# Math 244

## PSET 3

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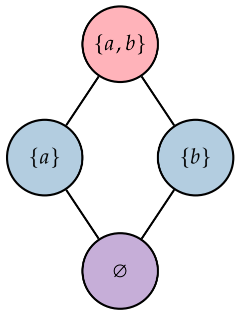
# Contents

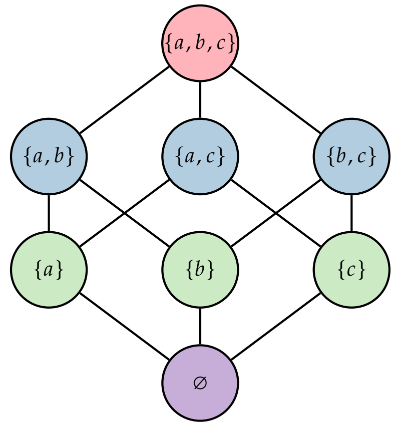
## 2.3 Prob 1

**Question**

How many linear extensions of $\mathcal{B}_2$ are there, and what about $\mathcal{B}_3$?




**$\mathbf{B_2}$**




**$\mathbf{B_3}$**

**Remark** For Prob 1

For $\mathcal{B}_2$ there are two linear extensions because the linear extension must respect and preserve the original partial order so the empty set must be first and the set $\{a, b\}$ must be at the top which means ther variation in ordering the remaining elements is 2! as those are the elements below $\{a, b\}$ and above the empty set.

For $\mathcal{B}_3$ there are 36 possible linear extensions. Since the linear extension needs to preserve teh original partial order than that means the unique top element and unique bottom element of the set $\{a, b, c\}$ and the empty set must be the last and first element of any linear extension respectively. For the remaining elements the sets with cardinlaties 1 must all come before the sets of cardinality 2 but there are 3 sets of cardinality 1 that can be ordered in 3! ways as there are 3 options for the first one, 2 for the second, and 1 for the last one. This logic also applies to the 3 sets of cardinality of 2 that must all go after the sets of cardinality 1 and before the unique top element of cardinality 3 so there are 3! ways to order those. This makes the total possible linear extensions the result of multiplying the possibilities of each possible way of ordering the elements so $1 \cdot 3! \cdot 3! \cdot 1$ which is 36.

## Bonus Problem

**Question**

Prove that not every finite poset admits an embedding into the poset $(\mathbb{N}^2, \preceq)$, where $(x_1, y_1) \preceq (x_2, y_2)$ if and only if $x_1 \leq x_2$ and $y_1 \leq y_2$.

## 2.4 Prob 3

**Question**

Find a sequence of real numbers of length 16 that contains no monotone subsequence of length 5.

| | | | | |
|---|---|---|---|---|
| $i=1$ | 4 | 8 | 12 | 16 |
| $i=2$ | 3 | 7 | 11 | 15 |
| $i=3$ | 2 | 6 | 10 | 14 |
| $i=4$ | 1 | 5 | 9 | 13 |
| | $j=1$ | $j=2$ | $j=3$ | $j=4$ |

Reading the cells row by row (from top row to bottom row) gives the sequence:

$$4, 8, 12, 16, \quad 3, 7, 11, 15, \quad 2, 6, 10, 14, \quad 1, 5, 9, 13.$$

**Example** (For Prob 2)

Array: [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16]

## 2.4 Prob 4

**Question**

Prove the following strengthening of Theorem 2.4.6: Let $k, \ell$ be natural numbers. Then every sequence of real numbers of length $k\ell + 1$ contains a nondecreasing subsequence of length $k + 1$ or a decreasing subsequence of length $\ell + 1$.

**Proof**

Let a sequence $(x_1, x_2, \ldots, x_{kl+1})$ be a sequence of real numbers and let $X = \{1, 2, \ldots, kl + 1\}$.
Define a relation $\preceq$ on $X$ by

$$i \preceq j \quad iff \quad i \leq j \text{ and } x_i \leq x_j$$

**Claim**

$\preceq$ is an ordering

reflexive: $i \preceq i$ for all $i$ since $i \leq i$ and $x_i \leq x_i$

antisymetric: Suppose $i \preceq j$ and $j \preceq i$. Then $i \leq j$ and $j \leq i$ and $x_i \leq x_j$ and $x_j \leq x_i$.
implies $i = j$ and $x_i = x_j$

transitive: suppose $i \preceq j$ and $j \preceq k$. Want to show $i \preceq k$.
$i \preceq j$ implies $i \leq j$ and $x_i \leq x_j$.
$j \preceq k$ implies $j \leq k$ and $x_j \leq x_k$. But then $i \leq k$ and $x_i \leq x_k$.
which implies $i \preceq k$

by the theorem that for every finite ordered set $P = (Xm \preceq)$ we have

$$\alpha(P) \cdot \omega(P) \geq |X|$$

we then have

$$\alpha(P) \cdot \omega(P) \geq |kl + 1|$$

So either

1. $\alpha(X, \preceq) > l + 1$

2. $\omega(X, \preceq) > k+1$

If 1, then $(X, \preceq)$ has an independent set of size $l+1$
We then have

$$i_1 < i_2, \ldots, i_l + 1$$

$$x_{i1} \geq x_{i_2}, \geq, x_{i_{k+1}}$$

If 2, then $(X, \preceq)$ has a chain for length $k+1$ We then have

$$i_1 < i_2, \ldots, i_k + 1$$

$$x_{i1} \leq x_{i_2}, \ldots, x_{i_{k+1}}$$

## 3.1 Prob 2

**Question**

Determine the number of ordered pairs $(A, B)$, where $A \subseteq B \subseteq \{1, 2, \ldots, n\}$.

## 3.1 Prob 6

**Question**

Show that a natural number $n \geq 1$ has an odd number of divisors (including 1 and itself) if and only if $\sqrt{n}$ is an integer. *The textbook has a hint to this problem in the back.*