1 Mathematical Formulation of the Multiplicative Extended Kalman Filter

This section provides a rigorous mathematical formulation of the Multiplicative Extended Kalman Filter (MEKF) used for simultaneous attitude and angular velocity estimation of a rigid body, based on noisy quaternion measurements from visual pose estimation. The MEKF leverages a quaternion representation for attitude and a random walk model for angular velocity, with an error-state approach to maintain quaternion unit norm constraints.

1.1 State Definition

The state vector comprises a unit quaternion representing the attitude and the angular velocity:

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \boldsymbol{\omega} \end{bmatrix},$$

where:

- $\mathbf{q} = [q_w, q_x, q_y, q_z]^T \in \mathbb{R}^4$ is the unit quaternion ($\|\mathbf{q}\|_2 = 1$) describing the rotation from the body frame to the inertial frame.
- $\boldsymbol{\omega} = [\omega_x, \omega_y, \omega_z]^T \in \mathbb{R}^3$ is the angular velocity in the body frame (rad/s).

The quaternion follows the convention q_w as the scalar component, with $\mathbf{q}_v = [q_x, q_y, q_z]^T$ as the vector part.

1.2 Process Model

The process model describes the continuous-time dynamics of the state.

1.2.1 Quaternion Kinematics

The quaternion evolves according to the kinematic equation:

$$\dot{\mathbf{q}}(t) = \frac{1}{2}\Omega(\boldsymbol{\omega}(t))\mathbf{q}(t),$$

where $\Omega(\boldsymbol{\omega})$ is the quaternion rate matrix:

$$\Omega(\boldsymbol{\omega}) = \begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix}.$$

In discrete time with step size Δt, the quaternion is propagated approximately as:

$$\mathbf{q}_{k+1} = \mathbf{q}_k \otimes \exp\left(\frac{1}{2}\boldsymbol{\omega}_k \Delta t\right),$$

where $\otimes$ denotes quaternion multiplication, and the exponential map for a small rotation vector $\mathbf{v} \in \mathbb{R}^3$ is:

$$\exp(\mathbf{v}) = \begin{bmatrix} \cos(\|\mathbf{v}\|) \\ \frac{\mathbf{v}}{\|\mathbf{v}\|}\sin(\|\mathbf{v}\|) \end{bmatrix}, \quad \text{if } \|\mathbf{v}\| > 0,$$

or $\exp(\mathbf{0}) = [1, 0, 0, 0]^T$. The quaternion is normalized after propagation to ensure $\|\mathbf{q}_{k+1}\|_2 = 1$.

1.2.2 Angular Velocity Dynamics

The angular velocity is modeled as a damped random walk:

$$\dot{\boldsymbol{\omega}}(t) = -\alpha\boldsymbol{\omega}(t) + \mathbf{w}_\omega(t),$$

where $\alpha \geq 0$ is the damping factor (s^{-1}), and $\mathbf{w}_\omega(t) \in \mathbb{R}^3$ is zero-mean Gaussian process noise with covariance:

$$\mathbb{E}[\mathbf{w}_\omega(t)\mathbf{w}_\omega^T(\tau)] = \mathbf{Q}_\omega\delta(t-\tau).$$

In discrete time, this becomes:

$$\boldsymbol{\omega}_{k+1} = e^{-\alpha\Delta t}\boldsymbol{\omega}_k + \mathbf{w}_{\omega,k},$$

where $\mathbf{w}_{\omega,k} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_\omega\Delta t)$.

1.2.3 Error State

To handle the quaternion's unit norm constraint, the MEKF uses an error-state vector:

$$\delta\mathbf{x} = \begin{bmatrix} \delta\boldsymbol{\theta} \\ \delta\boldsymbol{\omega} \end{bmatrix} \in \mathbb{R}^6,$$

where:

- $\delta\boldsymbol{\theta} \in \mathbb{R}^3$ is the small-angle rotation vector representing the error quaternion $\delta\mathbf{q} \approx [1, \frac{1}{2}\delta\boldsymbol{\theta}^T]^T$, such that $\mathbf{q}_{\text{true}} = \delta\mathbf{q} \otimes \mathbf{q}_{\text{est}}$.
- $\delta\boldsymbol{\omega} = \boldsymbol{\omega}_{\text{true}} - \boldsymbol{\omega}_{\text{est}} \in \mathbb{R}^3$ is the angular velocity error.

The error-state dynamics are linearized as:

$$\dot{\delta\mathbf{x}}(t) = \mathbf{F}(t)\delta\mathbf{x}(t) + \mathbf{w}(t),$$

where the state transition matrix is:

$$\mathbf{F}(t) = \begin{bmatrix} -[\boldsymbol{\omega}(t)]_\times & \mathbf{I}_{3\times3} \\ \mathbf{0}_{3\times3} & -\alpha\mathbf{I}_{3\times3} \end{bmatrix},$$

with $[\boldsymbol{\omega}]_\times$ the skew-symmetric matrix of $\boldsymbol{\omega}$:

$$[\boldsymbol{\omega}]_\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix},$$

and $\mathbf{w}(t) = [\mathbf{w}_\theta^T, \mathbf{w}_\omega^T]^T$ is process noise with covariance:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_\theta & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{Q}_\omega \end{bmatrix}.$$

The discrete-time transition matrix is approximated as:

$$\boldsymbol{\Phi}_k = \mathbf{I}_{6\times6} + \mathbf{F}_k\Delta t.$$

1.3 Measurement Model

The measurement is a noisy quaternion from visual pose estimation:

$$\mathbf{q}_{\text{meas},k} = \delta\mathbf{q}_{\text{meas},k} \otimes \mathbf{q}_{\text{true},k},$$

where $\delta\mathbf{q}_{\text{meas},k} \approx [1, \frac{1}{2}\mathbf{v}_k^T]^T$, and $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$ is the measurement noise in the rotation vector space, with covariance $\mathbf{R} \in \mathbb{R}^{3\times 3}$. The measurement residual is the error quaternion:

$$\delta\mathbf{q}_k = \mathbf{q}_{\text{meas},k} \otimes \mathbf{q}_{\text{est},k}^{-1} \approx \begin{bmatrix} 1 \\ \frac{1}{2}\delta\boldsymbol{\theta}_k \end{bmatrix},$$

yielding the measurement vector:

$$\mathbf{z}_k = \delta\boldsymbol{\theta}_k \approx 2[\delta\mathbf{q}_k]_v,$$

where $[\delta\mathbf{q}_k]_v$ is the vector part of $\delta\mathbf{q}_k$. The measurement model is:

$$\mathbf{z}_k = \mathbf{H}\delta\mathbf{x}_k + \mathbf{v}_k,$$

with:

$$\mathbf{H} = \begin{bmatrix} \mathbf{I}_{3\times 3} & \mathbf{0}_{3\times 3} \end{bmatrix},$$

and $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R})$.

1.4 MEKF Algorithm

The MEKF operates in two phases: propagation and update.

1.4.1 Propagation

The nominal state is propagated as:

$$\mathbf{q}_{k+1|k} = \mathbf{q}_{k|k} \otimes \exp\left(\frac{1}{2}\boldsymbol{\omega}_{k|k}\Delta t\right),$$

$$\boldsymbol{\omega}_{k+1|k} = e^{-\alpha\Delta t}\boldsymbol{\omega}_{k|k},$$

with $\mathbf{q}_{k+1|k}$ normalized. The error-state covariance is propagated:

$$\mathbf{P}_{k+1|k} = \boldsymbol{\Phi}_k\mathbf{P}_{k|k}\boldsymbol{\Phi}_k^T + \mathbf{Q}\Delta t,$$

and symmetrized: $\mathbf{P}_{k+1|k} = \frac{1}{2}(\mathbf{P}_{k+1|k} + \mathbf{P}_{k+1|k}^T)$.

1.4.2 Update

Given a measurement $\mathbf{q}_{\text{meas},k}$, compute the residual:

$$\delta\mathbf{q}_k = \mathbf{q}_{\text{meas},k} \otimes \mathbf{q}_{k|k-1}^{-1}, \quad \mathbf{z}_k = 2[\delta\mathbf{q}_k]_v.$$

Perform outlier rejection using the Mahalanobis distance:

$$d_k = \mathbf{z}_k^T\mathbf{S}_k^{-1}\mathbf{z}_k,$$

where $\mathbf{S}_k = \mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^T + \mathbf{R}$. If $d_k > \chi^2_{3,0.95} \approx 7.81$, skip the update. Otherwise, compute the Kalman gain:

$$\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}^T\mathbf{S}_k^{-1},$$

and update the error state:

$$\delta \mathbf{x}_{k|k} = \mathbf{K}_k \mathbf{z}_k = \begin{bmatrix} \delta \boldsymbol{\theta}_{k|k} \\ \delta \boldsymbol{\omega}_{k|k} \end{bmatrix}.$$

Update the nominal state:

$$\mathbf{q}_{k|k} = \begin{bmatrix} 1 \\ \frac{1}{2}\delta \boldsymbol{\theta}_{k|k} \end{bmatrix} \otimes \mathbf{q}_{k|k-1}, \quad \mathbf{q}_{k|k} \leftarrow \frac{\mathbf{q}_{k|k}}{\|\mathbf{q}_{k|k}\|_2},$$

$$\boldsymbol{\omega}_{k|k} = \boldsymbol{\omega}_{k|k-1} + \delta \boldsymbol{\omega}_{k|k}.$$

Update the covariance:

$$\mathbf{P}_{k|k} = (\mathbf{I}_{6\times 6} - \mathbf{K}_k \mathbf{H}) \mathbf{P}_{k|k-1} (\mathbf{I}_{6\times 6} - \mathbf{K}_k \mathbf{H})^T + \mathbf{K}_k \mathbf{R} \mathbf{K}_k^T,$$

and symmetrize: $\mathbf{P}_{k|k} = \frac{1}{2}(\mathbf{P}_{k|k} + \mathbf{P}_{k|k}^T)$.

2 Filter Tuning

The MEKF was tuned to estimate the attitude and angular velocity of a rigid body with time-varying angular velocity, based on noisy quaternion measurements from visual pose estimation. The testbench simulates sinusoidal angular velocities:

$$\boldsymbol{\omega}(t) = \begin{bmatrix} 0.5\sin(0.2t) \\ 0.3\cos(0.15t + 0.5) \\ 0.2\sin(0.3t + 1.0) \end{bmatrix} \text{ rad/s},$$

with measurement noise standard deviation of 0.05 rad and 5% outlier probability (scaled by 5). Initial tuning yielded good attitude estimation (mean error < 5 degrees) but high per-axis angular velocity errors (up to 1 rad/s). A convergence test with fading angular velocity ($\boldsymbol{\omega}(t) \to 0$) confirmed the filter's ability to converge, indicating that the errors were due to suboptimal tuning for dynamic conditions. The tuning process focused on reducing per-axis angular velocity errors to <0.2 rad/s while maintaining attitude accuracy.

2.1 Initial Covariance Tuning

The initial error-state covariance reflects uncertainty in the attitude and angular velocity estimates:

$$\mathbf{P}_0 = \text{diag}(0.01, 0.01, 0.01, 1.0, 1.0, 1.0).$$

The attitude components ($\delta\boldsymbol{\theta}$) were initialized with a variance of 0.01 rad^2 ($5.7°$), matching the expected initial quaternion error. The angular velocity components ($\delta\boldsymbol{\omega}$) used 1.0 rad^2/s^2, reflecting high uncertainty due to the noisy initial estimate derived from differencing quaternion measurements. To improve initial convergence, the angular velocity variance was increased to:

$$\mathbf{P}_0 = \text{diag}(0.01, 0.01, 0.01, 2.0, 2.0, 2.0),$$

allowing larger initial corrections and reducing early errors.

2.2 Process Noise Tuning

The process noise covariance accounts for unmodeled dynamics:

$$\mathbf{Q} = \mathrm{diag}(q_\theta, q_\theta, q_\theta, q_\omega, q_\omega, q_\omega).$$

Initially, $q_\theta = 10^{-4}$ rad^2/s and $q_\omega = 0.2$ rad^2/s^3 were used. The small q_θ ensured stable attitude propagation, as the quaternion kinematics are well-modeled. However, the angular velocity's sinusoidal variations (accelerations up to 0.15 rad/s^2) required a higher q_ω. After testing, q_ω was increased to:

$$q_\omega = 1.0\ \mathrm{rad}^2/\mathrm{s}^3,$$

enabling the filter to track rapid changes, reducing per-axis errors and improving correlations (>0.9).

2.3 Measurement Noise Tuning

The measurement noise covariance was set based on the quaternion noise standard deviation ($\sigma = 0.05$ rad):

$$\mathbf{R} = \sigma^2 \mathbf{I}_{3\times 3} = 0.0025 \mathbf{I}_{3\times 3}\ \mathrm{rad}^2.$$

This was found adequate, as the attitude error remained <5 degrees. Outlier rejection using a χ^2 threshold of 7.81 (3 DoF, 95% confidence) mitigated large measurement errors (5% outliers scaled by 5).

2.4 Damping Factor Tuning

The angular velocity's random walk model included a damping factor:

$$\dot{\boldsymbol{\omega}}(t) = -\alpha \boldsymbol{\omega}(t) + \mathbf{w}_\omega(t).$$

Initially, $\alpha = 0.1$ s^{-1} assumed slow angular velocity changes, but the testbench's frequencies (up to 0.3 rad/s) required a less restrictive model. The damping was reduced to:

$$\alpha = 0.01\ \mathrm{s}^{-1},$$

allowing faster variations, which improved tracking without introducing drift.

2.5 Diagnostics and Performance

The tuning was validated using Monte Carlo simulations ($N = 5$) with the testbench. Key metrics included:

- **Attitude Error**: Mean error <5 degrees, compared to 2.86 degrees for noisy measurements.
- **Angular Velocity Error**: Per-axis max errors targeted <0.2 rad/s, mean errors <0.05 rad/s.
- **Correlations**: Per-axis correlations between true and estimated $\boldsymbol{\omega}$ targeted >0.9.
- **Kalman Gain**: Mean norm of $\mathbf{K}[3:6,:] > 10^{-3}$, ensuring $\delta\boldsymbol{\omega}$ updates.
- **Outliers**: <5% of measurements rejected.

Running mean errors and time-windowed correlations were monitored to assess tracking stability. If errors exceeded targets, axis-specific $\mathbf{Q}$ adjustments (e.g., $q_{\omega,x} = 1.5$ for ω_x) were considered, given ω_x's higher amplitude (0.5 rad/s).

2.6 Future Tuning Considerations

If errors remain above 0.2 rad/s, further tuning may involve:

- Increasing q_ω to 2.0 rad^2/s^3 or removing damping ($\alpha = 0$).
- Testing with real visual pose estimation data to match actual dynamics.
- Extending the state to include angular acceleration, modeling $\dot{\boldsymbol{\omega}}$ as a random walk.