

HD notes:

Plane-wave expansion:

$$h_{ij}(t, \vec{x}) = \int df \int d^2\Omega_{\hat{n}} \sum_A h_A(f, \hat{n}) e_{ij}^A(\hat{n}) e^{i2\pi f(t - \hat{n} \cdot \vec{x}/c)}$$

Ensemble average (unpolarized, isotropic, stationary - Gaussian)
b.c. boundary round

$$\langle h_A(f, \hat{n}) \rangle = 0$$

$$\langle h_A(f, \hat{n}) h_{A'}^*(f', \hat{n}') \rangle = \frac{1}{16\pi} S_h(f) \delta(f-f') \delta_{AA'} \delta^2(\hat{n}, \hat{n}')$$

Detector response:

$$h_a(t) = \int df \int d^2\Omega_{\hat{n}} \sum_A h_A(f, \hat{n}) R_a^A(f, \hat{n}) e^{i2\pi ft}$$

$$\rightarrow \tilde{h}_a(f) = \int d^2\Omega_{\hat{n}} \sum_A h_A(f, \hat{n}) R_a^A(f, \hat{n})$$

Cross-spectrum (1-sided):

$$\langle \tilde{h}_a(f) \tilde{h}_b^*(f') \rangle = \frac{1}{2} \delta(f-f') C_{ab}(f)$$

$$= \frac{1}{2} \delta(f-f') \Gamma_{ab}(f) S_h(f) \quad \begin{matrix} \text{def'n of} \\ \Gamma_{ab}(f) \end{matrix}$$

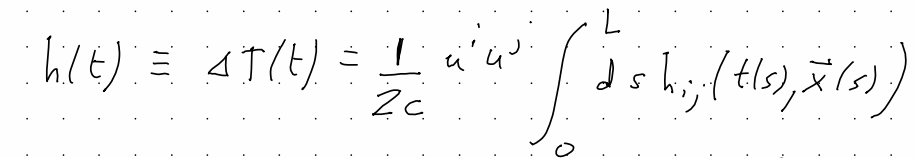
$$\begin{aligned}
 L_{115} &= \langle \tilde{h}_a(f) \tilde{h}_b^*(f') \rangle = \frac{1}{16\pi} S_h(f) \delta f \delta \\
 &= \int d^3\Omega_{\mathbb{H}} \int d^2\Omega_{\mathbb{H}} \sum_A \sum_{A'} \langle h_A(f, \hat{n}) h_{A'}^*(f', \hat{n}') \rangle \\
 &\quad R_a^A(f, \hat{n}) R_b^{A'*}(f', \hat{n}')
 \end{aligned}$$

$$\begin{aligned}
 &= \int d^2\Omega_{\mathbb{H}} \sum_A \frac{1}{16\pi} S_h(f) \delta(f-f') R_a^A(f, \hat{n}) R_b^{A'*}(f, \hat{n}) \\
 &= \frac{1}{2} \delta(f-f') S_h(f) \underbrace{\frac{1}{8\pi} \int d^2\Omega_{\mathbb{H}} \sum_A R_a^A(f, \hat{n}) R_b^{A'*}(f, \hat{n})}_{\Gamma_{ab}(f)}
 \end{aligned}$$

$$= \frac{1}{2} \delta(f-f') S_h(f) \Gamma_{ab}(f)$$

where

$$\Gamma_{ab}(f) \equiv \frac{1}{8\pi} \int d^2\Omega_{\mathbb{H}} \sum_A R_a^A(f, \hat{n}) R_b^{A'*}(f, \hat{n})$$



$$\begin{aligned}\vec{r}_X(s) &= \vec{r}_1 + (\vec{r}_2 - \vec{r}_1)s/L \\ &= \vec{r}_1 + \hat{u}_s\end{aligned}$$

$$= e^{i 2 \pi f \left[t - \frac{L}{c} - \frac{\hat{k} \cdot \vec{r}_1}{c} \right]} e^{i 2 \pi f \left(1 - \hat{k} \cdot \hat{q} \right) \frac{S}{c}}$$

$$h(t) = \frac{1}{2c} \int df \int d^2 \Omega_\pi \sum_A h_A(t, \hat{\pi}) e_{ij}^A(\hat{\pi}) u_i^j e^{i2\pi f [t - \frac{L}{c} - \frac{\hat{\pi} \cdot \vec{r}_1}{c}]} \times \int_0^L ds e^{i2\pi f (1 - \frac{\hat{\pi} \cdot \hat{u}}{c}) s/c}$$

$$\int_0^L ds \ e^{i2\pi f \frac{s}{c} (1 - \hat{k} \cdot \hat{u})} = \left(\frac{c}{i2\pi f} \right) \left(\frac{1}{1 - \hat{k} \cdot \hat{u}} \right) \left[e^{i2\pi f \frac{L}{c} (1 - \hat{k} \cdot \hat{u})} - 1 \right]$$

Thus,

$$h(t) = \frac{1}{2c} \int df \int d^2 \Omega_{\pi} \sum_A h_A(k, \hat{k}) \left(\frac{c}{i2\pi f} \right) \frac{u^i u^j e^A_{ij}(\hat{k})}{(1 - \hat{k} \cdot \hat{u})} e^{i2\pi f \left(t - \frac{L}{c} - \hat{k} \cdot \frac{\vec{r}_i}{c} \right)} \left[e^{i2\pi f \frac{L}{c} (1 - \hat{k} \cdot \hat{u})} - 1 \right]$$

Rewrite the exponential terms:

$$\begin{aligned} & e^{i2\pi f t} e^{-i2\pi f \frac{L}{c} (1 + \hat{k} \cdot \frac{\vec{r}_i}{L})} \left[e^{i2\pi f \frac{L}{c} (1 - \hat{k} \cdot \hat{u})} - 1 \right] \\ &= e^{i2\pi f t} e^{-i2\pi f \frac{L}{c} (1 + \hat{k} \cdot \frac{\vec{r}_i}{L})} e^{i2\pi f \frac{L}{c} (1 - \hat{k} \cdot \hat{u})} \left[1 - e^{-i2\pi f \frac{L}{c} (1 - \hat{k} \cdot \hat{u})} \right] \\ &= e^{i2\pi f t} e^{-i2\pi f \frac{L}{c} \hat{k} \cdot \left(\frac{\vec{r}_i}{L} + \hat{u} \right)} \left[1 - e^{-i2\pi f \frac{L}{c} (1 - \hat{k} \cdot \hat{u})} \right] \\ &= e^{i2\pi f t} e^{-i2\pi f \hat{k} \cdot \vec{r}_2 / c} \left[1 - e^{-i2\pi f \frac{L}{c} (1 - \hat{k} \cdot \hat{u})} \right] \end{aligned}$$

Using $\hat{u} = \frac{\vec{r}_2 - \vec{r}_i}{L}$

↑ h_{ν}

$$h(t) = \frac{1}{2\pi} \int df \int d^2\Omega_H \sum_A h_A(t, \hat{n}) \left(\frac{1}{i2\pi f} \right) \frac{u^i u^j e^A_{ij}(\hat{n})}{(1 - \hat{n} \cdot \hat{u})} e^{i2\pi f t} e^{-i2\pi f \hat{n} \cdot \vec{r}_2/c} \left[1 - e^{-i2\pi f \frac{L}{c} (1 - \hat{n} \cdot \hat{u})} \right]$$

$$= \int df \int d^2\Omega_H \sum_A h_A(t, \hat{n}) e^{i2\pi f t} e^{-i2\pi f \hat{n} \cdot \vec{r}_2/c} \left(\frac{1}{i2\pi f} \right) \frac{1}{2} \frac{u^i u^j e^A_{ij}(\hat{n})}{(1 - \hat{n} \cdot \hat{u})} \left[1 - e^{-i2\pi f \frac{L}{c} (1 - \hat{n} \cdot \hat{u})} \right]$$

so

$$R^A(t, \hat{n}) = \underbrace{\left(\frac{1}{i2\pi f} \right)}_{\uparrow \text{ for } \frac{\Delta v}{v}} \underbrace{\frac{1}{2} \frac{u^i u^j e^A_{ij}(\hat{n})}{(1 - \hat{n} \cdot \hat{u})}}_{F^A(\hat{n})} \underbrace{\left[1 - e^{-i2\pi f \frac{L}{c} (1 - \hat{n} \cdot \hat{u})} \right]}_{\text{pulsar term}} \underbrace{e^{-i2\pi f \hat{n} \cdot \vec{r}_2/c}}_{\text{this will be the same for radio measurements made from Eq. 1b}}$$

So for redshift measurements

$$R_a^A(t, \hat{k}) \simeq F_a^A(\hat{k}) \left[1 - e^{-i2\pi f \frac{L_a}{c} (1 + \hat{k} \cdot \hat{p}_a)} \right] \quad a=1, 2, \dots$$

where $F_a^A(\hat{k}) \equiv \frac{1}{2} \frac{\hat{p}_a^i \hat{p}_a^j}{1 + \hat{p}_a \cdot \hat{k}} e_{ij}^A(\hat{k})$, $\hat{u} = -\hat{p}$



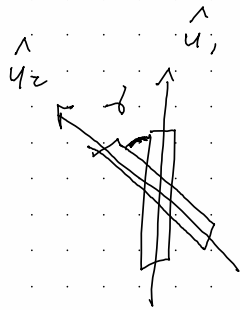
$$\Gamma_{ab}(f) = \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} R_a^A(t, \hat{k}) R_b^{A*}(t, \hat{k})$$

$$\simeq \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} F_a^A(\hat{k}) F_b^A(\hat{k}) \left[1 - e^{-i2\pi f \frac{L_a}{c} (1 + \hat{k} \cdot \hat{p}_a)} \right] \left[1 - e^{+i2\pi f \frac{L_b}{c} (1 + \hat{k} \cdot \hat{p}_b)} \right]$$

↑
except for auto-correlations

$$\rightarrow \Gamma_{ab}(f) \simeq \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} F_a^A(\hat{k}) F_b^A(\hat{k}) [1 + \delta_{ab}]$$

HW problem



$$r_a(t) = \hat{u}_a \cdot \vec{E}(t, \vec{x} = \vec{0}), \quad a=1,2$$

$$\vec{E}(t, \vec{x}) = \int df \int d^2 \Omega_{\vec{k}} \sum_{A=1,2} \tilde{E}_A(f, \hat{k}) \hat{e}_A(\hat{k}) e^{i 2\pi f (t - \vec{k} \cdot \vec{x}/c)}$$

$$\begin{aligned} r_a(t) &= \int df e^{i 2\pi f t} \int d^2 \Omega_{\vec{k}} \sum_A \tilde{E}_A(f, \hat{k}) \hat{u}_a \cdot \hat{e}_A(\hat{k}) \\ &= \int df e^{i 2\pi f t} \int d^2 \Omega_{\vec{k}} \sum_A \tilde{E}_A(f, \hat{k}) R_a^A(\hat{k}) \end{aligned}$$

So: $R_a^A(\hat{k}) = \hat{u}_a \cdot \hat{e}_A(\hat{k})$

$$\hat{e}_1(\hat{k}) = -\hat{\phi}, \quad \hat{e}_2(\hat{k}) = -\hat{\theta}, \quad \hat{k} = -\hat{r}$$

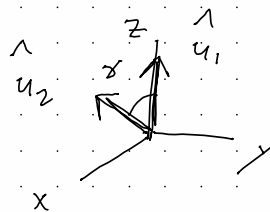
$$\hat{r} = \sin\theta \cos\phi \hat{x} + \sin\theta \sin\phi \hat{y} + \cos\theta \hat{z} = -\hat{k}$$

$$\hat{\theta} = \cos\theta \cos\phi \hat{x} + \cos\theta \sin\phi \hat{y} - \sin\theta \hat{z} = -\hat{m}$$

$$\hat{\phi} = -\sin\phi \hat{x} + \cos\phi \hat{y} = -\hat{\chi}$$

$$\hat{u}_1 = \hat{z}$$

$$\hat{u}_2 = \sin \gamma \hat{x} + \cos \gamma \hat{z}$$



Thus,

$$\Gamma_{12} = \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} \sum_A R_1^A(\hat{k}) R_2^A(\hat{k})$$

$$= \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} \sum_A \hat{u}_1 \cdot \hat{e}_A(\hat{k}) \hat{u}_2 \cdot \hat{e}_A(\hat{k})$$

$$\hat{u}_1 \cdot \hat{e}_1(\hat{k}) = -\hat{z} \cdot \hat{\phi} = 0$$

$$\hat{u}_1 \cdot \hat{e}_2(\hat{k}) = -\hat{z} \cdot \hat{\theta} = \sin \theta$$

$$\hat{u}_2 \cdot \hat{e}_1(\hat{k}) = -(\sin \gamma \hat{x} + \cos \gamma \hat{z}) \cdot \hat{\phi} = +\sin \gamma \sin \phi$$

$$\hat{u}_2 \cdot \hat{e}_2(\hat{k}) = -(\sin \gamma \hat{x} + \cos \gamma \hat{z}) \cdot \hat{\theta} = -\sin \gamma \cos \theta \cos \phi + \cos \gamma \sin \theta$$

$$\Gamma_{12} = \frac{1}{8\pi} \int d^2\Omega_{\hat{k}} \sin \theta (-\sin \gamma \cos \theta \cos \phi + \cos \gamma \sin \theta)$$

$$= -\sin \gamma \left(\frac{1}{8\pi} \right) \int d^2\Omega_{\hat{k}} \sin \theta \cos \theta \cos \phi + \cos \gamma \left(\frac{1}{8\pi} \right) \int d^2\Omega_{\hat{k}} \sin^2 \theta$$

Now:
$$\int d^2 \Omega_H \sin \theta \cos \theta \cos \phi = \int_0^{2\pi} d\phi \int_{-1}^1 d(\cos \theta) \sin \theta \cos \theta \cos \phi$$

$$= 0 \quad \left(\text{since } \int_0^{2\pi} d\phi \cos \phi = 0 \right)$$

$$\begin{aligned} \int d^2 \Omega_H \sin^2 \theta &= \int_0^{2\pi} d\phi \int_{-1}^1 d(\cos \theta) \sin^2 \theta \\ &= 2\pi \int_{-1}^1 dx (1-x^2) \\ &= 2\pi \left(x - \frac{x^3}{3} \right) \Big|_{-1}^1 \\ &= 4\pi \left(1 - \frac{1}{3} \right) \\ &= \frac{8\pi}{3} \end{aligned}$$

Thus

$$\Gamma_{12} = \cos \gamma \left(\frac{1}{8\pi} \right) \frac{8\pi}{3} = \boxed{\frac{1}{3} \cos \gamma}$$

Homework problem: Breathing mode / Scalar transverse

$$e_{ij}^B = \hat{l}_i \hat{l}_j + \hat{m}_i \hat{m}_j$$

$$e_{ij}^+ = \hat{l}_i \hat{l}_j - \hat{m}_i \hat{m}_j$$

$$e_{ij}^x = \hat{l}_i \hat{m}_j + \hat{m}_i \hat{l}_j$$

$$\Gamma_{ab}(t) = \frac{1}{8\pi} \int d^2 \Omega_{\hat{r}} R_a^B(t, \hat{r}) R_b^{B*}(t, \hat{r}) \simeq \frac{1}{8\pi}$$

$$\simeq \frac{1}{8\pi} \int d^2 \Omega_{\hat{r}} F_a^B(\hat{r}) F_b^B(\hat{r}) (1 + \delta_{ab})$$

$$F_a^B(\hat{r}) = \frac{1}{2} \frac{p_a^i p_a^j e_{ij}^B(\hat{r})}{(1 + \hat{p}_a \cdot \hat{r})}$$

$$\begin{aligned} \hat{r} &= -\hat{r} \\ \hat{l} &= -\hat{l} \\ \hat{m} &= -\hat{m} \end{aligned}$$

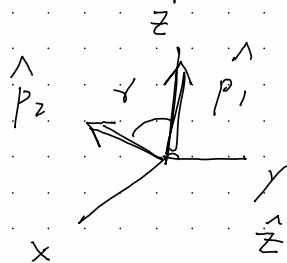


Take polar, a and b along \hat{z} -axis and in xy -plane:

$$F_1^B(\hat{r}) = \frac{1}{2} \frac{\hat{z}^i \hat{z}^j}{(1 + \hat{z} \cdot \hat{r})} e_{ij}^B(\hat{r})$$

$$\hat{z}^i \hat{z}^j e_{ij}^B(\hat{r}) = \hat{z}^i \hat{z}^j (\hat{l}_i \hat{l}_j + \hat{m}_i \hat{m}_j)$$

$$= (\hat{z} \cdot \hat{l})^2 + (\hat{z} \cdot \hat{m})^2$$



$$\hat{z} \cdot \hat{l} = 0$$

$$\rightarrow \hat{z} \cdot \hat{z} e_{ij}^B = \sin^2 \theta$$

$$\hat{z} \cdot \hat{m} = \sin \theta$$

$$\rightarrow F_1^B(\hat{k}) = \frac{1}{z} \frac{\sin^2 \theta}{1 - \cos \theta}$$

$$\hat{z} \cdot \hat{k} = -\cos \theta$$

$$= \frac{1}{z} \left(\frac{1 - \cos^2 \theta}{1 - \cos \theta} \right)$$

$$= \boxed{\frac{1}{z} (1 + \cos \theta)}$$

$$F_z^B(\hat{k}) = \frac{1}{z} \frac{(\cos \gamma \hat{z} \cdot \hat{l} + \sin \gamma \hat{x} \cdot \hat{l})(\cos \gamma \hat{z} \cdot \hat{m} + \sin \gamma \hat{x} \cdot \hat{m})}{1 + (\cos \gamma \hat{z} \cdot \hat{k} + \sin \gamma \hat{x} \cdot \hat{k})} e_{ij}^B(\hat{k})$$

$l_i l_j + m_i m_j$

Now, $\hat{z} \cdot \hat{l} = 0$, $\hat{z} \cdot \hat{m} = \sin \theta$, $\hat{z} \cdot \hat{k} = -\cos \theta$

$$\hat{x} \cdot \hat{l} = \sin \phi, \quad \hat{x} \cdot \hat{m} = -\cos \theta \cos \phi, \quad \hat{x} \cdot \hat{k} = -\sin \theta \cos \phi$$

Denominator: $1 - \cos \gamma \cos \theta - \sin \gamma \sin \theta \cos \phi$

$$\begin{aligned}
\text{Numerator} &= (\cos\gamma \hat{z}' + \sin\gamma \hat{x}') (\cos\gamma \hat{z}'' + \sin\gamma \hat{x}'') (l, l, + m, m,) \\
&= \left(\cos^2\gamma \hat{z}' \hat{z}'' + \sin^2\gamma \hat{x}' \hat{x}'' + \sin\gamma \cos\gamma (\hat{z}' \hat{x}'' + \hat{z}'' \hat{x}') \right) \\
&\quad (l, l, + m, m,) \\
&= \cos^2\gamma \underbrace{(\hat{z}' \hat{z}'')^2}_{\sin^2\phi} + \sin^2\gamma \underbrace{(\hat{x}' \hat{x}'')^2}_{\sin^2\phi} + 2 \sin\gamma \cos\gamma (\hat{z}' \hat{x}'') (\hat{x}' \hat{z}') \\
&\quad + \cos^2\gamma (\hat{z}' \hat{m}'')^2 + \sin^2\gamma (\hat{x}' \hat{m}'')^2 + 2 \sin\gamma \cos\gamma (\hat{z}' \hat{m}'') (\hat{x}' \hat{m}') \\
&= \sin^2\gamma \sin^2\phi + \cos^2\gamma \sin^2\theta + \sin^2\gamma \cos^2\theta \cos^2\phi \\
&\quad + 2 \sin\gamma \cos\gamma \sin\theta (-\cos\theta \cos\phi)
\end{aligned}$$

$$\begin{aligned}
&I_{h\nu}, \\
F_z^B(\hat{H}) &= \frac{1}{2} \frac{\sin^2\gamma \sin^2\phi + \cos^2\gamma \sin^2\theta + \sin^2\gamma \cos^2\theta \cos^2\phi - 2 \sin\gamma \cos\gamma \sin\theta \cos\theta \cos\phi}{(1 - \cos\gamma \cos\theta - \sin\gamma \sin\theta \cos\phi)}
\end{aligned}$$

No. 1,

Note:

$$(1 - \cos \gamma \cos \theta - \sin \gamma \sin \theta \cos \phi) (1 + \cos \gamma \cos \theta + \sin \gamma \sin \theta \cos \phi)$$

$$= 1 + \cancel{\cos \gamma \cos \theta} + \cancel{\sin \gamma \sin \theta \cos \phi} - \cancel{\cos \gamma \cos \theta} - \cos^2 \gamma \cos^2 \theta - \sin \gamma \cos \gamma \sin \theta \cos \theta \cos \phi - \cancel{\sin \gamma \sin \theta \cos \phi} - \sin \gamma \cos \gamma \sin \theta \cos \theta \cos \phi - \sin^2 \gamma \sin^2 \theta \cos^2 \phi$$

$$= 1 - \cos^2 \gamma \cos^2 \theta - \sin^2 \gamma \sin^2 \theta \cos^2 \phi - 2 \sin \gamma \cos \gamma \sin \theta \cos \theta \cos \phi$$

\uparrow \uparrow
 $(1 - \sin^2 \theta)$ $(1 - \cos^2 \theta)$

$$= \underline{1 - \cos^2 \gamma} + \cos^2 \gamma \sin^2 \theta - \sin^2 \gamma \cos^2 \phi + \sin^2 \gamma \cos^2 \theta \cos^2 \phi - 2 \sin \gamma \cos \gamma \sin \theta \cos \theta \cos \phi$$

$$= \sin^2 \gamma (1 - \cos^2 \phi) + \cos^2 \gamma \sin^2 \theta + \sin^2 \gamma \cos^2 \theta \cos^2 \phi - 2 \sin \gamma \cos \gamma \sin \theta \cos \theta \cos \phi$$

$$= \sin^2 \gamma \sin^2 \phi + \cos^2 \gamma \sin^2 \theta + \sin^2 \gamma \cos^2 \theta \cos^2 \phi - 2 \sin \gamma \cos \gamma \sin \theta \cos \theta \cos \phi$$

$$= \text{numerator}$$

$$\text{So } F_2^B(\hat{n}) = \frac{1}{2} (1 + \cos\gamma \cos\theta + \sin\gamma \sin\theta \cos\phi)$$

Recall, $F_1^B(\hat{n}) = \frac{1}{2} (1 + \cos\theta)$

thus,

$$\Gamma_{12} = \frac{1}{8\pi} \int d^2\Omega_n \frac{1}{4} (1 + \cos\theta) (1 + \cos\gamma \cos\theta + \sin\gamma \sin\theta \cos\phi)$$

$$= \frac{1}{32\pi} \int_0^{2\pi} d\phi \int_{-1}^1 dx (1+x) (1 + \cos\gamma x + \sin\gamma \sqrt{1-x^2} \cos\phi)$$

$x = \cos\theta$

when integrated
 $\int_0^{2\pi} d\phi$

$$= \frac{1}{32\pi} (2\pi) \int_{-1}^1 dx (1+x) (1 + \cos\gamma x)$$

$$= \frac{1}{16} \int_{-1}^1 dx (1 + (1 + \cos\gamma)x + \cos\gamma x^2)$$

$$= \frac{1}{16} \left[x + \frac{1}{2} + \cos\gamma \frac{x^3}{3} \right]_{-1}^1 = \frac{1}{8} \left[1 + \frac{1}{3} \cos\gamma \right]$$