Math 115 QR

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July 9 2024

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Chapter 1

1.1 Warm up 1

Question 1

Suppose we are excavating a hole that is 10ft long, 6ft wide, and 4ft deep. If the soil weighs 12 lb/ft^3, what is the total weight of soil removed

Solution: There is a uniform density

$$\text{volume: } (4)(6)(10) = 240\text{ft}^3$$
$$\text{weight: } (12\text{lb/ft}^3)(240\text{ft}^3) = 2880\text{lbs}$$

Question 2

Suppose instead the hole is 6ft long, 4ft wide, and 10 ft deep. Which hole takes more energy to dig?

Solution: more energy for (2) since we need to lift the soil out of the hole

1.2 In class notes

Note:-

In physics work = (force)(distance)
(ft * lbs) = (lbs)(ft)
(N * m) = (n)(M)

Example 1.2.1 (Q: For (2), how much work is required to remove the soil)

Slice the shape into layers parallel to the bottom of the hole.
Let x be the depth from the top of the hole ($0 \leqslant x \leqslant 10$) each layer has thicknes Δx.
Volume of the i$^{\text{th}}$ slice = $(4)(6)(\Delta x) = 24\Delta x$ ft^3
Weight of the i$^{\text{th}}$ slice = 12lb/ft^3($24\Delta x$)ft^3 = $288\Delta x$ lbs
Work need to lift i$^{\text{th}}$ $\approx$ ($288\Delta x$ lbs)($x_i$ft)
It is approximate and not exact because no all of the ith slice is lifted the same distance
Total work $\approx \sum_{i=1}^{n} 288 x_i \Delta x$(ft * lbs)
$\int_0^{10} 288x\,dx$ ft * lbs = $(144x^2)\Big|_0^{10}$ = 14400 ft * lbs

Example 1.2.2 (Q: How much work for a 10 by 6 by 4 hole)

Volume of the i$^{\text{th}}$ slice = $(10)(6)(\Delta x) = 60\Delta x$ ft^3
Weight of the i$^{\text{th}}$ slice = 12lb/ft^3($60\Delta x$)ft^3 = $720\Delta x$ lbs

Work need to lift $\text{i}^{\text{th}} \approx (720\Delta x \text{ lbs})(x_i\text{ft})$
It is approximate and not exact because no all of the i^{th} slice is lifted the same distance
Total work $\approx \sum_{i=1}^{n} 720x_i\Delta x(\text{ft} * \text{lbs})$
$\int_0^4 720xdx \text{ ft} * \text{lbs} = (360x^2)\Big|_0^4 = 5120 \text{ ft} * \text{lbs}$

Example 1.2.3 (Suppose the density of the soil is given by $p(x) = 8 + 2x$ lb/ft^3. How does the change the answer for the shallow soil?)
Volume of the i^{th} slice $= 60\Delta x\text{ft}^3$
Weight of the i^{th} slice $= (8 + 2x_i)60\Delta x$
Work needed to lift the i^{th} slice $\approx (8 + 2x + i)(60\Delta x)(x_i)$
Total work $\approx \sum_{i=1}^{n}(8 + 2x_i)(60x_i)\Delta x$
$\int_0^4 (8 + 2x_i)(60x)dx$

Example 1.2.4 (Circular corn field of radius 50 ft. The density of the corn in (ears/ft^2) is a function $f(y)$ where y is the distance from the center of the circle. Write an intergral to compute the total yield in ears)
area of i^{th} slice $= \pi y_i^2 - \pi(y_i - \Delta y)^2$
$= \pi y_i^2 - \pi(y_i^2 - 2y_i\Delta y + (\Delta y)^2)$
$\approx 2\pi y_i\Delta y$
ears in the i^{th} slice $= p(y_i)2\pi Y_i\Delta y$
total number of ears $\approx \sum_{i=1}^{n} p(y_i)2\pi y_i\Delta y$
total number of ears $= \int_0^{50} f(y)2\pi y\Delta y$

Chapter 2

2.1 Warm up

Question 3

Supppose the circular corn field instead has an irrigation ditch running along a diameter. As before the density of corn is a function of the distance from the water source. How must you slice up the region so that the desnity is (approx) constant in each slice?

Solution: You must slice parallel to the water source so it should be sliced vertically.

Question 4

If the density (in ears/ft^2) is given by the function g, write an integral to compute the total number of ears in the field

Solution:

$$\left(\frac{l_i}{2}\right)^2 + y_i = 50^2$$

$$\frac{l_i^2}{4} = 2500 - y_i^2$$

$$l_i = \sqrt{10000 - 4y_i^2}$$

number of ears in i^{th} slice $\approx g(y_i)\sqrt{10000 - 4y_i^2}\Delta y$

total number of ears $\approx 2\sum_{i=1}^{n} g(y_i)\sqrt{10000 - 4y_i^2}\Delta y$

total numebr of ears $= 2\int_0^{50} g(y_i)\sqrt{10000 - 4y_i^2}dy$

Example 2.1.1 (Conical tank full of sludge with density $f(2)$ kg/m^3, where z is depth. Find an integral to compute the towrk done (against gravity) in pumping all the sludge to height p 1 m aove the tank)

Volume of the i^{th} slice $\approx \pi r_i^2 \Delta x$

$\frac{r_i}{6-z_i} = \frac{3}{6}$

$r_i = \frac{1}{2}(6 - z_i) = 3 - \frac{z_i}{2}$

Volume of the i^{th} slice $\pi\left(3 - \frac{}{z}{}_i 2\right)^2 \Delta z$ m^3

mass of the i^{th} slice $\approx f(z_i)\pi\left(3 - \frac{z_i}{2}\right)^2 \Delta z$ kg

weight of the i^{th} slice $\approx 9.8f(z_i)\pi\left(3 - \frac{z_i}{2}\right)^2 \Delta z$ N

work for the i^{th} slice $\approx 9.8f(z_i)\pi\left(3 - \frac{z_i}{2}\right)^2 \Delta z(z_i + 1)$ J

total work = $\sum_{i=1}^{n} 9.8 f(z_i)\pi \left(3 - \frac{z_i}{2}\right)^2 (z_i + 1)\Delta z$ J
total work = $\int_0^6 9.8 f(z_i)\pi \left(3 - \frac{z}{2}\right)^2 (z + 1) dz$ J

Example 2.1.2 (Given a functino f(x), $a \leqslant x \leqslant b$ what is the length of the graph)
length of i^{th} piece $\approx \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \approx \sqrt{(\Delta x + (f'(x_i))\Delta x)^2} = \sqrt{1 + (f'(x_i))^2}\Delta x$
total length = $\sum_{i=1}^{n} \sqrt{1 + (f'(x_i))^2}\Delta x$
total length = $\int_a^b \sqrt{1 + (f'(x_i))^2} dx$

Chapter 3

3.1 Warm up

Question 5

Find the tangent line to $f(x) = \frac{1}{1+x^2}$ at $x = 2$

Solution:

$$f'(x) = \frac{0 \cdot (1+x^2) - 1(2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}$$

$$f'(2) = \frac{-2(2)}{(1+2^2)^2} = \frac{-2}{5^2} = \frac{-2}{25}$$

$$f(2) = \frac{1}{5}$$

$$y - \frac{1}{5} = -\frac{2}{25}(x-2)$$

$$y = \frac{1}{5} - \frac{4}{25}(x-2)$$

Question 6

If we use this line to aproximate f near $x = 2$, will we get an overestimate or an underestimate?

Solution:

$$f''(x) = \frac{-2(1+x^2)^2 - (-2x)2(1+x^2)(2x)}{(1+x^2)^4}$$

$$f''(x) = \frac{-2 - 2x^2 + 8x^2}{(1+x^2)^3}$$

$$f''(x) = \frac{6x^2 - 2}{(1+x^2)^3}$$

$$f'(x) = 0$$

$$0 = 6x^2 - 2$$

$$x = \pm\sqrt{\frac{1}{3}}$$

$L(x)$ underestimates $f(x)$ near $x = 2$

Example 3.1.1 (Find the quadratic function that best approximates $f(x) = \frac{1}{1+x^2}$ near $x = 2$)

Find $Q(x) = C_0 + C_1(x-2) + C_2(x-2)^2$ such that $Q(2) = f(2) = \frac{1}{5}$, $Q'(2) = f'(2) = \frac{-4}{25}$, and $G''(x) = f''(2) = \frac{22}{125}$

$$Q'(x) = C_1 + 2C_2(2-2)$$

$$Q''(x) = 2c_2$$

$$Q(2) = C_0 = \frac{1}{5}$$

$$Q'(2) = C_1 = \frac{-4}{25}$$

$$Q''(2) = 2C_2 = \frac{22}{125} \to C_2 = \frac{11}{125}$$

$$Q(x) = \frac{1}{5} - \frac{4}{25}(x-2) + \frac{11}{125}(x-2)^2$$

Note:-

General formula: $Q(x) = f(a) + f'(a)(x-a) + \frac{f''(x)}{2}(x-1)^2$

Def: The n^{th} degree Taylor polynomial of $f(x)$ based at $x = a$ is:

$P_n(x) = f(a) + f'(a)(x-1) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{6}(x-a)^3 + \frac{f^{(4)}(a)}{24}(x-a)^4 + \ldots \frac{f^{(n)}(a)}{n!}(x-a)^n$

$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ Define:

$f^{(0)}(x) = f(x)$ $0! = 1$

$(x-a)^0 = 1$ for all x

Example 3.1.2 (Let $f(x) = \frac{1}{1-x}$, based at $x = 0$. Find $P_4(x)$)

$$f(x) = \frac{1}{1-x}$$

$$f'(x) = \frac{-1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{2}{(1-x)^3}$$

$$f^{(3)}(x) = \frac{6}{(1-x)^4}$$

$$f^{(4)} = \frac{24}{(1-x)^5}$$