Math 244

PSET 2

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Contents

Section 1.5 Problem 5

Question

Prove the associativity of composing relations: If R, S, and T are relations such that $(R \circ S) \circ T$ is well defined, then $R \circ (S \circ T)$ is well defined and equal to $(R \circ S) \circ T$.

Proof

Suppose (w, z) lies in $(R \circ S) \circ T$
If $(w, y) \in R \circ S$, there is some x such that

$$(w, x) \in R \quad \text{and} \quad (x, y) \in S.$$

Since we also have $(y, z) \in T$, so $(x, z) \in S \circ T$.
We now have $(w, x) \in R$ and $(x, z) \in S \circ T$, which means that $(w, z) \in R \circ (S \circ T)$.
This means

$$(R \circ S) \circ T \subseteq R \circ (S \circ T).$$

Now, for the reverse inclusion, suppose $(w, z) \in R \circ (S \circ T)$. Then there exists some x such that

1 $(w, x) \in R$, and

2 $(x, z) \in S \circ T$.

Since $(x, z) \in S \circ T$, there is some y such that

$$(x, y) \in S \quad \text{and} \quad (y, z) \in T.$$

Since $(w, x) \in R$ and $(x, y) \in S$, we conclude that $(w, y) \in R \circ S$.
Since we also have $(y, z) \in T$, that means $(w, z) \in (R \circ S) \circ T$.

So

$$R \circ (S \circ T) \subseteq (R \circ S) \circ T.$$

So

$$(R \circ S) \circ T = R \circ (S \circ T).$$

Section 1.6 Problem 3

Question

Prove that a relation R is transitive if and only if $R \circ R \subseteq R$.

Key Idea

We need to show

$\Rightarrow$ If R is transitive, then, $R \circ R \subseteq R$

$\Leftarrow$ If $R \circ R \subseteq R$, then R is transitive.

Remark $\Rightarrow$

We assume that R is transitive.
By def or relational composition we know that for a pair $(a, c) \in R \circ R$ there exists some b $s.t.$

$$(a, b) \in R \quad \text{and} \quad (b, c) \in R$$

Since $(a, b) \in R$ and $(b, c) \in R$, and R is transitive

$$(a, c) \in R$$

So every pair in $R \circ R \subseteq R$

Remark $\Leftarrow$

We asume $R \circ R \subseteq R$
R is transitive *iff* for all a, b, c

$$(a, b) \in R \quad \text{and} \quad (b, c) \in R$$

so,

$$(a, c) \in R$$

Now we suppose $(a, b) \in R$ and (b, c) in R which would make b the intermediate element so

$$(a, c) \in R \circ R$$

from our assumption any pair in $R \circ R$ has to be in R

$$(a, c) \in R$$

Section 1.6 Problem 6

Question

Describe all relations on a set X that are equivalences and orderings at the same time.

Remark For Prob 3

For this to be the case reflexivity and transitivity are required but for equivalence symmetry is required while for ordering antisymmetry is required. IF R is symmetric and antisymmetric, for any $x, y \in X$

1 xRy and yRx for symmetry

2 $x = y$ for antisymmetry

Meaning that if xRy holds than x and y have to be the same element. So really the relation is relating an element to itself

$$R = \{(x, x) : x \in X\}$$

Section 2.1 Problem 4

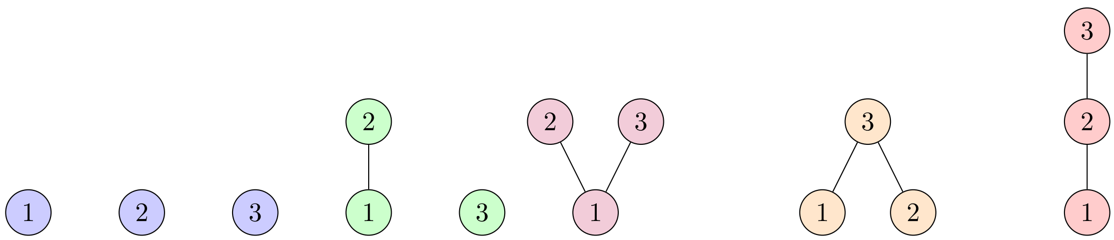
Question

Let $(X, \le)$, $(Y, \le)$ be ordered sets. We say that they are *isomorphic* if there exists a bijection $f : X \to Y$ such that for every $x, y \in X$, we have $x \le y$ if and only if $f(x) \le f(y)$.

a) Draw Hasse diagrams for all non-isomorphic ordered sets with 3 elements posets.

b) Prove that any two n-elements linearly ordered sets are isomorphic.

Prob 4.a



Prob 4.b

Key Idea

We want to show that if $(X, \le)$ and $(Y, \le)$ are two linearly ordered sets with n elements, then they must be isomorphic.

Base Case

$n = 1$
If X and Y both have exactly one element, say $X = \{x\}$ and $Y = \{y\}$, we can define the function

$$h(x) = y.$$

This is clearly a bijection since both sets have only one element, and there is no ordering to worry about. Since order is trivially preserved, X and Y are isomorphic.

Inductive Hypothesis

Assume that for any two linearly ordered sets with $n-1$ elements, there exists an order-preserving bijection (an isomorphism) between them.

Inductive Step

Suppose X and Y are two linearly ordered sets with n elements. Since the order is total, each set has a unique largest element. Let

$$x_{\max} = \text{largest element of } X, \quad y_{\max} = \text{largest element of } Y.$$

Now, consider the subsets

$$X' = X \setminus \{x_{\max}\}, \quad Y' = Y \setminus \{y_{\max}\}.$$

These are both linearly ordered sets with $n-1$ elements. By the inductive hypothesis, there exists an isomorphism

$$g : X' \to Y'.$$

Let $f : X \longrightarrow Y$ be

$$f(x) = \begin{cases} g(x) & \text{if } x \in X', \\ y_{\max} & \text{if } x = x_{\max}. \end{cases}$$

Since g is a bijection from X' to Y' and $y_{\max}$ is not in the image of g, the function f is one-to-one and onto.

If both x_1 and x_2 are in X', then $f(x_1) = g(x_1)$ and $f(x_2) = g(x_2)$, and since g is order preserving, the order is maintained.
If $x_1 \in X'$ and $x_2 = x_{\max}$, then in X we have $x_1 \leq x_{\max}$. In Y, since $y_{\max}$ is the largest, $g(x_1) \leq y_{\max}$, so the order is preserved.
Finally, if both $x_1 = x_{\max}$ and $x_2 = x_{\max}$, the order obviously holds.

Since f is a bijection and it preserves the order, it is an isomorphism.

By the principle of induction, the claim holds for all n.

Section 2.2 Problem 3

Question

a) Consider the set $\{1, 2, \ldots n\}$ ordered by the divisibility relation $|$. What is the maximum possible number of elements of a set $X \subseteq \{1, 2, \ldots n\}$ that is ordered linearly by the relation $|$

b) Solve the same question for the set $2^{\{1,2,\ldots n\}}$ ordered by the inclusion relation $\subseteq$.

Prob 5.a

Remark Prob 5.a

Since the smallest prime number is 2 then that means that each subsequent value after the first must be at minimum 2 times the previous making the longest divisibility chain $\lfloor \log_2(n) \rfloor + 1$. The plus 1 is because we can include the number 1 in the divisibility chain but we also have to floor the function because we can't have a chain with a non integer amount of elements.

Prob 5.b

Remark Prob 5.b

A chain in $2^{\{1,2,\ldots n\}}$ would be a collection of subsets $S_1 \subset S_2 \subset \ldots \subset S_k$ If two distinct subsets have the same number of elements, then none could be a proper subset of the other. So the sizes of subsequent subsets must be strictly increasing.

Since any subset of $\{1, 2, \ldots, n\}$ has a cardinality between 0 and n, there can be at most $n + 1$ different cardinalities.

Optionl Bonus Problem

Question

Let $\text{le}(X, \preceq)$ denote teh number of linear extensions of a partially ordered set $(X, \preceq)$. Prove:

a) $\text{le}(X, \preceq) = 1$ if and only if $(X, \preceq)$ is a linear ordering.

b) $\text{le}(X, \preceq) \leq n!$, where $n = |X|$

Bonus Prob Part a

Proof

($\Rightarrow$) Suppose that $\text{le}(X, \preceq) = 1$.

Assume, for contradiction, that $(X, \preceq)$ is not a linear order. Then there exist two distinct elements $a, b \in X$ such that neither $a \preceq b$ nor $b \preceq a$ holds. But in linear extensions every pair of elements must be comparable. So two cases

1. One linear extension where a comes before b
2. or one where b comes before a .

This would make two distinct linear extensions, contradicting the assumption. $\preceq$ must be a total order.

($\Leftarrow$) Suppose that $(X, \preceq)$ is a linear total order. We need to show that $\text{le}(X, \preceq) = 1$.

Since $\preceq$ is already a total order, it already orders every pair of elements in X. The only possible linear extension is the order $\preceq$ itself. So, there is exactly one linear extension. $\text{le}(X, \preceq) = 1$.

Bonus Prob Part B

Proof

A linear extension of $(X, \preceq)$ is a total order on X that respects the partial order. A total order on a finite set X with n elements would be a permutation of X and there are $n!$ permutations of a set with n elements. But not every permutation of X will be a linear extension of $(X, \preceq)$ because the permutation must respect the partial order $\preceq$. So, the set of linear extensions is a subset of the set of all permutations of X.

$$\text{le}(X, \preceq) \leq n!$$