Math 244

PSET 3

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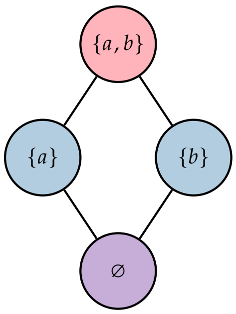
Contents

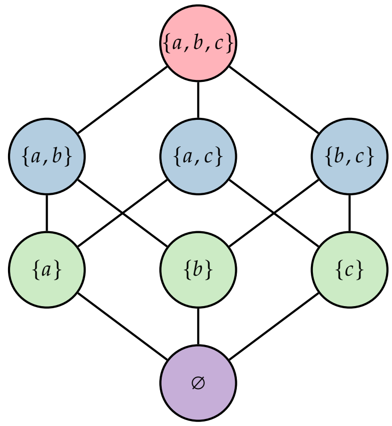
2.3 Prob 1

Question

How many linear extensions of $\mathcal{B}_2$ are there, and what about $\mathcal{B}_3$?



$\mathbf{B_2}$



$\mathbf{B_3}$

Remark For Prob 1

For $\mathcal{B}_2$ there are two linear extensions because the linear extension must respect and preserve the original partial order so the empty set must be first and the set $\{a, b\}$ must be at the top which means ther variation in ordering the remaining elements is 2! as those are the elements below $\{a, b\}$ and above the empty set.

For $\mathcal{B}_3$ there are 36 possible linear extensions. Since the linear extension needs to preserve teh original partial order than that means the unique top element and unique bottom element of the set $\{a, b, c\}$ and the empty set must be the last and first element of any linear extension respectively. For the remaining elements the sets with cardinlaties 1 must all come before the sets of cardinality 2 but there are 3 sets of cardinality 1 that can be ordered in 3! ways as there are 3 options for the first one, 2 for the second, and 1 for the last one. This logic also applies to the 3 sets of cardinality of 2 that must all go after the sets of cardinality 1 and before the unique top element of cardinality 3 so there are 3! ways to order those. This makes the total possible linear extensions the result of multiplying the possibilities of each possible way of ordering the elements so $1 \cdot 3! \cdot 3! \cdot 1$ which is 36.

Bonus Problem

Question

Prove that not every finite poset admits an embedding into the poset $(\mathbb{N}^2, \preceq)$, where $(x_1, y_1) \preceq (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$.

2.4 Prob 3

Question

Find a sequence of real numbers of length 16 that contains no monotone subsequence of length 5.

$i = 1$	4	8	12	16
$i = 2$	3	7	11	15
$i = 3$	2	6	10	14
$i = 4$	1	5	9	13
	$j = 1$	$j = 2$	$j = 3$	$j = 4$

Example (For Prob 2)

Reading the cells row by row (from top row to bottom row) gives the sequence:

$$4, 8, 12, 16, \quad 3, 7, 11, 15, \quad 2, 6, 10, 14, \quad 1, 5, 9, 13.$$

Why there can't be an increasing subsequence of length 5 This works because we split the sequence into a matrice of of 4×4 and by the pigeonhole principle, if you were to pick 5 numbers from the sequence at least of them would have to come from the same column but in each column, the numbers appear in decreasing order thus any two numbers comming from teh same cloum will be in decreasing order which prevents the entire subsequence from being increasing.

Why there can't be a strictly decreasing subsequence of length 5 A similar argument applies for the decreasing subsequence. Our grid has 4 rows and if you were to pick 5 numbers from teh sequence then by the pigeonhole principle at least two would have to come from the same row but the entries in each row are in increasing order preventing a nondecreasing sequence of length 5.

2.4 Prob 4

Question

Prove the following strengthening of Theorem 2.4.6: Let k, ℓ be natural numbers. Then every sequence of real numbers of length $k\ell + 1$ contains a nondecreasing subsequence of length $k+1$ or a decreasing subsequence of length $\ell + 1$.

Proof

Let $(x_1, x_2, \dots, x_{k\ell+1})$ be any sequence of real numbers and let

$$X = \{1, 2, \dots, k\ell + 1\}.$$

We define a relation $\preceq$ on X by

$$i \preceq j \iff i \le j \text{ and } x_i \le x_j.$$

Claim

The relation $\preceq$ is a partial order on X.

Proof of Claim.

- **Reflexivity:** For every $i \in X$, we have $i \le i$ and $x_i \le x_i$. Hence $i \preceq i$.
- **Antisymmetry:** Suppose $i \preceq j$ and $j \preceq i$. Then $i \le j$, $j \le i$, $x_i \le x_j$, and $x_j \le x_i$. Thus $i = j$ and $x_i = x_j$, so $i \preceq j$ and $j \preceq i$ imply $i = j$.
- **Transitivity:** Suppose $i \preceq j$ and $j \preceq k$. Then $i \le j \le k$ and $x_i \le x_j \le x_k$. Consequently $i \le k$ and $x_i \le x_k$, so $i \preceq k$.

Hence $\preceq$ is indeed a partial order.

We now observe that in this poset $(X, \preceq)$:

- A *chain* of size m corresponds to indices $i_1 < i_2 < \cdots < i_m$ such that $x_{i_1} \le x_{i_2} \le \cdots \le x_{i_m}$,
- An *antichain* of size m corresponds to indices $i_1 < i_2 < \cdots < i_m$ for which no two are comparable under $\preceq$. Equivalently, if $i_p < i_q$ in X then we cannot have $x_{i_p} \le x_{i_q}$. That forces $x_{i_1} > x_{i_2} > \cdots > x_{i_m}$,

By Dilworth's theorem every finite poset $P = (X, \preceq)$ satisfies

$$|X| \le \alpha(P) \cdot \omega(P),$$

where $\alpha(P)$ is the maximum size of any antichain in P and $\omega(P)$ is the maximum size of any chain. Applying this to our poset $(X, \preceq)$ yields

$$k\ell + 1 = |X| \le \alpha\big(X, \preceq\big) \cdot \omega\big(X, \preceq\big).$$

Consequently, it is *impossible* for both $\alpha(P) \le \ell$ and $\omega(P) \le k$ to hold, since that would force $\alpha(P) \cdot \omega(P) \le k\ell < k\ell + 1$. Therefore, at least one of the following must be true:

$$\alpha(P) \ge \ell + 1 \quad \text{or} \quad \omega(P) \ge k + 1.$$

- If $\alpha(P) \ge \ell + 1$, then there is an antichain of size $\ell + 1$, which corresponds to a *strictly decreasing* subsequence $x_{i_1} > x_{i_2} > \cdots > x_{i_{\ell+1}}$ of length $\ell + 1$.
- If $\omega(P) \ge k + 1$, then there is a chain of size $k+1$, which corresponds to a *nondecreasing* subsequence $x_{i_1} \le x_{i_2} \le \cdots \le x_{i_{k+1}}$ of length $k+1$.

3.1 Prob 2

Question

Determine the number of ordered pairs (A, B), where $A \subseteq B \subseteq \{1, 2, \dots, n\}$.

Proof

Let $X = \{1, 2, \dots, n\}$. We wish to count the number of ordered pairs (A, B) such that

$$A \subseteq B \subseteq X.$$

Key Observation. For each element $i \in X$, there are exactly three ways to place i with respect to the pair (A, B):

$$(1)\ i \notin B \text{ (hence } i \notin A), \quad (2)\ i \in B \setminus A, \quad (3)\ i \in A \subseteq B.$$

Hence, if we make this choice for every element $i \in X$, we uniquely determine a pair (A, B) with $A \subseteq B$. Conversely, every such pair arises by making one of these three choices for each i.

It follows that there is a 3-to-1 correspondence between all n-element subsets of these "choices" and the valid pairs (A, B). Since each of the n elements admits exactly three possibilities, the total number of ordered pairs (A, B) with $A \subseteq B \subseteq X$ is

$$3^n.$$

3.1 Prob 6

Question

Show that a natural number $n \geq 1$ has an odd number of divisors (including 1 and itself) if and only if $\sqrt{n}$ is an integer. *The textbook has a hint to this problem in the back.*

Proof

Let $n \geq 1$ be a natural number and denote by $d(n)$ the number of positive divisors of n (including 1 and n itself). We claim that $d(n)$ is odd if and only if $\sqrt{n}$ is an integer.

($\Longrightarrow$) Suppose $d(n)$ is odd. We will show that n must be a perfect square. Consider the divisors of n. Each divisor k can be paired with its "partner" $\frac{n}{k}$. Usually $k \neq \frac{n}{k}$, so divisors come in pairs. However, because $d(n)$ is odd, at least one divisor must be "self-paired," that is, satisfy $k = \frac{n}{k}$. Solving $k^2 = n$ shows that $k = \sqrt{n}$ is an integer. Thus n is a perfect square.

($\Longleftarrow$) Conversely, suppose $n = m^2$ for some integer $m \geq 1$. Then each positive divisor $k \neq m$ of n can be paired uniquely with $n/k = \frac{m^2}{k} \neq k$. Only the divisor $k = m$ is paired with itself. Hence all divisors except m come in distinct pairs, and the divisor m stands alone, contributing 1 more to the total count. Therefore $d(n)$ is an odd number.

Combining the two directions shows that n has an odd number of positive divisors if and only if n is a perfect square.