

Math 244

PSET 4

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3.2 Prob 1

Question

How many permutations of $\{1, 2, \dots, n\}$ have a single cycle?

Proof

There are $n!$ ways to arrange n distinct elements in a sequence. Any sequence

$$(a_1, a_2, \dots, a_n)$$

can be thought of as a candidate for an n -cycle when interpreted in cyclic notation as

$$(a_1, a_2, \dots, a_n).$$

But we can't count cycles where the order is simply rotated cyclically

$$(a_1, a_2, \dots, a_n)$$

is the same as the cycle

$$(a_2, a_3, \dots, a_n, a_1),$$

there are exactly n different ways to write the same cycle by cyclically rotating the elements.

If we count every sequence as a distinct cycle, we overcount each n -cycle n times.

To deal with the overcounting we divide the total number of sequences by n .

$$\frac{n!}{n} = (n - 1)!,$$



3.2 Prob 2

Question

For a permutation $p : X \rightarrow X$, let p^k denote the permutation arising by a k -fold composition of p , i.e., $p^1 = p$ and $p^k = p \circ p^{k-1}$. Define a relation \approx on the set X as follows: $i \approx j$ if and only if there exists a $k \geq 1$ such that $p^k(i) = j$. Prove that \approx is an equivalence relation on X , and that its classes are the cycles of p .

Proof

Want to show that \approx satisfies the three properties of an equivalence relation: reflexivity, symmetry, and transitivity.

For any $i \in X$, since p is a permutation meaning its bijective, i must belong to some cycle. So, there exists a smallest positive integer $k \geq 1$ such that

$$p^k(i) = i.$$

By the definition of \approx , we have $i \approx i$. So, \approx is reflexive.

Suppose $i \approx j$. Then there exists an integer $k \geq 1$ such that

$$p^k(i) = j.$$

Since p is a permutation, it has an inverse p^{-1} . Applying p^{-k} to both sides, we get

$$i = p^{-k}(j).$$

Since this is also a permutation, there exists some positive integer m such that $p^m(j) = i$. Thus, $j \approx i$, showing that \approx is symmetric.

Suppose $i \approx j$ and $j \approx k$. Then there exist positive integers m and n such that

$$p^m(i) = j \quad \text{and} \quad p^n(j) = k.$$

Now, consider the composition:

$$p^{m+n}(i) = p^n(p^m(i)) = p^n(j) = k.$$

Since $m + n \geq 1$, it follows that $i \approx k$. So, \approx is transitive.

In cycle notation, the cycle containing an element i is defined as:

$$\{i, p(i), p^2(i), \dots, p^{k-1}(i)\},$$

where k is the smallest positive integer such that $p^k(i) = i$.

By our definition of \approx , an element $j \in X$ is related to i if and only if there exists some $k \geq 1$ such that $p^k(i) = j$. This is exactly the description of the cycle of i . So, the equivalence class of i under \approx is precisely the set of all elements that can be reached from i by some power of p , which corresponds to the cycle of 

3.3 Prob 7

Question

How many functions $f : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ are there that are *monotonic*; that is, for $i < j$ we have $f(i) \leq f(j)$? *The textbook has a hint to this problem in the back.*

Proof

Define a new sequence g by

$$g(i) = f(i) + i - 1, \quad \text{for } i = 1, 2, \dots, n.$$

Since $f(1) \leq f(2) \leq \dots \leq f(n)$, it follows that

$$g(1) = f(1) \leq f(2) + 1 = g(2) \leq \dots \leq f(n) + n - 1 = g(n).$$

Because we are adding the strictly increasing sequence $0, 1, 2, \dots, n - 1$ to $f(i)$, the sequence $g(1), g(2), \dots, g(n)$ becomes strictly increasing:

$$g(1) < g(2) < \dots < g(n).$$

Since $f(i) \in \{1, 2, \dots, n\}$, we obtain:

- For $i = 1$: $g(1) = f(1) \geq 1$.
- For $i = n$: $g(n) = f(n) + n - 1 \leq n + n - 1 = 2n - 1$.

Thus, each $g(i)$ is a distinct element from the set $\{1, 2, \dots, 2n - 1\}$.

A strictly increasing sequence $g(1) < g(2) < \dots < g(n)$ chosen from $\{1, 2, \dots, 2n - 1\}$ is equivalent to choosing an n -element subset from a set of $2n - 1$ elements. There are

$$\binom{2n - 1}{n}$$

ways to select such a subset.

Since the transformation $f(i) \mapsto g(i) = f(i) + i - 1$ is a bijection between the set of monotonic functions f and the set of strictly increasing sequences g , the number of monotonic functions is given by:

$$\boxed{\binom{2n-1}{n}}.$$



3.3 Prob 21

Question

(optional bonus problem) Draw a triangle ABC . Draw n points lying on the side AB (but different from A and B) and connect them all by segments to the vertex C . Similarly, draw n points on the side AC and connect them to B .

1. How many intersections of the drawn segments are there? Into how many regions is the triangle ABC partitioned by the drawn segments?
2. Draw n points on the side BC and connect them to A . Assume that no 3 of the drawn segments intersect at a single point. How many intersections are there now?
3. How many regions are there in the situation of (b)?

3.7 Prob 3

Question

(Sieve of Eratosthenes) How many numbers are left in the set $\{1, 2, \dots, 1000\}$ after all multiples of 2, 3, 5, and 7 are crossed out?

Remark For Prob 4

Define:

- A_2 as the set of multiples of 2,
- A_3 as the set of multiples of 3,
- A_5 as the set of multiples of 5,
- A_7 as the set of multiples of 7.

We want the size of the complement of $A_2 \cup A_3 \cup A_5 \cup A_7$
We count the multiples of 2, 3, 5, 7 in the range of [1, 1000].

$$\begin{aligned} |A_2| &= \left\lfloor \frac{1000}{2} \right\rfloor = 500, \\ |A_3| &= \left\lfloor \frac{1000}{3} \right\rfloor = 333, \\ |A_5| &= \left\lfloor \frac{1000}{5} \right\rfloor = 200, \\ |A_7| &= \left\lfloor \frac{1000}{7} \right\rfloor = 142. \end{aligned}$$

count of multiples for pairs using least common multiple:

$$|A_2 \cap A_3| = \left\lfloor \frac{1000}{6} \right\rfloor = 166,$$

$$|A_2 \cap A_5| = \left\lfloor \frac{1000}{10} \right\rfloor = 100,$$

$$|A_2 \cap A_7| = \left\lfloor \frac{1000}{14} \right\rfloor = 71,$$

$$|A_3 \cap A_5| = \left\lfloor \frac{1000}{15} \right\rfloor = 66,$$

$$|A_3 \cap A_7| = \left\lfloor \frac{1000}{21} \right\rfloor = 47,$$

$$|A_5 \cap A_7| = \left\lfloor \frac{1000}{35} \right\rfloor = 28.$$

Count of multiples for triples

$$|A_2 \cap A_3 \cap A_5| = \left\lfloor \frac{1000}{30} \right\rfloor = 33,$$

$$|A_2 \cap A_3 \cap A_7| = \left\lfloor \frac{1000}{42} \right\rfloor = 23,$$

$$|A_2 \cap A_5 \cap A_7| = \left\lfloor \frac{1000}{70} \right\rfloor = 14,$$

$$|A_3 \cap A_5 \cap A_7| = \left\lfloor \frac{1000}{105} \right\rfloor = 9.$$

Count of multiples which all four share

$$|A_2 \cap A_3 \cap A_5 \cap A_7| = \left\lfloor \frac{1000}{210} \right\rfloor = 4.$$

Apply the Inclusion-Exclusion Principle

The count of numbers divisible by at least one of 2, 3, 5, 7 is:

$$\begin{aligned} |A_2 \cup A_3 \cup A_5 \cup A_7| &= |A_2| + |A_3| + |A_5| + |A_7| \\ &\quad - (|A_2 \cap A_3| + |A_2 \cap A_5| + |A_2 \cap A_7| + |A_3 \cap A_5| + |A_3 \cap A_7| + |A_5 \cap A_7|) \\ &\quad + (|A_2 \cap A_3 \cap A_5| + |A_2 \cap A_3 \cap A_7| + |A_2 \cap A_5 \cap A_7| + |A_3 \cap A_5 \cap A_7|) \\ &\quad - |A_2 \cap A_3 \cap A_5 \cap A_7|. \end{aligned}$$

Plugging in values:

$$\begin{aligned} |A_2 \cup A_3 \cup A_5 \cup A_7| &= 500 + 333 + 200 + 142 \\ &\quad - (166 + 100 + 71 + 66 + 47 + 28) \\ &\quad + (33 + 23 + 14 + 9) \\ &\quad - 4. \end{aligned}$$

Computing

$$500 + 333 + 200 + 142 = 1175,$$

$$166 + 100 + 71 + 66 + 47 + 28 = 478,$$

$$33 + 23 + 14 + 9 = 79.$$

So,

$$|A_2 \cup A_3 \cup A_5 \cup A_7| = 1175 - 478 + 79 - 4.$$

Evaluating:

$$\begin{aligned} 1175 - 478 &= 697, \\ 697 + 79 &= 776, \\ 776 - 4 &= 772. \end{aligned}$$

The count of numbers **not** divisible by 2, 3, 5, or 7 is:

$$1000 - 772 = 228.$$



3.8 Prob 4

Question

Prove the equation

$$D(n) = n! - nD(n-1) - \binom{n}{2}D(n-2) - \cdots - \binom{n}{n-1}D(1) - 1.$$

Proof

Suppose a permutation of $\{1, 2, \dots, n\}$ has exactly i fixed points. There are $\binom{n}{i}$ ways to choose these fixed points. The remaining $n - i$ elements must form a derangement, which can be done in $D(n - i)$ ways.

Therefore, the total number of permutations of n elements can be written as

$$\sum_{i=0}^n \binom{n}{i} D(n-i) = n!.$$

Let $j = n - i$. Then as i runs from 0 to n , so does j . Since the binomial coefficient is symmetric

$$\sum_{j=0}^n \binom{n}{j} D(j) = n!.$$

Expanding

$$\binom{n}{0}D(0) + \binom{n}{1}D(1) + \cdots + \binom{n}{n-1}D(n-1) + \binom{n}{n}D(n) = n!.$$

Since $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$, and $D(0) = 1$, we have:

$$1 \cdot 1 + \binom{n}{1}D(1) + \cdots + \binom{n}{n-1}D(n-1) + 1 \cdot D(n) = n!.$$

Solve for $D(n)$ by subtracting the contributions of $D(0), D(1), \dots, D(n-1)$ from both sides:

$$D(n) = n! - \left[\binom{n}{1}D(1) + \binom{n}{2}D(2) + \cdots + \binom{n}{n-1}D(n-1) + \binom{n}{0}D(0) \right].$$

Since $\binom{n}{0}D(0) = 1$

$$D(n) = n! - \binom{n}{1}D(n-1) - \binom{n}{2}D(n-2) - \cdots - \binom{n}{n-1}D(1) - 1.$$

