Math 115 QR

Alex Hernandez Juarez

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Chapter 1

1.1 Warm up 1

Question 1

Suppose we are excavating a hole that is 10ft long, 6ft wide, and 4ft deep. If the soil weighs 12 lb/ft^3, what is the total weight of soil removed

Solution: There is a uniform density

$$\text{volume: } (4)(6)(10) = 240\text{ft}^3$$
$$\text{weight: } (12\text{lb/ft}^3)(240\text{ft}^3) = 2880\text{lbs}$$

Question 2

Suppose instead the hole is 6ft long, 4ft wide, and 10 ft deep. Which hole takes more energy to dig?

Solution: more energy for (2) since we need to lift the soil out of the hole

1.2 In class notes

Note:-

In physics work = (force)(distance)
(ft * lbs) = (lbs)(ft)
(N * m) = (n)(M)

Example 1.2.1 (Q: For (2), how much work is required to remove the soil)

Slice the shape into layers parallel to the bottom of the hole.
Let x be the depth from the top of the hole ($0 \leqslant x \leqslant 10$) each layer has thicknes Δx.
Volume of the i$^{\text{th}}$ slice = $(4)(6)(\Delta x) = 24\Delta x$ ft^3
Weight of the i$^{\text{th}}$ slice = $12\text{lb/ft}^3(24\Delta x)\text{ft}^3 = 288\Delta x$ lbs
Work need to lift i$^{\text{th}}$ $\approx (288\Delta x \text{ lbs})(x_i\text{ft})$
It is approximate and not exact because no all of the ith slice is lifted the same distance
Total work $\approx \sum_{i=1}^{n} 288x_i\Delta x(\text{ft} * \text{lbs})$
$\int_0^{10} 288x dx \text{ ft} * \text{lbs} = (144x^2)\Big|_0^{10} = 14400 \text{ ft} * \text{lbs}$

Example 1.2.2 (Q: How much work for a 10 by 6 by 4 hole)

Volume of the i$^{\text{th}}$ slice = $(10)(6)(\Delta x) = 60\Delta x$ ft^3
Weight of the i$^{\text{th}}$ slice = $12\text{lb/ft}^3(60\Delta x)\text{ft}^3 = 720\Delta x$ lbs

Work need to lift i$^{\text{th}}$ $\approx$ (720Δx lbs)($x_i$ft)
It is approximate and not exact because no all of the ith slice is lifted the same distance
Total work $\approx \sum_{i=1}^{n} 720 x_i \Delta x$(ft * lbs)
$\int_0^4 720x dx$ ft * lbs = $(360x^2)\Big|_0^4$ = 5120 ft * lbs

Example 1.2.3 (Suppose the density of the soil is given by $p(x) = 8 + 2x$ lb/ft^3. How does the change the answer for the shallow soil?)
Volume of the ith slice = 60Δxft^3
Weight of the ith slice = $(8 + 2x_i)60\Delta x$
Work needed to lift the ith slice $\approx (8 + 2x + i)(60\Delta x)(x_i)$
Total work $\approx \sum_{i=1}^{n}(8 + 2x_i)(60x_i)\Delta x$
$\int_0^4 (8 + 2x_i)(60x)dx$

Example 1.2.4 (Circular corn field of radius 50 ft. The density of the corn in (ears/ft^2) is a function $f(y)$ where y is the distance from the center of the circle. Write an intergral to compute the total yield in ears)
area of i$^{\text{th}}$ slice = $\pi y_i^2 - \pi(y_i - \Delta y)^2$
$= \pi y_i^2 - \pi(y_i^2 - 2y_i\Delta y + (\Delta y)^2)$
$\approx 2\pi y_i \Delta y$
ears in the ith slice = $p(y_i)2\pi Y_i \Delta y$
total number of ears $\approx \sum_{i=1}^{n} p(y_i)2\pi y_i \Delta y$
total number of ears = $\int_0^{50} f(y)2\pi y \Delta y$

Chapter 2

2.1 Warm up

Question 3

Supppose the circular corn field instead has an irrigation ditch running along a diameter. As before the density of corn is a function of the distance from the water source. How must you slice up the region so that the desnity is (approx) constant in each slice?

Solution: You must slice parallel to the water source so it should be sliced vertically.

Question 4

If the density (in ears/ft^2) is given by the function g, write an integral to compute the total number of ears in the field

Solution:

$$\left(\frac{l_i}{2}\right)^2 + y_i = 50^2$$

$$\frac{l_i^2}{4} = 2500 - y_i^2$$

$$l_i = \sqrt{10000 - 4y_i^2}$$

number of ears in i^{th} slice $\approx g(y_i)\sqrt{10000 - 4y_i^2}\Delta y$

total number of ears $\approx 2\sum_{i=1}^{n} g(y_i)\sqrt{10000 - 4y_i^2}\Delta y$

total numebr of ears $= 2\int_0^{50} g(y_i)\sqrt{10000 - 4y_i^2}dy$

Example 2.1.1 (Conical tank full of sludge with density $f(2)$ kg/m^3, where z is depth. Find an integral to compute the towrk done (against gravity) in pumping all the sludge to height p 1 m aove the tank)

Volume of the i^{th} slice $\approx \pi r_i^2 \Delta x$

$\frac{r_i}{6-z_i} = \frac{3}{6}$

$r_i = \frac{1}{2}(6 - z_i) = 3 - \frac{z_i}{2}$

Volume of the i^{th} slice $\pi\left(3 - \frac{}{z}_i 2\right)^2 \Delta z$ m^3

mass of the i^{th} slice $\approx f(z_i)\pi\left(3 - \frac{z_i}{2}\right)^2 \Delta z$ kg

weight of the i^{th} slice $\approx 9.8 f(z_i)\pi\left(3 - \frac{z_i}{2}\right)^2 \Delta z$ N

work for the i^{th} slice $\approx 9.8 f(z_i)\pi\left(3 - \frac{z_i}{2}\right)^2 \Delta z(z_i + 1)$ J

total work = $\sum_{i=1}^{n} 9.8 f(z_i)\pi \left(3 - \frac{z_i}{2}\right)^2 (z_i + 1)\Delta z$ J
total work = $\int_0^6 9.8 f(z_i)\pi \left(3 - \frac{z}{2}\right)^2 (z + 1) dz$ J

Example 2.1.2 (Given a functino f(x), $a \leqslant x \leqslant b$ what is the length of the graph)
length of i^{th} piece $\approx \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \approx \sqrt{(\Delta x + (f'(x_i))\Delta x)^2} = \sqrt{1 + (f'(x_i))^2}\Delta x$
total length = $\sum_{i=1}^{n} \sqrt{1 + (f'(x_i))^2}\Delta x$
total length = $\int_a^b \sqrt{1 + (f'(x_i))^2} dx$

Chapter 3

3.1 Warm up

Question 5

Find the tangent line to $f(x) = \frac{1}{1+x^2}$ at $x = 2$

Solution:

$$f'(x) = \frac{0 \cdot (1+x^2) - 1(2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}$$
$$f'(2) = \frac{-2(2)}{(1+2^2)^2} = \frac{-2}{5^2} = \frac{-2}{25}$$
$$f(2) = \frac{1}{5}$$
$$y - \frac{1}{5} = -\frac{2}{25}(x-2)$$
$$y = \frac{1}{5} - \frac{4}{25}(x-2)$$

Question 6

If we use this line to aproximate f near $x = 2$, will we get an overestimate or an underestimate?

Solution:

$$f''(x) = \frac{-2(1+x^2)^2 - (-2x)2(1+x^2)(2x)}{(1+x^2)^4}$$
$$f''(x) = \frac{-2 - 2x^2 + 8x^2}{(1+x^2)^3}$$
$$f''(x) = \frac{6x^2 - 2}{(1+x^2)^3}$$
$$f'(x) = 0$$
$$0 = 6x^2 - 2$$
$$x = \pm\sqrt{\frac{1}{3}}$$

$L(x)$ underestimates $f(x)$ near $x = 2$

Example 3.1.1 (Find the quadratic function that best approximates $f(x) = \frac{1}{1+x^2}$ near $x = 2$)

Find $Q(x) = C_0 + C_1(x-2) + C_2(x-2)^2$ such that $Q(2) = f(2) = \frac{1}{5}$, $Q'(2) = f'(2) = \frac{-4}{25}$, and $G''(x) = f''(2) = \frac{22}{125}$

$$Q'(x) = C_1 + 2C_2(2-2)$$

$$Q''(x) = 2c_2$$

$$Q(2) = C_0 = \frac{1}{5}$$

$$Q'(2) = C_1 = \frac{-4}{25}$$

$$Q''(2) = 2C_2 = \frac{22}{125} \to C_2 = \frac{11}{125}$$

$$Q(x) = \frac{1}{5} - \frac{4}{25}(x-2) + \frac{11}{125}(x-2)^2$$

Note:-

General formula: $Q(x) = f(a) + f'(a)(x-a) + \frac{f''(x)}{2}(x-1)^2$
Def: The nth degree Taylor polynomial of $f(x)$ based at $x = a$ is:
$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{6}(x-a)^3 + \frac{f^{(4)}(a)}{24}(x-a)^4 + \ldots \frac{f^{(n)}(a)}{n!}(x-a)^n$
$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ Define:
$f^{(0)}(x) = f(x)$ $0! = 1$
$(x-a)^0 = 1$ for all x

Example 3.1.2 (Let $f(x) = \frac{1}{1-x}$, based at $x = 0$. Find $P_4(x)$)

$$f(x) = \frac{1}{1-x}$$

$$f'(x) = \frac{-1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{2}{(1-x)^3}$$

$$f^{(3)}(x) = \frac{6}{(1-x)^4}$$

$$f^{(4)} = \frac{24}{(1-x)^5}$$

Chapter 4

4.1 Warm Up

Question 7

Suppose for a cerntain function f we know that $|f'(x)| \leqslant f$ for all x.
(1) What is the largest possible value of $\left|f(4) - f(1)\right|$?
(2) What is the largest possible value of $|f(b) - f(a)|$ for a given interval $[a, b]$?
(3) Suppose we also know that f(1) = 10. Find upper and lower bound for $f(4)$.

Solution:
(1)
15
(2)
$(b-a)5$
(3)
Upper: 25
Lower: -5

Note:-

Mean value theorem: If f is continuous on $[a, b]$ and differentiable on the (a, b) then there exists a c on (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Taylor's Theorem: If f is continuous on $[a, b]$ and $(n+1)^{\text{st}}$ differentiable on (a, b) then there exists a c in (a, b) such that $f(b) = f(a) + f'(a)(b-a) + \frac{f'(a)}{2}(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$

Mean Value Theorem as the Taylor's Theorem at $n = 0$.

$$f(b) = f(a) + \frac{f'(c)}{1!}(b-a)^1$$

$$f(b) - f(a) = f'(c)(b-a)$$

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$P_n(b)$: Where $P_n(x)$ is n^{th} degree Taylor Poly for $f(x)$ based at $x = a$. So,

$$|f(b) - P_n(b)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}\right| = \frac{\left|f^{(n+1)}(c)\right|(b-a)^{n+1}}{(n+1)!}$$

Error Bound: $|f(x) - P_n(x)| \leqslant \frac{M}{(n+1)!}(x-a)^{n+1}$
n^{th} degree TP based at x = a: where $|f^{(n+1)}(z)| \leqslant M$ for all z between a and x.

Example 4.1.1 (In quiz, we wanted to approximate $\sqrt{3.95}$ using 1^{st} degree TP based at $x = 4$.)

$$P_1(x) = 2 + \frac{1}{4}(x-a)$$

$$\sqrt{3.95} \approx P_1(3.95) = 2 + \frac{1}{4}(3.95 - 4) = 2 - 0.00125 = 1.9875$$

$$f(x) = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$|f''(x)| = \left|\frac{1}{4}x^{-\frac{3}{1}}\right| = \frac{1}{4}x^{-\frac{3}{2}}$$

$$M = \frac{1}{4} : \left|\sqrt{3.95} - 1.9875\right| \leqslant \frac{\frac{1}{4}\,|3.95-4|^2}{2} \approx 0.00004$$

Example 4.1.2 (Find n so that the n^{th} defree TP for $f(x) = \cos x$ based at $x = 0$ approximate cos(0.03) to within 10^{-15})

$$f'(x) = -\sin(x)$$

$$f''(x) = -\cos(x)$$

$$f^{(3)}(x) = \sin(x)$$

$$f^{(4)}(x) = \cos(x)$$

$$\left|f^{(n+1)} \leqslant 1\right| \text{ eveywhere and for every n}$$

$$M = 1$$

$$\cos(0.03) - P_n(0.03) \leqslant \frac{1 \cdot (0.03-0)^{n+1}}{n+1} = \left(\frac{3}{1000}\right)^{n+1} \frac{1}{(n+1)!} \leqslant 10^{-15}$$

Chapter 5

5.1

Warm up

Question 8

(a) Find the 5^{th} degree Taylor Polynomial for $f(x) = e^x$ based at $x = 0$.
(b) Find $P_n(x)$.
(c) Find an upper bound (in terms of n) of the error $\left|f(x) - P_n(x)\right|$, for $x > 0$
(d) What is $\lim_{n\to\infty} |f(x) - P_n(x)$

Solution: (a)

$$f(x) = e^x$$
$$f'(x) = e^x$$
$$f''(x) = e^x$$
$$f^{(2)}(x) = e^x$$
$$f^{(3)}(x) = e^x$$
$$f^{(4)}(x) = e^x$$
$$f^{(5)}(x) = e^x$$
$$P_5(x) = e^0 + e^0(x) + \frac{e^0}{2}(x)^2 + \frac{e^0}{6}(x)^3 + \frac{e^0}{24}(x)^4 + \frac{e^0}{120}(x)^5$$
$$P_5(x) = 1 + (x) + \frac{1}{2}(x)^2 + \frac{1}{6}(x)^3 + \frac{1}{24}(x)^4 + \frac{1}{120}(x)^5$$

(b)

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!}(x)^k$$

(c)

$$|f(x) - P_n(x)| \leqslant \frac{M(x-a)^{n+1}}{(n+1)}$$

$$|f(x) - P_n(x)| \leqslant \frac{e^x(x)^{n+1}}{(n+1)!}, \text{where } |e^z| \leqslant M \text{ for all } z \text{ between } 0 \text{ and } x$$

(d)

$$\lim_{n\to\infty} |f(x) - P_n(x) =?$$

$$\lim_{n\to\infty} \frac{e^x \cdot x^{n+1}}{(n+1)!}$$

$$e^x \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$$

5.2 Infinite Series

Example 5.2.1 (Consider $x = 1$ in the warm up. We showed $\lim_{n\to\infty} |e^1 - P_n(1)| = 0$. ie $\lim_{n\to\infty} |e - \left(1 + 1\frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots + \frac{1}{n!}\right)| = 0$
$\lim_{n\to\infty} \left(1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots + + \frac{1}{n!}\right) = e$)
i.e. $\sum_{k=0}^{\infty} \frac{1}{k!} = e$

Note:-

Def: An infinite series is a sum of the form

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \ldots$$

Def: Given an infinite series $\sum_{k=0}^{\infty} a_k$ the n^{th} partial sum is

$$S_n = \sum_{k=0}^{n} a_k$$

Example 5.2.2 (Example 1)

$$\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \ldots$$

$$S_0 = \sum_{k=0}^{0} \frac{1}{k!} = \frac{1}{0!} = 1$$

$$S_1 = \sum_{k=0}^{1} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} = 2$$

$$S_2 = \sum_{k=0}^{2} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} = \frac{5}{2}$$

$$S_3 = \sum_{k=0}^{3} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} = \frac{8}{3}$$

Example 5.2.3 (Example 2)

$$\sum_{k=0}^{\infty}(-1)^k = 1 - 1 + 1 - 1 + 1 \dots$$

$$S_0 = 1$$
$$S_1 = 1 - 1 = 0$$
$$S_2 = S_1 + 1 = 1$$
$$S_3 = S_2 - 1 = 0$$

Example 5.2.4 (Example 3)

$$\sum_{k=2}^{\infty}\frac{1}{k^2+k}$$

$$S_2 = \frac{1}{6}$$
$$S_3 = S_2 + \frac{1}{3^2+3} = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$$
$$S_4 = S_3 + \frac{1}{16+4} = \frac{1}{12} + \frac{1}{20} = \frac{3}{10}$$

Note:-

Def: $\sum_{k=0}^{\infty} a_k$ converges if $\lim_{n\to\infty} S_n$ exists and is finite, in which case we write

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} S_n$$

if $\lim_{n\to\infty} S_n$ does not exist (including $\pm\infty$) the series diverges
Ex(1)

$$\sum_{k=0}^{\infty}\frac{1}{k!} \text{ converges because } \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{1}{k!} = e$$

Ex(2)