^A(\hat{k})F_b^A(\hat{k})\left[1 - e^{-i2\pi f\frac{L_a}{c}(1+\hat{k}\cdot\hat{p}_a)}\right]\left[1 - e^{+i2\pi f\frac{L_b}{c}(1+\hat{k}\cdot\hat{p}_b)}\right]$$

except for autocorrelations

$$\rightarrow \Gamma_{ab}(f) \simeq \frac{1}{8\pi}\int d^2\Omega_{\hat{k}}\, F_a^A(\hat{k})F_b^A(\hat{k})\left[1+\delta_{ab}\right]$$

HW problem



$$r_a(t) = \hat{u}_a \cdot \vec{E}(t, \vec{x} = \vec{0}) \quad , \quad a = 1,2$$

$$\vec{E}(t,\vec{x}) = \int df \int d^2\Omega_{\hat{k}} \sum_{A=1,2} \tilde{E}_A(f,\hat{k})\, \hat{\epsilon}_A(\hat{k})\, e^{i2\pi f(t - \hat{k}\cdot\vec{x}/c)}$$

$$r_a(t) = \int df\, e^{i2\pi f t} \int d^2\Omega_{\hat{k}} \sum_A \tilde{E}_A(f,\hat{k})\, \hat{u}_a \cdot \hat{\epsilon}_A(\hat{k})$$

$$= \int df\, e^{i2\pi f t} \int d^2\Omega_{\hat{k}} \sum_A \tilde{E}_A(f,\hat{k})\, R_a^A(\hat{k})$$

So: $R_a^A(\hat{k}) = \hat{u}_a \cdot \hat{\epsilon}_A(\hat{k})$

$$\hat{\epsilon}_1(\hat{k}) = -\hat{\phi} \quad , \quad \hat{\epsilon}_2(\hat{k}) = -\hat{\theta} \quad , \quad \hat{k} = -\hat{r}$$

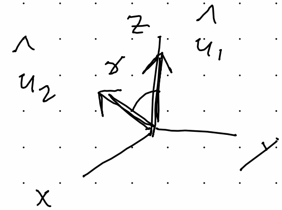
$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} \quad = -\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} \quad = -\hat{m}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \quad = -\hat{l}$$

$\hat{u}_1 = \hat{z}$

$\hat{u}_2 = \sin\gamma\,\hat{x} + \cos\gamma\,\hat{z}$



Thus,

$$\Gamma_{12} = \frac{1}{8\pi}\int d^2\Omega_{\hat{k}} \sum_A R_1^A(\hat{k})\, R_2^A(\hat{k})$$

$$= \frac{1}{8\pi}\int d^2\Omega_{\hat{k}} \sum_A \hat{u}_1 \cdot \hat{\epsilon}_A(\hat{k})\ \hat{u}_2 \cdot \hat{\epsilon}_A(\hat{k})$$

$$\hat{u}_1 \cdot \hat{\epsilon}_1(\hat{k}) = -\hat{z}\cdot\hat{\phi} = 0$$

$$\hat{u}_1 \cdot \hat{\epsilon}_2(\hat{k}) = -\hat{z}\cdot\hat{\theta} = \sin\theta$$

$$\hat{u}_2 \cdot \hat{\epsilon}_1(\hat{k}) = -(\sin\gamma\,\hat{x} + \cos\gamma\,\hat{z})\cdot\hat{\phi} = +\sin\gamma\sin\phi$$

$$\hat{u}_2 \cdot \hat{\epsilon}_2(\hat{k}) = -(\sin\gamma\,\hat{x} + \cos\gamma\,\hat{z})\cdot\hat{\theta} = -\sin\gamma\cos\theta\cos\phi + \cos\gamma\sin\theta$$

$$\Gamma_{12} = \frac{1}{8\pi}\int d^2\Omega_{\hat{k}}\ \sin\theta\left(-\sin\gamma\cos\theta\cos\phi + \cos\gamma\sin\theta\right)$$

$$= -\sin\gamma\left(\frac{1}{8\pi}\right)\int d^2\Omega_{\hat{k}}\ \sin\theta\cos\theta\cos\phi + \cos\gamma\left(\frac{1}{8\pi}\right)\int d^2\Omega_{\hat{k}}\ \sin^2\theta$$

Now:
$$\int d^2\Omega_k \sin\theta\cos\theta\cos\phi = \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \sin\theta\cos\theta\cos\phi$$

$$= 0 \quad \left(\text{since } \int_0^{2\pi} d\phi \cos\phi = 0\right)$$

$$\int d^2\Omega_k \sin^2\theta = \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \sin^2\theta$$

$$= 2\pi \int_{-1}^{1} dx \left(1-x^2\right)$$

$$= 2\pi \left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$$

$$= 4\pi \left(1 - \frac{1}{3}\right)$$

$$= \frac{8\pi}{3}$$

Thus
$$\Gamma_{12} = \cos\gamma \left(\frac{1}{8\pi}\right) \frac{8\pi}{3} = \boxed{\frac{1}{3}\cos\gamma}$$

Homework problem: Breathing mode / Scalar transverse

$$e^B_{ij} = \hat{l}_i \hat{l}_j + \hat{m}_i \hat{m}_j$$

$$e^+_{ij} = \hat{l}_i \hat{l}_j - \hat{m}_i \hat{m}_j$$

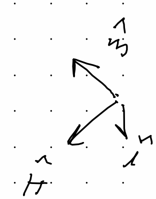
$$e^\times_{ij} = \hat{l}_i \hat{m}_j + \hat{m}_i \hat{l}_j$$

$$\Gamma_{ab}(t) = \frac{1}{8\pi} \int d^2\Omega_{\hat{k}}\, R^B_a(t,\hat{k})\, R^{B*}_b(t,\hat{k}) \simeq \frac{1}{8\pi}$$

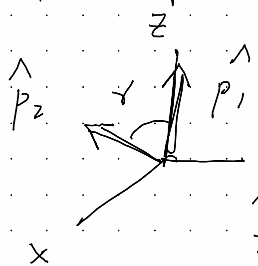
$$\simeq \frac{1}{8\pi} \int d^2\Omega_{\hat{k}}\, F^B_a(\hat{k})\, F^B_b(\hat{k}) \left(1+\delta_{ab}\right)$$

$$F^B_a(\hat{k}) = \frac{1}{2} \frac{p_a^i p_a^j}{(1+\hat{p}_a \cdot \hat{k})} e^B_{ij}(\hat{k}),$$

$$\hat{k} = -\hat{r}$$
$$\hat{l} = -\hat{\phi}$$
$$m = -\hat{\theta}$$



Take pulsars a and b along $\hat{z}$-axis and in xy-plane:



$$F^B_1(\hat{k}) = \frac{1}{2} \frac{\hat{z}^i \hat{z}^j}{(1+\hat{z}\cdot\hat{k})} e^B_{ij}(\hat{k})$$

$$\hat{z}^i \hat{z}^j e^B_{ij}(\hat{k}) = \hat{z}^i \hat{z}^j \left(\hat{l}_i \hat{l}_j + \hat{m}_i \hat{m}_j\right)$$
$$= (\hat{z}\cdot\hat{l})^2 + (\hat{z}\cdot\hat{m})^2$$

$$\hat{z}\cdot\hat{l} = 0 \rightarrow \hat{z}^i\hat{z}^j e^B_{ij} = \sin^2\theta$$

$$\hat{z}\cdot\hat{m} = \sin\theta \rightarrow F_1^B(\hat{k}) = \frac{1}{2}\frac{\sin^2\theta}{1-\cos\theta}$$

$$\hat{z}\cdot\hat{k} = -\cos\theta$$

$$= \frac{1}{2}\left(\frac{1-\cos^2\theta}{1-\cos\theta}\right)$$

$$= \boxed{\frac{1}{2}(1+\cos\theta)}$$

$$F_2^B(\hat{k}) = \frac{1}{2}\frac{(\cos\gamma\,\hat{z}^i + \sin\gamma\,\hat{x}^i)(\cos\gamma\,\hat{z}^j + \sin\gamma\,\hat{x}^j)}{1+(\cos\gamma\,\hat{z}\cdot\hat{k} + \sin\gamma\,\hat{x}\cdot\hat{k})}\underbrace{e^B_{ij}(\hat{k})}_{l_il_j + m_im_j}$$

<u>Now</u>: $\hat{z}\cdot\hat{l} = 0$, $\hat{z}\cdot\hat{m} = \sin\theta$, $\hat{z}\cdot\hat{k} = -\cos\theta$

$\hat{x}\cdot\hat{l} = \sin\phi$, $\hat{x}\cdot\hat{m} = -\cos\theta\cos\phi$, $\hat{x}\cdot\hat{k} = -\sin\theta\cos\phi$

<u>Denominator</u>: $1 - \cos\gamma\cos\theta - \sin\gamma\sin\theta\cos\phi$

$$\text{Numerator} = \left(\cos\gamma\,\hat{z}^i + \sin\gamma\,\hat{x}^i\right)\left(\cos\gamma\,\hat{z}^j + \sin\gamma\,\hat{x}^j\right)\left(l_i l_j + m_i m_j\right)$$

$$= \left(\cos^2\gamma\,\hat{z}^i\hat{z}^j + \sin^2\gamma\,\hat{x}^i\hat{x}^j + \sin\gamma\cos\gamma\left(\hat{z}^i\hat{x}^j + \hat{z}^j\hat{x}^i\right)\right)\left(l_i l_j + m_i m_j\right)$$

$$= \cos^2\gamma\,\cancelto{0}{\left(\hat{z}\cdot\hat{l}\right)^2} + \sin^2\gamma\,\underbrace{\left(\hat{x}\cdot\hat{l}\right)^2}_{\sin\phi} + 2\sin\gamma\cos\gamma\,\cancelto{0}{\left(\hat{z}\cdot\hat{l}\right)}\left(\hat{x}\cdot\hat{l}\right)$$

$$+ \cos^2\gamma\left(\hat{z}\cdot\hat{m}\right)^2 + \sin^2\gamma\left(\hat{x}\cdot\hat{m}\right)^2 + 2\sin\gamma\cos\gamma\left(\hat{z}\cdot\hat{m}\right)\left(\hat{x}\cdot\hat{m}\right)$$

$$= \sin^2\gamma\sin^2\phi + \cos^2\gamma\sin^2\theta + \sin^2\gamma\cos^2\theta\cos^2\phi$$

$$+ 2\sin\gamma\cos\gamma\sin\theta\left(-\cos\theta\cos\phi\right)$$

Thus,

$$F_z^B(\hat{k}) = \frac{1}{2}\,\frac{\sin^2\gamma\sin^2\phi + \cos^2\gamma\sin^2\theta + \sin^2\gamma\cos^2\theta\cos^2\phi - 2\sin\gamma\cos\gamma\sin\theta\cos\theta\cos\phi}{\left(1 - \cos\gamma\cos\theta - \sin\gamma\sin\theta\cos\phi\right)}$$

Now

Note:

$$\left(1-\cos\gamma\cos\theta-\sin\gamma\sin\theta\cos\phi\right)\left(1+\cos\gamma\cos\theta+\sin\gamma\sin\theta\cos\phi\right)$$

$$= 1 + \cancel{\cos\gamma\cos\theta} + \cancel{\sin\gamma\sin\theta\cos\phi}$$

$$- \cancel{\cos\gamma\cos\theta} - \cos^2\gamma\cos^2\theta - \sin\gamma\cos\gamma\sin\theta\cos\theta\cos\phi$$

$$- \cancel{\sin\gamma\sin\theta\cos\phi} - \sin\gamma\cos\gamma\sin\theta\cos\theta\cos\phi - \sin^2\gamma\sin^2\theta\cos^2\phi$$

$$= 1 - \cos^2\gamma\underbrace{\cos^2\theta}_{(1-\sin^2\theta)} - \sin^2\gamma\underbrace{\sin^2\theta}_{(1-\cos^2\theta)}\cos^2\phi - 2\sin\gamma\cos\gamma\sin\theta\cos\theta\cos\phi$$

$$= 1-\cos^2\gamma + \cos^2\gamma\sin^2\theta - \sin^2\gamma\cos^2\phi + \sin^2\gamma\cos^2\theta\cos^2\phi$$
$$- 2\sin\gamma\cos\gamma\sin\theta\cos\theta\cos\phi$$

$$= \sin^2\gamma\left(1-\cos^2\phi\right) + \cos^2\gamma\sin^2\theta + \sin^2\gamma\cos^2\theta\cos^2\phi$$
$$- 2\sin\gamma\cos\gamma\sin\theta\cos\theta\cos\phi$$

$$= \sin^2\gamma\sin^2\phi + \cos^2\gamma\sin^2\theta + \sin^2\gamma\cos^2\theta\cos^2\phi$$
$$- 2\sin\gamma\cos\gamma\sin\theta\cos\theta\cos\phi$$

$= \text{numerator}$

So $F_2^B(\hat{\Pi}) = \frac{1}{2}(1 + \cos\gamma\cos\theta + \sin\gamma\sin\theta\cos\phi)$

Recall: $F_1^B(\hat{\Pi}) = \frac{1}{2}(1+\cos\theta)$

Thus,

$$\Gamma_{12} = \frac{1}{8\pi}\int d^2\Omega_\pi \frac{1}{4}(1+\cos\theta)(1+\cos\gamma\cos\theta + \sin\gamma\sin\theta\cos\phi)$$

$$= \frac{1}{32\pi}\int_0^{2\pi} d\phi \int_{-1}^{1} dx\,(1+x)(1+\cos\gamma\, x + \sin\gamma\sqrt{1-x^2}\cos\phi)$$

$x = \cos\theta$

0 when integrated $\int_0^{2\pi} d\phi$

$$= \frac{1}{32\pi}\cdot 2\pi \cdot \int_{-1}^{1} dx\,(1+x)(1+\cos\gamma\, x)$$

$$= \frac{1}{16}\int_{-1}^{1} dx\,(1 + \cancel{(1+\cos\gamma)x}^{\,0} + \cos\gamma\, x^2)$$

$$= \frac{1}{16}\left[x\Big|_{-1}^{1} + \cos\gamma\,\frac{x^3}{3}\Big|_{-1}^{1}\right] = \frac{1}{8}\left[1 + \frac{1}{3}\cos\gamma\right]$$