Math 244

PSET 1

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Section 1.2 Problem 5

Question: Section 1.2 Problem 5

Is a "cancellation" possible for the Cartesion Product? That is, if $X \times Y = X \times Z$ holds for some sets, X, Y, and Z, does it follow that $Y = Z$?

Remark What is the Cartesian Product?

The Cartesian product of X and Y is the set of all ordered pairs of the form (x, y), where $x \in X$ and $y \in Y$.

Remark Asnwer

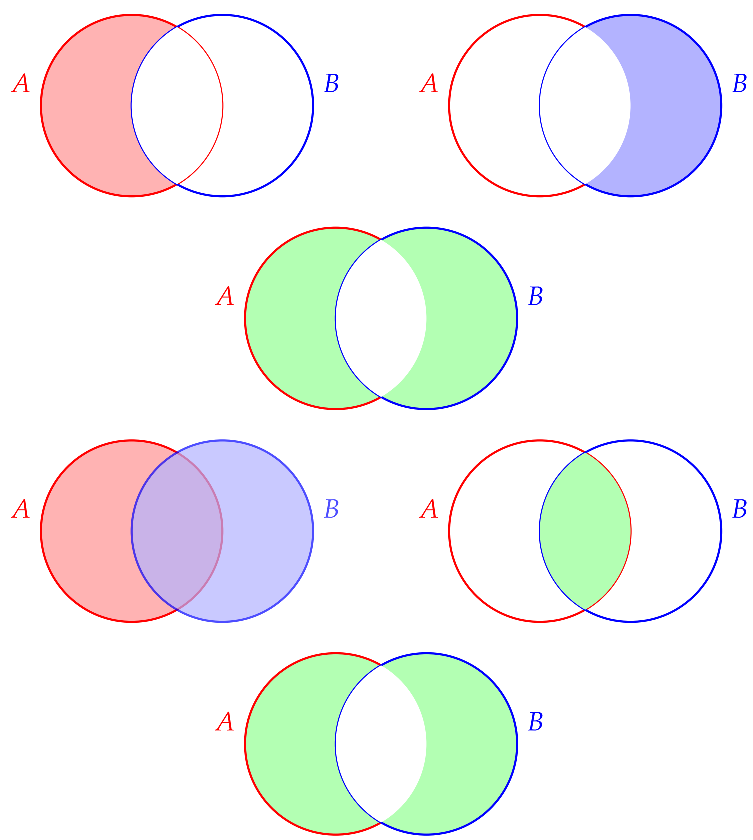
The "cancellation" is not possible for the Cartesian Product unless it is stated that X is not an empty set. For if X is an empty set, then the Cartesian Product of X and another set would always be the empty set. In this scenario, Y and Z could be different and their Cartesian Products with X would still be the empty set.

Section 1.2 Problem 6

Question: Section 1.2 Problem 6

Prove that for any two sets A, B we have

$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$$



Claim Claim 1 For Probelm 2

If $x \in (A \setminus B) \cup (B \setminus A)$, then $x \in A \setminus B$ or $x \in B \setminus A$.

Remark 1

If $x \in A \setminus B$, then $x \in A$ and $x \notin B$. This also implies $x \in A \cup B$ but $x \notin A \cap B$, so $x \in (A \cup B) \setminus (A \cap B)$.
If $x \in B \setminus A$, then $x \in B$ and $x \notin A$, which implies $x \in A \cup B$ but $x \notin A \cap B$, so $x \in (A \cup B) \setminus (A \cap B)$.

This proves that:

$$(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B).$$

Claim Claim 2 For Probelm 2

If x is in $(A \cup B) \setminus (A \cap B)$, then $x \in A \cup B$ and $x \notin A \cap B$.

Remark 2

If $x \in A \cup B$ but $x \notin A \cap B$, then x must belong to exactly one of A or B.
If $x \in A$ but $x \notin B$, then $x \in A \setminus B$.
If $x \in B$ but $x \notin A$, then $x \in B \setminus A$.

Therefore, $x \in (A \setminus B) \cup (B \setminus A)$.
This proves that:

$$(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A).$$

Section 1.3 Problem 2

Question: Section 1.3 Problem 2

The numbers F_0, F_1, F_2, F_3, ... are defined as follows:

$$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n \text{for } n = 0, 1, 2, \ldots$$

Prove that for any $n \geq 0$ we have $F_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$

Base Case

The formula $F_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$ holds for $n = 0$ since $F_0 = 0$ and $\left(\frac{1+\sqrt{5}}{2}\right)^{-1} = \frac{2}{1+\sqrt{5}}$.

The formula $F_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$ holds for $n = 1$ since $F_1 = 1$ and $\left(\frac{1+\sqrt{5}}{2}\right)^{0} = 1$.

Inductive Hypothesis

Let us suppose that the formula $F_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$ and $F_{n+1} \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n}$ holds for some $n \geq 0$.

Remark The Golden Ratio

ϕ is equal to $\frac{1+\sqrt{5}}{2}$.

Inductive Step

The numbers $F_0, F_1, F_2, \ldots$ are defined as follows:

$$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n \text{ for } n = 0, 1, 2, \ldots$$

$$F_{n+2} = F_{n+1} + F_n$$

Use the inductive hypothesis and subsitute

$$F_{n+2} \leq \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$$

Subsitute phi for $\left(\frac{1+\sqrt{5}}{2}\right)$

$$F_{n+2} = F_{n+1} + F_n \leq \phi^n + \phi^{n-1}$$

$$\phi^n + \phi^{n-1} = \phi^{n-1}(\phi + 1)$$

Key Idea

Theoretically reach a conlcusion that proves the formula $F_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$ holds for all $n \geq 0$.

1.4 Problem 2

Question: Section 1.4 Problem 2

Find an example of:

(a) A one-to-one function $f : \mathbb{N} \to \mathbb{N}$ that is not onto.

(b) A function $f : \mathbb{N} \to \mathbb{N}$ that is onto but not one-to-one.

1.4 Problem 6

Question

Prove that the following two statements about a function $g_1 : Z \to X$ and $g_2 : Z \to X$ the composed functions $f \circ g_1$ and $f \circ g_2$ are also distinct.