Math 115 QR

Alex Hernandez Juarez

July 9 2024

Contents

Chapter 1

1.1 Warm up 1

Question 1

Suppose we are excavating a hole that is 10ft long, 6ft wide, and 4ft deep. If the soil weighs 12 lb/ft^3, what is the total weight of soil removed

Solution: There is a uniform density

$$\text{volume: } (4)(6)(10) = 240\text{ft}^3$$

$$\text{weight: } (12\text{lb}/\text{ft}^3)(240\text{ft}^3) = 2880\text{lbs}$$

Question 2

Suppose instead the hole is 6ft long, 4ft wide, and 10 ft deep. Which hole takes more energy to dig?

Solution: more energy for (2) since we need to lift the soil out of the hole

1.2 In class notes

Note:-

In physics work = (force)(distance)
(ft * lbs) = (lbs)(ft)
(N * m) = (n)(M)

Example 1.2.1: Q: For (2), how much work is required to remove the soil

Slice the shape into layers parallel to the bottom of the hole.
Let x be the depth from the top of the hole ($0 \leqslant x \leqslant 10$) each layer has thicknes Δx.

- Volume of the i$^{\text{th}}$ slice = $(4)(6)(\Delta x) = 24\Delta x$ ft^3
- Weight of the i$^{\text{th}}$ slice = 12lb/ft^3$(24\Delta x)$ft^3 = $288\Delta x$lbs
- Work needed to lift i$^{\text{th}}$ $\approx$ $(288\Delta x$ lbs$)(x_i$ft$)$

It is approximate and not exact because no all of the ith slice is lifted the same distance

- Total work $\approx \sum_{i=1}^{n} 288x_i\Delta x$(ft * lbs)

$$\int_0^{10} 288x dx \text{ft} * \text{lbs} = (144x^2)\Big|_0^{10} = 14400\text{ft} * \text{lbs}$$

Example 1.2.2: Q: How much work for a 10 by 6 by 4 hole

- Volume of the i$^{\text{th}}$ slice = $(10)(6)(\Delta x) = 60\Delta x$ ft^3
- Weight of the i$^{\text{th}}$ slice = 12lb/ft^3$(60\Delta x)$ft^3 = $720\Delta x$ lbs
- Work need to lift i$^{\text{th}}$ $\approx$ $(720\Delta x$ lbs$)(x_i$ft$)$

It is approximate and not exact because no all of the ith slice is lifted the same distance

$$\text{Total work } \approx \sum_{i=1}^{n} 720x_i\Delta x(\text{ft} * \text{lbs})$$

$$\int_0^{4} 720x dx \text{ ft} * \text{lbs} = (360x^2)\Big|_0^{4} = 5120\text{ft} * \text{lbs}$$

Example 1.2.3: Suppose the density of the soil is given by $p(x) = 8 + 2x$ lb/ft^3. How does the change the answer for the shallow soil?

- Volume of the ithslice = $60\Delta x$ft^3
- Weight of the ithslice = $(8 + 2x_i)60\Delta x$
- Work needed to lift the ithslice $\approx (8 + 2x + i)(60\Delta x)(x_i)$
- Total work $\approx \sum_{i=1}^{n}(8 + 2x_i)(60x_i)\Delta$

$$\int_0^{4} (8 + 2x_i)(60x)dx$$

Example 1.2.4: Circular corn field of radius 50 ft. The density of the corn in (ears/ft^2) is a function $f(y)$ where y is the distance from the center of the circle. Write an intergral to compute the total yield in ears

- area of i$^{\text{th}}$ slice = $\pi y_i^2 - \pi(y_i - \Delta y)^2 = \pi y_i^2 - \pi(y_i^2 - 2y_i\Delta y + (\Delta y)^2)$
- ears in the ith slice = $p(y_i)2\pi Y_i\Delta y \approx 2\pi y_i \Delta y$
- total number of ears $\approx \sum_{i=1}^{n} p(y_i)2\pi y_i \Delta y$
- total number of ears = $\int_0^{50} f(y)2\pi y \Delta y$

Chapter 2

2.1 Warm up

Question 3

Supppose the circular corn field instead has an irrigation ditch running along a diameter. As before the density of corn is a function of the distance from the water source. How must you slice up the region so that the desnity is (approx) constant in each slice?

Solution: You must slice parallel to the water source so it should be sliced vertically.

Question 4

If the density (in ears/ft^2) is given by the function g, write an integral to compute the total number of ears in the field

Solution:

$$\left(\frac{l_i}{2}\right)^2 + y_i = 50^2$$

$$\frac{l_i^2}{4} = 2500 - y_i^2$$

$$l_i = \sqrt{10000 - 4y_i^2}$$

number of ears in i^{th} slice $\approx g(y_i)\sqrt{10000 - 4y_i^2}\Delta y$

total number of ears $\approx 2\sum_{i=1}^{n} g(y_i)\sqrt{10000 - 4y_i^2}\Delta y$

total numebr of ears $= 2\int_0^{50} g(y_i)\sqrt{10000 - 4y_i^2}dy$

Example 2.1.1: Conical tank full of sludge with density $f(2)$ kg/m^3, where z is depth. Find an integral to compute the towrk done (against gravity) in pumping all the sludge to height p 1 m aove the tank

- Volume of the i^{th} slice $\approx \pi r_i^2 \Delta x$
- $\frac{r_i}{6-z_i} = \frac{3}{6}$
- $r_i = \frac{1}{2}(6 - z_i) = 3 - \frac{z_i}{2}$
- Volume of the i^{th} slice $\pi \left(3 - \frac{}{z\,i}2\right)^2 \Delta z$ m^3
- mass of the i^{th} slice $\approx f(z_i)\pi \left(3 - \frac{z_i}{2}\right)^2 \Delta z$ kg
- weight of the i^{th} slice $\approx 9.8 f(z_i)\pi \left(3 - \frac{z_i}{2}\right)^2 \Delta z$ N
- work for the i^{th} slice $\approx 9.8 f(z_i)\pi \left(3 - \frac{z_i}{2}\right)^2 \Delta z(z_i + 1)$ J
- total work $= \sum_{i=1}^{n} 9.8 f(z_i)\pi \left(3 - \frac{z_i}{2}\right)^2 (z_i + 1)\Delta z$ J
- total work $= \int_0^6 9.8 f(z_i)\pi \left(3 - \frac{z}{2}\right)^2 (z + 1) dz$ J

Example 2.1.2: Given a functino f(x), $a \leqslant x \leqslant b$ what is the length of the graph

- length of i^{th} piece $\approx \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \approx \sqrt{(\Delta x + (f'(x_i))\Delta x)^2} = \sqrt{1 + (f'(x_i))^2}\Delta x$
- total length $= \sum_{i=1}^{n} \sqrt{1 + (f'(x_i))^2}\Delta x$
- total length $= \int_a^b \sqrt{1 + (f'(x_i))^2} dx$

Chapter 3

3.1 Warm up

Question 5

Find the tangent line to $f(x) = \frac{1}{1+x^2}$ at $x = 2$

Solution:

$$f'(x) = \frac{0 \cdot (1+x^2) - 1(2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}$$

$$f'(2) = \frac{-2(2)}{(1+2^2)^2} = \frac{-2}{5^2} = \frac{-2}{25}$$

$$f(2) = \frac{1}{5}$$

$$y - \frac{1}{5} = -\frac{2}{25}(x-2)$$

$$y = \frac{1}{5} - \frac{4}{25}(x-2)$$

Question 6

If we use this line to aproximate f near $x = 2$, will we get an overestimate or an underestimate?

Solution:

$$f''(x) = \frac{-2(1+x^2)^2 - (-2x)2(1+x^2)(2x)}{(1+x^2)^4}$$

$$f''(x) = \frac{-2 - 2x^2 + 8x^2}{(1+x^2)^3}$$

$$f''(x) = \frac{6x^2 - 2}{(1+x^2)^3}$$

$$f'(x) = 0$$

$$0 = 6x^2 - 2$$

$$x = \pm\sqrt{\frac{1}{3}}$$

$L(x)$ underestimates $f(x)$ near $x = 2$

Example 3.1.1: Find the quadratic function that best approximates $f(x) = \frac{1}{1+x^2}$ near $x = 2$

Find $Q(x) = C_0 + C_1(x-2) + C_2(x-2)^2$ such that $Q(2) = f(2) = \frac{1}{5}$, $Q'(2) = f'(2) = \frac{-4}{25}$, and $G''(x) = f''(2) = \frac{22}{125}$

$$Q'(x) = C_1 + 2C_2(2-2)$$

$$Q''(x) = 2c_2$$

$$Q(2) = C_0 = \frac{1}{5}$$

$$Q'(2) = C_1 = \frac{-4}{25}$$

$$Q''(2) = 2C_2 = \frac{22}{125} \longrightarrow C_2 = \frac{11}{125}$$

$$Q(x) = \frac{1}{5} - \frac{4}{25}(x-2) + \frac{11}{125}(x-2)^2$$

Note:-

General formula for Taylor Polynomial: $Q(x) = f(a) + f'(a)(x-a) + \frac{f''(x)}{2}(x-1)^2$

Definition 3.1.1: Taylor Polynomial

he nth degree Taylor polynomial of $f(x)$ based at $x = a$ is:
$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{6}(x-a)^3 + \frac{f^{(4)}(a)}{24}(x-a)^4 + \ldots \frac{f^{(n)}(a)}{n!}(x-a)^n$

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Define:

- $f^{(0)}(x) = f(x)$
- $0! = 1$
- $(x-a)^0 = 1$ for all x

Example 3.1.2: Let $f(x) = \frac{1}{1-x}$, based at $x = 0$. Find $P_4(x)$

$$f(x) = \frac{1}{1-x}$$

$$f'(x) = \frac{-1}{(1-x)^2}(-1) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{2}{(1-x)^3}$$

$$f^{(3)}(x) = \frac{6}{(1-x)^4}$$

$$f^{(4)} = \frac{24}{(1-x)^5}$$

Chapter 4

4.1 Warm Up

Question 7

Suppose for a certain function f we know that $|f'(x)| \leqslant f$ for all x.
(1) What is the largest possible value of $\left|f(4) - f(1)\right|$?
(2) What is the largest possible value of $|f(b) - f(a)|$ for a given interval $[a, b]$?
(3) Suppose we also know that f(1) = 10. Find upper and lower bound for $f(4)$.

Solution:
(1)
15
(2)
$(b - a)5$
(3)
Upper: 25
Lower: -5

Note:-

Mean value theorem: If f is continuous on $[a, b]$ and differentiable on the (a, b) then there exists a c on (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Taylor's Theorem: If f is continuous on $[a, b]$ and $(n+1)^{\text{st}}$ differentiable on (a, b) then there exists a c in (a, b) such that $f(b) = f(a) + f'(a)(b-a) + \frac{f'(a)}{2}(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$

Mean Value Theorem as the Taylor's Theorem at $n = 0$.

$$f(b) = f(a) + \frac{f'(c)}{1!}(b-a)^1$$

$$f(b) - f(a) = f'(c)(b-a)$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$P_n(b)$: Where $P_n(x)$ is n^{th} degree Taylor Poly for $f(x)$ based at $x = a$. So,

$$|f(b) - P_n(b)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}\right| = \frac{\left|f^{(n+1)}(c)\right|(b-a)^{n+1}}{(n+1)!}$$

Error Bound: $|f(x) - P_n(x)| \leqslant \frac{M}{(n+1)!}(x-a)^{n+1}$
n^{th} degree TP based at x = a: where $|f^{(n+1)}(z)| \leqslant M$ for all z between a and x.

Example 4.1.1: In quiz, we wanted to approximate $\sqrt{3.95}$ using 1^{st} degree TP based at $x = 4$.

$$P_1(x) = 2 + \frac{1}{4}(x - a)$$

$$\sqrt{3.95} \approx P_1(3.95) = 2 + \frac{1}{4}(3.95 - 4) = 2 - 0.00125 = 1.9875$$

$$f(x) = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$|f''(x)| = \left|\frac{1}{4}x^{-\frac{3}{1}}\right| = \frac{1}{4}x^{-\frac{3}{2}}$$

$$M = \frac{1}{4} : \left|\sqrt{3.95} - 1.9875\right| \leqslant \frac{\frac{1}{4}|3.95 - 4|^2}{2} \approx 0.00004$$

Example 4.1.2: Find n so that the n^{th} defree TP for $f(x) = \cos x$ based at $x = 0$ approximate $\cos(0.03)$ to within 10^{-15}

$$f'(x) = -\sin(x)$$

$$f''(x) = -\cos(x)$$

$$f^{(3)}(x) = \sin(x)$$

$$f^{(4)}(x) = \cos(x)$$

$$\left|f^{(n+1)} \leqslant 1\right| \text{ eveywhere and for every n}$$

$$M = 1$$

$$\cos(0.03) - P_n(0.03) \leqslant \frac{1 \cdot (0.03 - 0)^{n+1}}{n+1} = \left(\frac{3}{1000}\right)^{n+1} \frac{1}{(n+1)!} \leqslant 10^{-15}$$

Chapter 5

5.1

Warm up

Question 8

(a) Find the 5^{th} degree Taylor Polynomial for $f(x) = e^x$ based at $x = 0$.
(b) Find $P_n(x)$.
(c) Find an upper bound (in terms of n) of the error $\left|f(x) - P_n(x)\right|$, for $x > 0$
(d) What is $\lim_{n\to\infty} |f(x) - P_n(x)$

Solution: (a)

$$f(x) = e^x$$
$$f'(x) = e^x$$
$$f''(x) = e^x$$
$$f^{(2)}(x) = e^x$$
$$f^{(3)}(x) = e^x$$
$$f^{(4)}(x) = e^x$$
$$f^{(5)}(x) = e^x$$
$$P_5(x) = e^0 + e^0(x) + \frac{e^0}{2}(x)^2 + \frac{e^0}{6}(x)^3 + \frac{e^0}{24}(x)^4 + \frac{e^0}{120}(x)^5$$
$$P_5(x) = 1 + (x) + \frac{1}{2}(x)^2 + \frac{1}{6}(x)^3 + \frac{1}{24}(x)^4 + \frac{1}{120}(x)^5$$

(b)

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!}(x)^k$$

(c)

$$|f(x) - P_n(x)| \leqslant \frac{M(x-a)^{n+1}}{(n+1)}$$

$$|f(x) - P_n(x)| \leqslant \frac{e^x(x)^{n+1}}{(n+1)!}, \text{where } |e^z| \leqslant M \text{ for all } z \text{ between } 0 \text{ and } x$$

(d)

$$\lim_{n\to\infty} |f(x) - P_n(x) = ?$$

$$\lim_{n\to\infty} \frac{e^x \cdot x^{n+1}}{(n+1)!}$$

$$e^x \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$$

5.2 Infinite Series

Example 5.2.1: Consider $x = 1$ in the warm up. We showed $\lim_{n\to\infty} |e^1 - P_n(1)| = 0$. **ie** $\lim_{n\to\infty} |e - \left(1 + 1\frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots + \frac{1}{n!}\right)| = 0$
$\lim_{n\to\infty} \left(1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots + +\frac{1}{n!}\right) = e$

i.e. $\sum_{k=0}^{\infty} \frac{1}{k!} = e$

Note:-

Def: An infinite series is a sum of the form

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \ldots$$

Def: Given an infinite series $\sum_{k=0}^{\infty} a_k$ the n^{th} partial sum is

$$S_n = \sum_{k=0}^{n} a_k$$

Example 5.2.2: Example 1

$$\sum_{k=0}^{\infty} \frac{1}{k!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \ldots$$

$$S_0 = \sum_{k=0}^{0} \frac{1}{k!} = \frac{1}{0!} = 1$$

$$S_1 = \sum_{k=0}^{1} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} = 2$$

$$S_2 = \sum_{k=0}^{2} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} = \frac{5}{2}$$

$$S_3 = \sum_{k=0}^{3} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} = \frac{8}{3}$$

Example 5.2.3: Example 2

$$\sum_{k=0}^{\infty}(-1)^k = 1 - 1 + 1 - 1 + 1 \dots$$

$$S_0 = 1$$
$$S_1 = 1 - 1 = 0$$
$$S_2 = S_1 + 1 = 1$$
$$S_3 = S_2 - 1 = 0$$

Example 5.2.4: Example 3

$$\sum_{k=2}^{\infty}\frac{1}{k^2 + k}$$

$$S_2 = \frac{1}{6}$$
$$S_3 = S_2 + \frac{1}{3^2 + 3} = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}$$
$$S_4 = S_3 + \frac{1}{16 + 4} = \frac{1}{12} + \frac{1}{20} = \frac{3}{10}$$

Note:-

Def: $\sum_{k=0}^{\infty} a_k$ converges if $\lim_{n\to\infty} S_n$ exists and is finite, in which case we write

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} S_n$$

if $\lim_{n\to\infty} S_n$ does not exist (including $\pm\infty$) the series diverges
Ex(1)

$$\sum_{k=0}^{\infty}\frac{1}{k!} \text{ converges because } \lim_{n\to\infty}\sum_{k=0}^{\infty}\frac{1}{k!} = e$$

Ex(2)

Chapter 6

6.1

Warm up

Question 9

(a)Verify using algebra that $\frac{1}{k^2+k}$
(b) Because of (1), we can write the seris as $RS_n = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots \left(\frac{1}{n} - \frac{1}{n+1}\right)$ use this to form a close formula for the nth partial sum.
(c) Does the series converge?

Solution:

$$\frac{1}{k^2+k}$$

$$\frac{1}{k(k+1)}$$

$$\frac{A}{k} + \frac{B}{k+1} = \frac{1}{k(k+1)}$$

$$\frac{A(k+1) + B(k)}{k^2+k}$$

$$A(-1+1) + B(-1) = -B = 1$$

$$A(1+0) + B(0) = A = 1$$

(b)

$$S_n = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$S_n = \frac{1}{2} - \frac{1}{n+1}$$

(c) Yes to $\frac{1}{2}$

6.2 Geometric Series

Example 6.2.1: Suppose a patient takes 100mg of a certain druge once per day. In 24 hours the patients body eliminates 80% of the drug

(a) If the first dose is the $k = 0$ dose, how much of the drug is present immediately after the $k = 1$ dose

$$S_1 = 100 + 100(0.2)$$

(b) $k = 2$?

$$S_2 = 100 + 120(0.2) = 100 + 100(0.2) + 100(0.2)^2 = 124$$

(c) = 3

$$S_3 = 100 + 124(0.2) = 100 + 100(0.2) + 100(0.2)^2 + 100(0.2)^3 100 + 100(0.2) + 100(0.2)^2 = 124.8$$

$$S_n = 100 + 124(0.2) = 100 + 100(0.2) + 100(0.2)^2 + 100(0.2)^3 100 + 100(0.2) + 100(0.2)^2 + \ldots + 100(0.2)^n$$

$$S_n = \sum_{k=0}^{n} 100(0.2)^k$$

(d) What happens in the long term?

$$\sum_{k=0}^{\infty} 100(0.2)^k = \frac{100}{1 - 0.2} = \frac{100}{0.8} = 125mg$$

Note:-

In general a geometric series has the form

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2$$

a is the first term.
r is the common ratio between succesive terms.

$$S_n = a + ar + ar^2 + \ldots + ar^n$$

$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n+1}$$

$$(1 - r)S_n = a\left(1 - r^{n+1}\right)$$

$$S_n = a\left(\frac{1 - r^{n+1}}{1 - r}\right)$$

$r^{n+1} \longrightarrow 0$ if $|r| < 1$
$r^{n+1} \longrightarrow \infty$ if $|r| > 1$
If $|r| < 1$, the series $\sum_{k=0}^{\infty} ar^k$ converges to $\frac{a}{1-r}$. Otherwise, the series diverges

6.3 Iv

Note:-

Harmonic Series

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{5} + \dots$$

Chapter 7

7.1 Warm Up

Question 10

Which of the following statements must be true about a series and its partial sums, which must be false, and for which is it impossible to say
(1) If $a_k > 0$ for all k, then $S_n = \sum_{k=1}^{n} a_k$ is increasing
(2) If $\lim_{n\to\infty} a_k = 0$, then $lim_{n\to\infty} s_n$ exists and is finite
(3) If $S_n = \sum_{k=1}^{n} a_k = \frac{4n^2}{1+n^2}$, then $a_2 = \frac{6}{5}$
(4) If $S_n = \sum_{k=1}^{n} a_k = \frac{4n^2}{1+n^2}$ then the series converges.

Solution:
(1) true $s_{n+1} = s_n + a_{n+1} > s_n$ since $a_n > 0$
(2) its $\sum_{k=1}^{n} \frac{1}{k}$ diverges and $\sum_{k=1}^{n} = \frac{1}{k^2}$ converges
(3)

$$S_1 = \frac{4(1)^2}{1+1^2} = \frac{4}{2} = 2$$

$$a_2 = S_2 - S_1 = \frac{4(2)^2}{1+(2)^2} - 2 = \frac{16}{5} - 2$$

true.
(4) true, $\lim_{n\to\infty} \frac{4n^2}{1+n^2} = 4$

Example 7.1.1: Cconsider a bowl whose shpae is given by rotating the grahp of $y = x^2$, $o \leqslant x \leqslant 2$ around the y-axis

(a) Write an integral to compute the volume of the bowl.

$$\text{volume of the i}^{th} \text{ slice } \pi r_i^2 \Delta h$$

$$h_i = r_i^2$$

$$r_i = \sqrt{h_i}$$

$$\pi\left(\sqrt{h_i}\right)^2 \Delta h$$

$$\text{total volume } \approx \sum_{k=1}^{n} \pi h_i \Delta h$$

$$\text{total volume} = \int_0^4 \pi h dh \text{ cm}^3$$

(b) If density of oatmeal bowm $\rho(h)$ g/cm^3 where h is the veritcal distance from the bottom, $0 \leqslant h \leqslant 4$ write an integral to compute the mass of oatmeal in the bowl.

$$\text{mass of i}^{th} \text{ slice} = \rho(h_i)\pi h_i \Delta h$$

$$\text{total mass} = \int_0^4 \rho(h)\pi h dh$$

(c) Write an integral to compute work required to lift all oatmeal to height of 15 cm above top of bowl.

$$g = \frac{\rho(h_i)\pi\Delta h}{1000}$$

$$\text{work for i}^{th} \text{ slice } = \frac{9.8(\rho(h_i))\pi h_i \Delta h}{1000}\left(\frac{19 - h_i}{100}\right)$$

$$\text{total work} = \int_0^4 \frac{9.8\rho(h)\pi h(19 - h)}{100000} dh$$

Note:-

If f is increasing, $L_n < I < R_n$
If f is decreasing, $R_n < I < L_n$
If f is concave up, $M_n < I < T_n$
If f is concave down, $T_n < I < Mn$

Example 7.1.2: Let $I = \int_0^8 \frac{1}{x+3} dx$

(a) Is M_2 an over or under approx. for I?

$$f(x) = \frac{1}{x+3} = (x+8)^{-1}$$

$$f'(x) = -(x+3)^{-2}$$

$$f''(x) = 2(x+3)^{-3}$$

$2(x+3)^{-3} > 0$ on $[0, 8]$. underestimate. (b) Compute M_2

$$M_2 = f(2)\Delta x + f(6)\Delta x$$

$$M_2 = \frac{1}{5}(4) + \frac{1}{9}(4)$$

$$M_2 = \frac{56}{45}$$

(c) Find an upper bound on the error when using M_2 to approximate I.

$$|I - M_n \leqslant \frac{M(b-a)^3}{24n^2}, \text{ where } |f''(x)| \leqslant M_n \text{ on } [0, 8]$$

$$f''(x) = \frac{2}{(x+3)^3} \leqslant \frac{2}{27} \text{ on } [0, 8]$$

$$|I - M_n| \leqslant \frac{\frac{2}{27}(8-0)}{24(2)^2} = \frac{2^5}{3^4} = \frac{32}{81}$$