Math 244

PSET 4

Feb 21 2025

Contents

3.2 Prob 1

Question

How many permutations of $\{1, 2, \ldots, n\}$ have a single cycle?

Proof

There are $n!$ ways to arrange n distinct elements in a sequence. Any sequence

$$(a_1, a_2, \ldots, a_n)$$

can be thought of as a candidate for an n-cycle when interpreted in cyclic notation as

$$(a_1, a_2, \ldots, a_n).$$

But we can't count cycles where the order is simply rotated cyclically

$$(a_1, a_2, \ldots, a_n)$$

is the same as the cycle

$$(a_2, a_3, \ldots, a_n, a_1),$$

there are exactly n different ways to write the same cycle by cyclically rotating the elements.

If we count every sequence as a distinct cycle, we overcount each n-cycle n times.

To deal with the overcounting we divide the total number of sequences by n.

$$\frac{n!}{n} = (n-1)!.$$

3.2 Prob 2

Question

For a permutation $p : X \longrightarrow X$, let p^k denote the permutation arising by a k-fold composition of p, i.e., $p^1 = p$ and $p^k = p \circ p^{k-1}$. Define a relation $\approx$ on the set X as follows: $i \approx j$ if and only if there exists a $k \geq 1$ such that $p^k(i) = j$. Prove that $\approx$ is an equivalence relation on X, and that its classes are the cycles of p.

Proof

Want to show that $\approx$ satisfies the three properties of an equivalence relation: reflexivity, symmetry, and transitivity.

For any $i \in X$, since p is a permutation meaning its bijective, i must belong to some cycle. So, there exists a smallest positive integer $k \geq 1$ such that

$$p^k(i) = i.$$

By the definition of $\approx$, we have $i \approx i$. So, $\approx$ is reflexive.

Suppose $i \approx j$. Then there exists an integer $k \geq 1$ such that

$$p^k(i) = j.$$

Since p is a permutation, it has an inverse p^{-1}. Applying p^{-k} to both sides, we get

$$i = p^{-k}(j).$$

Since this is also a permutation, there exists some positive integer m such that $p^m(j) = i$. Thus, $j \approx i$, showing that $\approx$ is symmetric.

Suppose $i \approx j$ and $j \approx k$. Then there exist positive integers m and n such that

$$p^m(i) = j \quad \text{and} \quad p^n(j) = k.$$

Now, consider the composition:

$$p^{m+n}(i) = p^n(p^m(i)) = p^n(j) = k.$$

Since $m + n \geq 1$, it follows that $i \approx k$. So, $\approx$ is transitive.

In cycle notation, the cycle containing an element i is defined as:

$$\{i, p(i), p^2(i), \ldots, p^{k-1}(i)\},$$

where k is the smallest positive integer such that $p^k(i) = i$.

By our definition of $\approx$, an element $j \in X$ is related to i if and only if there exists some $k \geq 1$ such that $p^k(i) = j$. This is exactly the description of the cycle of i. So, the equivalence class of i under $\approx$ is precisely the set of all elements that can be reached from i by some power of p, which corresponds to the cycle of i.

3.3 Prob 7

Question

How many functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ are there that are *monotonic*; that is, for $i < j$ we have $f(i) \leq f(j)$? *The textbook has a hint to this problem in the back.*

Proof

Define a new sequence g by

$$g(i) = f(i) + i - 1, \quad \text{for } i = 1, 2, \ldots, n.$$

Since $f(1) \leq f(2) \leq \cdots \leq f(n)$, it follows that

$$g(1) = f(1) \leq f(2) + 1 = g(2) \leq \cdots \leq f(n) + n - 1 = g(n).$$

Because we are adding the strictly increasing sequence $0, 1, 2, \ldots, n-1$ to $f(i)$, the sequence $g(1), g(2), \ldots, g(n)$ becomes strictly increasing:

$$g(1) < g(2) < \cdots < g(n).$$

Since $f(i) \in \{1, 2, \ldots, n\}$, we obtain:

- For $i = 1$: $g(1) = f(1) \geq 1$.
- For $i = n$: $g(n) = f(n) + n - 1 \leq n + n - 1 = 2n - 1$.

Thus, each $g(i)$ is a distinct element from the set $\{1, 2, \ldots, 2n - 1\}$.

A strictly increasing sequence $g(1) < g(2) < \cdots < g(n)$ chosen from $\{1, 2, \ldots, 2n - 1\}$ is equivalent to choosing an n-element subset from a set of $2n - 1$ elements. There are

$$\binom{2n-1}{n}$$

ways to select such a subset.

Since the transformation $f(i) \mapsto g(i) = f(i) + i - 1$ is a bijection between the set of monotonic functions f and the set of strictly increasing sequences g, the number of monotonic functions is given by:

$$\boxed{\binom{2n-1}{n}}.$$

3.3 Prob 21

Question

(optional bonus problem) Draw a triangle ABC. Draw n points lying on the side AB (but different from A and B) and connect them all by segments to the vertex C. Similarly, draw n points on the side AC and connect them to B.

1. How many intersections of the drawn segments are there? Into how many regions is the triangle ABC partitioned by the drawn segments?
2. Draw n points on the side BC and connect them to A. Assume that no 3 of the drawn segments intersect at a single point. How many intersections are there now?
3. How many regions are there in the situation of (b)?

3.7 Prob 3

Question

(Sieve of Eratosthenes) How many numbers are left in the set $\{1, 2, \dots, 1000\}$ after all multiples of 2, 3, 5, and 7 are crossed out?

Remark For Prob 4

Define:

- A_2 as the set of multiples of 2,
- A_3 as the set of multiples of 3,
- A_5 as the set of multiples of 5,
- A_7 as the set of multiples of 7.

We want the size of the complement of $A_2 \cup A_3 \cup A_5 \cup A_7$
We cound the multiples of $2, 3, 5, 7$ in the range of [1,1000].

$$|A_2| = \left\lfloor \frac{1000}{2} \right\rfloor = 500,$$
$$|A_3| = \left\lfloor \frac{1000}{3} \right\rfloor = 333,$$
$$|A_5| = \left\lfloor \frac{1000}{5} \right\rfloor = 200,$$
$$|A_7| = \left\lfloor \frac{1000}{7} \right\rfloor = 142.$$

count of multiples for pairs using least common multiple:

$$
\begin{aligned}
|A_2 \cap A_3| &= \left\lfloor \frac{1000}{6} \right\rfloor = 166, \\
|A_2 \cap A_5| &= \left\lfloor \frac{1000}{10} \right\rfloor = 100, \\
|A_2 \cap A_7| &= \left\lfloor \frac{1000}{14} \right\rfloor = 71, \\
|A_3 \cap A_5| &= \left\lfloor \frac{1000}{15} \right\rfloor = 66, \\
|A_3 \cap A_7| &= \left\lfloor \frac{1000}{21} \right\rfloor = 47, \\
|A_5 \cap A_7| &= \left\lfloor \frac{1000}{35} \right\rfloor = 28.
\end{aligned}
$$

Count of multiples for triples

$$
\begin{aligned}
|A_2 \cap A_3 \cap A_5| &= \left\lfloor \frac{1000}{30} \right\rfloor = 33, \\
|A_2 \cap A_3 \cap A_7| &= \left\lfloor \frac{1000}{42} \right\rfloor = 23, \\
|A_2 \cap A_5 \cap A_7| &= \left\lfloor \frac{1000}{70} \right\rfloor = 14, \\
|A_3 \cap A_5 \cap A_7| &= \left\lfloor \frac{1000}{105} \right\rfloor = 9.
\end{aligned}
$$

Count of multiples which all four share

$$|A_2 \cap A_3 \cap A_5 \cap A_7| = \left\lfloor \frac{1000}{210} \right\rfloor = 4.$$

Apply the Inclusion-Exclusion Principle
The count of numbers divisible by at least one of $2, 3, 5, 7$ is:

$$
\begin{aligned}
|A_2 \cup A_3 \cup A_5 \cup A_7| = |A_2| + |A_3| + |A_5| + |A_7| \\
&- \Big(|A_2 \cap A_3| + |A_2 \cap A_5| + |A_2 \cap A_7| + |A_3 \cap A_5| + |A_3 \cap A_7| + |A_5 \cap A_7|\Big) \\
&+ \Big(|A_2 \cap A_3 \cap A_5| + |A_2 \cap A_3 \cap A_7| + |A_2 \cap A_5 \cap A_7| + |A_3 \cap A_5 \cap A_7|\Big) \\
&- |A_2 \cap A_3 \cap A_5 \cap A_7|.
\end{aligned}
$$

Plugging in values:

$$
\begin{aligned}
|A_2 \cup A_3 \cup A_5 \cup A_7| &= 500 + 333 + 200 + 142 \\
&- (166 + 100 + 71 + 66 + 47 + 28) \\
&+ (33 + 23 + 14 + 9) \\
&- 4.
\end{aligned}
$$

Computing

$$
\begin{aligned}
500 + 333 + 200 + 142 &= 1175, \\
166 + 100 + 71 + 66 + 47 + 28 &= 478, \\
33 + 23 + 14 + 9 &= 79.
\end{aligned}
$$

So,

$$|A_2 \cup A_3 \cup A_5 \cup A_7| = 1175 - 478 + 79 - 4.$$

Evaluating:

$$\begin{aligned} 1175 - 478 &= 697, \\ 697 + 79 &= 776, \\ 776 - 4 &= 772. \end{aligned}$$

The count of numbers **not** divisible by 2, 3, 5, or 7 is:

$$1000 - 772 = 228.$$

3.8 Prob 4

Question

Prove the equation

$$D(n) = n! - nD(n-1) - \binom{n}{2}D(n-2) - \cdots - \binom{n}{n-1}D(1) - 1.$$