Risk Quadrangles

Gersi Doko

Department of Computer Science
University of New Hampshire

Financial Analysts

Goal: We want $\mathcal{R} : \tilde{X} \to \mathbb{R}$ that represents the "hazard" for $\tilde{X} : \Omega \to \mathbb{R}$

Insurance Providers

Goal: We want $\mathcal{D} : \tilde{X} \to \mathbb{R}$ that represents the "uncertainty" for $\tilde{X} : \Omega \to \mathbb{R}$

ML Practitioners

Goal: We want $\mathcal{E} : \tilde{X} \to \mathbb{R}$ that represents the "non-zeroness" for $\tilde{X} : \Omega \to \mathbb{R}$

Risk Quadrangle

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Observation: The three goals are not independent

Risk Quadrangle

$$
\begin{array}{ccccccc}
 & & \text{risk } \mathcal{R} & \longleftrightarrow & \mathcal{D} \text{ deviation} & & \\
\text{optimization} & & \uparrow\downarrow & \mathcal{S} & \downarrow\uparrow & & \text{estimation} \\
 & & \text{regret } \mathcal{V} & \longleftrightarrow & \mathcal{E} \text{ error} & &
\end{array}
$$

Diagram 1: The Fundamental Risk Quadrangle

Risk Quadrangle

$\mathcal{R}(X)$ provides a numerical surrogate for the overall hazard in X,
$\mathcal{D}(X)$ measures the "nonconstancy" in X as its uncertainty,
$\mathcal{E}(X)$ measures the "nonzeroness" in X,
$\mathcal{V}(X)$ measures the "regret" in facing the mix of outcomes of X,
$\mathcal{S}(X)$ is the "statistic" associated with X through $\mathcal{E}$ and $\mathcal{V}$.

Diagram 2: The Quantifications in the Quadrangle.

Fundamental Theorem of Risk Quadrangles

$$\mathcal{R}(X) = EX + \mathcal{D}(X), \qquad \mathcal{D}(X) = \mathcal{R}(X) - EX$$

$$\mathcal{V}(X) = EX + \mathcal{E}(X), \qquad \mathcal{E}(X) = \mathcal{V}(X) - EX$$

$$\mathcal{R}(X) = \min_C \{\, C + \mathcal{V}(X - C) \,\}, \qquad \mathcal{D}(X) = \min_C \{\, \mathcal{E}(X - C) \,\}$$

$$\operatorname*{argmin}_C \{\, C + \mathcal{V}(X - C) \,\} = \mathcal{S}(X) = \operatorname*{argmin}_C \{\, \mathcal{E}(X - C) \,\}$$

Diagram 3: The General Relationships

Example Risk Quadrangle

Example 1: A Mean-Based Quadrangle (with $\lambda > 0$ as a scaling parameter)

$$\mathcal{S}(X) = EX = \mu(X) = \text{mean}$$

$$\mathcal{R}(X) = \mu(X) + \lambda\,\sigma(X) = \text{safety margin tail risk}$$

$$\mathcal{D}(X) = \lambda\,\sigma(X) = \text{standard deviation, scaled}$$

$$\mathcal{V}(X) = \mu(X) + \lambda||X||_2 = L^2\text{-regret, scaled}$$

$$\mathcal{E}(X) = \lambda||X||_2 = L^2\text{-error, scaled}$$

Example Risk Quadrangle

Example 2: A Quantile-Based Quadrangle (at any confidence level $\alpha \in (0,1)$)

$$\mathcal{S}(X) = \mathrm{VaR}_\alpha(X) = q_\alpha(X) = \text{quantile}$$

$$\mathcal{R}(X) = \mathrm{CVaR}_\alpha(X) = \bar{q}_\alpha(X) = \text{superquantile}$$

$$\mathcal{D}(X) = \mathrm{CVaR}_\alpha(X - EX) = \bar{q}_\alpha(X - EX) = \text{superquantile-deviation}$$

$$\mathcal{V}(X) = \tfrac{1}{1-\alpha} EX_+ = \text{average absolute loss, scaled}^{17}$$

$$\mathcal{E}(X) = E[\tfrac{\alpha}{1-\alpha} X_+ + X_-] = \text{normalized Koenker-Bassett error}$$

Elicitability

Risk measure $\mathcal{R}$ is **elicitable** if there exists a function S such that

$$\mathcal{R}(\tilde{X}) = \arg\min_{z \in \mathbb{R}} \mathbb{E}[S(\tilde{X}, z)]$$

Example: $\mathcal{R}(\tilde{X}) = \mathbb{E}(\tilde{X})$ is elicitable with $S(\tilde{X}, z) = (\tilde{X} - z)^2$

Regression

$$\text{minimize } \mathcal{E}(Z_f) \text{ over all } f \in \mathcal{C}, \text{ where } Z_f = Y - f(X_1, \ldots, X_n),$$

Regression

Regression Theorem. *Consider problem (5.6) for random variables $X_1, \dots, X_n$ and Y in the case of $\mathcal{E}$ being a regular measure of error and $\mathcal{C}$ being a class of functions $f : I\!R^n \to I\!R$ such that*

$$f \in \mathcal{C} \implies f + C \in \mathcal{C} \text{ for all } C \in I\!R. \tag{5.8}$$

Let $\mathcal{D}$ and $\mathcal{S}$ correspond to $\mathcal{E}$ as in the Quadrangle Theorem. Problem (5.6) is equivalent then to:

$$\text{minimize } \mathcal{D}(Z_f) \text{ over all } f \in \mathcal{C} \text{ such that } 0 \in \mathcal{S}(Z_f), \tag{5.9}$$

Elicitability

$$\mathrm{CVaR}_\alpha(X) = \min_C \left\{ C + \frac{1}{1-\alpha} E[X-C]_+ \right\},$$
$$\mathrm{VaR}_\alpha(X) = \operatorname*{argmin}_C \left\{ C + \frac{1}{1-\alpha} E[X-C]_+ \right\}.$$