

# Risk Quadrangles

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# Financial Analysts

*Goal:* We want  $\mathcal{R} : \tilde{X} \rightarrow \mathbb{R}$  that represents the "hazard" for  $\tilde{X} : \Omega \rightarrow \mathbb{R}$

# Insurance Providers

*Goal:* We want  $\mathcal{D} : \tilde{X} \rightarrow \mathbb{R}$  that represents the "uncertainty" for  $\tilde{X} : \Omega \rightarrow \mathbb{R}$

# ML Practitioners

*Goal:* We want  $\mathcal{E} : \tilde{X} \rightarrow \mathbb{R}$  that represents the "non-zeroness" for  $\tilde{X} : \Omega \rightarrow \mathbb{R}$

## Risk Quadrangle

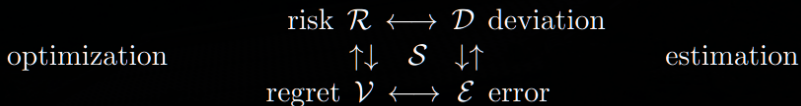
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**Observation: The three goals are not independent**

# Risk Quadrangle



**Diagram 1: The Fundamental Risk Quadrangle**

# Risk Quadrangle

$\mathcal{R}(X)$  provides a numerical surrogate for the overall hazard in  $X$ ,  
 $\mathcal{D}(X)$  measures the “nonconstancy” in  $X$  as its uncertainty,  
 $\mathcal{E}(X)$  measures the “nonzeroness” in  $X$ ,  
 $\mathcal{V}(X)$  measures the “regret” in facing the mix of outcomes of  $X$ ,  
 $\mathcal{S}(X)$  is the “statistic” associated with  $X$  through  $\mathcal{E}$  and  $\mathcal{V}$ .

**Diagram 2: The Quantifications in the Quadrangle.**

# Fundamental Theorem of Risk Quadrangles

$$\mathcal{R}(X) = EX + \mathcal{D}(X), \quad \mathcal{D}(X) = \mathcal{R}(X) - EX$$

$$\mathcal{V}(X) = EX + \mathcal{E}(X), \quad \mathcal{E}(X) = \mathcal{V}(X) - EX$$

$$\mathcal{R}(X) = \min_C \{ C + \mathcal{V}(X - C) \}, \quad \mathcal{D}(X) = \min_C \{ \mathcal{E}(X - C) \}$$

$$\operatorname{argmin}_C \{ C + \mathcal{V}(X - C) \} = \mathcal{S}(X) = \operatorname{argmin}_C \{ \mathcal{E}(X - C) \}$$

**Diagram 3: The General Relationships**



## Example Risk Quadrangle

**Example 1: A Mean-Based Quadrangle** (with  $\lambda > 0$  as a scaling parameter)

$$\mathcal{S}(X) = EX = \mu(X) = \text{mean}$$

$$\mathcal{R}(X) = \mu(X) + \lambda \sigma(X) = \text{safety margin tail risk}$$

$$\mathcal{D}(X) = \lambda \sigma(X) = \text{standard deviation, scaled}$$

$$\mathcal{V}(X) = \mu(X) + \lambda \|X\|_2 = L^2\text{-regret, scaled}$$

$$\mathcal{E}(X) = \lambda \|X\|_2 = L^2\text{-error, scaled}$$

## Example Risk Quadrangle

**Example 2: A Quantile-Based Quadrangle** (at any confidence level  $\alpha \in (0, 1)$ )

$$\mathcal{S}(X) = \text{VaR}_\alpha(X) = q_\alpha(X) = \text{quantile}$$

$$\mathcal{R}(X) = \text{CVaR}_\alpha(X) = \bar{q}_\alpha(X) = \text{superquantile}$$

$$\mathcal{D}(X) = \text{CVaR}_\alpha(X - EX) = \bar{q}_\alpha(X - EX) = \text{superquantile-deviation}$$

$$\mathcal{V}(X) = \frac{1}{1-\alpha} EX_+ = \text{average absolute loss, scaled}^{17}$$

$$\mathcal{E}(X) = E\left[\frac{\alpha}{1-\alpha} X_+ + X_-\right] = \text{normalized Koenker-Bassett error}$$

# Elicitability

Risk measure  $\mathcal{R}$  is **elicitable** if there exists a function  $S$  such that

$$\mathcal{R}(\tilde{X}) = \arg \min_{z \in \mathbb{R}} \mathbb{E}[S(\tilde{X}, z)]$$

**Example:**  $\mathcal{R}(\tilde{X}) = \mathbb{E}(\tilde{X})$  is elicitable with  $S(\tilde{X}, z) = (\tilde{X} - z)^2$

# Regression

minimize  $\mathcal{E}(Z_f)$  over all  $f \in \mathcal{C}$ , where  $Z_f = Y - f(X_1, \dots, X_n)$ ,

# Regression

**Regression Theorem.** Consider problem (5.6) for random variables  $X_1, \dots, X_n$  and  $Y$  in the case of  $\mathcal{E}$  being a regular measure of error and  $\mathcal{C}$  being a class of functions  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$f \in \mathcal{C} \implies f + C \in \mathcal{C} \text{ for all } C \in \mathbb{R}. \quad (5.8)$$

Let  $\mathcal{D}$  and  $\mathcal{S}$  correspond to  $\mathcal{E}$  as in the Quadrangle Theorem. Problem (5.6) is equivalent then to:

$$\text{minimize } \mathcal{D}(Z_f) \text{ over all } f \in \mathcal{C} \text{ such that } 0 \in \mathcal{S}(Z_f), \quad (5.9)$$

# Elicitability

$$\begin{aligned}\text{CVaR}_\alpha(X) &= \min_C \left\{ C + \frac{1}{1-\alpha} E[X - C]_+ \right\}, \\ \text{VaR}_\alpha(X) &= \operatorname{argmin}_C \left\{ C + \frac{1}{1-\alpha} E[X - C]_+ \right\}.\end{aligned}$$