Math 244

PSET 1

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Section 1.2 Problem 5

Question: Section 1.2 Problem 5

Is a "cancellation" possible for the Cartesion Product? That is, if $X \times Y = X \times Z$ holds for some sets, X, Y, and Z, does it follow that $Y = Z$?

Remark What is the Cartesian Product?

The Cartesian product of X and Y is the set of all ordered pairs of the form (x, y), where $x \in X$ and $y \in Y$.

Remark Asnwer

The "cancellation" is not possible for the Cartesian Product unless it is stated that X is not an empty set. For if X is an empty set, then the Cartesian Product of X and another set would always be the empty set. In this scenario, Y and Z could be different and their Cartesian Products with X would still be the empty set.

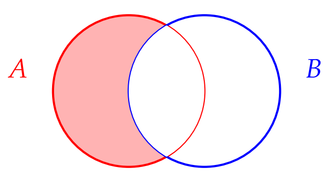
Section 1.2 Problem 6

Question: Section 1.2 Problem 6

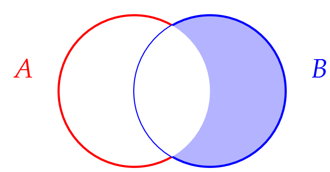
Prove that for any two sets A, B we have

$$(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$$

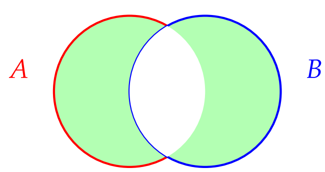
$A \setminus B$



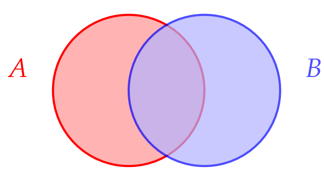
$B \setminus A$



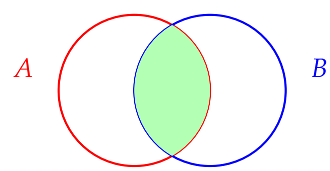
$(A \setminus B) \cup (B \setminus A)$



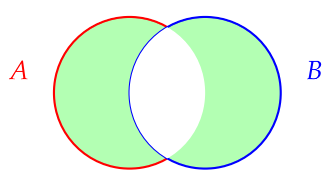
$A \cup B$



$A \cap B$



$(A \cup B) \setminus (A \cap B)$



Claim Claim 1 For Probelm 2

If $x \in (A \setminus B) \cup (B \setminus A)$, then $x \in A \setminus B$ or $x \in B \setminus A$.

Remark 1

If $x \in A \setminus B$, then $x \in A$ and $x \notin B$. This also implies $x \in A \cup B$ but $x \notin A \cap B$, so $x \in (A \cup B) \setminus (A \cap B)$.
If $x \in B \setminus A$, then $x \in B$ and $x \notin A$, which implies $x \in A \cup B$ but $x \notin A \cap B$, so $x \in (A \cup B) \setminus (A \cap B)$.

This proves that:

$$(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)\,.$$

Claim Claim 2 For Probelm 2

If x is in $(A \cup B) \setminus (A \cap B)$, then $x \in A \cup B$ and $x \notin A \cap B$.

Remark 2

If $x \in A \cup B$ but $x \notin A \cap B$, then x must belong to exactly one of A or B.
If $x \in A$ but $x \notin B$, then $x \in A \setminus B$.
If $x \in B$ but $x \notin A$, then $x \in B \setminus A$.

Therefore, $x \in (A \setminus B) \cup (B \setminus A)$.
This proves that:

$$(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)\,.$$

Section 1.3 Problem 2

Question: Section 1.3 Problem 2

The numbers F_0, F_1, F_2, F_3, ... are defined as follows:

$$F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n \text{for } n = 0, 1, 2, \ldots$$

Prove that for any $n \geq 0$ we have $F_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$

Base Case

The formula $F_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$ holds for $n = 0$ since $F_0 = 0$ and $\left(\frac{1+\sqrt{5}}{2}\right)^{-1} = \frac{2}{1+\sqrt{5}}$.

The formula $F_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$ holds for $n = 1$ since $F_1 = 1$ and $\left(\frac{1+\sqrt{5}}{2}\right)^{0} = 1$.

Inductive Hypothesis

Let us suppose that the formula $F_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$ and $F_{n+1} \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n}$ holds for some $n \geq 0$.

Remark The Golden Ratio

ϕ is equal to $\frac{1+\sqrt{5}}{2}$.

Inductive Step

Using the Fibonacci recurrence relation $F_{n+2} = F_{n+1} + F_n$ and the inductive hypothesis:

$$F_{n+2} \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}.$$

Factor out $\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$:

$$F_{n+2} \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \left(\frac{1+\sqrt{5}}{2} + 1\right).$$

Since $\frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^{2}$, we have:

$$F_{n+2} \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{2} = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1}.$$

$\therefore \forall n \geq 0$

$$F_n \leq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$$

Bonus Problem

Question: Bonus Problem

In ancient Egypt, fractions were written as sums of fractions with numerator 1. For instance, $\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$. Consider the following algorithm for writing a fraction $\frac{m}{n}$ in this form ($1 \leq m < n$): write the fraction $\frac{1}{\lceil n/m \rceil}$, calculate the fraction $\frac{m}{n} - \frac{1}{\lceil n/m \rceil}$, and if it is nonzero repeat the same step. Prove that this algorithm always finishes in a finite number of steps.

Claim For Bonus Problem

For every fraction $\frac{m}{n}$, with integers m and n satisfying $1 \leq m \leq n$ the algorithm will terminate in a finite number of steps.

Remark 1

Let $k = \lceil \frac{m}{n} \rceil$.

Base Case

$m = 1$

Here, $\frac{m}{n} = \frac{1}{n}$

$$k = \lceil \frac{n}{1} \rceil = n$$

$$\frac{1}{n} - \frac{1}{n} = 0$$

When m = 1 the algorithm terminates

Inductive Hypothesis

For every smaller numerator ℓ ($1 \leq \ell < m$), and for every denominator n' ($\ell < n'$), the fraction $\frac{\ell}{n'}$ can be completely reduced to a sum of unit fractions in finite steps.

Remark 2

Let $k = \lceil \frac{m}{n} \rceil$.
The fraction $\frac{m}{n} - \frac{1}{k}$ is:

$$\frac{m}{n} - \frac{1}{k} = \frac{mk - n}{kn}$$

Since $m < n$, $mk - n < n^2 - n = n(n-1)$.
Therefore, $mk - n < kn$.

1.4 Problem 2

Question: Section 1.4 Problem 2

Find an example of:

(a) A one-to-one function $f : \mathbb{N} \to \mathbb{N}$ that is not onto.

(b) A function $f : \mathbb{N} \to \mathbb{N}$ that is onto but not one-to-one.

Example (Example One-to-One Function)

We could make a function $f(x) = 2x$ with the domain $\mathbb{N}$. This is one-to-one because for every x there is a unique $2x$. However, this function is not onto because there are are inputs x that are impossible to receive as results of $f(x)$. For example the number 1 is impossible to get from $f(x) = 2x$ given the domain.

Example (Example Onto Function)

We could make a function $f(x) = \left\lceil \frac{x}{2} \right\rceil$ with the domain $\mathbb{N}$. This function is onto because for every x there is a unique $\left\lceil \frac{x}{2} \right\rceil$. However, this function is not one-to-one because there are multiple inputs x that map to the same output. For example, $f(1) = f(2) = 1$.

1.4 Problem 6

Question

Prove that the following two statements about a function $f : X \to Y$ are equivalent:

(i) f is one-to-one.

(ii) For any set Z and any two distinct functions $g_1 : Z \to X$ and $g_2 : Z \to X$ the composed functions $f \circ g_1$ and $f \circ g_2$ are distinct.

Claim Statement of Claim

Assume $f : X \to Y$ is one-to-one .

Key Idea

If $g_1 \neq g_2$, then there exists some $z \in Z$ such that $g_1(z) \neq g_2(z)$, since two functions are distinct if and only if they differ in at least one input.

Remark Argument

Since f is one-to-one, $g_1(z) \neq g_2(z)$ implies $f(g_1(z)) \neq f(g_2(z))$. At the input z $(f \circ g_1)(z) \neq (f \circ g_2)(z)$.

Claim Statement of Claim

Assume that for any set Z and any two distinct functions $g_1, g_2 : Z \longrightarrow X$, the compositions $f \circ g_1$ and $f \circ g_2$ are distinct.

Contrapositive Contrapositive Approach

Assume f is not one-to-one. Then there exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.

Remark Counterexample Construction

Let $Z = \{1, 2\}$. Definition of g_1 and g_2:

$$g_1(1) = x_1, \quad g_1(2) = x_2, \quad g_2(1) = x_2, \quad g_2(2) = x_1.$$

Here, $g_1 \neq g_2$ because they differ on at least one input.

$$g_1(1) \neq g_2(1), \quad \text{and} \quad g_1(2) \neq g_2(2).$$

Since $f(x_1) = f(x_2)$, we have:

$$(f \circ g_1)(1) = f(g_1(1)) = f(x_1), \quad (f \circ g_2)(1) = f(g_2(1)) = f(x_2),$$

$$(f \circ g_1)(2) = f(g_1(2)) = f(x_2), \quad (f \circ g_2)(2) = f(g_2(2)) = f(x_1).$$

Thus, $(f \circ g_1)(z) = (f \circ g_2)(z)$ for all $z \in Z$, implying $f \circ g_1 = f \circ g_2$.
This contradicts the assumption that $f \circ g_1 \neq f \circ g_2$ whenever $g_1 \neq g_2$. Therefore, f must be one-to-one.