Hyperplane Math

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Basics

- A hyperplane is defined by the following equation

$$a_1 x_1 + a_2 x_2 + \ldots + a_N x_N = b$$

$$\underline{a}^T \underline{x} = b = -a_o$$

$$a_o + a_1 x_1 + a_2 x_2 + \ldots + a_N x_N = 0$$

$$[a_o, \underline{a}]^T [1, \underline{x}] = 0$$

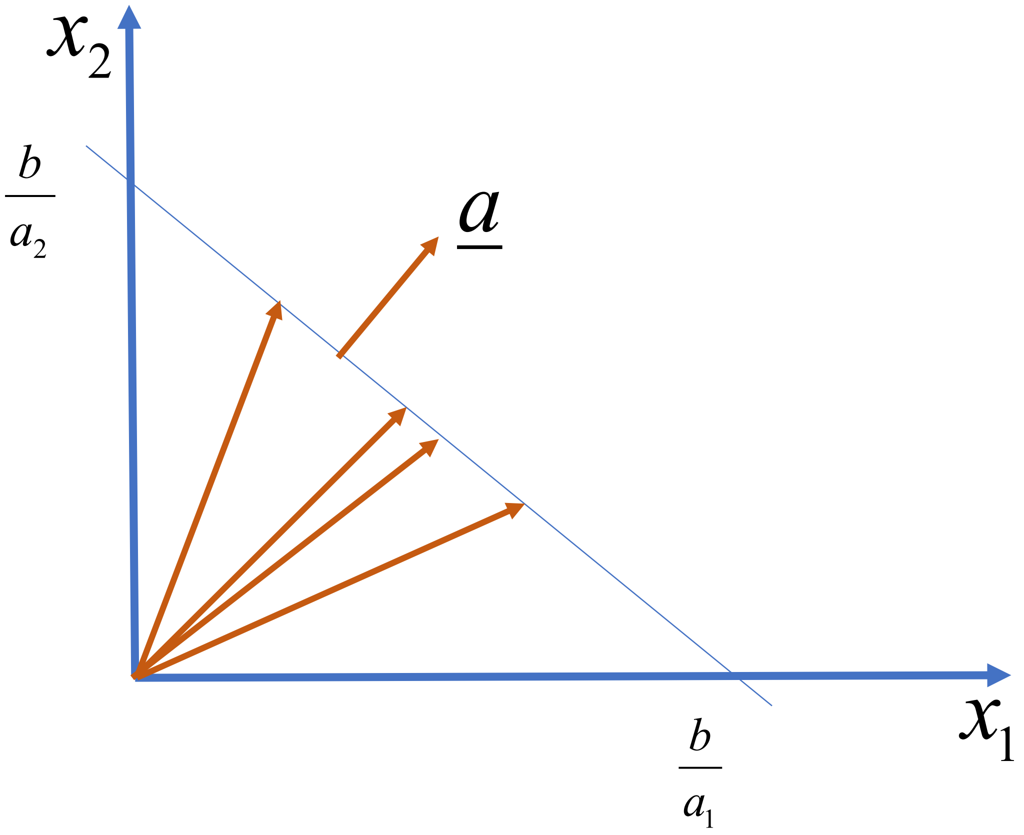
$$\underline{\widehat{a}}^T \underline{\widehat{x}} = 0$$

Vector a defines the hyperplane. It is perpendicular to it!

Compact formulation but careful:

$\underline{\widehat{a}}$ is not orthogonal to the hyperplane

Simple Example



Note that all points on such hyperplane, defined as vectors are usually neither along the hyperplane nor perpendicular to it.

$$a_1 x_1 + a_2 x_2 = b$$

$$\underline{a}^T \underline{x} = b = -a_o$$

$$a_o + a_1 x_1 + a_2 x_2 = 0$$

$$\left[a_o, \underline{a}\right]^T \left[1, \underline{x}\right] = 0$$

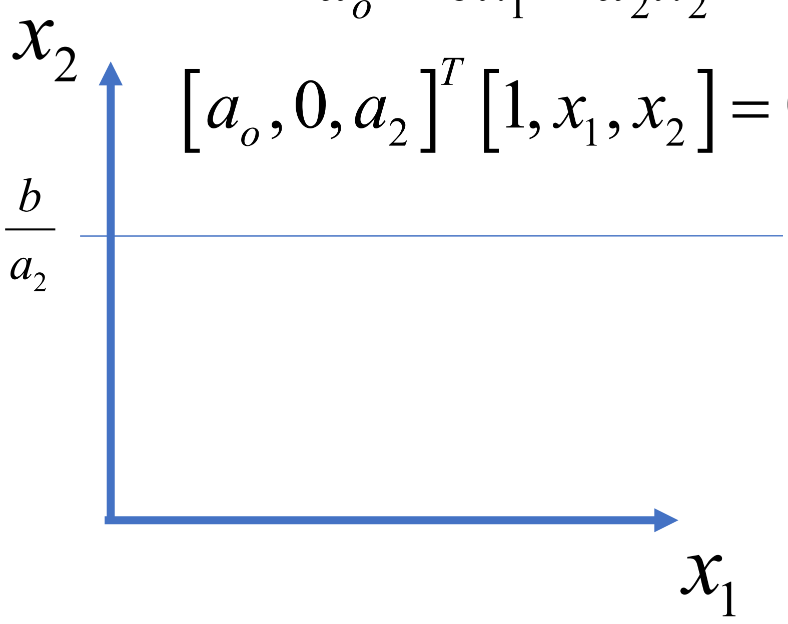
$$\underline{\overset{\frown}{a}}^T \underline{\overset{\frown}{x}} = 0$$

Simple Example: extreme cases

$$0x_1 + a_2 x_2 = b = -a_o$$

$$a_o + 0x_1 + a_2 x_2 = 0$$

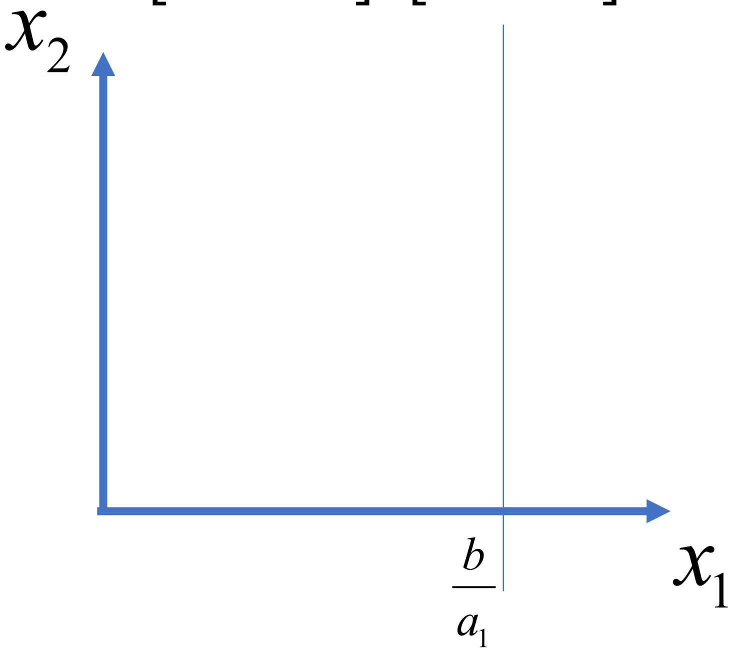
$$\left[a_o, 0, a_2\right]^T \left[1, x_1, x_2\right] = 0$$



$$a_1 x_1 + 0x_2 = b = -a_o$$

$$a_o + a_1 x_1 + a0x_2 = 0$$

$$\left[a_o, a_1, 0\right]^T \left[1, x_1, x_2\right] = 0$$



Orientation

- What does mean $\underline{a}^T \underline{x} > 0$?
- Answer:

$$a_1 x_1 + a_2 x_2 > b$$

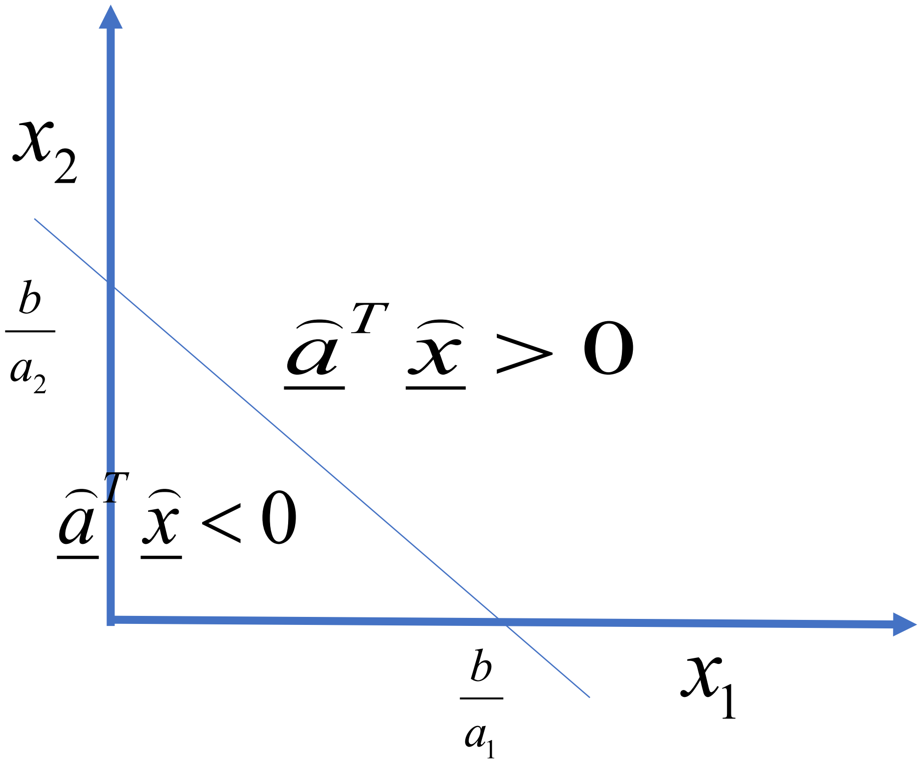
$$\underline{a}^T \underline{x} > b = -a_o$$

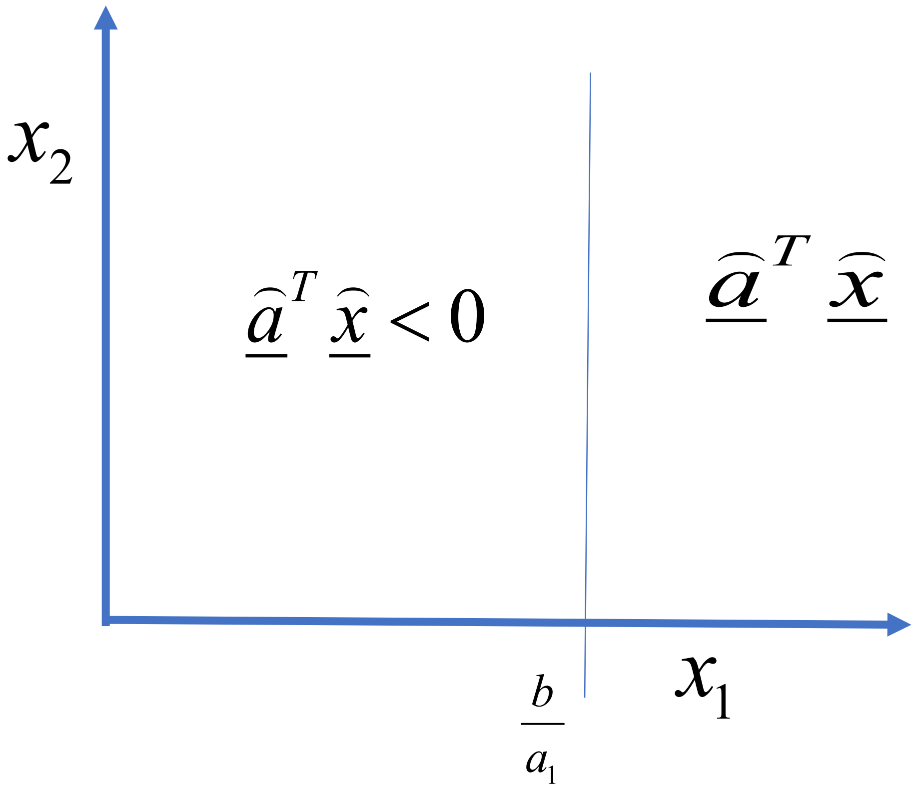
$$a_o + a_1 x_1 + a_2 x_2 > 0$$

$$\left[a_o, \underline{a}\right]^T \left[1, \underline{x}\right] > 0$$

With this we mean all points above the hyperplane

Simple Example





How many points do I need to span it?

- For a hyperplane of order N we need N distinct points $\underline{x}_1, \underline{x}_2, \ldots \underline{x}_N$.

$$\left(\begin{bmatrix} x_{1,1} \\ x_{1,2} \\ \\ x_{1,N} \end{bmatrix} \begin{bmatrix} x_{2,1} \\ x_{2,2} \\ \\ x_{2,N} \end{bmatrix} \qquad \begin{bmatrix} x_{1,N} \\ x_{2,N} \\ \\ x_{N,N} \end{bmatrix} \right) \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = b \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow \underline{a}$$

- “Distinct” means linearly independent vectors!

Points on the hyperplane

- Given N points on an N-dimensional hyperplane, we can find all points on such hyperplane by

$$\underline{x}_1+\alpha_2\left(\underline{x}_2-\underline{x}_1\right)+\alpha_3\left(\underline{x}_3-\underline{x}_1\right)+\ldots+\alpha_N\left(\underline{x}_N-\underline{x}_1\right)$$

- We can test if these points are on the hyperplane by:

$$\underline{a}^T\left\{\underline{x}_1+\alpha_2\left(\underline{x}_2-\underline{x}_1\right)+\alpha_3\left(\underline{x}_3-\underline{x}_1\right)+\ldots+\alpha_N\left(\underline{x}_N-\underline{x}_1\right)\right\}=$$

$$\underline{a}^T\underline{x}_1+\alpha_2\underbrace{\underline{a}^T\left(\underline{x}_2-\underline{x}_1\right)}_{0}+\alpha_3\underbrace{\underline{a}^T\left(\underline{x}_3-\underline{x}_1\right)}_{0}+\ldots+\alpha_N\underbrace{\underline{a}^T\left(\underline{x}_N-\underline{x}_1\right)}_{0}=b$$

Which distance has a given point to it?

- Starting from point p, vector a directly leads to the hyperplane. We thus have to find its length when it hits the hyperplane.

$$\underline{a}^T\left(\alpha\underline{a}+\underline{p}\right)=b$$

$$\alpha=\frac{b-\underline{a}^T\underline{p}}{\underline{a}^T\underline{a}}$$

$$\text{dist}=\frac{\left|b-\underline{a}^T\underline{p}\right|}{\sqrt{\underline{a}^T\underline{a}}}$$

Intersecting Hyperplanes

- Assume we have n<N hyperplanes each of dimension N
- Where do they intersect?

$$\begin{pmatrix} \underline{a}_1^T \\ \underline{a}_2^T \\ \vdots \\ \underline{a}_n^T \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = A\underline{x}$$

- We obtain an underdetermined set of equations, a minimum norm solution for $\underline{x}$ comes from the row space of matrix A added by all possible solutions coming from the nullspace of matrix A.
- In case n=N, only one point remains for intersection.

Colinear Hyperplanes

- have the same coefficients $\underline{a}$ but different values b_i.
- Take two points, one on each hyperplane:

$$\underline{a}^T \underline{x}_1 = b_1$$

$$\underline{a}^T \underline{x}_2 = b_2$$

- Both need to be connected by $\underline{x}_2 - \underline{x}_1 = \alpha\, \underline{a}$:

$$\underline{a}^T \left(\underbrace{\underline{x}_1 - \underline{x}_2}_{\alpha\, \underline{a}} \right) = b_1 - b_2$$

$$\text{dist} = \frac{|b_1 - b_2|}{\sqrt{\underline{a}^T \underline{a}}}$$