An Asian option written on an asset S_t and on an average value

$$A_t = \frac{1}{t}\int_0^t S_\tau \, d\tau$$

and payoff $\varphi(S, A)$ is worth $V_t = E[\mathrm{e}^{-r(T-t)}\varphi(S_T, A_T)]$. The price is also given by a function $V = V(t, S, A)$ such that

$$\begin{cases} -\partial_t V - \dfrac{1}{2}\sigma^2 S^2 \partial_{SS} V - rS\partial_S V - \dfrac{S-A}{t}\partial_A V + rV = 0, \\ V(T, S, A) = \varphi(S, A), \end{cases} \tag{1}$$

for $S > 0$, $A > 0$ and $t \in [0, T[$.

1. By considering the two-dimensional process $X_t := (S_t, A_t)$, recover the PDE (1).

2. We consider that the price of the option at the maturity date T is $\varphi = (A_T - K)_+$ (case of the fixed-strike call). Check that this corresponds to a right to buy at the price $K - A_T + S_T$ at maturity.

We make the change of variable $x := \dfrac{K - tA/T}{S}$ and we look for a particular solution in the form $V(t, S, A) = Sf(T - t, x)$. We shall start by showing that f must satisfy the following PDE:

$$\begin{cases} \partial_t f - \dfrac{1}{2}\sigma^2 x^2 \partial_{xx} f + \left(\dfrac{1}{T} + rx\right)\partial_x f = 0, \\ f(0, x) = x_- = \max(-x, 0). \end{cases} \tag{2}$$

The value sought is then $V(0, S, S) = Sf(T, x = K/S)$.

3. In the case where the initial condition (2) is $g(0, x) := -x$, determine an analytic solution of the equation of the form $g(t, x) = xa(t) + b(t)$ (compute explicitly $a(t)$ and $b(t)$). Next, for the case of $f(0, x) := x_-$, we shall admit that $\lim_{x\to+\infty} f(t, x) = 0$ and that $f(t, x) \sim g(t, x)$ when $x \to -\infty$. Deduce an approximate PDE on a spatial domain of the type $[X_{\min}, X_{\max}]$ (with $X_{\min} \le 0$, $X_{\max} > \bar{x} := K/S$), specifying the boundary conditions used.

4. Solve the PDE using finite differences: we shall be interested in particular in the explicit Euler, implicit Euler, and Crank–Nicolson schemes for the discretization in time, and with a centered scheme for the discretization in space.

Pay attention to the boundary conditions, in particular for $x = X_{\min}$, where the solution is not equal to zero. To check that the coded methods are correct, we can first simulate the cases of the initial condition $g(0, x) = -x$ for which an exact solution is available.

5. For each method, we can study stability. It is observed that the numerical methods introduced do not satisfy the stability condition at the nodes close to $x = 0$. What are the consequences from the point of view of the simulations?

6. We can also study whether the numerical order is consistent with the expected theoretical order. To study the numerical order, since the exact solution is not explicitly known, we can use as reference solution the numerical solution obtained with a very fine mesh in space and in time, and compare the values at a fixed point S.

7. Compare with the results of [15] (typically with $r = 0.09$ and $\sigma = 0.3$). (We can also consult [9] for more precise results.)