

Math 244

PSET 1

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## Section 1.2 Problem 5

### Question: Section 1.2 Problem 5

Is a "cancellation" possible for the Cartesian Product? That is, if  $X \times Y = X \times Z$  holds for some sets,  $X$ ,  $Y$ , and  $Z$ , does it follow that  $Y = Z$ ?

### Remark What is the Cartesian Product?

The Cartesian product of  $X$  and  $Y$  is the set of all ordered pairs of the form  $(x, y)$ , where  $x \in X$  and  $y \in Y$ .



### Remark Answer

The "cancellation" is not possible for the Cartesian Product unless it is stated that  $X$  is not an empty set. For if  $X$  is an empty set, then the Cartesian Product of  $X$  and another set would always be the empty set. In this scenario,  $Y$  and  $Z$  could be different and their Cartesian Products with  $X$  would still be the empty set.

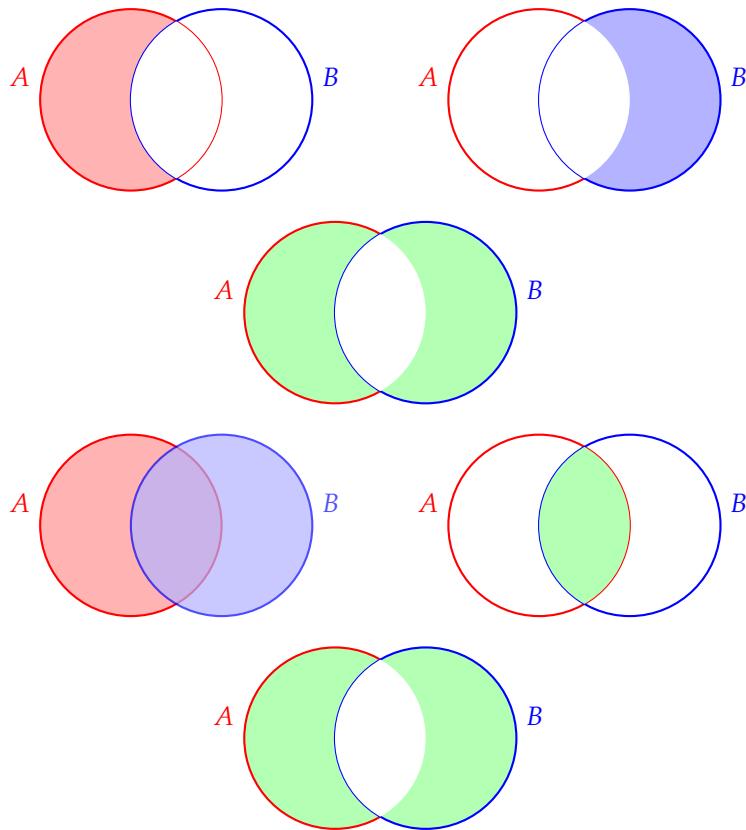


## Section 1.2 Problem 6

### Question: Section 1.2 Problem 6

Prove that for any two sets  $A$ ,  $B$  we have

$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$$



**Claim** Claim 1 For Problem 2

If  $x \in (A \setminus B) \cup (B \setminus A)$ , then  $x \in A \setminus B$  or  $x \in B \setminus A$ .

**Remark 1**

If  $x \in A \setminus B$ , then  $x \in A$  and  $x \notin B$ . This also implies  $x \in A \cup B$  but  $x \notin A \cap B$ , so  $x \in (A \cup B) \setminus (A \cap B)$ . If  $x \in B \setminus A$ , then  $x \in B$  and  $x \notin A$ , which implies  $x \in A \cup B$  but  $x \notin A \cap B$ , so  $x \in (A \cup B) \setminus (A \cap B)$ .

This proves that:

$$(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B).$$

**Claim** Claim 2 For Problem 2

If  $x$  is in  $(A \cup B) \setminus (A \cap B)$ , then  $x \in A \cup B$  and  $x \notin A \cap B$ .

**Remark 2**

If  $x \in A \cup B$  but  $x \notin A \cap B$ , then  $x$  must belong to exactly one of  $A$  or  $B$ .

If  $x \in A$  but  $x \notin B$ , then  $x \in A \setminus B$ .

If  $x \in B$  but  $x \notin A$ , then  $x \in B \setminus A$ .

Therefore,  $x \in (A \setminus B) \cup (B \setminus A)$ .

This proves that:

$$(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A).$$



## Section 1.3 Problem 2

### Question: Section 1.3 Problem 2

The numbers  $F_0, F_1, F_2, F_3, \dots$  are defined as follows:

$$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n \text{ for } n = 0, 1, 2, \dots$$

Prove that for any  $n \geq 0$  we have  $F_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$

**Base Case**

The formula  $F_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$  holds for  $n = 0$  since  $F_0 = 0$  and  $\left(\frac{1+\sqrt{5}}{2}\right)^{-1} = \frac{2}{1+\sqrt{5}}$ .

The formula  $F_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$  holds for  $n = 1$  since  $F_1 = 1$  and  $\left(\frac{1+\sqrt{5}}{2}\right)^0 = 1$ .

**Inductive Hypothesis**

Let us suppose that the formula  $F_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$  and  $F_{n+1} \leq \left(\frac{1+\sqrt{5}}{2}\right)^n$  holds for some  $n \geq 0$ .

**Remark The Golden Ratio**

$\phi$  is equal to  $\frac{1+\sqrt{5}}{2}$ .



### Inductive Step

The numbers  $F_0, F_1, F_2, \dots$  are defined as follows:

$$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n \text{ for } n = 0, 1, 2, \dots$$

$$F_{n+2} = F_{n+1} + F_n$$

Use the inductive hypothesis and substitute

$$F_{n+2} \leq \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$$

Substitute phi for  $\left(\frac{1+\sqrt{5}}{2}\right)$

$$F_{n+2} = F_{n+1} + F_n \leq \phi^n + \phi^{n-1}$$

$$\phi^n + \phi^{n-1} = \phi^{n-1}(\phi + 1)$$



### Key Idea

By the principle of mathematical induction, the statement  $P(n)$  is true for all  $n \geq 1$ .



## 1.4 Problem 2

### Question: Section 1.4 Problem 2

Find an example of:

- A one-to-one function  $f : \mathbb{N} \rightarrow \mathbb{N}$  that is not onto.
- A function  $f : \mathbb{N} \rightarrow \mathbb{N}$  that is onto but not one-to-one.

## 1.4 Problem 6

### Question

Prove that the following two statements about a function  $g_1 : Z \rightarrow X$  and  $g_2 : Z \rightarrow X$  the composed functions  $f \circ g_1$  and  $f \circ g_2$  are also distinct.