Math 244

PSET 4

Feb 21 2025

Contents

3.2 Prob 1

Question

How many permutations of $\{1, 2, \ldots, n\}$ have a single cycle?

Proof

There are $n!$ ways to arrange n distinct elements in a sequence. Any sequence

$$(a_1, a_2, \ldots, a_n)$$

can be thought of as a candidate for an n-cycle when interpreted in cyclic notation as

$$(a_1, a_2, \ldots, a_n).$$

But we can't count cycles where the order is simply rotated cyclically

$$(a_1, a_2, \ldots, a_n)$$

is the same as the cycle

$$(a_2, a_3, \ldots, a_n, a_1),$$

there are exactly n different ways to write the same cycle by cyclically rotating the elements.

If we count every sequence as a distinct cycle, we overcount each n-cycle n times.

To deal with the overcounting we divide the total number of sequences by n.

$$\frac{n!}{n} = (n-1)!.$$

3.2 Prob 2

Question

For a permutation $p : X \longrightarrow X$, let p^k denote the permutation arising by a k-fold composition of p, i.e., $p^1 = p$ and $p^k = p \circ p^{k-1}$. Define a relation $\approx$ on the set X as follows: $i \approx j$ if and only if there exists a $k \geq 1$ such that $p^k(i) = j$. Prove that $\approx$ is an equivalence relation on X, and that its classes are the cycles of p.

Proof

Want to show that $\approx$ satisfies the three properties of an equivalence relation: reflexivity, symmetry, and transitivity.

For any $i \in X$, since p is a permutation meaning its bijective, i must belong to some cycle. So, there exists a smallest positive integer $k \geq 1$ such that

$$p^k(i) = i.$$

By the definition of $\approx$, we have $i \approx i$. So, $\approx$ is reflexive.

Suppose $i \approx j$. Then there exists an integer $k \geq 1$ such that

$$p^k(i) = j.$$

Since p is a permutation, it has an inverse p^{-1}. Applying p^{-k} to both sides, we get

$$i = p^{-k}(j).$$

Since this is also a permutation, there exists some positive integer m such that $p^m(j) = i$. Thus, $j \approx i$, showing that $\approx$ is symmetric.

Suppose $i \approx j$ and $j \approx k$. Then there exist positive integers m and n such that

$$p^m(i) = j \quad \text{and} \quad p^n(j) = k.$$

Now, consider the composition:

$$p^{m+n}(i) = p^n(p^m(i)) = p^n(j) = k.$$

Since $m + n \geq 1$, it follows that $i \approx k$. So, $\approx$ is transitive.

In cycle notation, the cycle containing an element i is defined as:

$$\{i,\, p(i),\, p^2(i),\, \ldots,\, p^{k-1}(i)\},$$

where k is the smallest positive integer such that $p^k(i) = i$.

By our definition of $\approx$, an element $j \in X$ is related to i if and only if there exists some $k \geq 1$ such that $p^k(i) = j$. This is exactly the description of the cycle of i. So, the equivalence class of i under $\approx$ is precisely the set of all elements that can be reached from i by some power of p, which corresponds to the cycle of i.

3.3 Prob 7

Question

How many functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ are there that are *monotonic*; that is, for $i < j$ we have $f(i) \leq f(j)$? *The textbook has a hint to this problem in the back.*

Proof

A monotonic function

$$f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$

satisfies

$$f(1) \leq f(2) \leq \cdots \leq f(n).$$

So, the function is completely determined by the non-decreasing sequence

$$(f(1), f(2), \ldots, f(n)),$$

where each $f(i)$ is in $\{1, 2, \ldots, n\}$.
Define x_k as the number of times the value k appears in the sequence. Then we have

$$x_1 + x_2 + \cdots + x_n = n, \quad \text{with} \quad x_k \geq 0.$$

The number of solutions to this equation in nonnegative integers is given by the stars and bars formula:

$$\binom{n + n - 1}{n - 1} = \binom{2n - 1}{n - 1}.$$

3.3 Prob 21

Question

(optional bonus problem) Draw a triangle ABC. Draw n points lying on the side AB (but different from A and B) and connect them all by segments to the vertex C. Similarly, draw n points on the side AC and connect them to B.

1. How many intersections of the drawn segments are there? Into how many regions is the triangle ABC partitioned by the drawn segments?
2. Draw n points on the side BC and connect them to A. Assume that no 3 of the drawn segments intersect at a single point. How many intersections are there now?
3. How many regions are there in the situation of (b)?

Bonus Prob Part 1

Remark Part a

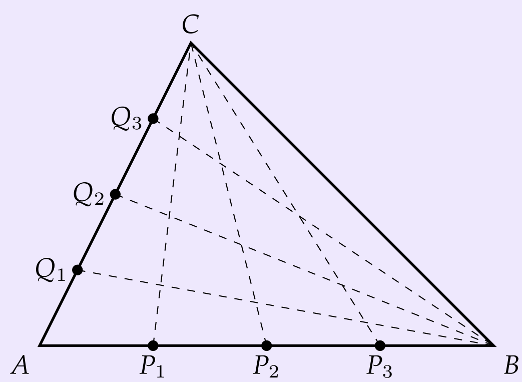
First, place n points $(P_1, P_2, \ldots, P_n)$ on side AB and n points $(Q_1, Q_2, \ldots, Q_n)$ on side AC. Draw segments CP_i for $i = 1, \ldots, n$ and segments BQ_j for $j = 1, \ldots, n$. Since every segment from C to a point on AB meets every segment from B to a point on AC exactly once, there are

$$n \times n = n^2$$

intersections. These segments divide the triangle into a grid-like pattern having $(n + 1)$ rows and $(n + 1)$ columns, so the total number of regions is

$$(n + 1)^2.$$

Diagram for (a):



Bonus Prob Part 2

Remark For Prob 2 and 3

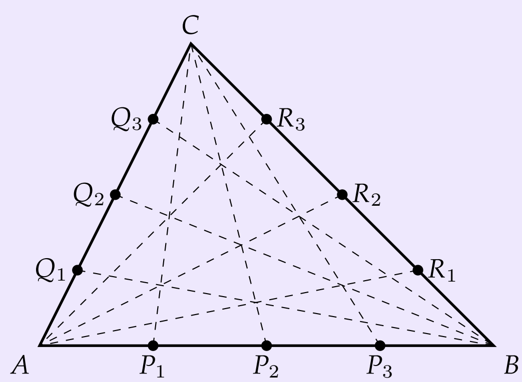
Now, we add n points ($R_1, R_2, \ldots, R_n$) on side BC and draw segments AR_i from each to vertex A. Under the assumption that no three segments are concurrent, each new segment AR_i intersects every segment from C to AB making n segement and every segment from B to AC making n segements. So, each segment AR_i contributes $2n$ new intersections. Since there are n such segments, the additional intersections amount to

$$n \cdot (2n) = 2n^2.$$

We add these to the orginal ^{2} intersections

$$n^2 + 2n^2 = 3n^2.$$

Diagram for (b) and (c):



Bonus Prob Part 3

Remark For part 3

When a new segment is added, if it is intersected k times, it is divided into $k+1$ pieces—each piece increasing the number of regions by 1. Each segment AR_i is intersected $2n$ times and splits into $2n+1$ pieces. So, these n segements add

$$n(2n+1)$$

regions. The total number of regions is

$$(n+1)^2 + n(2n+1) = 3n^2 + 3n + 1.$$

3.7 Prob 3

Question

(Sieve of Eratosthenes) How many numbers are left in the set $\{1, 2, \ldots, 1000\}$ after all multiples of 2, 3, 5, and 7 are crossed out?

Remark For Prob 4

Define:

- A_2 as the set of multiples of 2,
- A_3 as the set of multiples of 3,
- A_5 as the set of multiples of 5,
- A_7 as the set of multiples of 7.

We want the size of the complement of $A_2 \cup A_3 \cup A_5 \cup A_7$
We cound the multiples of $2, 3, 5, 7$ in the range of [1,1000].

$$|A_2| = \left\lfloor \frac{1000}{2} \right\rfloor = 500, \quad |A_3| = \left\lfloor \frac{1000}{3} \right\rfloor = 333, \quad |A_5| = \left\lfloor \frac{1000}{5} \right\rfloor = 200, \quad |A_7| = \left\lfloor \frac{1000}{7} \right\rfloor = 142.$$

count of multiples for pairs using least common multiple:

$$|A_2 \cap A_3| = \left\lfloor \frac{1000}{6} \right\rfloor = 166, \quad |A_2 \cap A_5| = \left\lfloor \frac{1000}{10} \right\rfloor = 100, \quad |A_2 \cap A_7| = \left\lfloor \frac{1000}{14} \right\rfloor = 71,$$

$$|A_3 \cap A_5| = \left\lfloor \frac{1000}{15} \right\rfloor = 66, \quad |A_3 \cap A_7| = \left\lfloor \frac{1000}{21} \right\rfloor = 47, \quad |A_5 \cap A_7| = \left\lfloor \frac{1000}{35} \right\rfloor = 28.$$

Count of multiples for triples

$$|A_2 \cap A_3 \cap A_5| = \left\lfloor \frac{1000}{30} \right\rfloor = 33, \quad |A_2 \cap A_3 \cap A_7| = \left\lfloor \frac{1000}{42} \right\rfloor = 23,$$

$$|A_2 \cap A_5 \cap A_7| = \left\lfloor \frac{1000}{70} \right\rfloor = 14, \quad |A_3 \cap A_5 \cap A_7| = \left\lfloor \frac{1000}{105} \right\rfloor = 9.$$

Count of multiples which all four share

$$|A_2 \cap A_3 \cap A_5 \cap A_7| = \left\lfloor \frac{1000}{210} \right\rfloor = 4.$$

Apply the Inclusion-Exclusion Principle
The count of numbers divisible by at least one of $2, 3, 5, 7$ is:

$$\begin{aligned}
|A_2 \cup A_3 \cup A_5 \cup A_7| = |A_2| + |A_3| + |A_5| + |A_7| \\
&- \Big(|A_2 \cap A_3| + |A_2 \cap A_5| + |A_2 \cap A_7| + |A_3 \cap A_5| + |A_3 \cap A_7| + |A_5 \cap A_7|\Big) \\
&+ \Big(|A_2 \cap A_3 \cap A_5| + |A_2 \cap A_3 \cap A_7| + |A_2 \cap A_5 \cap A_7| + |A_3 \cap A_5 \cap A_7|\Big) \\
&- |A_2 \cap A_3 \cap A_5 \cap A_7|.
\end{aligned}$$

$$\begin{aligned}
|A_2 \cup A_3 \cup A_5 \cup A_7| &= 500 + 333 + 200 + 142 \\
&- (166 + 100 + 71 + 66 + 47 + 28) \\
&+ (33 + 23 + 14 + 9) \\
&- 4.
\end{aligned}$$

$$\begin{aligned}
500 + 333 + 200 + 142 &= 1175, \\
166 + 100 + 71 + 66 + 47 + 28 &= 478, \\
33 + 23 + 14 + 9 &= 79.
\end{aligned}$$

$$|A_2 \cup A_3 \cup A_5 \cup A_7| = 1175 - 478 + 79 - 4.$$

$$1175 - 478 = 697, \quad 697 + 79 \quad = 776, \quad 776 - 4 = 772.$$

The count of numbers **not** divisible by 2, 3, 5, or 7 is:

$$1000 - 772 = 228.$$

3.8 Prob 4

Question

Prove the equation

$$D(n) = n! - nD(n-1) - \binom{n}{2}D(n-2) - \cdots - \binom{n}{n-1}D(1) - 1.$$

Proof

Suppose a permutation of $\{1, 2, \ldots, n\}$ has exactly i fixed points. There are $\binom{n}{i}$ ways to choose these fixed points. The remaining $n - i$ elements must form a derangement, which can be done in $D(n-i)$ ways.

Therefore, the total number of permutations of n elements can be written as

$$\sum_{i=0}^{n} \binom{n}{i} D(n-i) = n!.$$

Let $j = n - i$. Then as i runs from 0 to n, so does j. Since the binomial coefficient is symmetric

$$\sum_{j=0}^{n} \binom{n}{j} D(j) = n!.$$

Expanding

$$\binom{n}{0}D(0) + \binom{n}{1}D(1) + \cdots + \binom{n}{n-1}D(n-1) + \binom{n}{n}D(n) = n!.$$

Since $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$, and $D(0) = 1$, we have:

$$1 \cdot 1 + \binom{n}{1}D(1) + \cdots + \binom{n}{n-1}D(n-1) + 1 \cdot D(n) = n!.$$

Solve for $D(n)$ by subtractracting the contributions of $D(0), D(1), \ldots, D(n-1)$ from both sides:

$$D(n) = n! - \left[\binom{n}{1}D(1) + \binom{n}{2}D(2) + \cdots + \binom{n}{n-1}D(n-1) + \binom{n}{0}D(0)\right].$$

Since $\binom{n}{0}D(0) = 1$

$$D(n) = n! - \binom{n}{1}D(n-1) - \binom{n}{2}D(n-2) - \cdots - \binom{n}{n-1}D(1) - 1.$$