

# Math 115 QR

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# Chapter 1

## 1.1 Warm up 1

### Question 1

Suppose we are excavating a hole that is 10ft long, 6ft wide, and 4ft deep. If the soil weighs 12 lb/ft<sup>3</sup>, what is the total weight of soil removed

**Solution:** There is a uniform density

$$\text{volume: } (4)(6)(10) = 240\text{ft}^3$$

$$\text{weight: } (12\text{lb/ft}^3)(240\text{ft}^3) = 2880\text{lbs}$$

### Question 2

Suppose instead the hole is 6ft long, 4ft wide, and 10 ft deep. Which hole takes more energy to dig?

**Solution:** more energy for (2) since we need to lift the soil out of the hole

## 1.2 In class notes

### Note:-

In physics work = (force)(distance)  
(ft \* lbs) = (lbs)(ft)  
(N \* m) = (n)(M)



**Example 1.2.1: Q: For (2), how much work is required to remove the soil**

Slice the shape into layers parallel to the bottom of the hole.

Let  $x$  be the depth from the top of the hole ( $0 \leq x \leq 10$ ) each layer has thickness  $\Delta x$ .

- Volume of the  $i^{\text{th}}$  slice =  $(4)(6)(\Delta x) = 24\Delta x \text{ ft}^3$
- Weight of the  $i^{\text{th}}$  slice =  $12\text{lb}/\text{ft}^3(24\Delta x)\text{ft}^3 = 288\Delta x\text{lbs}$
- Work needed to lift  $i^{\text{th}} \approx (288\Delta x \text{ lbs})(x_i\text{ft})$

It is approximate and not exact because no all of the  $i^{\text{th}}$  slice is lifted the same distance

- Total work  $\approx \sum_{i=1}^n 288x_i\Delta x(\text{ft} * \text{lbs})$

$$\int_0^{10} 288x dx \text{ft} * \text{lbs} = (144x^2) \Big|_0^{10} = 14400\text{ft} * \text{lbs}$$



**Example 1.2.2: Q: How much work for a 10 by 6 by 4 hole**

- Volume of the  $i^{\text{th}}$  slice =  $(10)(6)(\Delta x) = 60\Delta x \text{ ft}^3$
- Weight of the  $i^{\text{th}}$  slice =  $12\text{lb}/\text{ft}^3(60\Delta x)\text{ft}^3 = 720\Delta x \text{ lbs}$
- Work need to lift  $i^{\text{th}} \approx (720\Delta x \text{ lbs})(x_i\text{ft})$

It is approximate and not exact because no all of the  $i^{\text{th}}$  slice is lifted the same distance

$$\text{Total work} \approx \sum_{i=1}^n 720x_i\Delta x(\text{ft} * \text{lbs})$$

$$\int_0^4 720x dx \text{ft} * \text{lbs} = (360x^2) \Big|_0^4 = 5120\text{ft} * \text{lbs}$$



**Example 1.2.3: Suppose the density of the soil is given by  $p(x) = 8 + 2x \text{ lb}/\text{ft}^3$ . How does the change the answer for the shallow soil?**

- Volume of the  $i^{\text{th}}$  slice =  $60\Delta x\text{ft}^3$
- Weight of the  $i^{\text{th}}$  slice =  $(8 + 2x_i)60\Delta x$
- Work needed to lift the  $i^{\text{th}}$  slice  $\approx (8 + 2x + i)(60\Delta x)(x_i)$
- Total work  $\approx \sum_{i=1}^n (8 + 2x_i)(60x_i)\Delta$

$$\int_0^4 (8 + 2x_i)(60x) dx$$



**Example 1.2.4:** Circular corn field of radius 50 ft. The density of the corn in (ears/ft<sup>2</sup>) is a function  $f(y)$  where  $y$  is the distance from the center of the circle. Write an integral to compute the total yield in ears

- area of  $i^{\text{th}}$  slice =  $\pi y_i^2 - \pi(y_i - \Delta y)^2 = \pi y_i^2 - \pi(y_i^2 - 2y_i\Delta y + (\Delta y)^2)$
- ears in the  $i^{\text{th}}$  slice =  $p(y_i)2\pi y_i\Delta y \approx 2\pi y_i\Delta y$
- total number of ears  $\approx \sum_{i=1}^n p(y_i)2\pi y_i\Delta y$
- total number of ears =  $\int_0^{50} f(y)2\pi y\Delta y$



# Chapter 2

## 2.1 Warm up

### Question 3

Suppose the circular corn field instead has an irrigation ditch running along a diameter. As before the density of corn is a function of the distance from the water source. How must you slice up the region so that the density is (approx) constant in each slice?

**Solution:** You must slice parallel to the water source so it should be sliced vertically.

### Question 4

If the density (in ears/ft<sup>2</sup>) is given by the function  $g$ , write an integral to compute the total number of ears in the field

**Solution:**

$$\left(\frac{l_i}{2}\right)^2 + y_i^2 = 50^2$$

$$\frac{l_i^2}{4} = 2500 - y_i^2$$

$$l_i = \sqrt{10000 - 4y_i^2}$$

$$\text{number of ears in } i^{\text{th}} \text{ slice} \approx g(y_i) \sqrt{10000 - 4y_i^2} \Delta y$$

$$\text{total number of ears} \approx 2 \sum_{i=1}^n g(y_i) \sqrt{10000 - 4y_i^2} \Delta y$$

$$\text{total number of ears} = 2 \int_0^{50} g(y) \sqrt{10000 - 4y^2} dy$$

**Example 2.1.1:** Conical tank full of sludge with density  $f(z)$  kg/m<sup>3</sup>, where  $z$  is depth. Find an integral to compute the work done (against gravity) in pumping all the sludge to height  $p$  1 m above the tank

- Volume of the  $i^{th}$  slice  $\approx \pi r_i^2 \Delta z$
- $\frac{r_i}{6-z_i} = \frac{3}{6}$
- $r_i = \frac{1}{2}(6 - z_i) = 3 - \frac{z_i}{2}$
- Volume of the  $i^{th}$  slice  $\pi \left(3 - \frac{z_i}{2}\right)^2 \Delta z$  m<sup>3</sup>
- mass of the  $i^{th}$  slice  $\approx f(z_i) \pi \left(3 - \frac{z_i}{2}\right)^2 \Delta z$  kg
- weight of the  $i^{th}$  slice  $\approx 9.8 f(z_i) \pi \left(3 - \frac{z_i}{2}\right)^2 \Delta z$  N
- work for the  $i^{th}$  slice  $\approx 9.8 f(z_i) \pi \left(3 - \frac{z_i}{2}\right)^2 \Delta z (z_i + 1)$  J
- total work  $= \sum_{i=1}^n 9.8 f(z_i) \pi \left(3 - \frac{z_i}{2}\right)^2 (z_i + 1) \Delta z$  J
- total work  $= \int_0^6 9.8 f(z) \pi \left(3 - \frac{z}{2}\right)^2 (z + 1) dz$  J



**Example 2.1.2:** Given a function  $f(x)$ ,  $a \leq x \leq b$  what is the length of the graph

- length of  $i^{th}$  piece  $\approx \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \approx \sqrt{(\Delta x + (f'(x_i))\Delta x)^2} = \sqrt{1 + (f'(x_i))^2} \Delta x$
- total length  $= \sum_{i=1}^n \sqrt{1 + (f'(x_i))^2} \Delta x$
- total length  $= \int_a^b \sqrt{1 + (f'(x))^2} dx$



# Chapter 3

## 3.1 Warm up

### Question 5

Find the tangent line to  $f(x) = \frac{1}{1+x^2}$  at  $x = 2$

*Solution:*

$$f'(x) = \frac{0 \cdot (1+x^2) - 1(2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}$$

$$f'(2) = \frac{-2(2)}{(1+2^2)^2} = \frac{-2}{5^2} = \frac{-2}{25}$$

$$f(2) = \frac{1}{5}$$

$$y - \frac{1}{5} = -\frac{2}{25}(x - 2)$$

$$y = \frac{1}{5} - \frac{4}{25}(x - 2)$$

### Question 6

If we use this line to approximate  $f$  near  $x = 2$ , will we get an overestimate or an underestimate?

*Solution:*

$$f''(x) = \frac{-2(1+x^2)^2 - (-2x)2(1+x^2)(2x)}{(1+x^2)^4}$$

$$f''(x) = \frac{-2 - 2x^2 + 8x^2}{(1+x^2)^3}$$

$$f''(x) = \frac{6x^2 - 2}{(1+x^2)^3}$$

$$f'(x) = 0$$

$$0 = 6x^2 - 2$$

$$x = \pm \sqrt{\frac{1}{3}}$$



$L(x)$  underestimates  $f(x)$  near  $x = 2$

**Example 3.1.1:** Find the quadratic function that best approximates  $f(x) = \frac{1}{1+x^2}$  near  $x = 2$

Find  $Q(x) = C_0 + C_1(x - 2) + C_2(x - 2)^2$  such that  $Q(2) = f(2) = \frac{1}{5}$ ,  $Q'(2) = f'(2) = \frac{-4}{25}$ , and  $Q''(2) = f''(2) = \frac{22}{125}$

$$Q'(x) = C_1 + 2C_2(x - 2)$$

$$Q''(x) = 2C_2$$

$$Q(2) = C_0 = \frac{1}{5}$$

$$Q'(2) = C_1 = \frac{-4}{25}$$

$$Q''(2) = 2C_2 = \frac{22}{125} \rightarrow C_2 = \frac{11}{125}$$

$$Q(x) = \frac{1}{5} - \frac{4}{25}(x - 2) + \frac{11}{125}(x - 2)^2$$



**Note:-**

General formula for Taylor Polynomial:  $Q(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$



### Definition 3.1.1: Taylor Polynomial

The  $n^{th}$  degree Taylor polynomial of  $f(x)$  based at  $x = a$  is:

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2 + \frac{f^{(3)}(a)}{6}(x - a)^3 + \frac{f^{(4)}(a)}{24}(x - a)^4 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

$$\sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x - a)^k$$

Define:

- $f^{(0)}(x) = f(x)$
- $0! = 1$
- $(x - a)^0 = 1$  for all  $x$



**Example 3.1.2:** Let  $f(x) = \frac{1}{1-x}$ , based at  $x = 0$ . Find  $P_4(x)$

$$f(x) = \frac{1}{1-x}$$

$$f'(x) = \frac{-1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{2}{(1-x)^3}$$

$$f^{(3)}(x) = \frac{6}{(1-x)^4}$$

$$f^{(4)}(x) = \frac{24}{(1-x)^5}$$



# Chapter 4

## 4.1 Warm Up

### Question 7

Suppose for a certain function  $f$  we know that  $|f'(x)| \leq f$  for all  $x$ .

- (1) What is the largest possible value of  $|f(4) - f(1)|$ ?
- (2) What is the largest possible value of  $|f(b) - f(a)|$  for a given interval  $[a, b]$ ?
- (3) Suppose we also know that  $f(1) = 10$ . Find upper and lower bound for  $f(4)$ .

**Solution:**

(1)

15

(2)

$(b - a)5$

(3)

Upper: 25

Lower: -5

### Note:-

Mean value theorem: If  $f$  is continuous on  $[a, b]$  and differentiable on the  $(a, b)$  then there exists a  $c$  on  $(a, b)$  such that  $f'(c) = \frac{f(b) - f(a)}{b - a}$

Taylor's Theorem: If  $f$  is continuous on  $[a, b]$  and  $(n + 1)^{\text{st}}$  differentiable on  $(a, b)$  then there exists a  $c$  in  $(a, b)$  such that  $f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2}(b - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(b - a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b - a)^{n+1}$

Mean Value Theorem as the Taylor's Theorem at  $n = 0$ .

$$f(b) = f(a) + \frac{f'(c)}{1!}(b - a)^1$$

$$f(b) - f(a) = f'(c)(b - a)$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$P_n(b)$ : Where  $P_n(x)$  is  $n^{\text{th}}$  degree Taylor Poly for  $f(x)$  based at  $x = a$ . So,

$$|f(b) - P_n(b)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}(b - a)^{n+1} \right| = \frac{|f^{(n+1)}(c)|(b - a)^{n+1}}{(n+1)!}$$

Error Bound:  $|f(x) - P_n(x)| \leq \frac{M}{(n+1)!}(x-a)^{n+1}$   
 $n^{th}$  degree TP based at  $x = a$ : where  $|f^{(n+1)}(z)| \leq M$  for all  $z$  between  $a$  and  $x$ .



**Example 4.1.1:** In quiz, we wanted to approximate  $\sqrt{3.95}$  using 1<sup>st</sup> degree TP based at  $x = 4$ .

$$P_1(x) = 2 + \frac{1}{4}(x - 4)$$

$$\sqrt{3.95} \approx P_1(3.95) = 2 + \frac{1}{4}(3.95 - 4) = 2 - 0.00125 = 1.9875$$

$$f(x) = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$|f''(x)| = \left| \frac{1}{4}x^{-\frac{3}{2}} \right| = \frac{1}{4}x^{-\frac{3}{2}}$$

$$M = \frac{1}{4} : \left| \sqrt{3.95} - 1.9875 \right| \leq \frac{\frac{1}{4}|3.95 - 4|^2}{2} \approx 0.00004$$



**Example 4.1.2:** Find  $n$  so that the  $n^{th}$  degree TP for  $f(x) = \cos x$  based at  $x = 0$  approximate  $\cos(0.03)$  to within  $10^{-15}$

$$f'(x) = -\sin(x)$$

$$f''(x) = -\cos(x)$$

$$f^{(3)}(x) = \sin(x)$$

$$f^{(4)}(x) = \cos(x)$$

$$|f^{(n+1)}| \leq 1 \text{ everywhere and for every } n$$

$$M = 1$$

$$\cos(0.03) - P_n(0.03) \leq \frac{1 \cdot (0.03 - 0)^{n+1}}{n+1} = \left( \frac{3}{1000} \right)^{n+1} \frac{1}{(n+1)!} \leq 10^{-15}$$



# Chapter 5

## 5.1

Warm up

### Question 8

- (a) Find the 5<sup>th</sup> degree Taylor Polynomial for  $f(x) = e^x$  based at  $x = 0$ .
- (b) Find  $P_n(x)$ .
- (c) Find an upper bound (in terms of  $n$ ) of the error  $|f(x) - P_n(x)|$ , for  $x > 0$
- (d) What is  $\lim_{n \rightarrow \infty} |f(x) - P_n(x)|$

**Solution:** (a)

$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f''(x) = e^x$$

$$f^{(2)}(x) = e^x$$

$$f^{(3)}(x) = e^x$$

$$f^{(4)}(x) = e^x$$

$$f^{(5)}(x) = e^x$$

$$P_5(x) = e^0 + e^0(x) + \frac{e^0}{2}(x)^2 + \frac{e^0}{6}(x)^3 + \frac{e^0}{24}(x)^4 + \frac{e^0}{120}(x)^5$$

$$P_5(x) = 1 + (x) + \frac{1}{2}(x)^2 + \frac{1}{6}(x)^3 + \frac{1}{24}(x)^4 + \frac{1}{120}(x)^5$$

(b)

$$P_n(x) = \sum_{k=0}^n \frac{1}{k!}(x)^k$$

(c)

$$|f(x) - P_n(x)| \leq \frac{M(x-a)^{n+1}}{(n+1)!}$$

$$|f(x) - P_n(x)| \leq \frac{e^x(x)^{n+1}}{(n+1)!}, \text{ where } |e^z| \leq M \text{ for all } z \text{ between } 0 \text{ and } x$$

(d)

$$\lim_{n \rightarrow \infty} |f(x) - P_n(x)| = ?$$

$$\lim_{n \rightarrow \infty} \frac{e^x \cdot x^{n+1}}{(n+1)!}$$

$$e^x \lim_{n \rightarrow \infty} \frac{x^{n+1}}{(n+1)!} = 0$$

## 5.2 Infinite Series

**Example 5.2.1:** Consider  $x = 1$  in the warm up. We showed  $\lim_{n \rightarrow \infty} |e^1 - P_n(1)| = 0$ . ie  
 $\lim_{n \rightarrow \infty} |e - (1 + 1\frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots + \frac{1}{n!})| = 0$   
 $\lim_{n \rightarrow \infty} (1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots + \frac{1}{n!}) = e$

i.e.  $\sum_{k=0}^{\infty} \frac{1}{k!} = e$



### Note:-

Def: An infinite series is a sum of the form

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \dots$$

Def: Given an infinite series  $\sum_{k=0}^{\infty} a_k$  the  $n^{th}$  partial sum is

$$S_n = \sum_{k=0}^n a_k$$



### Example 5.2.2: Example 1

$$\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots$$

$$S_0 = \sum_{k=0}^0 \frac{1}{k!} = \frac{1}{0!} = 1$$

$$S_1 = \sum_{k=0}^1 \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} = 2$$

$$S_2 = \sum_{k=0}^2 \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} = \frac{5}{2}$$

$$S_3 = \sum_{k=0}^3 \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} = \frac{8}{3}$$



Example 5.2.3: Example 2

$$\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + 1 \dots$$

$$S_0 = 1$$

$$S_1 = 1 - 1 = 0$$

$$S_2 = S_1 + 1 = 1$$

$$S_3 = S_2 - 1 = 0$$



Example 5.2.4: Example 3

$$\sum_{k=2}^{\infty} \frac{1}{k^2 + k}$$

$$S_2 = \frac{1}{6}$$

$$S_3 = S_2 + \frac{1}{3^2 + 3} = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$$

$$S_4 = S_3 + \frac{1}{16 + 4} = \frac{1}{12} + \frac{1}{20} = \frac{3}{10}$$



**Note:-**

Def:  $\sum_{k=0}^{\infty} a_k$  converges if  $\lim_{n \rightarrow \infty} S_n$  exists and is finite, in which case we write

$$\sum_{k=0}^{\infty} a_k = \lim_{n \rightarrow \infty} S_n$$

if  $\lim_{n \rightarrow \infty} S_n$  does not exist (including  $\pm\infty$ ) the series diverges

Ex(1)

$$\sum_{k=0}^{\infty} \frac{1}{k!} \text{ converges because } \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{k!} = e$$

Ex(2)



# Chapter 6

## 6.1

Warm up

### Question 9

- (a) Verify using algebra that  $\frac{1}{k^2+k}$   
(b) Because of (1), we can write the series as  $RS_n = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$  use this to form a close formula for the  $n^{th}$  partial sum.  
(c) Does the series converge?

**Solution:**

$$\frac{1}{k^2 + k}$$

$$\frac{1}{k(k+1)}$$

$$\frac{A}{k} + \frac{B}{k+1} = \frac{1}{k(k+1)}$$

$$\frac{A(k+1) + B(k)}{k^2 + k}$$

$$A(-1+1) + B(-1) = -B = 1$$

$$A(1+0) + B(0) = A = 1$$

(b)

$$S_n = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$S_n = \frac{1}{2} - \frac{1}{n+1}$$

(c) Yes to  $\frac{1}{2}$



## 6.2 Geometric Series

**Example 6.2.1:** Suppose a patient takes 100mg of a certain drug once per day. In 24 hours the patient's body eliminates 80% of the drug

(a) If the first dose is the  $k = 0$  dose, how much of the drug is present immediately after the  $k = 1$  dose

$$S_1 = 100 + 100(0.2)$$

(b)  $k = 2$ ?

$$S_2 = 100 + 120(0.2) = 100 + 100(0.2) + 100(0.2)^2 = 124$$

(c)  $k = 3$

$$S_3 = 100 + 124(0.2) = 100 + 100(0.2) + 100(0.2)^2 + 100(0.2)^3 = 124.8$$

$$S_n = 100 + 124(0.2) = 100 + 100(0.2) + 100(0.2)^2 + 100(0.2)^3 + \dots + 100(0.2)^n$$

$$S_n = \sum_{k=0}^n 100(0.2)^k$$

(d) What happens in the long term?

$$\sum_{k=0}^{\infty} 100(0.2)^k = \frac{100}{1 - 0.2} = \frac{100}{0.8} = 125 \text{mg}$$



### Note:-

In general a geometric series has the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \dots$$

$a$  is the first term.

$r$  is the common ratio between successive terms.

$$S_n = a + ar + ar^2 + \dots + ar^n$$

$$rS_n = ar + ar^2 + ar^3 + \dots + ar^{n+1}$$

$$(1 - r)S_n = a(1 - r^{n+1})$$

$$S_n = a \left( \frac{1 - r^{n+1}}{1 - r} \right)$$

$$r^{n+1} \rightarrow 0 \text{ if } |r| < 1$$

$$r^{n+1} \rightarrow \infty \text{ if } |r| > 1$$

If  $|r| < 1$ , the series  $\sum_{k=0}^{\infty} ar^k$  converges to  $\frac{a}{1-r}$ . Otherwise, the series diverges



## 6.3 Iv

**Note:-**

Harmonic Series

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{5} + \dots$$

# Chapter 7

## 7.1 Warm Up

### Question 10

Which of the following statements must be true about a series and its partial sums, which must be false, and for which is it impossible to say

- (1) If  $a_k > 0$  for all  $k$ , then  $S_n = \sum_{k=1}^n a_k$  is increasing
- (2) If  $\lim_{n \rightarrow \infty} a_k = 0$ , then  $\lim_{n \rightarrow \infty} S_n$  exists and is finite
- (3) If  $S_n = \sum_{k=1}^n a_k = \frac{4n^2}{1+n^2}$ , then  $a_2 = \frac{6}{5}$
- (4) If  $S_n = \sum_{k=1}^n a_k = \frac{4n^2}{1+n^2}$  then the series converges.

### **Solution:**

- (1) true  $s_{n+1} = s_n + a_{n+1} > s_n$  since  $a_n > 0$
- (2) its  $\sum_{k=1}^n \frac{1}{k}$  diverges and  $\sum_{k=1}^n \frac{1}{k^2}$  converges
- (3)

$$S_1 = \frac{4(1)^2}{1+1^2} = \frac{4}{2} = 2$$

$$a_2 = S_2 - S_1 = \frac{4(2)^2}{1+(2)^2} - 2 = \frac{16}{5} - 2$$

true.

- (4) true,  $\lim_{n \rightarrow \infty} \frac{4n^2}{1+n^2} = 4$

**Example 7.1.1:** Consider a bowl whose shape is given by rotating the graph of  $y = x^2$ ,  $0 \leq x \leq 2$  around the y-axis

(a) Write an integral to compute the volume of the bowl.

volume of the  $i^{th}$  slice  $\pi r_i^2 \Delta h$

$$h_i = r_i^2$$

$$r_i = \sqrt{h_i}$$

$$\pi (\sqrt{h_i})^2 \Delta h$$

$$\text{total volume} \approx \sum_{k=1}^n \pi h_i \Delta h$$

$$\text{total volume} = \int_0^4 \pi h dh \text{ cm}^3$$

(b) If density of oatmeal bowl  $\rho(h)$  g/cm<sup>3</sup> where  $h$  is the vertical distance from the bottom,  $0 \leq h \leq 4$  write an integral to compute the mass of oatmeal in the bowl.

mass of  $i^{th}$  slice =  $\rho(h_i) \pi h_i \Delta h$

$$\text{total mass} = \int_0^4 \rho(h) \pi h dh$$

(c) Write an integral to compute work required to lift all oatmeal to height of 15 cm above top of bowl.

$$g = \frac{\rho(h_i) \pi \Delta h}{1000}$$

$$\text{work for } i^{th} \text{ slice} = \frac{9.8(\rho(h_i)) \pi h_i \Delta h}{1000} \left( \frac{19 - h_i}{100} \right)$$

$$\text{total work} = \int_0^4 \frac{9.8 \rho(h) \pi h (19 - h)}{100000} dh$$



**Note:-**

If  $f$  is increasing,  $L_n < I < R_n$

If  $f$  is decreasing,  $R_n < I < L_n$

If  $f$  is concave up,  $M_n < I < T_n$

If  $f$  is concave down,  $T_n < I < M_n$



**Example 7.1.2:** Let  $I = \int_0^8 \frac{1}{x+3} dx$

(a) Is  $M_2$  an over or under approx. for  $I$ ?

$$f(x) = \frac{1}{x+3} = (x+3)^{-1}$$

$$f'(x) = -(x+3)^{-2}$$

$$f''(x) = 2(x+3)^{-3}$$

$2(x+3)^{-3} > 0$  on  $[0, 8]$ . underestimate. (b) Compute  $M_2$

$$M_2 = f(2)\Delta x + f(6)\Delta x$$

$$M_2 = \frac{1}{5}(4) + \frac{1}{9}(4)$$

$$M_2 = \frac{56}{45}$$

(c) Find an upper bound on the error when using  $M_2$  to approximate  $I$ .

$$|I - M_n| \leq \frac{M(b-a)^3}{24n^2}, \text{ where } |f''(x)| \leq M_n \text{ on } [0, 8]$$

$$f''(x) = \frac{2}{(x+3)^3} \leq \frac{2}{27} \text{ on } [0, 8]$$

$$|I - M_n| \leq \frac{\frac{2}{27}(8-0)}{24(2)^2} = \frac{2^5}{3^4} = \frac{32}{81}$$

